

## Flow models and Observables

Thermal boundary layers, Geoid, Topography, Viscosity

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# Energy Sources

## Geological change requires an energy source.

What energy sources might be available to be turned into mechanical motion at depth ?

Solar energy ?

- Drives the weather & life — therefore influences sedimentation / erosion.
- $1.75 \times 10^{17}$  W available at the surface

Tidal dissipation ?

- Total power  $4 \times 10^{12}$  W - 60% dissipated within the planet
- Io & early Mercury

Impacts ?

- Kinetic energy of KT boundary impactor  $\sim 10^{23}$  J — one per 100Myr corresponds to  $3 \times 10^7$  W
- Early Earth history

# Energy Sources ...

Heating / Cooling of the interior

- Total heat loss from within the Earth is  $4.4 \times 10^{13}$  W.
- Secular cooling, and / or radioactive decay.
- Shallow sources / deep ?

Chemical differentiation

- Gravitational settling of core  $\sim 10^{32}$  J
- $10^{14}$  W averaged over age of Earth
- Early Earth process

Chemical reactions / phase changes

- Solidification of inner core (Geodynamo ?)

**But how does this Energy produce Geological Change ?**

# Thought experiment

Which works better, boiling water on the stove, or under the grill ?

- Why ?
- What stirs the water ?



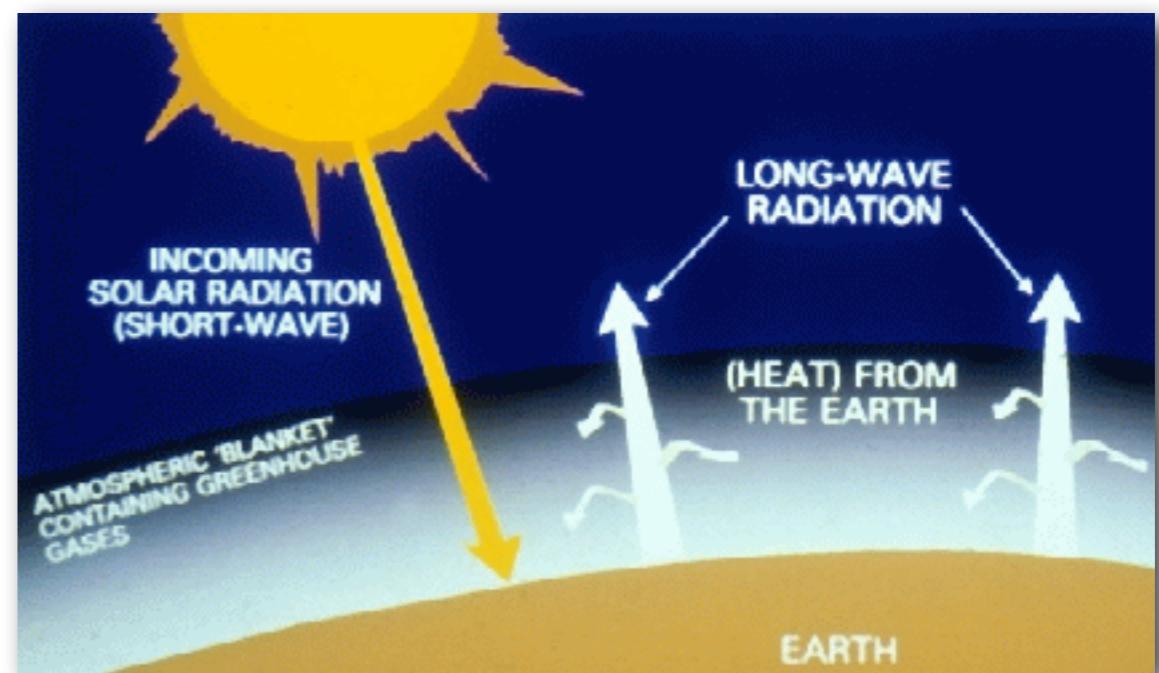
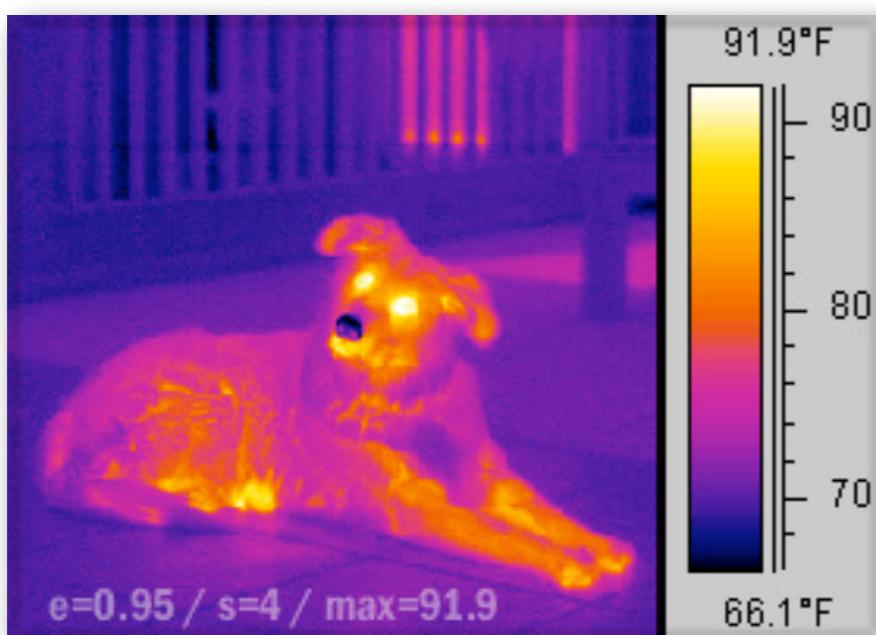
# Radiation



## Definition



- The transfer of (heat) energy by photons from one material to another.
- Does not require physical contact, or intermediary material (i.e. can transfer heat across a vacuum).
- Range of operation depends on how translucent the intermediary material is (to infra red)



# Conduction

## Definition:

- Conduction is the transfer of heat (or electric current) from one substance to another by direct contact (lattice vibrations / electrons)
- The transfer is always from a higher temperature to a lower temperature. Denser substances are usually better conductors; metals are excellent conductors.

$$Q = -\frac{kA\Delta T}{L}$$

## Fourier's Law:

- Q is the heat transferred per unit time, A is the cross sectional area of the conducting sample, L is its width, k is a thermal conductivity ("constant") dependent on the nature of the material and often also on the temperature, and  $\Delta T$  is the temperature difference across the sample (NOTE the minus sign ... why is that there ?)

