

Notes on Charles Pinter's 'A Book Of Abstract Algebra'

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July 10, 2021

Personal notes taken while studying Charles Pinter's 'A Book Of Abstract Algebra'

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Operations

Question 1. What is an *operation* on a set A ?

Definitions 2 (Informal definition). An operation is any rule which assigns to each ordered pair of elements of A a unique element in A .

Definitions 3 (Formal definition). Let A be any set:

An operation $*$ on A is a rule which assigns to each ordered pairs (a, b) of elements of A exactly one $a * b$ in A , such that:

- $a * b$ is defined for *every* ordered pair (a, b) of elements of A .¹
- $a * b$ must be *uniquely* defined.²
- If $a, b \in A$, then $a * b \in A$.³

Definitions 4 (Commutativity). An operation $*$ is said to be *commutative* if it satisfies

$$a * b = b * a \quad (1)$$

for any two elements a and b in A .

Definitions 5 (Associativity). An operation $*$ is said to be *associative* if it satisfies

$$(a * b) * c = a * (b * c) \quad (2)$$

for any three elements a, b and c in A .

Definitions 6 (Identity element). The *identity* element e with respect to the operation $*$ has the property that:

$$e * a = a \quad \text{and} \quad a * e = a \quad (3)$$

is true for every element a in A .

Definitions 7 (Inverses). The inverse of any element a , item denoted by a^{-1} has the property that:

$$a * a^{-1} = e \quad \text{and} \quad a^{-1} * a = e \quad (4)$$

¹ In \mathbb{R} , division does not qualify as operation since it does not satisfy this condition. i.e. the ordered pair $(a, 0)$ has undefined quotient $a/0$.

² If \circ is defined on (a, b) to be the number whose square is ab . In \mathbb{R} , \circ does not qualify as an operation since $2 \circ 2$ could be either 2, or $+2$.

³ A is closed under the operation $*$

The Definitions of Groups

Question 8. *What is a group?*

Definitions 9 (Informal definition). A group is defined to be a set with an operation $(*)$ which is associative, has an identity element, and each element in the set has an inverse.

Definitions 10 (Formal definition). A group is a set G , together with an operation $*$ which satisfies:

- $*$ is associative.
- There exists an element e in G such that $a * e = a$ and $e * a = a$ for every element a in G .
- For every element a in G , there is an element a^{-1} in G such that $a * a^{-1} = e$ and $a^{-1} * a = e$.

A group as defined above is usually denoted by the pair symbol $(G, *)$, which denotes that a group is a set G together with the operation $*$.⁴

Example 13 (Finite groups: Groups of integers modulo n). The group of integers modulo $n > 1$ consists of the set

$$\{0, 1, 2, \dots, n-1\} \quad (5)$$

together with the operation of addition modulo n

⁴ If there is no chance of ambiguity, the group is usually denoted with just the letter G .

Remark 11. The set of integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is a group with the operation of addition, denoted by $(\mathbb{Z}, +)$. Similarly, the set of rational numbers and addition $(\mathbb{Q}, +)$, and the set of real numbers $(\mathbb{R}, +)$.

Remark 12. Many a times, algebraic structures apparent in the study of natural phenomena (that is to say in physics) are groups, *i.e.* quantum spin, angular momentum

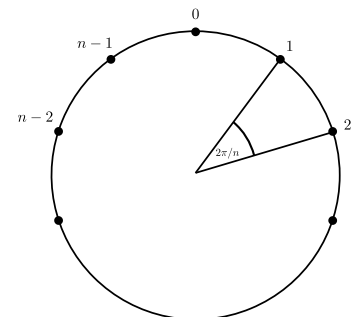


Figure 1