

# Correctness of bbchallenge’s deciders

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## Abstract

The Busy Beaver Challenge (or bbchallenge) aims at collaboratively solving the following conjecture: “ $BB(5) = 47,176,870$ ” [Aaronson, 2020]. This goal amounts to decide whether or not 88,664,064 Turing machines with 5-state halt or not – starting from all-0 tape. In order to decide the behavior of these machines we write *deciders*. A decider is a program that takes as input a Turing machine and outputs **true** if it is able to tell whether the machine halts or not. Each decider is specialised in recognising a particular type of behavior that can be decided.

In this document we are concerned with proving the correctness of these deciders programs. More context and information about this methodology are available at <https://bbchallenge.org>.

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## 1 Conventions

	0	1
A	1RB	1LC
B	1RC	1RB
C	1RD	0LE
D	1LA	1LD
E	- - -	0LE

Table 1: Transition table of the current 5-state busy beaver champion: it halts after 47,176,870 steps. <https://bbchallenge.org/1RB1LC1RC1RB1RD0LE1LA1LD---0LA&status=halt>

The set  $\mathbb{N}$  denotes  $\{0, 1, 2 \dots\}$ .

**Turing machines.** The Turing machines that are studied in the context of bbchallenge use a binary alphabet and a single bi-infinite tape. Machine transitions are either undefined (in which case the machine halts) or given by (a) a symbol to write (b) a direction to move (right or left) and (c) a state to go to. Table 1 gives the transition table of the current 5-state busy beaver champion. The machine halts after 47,176,870 steps (starting from all-0 tape) when it reads a 0 in state E, which is undefined.

A *configuration* of a Turing machine is defined by the 3-tuple: (i) state (ii) position of the head (iii) content of the memory tape. In the context of bbchallenge, *the initial configuration* of a machine is always (i) state is 0, i.e. the first state to appear in the machine’s description (ii) head’s position is 0 (iii) the initial tape is all-0 – i.e. each memory cell is containing 0. We write  $c_1 \vdash_{\mathcal{M}} c_2$  if a configuration  $c_2$  is obtained from  $c_1$  in one computation step of machine  $\mathcal{M}$ . We omit  $\mathcal{M}$  if it is clear from context. We let  $c_1 \vdash^s c_2$  denote a sequence of  $s$  computation steps, and let  $c_1 \vdash^* c_2$  denote zero or more computation steps. We write  $c_1 \vdash \perp$  if the machine halts after executing one computation step from configuration  $c_1$ . In the

context of `bbchallenge`, halting happens when an undefined machine transition is met i.e. no instruction is given for when the machine is in the state, tape position and tape corresponding to configuration  $c_1$ .

**Space-time diagram.** We use space-time diagrams to give a visual representation of the behavior of a given machine. The space-time diagram of machine  $\mathcal{M}$  is an image where the  $i^{\text{th}}$  row of the image gives:

1. The content of the tape after  $i$  steps (black is 0 and white is 1).
2. The position of the head is colored to give state information using the following colours for 5-state machines: A, B, C, D, E.