# Convection

## Definition

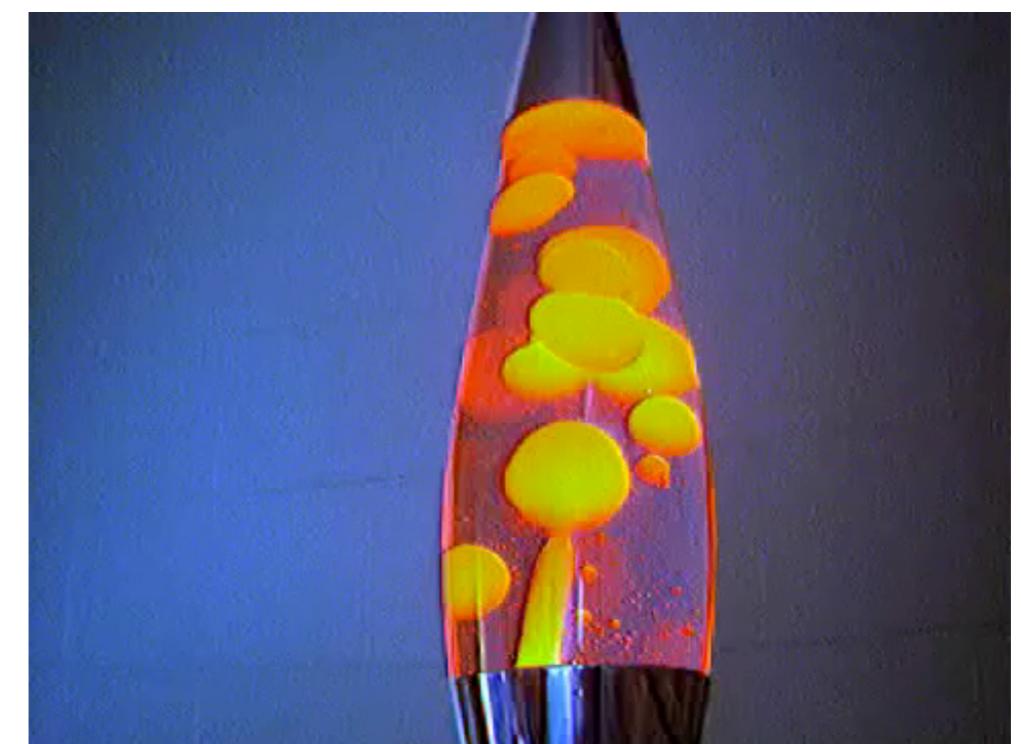
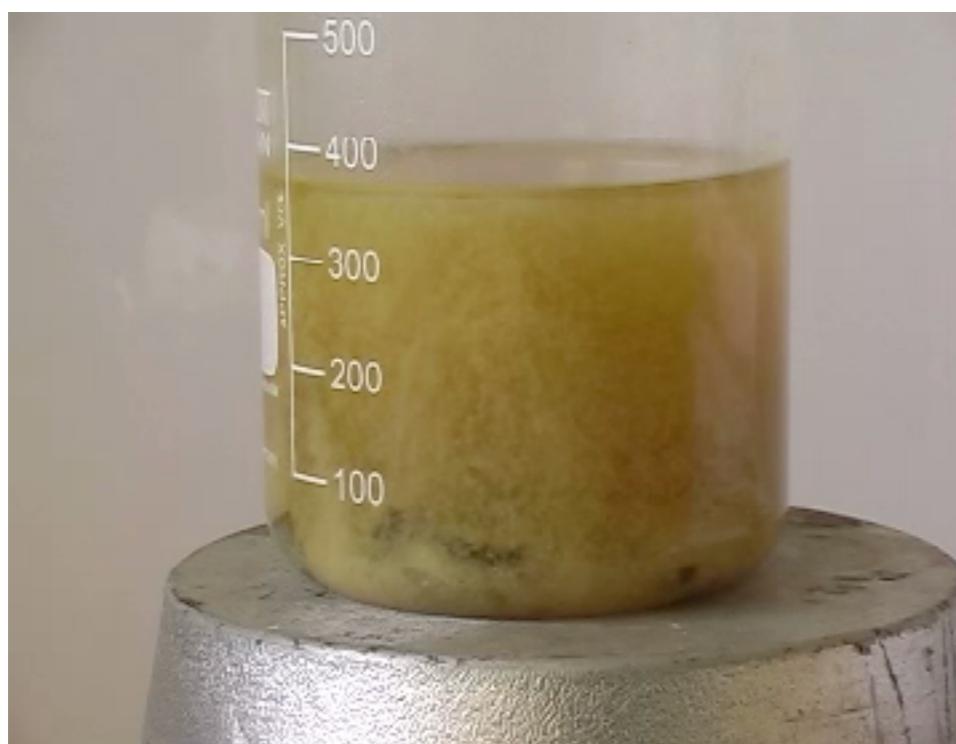
- Convection is the transfer of heat by the self-organised movement of a fluid. *Free convection* is when the fluid is stirred entirely by rising buoyant material and sinking negatively-buoyant material. *Forced convection* is produced when the fluid is stirred mechanically.
- An important process in convection is **advection** which is the transport of heat by the fluid as it moves (think of the blobs of wax in a lava lamp, or a bucket of hot water being carried from place to place).
- In most cases where convection occurs, heat is transported in and out of the fluid by conduction. Advection adds to the rate heat transport so that the heat flow is always more efficient by convection than by conduction.
- Where conduction is important, a narrow “thermal boundary layer” often develops.

# Convection

There is a similar formula for the heat transferred by a convecting fluid layer

$$Q \propto Q_c \left[ \frac{g\rho\alpha\Delta T d^3}{\kappa\eta} \right]^{\frac{1}{3}}$$

$Q_c$  is the heat that would be carried by conduction,  $g$  is gravitational acceleration,  $\rho$  is density,  $\alpha$  thermal expansivity,  $d$  is depth of the layer (perpendicular to gravity),  $\kappa$  is thermal diffusivity,  $\eta$  is viscosity.



# Conduction v. Advection

Heat conduction:

- Conduction is the transfer of heat from one substance to another by direct contact (lattice vibrations / electrons)
- The transfer is always from a higher temperature to a lower temperature. Denser substances are usually better conductors; metals are excellent conductors.

