

Exercises from Charles Pinter's 'A Book of Abstract Algebra'

Unathi Skosana

July 18, 2021

Exercises accompanying the notes from Charles Pinter's 'A Book of Abstract Algebra'

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Operations **1**

Operations

Exercise 1. For each rule, is it an operation, if not, why?

1. $a * b = \sqrt{|ab|}$, on the set \mathbb{Q}
No. If $a = b = 1$, then $1 * 1 = \sqrt{2} \notin \mathbb{Q}$
2. $a * b = a \ln b$, on the set $\{x \in \mathbb{R} : x > 0\}$.
No. It is not closed under $*$, e.g. $0 < b < 1$, then $a * b = a \ln b < 0$
3. $a * b$ is a root of the equation, $x^2 = a^2 b^2$, on the set \mathbb{R}
No. The operation isn't uniquely defined. $x^2 = a^2 b^2$ has two roots, namely $+ab$ and $-ab$
4. Subtraction, on the set \mathbb{Z} .
Yes.
5. Subtraction, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.
No. e.g. $b = a + 1$ then $a * b = -1$ which is not in the set.
6. $a * b = |a - b|$, on the set $\{n \in \mathbb{Z} : n \geq 0\}$
Yes. (distance metric)

Exercise 2. Indicate whether or not :

- I it is commutative,
 - II it is associative,
 - III \mathbb{R} has an identity element with respect to $*$
 - IV every $x \in \mathbb{R}$ has an inverse with respect to $*$
1. $x * y = x + 2y + 4$

I Commutativity:

$$y * x = y + 4x + 4 \neq x + 4y + 4 \quad \forall x, y \in \mathbb{R} \quad (1)$$

Not commutative

II Associativity:

$$\begin{aligned}
 x * (y * z) &= x * (y + 2z + 4) = x + 2(y + 2z + 4) + 4 \\
 (x * y) * z &= (x + 2y + 4) * z = x + 2y + 4 + 2z + 4 \\
 x * (y * z) &\neq (x * y) * z
 \end{aligned} \tag{2}$$

Not associative

III Existence of identity:

$$\begin{aligned}
 x * e &= x + 2e + 4 = x \\
 2e &= 4 \implies e = -2
 \end{aligned} \tag{3}$$

Check

$$\begin{aligned}
 x * 2 &= x + -2 * 2 + 4 = x \\
 2 * x &= -2 + 2x + 4 = 2 * (x + 1)
 \end{aligned} \tag{4}$$

No identity element, and thus no inverses.

$$2. \ x * y = x + 2y - xy$$

I Commutativity:

$$y * x = y + 2x - yx \neq x + 2y - xy \quad \forall x, y \in \mathbb{R} \tag{5}$$

II Associativity:

$$\begin{aligned}
 x * (y * z) &= x * (y + 2z - yz) = x + 2(y + 2z - yz) - x(y + 2z - yz) \\
 &= x + 2y + 4z - 2yz - xy - 2xz + xyz \\
 (x * y) * z &= (x + 2y - yx) * z = x + 2y - yx + 2z - (x + 2y - yx)z \\
 &= x + 2y - yx + 2z - xz - 2yz + yxz \\
 x * (y * z) &\neq (x * y) * z
 \end{aligned} \tag{6}$$

III Existence of identity:

$$\begin{aligned}
 x * e &= x + 2e - xe = x \\
 2e - xe &= e(2 - x) = 0 \implies e = 0
 \end{aligned} \tag{7}$$

Check

$$\begin{aligned}
 x * 0 &= x + 0 - 0 = x \\
 0 * x &= 0 + 2x - 0 = 2x
 \end{aligned} \tag{8}$$

No identity element, thus no inverses.

$$3. \ x * y = |x + y|$$

I Commutativity:

$$y * x = |y + x| = |x + y| = x * y \quad (9)$$

II Associativity:

$$\begin{aligned} x * (y * z) &= x * |y + z| = |x + |y + z|| \\ (x * y) * z &= |x + y| * z = ||x + y| + z| \end{aligned} \quad (10)$$

$x * (y * z) \neq (x * y) * z$, to see this set $x = 1, y = -2, z = 0$

$$|1 + |-2|| \neq ||1 - 2||$$

III Existence of identity:

$$x * e = |x + e| = x \implies e = 0 \quad (11)$$

Check

$$\begin{aligned} x * 0 &= |x + 0| = x \\ 0 * x &= |0 + x| = x \end{aligned} \quad (12)$$

IV Existence of inverses:

$$x * x' = |x + x'| = 0 \implies x' = -x \quad (13)$$

Check

$$\begin{aligned} x * (-x) &= |x + (-x)| = 0 \\ (-x) * x &= |(-x) + x| = 0 \end{aligned} \quad (14)$$

