

**Exercise 1.2.5 (De Morgan's Laws)** Let  $A$  and  $B$  be subsets of  $\mathbb{R}$ .

1. If  $x \in A \cap B$ , explain why  $x \in A^c \cup B^c$ . This shows that  $(A \cap B)^c \subseteq A^c \cup B^c$ .
2. Prove the reverse inclusion  $(A \cap B)^c \supseteq A^c \cup B^c$  and conclude that  $(A \cup B)^c = A^c \cap B^c$ .
3. Show  $(A \cup B)^c = A^c \cap B^c$  by demonstrating inclusion both ways.

**Proof:**

1. If  $x \in (A \cap B)^c$ , this implies that  $x \notin A \cap B$ . By the definition of the set intersection, this means that  $x \notin A$  or  $x \notin B$ . This means that  $x \in A^c$  or  $x \in B^c$ . If  $x \in A^c$ , this means that  $x \in A^c \cup B^c$ . If  $x \in B^c$ , then  $x \in B^c \cup A^c$ , which is the same thing as  $x \in A^c \cup B^c$ . This implies that  $(A \cap B)^c \subseteq A^c \cup B^c$   $\square$
2. If  $x \in A^c \cup B^c$ , this means that  $x \in A^c$  or  $x \in B^c$ . This also means that  $x \notin A$  or  $x \notin B$ , which further implies that  $x \notin A \cap B$ . This finally implies that  $x \in (A \cap B)^c$ , showing as desired that  $(A \cap B)^c \supseteq A^c \cup B^c$ . As both inclusions have been shown,  $(A \cup B)^c = A^c \cap B^c$ .  $\square$
3. We shall proceed to show inclusion by cases.

$\Rightarrow$  For the forward inclusion, let  $x \in (A \cup B)^c$ , which implies that  $x \notin A \cup B$ , which means that  $x \notin A$  and  $x \notin B$ . This means that  $x \in A^c$  and  $x \in B^c$ , which is the same as  $x \in A^c \cap B^c$ . This finally implies that  $(A \cup B)^c \subseteq A^c \cap B^c$ .

$\Leftarrow$  For the reverse inclusion, let  $x \in A^c \cap B^c$ , which means that  $x \in A^c$  and  $x \in B^c$  by the definition of the set intersection. This implies that  $x \notin A$  and  $x \notin B$ , which means that  $x \notin A \cup B$ , which is the same thing as saying that  $x \in (A \cup B)^c$ . This finally implies that  $A^c \cap B^c \subseteq (A \cup B)^c$ .

**Conclusion** As we have shown that  $(A \cup B)^c \subseteq A^c \cap B^c$  and  $A^c \cap B^c \subseteq (A \cup B)^c$ , we have proven that  $(A \cup B)^c = A^c \cap B^c$ .  $\square$

**Exercise 1.2.6 (Triangle Inequality)**

1. Verify the triangle inequality in the special case where  $a$  and  $b$  have the same sign.
2. Find an efficient proof for all the cases at once by first demonstrating that  $(a + b)^2 \leq (|a| + |b|)^2$ .
3. Prove  $|a - b| \leq |a - c| + |c - d| + |d - b|$  for all  $a, b, c$ , and  $d$ .
4. Prove  $||a| - |b|| \leq |a - b|$ .

**Proof:**

1. We shall proceed by cases.

$a, b \in \mathbb{R}_{\geq 0}$  In the case that both  $a$  and  $b$  are positive,  $|a| = a$  and  $|b| = b$ , so we may rewrite the triangle inequality in this case as  $|a + b| \leq a + b$ . It follows that  $|a + b|$  can be written as  $a + b$  as both  $a$  and  $b$  are positive numbers and therefore their sum will be a positive number. Substituting the alternate forms we have just derived back into the triangle inequality, we get the equation  $a + b \leq a + b$ , which is true for all  $a, b \in \mathbb{R}_{\geq 0}$ .

$a, b \in \mathbb{R}_{< 0}$  In the case that both  $a$  and  $b$  are negative, we have from the definition of the absolute value that  $|a| = -a$  and  $|b| = -b$ . This means that  $|a| + |b| = -a + (-b) = -a - b$  and that  $|a + b| = -(a + b) = (-a - b)$ . Simplifying, we arrive at the equation  $-a - b \leq -a - b$ , which is true for all  $a, b \in \mathbb{R}_{\leq 0}$ .

