CSC 384
Introduction to Artificial Intelligence

Knowledge Representation 2

Alice Gao and Randy Hickey

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Learning Goals

By the end of this lecture, you should be able to

1. Define logical consequence.
2. Define soundness and completeness.
3. Describe the resolution rule.
4. Describe the two proof approaches (forward chaining and refutation proof).
5. Convert a formula into clausal form.

Outline

1. Logical Consequence
2. Resolution Proof
3. Convert Formula into Clausal Form

LOGICAL CONSEQUENCE

Models

- KB is a knowledge base consisting of multiple formulas.
- f is a formula. M is a model/interpretation.

- $M \vDash f$ means that f is true under M.
- M satisfies KB ($M \vDash KB$), if and only if

  every formula $f \in KB$ is true under M or

  $\forall f \in KB, M \vDash f$.

- One KB typically has many models.
- We intend the real world to be one model of KB.

Deducing Logical Consequences

- One KB does not contain all the true statements.
- How can we deduce true statements that are consistent with the KB but not in KB?

KB entails f or f is a logical consequence of KB

$$\Longleftrightarrow$$

$$(M \vDash KB) \rightarrow (M \vDash f)$$

f is true in every model of KB.

RESOLUTION PROOF

Computing Logical Consequences

- Want to reason with our knowledge

1. Represent the knowledge as logical formulas
2. Apply procedures to generate logical consequences
3. Implement these procedures in programs.

- We will develop a proof procedure called resolution.

Proof Procedures

- They manipulate formulas syntactically.
- They do not know or care about the meanings.
- Yet, they respect the meanings of the formulas.

- $KB \vdash f$: we can prove f from KB.
  - through a purely syntactic manipulation of the formulas in KB.

Properties of Proof Procedures

- Soundness

$$KB \vdash f \rightarrow KB \vDash f$$

  - If we can prove f from KB, then f is a logical consequence of KB.
  - "If we can prove f, then f is true."

- Completeness

$$KB \vDash f \rightarrow KB \vdash f$$

  - If f is a logical consequence of KB, then we can prove f from KB.
  - "If f is true, then we can prove f."

- Resolution is sound and complete!

Clausal Form

- To construct a resolution proof, we must convert each formula to clausal form.

- A literal is an atomic formula or the negation of an atomic formula.
  - e.g. $p(X)$ or $\neg p(X)$
- A clause is a disjunction (OR) of literals.
  - e.g. $p(X) \vee q(Y) \vee \neg r(Y, Z)$
  - We rewrite it: $(p(X), q(Y), \neg r(Y, Z))$

Resolution Rule

Resolution proof has one rule:

From the two clauses

$$(P, Q_1, \dots, Q_k)$$
$$(\neg P, R_1, \dots, R_n)$$

We infer the new clause

$$(Q_1, \dots, Q_k, R_1, \dots, R_n)$$

Two Proof Approaches

Suppose we want to prove f from KB.

- Forward Chaining (or Direct Proof)
  - Start with the clauses from the KB.
  - Generate a new clause using the resolution rule.
  - Stop when we generate f.

- Refutation Proof (or Proof by Contradiction)
  - Start with the clauses from KB and a new clause $\neg f$.
  - Generate a new clause using the resolution rule.
  - Stop when we generate the empty clause.

Forward Chaining Example

Want to prove likes(C,peanuts) from:

1. $(\text{elephant}(C), \text{giraffe}(C))$
2. $(\neg\text{elephant}(C), \text{likes}(C, \text{peanuts}))$
3. $(\neg\text{giraffe}(C), \text{likes}(C, \text{leaves}))$
4. $\neg\text{likes}(C, \text{leaves})$

Forward Chaining Proof:

Step	**Clauses to combine**	**Resulting clause**
5	3 and 4	¬giraffe(C)
6	1 and 5	elephant(C)
7	2 and 6	likes(C,peanuts)

Refutation Proof Example

Want to prove likes(C,peanuts) from:

1. $(\text{elephant}(C), \text{giraffe}(C))$
2. $(\neg\text{elephant}(C), \text{likes}(C, \text{peanuts}))$
3. $(\neg\text{giraffe}(C), \text{likes}(C, \text{leaves}))$
4. $\neg\text{likes}(C, \text{leaves})$

Refutation Proof:

Step	**Clauses to combine**	**Resulting clause**
5	Refutation clause	¬likes(C,peanuts)
6	2 and 5	¬elephant(C)
7	1 and 6	giraffe(C)
8	3 and 7	likes(C,leaves)
9	4 and 8	Empty clause

CONVERT A FORMULA INTO CLAUSAL FORM

Converting a Formula to Clausal Form

We follow the 8 steps below.