## 2 Decider for “Cyclers”

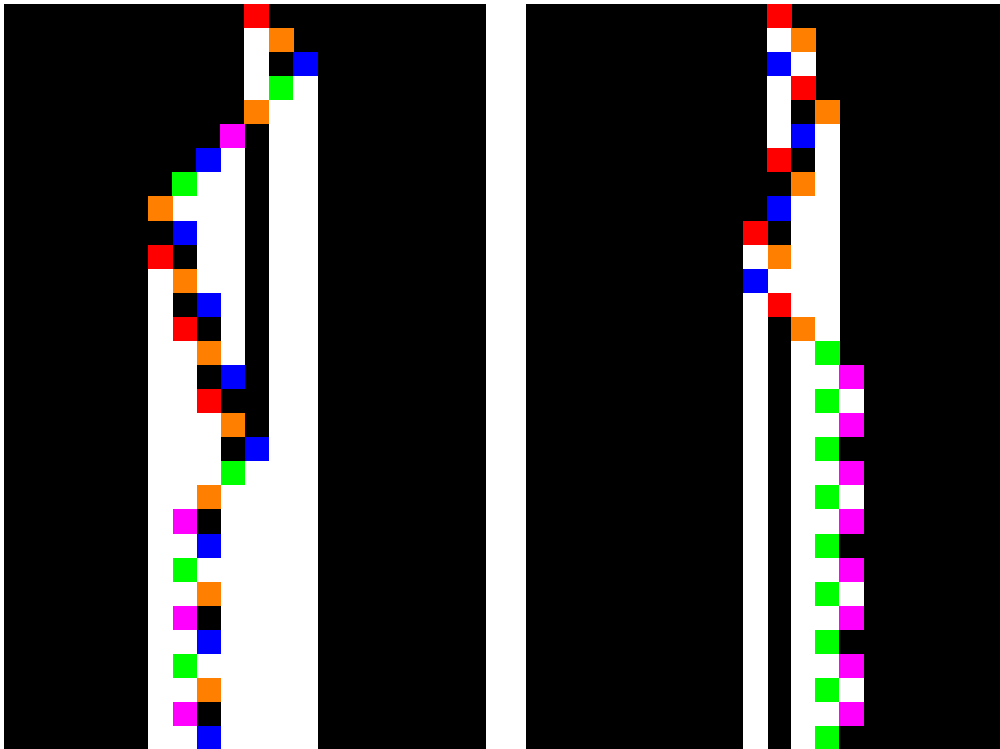


Figure 1: Space-time diagrams of the 30 first steps of `bbchallenge`’s machines #279,081 (left) and #4,239,083 (right) which are both “Cyclers”: they eventually repeat the same configuration for ever. Access the machines at <https://bbchallenge/279081> and <https://bbchallenge/4239083>.

The goal of this decider is to recognise Turing machines that cycle through the same configurations for ever. Such machines never halt. The method is simple: remember every configuration seen by a machine and return `true` if one is visited twice. A time limit (maximum number of steps) is also given for running the test in practice: the algorithm recognises any machine whose cycle fits within this limit<sup>1</sup>.

**Example 1.** Figure 1 gives the space-time diagrams of the 30 first iterations of two “Cyclers” machines: `bbchallenge`’s machines #279,081 (left) and #4,239,083 (right). Refer to <https://bbchallenge/279081> and <https://bbchallenge/4239083> for their transition tables. From these space-time diagrams we see that the machines eventually repeat the same configuration.

<sup>1</sup>In practice, for machines with 5 states the decider was run with 1000 steps time limit.

## 2.1 Pseudocode

We assume that we are given a Turing Machine type **TM** that encodes the transition table of a machine as well as a procedure **TuringMachineStep**(machine,configuration) which computes the next configuration of a Turing machine from the given configuration or **nil** if the machine halts at that step.

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### Algorithm 1 DECIDER-CYCLERS

---

```

1: struct Configuration {
2:   int state
3:   int headPosition
4:   int  $\rightarrow$  int tape
5: }
6: procedure bool DECIDER-CYCLERS(TM machine,int timeLimit)
7:   Configuration currConfiguration = {state = 0, headPosition = 0, .tape = {0:0}}
8:   Set<Configuration> configurationsSeen = {}
9:   int currTime = 0
10:  while currTime < timeLimit do
11:    if currConfiguration in configurationsSeen then
12:      return true
13:    configurationsSeen.insert(currConfiguration)
14:    currConfiguration = TuringMachineStep(machine,currConfiguration)
15:    currTime += 1
16:    if currConfiguration == nil then
17:      return false //machine has halted, it is not a Cyclor
18:  return false

```

---

## 2.2 Correctness

**Theorem 2.** Let  $\mathcal{M}$  be a Turing machine and  $t \in \mathbb{N}$  a time limit. Let  $c_0$  be the initial configuration of the machine. There exists  $i \in \mathbb{N}$  and  $j \in \mathbb{N}$  such that  $c_0 \vdash^i c_i \vdash^j c_i$  with  $i + j \leq t$  if and only if  $\text{DECIDER-CYCLERS}(\mathcal{M}, t)$  returns **true** (Algorithm 1).

*Proof.* This follows directly from the behavior of  $\text{DECIDER-CYCLERS}(\mathcal{M}, t)$ : all intermediate configurations below time  $t$  are recorded and the algorithm returns **true** if and only if one is visited twice. This mathematically translates to there exists  $i \in \mathbb{N}$  and  $j \in \mathbb{N}$  such that  $c_0 \vdash^i c_i \vdash^j c_i$  with  $i + j \leq t$ , which is what we want. Index  $i$  corresponds to the first time that  $c_i$  is seen (l.13 in Algorithm 1) while index  $j$  corresponds to the second time that  $c_i$  is seen (l.11 in Algorithm 1).  $\square$

**Corollary 3.** Let  $\mathcal{M}$  be a Turing machine and  $t \in \mathbb{N}$  a time limit. If  $\text{DECIDER-CYCLERS}(\mathcal{M}, t)$  returns **true** then the behavior of  $\mathcal{M}$  from all-0 tape has been decided:  $\mathcal{M}$  does not halt.