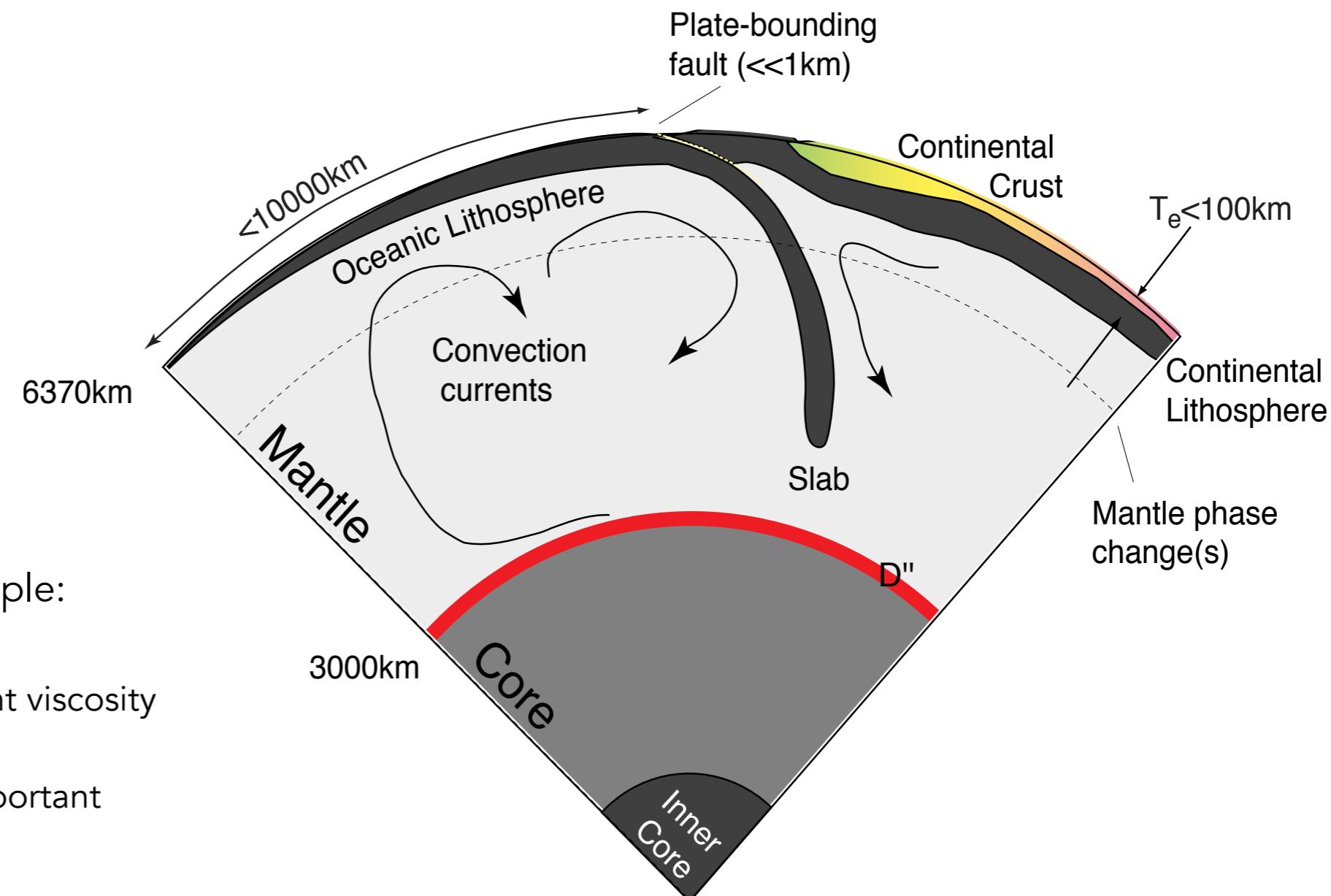
Heat advection:

- Physical transport of heat by motion of the material from place to place which carries the heat along before it can be conducted away

Convection is actually a balance between **conduction** of heat through the mantle rocks, and **advection**.

# Convection in solid planets

The Earth's mantle is solid but over geological time it deforms in precisely the same way as a very viscous fluid. This occurs by reorganization of atoms in the crystal lattice. The mantle remains solid the entire time.



The lava-lamp picture is over simple:

- Cool / warm have a very different viscosity
- Elastic and brittle effects are important at low temperature
- This produces cold plates instead of blobs

# Preliminaries

Before we can understand geological deformation produced by heat trying to escape from the Earth we have to know more about deformation of solids in general when they are stressed.

- Elastic deformation
- Viscous / ductile deformation
- Brittle deformation

We also need an unambiguous definition of the forces / stresses acting on a sample of material including

- Pressure (compression / dilation)
- Deviatoric stresses (shear)

# Stress and strain

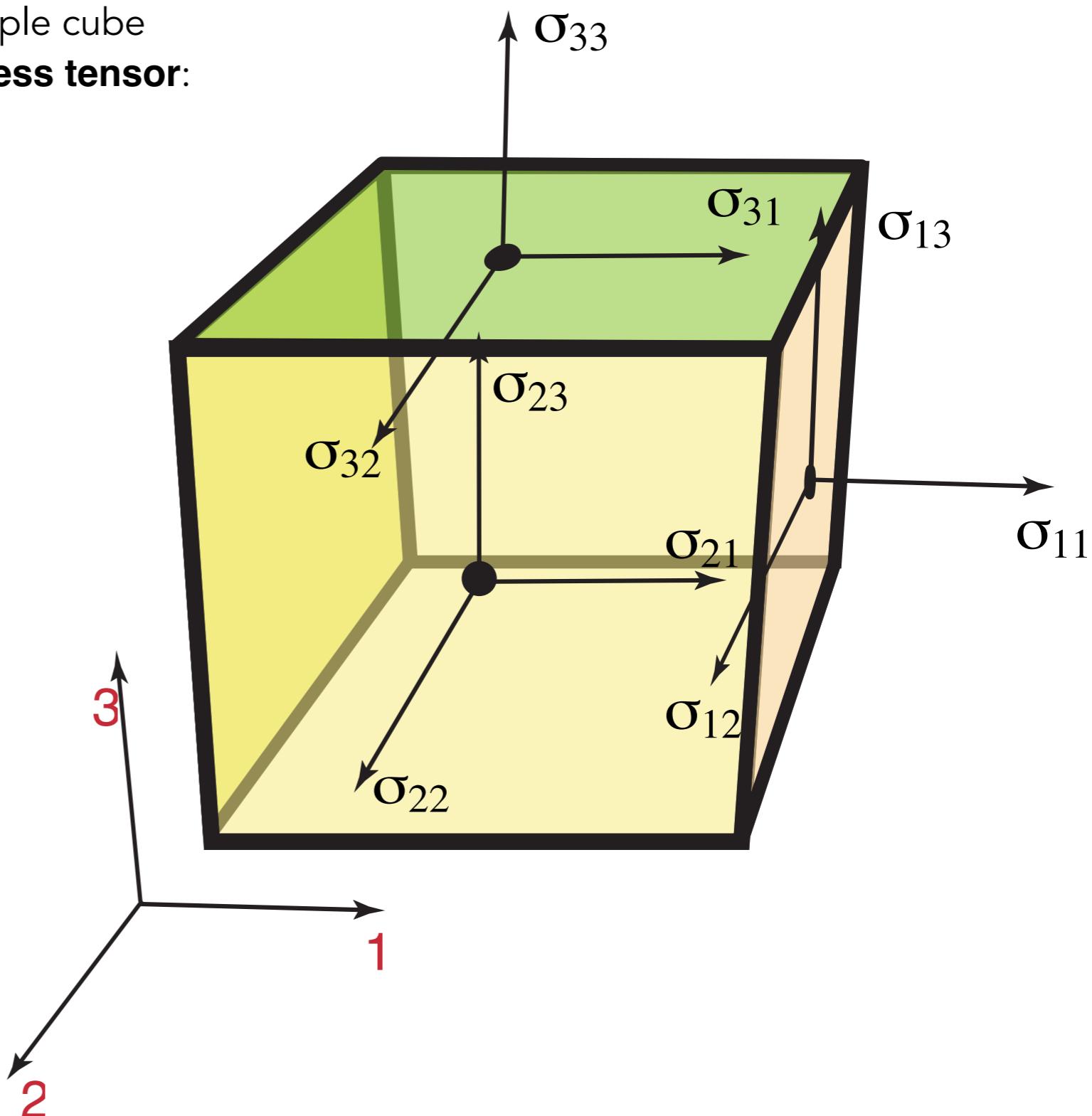
All possible forces and shears acting on a sample cube (assuming symmetry) can be written into a **stress tensor**:

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

**Symmetry:**  $\sigma_{ij} = \sigma_{ji}$

In general, stress/strain relations are tensor equations

In 1D, it's considerably easier

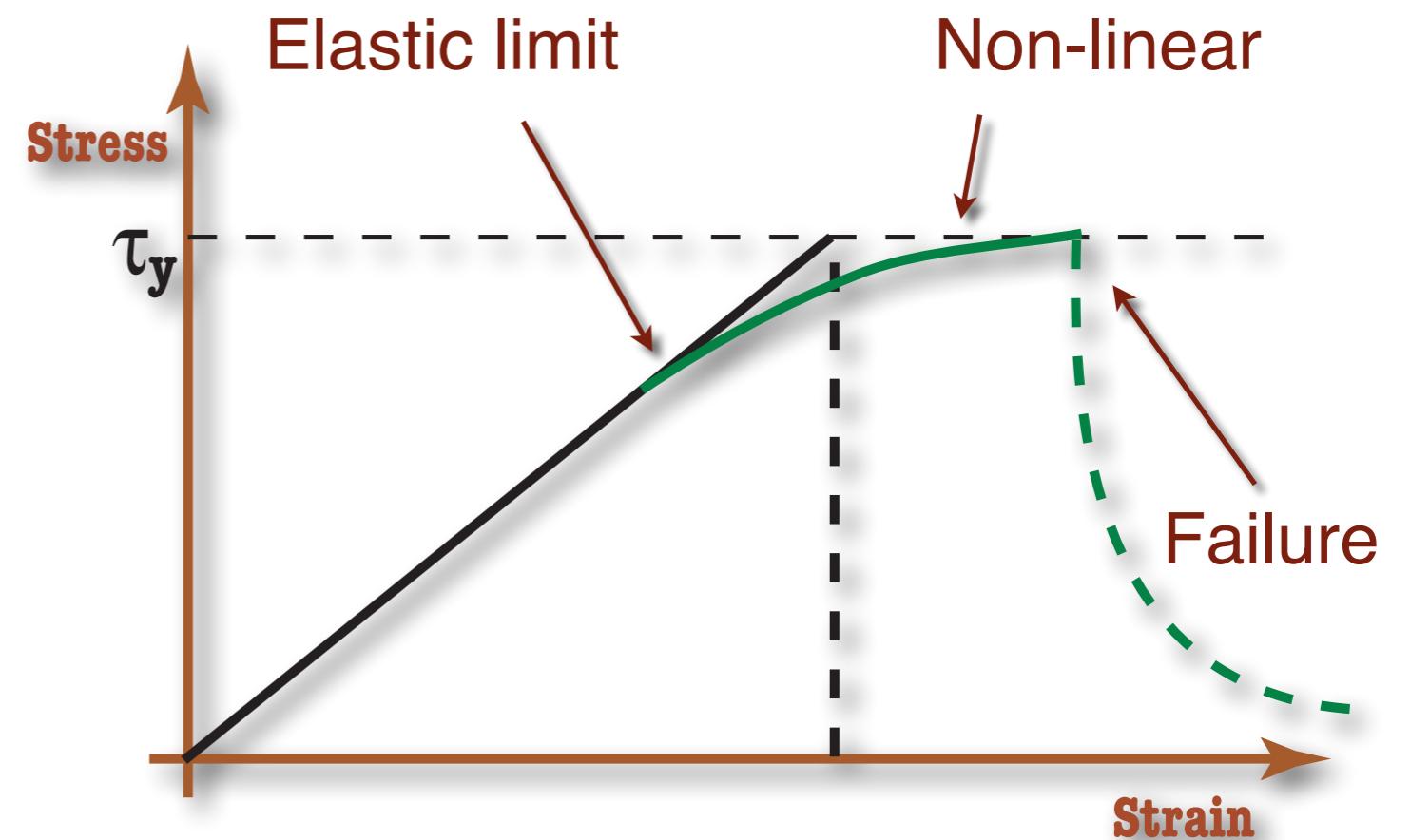


# What is elasticity ?

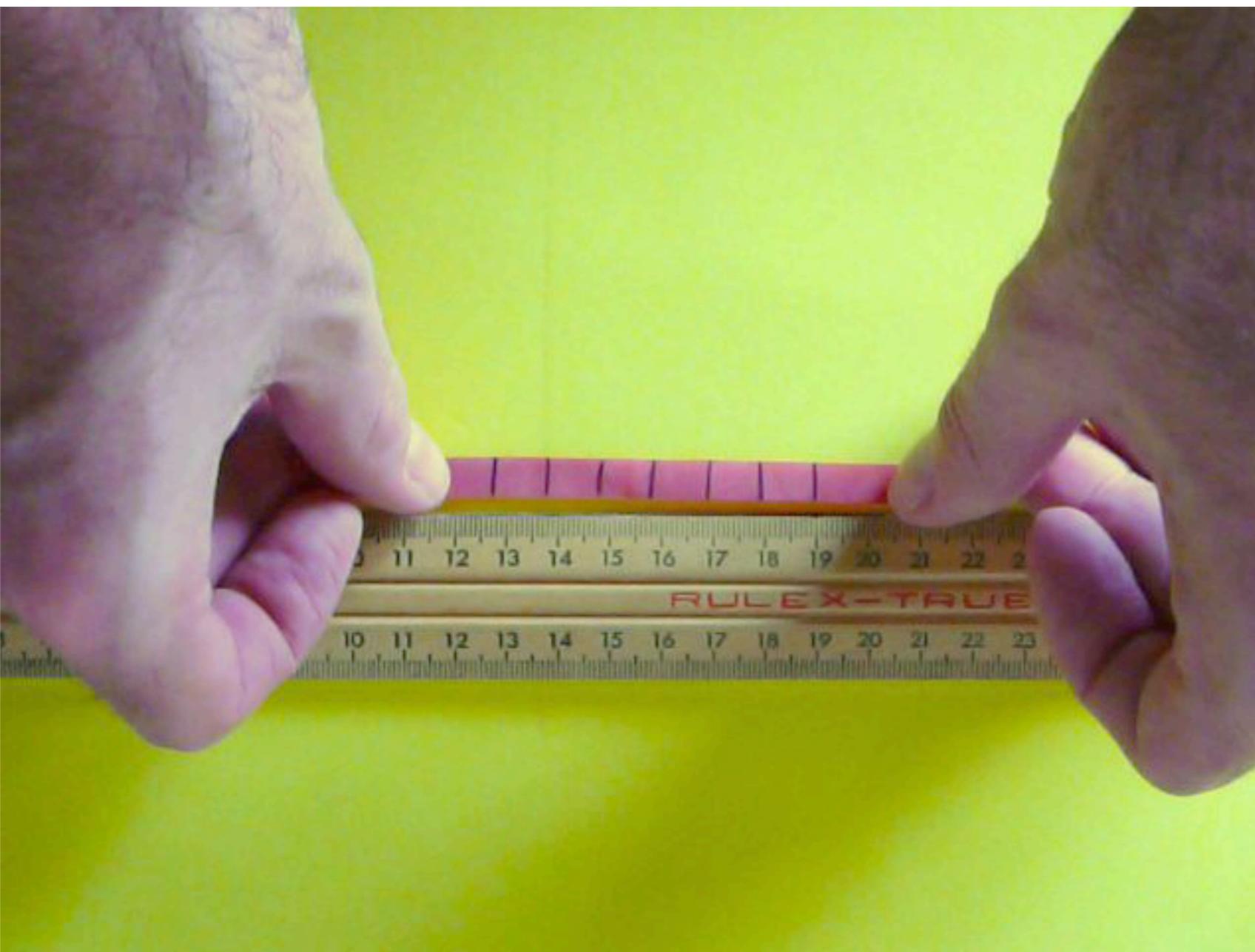
Pure elastic deformation is reversible - it doesn't matter how fast the deformation occurs, the behaviour is almost the same. However, there is an elastic limit — if the strain is larger than this, deformation is irreversible

