# math notes

matt baird

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# Chapter 1

# Chapter 2

## 2.1

- A set S is a collection of items $\{2, chair, imortality, \{1\}\}$
- 2 E S denotes that 2 is a member of set S. "$xES$"
  "$xED$"
- Important sets
  IR = set of all real numbers
  Z = set of ints
  Q = set of rational numbers
  N = set of natural numbers
- Def y A B are sets, and every element of A is inside B , then A is a subset of B $A \leq B$
- $N \leq Q$
  $N \leq R$
  $N \leq Z$
  $Z \leq Q$
  $Z \leq R$
  $Znot \leq N$
  $Q \leq R$

# Chapter 3

# Set Theory

## 3.1 Predicate and quantified statments

- Def a predicate is a sentence that contains a finite number of variables and that becomes a statement when specific values are substituted for the variable
  $P(x) = "x \geq 2"$
  $P(1) = 1 > 2" true$
  $P(1) = "1 > 2" false$

- The domain of the predicate is the set of all values that may be substituted for a variable.
  $D = R$
  $D = \{1, 2, 3\}$
  $D = Z^+$ (set of positive ints)

- The truth set of P(x) is the set of elements in D such that P(x) is true.
  $\{x \epsilon D \colon P(x)\}$
  $\{x = 1, 2, 3 \colon x > 2\} = \{3\}$

- Ex finding the truth set
  $Q(n) =$"n is a factor of g"
  a) $D = Z^+$
  $\{$ $n \epsilon Z^+ \colon$ n is a factor of 8 $\}$
  $= \{1, 2, 4, 8\}$
  b) $D = Z$
  $\{n \epsilon Z \colon n$ divides 8 evenly $\}$
  $= \{1, 2, 4, 8, -1, -2, -5, -8\}$

- Def The symbol $\forall$ denotes " for all" (for every, for any) and is called the universal quantifier.
  ' all humans are mortal"
  "$\forall$ humans h , h is mortal"
  define H = all humans " $/forallh\epsilon H$, h is mortal "

- Def A universal statement has the form " $\forall x \epsilon D, Q(x)$" where D is a domain and Q(x) a predicate.
  The statement is true if and only if Q(x) is true for each $x \epsilon D$. The statement is false id there is at least one $x \epsilon D$ such that Q(x) is false.

- $\forall x \epsilon D, x^2 \geq x$
  $a) D = \{1, 2, 3, 4\}$
  $1 \geq 1, 4 \geq 2, 9 \geq 3, 16 \geq 4$
  therefore the statement is true.
  b) $D = R$
  $let x = .5$counter example
  then $x^2 = .25$
  $.25 < .5$

- Def the symbol "$\exists$" denotes " there exist" ( there is one ... , there are some..) and is called the existential quantifier.
  "there is a cat on the fridge'
  $C = \{all\ cats\}$
  $\exists x \epsilon C$, c is on the fridge.

- def an existential "$\exists x \epsilon D$ such that $Q(x)$"
  "$\exists x \epsilon D$ such that Q(x)"
  is true when at least one x in D makes Q(x) true
  is false when every $x \epsilon D$ makes Q(x) false.
  Consider $\exists m \epsilon Z^+$ such that $m^2 = m$
  $\frac{m^2}{m} = \frac{m}{m} \Rightarrow m = 1$
  which in words is " there exits a positive int such that is is equal to its square."
  "some positive integer equals its square"

- " no dogs have wings"
  All dogs don't have wings"
  D = a set of dogs
  $\forall d \epsilon D$, d doesn't have wings.
  all, every, each ,none, no ; are universal words
  at least one , some, there is / exists; are existential words

- Universal conditional statements
  $\forall x, if P(x) then Q(x)$
  $\forall x R, if x > 2, then x^2 > 4$
  $Ex:$ No sloths are fast.
  $\forall s \epsilon U$, if s is a sloth , then s is not fast. $\equiv \forall s \epsilon S$, s is not fast
  P(x) = s is a sloth"
  $\{x \epsilon u$ , if P(x) , then Q(x)
  $\{x \epsilon u \colon P(x)\} \colon = D = \{sloths\}/equiv \forall x \epsilon D, Q(x)$.

