Exercises from Charles Pinter's 'A Book of Abstract Algebra'

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Exercises accompanying the notes from Charles Pinter's 'A Book of Abstract Algebra'

Contents

Operations

Exercise 1. For each rule, is it an operation, if not, why?

1. $a * b = \sqrt{|ab|}$, on the set $\mathbb{Q}$
   No. If $a = b = 1$, then $1 * 1 = \sqrt{2} \notin \mathbb{Q}$

2. $a * b = a \ln b$, on the set $\{x \in \mathbb{R} : x > 0\}\}$.
   No. It is not closed under $*$, e.g $0 < b < 1$, then $a * b = a \ln b < 0$

3. $a * b$ is a root of the equation, $x^2 = a^2 b^2$, on the set $\mathbb{R}$
   No. The operation isn't uniquely defined. $x^2 = a^2 b^2$ has to two roots, namely $+ab$ and $-ab$

4. Subtraction, on the set $\mathbb{Z}$.
   Yes.

5. Subtraction, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.
   No. e.g. $b = a + 1$ then $a * b = -1$ which is not in the set.

6. $a * b = |a - b|$, on the set $\{n \in \mathbb{Z} : n \geq 0\}$
   Yes. (distance metric)

Exercise 2. Indicate whether or not :

- I it is commutative,
- II it is associative,
- III $\mathbb{R}$ has an identity element with respect to $*$
- IV every $x \in \mathbb{R}$ has an inverse with respect to $*$

1. $x * y = x + 2y + 4$

   I Commutativity:

$$y * x = y + 4x + 4 \neq x + 4y + 4 \quad \forall x, y \in \mathbb{R} \tag{1}$$

   Not commutative

II Associativity:

$$\begin{aligned} x * (y * z) &= x * (y + 2z + 4) = x + 2(y + 2z + 4) + 4 \\ (x * y) * z &= (x + 2y + 4) * z = x + 2y + 4 + 2z + 4 \\ x * (y * z) &\neq (x * y) * z \end{aligned} \tag{2}$$

Not associative

III Existence of identity:

$$\begin{aligned} x * e &= x + 2e + 4 = x \\ 2e &= 4 \implies e = -2 \end{aligned} \tag{3}$$

Check

$$\begin{aligned} x * 2 &= x + -2 * 2 + 4 = x \\ 2 * x &= -2 + 2x + 4 = 2 * (x + 1) \end{aligned} \tag{4}$$

No identity element, and thus no inverses.

2. $x * y = x + 2y - xy$

I Commutativity:

$$y * x = y + 2x - yx \neq x + 2y - xy \quad \forall x, y \in \mathbb{R} \tag{5}$$

II Associativity:

$$\begin{aligned} & x * (y * z) = x * (y + 2z - yz) = x + 2(y + 2z - yz) - x(y + 2z - yz) \\ & = x + 2y + 4z - 2yz - xy - 2xz + xyz \\ & (x * y) * z = (x + 2y - yx) * z = x + 2y - yx + 2z - (x + 2y - yx)z \\ & = x + 2y - yx + 2z - xz - 2yz + yxz \\ & x * (y * z) \neq (x * y) * z \end{aligned} \tag{6}$$

III Existence of identity:

$$\begin{aligned} x * e &= x + 2e - xe = x \\ 2e - xe = e(2 - x) &= 0 \implies e = 0 \end{aligned} \tag{7}$$

Check

$$x * 0 = x + 2 * 0 - x * 0 = x \tag{8}$$

$$0 * x = 0 + 2x - 0 * x = 2x \tag{9}$$

No identity element, thus no inverses.

3. $x * y = |x + y|$

I Commutativity:

$$y * x = |y + x| = |x + y| = x * y \tag{10}$$

II Associativity:

$$\begin{aligned} x * (y * z) &= x * |y + z| = |x + |y + z|| \\ (x * y) * z &= |x + y| * z = ||x + y| + z| \end{aligned} \tag{11}$$

$x * (y * z) \neq (x * y) * z$, to see this set $x = 1, y = -2, z = 0$

$$|1 + |-2|| \neq ||1 - 2||$$

III Existence of identity:

$$x * e = |x + e| = x \implies e = 0 \tag{12}$$

Check

$$\begin{aligned} x * 0 &= |x + 0| = x \\ 0 * x &= |0 + x| = x \end{aligned} \tag{13}$$

IV Existence of inverses:

$$x * x' = |x + x'| = 0 \implies x' = -x \tag{14}$$

Check

$$\begin{aligned} x * (-x) &= |x + (-x)| = 0 \\ (-x) * x &= |(-x) + x| = 0 \end{aligned} \tag{15}$$