CSC 384 Introduction to Artificial Intelligence

Uncertainty 4
Variable Elimination Algorithm

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Learning Goals

By the end of this lecture, you should be able to

1. Define factors. Manipulate factors using operations restrict, sum out, multiply and normalize.
2. Describe/trace/implement the variable elimination algorithm for calculating a prior or posterior probability given a Bayesian network.

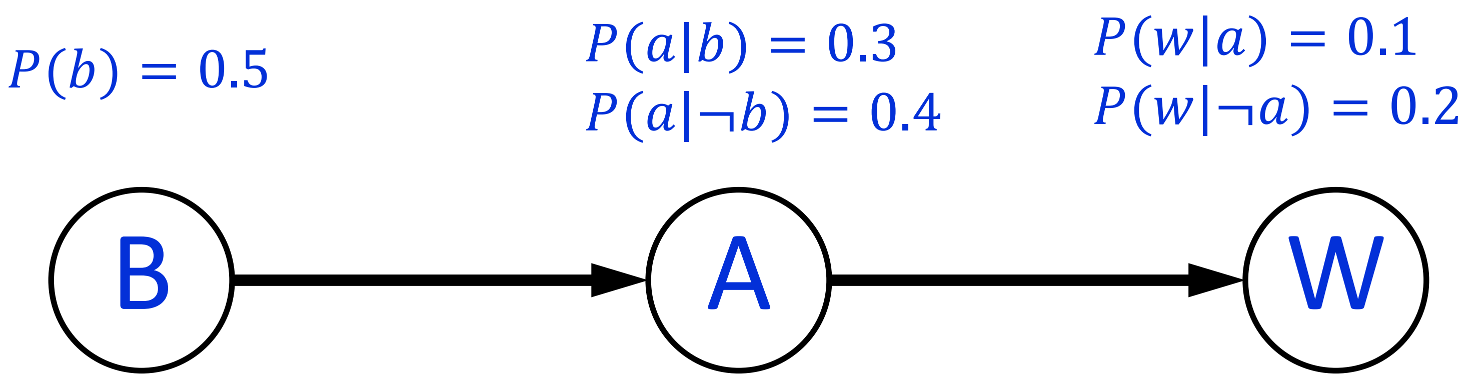
Outline

AN EXAMPLE TO MOTIVATE VEA

An Example to Motivate VEA

Calculate $P(A|\neg w)$

If Dr. Watson does not call
what is the probability that the alarm is ringing?



The Probability Tables

Goal is to calculate $P(A|\neg w)$.

$P(B)$

$P(b)$	0.5
$P(\neg b)$	0.5

$P(A|B)$

$P(a|b)$	0.3
$P(\neg a|b)$	0.7
$P(a|\neg b)$	0.4
$P(\neg a|\neg b)$	0.6

$P(W|A)$

$P(w|a)$	0.1
$P(\neg w|a)$	0.9
$P(w|\neg a)$	0.2
$P(\neg w|\neg a)$	0.8

Restrict W

Goal is to calculate $P(A|\neg w)$.

$P(b)$	0.5
$P(\neg b)$	0.5

$P(a|b)$	0.3
$P(\neg a|b)$	0.7
$P(a|\neg b)$	0.4
$P(\neg a|\neg b)$	0.6

~~$P(w|a)$~~	~~0.1~~
$P(\neg w|a)$	0.9
~~$P(w|\neg a)$~~	~~0.2~~
$P(\neg w|\neg a)$	0.8

$\Rightarrow$

$P(\neg w|a)$	0.9
$P(\neg w|\neg a)$	0.8

Calculate the Joint Probabilities

Goal is to calculate $P(A|\neg w)$.

