# Additive Synthesis

# Harmonic Functions

- Sine and cosine functions form the basis of harmony
- The cosine is a *phase shift* of the sine by *π/2*

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$$

- A sine wave with *frequency f* has **angular frequency** *ω = 2πf*

$$y(t) = \sin(2\pi f t) = \sin(\omega t)$$

# Discretized Sine Function

- If *R* is the sampling rate (so that *T = 1/R* is the time between samples), the *n*-th sample of a sine wave is

$$y_n = \sin\left(\frac{2\pi f n}{R}\right)$$

  where $y_n = y(n/R)$

- However, this involves the computation of the sine function (which can be expensive) for each output sample

# Recurrence Relation

- The *n*-th sample of a discretized sine function can be accurately be computed using the *recurrence relation:*

$$y_0 = 0 \qquad y_1 = \sin\left(\frac{2\pi f}{R}\right)$$

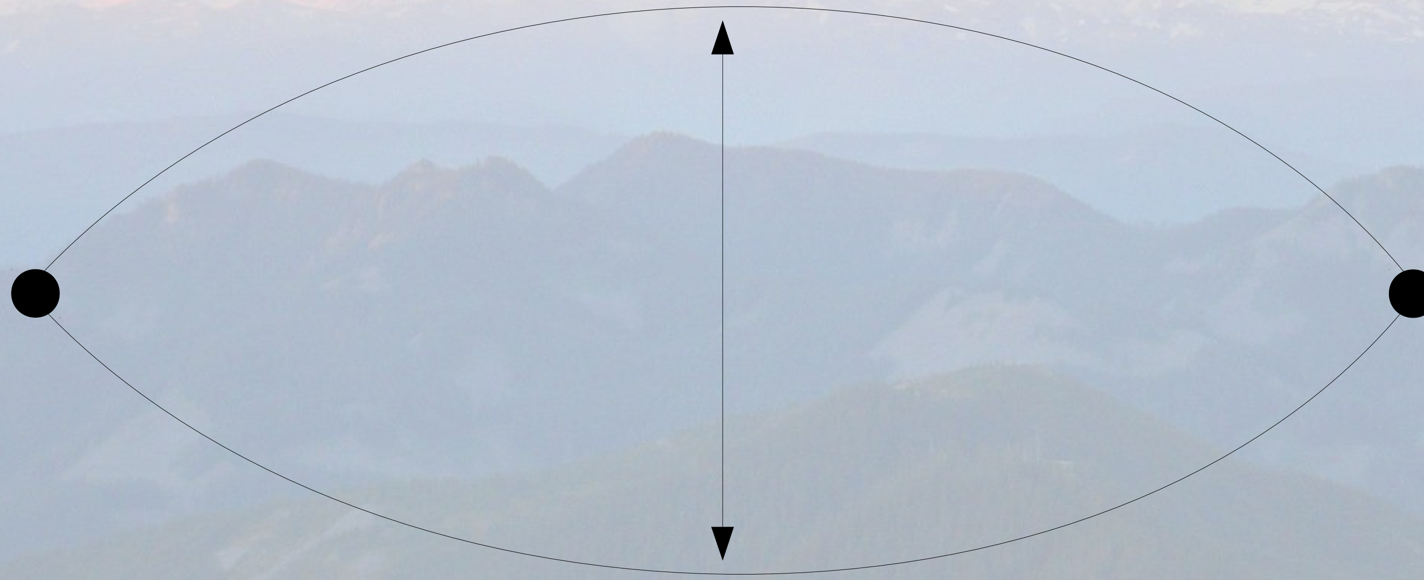
$$y_n = a\, y_{n-1} - y_{n-2}$$

where

$$a = 2\cos\left(\frac{2\pi f}{R}\right)$$

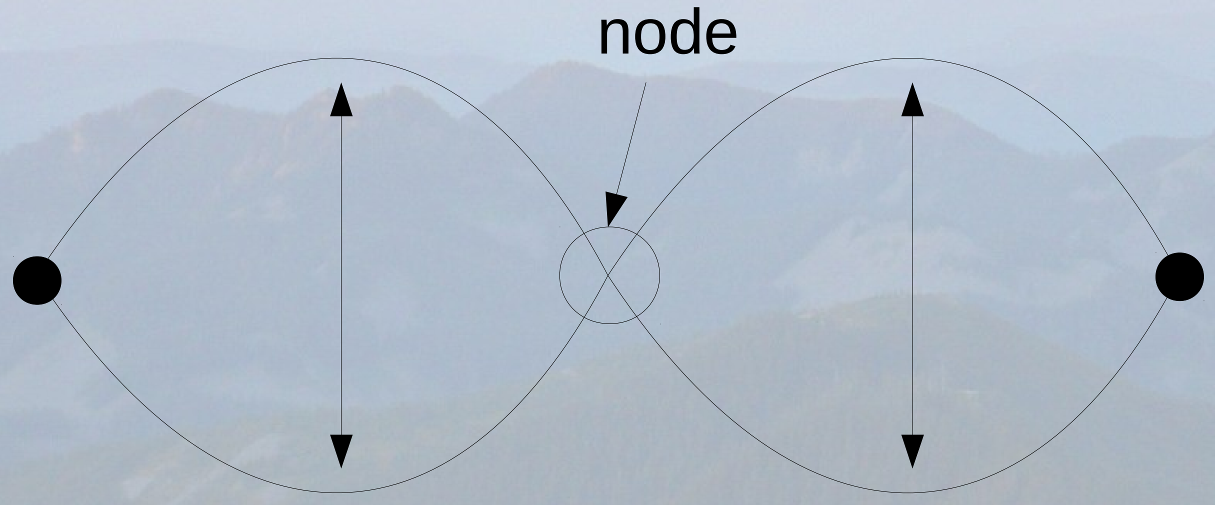
# Strings (1)

- A vibrating string has different modes of oscillation (standing waves)
- The first is the **fundamental**, or **first harmonic**



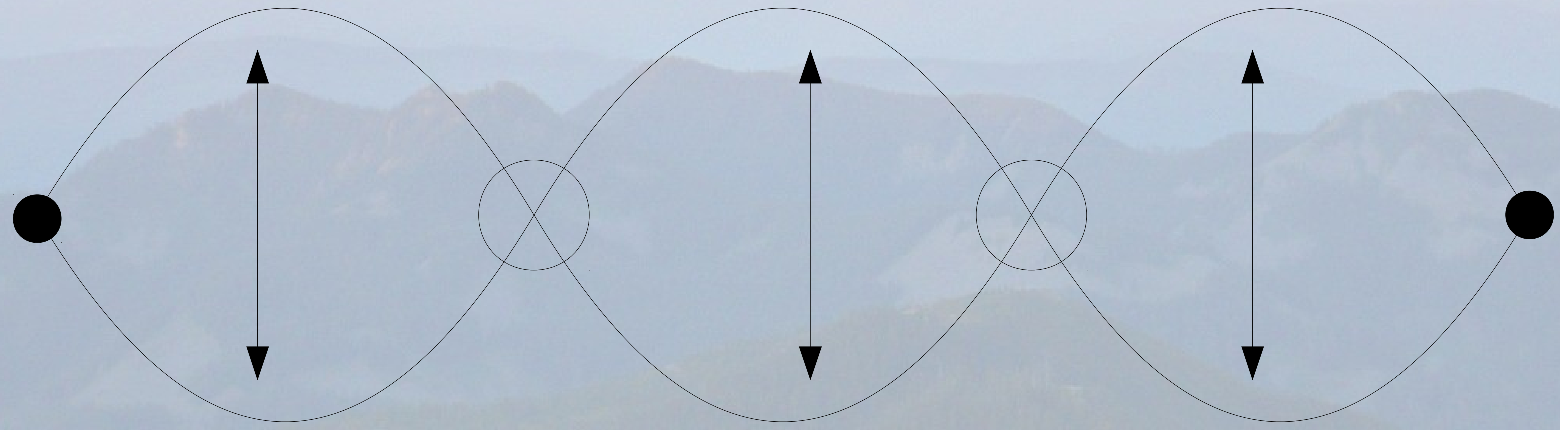
# Strings (2)