1. Eliminate implications.
2. Move negations inwards (and simplify double negations).
3. Standardize variables.
4. Skolemize.
5. Convert to Prenix Form.
6. Distribute conjunctions over disjunctions.
7. Flatten nested conjunctions and disjunctions.
8. Convert to clauses.

Step 1: Eliminate Implications

Recall the four logical connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$.

The implication is convenient but redundant.

$$p(x) \rightarrow q(y) \implies \neg p(x) \vee q(y)$$

Step 2: Move Negations Inwards

Move each negation inwards as much as possible.

$$\neg\big(f(X) \land g(X)\big) \Longrightarrow \big(\neg f(X) \lor \neg g(X)\big),$$
$$\neg\forall X\, f(X) \;\Longrightarrow\; \exists X\, \neg f(X)$$

$$\neg\big(f(X) \lor g(X)\big) \Longrightarrow \big(\neg f(X) \land \neg g(X)\big),$$
$$\neg\exists X\, f(X) \;\Longrightarrow\; \forall X\, \neg f(X)$$

Also, simplify double negations.

$$\neg\neg g(X) \Longrightarrow g(X)$$

Step 3: Standardize Variables

Rename variables so that

each quantified variable is unique.

Example:

$$\big(\forall X\ p(X)\big) \land \big(\forall X\ q(X)\big)$$
$$\Rightarrow$$
$$\big(\forall X\ p(X)\big) \land \big(\forall Y\ q(Y)\big)$$

Step 4: Skolemize

A.k.a eliminate existential quantifiers

by introducing new function symbols.

Example 1:

$$\exists X\, p(X) \Longrightarrow p(a)$$

Example 2:

$$\forall X \forall Y \exists Z \forall W\, p(X, Y, Z, W)$$
$$\Longrightarrow$$
$$\forall X \forall Y \forall W\, p(X, Y, f(X, Y), W)$$

Skolemize Case 1

If the existential quantifier

does not have any universal quantifier outside of it,

Example:

$$\exists Y \ diligent(Y)$$

- Asserts that at least one individual is diligent.
- To remove $\exists$, we invent a name for this individual, say a constant symbol a to obtain

$$diligent(a)$$

- a must be new,
  - New: not equal to any previous constant symbol.
  - We must know nothing about a.

Skolemize Case 1

What could go wrong if a is not new?

Example:

$$rabbit(a),$$
$$\exists Y\ diligent(Y)$$

If we convert $\exists Y\ diligent(Y)$ to $diligent(a)$,

what goes wrong?

$$rabbit(a),$$
$$diligent(a)$$

Skolemize Case 2

If the existential quantifier

is within one or more universal quantifiers,

Example:

$$\forall X \exists Y \; likes(X, Y)$$

Translate this formula:

What if we replace Y with a new constant symbol a?

$$\forall X \; likes(X, a)$$

Translate this formula:

Skolemize Case 2

If the existential quantifier

is within one or more universal quantifiers,

Example:

$$\forall X \exists Y \; likes(X, Y)$$

- Each Y must be a function of the chosen X.

$$\forall X \; likes(X, g(X))$$

- g must be a new function symbol.
- g must be a function of all the universally quantified variables chosen before it.

Skolemize Exercises

$$\forall X \forall Y \forall Z \exists W \; p(X, Y, Z, W)$$

$$\forall X \forall Y \exists W \; p\big(X, Y, g(W)\big)$$

$$\forall X \forall Y \exists W \forall Z \; (p(X, Y, W) \wedge q(Z, W))$$

Skolemize Exercise Solutions

$$\forall X \forall Y \forall Z \exists W \; p(X, Y, Z, W)$$
$$\rightarrow \forall X \forall Y \forall Z \; p(X, Y, Z, h1(X, Y, Z))$$

$$\forall X \forall Y \exists W \; p\big(X, Y, g(W)\big)$$
$$\rightarrow \forall X \forall Y \; p\big(X, Y, g(h2(X, Y))\big)$$

$$\forall X \forall Y \exists W \forall Z \; (p(X, Y, W) \wedge q(Z, W))$$
$$\rightarrow \forall X \forall Y \forall Z \; (p(X, Y, h3(X, Y)) \wedge q(Z, h3(X, Y)))$$

Step 5: Convert to Prenix Form

Bring all universal quantifiers to the front.

