Exercises from Charles Pinter's 'A Book of Abstract Algebra'

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Exercises accompanying the notes from Charles Pinter's 'A Book of Abstract Algebra'

Contents

Operations

Exercise 1. For each rule, is it an operation, if not, why?

1. $a * b = \sqrt{|ab|}$, on the set $\mathbb{Q}$
   No. If $a = b = 1$, then $1 * 1 = \sqrt{2} \notin \mathbb{Q}$

2. $a * b = a \ln b$, on the set $\{x \in \mathbb{R} : x > 0\}\}$.
   No. It is not closed under $*$, e.g $0 < b < 1$, then $a * b = a \ln b < 0$

3. $a * b$ is a root of the equation, $x^2 = a^2 b^2$, on the set $\mathbb{R}$
   No. The operation isn't uniquely defined. $x^2 = a^2 b^2$ has to two roots, namely $+ab$ and $-ab$

4. Subtraction, on the set $\mathbb{Z}$.
   Yes.

5. Subtraction, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.
   No. e.g. $b = a + 1$ then $a * b = -1$ which is not in the set.

6. $a * b = |a - b|$, on the set $\{n \in \mathbb{Z} : n \geq 0\}$
   Yes. (distance metric)

Exercise 2. Indicate whether or not :

I it is commutative,

II it is associative,

III $\mathbb{R}$ has an identity element with respect to $*$

IV every $x \in \mathbb{R}$ has an inverse with respect to $*$

1. $x * y = x + 2y + 4$

   I Commutativity:

$$y * x = y + 4x + 4 \neq x + 4y + 4 \quad \forall x, y \in \mathbb{R} \tag{1}$$

   Not commutative

II Associativity:

$$
\begin{aligned}
&x * (y * z) = x * (y + 2z + 4) = x + 2(y + 2z + 4) + 4 \\
&(x * y) * z = (x + 2y + 4) * z = x + 2y + 4 + 2z + 4 \\
&x * (y * z) \neq (x * y) * z
\end{aligned}
\tag{2}
$$

Not associative

III Existence of identity:

$$
\begin{aligned}
&x * e = x + 2e + 4 = x \\
&2e = 4 \implies e = -2
\end{aligned}
\tag{3}
$$

Check

$$
\begin{aligned}
&x * 2 = x + -2 * 2 + 4 = x \\
&2 * x = -2 + 2x + 4 = 2 * (x + 1)
\end{aligned}
\tag{4}
$$

No identity element, and thus no inverses.

2. $x * y = x + 2y - xy$

I Commutativity:

$$
y * x = y + 2x - yx \neq x + 2y - xy \quad \forall x, y \in \mathbb{R}
\tag{5}
$$

II Associativity:

$$
\begin{aligned}
&x * (y * z) = x * (y + 2z - yz) = x + 2(y + 2z - yz) - x(y + 2z - yz) \\
&= x + 2y + 4z - 2yz - xy - 2xz + xyz \\
&(x * y) * z = (x + 2y - yx) * z = x + 2y - yx + 2z - (x + 2y - yx)z \\
&= x + 2y - yx + 2z - xz - 2yz + yxz \\
&x * (y * z) \neq (x * y) * z
\end{aligned}
\tag{6}
$$

III Existence of identity:

$$
\begin{aligned}
x * e = x + 2e - xe = x \\
2e - xe = e(2 - x) = 0 \implies e = 0
\end{aligned}
\tag{7}
$$

Check

$$
\begin{aligned}
x * 0 = x + 0 - 0 = x \\
0 * x = 0 + 2x - 0 = 2x
\end{aligned}
\tag{8}
$$

No identity element, thus no inverses.

3. $x * y = |x + y|$

I Commutativity:

$$y * x = |y + x| = |x + y| = x * y \tag{9}$$

II Associativity:

$$\begin{aligned} x * (y * z) &= x * |y + z| = |x + |y + z|| \\ (x * y) * z &= |x + y| * z = ||x + y| + z| \end{aligned} \tag{10}$$

$x * (y * z) \neq (x * y) * z$, to see this set $x = 1, y = -2, z = 0$

$$|1 + |-2|| \neq ||1 - 2||$$

III Existence of identity:

$$x * e = |x + e| = x \implies e = 0 \tag{11}$$

Check

$$\begin{aligned} x * 0 &= |x + 0| = x \\ 0 * x &= |0 + x| = x \end{aligned} \tag{12}$$

IV Existence of inverses:

$$x * x' = |x + x'| = 0 \implies x' = -x \tag{13}$$

Check

$$\begin{aligned} x * (-x) &= |x + (-x)| = 0 \\ (-x) * x &= |(-x) + x| = 0 \end{aligned} \tag{14}$$

