Exercise 1.2.5 (De Morgan's Laws) Let A and B be subsets of $\mathbb{R}$.

1. If $x \in A \cap B$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.
2. Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$ and conclude that $(A \cup B)^c = A^c \cap B^c$.
3. Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Proof:

1. If $x \in (A \cap B)^c$, this implies that $x \notin A \cap B$. By the definition of the set intersection, this means that $x \notin A$ or $x \notin B$. This means that $x \in A^c$ or $x \in B^c$. If $x \in A^c$, this means that $x \in A^c \cup B^c$. If $x \in B^c$, then $x \in B^c \cup A^c$, which is the same thing as $x \in A^c \cup B^c$. $\square$
2. If $x \in A^c \cup B^c$, this means that $x \in A^c$ or $x \in B^c$. This also means that $x \notin A$ or $x \notin B$, which further implies that $x \notin A \cap B$. This finally implies that $x \in (A \cap B)^c$ $\square$
3. Let $x \in (A \cup B)^c$, which implies that $x \notin A \cup B$, which means that $x \notin A$ and $x \notin B$. This means that $x \in A^c$ and $x \in B^c$, which is the same as $x \in A^c \cap B^c$. For the reverse inclusion, let $x \in A^c \cap B^c$, which means that $x \in A^c$ and $x \in B^c$ by the definition of the set intersection. This implies that $x \notin A$ and $x \notin B$, which means that $x \notin A \cup B$, which is the same thing as saying that $x \in (A \cup B)^c$. $\square$

Exercise 1.2.6 (Triangle Inequality)

1. Verify the triangle inequality in the special case where a and b have the same sign.
2. Find an efficient proof for all the cases at once by first demonstrating that $(a+b)^2 \leq (|a|+|b|)^2$.
3. Prove $|a-b| \leq |a-c|+|c-d|+|d-b|$ for all a, b, c, and d.
4. Prove $||a|-|b|| \leq |a-b|$.

Proof:

1. We shall proceed by cases. In the case that both a and b are positive, $|a| = a$ and $|b| = b$, so we may rewrite the triangle inequality in this case as $|a+b| \leq a+b$. It follows that $|a+b|$ can be written as $a+b$ as both a and b are positive numbers and therefore their sum will be a positive number. Substituting the alternate forms we have just derived back into the triangle inequality, we get the equation $a+b \leq a+b$, which is true for all $a, b \in \mathbb{R}_{\geq 0}$. In the case that both a and b are negative, we have from the definition of the absolute value that $|a| = -a$ and $|b| = -b$. This means that $|a|+|b| = -a+(-b) = -a-b$ and that $|a+b| = -(a+b) = (-a-b)$. Simplifying, we arrive at the equation $-a-b \leq -a-b$, which is true for all $a, b \in \mathbb{R}_{\leq 0}$. $\square$
2. We shall begin by expanding both sides of the equation. The left side expands to $a^2+2ab+b^2$ and the right side expands to $a^2+2|a||b|+b^2$. Note that we can drop the absolute value bars for a^2 and b^2 on the right as the square of any number, positive or negative is always positive, and by the definition of the absolute value, if a number a is positive, then $|a| = a$. Simplifying yields the inequality $2ab \leq 2|a||b|$, which is true because $2ab$ can be negative (consider the case where either a or b is negative but not both) but $2|a||b|$ cannot be negative due to the absolute value of any real number always being positive. As squaring preserves the inequality, it follows that $|a+b| \leq |a|+|b|$. $\square$
3. We begin by noticing that $|a-b|$ is equivalent to $|(a-c)+(c-d)+(d-b)|$. Now subtituting that into the original equation, we get $|(a-c)+(c-d)+(d-b)| \leq |a-c|+|c-d|+|d-b|$ Multiple applications of the multivariable triangle inequality, defined to be $(a-b)+c \leq |a-b|+|c|$, yields the desired result. $\square$
4. As $||a|-|b|| = ||b|-|a||$ for all $a, b \in \mathbb{R}$, we can assume that $|a| > |b|$. Then by application of the triangle inequality, $||a|-|b|| = |a|-|b| = |(a-b)+b|-|b| \leq |a-b|+|b|-|b| = |a-b|$. $\square$