- Ex $\forall x \epsilon R, if x \epsilon Z, then x \epsilon Q \equiv \forall x \epsilon Z, x \epsilon Q$.
  N : check 1,2,3
  Z : check negative
  Q : check $\frac{1}{2}, \frac{3}{4}$
  R : check $\pi, e, \sqrt{2}$

- Ex there exits a number that is both even and prime
  $\exists n$ such that $P(n) \wedge E(n)$.
  $\forall$ a prime number such that E(n)
  $\forall$ an even number such that P(n).
  " implicit quantification"
  y a number is an int , then ...
  $\forall n \epsilon Z$
  indefinite article "a"
  "24 can be written as the sum of two ints"
  $\forall m, n \epsilon Z$ such that $m + n = 24$
  $P(x) \Leftrightarrow Q(x)$.
  $\{x \colon Px(x)\} = \{x \colon Q(x)\}$

- homework 3.1 2,3,9,11,13,19

## 3.2 Negations of quantified statments

- $(\forall x \epsilon D, Q(x)) \equiv \exists x \epsilon D,\ such\ that\ \ Q(x)$
  $(\exists x \epsilon D\ such\ that\ Q(x)) \equiv \forall x \epsilon D,\ \ Q(x)$.
  Negate "some dogs are bad dogs"

" all dogs are god dogs"

$(\forall x\ if\ P(x)\ then\ Q(x)) \equiv \exists x\ such\ that\ P(x)\ and\ \ Q(x)$

$negating\ \forall$

$gives\ \exists$

$negating\ \exists$

$gives\ \forall$

- Universal statments that are vacuously true
  Consider an empty jar
  "all the grapes in the jar are red"
  $\forall g \epsilon J$
  negating that =
  "some grapes in the jar are not red"
  $\exists\ g\epsilon J\ Such\ that\ g\ is\ not\ red.$
- Universal Conditional
  $\forall x\epsilon D,\ if P(x)\ then\ Q(x)$

## 3.3 Statments sith multiple quantifiers

**from quiz**
**any interger equals twice some interger**
**which is "**$\forall n\epsilon z$

- "everyone has someone to love"
  "for every person, there is another person that they love "
  $\forall p\epsilon H, \exists r\epsilon H\ such\ that\ p\ loves\ r.$
- ($\forall x\epsilon D, \exists y\epsilon E$ such that P(x,y))
  $\equiv \exists x\epsilon D\ such\ that\ \forall y\epsilon E,\ \ P(x,y)$
  $(\exists x\epsilon D\ ; such\ that\ \forall y\ in\ E,\ P(x,y))$
  $\equiv \forall x\epsilon D, \exists y\epsilon E\ such\ that\ \ P(x,y).$

## 3.4 arguments with quantified statments

- Rule of universal instantiation if a property is true of everything is a set, it is true for any partculur member of the set.
- hello

# Chapter 4

# 4

## 4.1 4.1

## 4.2 4.2

- hello

## 4.3 4.3

**from quiz**
**1 for all ints m if m is even then** $3m + 5$ **is odd**
$m = 2k$ **for some** $k\epsilon Z$ $//3m + 5 = 3(2k) + 5 = 6k + 5$
**want** $6k + 5 = 2j + 1$ **for some** $j\epsilon Z$ **find the j**
$6k + 5 = 2j + 1$ $6k + 4 = 2j$ $3k + 2 = j$
**set** $j = 3k + 2$ **since** $k\epsilon Z, j\epsilon Z$
**we have** $3m + 5 = 2j + 1$ **for** $j\epsilon Z$
**2 for all ints a,b,and c if a—b and a—b then a—(b-c)**
$a|b \Leftrightarrow b = ar$ **for some** $r\epsilon Z$
$a|c \Leftrightarrow c = as$ **for some** $s\epsilon Z$
$b - c = ar - as == a(r - s)$
**since** $r, s\epsilon Z, r - s\epsilon Z$
$a|(b - c)$

## 4.4 Division into cases and the quotient-remainder theorem

- (quotient-remainder)
  given any interger n and interger $d > 0$ , there exist unique interger q,r such that $n = dq + r$ where $0 \leq r < d$

- given $n = qd + r$ we say " n div d" is the integer quotient q obtained when n is divided by , and " n mod d" is the integer remainder obtained when n is divided by d.

