Correctness of the decider for cyclers
bbchallenge

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Abstract

We give the pseudocode of the decider for the “Cyclers” family of bbchallenge and we prove its correctness. For more context please refer to `https://bbchallenge.org/`. The implementation of this decider is available at: `https://github.com/bbchallenge/bbchallenge-deciders/`.

1 Pseudocode

The goal of this decider is to recognise Turing machines that cycle through the same configurations for ever where a configuration is defined by the 3-tuple: (i) state (ii) position of the head (iii) content of the memory tape. Such machines never halt. The method is simple: remember every configuration seen by a machine and return `true` if one is visited twice. A time limit (maximum number of steps) is also given for running the test in practice: the algorithm recognises any machine whose cycle fits within this limit[1].

We assume that we are given a procedure **TuringMachineStep**(machine,configuration) which computes the next configuration of a machine from the given configuration or **nil** if the machine halts at that step.

Algorithm 1 DECIDER-CYLERS

```
struct Configuration {
    int state
    int headPosition
    int → int tape
}
procedure DECIDER-CYLERS(machine,timeLimit)
    Configuration currConfiguration = {.state = 0, .headPosition = 0, .tape = {0:0}}
    Set<Configuration> configurationsSeen = {}
    int currTime = 0
    while currTime < timeLimit do
        if currConfiguration in configurationsSeen then
            return true
        configurationsSeen.insert(currConfiguration)
        currConfiguration := TuringMachineStep(machine,currConfiguration)
        currTime += 1
        if currConfiguration == nil then
            return false //machine has halted
    return false
```

2 Correctness

The set $\mathbb{N}$ denotes $\{0, 1, 2 \dots\}$. The Turing machines that are studied in the context of bbchallenge use a binary alphabet and a single bi-infinite tape. A *configuration* is defined by the 3-tuple: (i) state (ii) position of the head (iii) content of the memory tape. In the context of bbchallenge, *the initial configuration* of a machine is always (i) state is 0, i.e. the first state to appear in the machine's description (ii) head's position is 0 (iii) the initial tape is all-0 – i.e. each memory cell is containing 0. In one step,

[1]In practice, for machines with 5 states the decider was run with 1000 steps time limit.

machine $\mathcal{M}$ transitions from configuration c to c' and we write $c \to_{\mathcal{M}} c'$. If the machine halts during that step we write $c \to_{\mathcal{M}} \bot$. By convention, $\bot \to_{\mathcal{M}} \bot$ is a valid transition for any machine $\mathcal{M}$. The operator $\to_{\mathcal{M}}^{n}$ is $\to_{\mathcal{M}}$ applied $n \in \mathbb{N}$ times. We write $c \to c'$ when the machine is clear from context.

Theorem 1. Let $\mathcal{M}$ be a Turing machine and $t \in \mathbb{N}$ a time limit. Let c_0 be the initial configuration of the machine. There exists $i \in \mathbb{N}$ and $j \in \mathbb{N}$ such that $c_0 \to^i c_i \to^j c_i$ with $i + j \leq t$ if and only if DECIDER-CYCLERS($\mathcal{M}$,t) returns true.

Proof. This follows directly from the behavior of DECIDER-CYCLERS($\mathcal{M}$,t): all intermediate configurations below time t are recorded and the algorithm returns true if and only if one is visited twice. This mathematically translates to there exists $i \in \mathbb{N}$ and $j \in \mathbb{N}$ such that $c_0 \to^i c_i \to^j c_i$ with $i + j \leq t$, which is what we want. $\square$

Corollary 2. Let $\mathcal{M}$ be a Turing machine and $t \in \mathbb{N}$ a time limit. If DECIDER-CYCLERS($\mathcal{M}$,t) returns true then the behavior of $\mathcal{M}$ from all-0 tape has been decided: $\mathcal{M}$ does not halt.

Proof. By Theorem 1, there exists $i \in \mathbb{N}$ and $j \in \mathbb{N}$ such that $c_0 \to^i c_i \to^j c_i$ with $i + j \leq t$. It follows that for all $k \in \mathbb{N}$, $c_0 \to^{i+kj} c_i$. The machine never halts as it will visit c_i infinitely often. $\square$