Exercise 1.2.5 (De Morgan's Laws) Let A and B be subsets of $\mathbb{R}$.

1. If $x \in A \cap B$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.
2. Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$ and conclude that $(A \cup B)^c = A^c \cap B^c$.
3. Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Proof:

1. If $x \in (A \cap B)^c$, this implies that $x \notin A \cap B$. By the definition of the set intersection, this means that $x \notin A$ or $x \notin B$. This means that $x \in A^c$ or $x \in B^c$. If $x \in A^c$, this means that $x \in A^c \cup B^c$. If $x \in B^c$, then $x \in B^c \cup A^c$, which is the same thing as $x \in A^c \cup B^c$. This implies that $(A \cap B)^c \subseteq A^c \cup B^c \square$
2. If $x \in A^c \cup B^c$, this means that $x \in A^c$ or $x \in B^c$. This also means that $x \notin A$ or $x \notin B$, which further implies that $x \notin A \cap B$. This finally implies that $x \in (A \cap B)^c$, showing as desired that $(A \cap B)^c \supseteq A^c \cup B^c$. As both incusions have been shown, $(A \cup B)^c = A^c \cap B^c$. $\square$
3. We shall proceed to show incluion by cases.

   ($\Rightarrow$) For the forward inclusion, let $x \in (A \cup B)^c$, which implies that $x \notin A \cup B$, which means that $x \notin A$ and $x \notin B$. This means that $x \in A^c$ and $x \in B^c$, which is the same as $x \in A^c \cap B^c$. This finally implies that $A \cup B)^c \subseteq A^c \cap B^c$.

   ($\Leftarrow$) For the reverse inclusion, let $x \in A^c \cap B^c$, which means that $x \in A^c$ and $x \in B^c$ by the definition of the set intersection. This implies that $x \notin A$ and $x \notin B$, which means that $x \notin A \cup B$, which is the same thing as saying that $x \in (A \cup B)^c$. This finally implies that $A^c \cap B^c \subseteq (A \cup B)^c$.

Conclusion As we have shown that $A \cup B)^c \subseteq A^c \cap B^c$ and $A^c \cap B^c \subseteq (A \cup B)^c$, we have proven that $(A \cup B)^c = A^c \cap B^c$. $\square$

Exercise 1.2.6 (Triangle Inequality)

1. Verify the triangle inequality in the special case where a and b have the same sign.
2. Find an efficient proof for all the cases at once by first demonstrating that $(a+b)^2 \leq (|a|+|b|)^2$.
3. Prove $|a-b| \leq |a-c| + |c-d| + |d-b|$ for all a, b, c, and d.
4. Prove $||a|-|b|| \leq |a-b|$.

Proof:

1. We shall proceed by cases.

$a, b \in \mathbb{R}_{\geq 0}$ In the case that both a and b are positive, $|a| = a$ and $|b| = b$, so we may rewrite the triangle inequality in this case as $|a+b| \leq a+b$. It follows that $|a+b|$ can be written as $a+b$ as both a and b are positive numbers and therefore their sum will be a positive number. Substituting the alternate forms we have just derived back into the triangle inequality, we get the equation $a+b \leq a+b$, which is true for all $a, b \in \mathbb{R}_{\geq 0}$.

$a, b \in \mathbb{R}_{<0}$ In the case that both a and b are negative, we have from the definition of the absolute value that $|a| = -a$ and $|b| = -b$. This means that $|a|+|b| = -a+(-b) = -a-b$ and that $|a+b| = -(a+b) = (-a-b)$. Simplifying, we arrive at the equation $-a-b \leq -a-b$, which is true for all $a, b \in \mathbb{R}_{\leq 0}$.

Conclusion We have shown that both cases reduce to the standard triangle inequality, showing it to be valid for the cases where both numbers are either positive or negative. $\square$

2. We shall begin by expanding both sides of the equation. The left side expands to $a^2 + 2ab + b^2$ and the right side expands to $a^2 + 2|a||b| + b^2$. Note that we can drop the absolute value bars for a^2 and b^2 on the right as the square of any number, positive or negative is always positive, and by the definition of the absolute value, if a number a is positive, then $|a| = a$. Simplifying yields the inequality $2ab \leq 2|a||b|$, which is true because $2ab$ can be negative (consider the case where either a or b is negative but not both) but $2|a||b|$ cannot be negative due to the absolute value of any real number always being positive. As squaring preserves the inequality, it follows that $|a+b| \leq |a| + |b|$. $\square$

3. We begin by noticing that $|a-b|$ is equivalent to $|(a-c)+(c-d)+(d-b)|$. Now subtituting that into the original equation, we get $|(a-c)+(c-d)+(d-b)| \leq |a-c|+|c-d|+|d-b|$ Multiple applications of the multivariable triangle inequality, defined to be $(a-b)+c \leq |a-b|+|c|$, yields the desired result. $\square$

4. As $||a|-|b|| = ||b|-|a||$ for all $a, b \in \mathbb{R}$, we can assume that $|a| > |b|$. Then by application of the triangle inequality, $||a|-|b|| = |a|-|b| = |(a-b)+b|-|b| \leq |a-b|+|b|-|b| = |a-b|$. $\square$

Exercise 1.2.7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x)\colon x \in A\}$.

1. Let $f(x) = x^2$. If $A := [0, 2]$ and $B := [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?
2. Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.
3. Show that, for an arbitrary function $g\colon \mathbb{R} \to \mathbb{R}$ it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.
4. Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Solutions:

1. $f(A) := [0, 4], f(B) := [1, 16], f(A \cap B) := [1, 4], f(A) \cap f(B) := [1, 4], f(A \cup B) := [0, 16], f(A) \cup f(B) = [0, 16]$
   This shows that $f(A \cap B) = f(A) \cap f(B)$ and $f(A \cup B) = f(A) \cup f(B)$
2. Let $A := \{-2\}$ and let $B := \{2\}$. Then $A \cap B = \{\emptyset\}$ and $f(A) \cap f(B) = \{4\}$.
3. Suppose that $y \in g(A \cap B)$. Then by the definition of a function, $\exists x \in A \cap B$ such that $g(x) = y$. If $x \in A \cap B$, then $x \in A$ or $x \in B$, implying that $y \in g(A)$ or $x \in g(B)$, which is the same thing as $y \in g(A) \cup g(B)$. This means that $g(A) \cap g(B) \subseteq g(A \cap B)$. $\square$
4. We conjecture that $g(A \cup B) = g(A) \cup g(B)$. We shall prove this by showing inclusion in both directions via cases.

   ($\Rightarrow$) For the forward direction, suppose that $y \in g(A \cup B)$, this means that $y \in g(A)$ or $y \in g(B)$. This implies that $y \in g(A) \cup g(B)$. This shows that $g(A \cup B) \subseteq g(A) \cup g(B)$.

   ($\Leftarrow$) For the reverse direction, suppose that $y \in g(A) \cup g(B)$. This means that $y \in g(A)$ or $y \in (B)$, which is the same as $y \in g(A \cup B)$ This shows that $g(A) \cup g(B) \subseteq g(A \cup B)$.

Conclusion As we have demonstrated that $g(A \cup B) \subseteq g(A) \cup g(B)$ and $g(A) \cup g(B) \subseteq g(A \cup B)$, $g(A \cup B) = g(A) \cup g(B)$. $\square$