$$P(B \wedge A \wedge \neg w) = P(B)P(A|B)P(\neg w|A)$$

$P(b \wedge \neg a \wedge \neg w)$

$= P(b)\, P(\neg a|b)\, P(\neg w|\neg a)$

$= 0.5 * 0.7 * 0.8 = 0.28$

$P(b)$	0.5
$P(\neg b)$	0.5

$\times$

$P(a\|b)$	0.3
$P(\neg a\|b)$	0.7
$P(a\|\neg b)$	0.4
$P(\neg a\|\neg b)$	0.6

$\times$

$P(\neg w\|a)$	0.9
$P(\neg w\|\neg a)$	0.8

$\Rightarrow$

$P(b \wedge a \wedge \neg w)$	0.135
$P(b \wedge \neg a \wedge \neg w)$	0.28
$P(\neg b \wedge a \wedge \neg w)$	0.18
$P(\neg b \wedge \neg a \wedge \neg w)$	0.24

Sum Out B

Goal is to calculate $P(A|\neg w)$.

$$P(A \wedge \neg w) = \sum_B P(B)P(A|B)P(\neg w|A)$$

$$P(a \wedge \neg w) = P(b \wedge a \wedge \neg w) + P(\neg b \wedge a \wedge \neg w)$$
$$= 0.135 + 0.18 = 0.315$$

$P(b \wedge a \wedge \neg w)$	0.135
$P(b \wedge \neg a \wedge \neg w)$	0.28
$P(\neg b \wedge a \wedge \neg w)$	0.18
$P(\neg b \wedge \neg a \wedge \neg w)$	0.24

$P(a \wedge \neg w)$	0.315
$P(\neg a \wedge \neg w)$	0.52

Normalize the Probabilities

$$P(a|\neg w) = \frac{P(a \wedge \neg w)}{P(a \wedge \neg w) + P(\neg a \wedge \neg w)} \text{ and } = \frac{0.315}{0.315 + 0.52} = 0.377$$

$$P(\neg a|\neg w) = \frac{P(\neg a \wedge \neg w)}{P(a \wedge \neg w) + P(\neg a \wedge \neg w)} = 1 - P(a|\neg w)$$

$P(a \wedge \neg w)$	0.315
$P(\neg a \wedge \neg w)$	0.52

$P(a|\neg w)$	0.377
$P(\neg a|\neg w)$	0.623

A Summary of the Calculations

$$P(A|\neg w) = \frac{P(A \wedge \neg w)}{P(a \wedge \neg w) + P(\neg a \wedge \neg w)}$$

To calculate $P(A|\neg w)$, it is sufficient to

4) Calculate the joint probabilities $P(A \wedge \neg w)$ and

5) Normalize them.

$$P(A \wedge \neg w) = \sum_B P(B) P(A|B) P(\neg w|A)$$

To calculate $P(A \wedge \neg w)$, we need to

1) Restrict the observed variables,

2) Calculate the joint probability of all variables, and

3) Sum out the hidden variables.

A Summary of the Steps

1) Write down the probability tables.
2) Restrict the observed variables.
3) Calculate the joint probability of all variables.
4) Sum out the hidden variables.
5) Normalize the probabilities.

VARIABLE ELIMINATION ALGORITHM

Variable Elimination Algorithm

To compute $P(Q | e_1 \wedge \cdots \wedge e_N)$

- Categorize the variables: query, evidence, and hidden.
- Create a factor for each node in the Bayesian network.
- Restrict each evidence variable to its observed value.
- Eliminate the hidden variables $h_1, \ldots, h_j, \ldots, h_M$.
  - For each hidden variable h_j
  - Multiply all the factors that contain h_j to create a new factor f_j.
  - Sum out h_j from factor f_j.
- Multiply the remaining factors.
- Normalize the remaining factor.

Create a Factor

Convert each conditional probability table to a factor.