*Proof.* By Theorem 2, there exists  $i \in \mathbb{N}$  and  $j \in \mathbb{N}$  such that  $c_0 \vdash^i c_i \vdash^j c_i$  with  $i + j \leq t$ . It follows that for all  $k \in \mathbb{N}$ ,  $c_0 \vdash^{i+kj} c_i$ . The machine never halts as it will visit  $c_i$  infinitely often.  $\square$

### 3 Decider for “Translated cyclers”

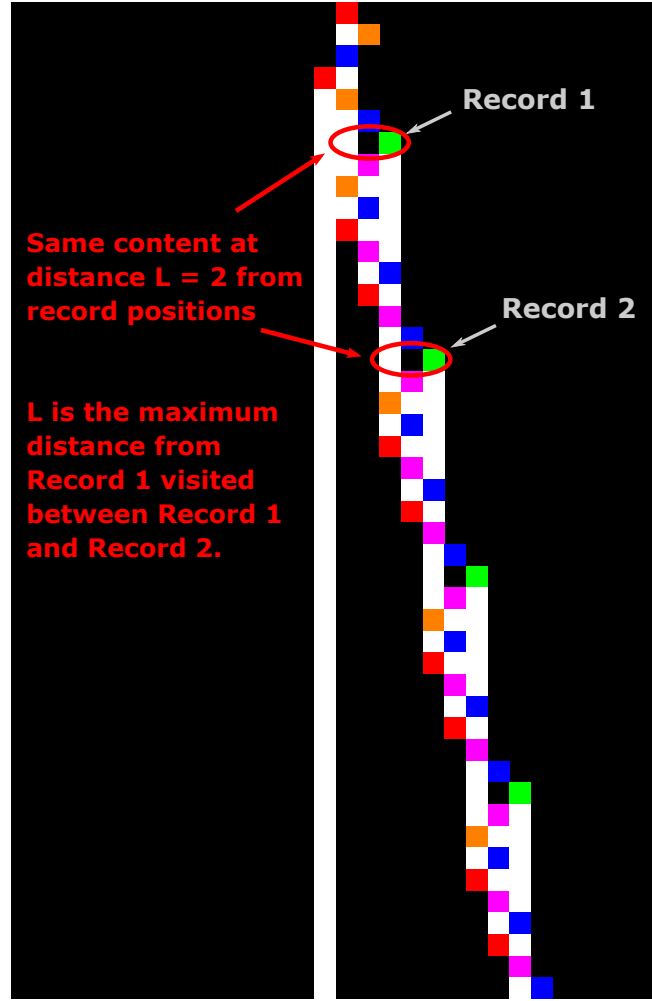


Figure 2: Example “Translated cycler”: 45-step space-time diagram of bbchallenge’s machine #44,394,115. See <https://bbchallenge.org/44394115>. The same bounded pattern is being translated to the right for ever. The text annotations illustrate the main idea recognising “Translated Cyclers”: find two configurations that break a record (i.e. visit a memory cell that was never visited before) in the same state (here state **D**) such that the content of the memory tape at distance  $L$  from the record positions is the same in both record configurations. Distance  $L$  is defined as being the maximum distance to record position 1 that was visited between the configuration of record 1 and record 2.

The goal of this decider is to recognise Turing machines that translate a bounded pattern for ever. We call such machines “Translated cyclers”. They are close to “Cyclers” (Section 2) in the sense that they are only repeating a pattern but there is added complexity as they are able to translate the pattern in space at the same time, hence the decider for Cyclers cannot directly apply here.

The main idea for this decider is illustrated in Figure 2 which gives the space-time diagram of a “Translated cycler”: bbchallenge’s machine #44,394,115 (c.f. <https://bbchallenge.org/44394115>). The idea is to find two configurations that break a record (i.e. visit a memory cell that was never visited before) in the same state (here state **D**) such that the content of the memory tape at distance  $L$  from the record positions is the same in both record configurations. Distance  $L$  is defined as being the maximum distance to record position 1 that was visited between the configuration of record 1 and record 2. In those conditions, we can prove that the machine will never halt.

