TW364: Applied Fourier Analysis

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Lecture 4

Previously...

Fourier series of f on $[-p, p]$: $\quad f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right]$

$$\text{with} \quad a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\,\cos\frac{n\pi x}{p}\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\,\sin\frac{n\pi x}{p}\,dx$$

- Convergence at point continuities and point discontinuities
- Periodic extension
- Partial sum approximations

11.3 Cosine series and sine series

Even and odd functions

- a function f is **even** when $f(-x) = f(x)$
- a function f is **odd** when $f(-x) = -f(x)$

Examples

$f(x) = x^2$ is even; $g(x) = x^3$ is odd; $h(x) = e^x$ is neither even nor odd

Note that:

$cos(kx)$ is even for any $k \in \mathbb{R}$

$sin(kx)$ is odd for any $k \in \mathbb{R}$

Properties of even and odd functions

- the product of two even functions is even
- the product of two odd functions is even
- the product of an even and odd function is odd
- the sum (difference) of two even functions is even
- the sum (difference) of two odd functions is odd
- if f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
- if f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$

Fourier series of an even function

If f is **even** on $[-p, p]$, its Fourier coefficients become

$$a_0 = \tfrac{2}{p} \int_0^p f(x)\, dx$$

$$a_n = \tfrac{2}{p} \int_0^p f(x) \cos \tfrac{n\pi x}{p}\, dx$$

$$b_n = 0$$

$\therefore$ for an even function f, we get the **cosine series**:

$$f(x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \tfrac{n\pi x}{p}$$

Fourier series of an odd function

If f is **odd** on $[-p, p]$, its Fourier coefficients become

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi x}{p} \, dx$$

$\therefore$ for an odd function f, we get the **sine series**:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{p}$$

Examples

(a) Find the Fourier series of $f(x) = x, \quad -2 < x < 2$

We note that this function is odd, so we expand in a sine series.

$$f(x) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi}{2} x\right)$$

(b) Find the Fourier series of $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 2, & 0 \le x < \pi \end{cases}$

We note that $f(x) - 1$ is odd, so we expand it in a sine series.

$$f(x) - 1 = \sum_{n=1}^{\infty} \frac{2-2(-1)^n}{n\pi} \sin(nx)$$

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{2-2(-1)^n}{n\pi} \sin(nx)$$

Half-range expansions

Suppose a function f is defined on an interval $[0, L]$.

We may now **choose** a definition of f on $[-L, 0)$, in order to find the Fourier series.

Some options:

- even reflection, which will lead to a cosine series
- odd reflection, which will lead to a sine series
- identify shift, which will lead to a Fourier series (in general)

Each of these will produce a different series expansion on $[-L, L]$, with different periodic extensions, but all will converge to f on $[0, L]$.

Example

Expand $f(x) = x^2, \ 0 < x < \pi$ in

(a) a cosine series

(b) a sine series

(c) a Fourier series

(a) $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)$

(b) $f(x) = \sum_{n=1}^{\infty} \left[\frac{4(-1)^n - 4}{n^3 \pi} - \frac{2\pi(-1)^n}{n} \right] \sin(nx)$

(c) $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{2+2(-1)^n}{n^2} \cos(nx) - \frac{\pi+\pi(-1)^n}{n} \sin(nx) \right]$