

math notes

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# Chapter 1

# Chapter 2

## 2.1

- A set  $S$  is a collection of items  $\{2, chair, immortality, \{1\}\}$
- $2 \in S$  denotes that 2 is a member of set  $S$ . " $x \in S$ "  
" $x \in D$ "
- Important sets  
     $\mathbb{R}$  = set of all real numbers  
     $\mathbb{Z}$  = set of ints  
     $\mathbb{Q}$  = set of rational numbers  
     $\mathbb{N}$  = set of natural numbers
- Def y  $A \subseteq B$  are sets, and every element of  $A$  is inside  $B$ , then  $A$  is a subset of  $B$   $A \subseteq B$
- $\mathbb{N} \subseteq \mathbb{Q}$   
     $\mathbb{N} \subseteq \mathbb{R}$   
     $\mathbb{N} \subseteq \mathbb{Z}$   
     $\mathbb{Z} \subseteq \mathbb{Q}$   
     $\mathbb{Z} \subseteq \mathbb{R}$   
     $\mathbb{Z} \not\subseteq \mathbb{N}$   
     $\mathbb{Q} \subseteq \mathbb{R}$

# Chapter 3

## Set Theory

### 3.1 Predicate and quantified statements

- Def a predicate is a sentence that contains a finite number of variables and that becomes a statement when specific values are substituted for the variable  
 $P(x) = "x \geq 2"$   
 $P(1) = 1 > 2$  *true*  
 $P(1) = "1 > 2"$  *false*
- The domain of the predicate is the set of all values that may be substituted for a variable.  
 $D = R$   
 $D = \{1, 2, 3\}$   
 $D = Z^+$  (set of positive ints)
- The truth set of  $P(x)$  is the set of elements in  $D$  such that  $P(x)$  is true.  
 $\{x \in D : P(x)\}$   
 $\{x = 1, 2, 3 : x > 2\} = \{3\}$
- Ex finding the truth set  
 $Q(n) = "n \text{ is a factor of } 8"$   
a)  $D = Z^+$   
 $\{n \in Z^+ : n \text{ is a factor of } 8\}$   
 $= \{1, 2, 4, 8\}$   
b)  $D = Z$   
 $\{n \in Z : n \text{ divides } 8 \text{ evenly}\}$   
 $= \{1, 2, 4, 8, -1, -2, -4, -8\}$
- Def The symbol  $\forall$  denotes "for all" (for every, for any) and is called the universal quantifier.  
'all humans are mortal'  
" $\forall$  humans  $h$ ,  $h$  is mortal"  
define  $H = \text{all humans}$  " *forall*  $h \in H$ ,  $h$  is mortal "
- Def A universal statement has the form " $\forall x \in D, Q(x)$ " where  $D$  is a domain and  $Q(x)$  a predicate.  
The statement is true if and only if  $Q(x)$  is true for each  $x \in D$ . The statement is false if there is at least one  $x \in D$  such that  $Q(x)$  is false.
- $\forall x \in D, x^2 \geq x$   
a)  $D = \{1, 2, 3, 4\}$   
 $1 \geq 1, 4 \geq 2, 9 \geq 3, 16 \geq 4$   
therefore the statement is true.  
b)  $D = R$   
*let*  $x = .5$  counter example  
then  $x^2 = .25$   
 $.25 < .5$

