Project II

Unathi Skosana

08 June 2017

Description

Consider an object with mass m moving in two dimensions, starting from a prescribed point A and ending at another prescribed point B falling under the influence of gravity only. The problem of finding the path or trajectory of the object from A to B that minimizes the time taken for the object to traverse from A to B is the main focus of this project. The problem is formally known as the brachistochrone curve or the curve of shortest descent and it has a very rich and fascinating history.

Mathematical characterization

For matters simple, we will consider the point A be on origin of our coordinate axis and take the downward direction to be positive and also I will denote the path along which the object moves as $s(x)$.

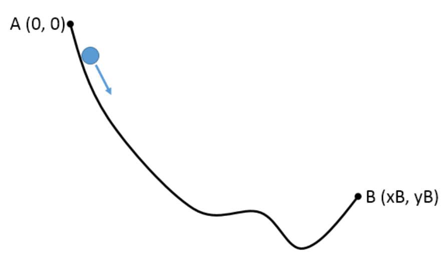

Figure 1: Problem characterization

Since the object is only under the influence of gravity it's energy is conversed that is, the sum of potential energy and kinetic energy is the same at any point along the trajectory of the object. Thus the energy difference between any two points A and B is 0.

$$E_A - E_B = 0 \implies E_A = E_B \implies K_A + U_A = K_B + U_B$$

Assuming the object starts from rest at A, we have.

$$\frac{1}{2}m(0)^2 + mg(0) = \frac{1}{2}mv^2 + mg(-y) \implies v^2 = 2gy \implies v = \sqrt{2gy}$$

We can write the velocity of the object as

$$v = \frac{ds(x)}{dt} = \frac{ds(x)}{dx}\frac{dx}{dt}$$

Therefore

$$\sqrt{2gy} = \frac{ds(x)}{dx}\frac{dx}{dt} \qquad (1)$$

Using Pythagoras' theorem we can write $ds(x)$ as.

$$ds(x) = \sqrt{dx^2 + dy^2} = \sqrt{dx^2(1 + \frac{dy^2}{dx^2})} = \sqrt{1 + (\frac{dy}{dx})^2}dx$$

Substituting into equation (1),

$$\sqrt{2gy} = \sqrt{1 + (\frac{dy}{dx})^2}\frac{dx}{dt} \implies dt = \frac{\sqrt{1 + (\frac{dy}{dx})^2}}{\sqrt{2gy}}dx$$

Taking the integral from 0 to $x(B)$

$$\int_0^{x(B)} dt = \tau = \frac{1}{\sqrt{2g}}\int_0^{x(B)} \frac{\sqrt{1 + \dot{y}^2}}{\sqrt{y}}dx$$

Since we are looking to minimize this integral we will apply the Euler-Lagrange formalism, writing the integral as

$$\tau = \int_0^{x(B)} L(\dot{y}, y)dx$$

with the Lagrangian $L(\dot{y}, y) = \frac{\sqrt{1+\dot{y}^2}}{\sqrt{y}}$

Euler-Lagrange formalism

The Euler-Lagrange equations take the form.

$$0 = \frac{d}{dx}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \quad i = 1, 2, 3, ...N$$

with q being the general coordinate and $\dot{q}$ it's derivative. In our case, it takes the form.

$$0 = \frac{d}{dx}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y}$$

With

$$\frac{\partial}{\partial y}\frac{\sqrt{1+\dot{y}^2}}{\sqrt{y}} = -\frac{\sqrt{1+\dot{y}^2}}{y^{3/2}}$$

$$\frac{\partial}{\partial \dot{y}}\frac{\sqrt{1+\dot{y}^2}}{\sqrt{y}} = \frac{\dot{y}}{(y+\dot{y}^2 y)^{1/2}}$$

Thus after taking the full derivative of above expression, and going through the motions to get the equation in a pleasant form. We finally have.

$$\frac{d}{dx}\frac{\dot{y}}{(y+\dot{y}^2 y)^{1/2}} = \frac{2\ddot{y}y - \dot{y}^2 - \dot{y}^4}{(y+\dot{y}^2 y)^{3/2}}$$

Putting everything together, we finally get second order differential equation for the path that minimizes the time taken by the object.

$$\frac{2\ddot{y}y - \dot{y}^2 - \dot{y}^4}{(y+\dot{y}^2 y)^{3/2}} + \frac{\sqrt{1+\dot{y}^2}}{y^{3/2}} = 0 \implies 2\ddot{y}y - \dot{y}^2 - \dot{y}^4 = -(1+\dot{y}^2)^2 \implies \ddot{y} = -\frac{(1+\dot{y}^2)}{2y}$$

Thus

$$\frac{d^2 y}{d^2 x} = -\frac{(1+(\frac{dy}{dx})^2)}{2y}$$

Program

The second order differential equation was solved using a modified Runge-Kutta for second order differential equations. The Runge-Kutta algorithm outputs the first derivative of $y(x)$ and $y(x)$ these functions had to further put into the integral in order to calculate the time taken τ for that particular trajectory $y(x)$. Due to the singularity of the integrand as $y \to 0$, I had to make the initial value of $y(x)$ very close to 0. The fact that integrand had a singularity at $y = 0$ made using adaptive step size control necessary.