1D response is quite simple to follow: stress is proportional to strain ...

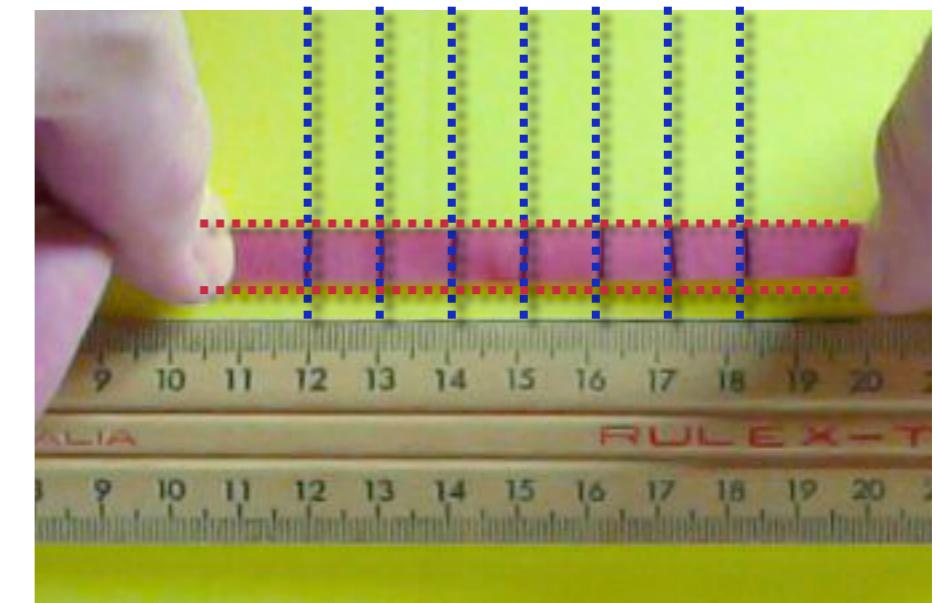
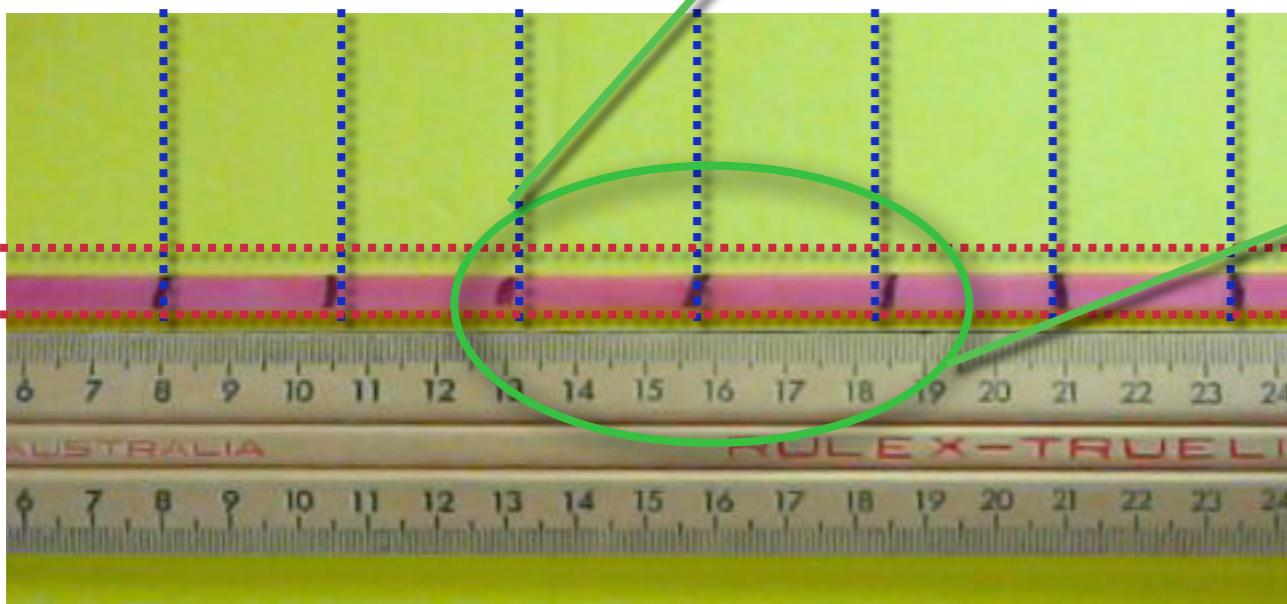
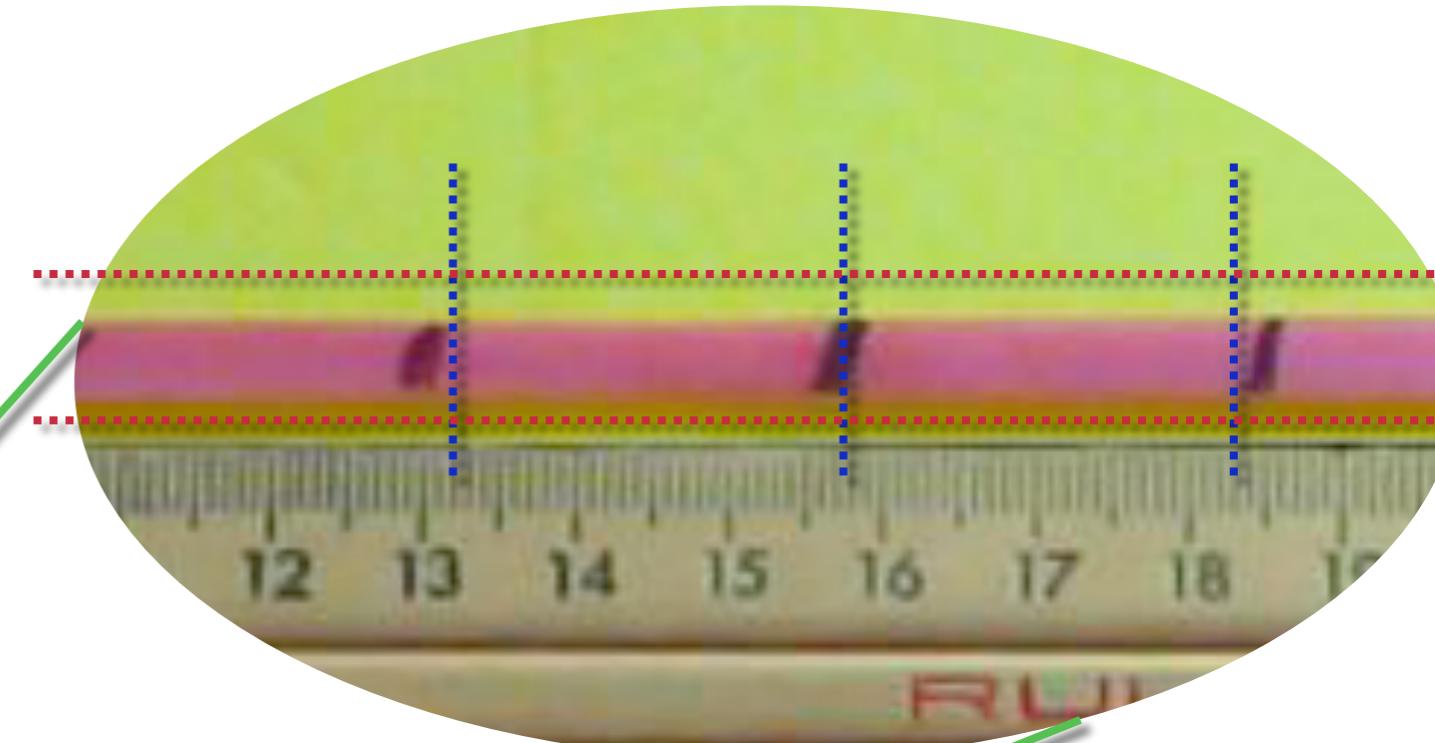
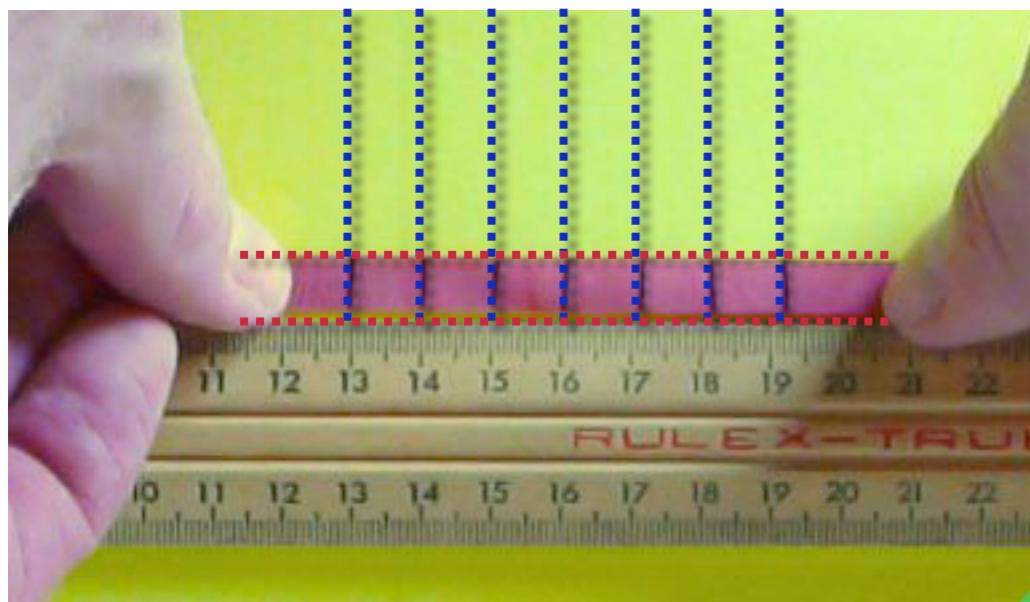
$$\sigma = E\epsilon$$