- Def the symbol " $\exists$ " denotes "there exist" (there is one ... , there are some..) and is called the existential quantifier.  
 "there is a cat on the fridge"  
 $C = \{all\ cats\}$   
 $\exists x \in C, c$  is on the fridge.
- def an existential " $\exists x \in D$  such that  $Q(x)$ "  
 " $\exists x \in D$  such that  $Q(x)$ "  
 is true when at least one  $x$  in  $D$  makes  $Q(x)$  true  
 is false when every  $x \in D$  makes  $Q(x)$  false.  
 Consider  $\exists m \in \mathbb{Z}^+$  such that  $m^2 = m$   
 $\frac{m^2}{m} = \frac{m}{m} \Rightarrow m = 1$   
 which in words is "there exists a positive int such that it is equal to its square."  
 "some positive integer equals its square"
- "no dogs have wings"  
 All dogs don't have wings"  
 $D$  = a set of dogs  
 $\forall d \in D, d$  doesn't have wings.  
 all, every, each, none, no ; are universal words  
 at least one, some, there is / exists; are existential words
- Universal conditional statements  
 $\forall x, if P(x) then Q(x)$   
 $\forall x \in \mathbb{R}, if x > 2, then x^2 > 4$   
 $Ex$ : No sloths are fast.  
 $\forall s \in U$ , if  $s$  is a sloth, then  $s$  is not fast.  $\equiv \forall s \in S, s$  is not fast  
 $P(x) = s$  is a sloth"  
 $\{x \in U, if P(x), then Q(x)\}$   
 $\{x \in U: P(x)\} : D = \{sloths\} / equiv \forall x \in D, Q(x).$
- $Ex \forall x \in R, if x \in Z, then x \in Q \equiv \forall x \in Z, x \in Q.$   
 $N$  : check 1,2,3  
 $Z$  : check negative  
 $Q$  : check  $\frac{1}{2}, \frac{3}{4}$   
 $R$  : check  $\pi, e, \sqrt{2}$
- $Ex$  there exists a number that is both even and prime  
 $\exists n$  such that  $P(n) \wedge E(n).$   
 $\forall$  a prime number such that  $E(n)$   
 $\forall$  an even number such that  $P(n).$   
 "implicit quantification"  
 $y$  a number is an int, then ...  
 $\forall n \in \mathbb{Z}$   
 indefinite article "a"  
 "24 can be written as the sum of two ints"  
 $\forall m, n \in \mathbb{Z}$  such that  $m + n = 24$   
 $P(x) \Leftrightarrow Q(x).$   
 $\{x: P(x)\} = \{x: Q(x)\}$
- homework 3.1 2,3,9,11,13,19

## 3.2 Negations of quantified statements

- $(\forall x \in D, Q(x)) \equiv \exists x \in D, such\ that\ Q(x)$   
 $(\exists x \in D such\ that\ Q(x)) \equiv \forall x \in D, \neg Q(x).$   
 Negate "some dogs are bad dogs"

” all dogs are god dogs”

$(\forall x \text{ if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } Q(x)$

*negating*  $\forall$

*gives*  $\exists$

*negating*  $\exists$

*gives*  $\forall$

- Universal statements that are vacuously true  
Consider an empty jar  
"all the grapes in the jar are red"  
 $\forall g \in J$   
negating that =  
"some grapes in the jar are not red"  
 $\exists g \in J$  Such that  $g$  is not red.
- Universal Conditional  
 $\forall x \in D, \text{ if } P(x) \text{ then } Q(x)$

### 3.3 Statments sith multiple quantifiers

from quiz

any interger equals twice some interger

which is " $\forall n \in \mathbb{Z}$

- "everyone has someone to love"  
"for every person, there is another person that they love "  
 $\forall p \in H, \exists r \in H$  such that  $p$  loves  $r$ .
- $(\forall x \in D, \exists y \in E \text{ such that } P(x, y))$   
 $\equiv \exists x \in D \text{ such that } \forall y \in E, P(x, y)$   
 $(\exists x \in D ; \text{ such that } \forall y \text{ in } E, P(x, y))$   
 $\equiv \forall x \in D, \exists y \in E \text{ such that } P(x, y).$

### 3.4 arguments with quantified statments

- Rule of universal instantiation if a property is true of everything in a set, it is true for any particular member of the set.
- hello

# Chapter 4

## 4

### 4.1 4.1

### 4.2 4.2

- hello

### 4.3 4.3

from quiz

**1 for all ints m if m is even then  $3m + 5$  is odd**

$m = 2k$  for some  $k \in \mathbb{Z}$   $\therefore 3m + 5 = 3(2k) + 5 = 6k + 5$

