Notes on Charles Pinter's book of abstract algebra

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These are notes taken while reading Charles Pinter's 'A Book Of Abstract Algebra'

Contents

Operations

Question 1. *What is an* ***operation*** *on a set A?*

Definitions 2 (Informal definition)**.** An operation is any rule which assigns to each ordered pair of elements of A a unique element in A.

Definitions 3 (Formal definition)**.** Let A be any set:
An operation $*$ on A is a rule which assigns to each ordered pairs (a, b) of elements of A exactly one $a * b$ in A

- $a * b$ is defined for *every* ordered pair (a, b) of elements of A. [1]
- $a * b$ must be *uniquely* defined. [2]
- If $a, b \in A$, then $a * b \in A$. [3]

Definitions 4 (Superfluous properties)**.**
- An operation $*$ is said to be *commutative* if it satisfies

$$a * b = b * a \tag{1}$$

for any two elements a and b in A.

- An operation $*$ is said to be *associative* if it satisfies

$$(a * b) * c = a * (b * c) \tag{2}$$

for any three elements a, b and c in A.

- The *identity* element e with respect to the operation $*$ has the property that:

$$e * a = a \quad \text{and} \quad a * e = a \tag{3}$$

for every element in a in A.

- The inverse of any element a, denoted by a^{-1} has the property that;

$$a * a^{-1} = e \quad \text{and} \quad a^{-1} * a = e \tag{4}$$

[1] In $\mathbb{R}$, division does not qualify as operation since it does not satisfy this condition. i.e. the ordered pair $(a, 0)$ has undefined quotient $a/0$.

[2] If $\diamond$ is defined on (a, b) to be the number whose square is ab. In $\mathbb{R}$, $\diamond$ does not qualify as an operation since $2 \diamond 2$ could be either 2, or $+2$

[3] A is closed under the operation $*$

Exercises

A. For each rule, is it an operation, if not, why?

1. $a * b = \sqrt{|ab|}$, on the set $\mathbb{Q}$
   No. If $a = b = 1$, then $1 * 1 = \sqrt{2} \notin \mathbb{Q}$

2. $a * b = a \ln b$, on the set $\{x \in \mathbb{R} : x > 0\}\}$.
   No. It is not closed under $*$, e.g $0 < b < 1$, then $a * b = a \ln b < 0$

3. $a * b$ is a root of the equation, $x^2 = a^2b^2$, on the set $\mathbb{R}$
   No. The operation isn't uniquely defined. $x^2 = a^2b^2$ has to two roots, namely $+ab$ and $-ab$

4. Subtraction, on the set $\mathbb{Z}$.
   Yes.

5. Subtraction, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.
   No. e.g. $b = a + 1$ then $a * b = -1$ which is not in the set.

6. $a * b = |a - b|$, on the set $\{n \in \mathbb{Z} : n \geq 0\}$
   Yes. (distance metric)