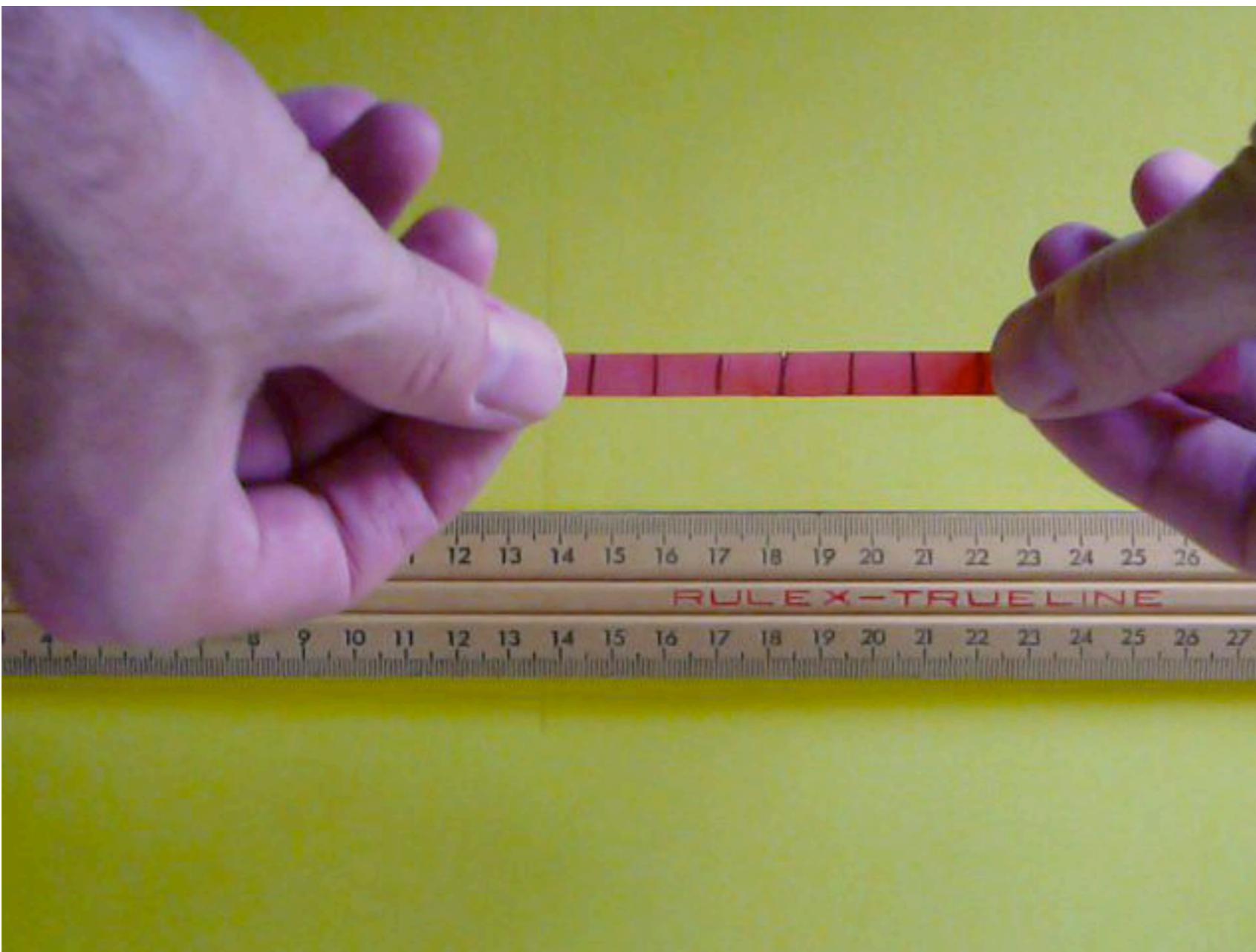
# What is elasticity ?