Results

The plot of the results below was obtained with the following initial conditions; $y(0) = -0.001$, $\frac{dy}{dx}|_{x=0} = -10000$, $x(0) = \frac{2}{3}\frac{y(0)}{y'(0)}$ and $x(E) = 100$

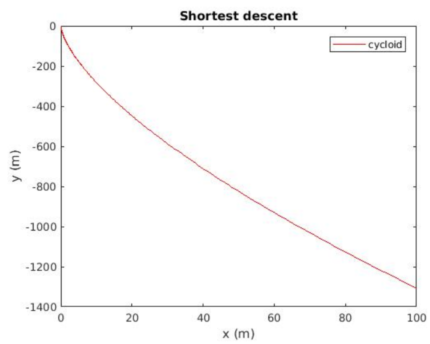

Figure 2: Trajectory of shortest descent

The time taken for this particular trajectory(cycloid) was approximately 16.3532 seconds. As a reliability test to see that indeed this is the trajectory the object can take while minimizing the time taken for traversal between A and B, I sought out to compare it with linear curve; A straight line connecting the initial point and end point, and compute the time taken for such a curve. Similar initial conditions were considered, the following plot was produced.

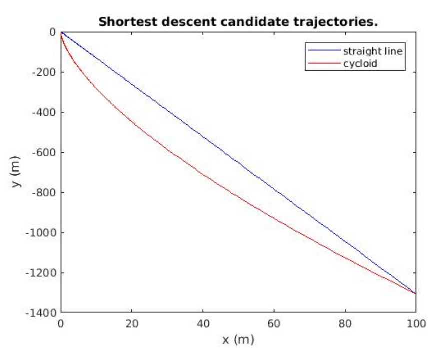

Figure 3: cycloid and a straight line

The time taken for the object to traverse along the straight line trajectory from the initial point to the end point was roughly 16.3648 seconds. This is more than the time taken by cycloid connecting the initial point and the end point by 0.0116 seconds !

Next we considered a small variation to the curve of shortest descent y, since the cycloid is assumed to be the curve of shortest descent any variation added to it should result in a curve increasing the time taken. We will consider a new function $y^*(x)$, such that

$$y^*(x) = y(x) + \eta(x)$$

The only condition imposed on variation function $\eta(x)$ is that the initial and end points are zero, $\eta(0) = 0$ and also $\eta(x_e) = 0$. The following variation functions that satisfies the those conditions was chosen

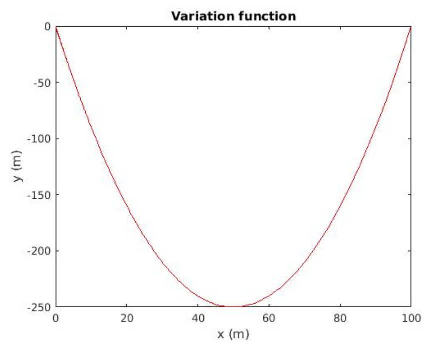

Figure 4: Variation function 1

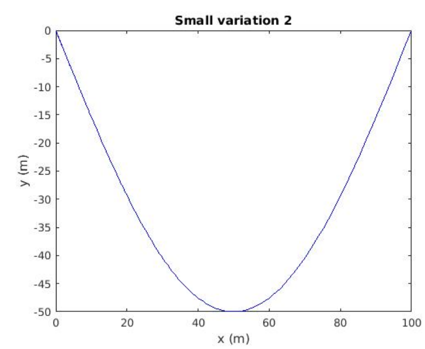

Figure 5: Variation function 2

This first variation function is mathematically defined by $\eta(x) = x^2 - (x_e)x$ and second variation function is defined to be $\eta(x) = sin(\pi \frac{x}{x_e})$ these two functions clearly satisfy the conditions for a variation function as they both have roots at $x = 0$ and $x = x_e$.

The initial conditions are still unchanged, the following plot compares the four different paths for the object considered thus far.

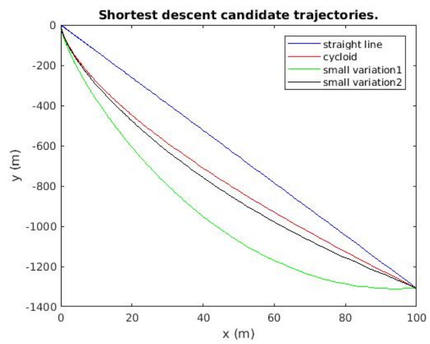

Figure 6: The four trajectories

The first of the newly considered trajectories takes the most time out of the four, the object takes roughly 16.443 seconds to traverse this trajectory; The object takes 0.0789 seconds more than it does in traversing the straight line, takes 0.0900 more seconds to traverse the trajectory obtained from the second variation function and it takes 0.0905 seconds more than it does in traversing the cycloid. The trajectory formed by adding the second variation to the original function $y(x)$ takes roughly 16.3537 seconds. These variations are all greater than the time taken by cycloid connecting the points A and B, these provides quantitative proof that the cycloid is indeed the trajectory of shortest descent.

Discussion and conclusion

The results considered in the section above provide quantitative proof that the cycloid is indeed, the trajectory of shortest descent between A and B. This may come as surprise that the shortest path between A and B; a straight line, doesn't yield the shortest descent. We can try to explain why the cycloid is minimizes the time taken by the object to descent by looking at it's structure compared to the straight line and also taking the energy conversation into consideration.

As the object falls from A to B, there is a trade off between the potential energy and the kinetic energy of the object, such that their sum remains constant. This means that the lower the objects get close to the ground, there will be an decrease in potential energy compensated for by increase in kinetic energy, the kinetic energy increases as a direct consequence of it's speed increasing.

To get from point A to point B in the shortest time there should be a combination of fast speed and a short distance to traverse, looking at shape of the cycloid we see there is a fast drop in height just after point A, this will result in rapid increase in falling speed. We see that before the curve gets to point B, and it been starting to curve up and resemble a straight line. These two traits of the cycloid combine for a fast speed and a short distance to cover.

The points stated this suffice to explain why the cycloid is preferred over a straight line, it is because it has perfect trade-off between the making the object gain speed fast enough over a short distance such that it results in the shortest descent.