- The second is called the **second harmonic**
  - Frequency of oscillation is twice that of the fundamental: $f = 2\,f_0$
  - One vibrational node



# Strings (3)

- The *n*-th harmonic
  - Has *(n-1)* vibrational nodes
  - Frequency of oscillation is *n* times the fundamental frequency: $f = n\, f_0$



3rd harmonic

# Strings (4)

- A plucked string is a superposition of many harmonics

$$y(t) = \sum_{n=1}^{\infty} A_n \sin(n \omega_0 t)$$

  where $\omega_0 = 2\pi f_0$

- The **amplitude** $A_n$ of the $n$-th harmonic determines the overall sound of the string
  - May depend on how the string is plucked

# Harmonic Series (1)

- For a given fundamental angular frequency $\omega_0$, the sequence of integral multiples

$$\omega_0, 2\omega_0, 3\omega_0, 4\omega_0, \ldots$$

  is called a **harmonic series**

- Aside: the so-called harmonic sequence in mathematics *1, 1/2, 1/3, 1/4, …* is related the above harmonic series since a string, when shortened to *1/n*-th of its original length, vibrates at n times its original frequency

# Harmonic Series (2)

- The individual terms in the harmonic series are called **partials**
- A harmonic series, along with a collection of coefficients $A_n$, can be used to additively construct a signal

$$y(t) = \sum_{n=1}^{\infty} A_n \sin(n \omega_0 t)$$

that can be identified as having a “pitch” value whose frequency is $f_0 = \omega_0 / 2\pi$

# Inharmonic series

- More generally, we may consider arbitrary ordered sequences of frequences

$$\omega_1 \leq \omega_2 \leq \omega_3 \leq \omega_4 \leq \ldots$$

- If $\omega_n \neq n\,\omega_1$ for some $n$, then the sequence is said to be **inharmonic**
- We may again construct a signal additively

$$y(t) = \sum_{n=1}^{\infty} A_n \sin(\omega_n t)$$

# Fourier Sine Series (1)

- Every continuous function on *[0,L]* can be expressed as a sum of sines

$$f(t) = \sum_{n=1}^{\infty} A_n \sin(n\pi t/L)$$

provided that *f(0) = 0* and *f(L) = 0*

- Indeed, the coefficients are given (uniquely) by

$$A_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx$$

# Fourier Sine Series (2)

- This follows from the basic orthogonality relation for sine functions

$$\int_0^L \sin(m\pi x/L)\sin(n\pi x/L)\,dx = \frac{L}{2}\delta_{mn}$$

- For a function of *L* = *N/R* seconds sampled at a rate *R*

$$A_n \approx \frac{2}{N}\sum_{k=0}^{N-1} f(k/R)\sin(\pi nk/N)$$

# Fourier Sine Series (3)

- The sum may require too many terms to be an accurate approximation
- Inefficient way to construct anything other than simple string-like sounds

# Inharmonic Deconstruction

- Given an audio signal, it is possible to extract the dominant frequencies and the amplitudes of the corresponding partials
  - The frequencies and amplitudes can be obtained using the Fourier Transform
- The original signal can then be approximated by an inharmonic sum of sinusoids
- Fewer partials are need to construct a more complex sound

