Notes on Matrix Analysis

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Personal notes taken while learning topics from matrix analysis

Contents

Introduction

Definitions 1 (Informal definition). An operation is any rule which assigns to each ordered pair of elements of A a unique element in A.

Definitions 2 (Formal definition). For a set A, an operation $*$ on A is a rule which assigns to each ordered pairs (a, b) of elements of A exactly one $a * b$ in A, such that:

- $a * b$ is defined for *every* ordered pair (a, b) of elements of A. [1]
- $a * b$ must be *uniquely* defined. [2]
- If $a, b \in A$, then $a * b \in A$. [3]

Definitions 3 (Commutativity). An operation $*$ is said to be *commutative* if it satisfies

$$a * b = b * a \tag{1}$$

for any two elements a and b in A.

Definitions 4 (Associativity). An operation $*$ is said to be *associative* if it satisfies

$$(a * b) * c = a * (b * c) \tag{2}$$

for any three elements a, b and c in A.

Definitions 5 (Identity element). The *identity* element e with respect to the operation $*$ has the property that:

$$e * a = a \quad \text{and} \quad a * e = a \tag{3}$$

is true for every element a in A.

Definitions 6 (Inverses). The inverse of any element a, item denoted by a^{-1} has the property that:

$$a * a^{-1} = e \quad \text{and} \quad a^{-1} * a = e \tag{4}$$

Definitions 7 (Scalar field). A *scalar field* is a set of scalars A, together with two operations, which we call addition $(+)$ and multiplication $(*)$. Both operations must be commutative, associative, have an identity in the set A, all elements in the set have inverses for both operations except the additive identity under multiplication, and multiplication must be distributive over addition.

Definitions 8 (Vector space). A *vector space* V over a field $\mathbb{F}$ is a set V of objects (called vectors) that is closed under a binary operation that is associative and commutative and has an identity (the zero vector, denoted by $\mathbf{0}$) and additive inverses in the set. The said set V is also closed under scalar multiplication of the vectors by elements of the underlying scalar field $\mathbb{F}$, satisfying the following properties for all $a, b \in \mathbb{F}$ and all $, u \in V$: $a(v + u) = av + au$, $(a + b)v = av + bv$, $a(bv) = (ab)v$ and $ev = v$ for the multiplication identity in $\mathbb{F}$.

Definitions 9 (Subspace). A *subspace* of a vector space V over a field $\mathbb{F}$ is a subset of V that is, by itself, a vector space over $\mathbb{F}$ using the same operations of vector addition and scalar multiplication as in V.

- The subsets $\{\mathbf{0}\}$ and V are called *trivial subspaces* of V. A *non-trivial subspace* is one that is not $\{0\}$ or V.
- A *proper subspace* is a non-trivial subspace not equal to V, and $\{\mathbf{0}\}$ is the *zero vector space*

[1] In $\mathbb{R}$, division does not qualify as operation since it does not satisfy this condition. i.e. the ordered pair $(a, 0)$ has undefined quotient $a/0$.
[2] If $\diamond$ is defined on (a, b) to be the number whose square is ab. In $\mathbb{R}$, $\diamond$ does not qualify as an operation since $2 \diamond 2$ could be either 2, or +2.
[3] A is closed under the operation $*$

Definitions 10 (Inner product). An *inner product* over a vector space V is a map that takes a pair of vectors in V to a scalar in the underlying field $\mathbb{F}$, denoted by $\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{C}$ with satisfying the following properties for all vectors $u, v, w \in V$ and all scalars $a, b \in F$:

- $\langle v, u \rangle = \overline{\langle v, u \rangle}$
- $\langle u + bv, w \rangle = \overline{a}\langle u, w \rangle + \overline{b}\langle v, w \rangle$[4]
- $\langle w, au + bv \rangle = a\langle w, u \rangle + b\langle w, v \rangle$
- $\langle u, u \rangle = 0 \implies u = \mathbf{0}$

[4] Such a defined bilinear form, that is linear in the second argument and conjugate linear in the second argument is said to be a sesquilinear form

Definitions 11 (Norm). The standard inner product on a finite-dimensional vector space V defined as $\langle v, u \rangle = v^* u$, corresponding to multiplication of a row vector with column vector. The said inner product induces a *norm* on V denoted by $\|\cdot\| : V \mapsto \mathbb{F}$ and defined as $\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{u^* u}$ with following properties for all $v \in V$ and $a \in F$: $\|v\| > 0 \ \forall v \neq \mathbf{0}$ and $\|av\| = \|a\|\|v\|$.

Definitions 12 (Hilbert space). A *Hilbert space* is a vector space endowed with a inner product; the pair $(V, \langle \cdot, \cdot \rangle)$ where V is real or complex vector space and $\langle \cdot, \cdot \rangle$ is an inner product on that space.

Definitions 13 (Span). If S is a subset of a vector space V over a field $\mathbb{F}$, the *span* of S, denoted by $\text{span}(S)$, is the intersection of all subspaces of V that contain S.