The translated cycler of Figure 2 features a relatively simple repeating pattern and transient pattern (pattern occurring before the repeating patterns starts). These can get significantly more complex, bbchallenge’s machine #59,090,563 is an example see Figure 3 and <https://bbchallenge.org/59090563>. The method for detecting the behavior is the same but more resources are needed.



Figure 3: More complex “Translated cycler”: 10,000-step space-time diagram (no state colours) of bbchallenge’s machine #59,090,563. See <https://bbchallenge.org/59090563>.

### 3.1 Pseudocode

We assume that we are given a Turing Machine type **TM** that encodes the transition table of a machine as well as a procedure **TuringMachineStep**(machine,configuration) which computes the next configuration of a Turing machine from the given configuration or **nil** if the machine halts at that step.

One minor complication of the technique described above is that one has to track record-holding configurations on both sides of the tape: a configuration can break a record on the right or on the left. Also, in order to compute distance  $L$  (see above or Definition 5) it is useful to add to memory cells the information of the last time step at which it was visited.

We also assume that we are given a routine GET-EXTREME-POSITION(tape,sideOfTape) which gives us the rightmost or leftmost position of the given tape (well defined as we always manipulate finite tapes).

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#### Algorithm 2 DECIDER-TRANSLATED-CYLERS

---

```

1: const int RIGHT, LEFT = 0, 1
2: struct ValueAndLastTimeVisited {
3:   int value
4:   int lastTimeVisited
5: }
6: struct Configuration {
7:   int state
8:   int headPosition
9:   int → ValueAndLastTimeVisited tape
10: }
11: procedure bool DECIDER-TRANSLATED-CYLERS(TM machine,int timeLimit)
12:   Configuration currConfiguration = {state = 0, .headPosition = 0, .tape = {0:{.value = 0,
    .lastTimeVisited = 0}}}}
13:   // 0: right records, 1: left records
14:   List<Configuration> recordHoldingConfigurations[2] = [[],[]]
15:   int extremePositions[2] = [0,0]
16:   int currTime = 0
17:   while currTime < timeLimit do
18:     int headPosition = currConfiguration.headPosition
19:     currConfiguration.tape[headPosition].lastTimeVisited = currTime
20:     if headPosition > extremePositions[RIGHT] or headPosition < extremePositions[LEFT] then
21:       int recordSide = (headPosition > extremePositions[RIGHT]) ? RIGHT : LEFT
22:       extremePositions[recordSide] = headPosition
23:       if CHECK-RECORDS(currConfiguration, recordHoldingConfigurations[recordSide], record-
    Side) then
24:         return true
25:       recordHoldingConfigurations[recordSide].append(currConfiguration)
26:       currConfiguration = TuringMachineStep(machine,currConfiguration)
27:       currTime += 1
28:       if currConfiguration == nil then
29:         return false //machine has halted, it is not a Translated Cyclor
30:   return false

```

---

### 3.2 Correctness

**Definition 4** (Record-holding configurations). Let  $\mathcal{M}$  be a Turing machine and  $c_0$  its busy beaver initial configuration (i.e. state is 0, head position is 0 and tape is all-0). Let  $c$  be a configuration reachable from  $c_0$ , i.e.  $c_0 \vdash^* c$ . Then  $c$  is said to be *record-holding* if the current head position had never been visited before. Records can be broken to the *right* (positive head position) or to the left (negative head position).

**Definition 5** (Distance  $L$  between record-holding configurations). Let  $\mathcal{M}$  be a Turing machine and  $r_1, r_2$  be two record-holding configurations on the same side of the tape at respective times  $t_1$  and  $t_2$  with  $t_1 < t_2$ . Let  $p_1$  and  $p_2$  be the tape positions of these records. Then, distance  $L$  between  $r_1$  and  $r_2$  is defined as  $\max\{|p_1 - p| \}$  with  $p$  any position visited by  $\mathcal{M}$  between  $t_1$  and  $t_2$  that is not beating record  $p_1$  (i.e.  $p \leq p_1$  for a record on the right and  $p \geq p_1$  for a record on the left).