# Using Envelopes

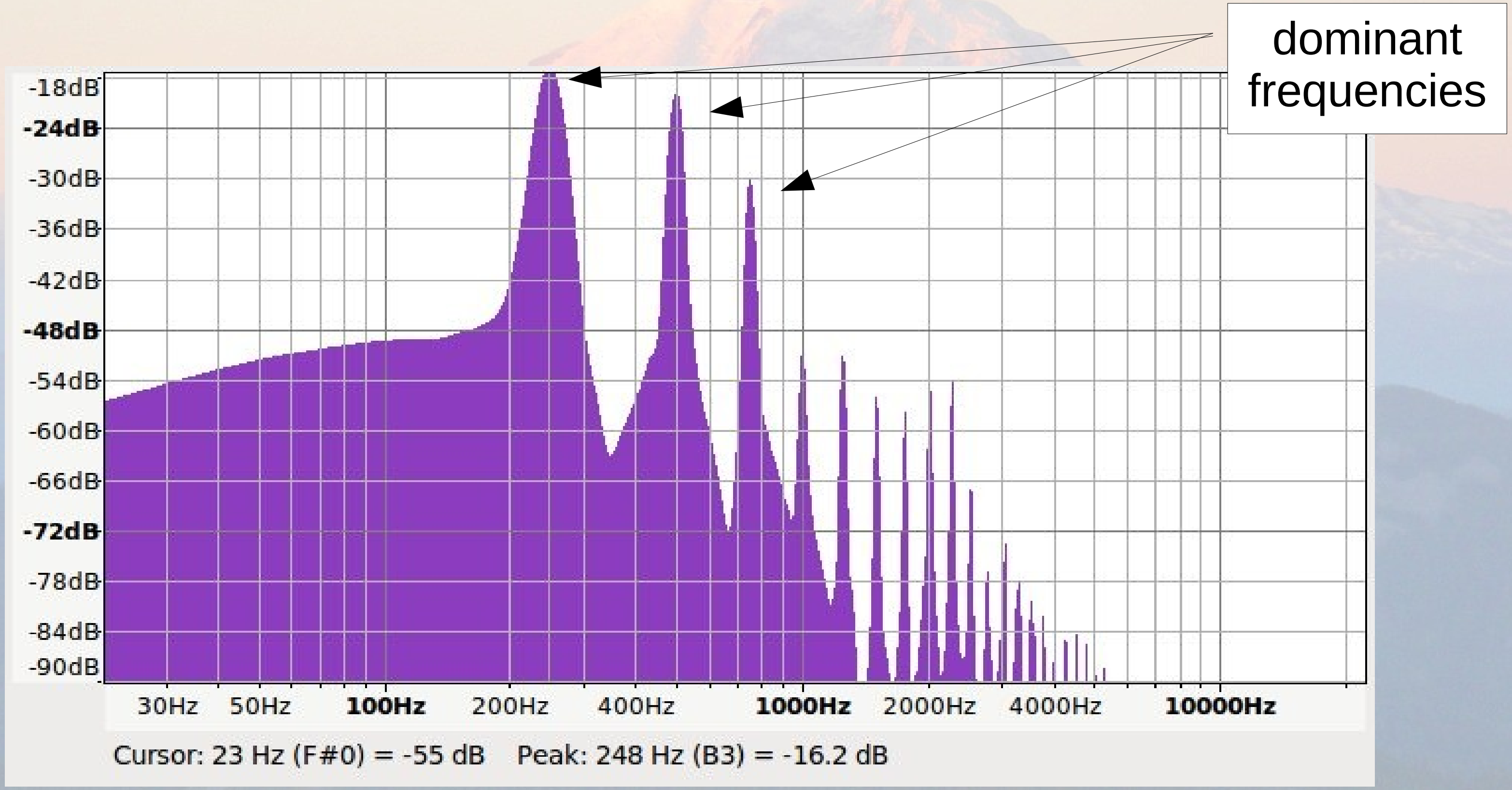
- We can synthesize an even wider variety of sounds with only a few sinusoids if we allow the coefficients in the sum of sinusoids to vary with time