# Elastic deformation in real life



## Elastic limit / failure



Stored energy release is very rapid.  
Deformation occurs in less than 1/30 second

# Elasticity

Lamé parameters describe elastic response.

In 1D, strain (length change) is proportional to the applied force. In general, there may also be a change in shape (other directions change length as well as the one being pulled / pushed)

$$\sigma_{xx} = 2\mu\epsilon_{xx} + \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})$$

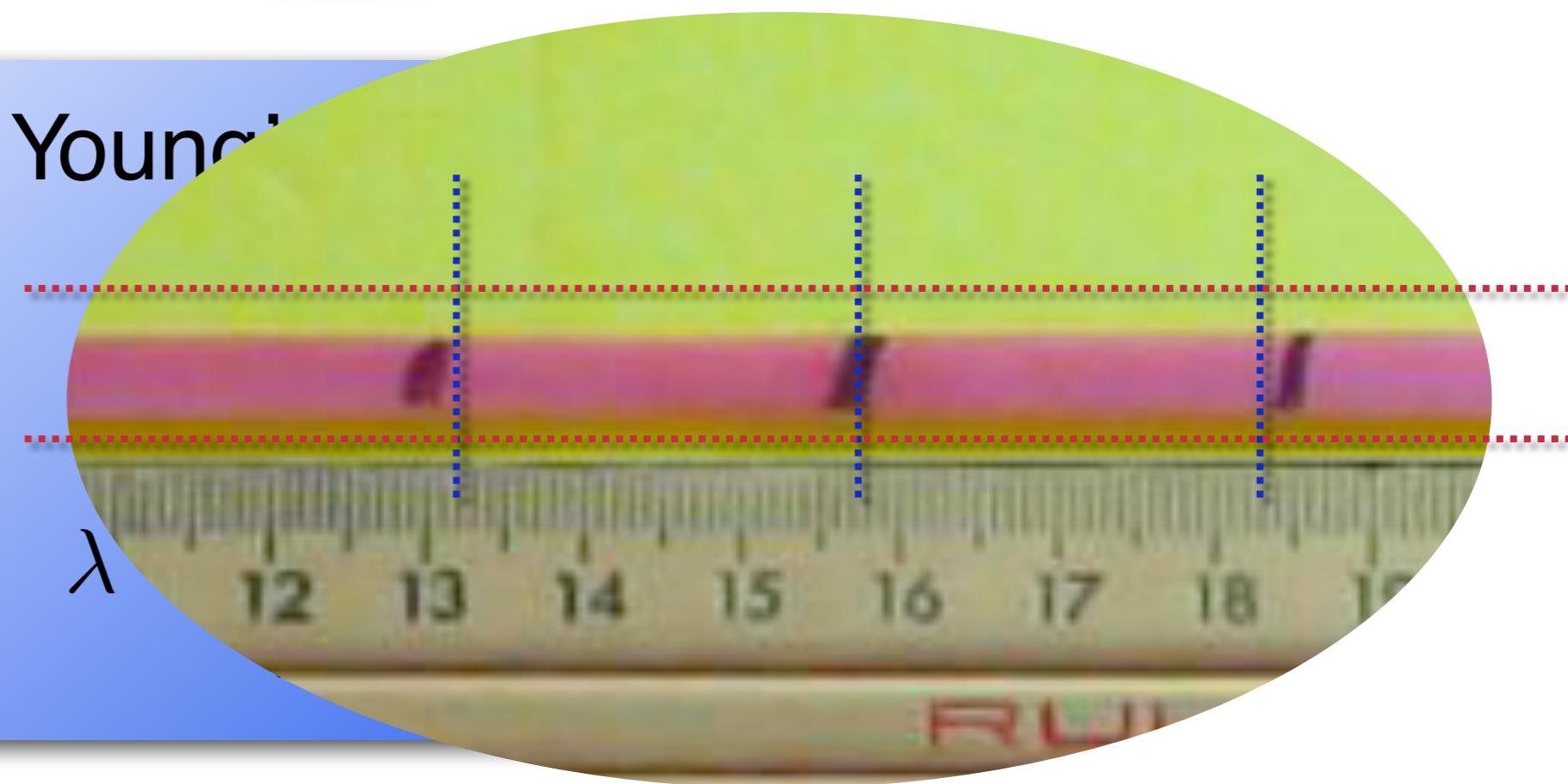
Stress –  $\sigma$

