TW364: Applied Fourier Analysis

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Lecture 5

Periodic driving force

The Fourier series can be useful in finding a solution of a differential equation (DE) describing a system with periodic driving force.

Undamped oscillator with external (driving) force $f(t)$, mass m, and spring constant k is described by the DE:

$$m\frac{d^2x}{dt^2} + kx = f(t)$$

Suppose $m = \frac{1}{16}$, $k = 4$, and $f(t)$ is the periodic extension of

$$f(t) = \pi t, \quad -1 < t < 1$$

Note: in a practical sense we only consider $f(t)$ for $t \geq 0$, but when applying the Fourier series we'll also regard $t < 0$.

$f(t)$ is the odd extension of πt, $0 < t < 1$, so let's get the sine series:

$$f(t) = \sum_{n=1}^{\infty} \tfrac{2(-1)^{n+1}}{n} \sin(n\pi t)$$

The DE becomes

$$\tfrac{1}{16} \frac{d^2x}{dt^2} + 4x = \sum_{n=1}^{\infty} \tfrac{2(-1)^{n+1}}{n} \sin(n\pi t)$$

This DE is linear and non-homogeneous. By the method of undetermined coefficients, we assume a particular solution similar in form to the RHS and its derivatives, i.e.

$$x_p = \sum_{n=1}^{\infty} \left[A_n \cos(n\pi t) + B_n \sin(n\pi t)\right]$$

Plug x_p into the LHS of the DE:

$$\sum_{n=1}^{\infty} \left[\left(\frac{-n^2\pi^2}{16} + 4\right) A_n \cos(n\pi t) + \left(\frac{-n^2\pi^2}{16} + 4\right) B_n \sin(n\pi t) \right] = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(n\pi t)$$

Solve for A_n and B_n:

$$A_n = 0, \quad B_n = \frac{32(-1)^{n+1}}{n(64-n^2\pi^2)}$$

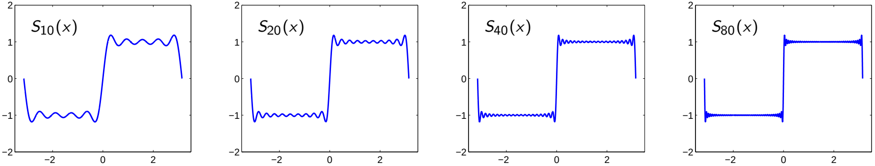
$$\therefore \quad x_p = \sum_{n=1}^{\infty} \frac{32(-1)^{n+1}}{n(64-n^2\pi^2)} \sin(n\pi t)$$

The Gibbs phenomenon

In Lecture 4 we found the Fourier series of $f(x) = \begin{cases} -1, & -\pi \leq x < 0 \\ 1, & 0 \leq x \leq \pi \end{cases}$

$$f(x) = \sum_{n=1}^{\infty} \frac{2-2(-1)^n}{n\pi} \sin(nx)$$

Partial sum approximations:



The oscillations exhibited by these approximations around function discontinuities (also in the periodic extension of f) are referred to as the Gibbs phenomenon.

The magnitude of the "overshoot" in $S_N(x)$ does not disappear as $N \to \infty$, but approaches a finite limit.

It turns out that this limit is about 8.95% of the magnitude of the particular jump discontinuity.

MATLAB DEMO

Observing this percentage for the previous example...

If the Gibbs overshoot does not vanish in the limit, does that not contradict the convergence theorem of the Fourier series?

No! The theorem describes point-wise convergence.

Since the region where oscillations occur get narrower as N increases, the oscillations will disappear in the neighbourhood of any x where $f(x)$ is continuous, even for x arbitrarily close to the discontinuity.