$P(w\|a)$	0.8
$P(\neg w\|a)$	0.2
$P(w\|\neg a)$	0.4
$P(\neg w\|\neg a)$	0.6

$\rightarrow$

w	a	0.8
$\neg w$	a	0.2
w	$\neg a$	0.4
$\neg w$	$\neg a$	0.6

$P(b)$	0.1
$P(\neg b)$	0.9

$\rightarrow$

b	0.1
$\neg b$	0.9

Restrict a Factor

Restrict each evidence variable to its observed value.

w	a	0.8
$\neg w$	a	0.2
~~w~~	~~$\neg a$~~	~~0.4~~
~~$\neg w$~~	~~$\neg a$~~	~~0.6~~

Restrict A to be true

~~w~~	~~0.8~~
$\neg w$	0.2

Restrict W to be false

0.2

Restrict a Factor

Restrict each evidence variable to its observed value.

w	a	0.8
$\neg w$	a	0.2
w	$\neg a$	0.4
$\neg w$	$\neg a$	0.6

Restrict A to be true

w	0.8
$\neg w$	0.2

Restrict W to be false

0.2

Sum out a Variable

Sum out a hidden variable from a factor.

w	a	0.8
$\neg w$	a	0.2
w	$\neg a$	0.4
$\neg w$	$\neg a$	0.6

Sum out A $0.8+0.4=1.2$

w	1.2
$\neg w$	0.8

$0.2+0.6=0.8$

Sum out W

0.6

Sum out a Variable

Sum out a hidden variable from a factor.

w	a	0.8
$\neg w$	a	0.2
w	$\neg a$	0.4
$\neg w$	$\neg a$	0.6

Sum out A

w	1.2
$\neg w$	0.8

Sum out W

2.0

Multiply Factors

Multiply factors in an element-wise fashion.

w	a	0.8
$\neg w$	a	0.2
w	$\neg a$	0.4
$\neg w$	$\neg a$	0.6

g	a	0.4
$\neg g$	a	0.6
g	$\neg a$	0.04
$\neg g$	$\neg a$	0.96

w	a	g	
$\neg w$	a	g	
w	$\neg a$	g	0.4×0.04
$\neg w$	$\neg a$	g	
w	a	$\neg g$	
$\neg w$	a	$\neg g$	
w	$\neg a$	$\neg g$	
$\neg w$	$\neg a$	$\neg g$	

Multiply Factors

Multiply two factors in an element-wise fashion.

w	a	0.8
$\neg w$	a	0.2
w	$\neg a$	0.4
$\neg w$	$\neg a$	0.6

g	a	0.4
$\neg g$	a	0.6
g	$\neg a$	0.04
$\neg g$	$\neg a$	0.96

w	a	g	0.32
$\neg w$	a	g	0.08
w	$\neg a$	g	0.016
$\neg w$	$\neg a$	g	0.024
w	a	$\neg g$	0.48
$\neg w$	a	$\neg g$	0.12
w	$\neg a$	$\neg g$	0.384
$\neg w$	$\neg a$	$\neg g$	0.576

Normalize a Factor

Convert a factor to a probability distribution.

w	1.2
$\neg w$	0.8

$1.2/(1.2+0.8) = 0.6$

w	0.6
$\neg w$	0.4

Normalize a Factor

Convert a factor to a probability distribution.

w	1.2
$\neg w$	0.8

w	0.6
$\neg w$	0.4

A FULL VEA EXAMPLE

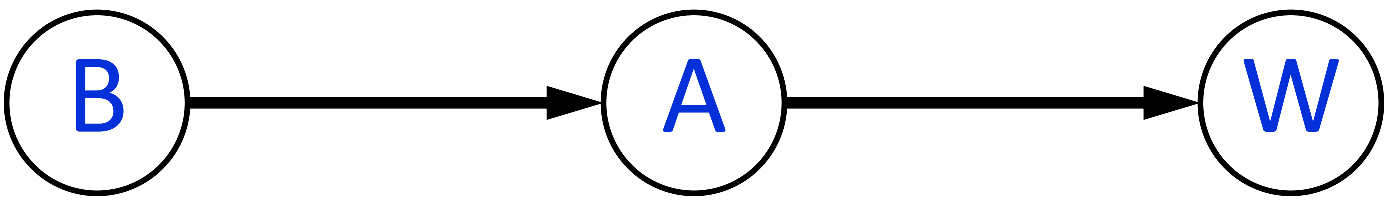
A VEA Example

Calculate $P(A|\neg w)$ using variable elimination algorithm.