**want  $6k + 5 = 2j + 1$  for some  $j \in \mathbb{Z}$  find the j**

$6k + 5 = 2j + 1$   $6k + 4 = 2j$   $3k + 2 = j$

set  $j = 3k + 2$  since  $k \in \mathbb{Z}, j \in \mathbb{Z}$

**we have  $3m + 5 = 2j + 1$  for  $j \in \mathbb{Z}$**

**2 for all ints a,b, and c if  $a \mid b$  and  $a \mid c$  then  $a \mid (b+c)$**

$a \mid b \Leftrightarrow b = ar$  for some  $r \in \mathbb{Z}$

$a \mid c \Leftrightarrow c = as$  for some  $s \in \mathbb{Z}$

$b + c = ar + as = a(r + s)$

since  $r, s \in \mathbb{Z}, r + s \in \mathbb{Z}$

$a \mid (b + c)$

### 4.4 Division into cases and the quotient-remainder theorem

- (quotient-remainder)  
given any integer n and integer  $d > 0$ , there exist unique integers q, r such that  $n = dq + r$  where  $0 \leq r < d$
- given  $n = dq + r$  we say "n div d" is the integer quotient q obtained when n is divided by d, and "n mod d" is the integer remainder obtained when n is divided by d.
- representations of integers  
we can prove all integers are even or odd using the QR  
given any n,  $2 \mid d$  then  $n = 2q + r$  where  $0 \leq r < 2$   
so  $r = 0$  or  $r = 1$   
two cases  
 $n = 2q + 0$  or  $n = 2q + 1$   
but  $q \in \mathbb{Z}$   
so n is even or n is odd
- prove that any two consecutive integers have opposite parity (that is one is even and one is odd).  
let m,  $m + 1$  be two consecutive integers  
case 1



$m$  is even therefore  $m = 2k = 2k + 1$  for the same  $k \in \mathbb{Z}$  therefore  $m + 1$  is odd

case 2

therefore  $m = 2k + 1$  for some  $k \in \mathbb{Z}$  then  $m + 1 = 2k + 1 + 1$

$$= 2k + 2$$

$$2(k + 1)$$

therefore  $(k + 1) \in \mathbb{Z}$  therefore  $m + 1$  is even

- can have more than 2 cases
- show that any integer  $n$  can be written in one of the four forms

$$n = 4q$$

$$n = 4q + 1$$

$$n = 4q + 2$$

$$n = 4q + 3$$

for some  $q \in \mathbb{Z}$

proof

QR - ther  $6 + d = 4$  then for all integers ,  $n$  ,  $n = 4q + r$  where  $0 \leq r < 4$

four cases  $r = 0, 1, 2, 3$

$$n = 4q$$

$$n = 4q + 1$$

$$n = 4q + 2$$

$$n = 4q + 3$$

## 4.5

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# Chapter 5

## Sequences and math induction

### 5.1 sequences

- def a sequence ins a function whose domain is either all the integers or all the integers greater than or equal to an integer.
- $a_0, a_1, a_2, a_n$   
an element  $a_k$  ( " a sub k " ) is called a term of the sequence and  $k$  is its index
- when the sequence doesn't have a final term it is an infinite sequence

- Explicit formulae for sequences

$$a_k = \frac{k}{k+1} \text{ for all integers } k \geq 1$$

$$a_1 = \frac{1}{2}$$

$$b_i = \frac{i-1}{i} \text{ for all integers } i \geq 2$$

$$b_2 = 1/2$$

$$b_3 = 2/3$$

$$b_4 = 3/4 \text{ etc}$$

$$a_k = b_i \text{ where } k = i - 1$$

- alternating sequences

$$c_j = (-1)^j \text{ for } j \geq 0$$

$$c_0 = (-1)^0 = 1$$

$$c_1 = (-1)^1 = -1$$

$$c_2 = (-1)^2 = 1$$

$$c_3 = (-1)^3 = -1$$

example

$$1, -1/4, 1/9, -1/16, 1/25, -1/36$$

1 in numerator

signs alternate

perfect squares in denominator

list  $k = 1, 2, 3, \text{etc}$  for each term

$$a_k = \frac{(-1)^{k+1}}{k^2}$$

positive  $a_k$  when  $k$  is odd

- summation notation

want to take the sum of all the elements in a sequence  $(a_1 + a_2 + a_3 + \dots)$

$$\sum_{k=1}^n a_k$$

let  $a_1 = -2, a_2 = -1, a_3 = 0, a_4 = 1, a_5 = 2$

compute

$$\sum_{k=1}^5 ak = a_1 + a_2 + a_3 + a_4 + a_5 = -2, + -1 + 0_1 + 2 = 0$$

$$\sum_{k=2}^2 a_k = a_2 = -1$$

$$\sum_{k=1}^2 a_2 k = a_2(1) + a_2(2) = a_2 + a_4 = -1 + 1 = 0$$

$$\sum_{k=1}^5 k^2 \text{ or } \sum_{i=0}^8 \frac{(-1)^i}{i+1}$$

$$\sum_{i=0}^n \frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \dots + \frac{(-1)^n}{n+1}$$