4. $x * y = |x - y|$

I Commutativity:

$$y * x = |y - x| = |-(x - y)| = |x - y| \quad (15)$$

II Associativity:

$$\begin{aligned} x * (y * z) &= x * |y - z| = |x - |y - z|| \\ (x * y) * z &= |x - y| * z = ||x - y| - z| \end{aligned} \quad (16)$$

$x * (y * z) \neq (x * y) * z$, to see this set $x = 1, y = -2, z = 0$

III Existence of identity:

$$x * e = |x - e| = x \implies e = 0 \quad (17)$$

Check

$$\begin{aligned} x * e &= |x - 0| = x \\ e * x &= |0 - x| = x \end{aligned} \quad (18)$$

IV Existence of inverses:

$$x * x' = |x - x'| = 0 \implies x' = x \quad (19)$$

Check

$$\begin{aligned} x * x' &= |x - x'| = |x - x| = 0 \\ x' * x &= |x' - x| = |x - x| = 0 \end{aligned} \quad (20)$$

5. $x * y = xy + 1$

i Commutativity:

$$y * x = yx + 1 = xy + 1 = x * y \quad (21)$$

ii Associativity:

$$\begin{aligned} x * (y * z) &= x * (yz + 1) = x(yz + 1) + 1 = xyz + x + 1 \\ (x * y) * z &= (xy + 1) * z = (xy + 1)z + 1 = xyz + z + 1 \end{aligned} \quad (22)$$

$$(x * y) * z \neq x * (y * z)$$

iii Existence of identity:

$$\begin{aligned} x * e &= xe + 1 = x \\ xe &= x - 1 \\ e &= \frac{x - 1}{x} \end{aligned} \quad (23)$$

Check

$$\begin{aligned} x * e &= x \frac{x - 1}{x} + 1 = x - 1 + 1 = x \\ e * x &= \frac{x - 1}{x} x + 1 = x - 1 + 1 = x \end{aligned} \quad (24)$$

6. $x * y = \max \{x, y\}$

I Commutativity:

$$y * x = \max \{y, x\} = x * y \quad (25)$$

II Associativity:

$$\begin{aligned} x * (y * z) &= x * \max \{y, z\} = \max \{x, \max \{y, z\}\} \\ (x * y) * z &= \max \{x, y\} * z = \max \{\max \{x, y\}, z\} \end{aligned} \quad (26)$$

Regardless of the how three numbers are compose with *, they produce the same result; that is larger of three numbers. Thus $x * (y * z) = (x * y) * z$.

III Existence of identity:

$$x * e = \max \{x, e\} = x \implies e \leq x \quad \forall x \in \mathbb{R} \quad (27)$$

No such element in \mathbb{R} unless you extend it to include $-\infty$ and ∞ . By this no inverses.

7. $x * y = \frac{xy}{x+y+1}$

i Commutativity:

$$y * x = \frac{yx}{y+x+1} = \frac{xy}{x+y+1} = x * y \quad (28)$$

ii Associativity:

$$\begin{aligned} x * (y * z) &= x * \frac{yz}{y+z+1} = \frac{\frac{xyz}{y+z+1}}{x + \frac{yz}{y+z+1} + 1} = \frac{xyz}{xy + xz + x + yz + y + z + 1} \\ (x * y) * z &= \frac{xy}{x+y+1} * z = \frac{\frac{xyz}{x+y+1}}{\frac{xy}{x+y+1} + z + 1} = \frac{xyz}{xy + xz + yz + z + x + y + 1} \end{aligned} \quad (29)$$

$$(x * y) * z = x * (y * z)$$

iii Existence of Identity:

$$\begin{aligned} x * e &= \frac{xe}{x+e+1} = x \\ x^2 + xe + x &= xe \\ x(x+1) &= 0 \end{aligned} \quad (30)$$

Only holds for $x = 0, x = -1$, thus no solution on the set of positive real numbers.
Thus no inverses as well.

Exercise 3. Let \mathcal{A} be the two-element set $\mathcal{A} = \{a, b\}$

1. Write the tables of all operations on \mathcal{A} and label these operations. There are four possible ordered pairs and for each ordered, we either assign it to a or b ; thus there are $2 \times 2 \times 2 \times 2 = 16$ operations possible.