4. $x * y = |x - y|$

I Commutativity:

$$y * x = |y - x| = |-(x - y)| = |x - y| \tag{15}$$

II Associativity:

$$\begin{aligned} x * (y * z) &= x * |y - z| = |x - |y - z|| \\ (x * y) * z &= |x - y| * z = ||x - y| - z| \end{aligned} \tag{16}$$

$x * (y * z) \neq (x * y) * z$, to see this set $x = 1, y = -2, z = 0$

III Existence of identity:

$$x * e = |x - e| = x \implies e = 0 \tag{17}$$

Check

$$\begin{aligned} x * e &= |x - 0| = x \\ e * x &= |0 - x| = x \end{aligned} \tag{18}$$

IV Existence of inverses:

$$x * x' = |x - x'| = 0 \implies x' = x \tag{19}$$

Check

$$\begin{aligned} x * x' &= |x - x'| = |x - x| = 0 \\ x' * x &= |x' - x| = |x - x| = 0 \end{aligned} \tag{20}$$

5. $x * y = xy + 1$

i Commutativity:

$$y * x = yx + 1 = xy + 1 = x * y \tag{21}$$

ii Associativity:

$$\begin{aligned} x * (y * z) &= x * (yz + 1) = x(yz + 1) + 1 = xyz + x + 1 \\ (x * y) * z &= (xy + 1) * z = (xy + 1)z + 1 = xyz + z + 1 \end{aligned} \tag{22}$$

$(x * y) * z \neq x * (y * z)$

iii Existence of identity:

$$\begin{aligned} x * e = xe + 1 &= x \\ xe &= x - 1 \\ e &= \frac{x-1}{x} \end{aligned} \tag{23}$$

Check

$$\begin{aligned} x * e &= x\frac{x-1}{x} + 1 = x - 1 + 1 = x \\ e * x &= \frac{x-1}{x}x + 1 = x - 1 + 1 = x \end{aligned} \tag{24}$$

6. $x * y = \max\{x, y\}$

I Commutativity:

$$y * x = \max\{y, x\} = x * y \tag{25}$$

II Associativity:

$$\begin{aligned} x * (y * z) &= x * \max\{y, z\} = \max\{x, \max\{y, z\}\} \\ (x * y) * z &= \max\{x, y\} * z = \max\{\max\{x, y\}, z\} \end{aligned} \tag{26}$$

Regardless of the how three numbers are compose with $*$, they produce the same result; that is larger of three numbers. Thus $x * (y * z) = (x * y) * z$.

III Existence of identity:

$$x * e = \max\{x, e\} = x \implies e \leq x \quad \forall x \in \mathbb{R} \tag{27}$$

No such element in $\mathbb{R}$ unless you extend it to include $-\infty$ and ∞. By this no inverses.

7. $x * y = \frac{xy}{x+y+1}$

   i Commutativity:

$$y * x = \frac{yx}{y+x+1} = \frac{xy}{x+y+1} = x * y \tag{28}$$

   ii Associativity:

$$\begin{aligned} x * (y * z) &= x * \frac{yz}{y+z+1} = \frac{\frac{xyz}{y+z+1}}{x + \frac{yz}{y+z+1} + 1} = \frac{xyz}{xy + xz + x + yz + y + z + 1} \\ (x * y) * z &= \frac{xy}{x+y+1} * z = \frac{\frac{xyz}{x+y+1}}{\frac{xy}{x+y+1} + z + 1} = \frac{xyz}{xy + xz + yz + z + x + y + 1} \end{aligned} \tag{29}$$

   $(x * y) * z = x * (y * z)$

   iii Existence of Identity:

$$\begin{aligned} x * e &= \frac{xe}{x+e+1} = x \\ x^2 &+ xe + x = xe \\ x(x &+ 1) = 0 \end{aligned} \tag{30}$$

   Only holds for $x = 0, x = -1$, thus no solution on the set of positive real numbers. Thus no inverses as well.

Exercise 3. Let A be the two-element set $A = \{a, b\}$

1. Write the tables of all operations on A and label these operations. There are four possible ordered pairs and for each ordered, we either assign it to a or b; thus there are $2{\times}2{\times}2{\times}2 = 16$ operations possible.