- representations of integers
  we can prove all integers are even or odd using the QR
  given any n , $6 + d = 2$ then $n = 2q + r$ where $0 \leq r < 2$
  so $r = 0$ or $r = 1$
  two cases
  $n = 2q + 0$ or $n = 2q + 1$
  but $q\epsilon Z$
  so n is even or n is odd

- prove that any two consectitive integers have opposite parity ( that is one is even and one is odd).
  let m ,$m + 1$ be two consective integers
  case 1

m is even therefore $m = 2k = 2k + 1$ for the same $k \epsilon Z$ therefore $m + 1$ is odd
case 2
therefore $m = 2k + 1$ for some $k \epsilon Z$ then $m + 1 = 2k + 1 + 1$
$= 2k + 2$
$2(k + 1)$
therefore $(k + 1) \epsilon Z$ therefore $m + 1$ is even

- can have more than 2 cases

- show that any integer n can be written in one of the four forms
$n = 4q$
$n = 4q + 1$
$n = 4q + 2$
$n = 4q + 3$
for some $q \epsilon Z$
proof
QR - ther $6 + d = 4$ then for all integers , n , $n = 4q + r$ where $0 \leq r < 4$
four cases $r = 0, 1, 2, 3$
$n = 4q$
$n = 4q + 1$
$n = 4q + 2$
$n = 4q + 3$

## 4.5

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# Chapter 5

# Sequences and math induction

## 5.1 sequences

- def a seqence ins a function whose domain is rither all the intergers or all the intergers greater that or equal to an integer.
- $a_0, a_1, a_2, a_n$
  an element $a_k$ ( " a sub k " ) is called a term of the sequence and k is its index
- when the sequence donsnt have a final term it is a n infinite sequence
- Explicit formulae for sequences
  $a_k = \frac{k}{k+1}$ for all intsegers $k \leq 1$
  $a_1 = \frac{1}{2}$
  $b_i = \frac{i-1}{i}$ for all ints $i \leq 2$
  $b_2 = 1/2$
  $b_3 = 2/3$
  $b_4 = 3/4$ etc
  $a_k = b_i$ where $k = i - 1$
- alternating sequences
  $c_j = (-1)^j \; for \; j \leq 0$
  $c_0 = (-1)^0 = 1$
  $c_1 = (-1)^1 = -1$
  $c_2 = (-1)^2 = 1$
  $c_3 = (-1)^3 = -1$
  example
  $1, -1/4, 1/9, -1/16, 1/25, -1/36$
  1 in numerator
  signs alternate
  perfict squares in denominator
  list$k = 1, 2, 3, etc$ for each term
  $a_k = \frac{(-1)^(k+1)}{k^2}$
  positive $a_k$ when $k$ is odd
- summation notation
  want to take the sum of all the elements in a sequence $(a_1 + a_2 + a_3 + ...)$
  $\sum_{k=1}^{n} a_k$
  let $a_1 = -2, a_2 = -1, a_3 = 0, a_4 = 1, a_5 = 2$
  compute

$\sum_{k=1}^{5} ak = a_1 + a_2 + a_3 + a_4 + a_5 = -2, + -1 + 0_1 + 2 = 0$
$\sum_{k=2}^{2} a_k = a_2 = -1$
$\sum_{k=1}^{2} a_2 k = a_2(1) + a_2(2) = a_2 + a_4 = -1 + 1 = 0$
$\sum_{k=1}^{5} k^2 \; or \; \sum_{i=0}^{8} \frac{(-1)^i}{i+1}$
$\sum_{i=0}^{n} \frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + ... + \frac{(-1)^n}{n+1}$
$= 1 - 1/2 + 1/3 + ... + \frac{(-1)^n}{n+1}$
$1/n + 2/(n+1) + 3/(n+3) + ... + (n+1)/(2n)$
the general term $a_k = (k+1)/(n+k)$ $2n = n + n$
the ints range from $(k = 0)$ to $(k = n)$
$\sum_{k=0}^{n} \frac{k+1}{n+k}$