Example

$$\forall X \ \left(p(X) \lor \left(\forall Y\ q(X,Y) \lor r(Y)\right)\right)$$

$$\Longrightarrow$$

$$\forall X \forall Y \left(p(X) \lor \left(q(X,Y) \lor r(Y)\right)\right)$$

Step 6: Conjunctions over Disjunctions

Convert the formula into

a conjunction (AND) of disjunctions (OR).

Example:

$$p(X) \vee \left(q(X) \wedge r(X)\right)$$
$$\Longrightarrow$$
$$(p(X) \vee q(X)) \wedge (p(X) \vee r(X))$$

This step is like step 5. Do you see why?

Step 7: Flatten Nested Conjunctions and Disjunctions

Remove brackets (wherever possible)!

Examples:

$$p(X) \vee \big(q(X) \vee r(X)\big) \implies p(X) \vee q(X) \vee r(X)$$

$$p(X) \wedge \big(q(X) \wedge r(X)\big) \implies p(X) \wedge q(X) \wedge r(X)$$

Step 8: Convert to Clauses

Remove the universal quantifiers.

Break apart the conjunctions.

Example:

$$\forall X \ \left(p(X) \wedge \left(q(X) \vee r(Y)\right)\right)$$

Clause 1: $p(X)$

Clause 2: $(q(X), r(Y))$

Converting a Formula to Clausal Form

Rephrasing the 8 steps below.

1. Eliminate implications.
2. Move negations inwards (and simplify double negations).
3. Rename variables.
4. Eliminate existential quantifiers.
5. Move universal quantifiers to the front.
6. Distribute conjunctions over disjunctions.
7. Remove brackets wherever possible.
8. Break apart the formula at the conjunctions into clauses.

Example: Converting a Formula into Clausal Form

$$\forall X \left(p(X) \right.$$

$$\rightarrow \left(\left(\forall Y \ \left(p(Y) \rightarrow p\left(f(X, Y)\right)\right)\right)\right.$$

$$\left.\left. \land \neg \left(\forall Y \left(\neg q(X, Y) \land p(Y) \right)\right)\right)\right)$$

Step 1: Eliminate implications

$$\forall X \left(\neg p(X) \right.$$
$$\left. \vee \left(\left(\forall Y \left(\neg p(Y) \vee p\big(f(X,Y)\big) \right) \right) \right. \right.$$
$$\left. \left. \wedge \neg \left(\forall Y \big(\neg q(X,Y) \wedge p(Y) \big) \right) \right) \right)$$

Step 2: Move negations inwards

$$\forall X \left(\neg p(X) \right.$$
$$\vee \left(\left(\forall Y \left(\neg p(Y) \vee p\big(f(X,Y)\big)\right)\right) \right.$$
$$\left.\left. \wedge \left(\exists Y \big(q(X,Y) \vee \neg p(Y) \big) \right) \right) \right)$$

Step 3: Rename variables

$$\forall X \left(\neg p(X) \vee \left(\left(\forall Y \left(\neg p(Y) \vee p\big(f(X,Y)\big) \right) \right) \wedge \left(\exists Z \big(q(X,Z) \vee \neg p(Z) \big) \right) \right) \right)$$

Step 4: Skolemize

$$\forall X \left(\neg p(X) \vee \left(\left(\forall Y \left(\neg p(Y) \vee p\big(f(X,Y)\big) \right) \right) \wedge \left(q(X, g(X)) \vee \neg p\big(g(X)\big) \right) \right) \right)$$

Step 5: Move all the universals to the front

$$\forall X \forall Y \left(\neg p(X) \right.$$

$$\left. \vee \left(\left(\neg p(Y) \vee p\big(f(X,Y)\big) \right) \wedge \left(q(X, g(X)) \vee \neg p\big(g(X)\big) \right) \right) \right)$$

Step 6: Conjunctions over disjunctions

$$\forall X \forall Y \left(\left(\neg p(X) \vee \left(\neg p(Y) \vee p\big(f(X,Y)\big) \right) \right) \wedge \left(\neg p(X) \vee \left(q\big(X, g(X)\big) \vee \neg p\big(g(X)\big) \right) \right) \right)$$

Step 7: Flatten nested conjunctions and disjunctions

$$\forall X \forall Y \left(\left(\neg p(X) \vee \neg p(Y) \vee p\big(f(X, Y)\big) \right) \right.$$
$$\left. \wedge \left(\neg p(X) \vee q(X, g(X)) \vee \neg p(g(X)) \right) \right)$$

Step 8: Convert to clauses

Clause 1:

$$\left(\neg p(X), \neg p(Y), p\left(f(X, Y)\right)\right)$$

Clause 2:

$$\left(\neg p(X) \lor, q\left(X, g(X)\right), \neg p\left(g(X)\right)\right)$$