$$y(t)=\sum_{n=1}^{\infty} A_n(t)\sin(\omega_n t)$$

- That is, we may multiply each partial by a time-varying envelope

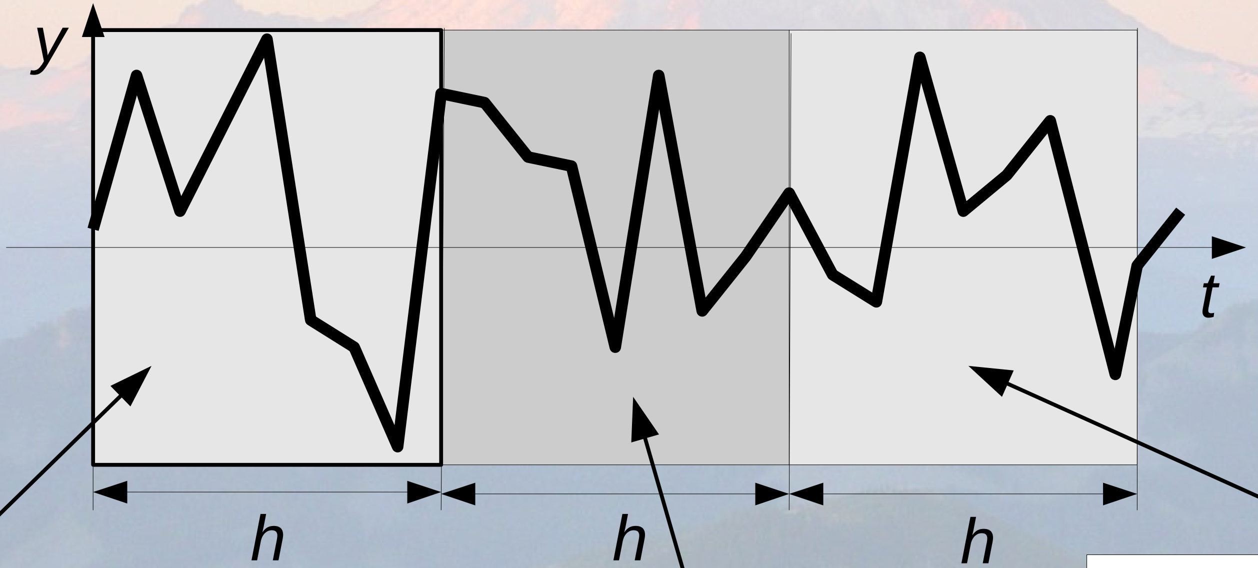
# Sound Modeling (1)

- Identify the dominant frequencies and amplitudes from the frequency spectrum



# Sound Modeling (2)

- Approximate the envelope $A_n(t)$ for each dominant partial by computing the Fourier coefficient $A_n$ on small time intervals