$(x, y)_{0_1}$	$x * y$	$(x, y)_{0_2}$	$x * y$	$(x, y)_{0_3}$	$x * y$	$(x, y)_{0_4}$	$x * y$
(a, a)	a	(a, a)	a	(a, a)	a	(a, a)	a
(a, b)	a	(a, b)	a	(a, b)	a	(a, b)	b
(b, a)	a	(b, a)	a	(b, a)	b	(b, a)	a
(b, b)	a	(b, b)	b	(b, b)	a	(b, b)	a

$(x, y)_{0_5}$	$x * y$	$(x, y)_{0_6}$	$x * y$	$(x, y)_{0_7}$	$x * y$	$(x, y)_{0_8}$	$x * y$
(a, a)	b	(a, a)	b	(a, a)	b	(a, a)	b
(a, b)	a	(a, b)	b	(a, b)	a	(a, b)	a
(b, a)	a	(b, a)	a	(b, a)	b	(b, a)	a
(b, b)	a	(b, b)	a	(b, b)	a	(b, b)	b

$(x, y)_{0_9} \mid x * y$	$(x, y)_{0_{10}} \mid x * y$	$(x, y)_{0_{11}} \mid x * y$	$(x, y)_{0_{12}} \mid x * y$
$(a, a) \mid a$	$(a, a) \mid a$	$(a, a) \mid a$	$(a, a) \mid b$
$(a, b) \mid b$	$(a, b) \mid a$	$(a, b) \mid b$	$(a, b) \mid b$
$(b, a) \mid a$	$(b, a) \mid b$	$(b, a) \mid b$	$(b, a) \mid b$
$(b, b) \mid b$	$(b, b) \mid b$	$(b, b) \mid a$	$(b, b) \mid a$

$(x, y)_{0_{13}} \mid x * y$	$(x, y)_{0_{14}} \mid x * y$	$(x, y)_{0_{15}} \mid x * y$	$(x, y)_{0_{16}} \mid x * y$
$(a, a) \mid b$	$(a, a) \mid b$	$(a, a) \mid a$	$(a, a) \mid b$
$(a, b) \mid b$	$(a, b) \mid a$	$(a, b) \mid b$	$(a, b) \mid b$
$(b, a) \mid a$	$(b, a) \mid b$	$(b, a) \mid b$	$(b, a) \mid b$
$(b, b) \mid b$	$(b, b) \mid b$	$(b, b) \mid b$	$(b, b) \mid b$

2. Which of the operations are commutative?

$0_1, 0_2, 0_5, 0_8, 0_{11}, 0_{12}, 0_{15}, 0_{16}$

3. Which of the operations are associative?

$0_1, 0_3, 0_9, 0_{10}, 0_{11}, 0_{12}, 0_{15}, 0_{16}$

4. Which of the operations has an identity element?

$0_2, 0_8, 0_{11}, 0_{15}$

5. Which of the operations does every element have an inverse?

$0_8, 0_{11}$

Exercise 4. Let \mathcal{A}^* be the set of all sequences of symbols in the alphabet \mathcal{A} . Define an operation on \mathcal{A}^* called concatenation: If \mathbf{a} and \mathbf{b} are in \mathcal{A}^* , say $\mathbf{a} = a_1 a_2 \dots a_n$ and $\mathbf{b} = b_1 b_2 \dots b_m$. *e.g.*, in the alphabet $\mathcal{A} = \{0, 1\}$, if $\mathbf{a} = 1001$ and $\mathbf{b} = 010$, then $\mathbf{ab} = 1001010$. Let the symbol λ denote the empty sequence

1. Show that the operation is associative

Let $\mathbf{a} = a_1 a_2 \dots a_n$, $\mathbf{b} = b_1 b_2 \dots b_m$, and $\mathbf{c} = c_1 c_2 \dots c_k$

$$\begin{aligned} \mathbf{a}(\mathbf{bc}) &= a_1 a_2 \dots a_n (b_1 b_2 \dots b_m c_1 c_2 \dots c_k) = a_1 a_2 \dots a_n b_1 b_2 \dots b_m c_1 c_2 \dots c_k \\ (\mathbf{ab})\mathbf{c} &= (a_1 a_2 \dots a_n b_1 b_2 \dots b_m) c_1 c_2 \dots c_k = a_1 a_2 \dots a_n b_1 b_2 \dots b_m c_1 c_2 \dots c_k \end{aligned} \quad (31)$$

2. Explain why the operation is not commutative?

For this operation to be commutative, we demand that every input sequence \mathbf{a}, \mathbf{b} in the alphabet \mathcal{A} ,

$$a_1 a_2 \dots a_n b_1 b_2 \dots b_m = b_1 b_2 \dots b_m a_1 a_2 \dots a_n \quad (32)$$

Consider the case $n > m$, this would mean

$$\begin{aligned}
a_i &= b_i & i &= 0 \dots m \\
a_i &= a_i & j &= m + 1 \dots n
\end{aligned}
\tag{33}$$

certainly this can't be true for all **a**, **b** in the alphabet A . *e.g.* **a** = 11, **b** = 0

3. Prove that there is an identity element for this operation

$$\begin{aligned}
\mathbf{a}(\lambda) &= a_1 a_2 \dots a_n () = \mathbf{a} \\
\lambda \mathbf{a} &= () a_1 a_2 \dots a_n = \mathbf{a}
\end{aligned}
\tag{34}$$

Thus the empty sequence is the identity element.