$P(b) = 0.5$

$P(a|b) = 0.3$
$P(a|\neg b) = 0.4$

$P(w|a) = 0.1$
$P(w|\neg a) = 0.2$



Categorize the Variables

Calculate $P(A|\neg w)$

- Query variables: A
- Evidence variables: W
- Hidden variables: B

Define the Factors

Define factor $f_1(B)$

b	0.5
$\neg b$	0.5

Define factor $f_2(B, A)$

b	a	0.3
b	$\neg a$	0.7
$\neg b$	a	0.4
$\neg b$	$\neg a$	0.6

Define factor $f_3(A, W)$

a	w	0.1
a	$\neg w$	0.9
$\neg a$	w	0.2
$\neg a$	$\neg w$	0.8

Restrict the Factors

Restrict factor $f_3(A, W)$ to $\neg w$ produce factor $f_4(A)$

a	w	0.1
a	$\neg w$	0.9
$\neg a$	w	0.2
$\neg a$	$\neg w$	0.8

a	0.9
$\neg a$	0.8

The remaining factors are $f_1(B)$, $f_2(B, A)$, $f_4(A)$.

b	0.5
$\neg b$	0.5

b	a	0.3
b	$\neg a$	0.7
$\neg b$	a	0.4
$\neg b$	$\neg a$	0.6

a	0.9
$\neg a$	0.8

Eliminate the Hidden Variable B

Multiply $f_1(B)$ and $f_2(B, A)$ to produce factor $f_5(B, A)$.

b	0.5
$\neg b$	0.5

$\times$

b	a	0.3
b	$\neg a$	0.7
$\neg b$	a	0.4
$\neg b$	$\neg a$	0.6

$\Rightarrow$

b	a	0.15
b	$\neg a$	0.35
$\neg b$	a	0.2
$\neg b$	$\neg a$	0.3

The remaining factors are $f_4(A)$, $f_5(B, A)$.

Eliminate the Hidden Variable B

Sum out B from $f_5(B, A)$ to produce factor $f_6(A)$.

b	a	0.15
b	$\neg a$	0.35
$\neg b$	a	0.2
$\neg b$	$\neg a$	0.3

a	0.35
$\neg a$	0.65

The remaining factors are $f_4(A)$, $f_6(A)$.

a	0.9
$\neg a$	0.8

a	0.35
$\neg a$	0.65

Multiply the Remaining Factors

Multiply $f_4(A)$ and $f_6(A)$ to produce factor $f_7(A)$.

a	0.9
$\neg a$	0.8

$\times$

a	0.35
$\neg a$	0.65

$\Rightarrow$

a	0.315
$\neg a$	0.52

The remaining factor is $f_7(A)$.

Normalize the Factor

Normalize $f_7(A)$ to produce factor $f_8(A)$.

a	0.315
$\neg a$	0.52

$\Rightarrow$

a	0.377
$\neg a$	0.623

What Did VEA Really Do?

① Restrict ② Multiply ③ Sum out

$$P(A|\neg w) = P(\neg w|A)\left(\sum_{B} P(B)P(A|B)\right)$$

④ Multiply ⑤ Normalize.