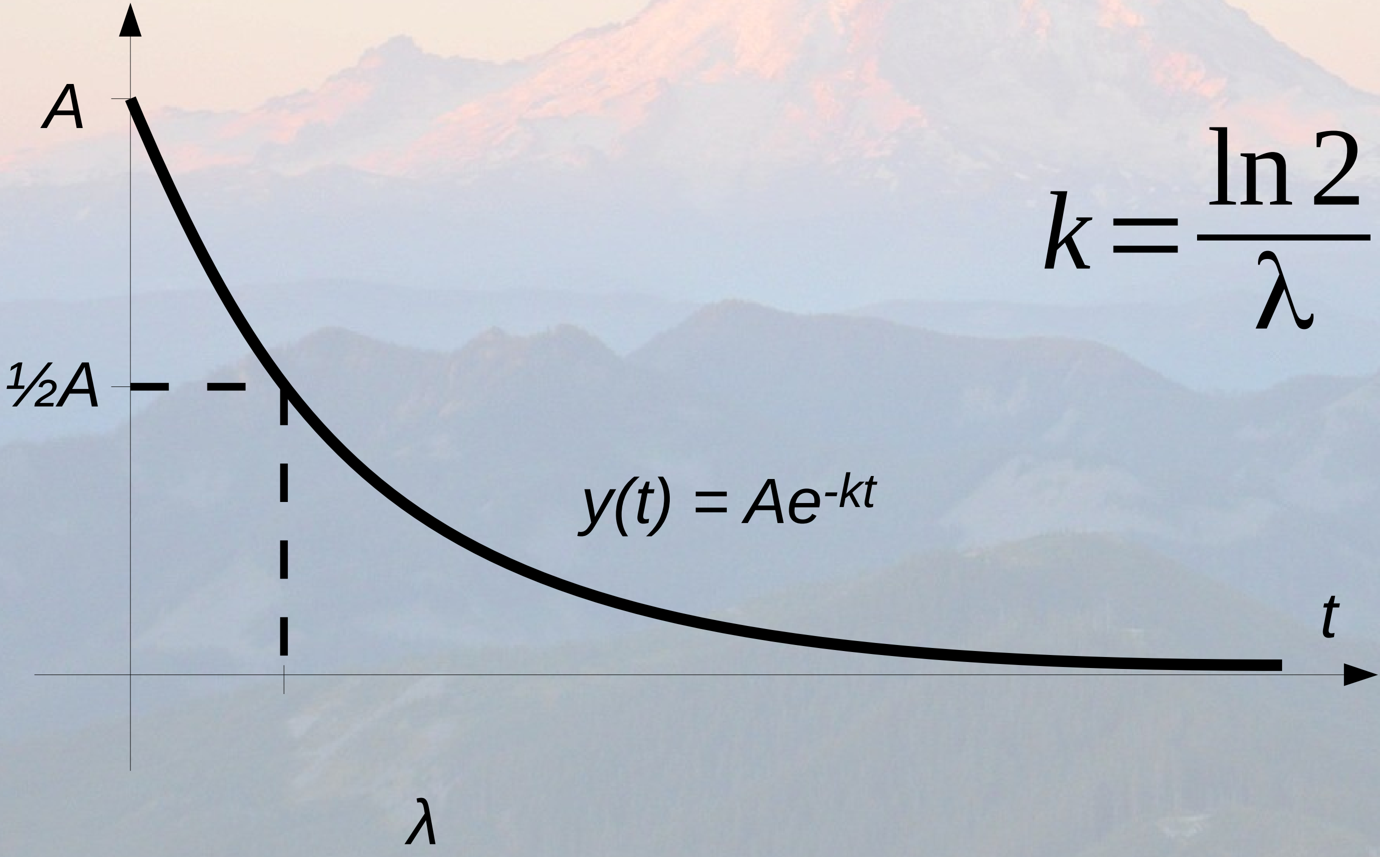

$$A_n(0) = \frac{2}{h}\int_0^h f(x)\sin\left(\frac{n\pi x}{h}\right)\,dx$$

$$A_n(h) = \frac{2}{h}\int_h^{2h} f(x)\sin\left(\frac{n\pi x}{h}\right)\,dx$$

$$A_n(2h) = \frac{2}{h}\int_{2h}^{3h} f(x)\sin\left(\frac{n\pi x}{h}\right)\,dx$$

