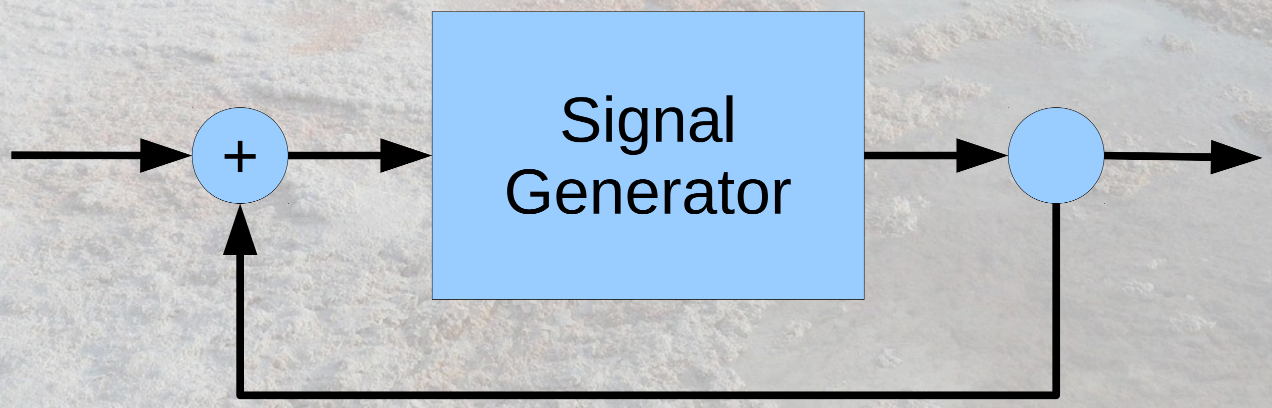
Frequency Modulation

Feedback

- Consider a system where an output signal is generated from an input signal
- **Feedback** – an attenuated, time-delayed copy of the output signal is mixed in with the input signal



Simple Feedback Model

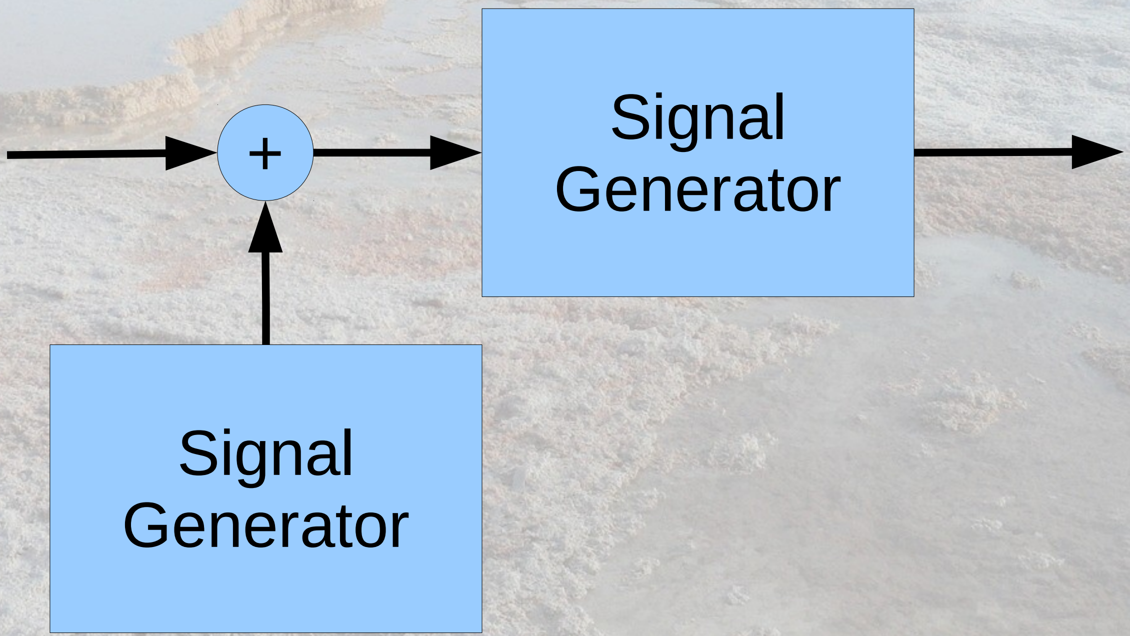
- Let $y = f(t)$ be the signal without feedback
- With feedback:

$$y(t) = f(t + \alpha f(t - \delta))$$

  where
  - α controls the amount of feedback
  - δ is the delay
- For example, take *f* to be a sinusoid

Frequency Modulation (FM)

- Similar to feedback, except a different signal is mixed into the system's input