$(x,y)_{0_1}$	$x * y$
(a,a)	a
(a,b)	a
(b,a)	a
(b,b)	a

$(x,y)_{0_2}$	$x * y$
(a,a)	a
(a,b)	a
(b,a)	a
(b,b)	b

$(x,y)_{0_3}$	$x * y$
(a,a)	a
(a,b)	a
(b,a)	b
(b,b)	a

$(x,y)_{0_4}$	$x * y$
(a,a)	a
(a,b)	b
(b,a)	a
(b,b)	a

$(x,y)_{0_5}$	$x * y$
(a,a)	b
(a,b)	a
(b,a)	a
(b,b)	a

$(x,y)_{0_6}$	$x * y$
(a,a)	b
(a,b)	b
(b,a)	a
(b,b)	a

$(x,y)_{0_7}$	$x * y$
(a,a)	b
(a,b)	a
(b,a)	b
(b,b)	a

$(x,y)_{0_8}$	$x * y$
(a,a)	b
(a,b)	a
(b,a)	a
(b,b)	b

$(x,y)_{0_9}$	$x*y$
(a,a)	a
(a,b)	b
(b,a)	a
(b,b)	b

$(x,y)_{0_{10}}$	$x*y$
(a,a)	a
(a,b)	a
(b,a)	b
(b,b)	b

$(x,y)_{0_{11}}$	$x*y$
(a,a)	a
(a,b)	b
(b,a)	b
(b,b)	a

$(x,y)_{0_{12}}$	$x*y$
(a,a)	b
(a,b)	b
(b,a)	b
(b,b)	a

$(x,y)_{0_{13}}$	$x*y$
(a,a)	b
(a,b)	b
(b,a)	a
(b,b)	b

$(x,y)_{0_{14}}$	$x*y$
(a,a)	b
(a,b)	a
(b,a)	b
(b,b)	b

$(x,y)_{0_{15}}$	$x*y$
(a,a)	a
(a,b)	b
(b,a)	b
(b,b)	b

$(x,y)_{0_{16}}$	$x*y$
(a,a)	b
(a,b)	b
(b,a)	b
(b,b)	b

2. Which of the operations are commutative?

   $0_1, 0_2, 0_5, 0_8, 0_{11}, 0_{12}, 0_{15}, 0_{16}$

3. Which of the operations are associative?

   $0_1, 0_3, 0_9, 0_{10}, 0_{11}, 0_{12}, 0_{15}, 0_{16}$

4. Which of the operations has an identity element?

   $0_2, 0_8, 0_{11}, 0_{15}$

5. Which of the operations does every element have an inverse?

   $0_8, 0_{11}$

Exercise 4. Let A^* be the set of all sequences of symbols in the alphabet A. Define an operation on A^* called concatenation: If $\mathbf{a}$ and $\mathbf{b}$ are in A^*, say $\mathbf{a} = a_1a_2 \dots a_n$ and $\mathbf{b} = b_1b_2 \dots b_m$. *e.g.*, in the alphabet $A = \{0, 1\}$, if $\mathbf{a} = 1001$ and $\mathbf{b} = 010$, then $\mathbf{ab} = 1001010$. Let the symbol λ denote the empty sequence

1. Show that the operation is associative

   Let $\mathbf{a} = a_1a_2 \dots a_n$, $\mathbf{b} = b_1b_2 \dots b_m$, and $\mathbf{c} = c_1c_2 \dots c_k$

$$\begin{aligned}\mathbf{a}(\mathbf{bc}) &= a_1a_2 \dots a_n(b_1b_2 \dots b_m c_1c_2 \dots c_k) = a_1a_2 \dots a_n b_1b_2 \dots b_m c_1c_2 \dots c_k \\ (\mathbf{ab})\mathbf{c} &= (a_1a_2 \dots a_n b_1b_2 \dots b_m)c_1c_2 \dots c_k = a_1a_2 \dots a_n b_1b_2 \dots b_m c_1c_2 \dots c_k \end{aligned} \tag{31}$$

2. Explain why the operation is not commutative?

   For this operation to be commutative, we demand that every input sequence $\mathbf{a}, \mathbf{b}$ in the alphabet A,

$$a_1a_2 \dots a_n b_1b_2 \dots b_m = b_1b_2 \dots b_m a_1a_2 \dots a_n \tag{32}$$

   Consider the case $n > m$, this would mean

$$
\begin{aligned}
a_i &= b_i \quad i = 0 \dots m \\
a_i &= a_i \quad j = m + 1 \dots n
\end{aligned} \tag{33}
$$

certainly this can't be true for all $\mathbf{a}, \mathbf{b}$ in the alphabet A. *e.g.* $\mathbf{a} = 11, \mathbf{b} = 0$

3. Prove that there is an identity element for this operation

$$
\begin{aligned}
\mathbf{a}(\lambda) &= a_1 a_2 \dots a_n() = \mathbf{a} \\
\lambda\mathbf{a} &= () a_1 a_2 \dots a_n = \mathbf{a}
\end{aligned} \tag{34}
$$

Thus the empty sequence is the identity element.