Revisiting the Learning Goals

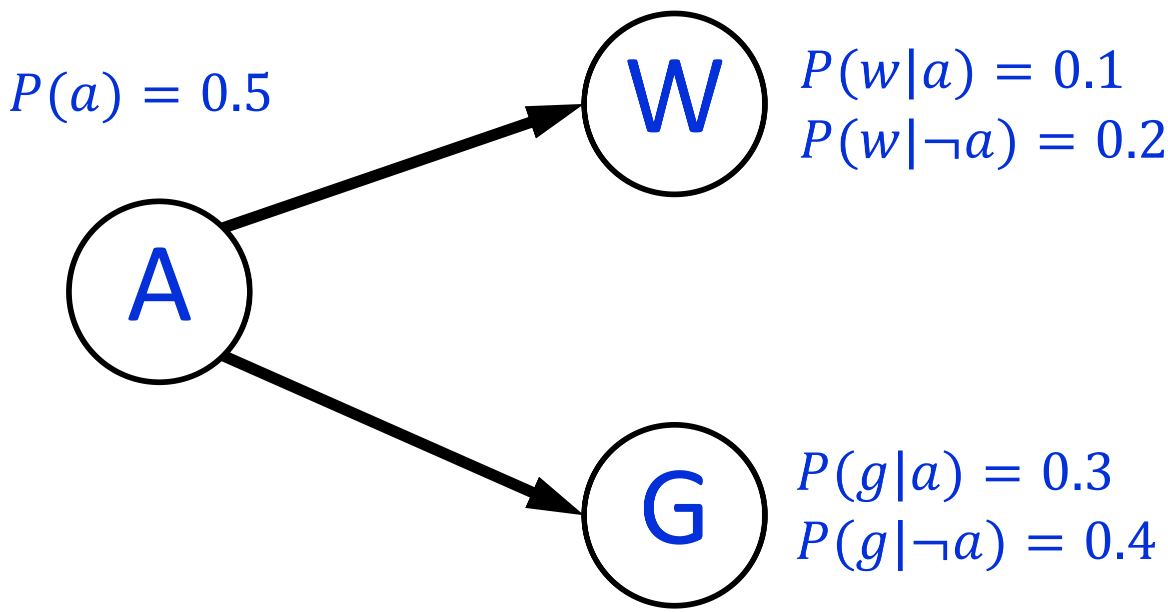
By the end of this lecture, you should be able to

1. Define factors. Manipulate factors using operations restrict, sum out, multiply and normalize.
2. Describe/trace/implement the variable elimination algorithm for calculating a prior or posterior probability given a Bayesian network.

AN EXTRA EXAMPLE ON VEA

A VEA Example

Calculate $P(W|\neg g)$ using the variable elimination algorithm.



Categorize the Variables

Calculate $P(W|\neg g)$

- Query variables: W
- Evidence variables: G
- Hidden variables: A

Define the Factors

Define factor $f_1(A)$

a	0.5
$\neg a$	0.5

Define factor $f_2(A, W)$

a	w	0.1
a	$\neg w$	0.9
$\neg a$	w	0.2
$\neg a$	$\neg w$	0.8

Define factor $f_3(A, G)$

a	g	0.3
a	$\neg g$	0.7
$\neg a$	g	0.4
$\neg a$	$\neg g$	0.6

Restrict the Factors

Restrict factor $f_3(A, G)$ to $\neg g$ produce factor $f_4(A)$

a	0.7
$\neg a$	0.6

The remaining factors are $f_1(A)$, $f_2(A, W)$, $f_4(A)$.

Eliminate the Hidden Variable A

Multiply $f_1(A)$, $f_2(A, W)$, and $f_4(A)$ to produce factor $f_5(A, W)$.

a	w	0.035
a	$\neg w$	0.315
$\neg a$	w	0.06
$\neg a$	$\neg w$	0.24

Sum out A from $f_5(A, W)$ to produce factor $f_6(W)$.

w	0.095
$\neg w$	0.555

The remaining factor is $f_6(W)$.

Multiply and Normalize

No need to multiply since there is one factor remaining.

Normalize factor $f_6(W)$ to get factor $f_7(W)$.

w	0.146
$\neg w$	0.854

That is, $P(w|\neg g) = 0.146$ and $P(\neg w|\neg g) = 0.854$.