---

**Algorithm 3** COMPUTE-DISTANCE-L and AUX-CHECK-RECORDS

---

```
1: procedure int COMPUTE-DISTANCE-L(Configuration currRecord, Configuration olderRecord,
   int recordSide)
2:   int olderRecordPos = olderRecord.headPosition
3:   int olderRecordTime = olderRecord.tape[olderRecordPos].lastTimeVisited
4:   int currRecordTime = currRecord.tape[currRecord.headPosition].lastTimeVisited
5:   int distanceL = 0
6:   for int pos in currRecord.tape do
7:     if pos > olderRecordPos and recordSide == RIGHT then continue
8:     if pos < olderRecordPos and recordSide == LEFT then continue
9:     int lastTimeVisited = currRecord.tape[pos].lastTimeVisited
10:    if lastTimeVisited ≥ olderRecordTime and lastTimeVisited ≤ currRecordTime then
11:      distanceL = max(distanceL, abs(pos-olderRecordPos))
12:  return distanceL
13: procedure bool AUX-CHECK-RECORDS (Configuration currRecord, List<Configuration> older-
   Records, int recordSide)
14:   for Configuration olderRecord in olderRecords do
15:     if currRecord.state != olderRecord.state then
16:       continue
17:     int distanceL = COMPUTE-DISTANCE-L(currRecord, olderRecord, recordSide)
18:     int currExtremePos = GET-EXTREME-POSITION(currRecord.tape, recordSide)
19:     int olderExtremePos = GET-EXTREME-POSITION(olderRecord.tape, recordSide)
20:     int step = (recordSide == RIGHT) ? -1 : 1
21:     bool isSameLocalTape = true
22:     for int offset = 0; abs(offset) < distanceL; offset += step do
23:       if currRecord.tape[currExtremePos+offset] != olderRecord.tape[olderExtremePos+offset]
24:         then
25:           isSameLocalTape = false
26:           break
27:   if isSameLocalTape then
28:     return true
29:   return false
```

---

**Lemma 6.** Let  $\mathcal{M}$  be a Turing machine. Let  $r_1$  and  $r_2$  be two configurations that broke a record in the same state and on the same side of the tape at respective times  $t_1$  and  $t_2$  with  $t_1 < t_2$ . Let  $p_1$  and  $p_2$  be the tape positions of these records. Let  $L$  be the distance between  $r_1$  and  $r_2$  (Definition 5). If the content of tape in  $r_1$  at distance  $L$  of  $p_1$  is the same than the content of the tape in  $r_2$  at distance  $L$  of  $p_2$  then  $\mathcal{M}$  never halts. Furthermore, by Definition 5, we know that distance  $L$  is the maximum distance that  $\mathcal{M}$  can travel to the left of  $p_1$  between times  $t_1$  and  $t_2$ .

*Proof.* Let's suppose that the record-holding configurations are on the right-hand side of the tape. By the hypotheses, we know the machine is in the same state in  $r_1$  and  $r_2$  and that the content of the tape at distance  $L$  to the left of  $p_1$  in  $r_1$  is the same as the content of the tape at distance  $L$  to the left of  $p_2$  in  $r_2$ . Note that the content of the tape to the right of  $p_1$  and  $p_2$  is the same: all-0 since they are record positions. Hence that after  $r_2$ , since it will read the same tape content the machine will reproduce the same behavior than it did after  $r_1$  but translated at position  $p_2$ : there will a record-holding configuration  $r_3$  such that the distance between record-holding configurations  $r_2$  and  $r_3$  is also  $L$  (Definition 5). Hence the machine will keep breaking records to the right for ever and will not halt. Analogous proof for records that are broken to the left.  $\square$

**Theorem 7.** Let  $\mathcal{M}$  be a Turing machine and  $t$  a time limit. The conditions of Lemma 6 are met before time  $t$  if and only if `DECIDER-TRANSLATED-CYCLERS`( $\mathcal{M}, t$ ) outputs `true` (Algorithm 2).

**Corollary 8.** Let  $\mathcal{M}$  be a Turing machine and  $t \in \mathbb{N}$  a time limit. If `DECIDER-TRANSLATED-CYCLERS`( $\mathcal{M}, t$ ) returns `true` then the behavior of  $\mathcal{M}$  from all-0 tape has been decided:  $\mathcal{M}$  does not halt.

*Proof.* Immediate by combining Lemma 6 and Theorem 7.  $\square$

## References

- [1] S. Aaronson. The busy beaver frontier. *SIGACT News*, 51(3):32–54, Sept. 2020. <https://www.scottaaronson.com/papers/bb.pdf>.