- telescoping sum
  $\sum_{k=1}^{n} \frac{1}{k(k+1)}$
  $1/k - 1/(k+1) = (k+1)/(k(k+1)) = (k)/(k(k+1)) = 1/(k(k+1))$
  $= \sum_{k=1}^{n} \frac{1/k}{1/(k+1)} = ((1/1)-(1/2))+((1/2)+(1/3))+((1/3)-(1/4))+...+((1/(n-1))-(1/n))+((1/n)-(1/(n+1)))$
- product notation
  $\prod_{k=1}^{n} a_k = (a_1)(a_2)(a_3)...(a_n)$
  $\prod_{k=1}^{5} k = 1 * 2 * 3 * 4 * 5 = 120$
  $\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} (a_k + b_k)$
  $\sum_{k=0}^{n} a_k = a_0 + \sum_{k=1}^{n} a_k$
  $2\ c * \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_k * c$
  $(\prod_{k=1}^{n} a_k)(\prod_{k=1}^{n} b_k) = \prod_{k=1}^{n} (a_k * b_k)$
- factorials
  $n! = n(n-1)(n-2)...3 * 2 * 1 = n(n-1)!$
  $n! = \prod_{k=1}^{n} k$
  we define $0! = 1$ $(8!)/(7!) = (8 * 7!)/(7!) = 8$
  $n, r \epsilon Z$ $0 \leq r \leq n$

**homework**
**section 5.1**
**3,12,19,31,40,49,55,65,75**

**test october 9th chapters 2-5**
**defs, proofs 4 questions 3 to do and bonus**

## 5.2 Math induction

- priciple of math induction
  let P(n) be a property defined for intergers n and let a be a fixed int

suppose the following to be true
P(a) is true
For all ints $k \geq a$, if P(k) is true, then $P(k+1)$ is true.
then the statment $\forall intsn \geq a, P(n)$ is true

- method of proof by math induction
  show that P(a) is true
  suppose P(k) is true for $k \geq a$ ,k use this to show $P(k+1)$ true

- example
  $1+2+3+...+n = \frac{n(n+1)}{2}$ for all ints $n \geq 1$ which is P(n)
  1 show P(1) is true $n = 1$
  $1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$
  2 assume P(k) is true
  $\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$
  show $P(k+1)$ $\sum_{j=1}^{(} k+1)j = \sum_{j=1}^{(} k)j + (k+1) = \frac{k(K+1)}{2} + k + 1(\frac{2}{2}) = \frac{k^2+k+2k+2}{2} = \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}$ so $P(k+1)$ is true

## 5.3 more math induction

1 showP(a) true
2: Assume P(k) is true for some $k \geq a$ use this to prove $P(k+1)$ is true
For all intergers $n \geq 0, 2^(2n) - 1$ is divisible by 3
pf: show that P(0) is true
$n = 0 : 2 - 1$ is divisible by 3
$2^0 - 1 = 1 - 1 = 0.0^n = 3^d * 0^k$
so 0 is divisible by 3
2. Assume p(k) iis true for $k \geq 0$
know : $2^(2k) - 1$ is divisible by 3 for $k \geq 0$
So $2^(2k) - 1 = 3r$ for some $r \epsilon Z$
$2^(2(k+1)) - 1 = 2^(2k+2) - 1 = 2^2k * 2^2 - 1 = 4 * 2^2k - 1 = 3 * 2^2k + 2^2k - 1 = 3 * 2^2k + 3r = 3(2^2k + r)$
let $s = 2^2k + r$ since $k, r \epsilon Z, s \epsilon Z$ Then $2^2(k+1) - 1 = 3s, s \epsilon Z$ So $2^2(k+1)$ is divisible by 3 Thus $P(k+1)$ is true So $P(n)$ is true for all integers $n \geq 0$