Simple FM Model

- Simple form of FM synthesis

$$y(t) = \sin(2\pi f_c t + \alpha_m \sin(2\pi f_m t))$$

where

  - f_c is the **carrier frequency**
  - α_m is the **amplitude of modulation**
  - f_m is the **modulation frequency**
- Waveforms other than sinusoids may be used

Alternate Formulation

- Define the **index of modulation** as

$$\mu_m = \frac{\alpha_m}{2\pi f_c}$$

- The our model for FM synthesis can then be written as

$$y(t) = \sin(2\pi f_c (t + \mu_m \sin(2\pi f_m t)))$$

Effect of Modulation Parameters

- Modulation frequencies below the limit of human hearing (< 20 Hz) give a vibrato effect
  - Amplitude/index of modulation controls the depth of the vibrato
- Frequencies within the range of human hearing introduce harmonics
  - Amplitude (or index) of modulation controls the amount of new harmonics

Digression: Bessel Functions (1)

- The **Bessel function** of integer order *n* may be defined as

$$J_n(t) = \frac{1}{\pi} \int_0^{\pi} \cos(nx - t \sin x)\, dx$$

- $y(t) = J_n(t)$ solves the differential equation

$$t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + (t^2 - n^2)\, y = 0$$

Digression: Bessel Functions (2)

- The following relations are satisfied

$$J_{-n}(t)=(-1)^n J_n(t)$$

$$J_{n+1}(t)=\frac{2n}{t}J_n(t)-J_{n-1}(t)$$

- In particular, J_0 and J_1 determine all other Bessel functions of integral order

Digression: Bessel Functions (3)

- For small values of *t*

$$J_n(t) \approx \frac{t^n}{2^n n!}$$

- Bessel functions are asymptotically harmonic:

$$J_n(t) \approx \sqrt{\frac{2}{\pi t}} \cos\left(t - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

for *t* large

Harmonic Effect of Modulation

- Modulation of a sinusoid adds harmonics
  - Multiples of the modulation frequency are added to the carrier frequency
  - The amplitudes of the harmonics are governed by Bessel functions

$$\sin\left(2\pi f_c t + \alpha_m \sin\left(2\pi f_m t\right)\right)$$

$$= \sum_{n\in\mathbb{Z}} J_n(\alpha_m) \sin\left(2\pi\left(f_c + n f_m\right) t\right)$$

Computing Simple FM Partials (1)

- Suppose: $f_c = 200$ Hz, $f_m = 150$ Hz, $\alpha_m = 1$

  Given: $J_0(1) \approx 0.7652$, $J_1(1) \approx 0.4401$

- $n = 0$ term

  amplitude: $J_0(1) \approx 0.7652$ (-2.32 dB)

  frequency: = $200 + 0(150) = 200$ Hz

- $n = 1$ term

  amplitude: $J_1(1) \approx 0.4401$ (-7.13 dB)

  frequency: $200 + 1(150) = 350$ Hz

Computing Simple FM Partials (2)

- *n = -1* term

  amplitude: $J_{-1}(1) = (-1)^1\, J_1(1) \approx -0.4401$

  (absolute value of amplitude: -7.13 dB)

  frequency: *200 – 1(150) = 50 Hz*

- *n = 2* term

$$J_2(1) = \frac{2(1)}{1} J_1(1) - J_0(1)$$
$$\approx 2(0.4401) - (0.7652) \approx 0.1150$$

  amplitude: $J_2(1) \approx 0.1150$ *(-18.8 dB)*

  frequency: *200 + 2(150) = 500 Hz*

Computing Simple FM Partials (3)

- *n = -2* term

  amplitude: $J_{-2}(1) = (-1)^2 J_2(1) \approx 0.1150$ *(-18.8 dB)*

  frequency: |*200 – 2(150)*| = |*-100*| = *100 Hz*

- *n = 3* term

$$J_3(1) = \frac{2(2)}{1} J_2(1) - J_1(1)$$
$$\approx 4(0.7652) - (0.1150) \approx 0.0199$$

  amplitude: $J_3(1) \approx 0.0199$ *(-34.0 dB)*

  frequency: *200 + 3(150) = 650 Hz*

Computing Simple FM Partials (4)

- *n* = *-3* term

  amplitude: $J_{-3}(1) = (-1)^3 J_3(1) \approx -0.0199$

  (absoluted value of amplitude: -34.0 dB)

  frequency: $|200 - 3(150)| = |-250| = 250$ *Hz*

Complex FM Synthesis (1)

- The simple FM synthesis model can be used as a building block to produce more complex sounds
- An envelope can be applied to the modulated sinusoid to shape the signal
- A shaped FM signal can be used to modulate another sinusoid
- FM signals can be combined additively

Complex FM Synthesis (2)