# Sound Modeling (3)

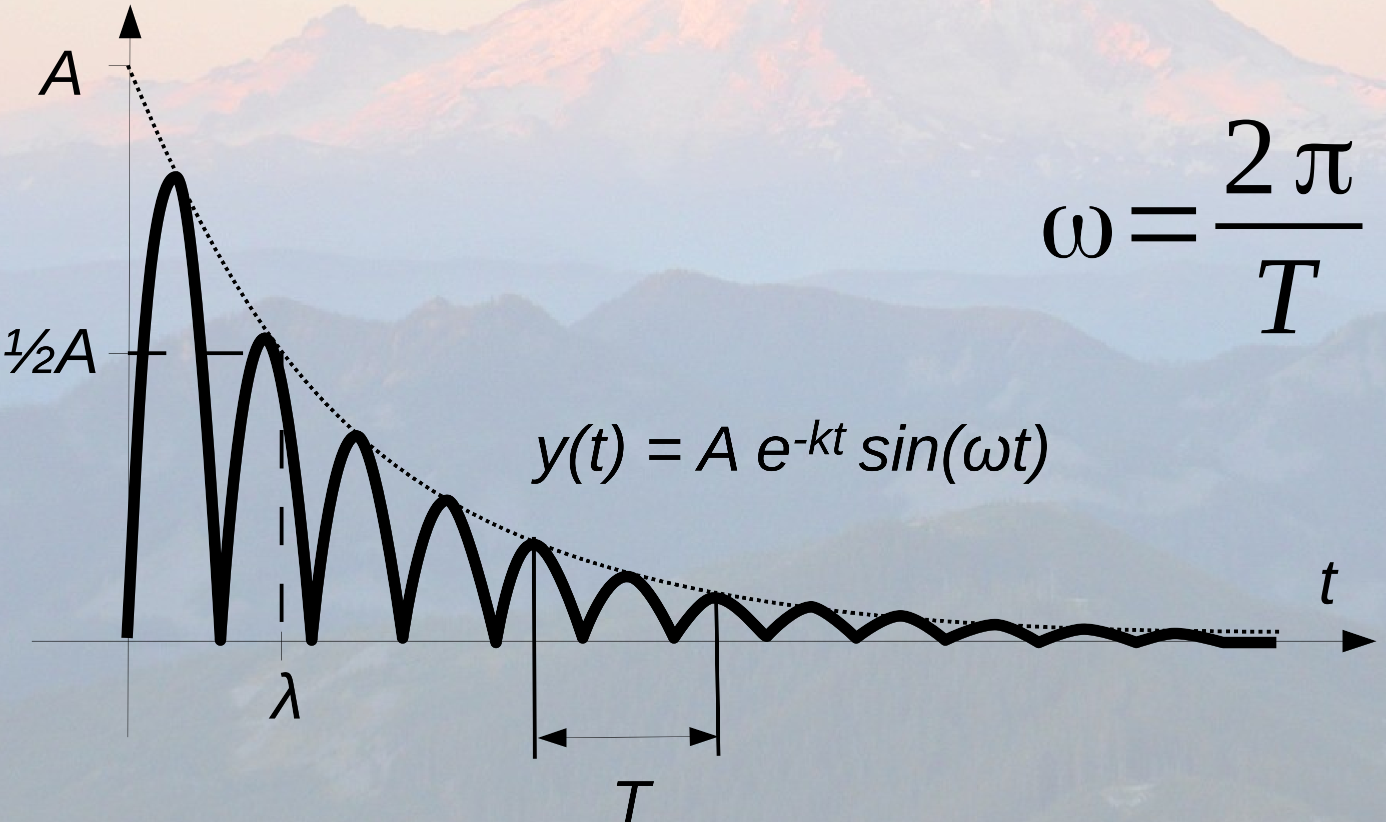
- The maximum amplitude *A* and half-life *λ* determine an exponential envelope