**Conclusion** We have shown that both cases reduce to the standard triangle inequality, showing it to be valid for the cases where both numbers are either positive or negative.  $\square$

2. We shall begin by expanding both sides of the equation. The left side expands to  $a^2 + 2ab + b^2$  and the right side expands to  $a^2 + 2|a||b| + b^2$ . Note that we can drop the absolute value bars for  $a^2$  and  $b^2$  on the right as the square of any number, positive or negative is always positive, and by the definition of the absolute value, if a number  $a$  is positive, then  $|a| = a$ . Simplifying yields the inequality  $2ab \leq 2|a||b|$ , which is true because  $2ab$  can be negative (consider the case where either  $a$  or  $b$  is negative but not both) but  $2|a||b|$  cannot be negative due to the absolute value of any real number always being positive. As squaring preserves the inequality, it follows that  $|a + b| \leq |a| + |b|$ .  $\square$

3. We begin by noticing that  $|a - b|$  is equivalent to  $|(a - c) + (c - d) + (d - b)|$ . Now substituting that into the original equation, we get  $|(a - c) + (c - d) + (d - b)| \leq |a - c| + |c - d| + |d - b|$ . Multiple applications of the multivariable triangle inequality, defined to be  $(a - b) + c \leq |a - b| + |c|$ , yields the desired result.  $\square$
4. As  $||a| - |b|| = ||b| - |a||$  for all  $a, b \in \mathbb{R}$ , we can assume that  $|a| > |b|$ . Then by application of the triangle inequality,  $||a| - |b|| = |a| - |b| = |(a - b) + b| - |b| \leq |a - b| + |b| - |b| = |a - b|$ .  $\square$

**Exercise 1.2.7**

Given a function  $f$  and a subset  $A$  of its domain, let  $f(A)$  represent the range of  $f$  over the set  $A$ ; that is,  $f(A) = \{f(x) : x \in A\}$ .

1. Let  $f(x) = x^2$ . If  $A := [0, 2]$  and  $B := [1, 4]$ , find  $f(A)$  and  $f(B)$ . Does  $f(A \cap B) = f(A) \cap f(B)$  in this case? Does  $f(A \cup B) = f(A) \cup f(B)$ ?
2. Find two sets  $A$  and  $B$  for which  $f(A \cap B) \neq f(A) \cap f(B)$ .
3. Show that, for an arbitrary function  $g: \mathbb{R} \rightarrow \mathbb{R}$  it is always true that  $g(A \cap B) \subseteq g(A) \cap g(B)$  for all sets  $A, B \subseteq \mathbb{R}$ .
4. Form and prove a conjecture about the relationship between  $g(A \cup B)$  and  $g(A) \cup g(B)$  for an arbitrary function  $g$ .

**Proof:**

1.  $f(A) := [0, 4]$ ,  $f(B) := [1, 16]$ ,  $f(A \cap B) := [1, 4]$ ,  $f(A) \cap f(B) := [1, 4]$ ,  $f(A \cup B) := [0, 16]$ ,  $f(A) \cup f(B) = [0, 16]$   
This shows that  $f(A \cap B) = f(A) \cap f(B)$  and  $f(A \cup B) = f(A) \cup f(B)$
2. Let  $A := \{-2\}$  and let  $B := \{2\}$ . Then  $A \cap B = \{\emptyset\}$  and  $f(A) \cap f(B) = \{4\}$ .
3. Suppose that  $y \in g(A \cap B)$ . Then by the definition of a function,  $\exists x \in A \cap B$  such that  $g(x) = y$ . If  $x \in A \cap B$ , then  $x \in A$  or  $x \in B$ , implying that  $y \in g(A)$  or  $y \in g(B)$ , which is the same thing as  $y \in g(A) \cup g(B)$ . This means that  $g(A \cap B) \subseteq g(A) \cup g(B)$ .  
 $\square$
4. We conjecture that  $g(A \cup B) = g(A) \cup g(B)$ . We shall prove this by showing inclusion in both directions via cases.
  - $\Rightarrow$  For the forward direction, suppose that  $y \in g(A \cup B)$ , this means that  $y \in g(A)$  or  $y \in g(B)$ . This implies that  $y \in g(A) \cup g(B)$ . This shows that  $g(A \cup B) \subseteq g(A) \cup g(B)$ .
  - $\Leftarrow$  For the reverse direction, suppose that  $y \in g(A) \cup g(B)$ . This means that  $y \in g(A)$  or  $y \in g(B)$ , which is the same as  $y \in g(A \cup B)$ . This shows that  $g(A) \cup g(B) \subseteq g(A \cup B)$ .

**Conclusion** As we have demonstrated that  $g(A \cup B) \subseteq g(A) \cup g(B)$  and  $g(A) \cup g(B) \subseteq g(A \cup B)$ ,  
 $g(A \cup B) = g(A) \cup g(B)$ .  $\square$