$$= 1 - 1/2 + 1/3 + \dots + \frac{(-1)^n}{n+1}$$

$$1/n + 2/(n+1) + 3/(n+3) + \dots + (n+1)/(2n)$$

$$\text{the general term } a_k = (k+1)/(n+k) \quad 2n = n+n$$

$$\text{the ints range from } (k=0) \text{ to } (k=n)$$

$$\sum_{k=0}^n \frac{k+1}{n+k}$$

- telescoping sum

$$\sum_{k=1}^n \frac{1}{k(k+1)}$$

$$1/k - 1/(k+1) = (k+1)/(k(k+1)) = (k)/(k(k+1)) = 1/(k(k+1))$$

$$= \sum_{k=1}^n \frac{1/k}{1/(k+1)} = ((1/1)-(1/2))+((1/2)+(1/3))+((1/3)-(1/4))+\dots+((1/(n-1))-(1/n))+((1/n)-(1/(n+1)))$$

- product notation

$$\prod_{k=1}^n a_k = (a_1)(a_2)(a_3)\dots(a_n)$$

$$\prod_{k=1}^5 k = 1 * 2 * 3 * 4 * 5 = 120$$

$$\sum_{k=1}^n a_k + \sum_{k=1}^n b_k = \sum_{k=1}^n (a_k + b_k)$$

$$\sum_{k=0}^n a_k = a_0 + \sum_{k=1}^n a_k$$

$$2 \ c * \sum_{k=1}^n a_k = \sum_{k=1}^n a_k * c$$

$$(\prod_{k=1}^n a_k)(\prod_{k=1}^n b_k) = \prod_{k=1}^n (a_k * b_k)$$

- factorials

$$n! = n(n-1)(n-2)\dots 3 * 2 * 1 = n(n-1)!$$

$$n! = \prod_{k=1}^n k$$

$$\text{we define } 0! = 1 \quad (8!)/(7!) = (8 * 7!)/(7!) = 8$$

$$n, r \in \mathbb{Z} \quad 0 \leq r \leq n$$

## homework

### section 5.1

3,12,19,31,40,49,55,65,75

test october 9th chapters 2-5

defs, proofs 4 questions 3 to do and bonus

## 5.2 Math induction

- principle of math induction

let  $P(n)$  be a property defined for integers  $n$  and let  $a$  be a fixed int

suppose the following to be true

$P(a)$  is true

For all ints  $k \geq a$ , if  $P(k)$  is true, then  $P(k+1)$  is true.

then the statement  $\forall \text{ints } n \geq a, P(n)$  is true

- method of proof by math induction

show that  $P(a)$  is true

suppose  $P(k)$  is true for  $k \geq a$ , use this to show  $P(k+1)$  true

- example

$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$  for all ints  $n \geq 1$  which is  $P(n)$

1 show  $P(1)$  is true  $n = 1$

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

2 assume  $P(k)$  is true

$$\sum_{j=1}^k j = \frac{k(k+1)}{2}$$

$$\begin{aligned} \text{show } P(k+1) \quad \sum_{j=1}^{k+1} (k+1)j &= \sum_{j=1}^k k j + (k+1) = \frac{k(K+1)}{2} + k + 1 = \frac{k^2 + k + 2k + 2}{2} = \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2} \text{ so } P(k+1) \text{ is true} \end{aligned}$$

### 5.3 more math induction

1 show  $P(a)$  true

2: Assume  $P(k)$  is true for some  $k \geq a$  use this to prove  $P(k+1)$  is true

For all integers  $n \geq 0, 2^{(2n)} - 1$  is divisible by 3

pf: show that  $P(0)$  is true

$n = 0 : 2 - 1$  is divisible by 3

$$2^0 - 1 = 1 - 1 = 0. 0^n = 3^d * 0^k$$

so 0 is divisible by 3

2. Assume  $P(k)$  is true for  $k \geq 0$

know :  $2^{(2k)} - 1$  is divisible by 3 for  $k \geq 0$

So  $2^{(2k)} - 1 = 3r$  for some  $r \in \mathbb{Z}$

$$2^{(2(k+1))} - 1 = 2^{(2k+2)} - 1 = 2^{2k} * 2^2 - 1 = 4 * 2^{2k} - 1 = 3 * 2^{2k} + 2^{2k} - 1 = 3 * 2^{2k} + 3r = 3(2^{2k} + r)$$

let  $s = 2^{2k} + r$  since  $k, r \in \mathbb{Z}, s \in \mathbb{Z}$  Then  $2^{(2(k+1))} - 1 = 3s, s \in \mathbb{Z}$  So  $2^{(2(k+1))}$  is divisible by 3 Thus  $P(k+1)$  is true So  $P(n)$  is true for all integers  $n \geq 0$