**Prove an inequality**

For all integers $n \geq 3, , 2n+1 < 2^n$
show that $P(3)$ is true: $n = 3 : 2(3) + 1 = 72^3 = 8.7 < 8$
assume $2k + 1 < 2^k \ for \ k \geq 3$ then $2(k+1) + 1 = 2k + 2 + 1 = (2k+1) + 2 < 2^k + 2'$ we know $2 < 2^n \forall n > 1$
remind: if $a < b \ then \ a + c < b + c$
In particuler, for $k \geq 3, 2 < 2^k$
then $2(k+1) + 1 < 2^k + 2 < 2^k + 2^k$ Now $2^k + 2^k = 2 * 2^k = 2^(k+1)$
thus $2(k+1) + 1 < 2^(k+1)$
define a sequence $a_1, a_2, a_3, ...$ as follows : $a_1 = 2a_k = 5 * a_k - 1, \forall k \geq 2$
$(a_2 = 5a, = 5 * 2 = 10)$ prove that for $n \geq 1, a_n = 2 * 5^(n-1)$
pf: $P(1) : a_1 = 2 * 5^(1-1)(check)$ $a_1 = 2.2 * 5^(1-1) = 2 * 5^0 = 2$ so P(1) is true
Assume P(k) is true. then $a_k = 2 * 5^k - 1 \ for \ k \geq 1. a_k + 1 = 5a_k = 5 * 2 * 5^k - 1 = 2 * (5 * 5^k - 1) = 2 * 5^k - 1 + 1 = 2 * 5^k \ So \ a_k + 1 = 2 * 5^k \ So \ P(k+1)$ is true

**MidTerm**
write the definitions for even, odd, prime, composite, divisibility
ex: even $h = 2k$ for some $k \epsilon Z$

truth tables for $p \Rightarrow q, p^q, p \vee q$

determine whether an argument is valid
indentify the major and minor premised the conclusion and any critical rows
.
T/F ch 2 thru 3 : negations of logical statments ; contrapositives of logical statments; valid and invalid argument forms ; translating between formal and informal language pg 121
.
four proofs pick three 2 are direct proofs pg 180
1 is by contradiction or contraposition pg 198
1 by math induction pg 244

memorize all the things

5.3 homework
6,8,19,24

## 5.4 strong math induction

Let P(n) be a property defined for integration, and let a and b be fixed ints with $a \leq b$
Supose the following statments are true
1 $P(a), P(a+1), ..., P(b)$ are all true
2 For ant in $k \geq b$ if P(iP is rue is true for all $i, a \leq i \leq k$, then $P(k+1)$ is true
then P(n) is true for $n \geq a$

Method
1 Prove $P(a), P(a+1), ..., P(b)$ are true
2 Suppose for all i such that $a \leq i \leq k$ P(k) is true and use this to show $P(k+1)$ is true

example :
Any int $n > 1$ is divisible by a prime number.
P(n) : " n is divisible by a prime number " "$\exists$ *prime p such that* $p|n$."
Show P(2) is true
2 is a prime number and 2—2

Suppose that for some $k \geq 2, P(i)$ is true for $2 \leq i \leq k$
Want to show : $k+1$ is divisible by a prime
Case 1 $k+1$ is prime. Then $k+1|k+1$ so $\exists p$ such that $p|k+1$
case 2 ; $K = 1$ is composite then $\exists a, b \epsilon N$ such that $k+1 = ab$. $a < a < k+1$ and $1 < b < k+1$ then $2 \leq a \leq k$ so P(a) is true

# Chapter 6

# Set Theory

**6.1 definitions and the elements method of proof** if S is a set and P(x) is a property that elements of S may or may not satisfy, we define $A = x\epsilon S|P(x)$ "the set of all x in s such that P(x) is true" $N = n\epsilon Z : n > 0$
$N \leq Z$
Subsets : Def : $A \leq B$ is $x\epsilon A\ then\ x \leq B$ Negation : $A \nleq B\ if\ \exists x\ such\ that\ x\epsilon$

**element method of proof** To prove a set is a subset of another let x,y be sets Prove $x \leq y$ :
1) suppose $x\epsilon X$ is a particular but arbatary element of X
2) show that $x\epsilon y$

