TW364: Applied Fourier Analysis

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Lecture 1

Module information

- Lecturer: Dr Willie Brink (`wbrink@sun.ac.za`)
- Web presence: `http://learn.sun.ac.za`
- Book: Zill, *Differential Equations with Boundary-Value Problems*, 8^{th} or 9^{th} ed.
- Module content:
  - chapter 11: Fourier series on a finite interval
  - chapter 12: solving PDEs using Fourier series
  - chapter 13: other coordinate systems
  - chapter 14: Fourier integrals and the Fourier transform
  - *additional*: the discrete Fourier transform
  - *additional*: applications of the DFT in signal processing
  - *additional*: the fast Fourier transform

Module information

- Tutorials:
  - traditional pen-and-paper tut in **A407**, followed by tut-test
    or
  - work on computer assignment in **Narga B**, submit one week later

- How your final mark will be calculated:
  - **30%**: tutorial tests and computer assignments
  - **35%**: test A1, on the work covered in Term 3
  - **35%**: test A2, on the work covered in Term 4
  - optional test A3, on all the work, replaces the lowest of A1 and A2
  - no sick tests (if you miss A1 or A2, you have to write A3)

Module information

- ~~Test dates and times:~~
  - ~~**test A1**: 6 Sep at 14:00 (in a tut period; not 3 Oct)~~
  - ~~**test A2**: 3 Nov at 14:00 (as scheduled by the exams office)~~
  - ~~**test A3**: 27 Nov at 14:00 (as suggested by the exams office)~~
- ~~If I receive no legitimate problems with these dates by next Monday (30 July), these will be the dates!~~

11.1 Orthogonal functions

Recall...

The inner product of vectors

$$(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i, \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$$

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $(\mathbf{u}, \mathbf{v}) = 0$.

The norm of $\mathbf{u}$ is defined as $\sqrt{(\mathbf{u}, \mathbf{u})}$.

Inner product of functions

The inner product of two functions f_1 and f_2 on an interval $[a, b]$ is defined as

$$(f_1, f_2) = \int_a^b f_1(x)\, f_2(x)\, dx.$$

Properties (easily proved)

- $(f_1, f_2) = (f_2, f_1)$
- $(kf_1, f_2) = k(f_1, f_2)$ with k a constant
- $(f, f) = 0$ if f is the zero function
- $(f, f) > 0$ if f is not the zero function

Two functions f_1 and f_2 are orthogonal on $[a, b]$ if $(f_1, f_2) = 0$.

Examples

(a) Are $f(x) = x^2$ and $g(x) = x^3$ orthogonal on $[-1, 1]$?

(b) Are $f(x) = x^2$ and $h(x) = x^4$ orthogonal on $[-1, 1]$?

(a) yes

(b) no

Orthogonal set of functions

An infinite set of functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is orthogonal on $[a, b]$ if $(\phi_m, \phi_n) = 0$ for $m \neq n$.

Orthonormal set

The norm of a function f on $[a, b]$ is $\sqrt{(f, f)} = \sqrt{\int_a^b \left[f(x)\right]^2 dx}$.

An orthogonal set $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ with the property that $\sqrt{(\phi_n, \phi_n)} = 1$ for $n = 0, 1, 2, \ldots$, is said to be orthonormal.

Example

Consider the set $\{1, \cos(x), \cos(2x), \cos(3x), \ldots\}$ on the interval $[-\pi, \pi]$.
Is the set orthogonal?

Let $\phi_0(x) = 1,\ \phi_1(x) = \cos(x),\ \phi_2(x) = \cos(2x),$ etc.

$$(\phi_0, \phi_n) = \int_{-\pi}^{\pi} \cos(nx)\, dx = 0, \quad n \neq 0$$

$$(\phi_m, \phi_n) = \int_{-\pi}^{\pi} \cos(mx)\, \cos(nx)\, dx = 0, \quad m \neq n$$

$\therefore$ yes, the set is indeed orthogonal

Question: how can we scale each function so that the set becomes orthonormal?