#### Prove an inequality

For all integers  $n \geq 3, 2n + 1 < 2^n$

show that  $P(3)$  is true:  $n = 3 : 2(3) + 1 = 7, 2^3 = 8, 7 < 8$

assume  $2k + 1 < 2^k$  for  $k \geq 3$  then  $2(k+1) + 1 = 2k + 2 + 1 = (2k + 1) + 2 < 2^k + 2^1$  we know  $2 < 2^n \forall n > 1$

remind: if  $a < b$  then  $a + c < b + c$

In particular, for  $k \geq 3, 2 < 2^k$

then  $2(k+1) + 1 < 2^k + 2 < 2^k + 2^k$  Now  $2^k + 2^k = 2 * 2^k = 2^{(k+1)}$

thus  $2(k+1) + 1 < 2^{(k+1)}$

define a sequence  $a_1, a_2, a_3, \dots$  as follows :  $a_1 = 2a_k = 5 * a_k - 1, \forall k \geq 2$

( $a_2 = 5a_1 = 5 * 2 = 10$ ) prove that for  $n \geq 1, a_n = 2 * 5^{(n-1)}$

pf:  $P(1) : a_1 = 2 * 5^{(1-1)}$  (check)  $a_1 = 2 * 5^0 = 2$  so  $P(1)$  is true

Assume  $P(k)$  is true. then  $a_k = 2 * 5^k - 1$  for  $k \geq 1. a_{k+1} = 5a_k = 5 * 2 * 5^k - 1 = 2 * (5 * 5^k - 1) = 2 * 5^{k+1} - 1 + 1 = 2 * 5^{k+1}$  So  $a_{k+1} = 2 * 5^{k+1}$  So  $P(k+1)$  is true

## MidTerm

write the definitions for even, odd, prime, composite, divisibility

ex: even  $h = 2k$  for some  $k \in \mathbb{Z}$

truth tables for  $p \Rightarrow q, p^q, p \vee q$

determine whether an argument is valid

identify the major and minor premises, the conclusion and any critical rows

T/F ch 2 thru 3 : negations of logical statements ; contrapositives of logical statements; valid and invalid argument forms ; translating between formal and informal language

four proofs pick three 2 are direct proofs

1 is by contradiction or contraposition

1 by math induction

memorize

all

the

things

5.3 homework

6,8,19,24

## 5.4 strong math induction

Let  $P(n)$  be a property defined for integration, and let  $a$  and  $b$  be fixed ints with  $a \leq b$

Suppose the following statements are true

1  $P(a), P(a+1), \dots, P(b)$  are all true

2 For any  $k \geq b$  if  $P(i)$  is true for all  $i, a \leq i \leq k$ , then  $P(k+1)$  is true

then  $P(n)$  is true for  $n \geq a$

Method

1 Prove  $P(a), P(a+1), \dots, P(b)$  are true

2 Suppose for all  $i$  such that  $a \leq i \leq k$   $P(i)$  is true and use this to show  $P(k+1)$  is true

example :

Any int  $n > 1$  is divisible by a prime number.

$P(n)$  : "  $n$  is divisible by a prime number " " $\exists$  prime  $p$  such that  $p|n$ ."

Show  $P(2)$  is true

2 is a prime number and  $2 \nmid 2$

Suppose that for some  $k \geq 2, P(i)$  is true for  $2 \leq i \leq k$

Want to show :  $k+1$  is divisible by a prime

Case 1  $k+1$  is prime. Then  $k+1|k+1$  so  $\exists p$  such that  $p|k+1$

case 2 ;  $k+1$  is composite then  $\exists a, b \in \mathbb{N}$  such that  $k+1 = ab$ .  $a < k+1$  and  $1 < b < k+1$  then  $2 \leq a \leq k$  so  $P(a)$  is true