**example** $A = m\epsilon|m = 6r + 12\ for\ some\ r\epsilon Z$
$B = n\epsilon Z|n = 3s\ for\ some; s\epsilon Z$
a) Prove $A \leq B$ Let k be a particular but arbitary member of A . then $k\epsilon Z$ and there exists $r\epsilon Z$ such that $k = 6r+12$
Want : $k \leq B \Leftrightarrow \exists s\epsilon Z$ such that $k = 3s$
$k = 5r + 12 = 3(2r + 4)$
Let $s = 2r + 4$ Since $r\epsilon Z, s\epsilon Z$ So $k = 3s, s\epsilon Z\ and\ k\epsilon Z$
$\therefore k\epsilon B$
$\therefore A \leq B$
b) disprove $B \leq A$ ( prove $B \nleq A$)
$\exists x\epsilon B$ such that $x \notin A$ $x \in B$ so $x = 3s$ for some integer s
Let $s = 1$ then $x = 3$ if $3 = 6r + 12$ then $6r = -9$ $r = -3/2 \in Z$
$\therefore x \notin A$
$\therefore B \nleq A$ ■

**Set equality** $A = B$ iff $A \leq BandB \leq A$
$A = [m \in Z|m = 3a$ for some $a \in Z]$
$B = [n \in Z|n = 3b - 3$ for some $b \in Z]$
1) Prove $A \leq B$ Let x$x \in A$ then $x \in Z$ and $\exists a \in Z$ such that $x = 3a$ S ince $a \in Z, (b - 1) \in Z$ Let $b - 1 = a$ Then $x = 3a = 3(b - 1)$ $x = 3b - 3, b \in Z$ So $x \in B$
$\therefore A \leq B$
Prove $B \leq A$ : Let $y \in B$
homework 6.1
1,(a,c,e),2 , 3(a,), 5

**home work 6.1 number 5** $C = n\epsilon Z|n = 6r - 5$ for some $r\epsilon Z$
$D = m\epsilon Z|m = 3s + 1$ for some $s\epsilon Z$
a: $C \leq D$ Let $n\epsilon C$ Then $\exists r\epsilon Z$ such that $n = 6r - 5$
$6r - 5 = 3s + 1$
$6r - 6 = 3s$
$2r - 2 = s$
Let $s = 2r - 2$ since $r\epsilon Z, s\epsilon Z$
then $3s = 6r - 6\ and\ 3s + 1 = 6r - 5$ then $n = 6r - 5 == 3s + 1//$So $n\epsilon D$ So $C \leq D$

**operations on sets** Let A, B be subsets of set U

1. The union of A and B , $A \cup B$ is the set of all element that are in A or B
2. the intersection of A and B $A \cap B$ is the set of all elements that are in A and B

the difference $B - A$ is the set of all elements in B that are not in A
The complement of A , $A^c$, is the set of all elements not in A
$A \cup B = [x \epsilon U | x \epsilon A \ or \ x \epsilon B]$
$A \cap B = [x \epsilon U | x \epsilon A \ and \ x \epsilon B]$
$B - A = [x \epsilon U : x \notin A \ and \ x \epsilon B]$
$A^c = [x \epsilon U | x \notin A]$

**t** $(a, b] = [x \in R | a < x \leq b]$
$[a, b] = [x \in R | a \leq x \leq b]$
$A = (-1, 0] = [xR| - 1 < x <\leq o]$
$B = [0, 1) = [x <\in R | o \leq x < 1]$
$A \cup B = [x \epsilon R| - 1 < x < 1]$
$A \cap B = [x \in R | x = 0] = [0]$
$B - A = [x \in R | 0 < x > 1] = (0, 1)$
$A^c = [x \in R | x \leq -1 or x > 0] = (-\infty, -1] \cup (o, \infty)$
$[B_i]_{i \geq 1}$ is an infident squence of subsets of U
$\infty \cup_{i=1} \ B_i \ = [x \in U | x \in B_i$ for some $i \geq 1]$
$\infty \cap_{i=1} \ B_i \ = [x \in U | x \in B_i$ for all $i \geq 1]$
DEF We Define $null$, the empty set ( the null set ) as the set containd no elements $[o, 1] \cap [2, 4] = null$
Def A and B are disjoint if $A \cap B = null$
Def A, ,...,$A_n$ are mutually disjoint if $A_i \cap A_j = null$ whenever $i! = j$

homework. 6.1 10,11,19,21,24