Digital Signal Processing I

Time Series

- Sampled audio signal is modeled by a (bi-infinite) sequence of real numbers

$$\boldsymbol{x} : \mathbb{Z} \rightarrow \mathbb{R}$$

- I.e., audio samples are represented as

$$\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$$

- In practice, we only use a finite subsequence

$$x_0, x_1, \ldots, x_{n-1}$$

and assume that $x_k = 0$ if $k < 0$ or $k \geq n$

Digital Signal Processing (DSP)

- The collection of all possible time series can be viewed as a (real) infinite dimensional vector space indexed over the integers:

$$\mathbb{R}(\mathbb{Z}) = (\mathit{set\ of\ all\ time\ series})$$

- A **DSP** is a transformation

$$F : \mathbb{R}(\mathbb{Z}) \rightarrow \mathbb{R}(\mathbb{Z})$$

  (not necessarily linear)

- Notation
  - Input signal: ***x***
  - Output signal: ***y***

Z-Transform

- Provides an alternate representation of a time series
- The **z-transform** of the time series

$$\boldsymbol{x} = \{x_n\}_{n \in \mathbb{Z}}$$

is the formal (Laurent) series

$$Z[\boldsymbol{x}] = \sum_{n \in \mathbb{Z}} x_n z^{-n}$$
$$= \ldots + x_{-2} z^2 + x_{-1} z + x_0 + x_1 z^{-1} + x_2 z^{-2} + \ldots$$

Z-transform Example

- Suppose ***x*** is the time series

$$x_{-1}=5, x_0=-3, x_1=8, x_2=4$$

(all other terms are *0*)

- The z-transform of ***x*** is

$$\mathrm{Z}[\boldsymbol{x}]=5\mathrm{z}-3+8\mathrm{z}^{-1}+4\mathrm{z}^{-2}$$

Time Delay

- A delay of k samples shifts the index
  - Input

$$\boldsymbol{x} : x_n$$

  - Output

$$\boldsymbol{y} : y_n = x_{n-k}$$

- This multiplies the z-transform by z^{-k}

$$Z[\boldsymbol{y}] = z^{-k} Z[\boldsymbol{x}]$$

Properties of the Z-transform

- Linearity

$$Z[\alpha\,\boldsymbol{x}+\beta\,\boldsymbol{y}]=\alpha\,Z[\boldsymbol{x}]+\beta\,Z[\boldsymbol{y}]$$

- Convolution

$$Z[\boldsymbol{x}]\,Z[\boldsymbol{y}]=Z[\boldsymbol{x}*\boldsymbol{y}]$$

where

$$(\boldsymbol{x}*\boldsymbol{y})_n=\sum_{k\in\mathbb{Z}} x_k\,y_{n-k}$$

Transfer Function

- Input-independent description of a filter
  - Input ***x***
  - Output ***y***

$$H(z) = \frac{Z[\boldsymbol{y}]}{Z[\boldsymbol{x}]}$$

- Time delay transfer function

$$H(z) = z^{-k}$$

Transfer Function Example (1)

- Suppose the *n*-th sample of a filtered signal is

$$y_n = 4x_n - x_{n-2}$$

- Using the linearity and time delay properties

$$\begin{aligned} Z[\vec{y}] &= 4Z[\vec{x}] - z^{-2} Z[\vec{x}] \\ &= \left(4 - z^{-2}\right) Z[\vec{x}] \\ \Rightarrow H(z) &= \frac{Z[\vec{y}]}{Z[\vec{x}]} = 4 - z^{-2} \end{aligned}$$

Transfer Function Example (2)

- Now suppose the transfer function is

$$H(z) = -1 + 2z^{-3}$$

- We find *n*-th sample of the filtered signal:

$$\frac{Z[\vec{y}]}{Z[\vec{x}]} = -1 + 2z^{-3}$$

$$\begin{aligned} Z[\vec{y}] &= \left(-1 + 2z^{-3}\right) Z[\vec{x}] \\ &= -Z[\vec{x}] + 2z^{-3} Z[\vec{x}] \end{aligned}$$

$$\Rightarrow y_n = -x_n + 2x_{n-3}$$

Impulse Response

- Impulse signal

$$\boldsymbol{x} : x_0 = 1$$

  ($x_n = 0$ if $n \neq 0$)

- Z-transform of impulse

$$Z[\boldsymbol{x}] = 1$$

- **Impulse response** (IR): output of a filter when applied to an impulse signal

Impulse Response Example

- Transfer function:

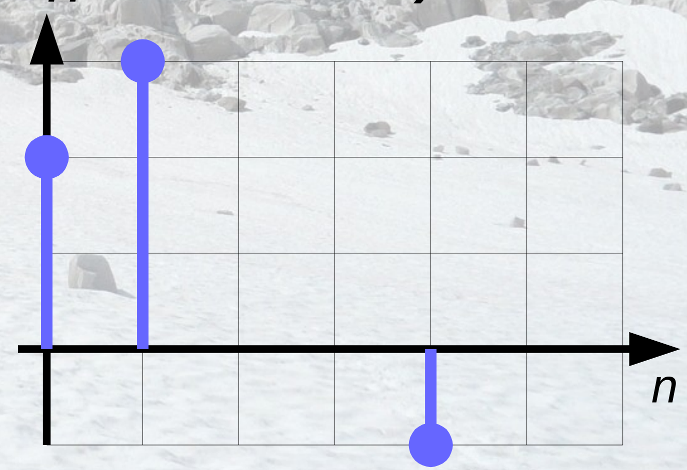
$$H(z) = 2 + 3z^{-1} - z^{-4}$$

- Recurrence relation:

$$y_n = 2x_n + 3x_{n-1} - x_{n-4}$$

- IR values (using $x_0=1$ and $x_n=0$ if $n \neq 0$)

$$y_0 = 2x_0 + 3x_{-1} - x_{-4} = 2$$
$$y_1 = 2x_1 + 3x_0 - x_{-3} = 3$$
$$y_4 = 2x_4 + 3x_3 - x_0 = -1$$



Multi-tap Echo

- Recurrence relation

$$y_n = x_n + a_1 x_{n-k_1} + a_2 x_{n-k_2} + \ldots$$

Typically, $k_m = mk$ (fixed delay k) and $a_m = a^m$

- Transfer function

$$H(z) = 1 + a_1 z^{-k_1} + a_2 z^{-k_2} + \ldots$$

- Impulse response

$y_0 = 1$ (dry signal), $y_{k_1} = a_1,\ y_{k_2} = a_2,\ \ldots$

Convolution Reverb

- Compute convolution of input signal with an impulse response (IR) function
- Different IR's can be used to simulate different acoustic environments
- IR is obtained by recording a sharp sound, such as a gun shot, in the desired acoustic environment
- Realistic, but computationally very expensive