TW364: Applied Fourier Analysis

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Lecture 2

- ~~New test dates and times:~~
  - ~~**test A1**: 20 Sep at 14:00 (in our tut period; not 3 Oct)~~
  - ~~**test A2**: 3 Nov at 14:00 (as scheduled by the exams office)~~
  - ~~**test A3**: 27 Nov at 14:00 (as suggested by the exams office)~~
- ~~If I receive no legitimate problems with these dates by next Wednesday (1 Aug), they will be fixed!~~

Previously...

Inner product of f_1 and f_2 on $[a, b]$: $(f_1, f_2) = \int_a^b f_1(x)\, f_2(x)\, dx$

Functions f_1 and f_2 are said to be orthogonal on $[a, b]$ if $(f_1, f_2) = 0$.

$\{\phi_0(x), \phi_1(x), \phi_2(x) \ldots\}$ is an orthogonal set if $(\phi_m, \phi_n) = 0$ for $m \neq n$.

Recall...

Orthogonal vector expansion

Three mutually orthogonal vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ form a basis for $\mathbb{R}^3$, such that any vector $\mathbf{u} \in \mathbb{R}^3$ can be expressed as

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

Take inner product with $\mathbf{v}_1$ on both sides:

$$(\mathbf{u}, \mathbf{v}_1) = c_1(\mathbf{v}_1, \mathbf{v}_1) + c_2 \cdot 0 + c_3 \cdot 0$$

$$\therefore \quad c_1 = \frac{(\mathbf{u}, \mathbf{v}_1)}{(\mathbf{v}_1, \mathbf{v}_1)}$$

Similarly, $\quad c_2 = \dfrac{(\mathbf{u}, \mathbf{v}_2)}{(\mathbf{v}_2, \mathbf{v}_2)}$ and $c_3 = \dfrac{(\mathbf{u}, \mathbf{v}_3)}{(\mathbf{v}_3, \mathbf{v}_3)}$

Orthogonal series expansion of functions

Let $\{\phi_0(x), \phi_1(x), \phi_2(x) \dots\}$ be an orthogonal set on $[a, b]$, and suppose $f(x)$ is defined on $[a, b]$.

We're looking for coefficients $c_0, c_1, c_2, \dots$ such that

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \dots$$

Multiply both sides by $\phi_n(x)$ and integrate over $[a, b]$:

$$\begin{aligned}(f, \phi_n) &= c_0(\phi_0, \phi_n) + c_1(\phi_1, \phi_n) + c_2(\phi_2, \phi_n) + \dots \\ &= c_n(\phi_n, \phi_n)\end{aligned}$$

$$\therefore \; c_n = \frac{(f, \phi_n)}{(\phi_n, \phi_n)}, \quad n = 0, 1, 2, \dots$$

Complete orthogonal sets

In order to expand f as a combination of orthogonal functions $\{\phi_0(x), \phi_1(x), \ldots\}$, f must not be orthogonal to all ϕ_n. (Why not?)

We'll assume that an orthogonal set is always complete. That is, the only non-member orthogonal to each member of the set is the zero function.

Convergence

Does a series expansion in terms of orthogonal functions actually exist, and will it converge to $f(x)$ for all $x \in [a, b]$?

We'll return to these questions...

11.2 Fourier series

Consider the set

$$\left\{1, \cos\frac{\pi x}{p}, \cos\frac{2\pi x}{p}, \cos\frac{3\pi x}{p}, \ldots, \sin\frac{\pi x}{p}, \sin\frac{2\pi x}{p}, \sin\frac{3\pi x}{p}, \ldots\right\}$$

which is orthogonal on the interval $[-p, p]$.

Tut 1, prob. 2 asks you to prove orthogonality for $p = \pi$. The extension is straightforward.

Suppose a given f is defined on $[-p, p]$, and can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right]$$

Following the procedure on slide 5, using the orthogonal set above, we find the coefficients in this series expansion.

The Fourier series of a function

Suppose f is defined on $[-p, p]$, and can be expressed as

$$f(x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \tfrac{n\pi x}{p} + b_n \sin \tfrac{n\pi x}{p} \right].$$

Then $a_0 = \frac{1}{p} \int_{-p}^{p} f(x)\, dx$

$$a_n = \tfrac{1}{p} \int_{-p}^{p} f(x) \cos \tfrac{n\pi x}{p}\, dx$$

$$b_n = \tfrac{1}{p} \int_{-p}^{p} f(x) \sin \tfrac{n\pi x}{p}\, dx$$

Conditions for convergence

Theorem 11.2.1, p. 433

> If f and f' are both piecewise continuous on $[-p, p]$, the Fourier series of f converges to $f(x)$ at all point continuities. At any point discontinuity x it converges to $\frac{1}{2}\big(f(x^+) + f(x^-)\big)$.

Example

Find the Fourier series of $f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ \pi - x, & 0 \leq x \leq \pi \end{cases}$

Note: f and f' are both piecewise continuous on $[-\pi, \pi]$.

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n^2\pi} \cos(nx) + \frac{1}{n} \sin(nx) \right]$$

Note: $x = 0$ is a point discontinuity of f, so at $x = 0$ the Fourier series (righthandside of above) will converge to $\frac{1}{2}(f(0^+) + f(0^-)) = \frac{\pi}{2}$.