$$k = \frac{\ln 2}{\lambda}$$

# Sound Modeling (4)

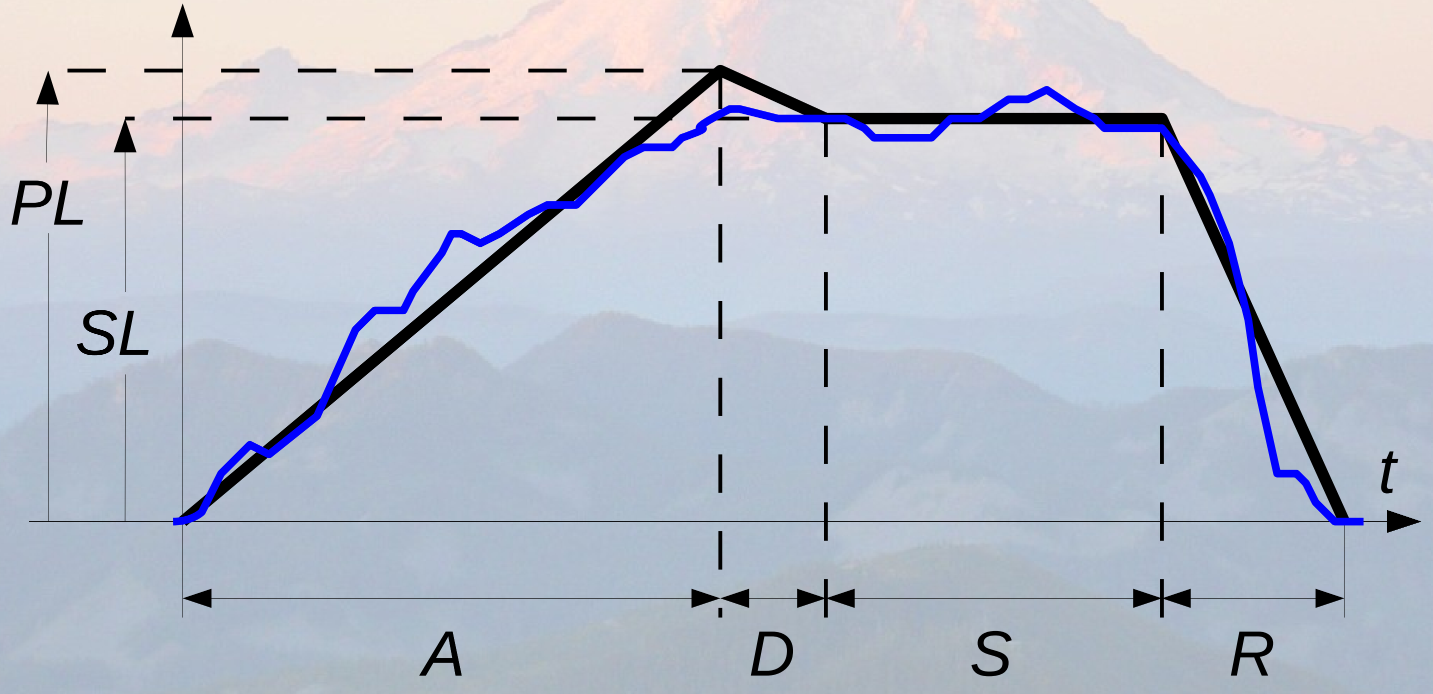
- The maximum amplitude *A*, the half-life *λ*, and the period *T* determine an exponential-sine envelope




$$\omega = \frac{2\pi}{T}$$

# Sound Modeling (5)

- Other envelopes may be approximated by an ADSR envelope



# Tonal Transposition

- Once we have modeled a sound, we can produce different pitches by scaling the frequencies:

$$y(t)=\sum_{n=1}^{\infty} A_n(t)\sin(\alpha\,\omega_n t)$$

produces a sound whose fundamental frequency is $\alpha f_1$, where $\omega_1 = 2\pi f_1$

# Other Deconstructions

- Instead of sinusoids, other functions can be used to construct and deconstruct sounds
- For deconstruction, we need a (complete) collection of functions satisfying an orthogonality relation
- For construction, orthogonality (and completeness) is not necessary
  - Amounts to mixing of sounds

# Amusement: Chebyshev Polynomials

- Chebyshev polynomials of the first kind are a complete collection of orthogonal polynomials $T_0, T_1, T_2, \ldots$ on the interval $[-1,1]$
- The polynomials are solutions of

$$\frac{d^2}{dx^2} T_n - x \frac{d}{dx} T_n + n^2 T_n = 0$$

for $n = 0, 1, 2, \ldots$

# Chebyshev Polynomials (2)

- Alternatively, the Chebyshev polynomials satisfy the recurrence relation

$$T_0(x)=1,\ \ T_1(x)=x,$$
$$T_n(x)=2\mathrm{xT}_{n-1}(x)-T_{n-2}(x)$$

- Orthogonality relation

$$\int_{-1}^{1} T_m(x)T_n(x)\,\frac{dx}{\sqrt{1-x^2}}=\frac{\pi}{2}\,\delta_{mn}$$

# Chebyshev Polynomials (3)

- Every function *f* on *(-1,1)* can be expressed as a sum of Chebyshev Polynomials

$$f(x)=\sum_{n=0}^{\infty} A_n T_n(x)$$

- The coefficients are determined by

$$A_n=\frac{2}{\pi}\int_{-1}^{1} f(x)T_n(x)\frac{dx}{\sqrt{1-x^2}}$$