TW364: Applied Fourier Analysis

Dr Willie Brink
Applied Mathematics, Stellenbosch University

Lecture 3

- **Finalized** test dates and times:
  - **test A1**: 3 Oct at 17:30 (as scheduled by the exams office)
  - **test A2**: 3 Nov at 14:00 (as scheduled by the exams office)
  - **test A3**: 27 Nov at 14:00 (as suggested by the exams office)

Previously...

Fourier series of f on $[-p, p]$: $f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right]$

with $a_0 = \frac{1}{p} \int_{-p}^{p} f(x)\, dx$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p}\, dx$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi x}{p}\, dx$$

If f and f' are both piecewise continuous on $[-p, p]$, the Fourier series of f converges to $f(x)$ at all point continuities. At any point discontinuity x it converges to $\frac{1}{2}\big(f(x^+) + f(x^-)\big)$.

Period of a function

A function f is periodic with period T (or T-periodic for short) if $f(x+T) = f(x)$, for all x.

Example: $\sin(x)$ is 4π-periodic, since $\sin(x + 4\pi) = \sin(x)$

$\sin(x)$ is also 2π-periodic, since $\sin(x + 2\pi) = \sin(x)$

Fundamental period

The smallest $T > 0$ such that $f(x+T) = f(x)$, for all x.

Example: the fundamental period of $\sin(x)$ is 2π.

Fundamental period of a Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right]$$

Fundamental period of both $\cos\frac{n\pi x}{p}$ and $\sin\frac{n\pi x}{p}$ is $\frac{2p}{n}$.

Note: any positive integer multiple of a period is also a period.

$\therefore$ all the Fourier basis functions have the period $2p$ in common, and this is the smallest period common to them.

So the Fourier series is periodic with fundamental period $2p$.

The Fourier series does not only represent f on $[-p, p]$,
but also generates the periodic extension of f outside $[-p, p]$.

Example

In Lecture 2 we found the Fourier series of $f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ \pi - x, & 0 \leq x \leq \pi \end{cases}$

to be $f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n^2\pi} \cos(nx) + \frac{1}{n} \sin(nx) \right]$.

The RHS converges to the periodic extension of f, with period 2π.

Graph of this periodic extension...

At $x = 0, \pm 2\pi, \pm 4\pi, \ldots$ the series converges to $\frac{\pi}{2}$.

At $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ the series converges to 0.

Partial sum approximations

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right]$$

The partial sum

$$S_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N} \left[a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right]$$

approximates the periodic extension $f(x)$, and this approximation improves for larger N.

Example

In Tutorial 1 we found the Fourier series of $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ x^2, & 0 \le x \le \pi \end{cases}$

to be $f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left[\frac{2(-1)^n}{n^2} \cos(nx) + \left(\frac{2(-1)^n - 2}{n^3 \pi} - \frac{(-1)^n \pi}{n} \right) \sin(nx) \right]$.

A few partial sum approximations:

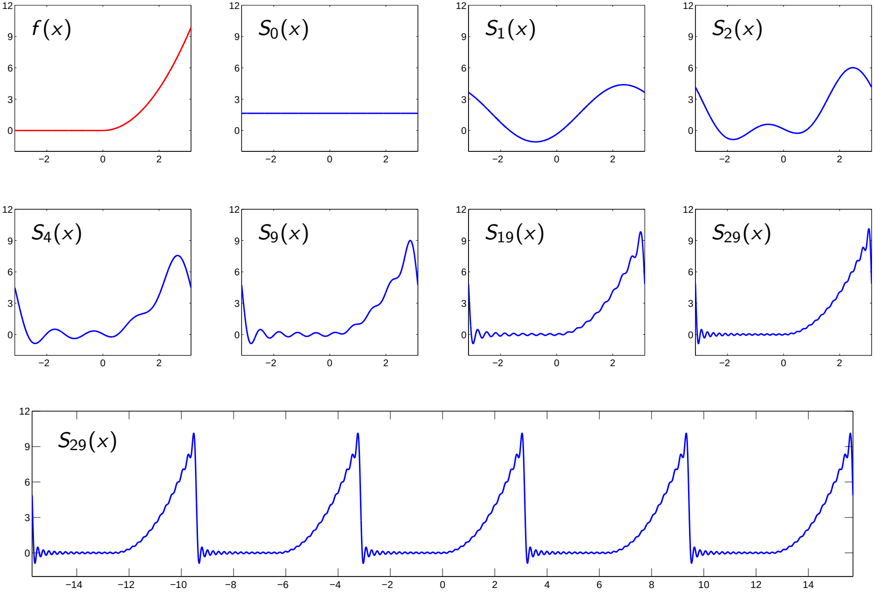
$$S_0(x) = \frac{\pi^2}{6}$$

$$S_1(x) = \frac{\pi^2}{6} - 2\cos(x) + \left(\pi - \frac{4}{\pi}\right)\sin(x)$$

$$S_2(x) = \frac{\pi^2}{6} - 2\cos(x) + \left(\pi - \frac{4}{\pi}\right)\sin(x) + \frac{1}{2}\cos(2x) - \frac{\pi}{2}\sin(2x)$$

$$\vdots$$

MATLAB DEMO



$f(x)$
$S_0(x)$
$S_1(x)$
$S_2(x)$
$S_4(x)$
$S_9(x)$
$S_{19}(x)$
$S_{29}(x)$
$S_{29}(x)$