Strain –  $\epsilon$

$$\mu = \frac{E}{2(1 + \nu)}$$

Young's

$\lambda$



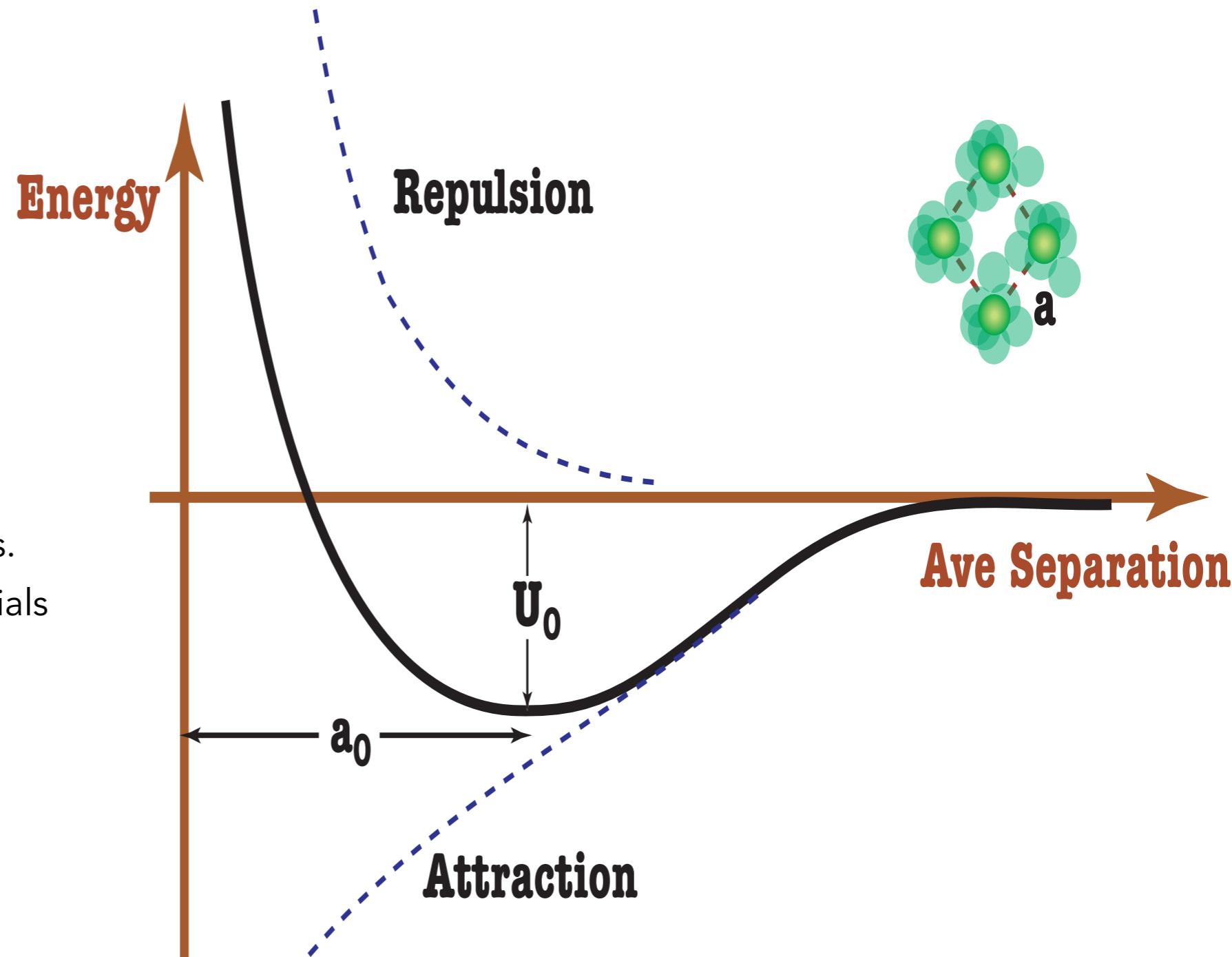
# Origin of elasticity

Elastic effects are most important at low strains, modest stresses and short timescales.

The mechanism for elastic deformation comes directly from the interatomic forces in solids. In crystals the lattice energy potentials have a minimum.

Equilibrium spacing is associated with this minimum.

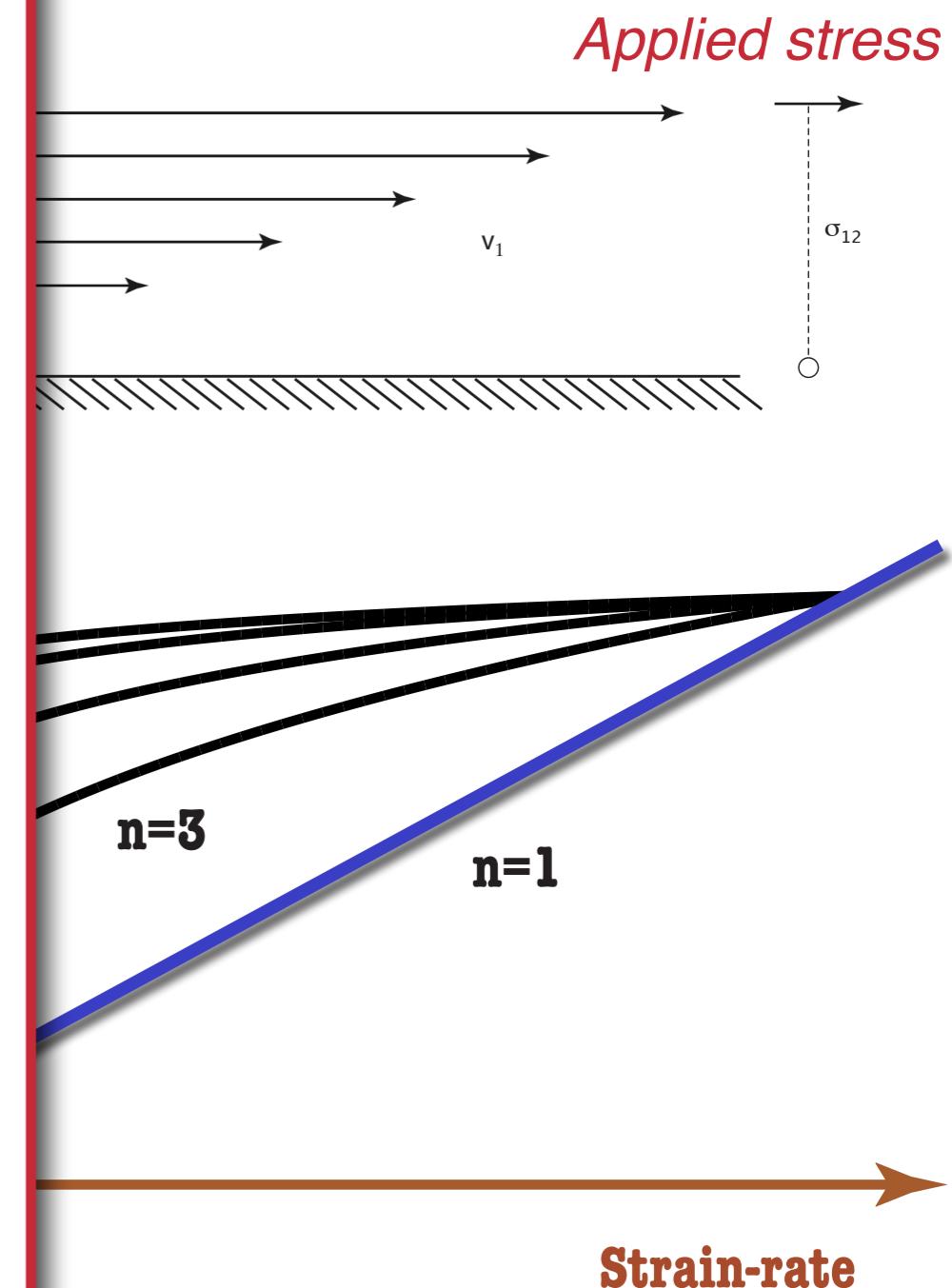
Asymmetry of the potential well is usually cited as the basis for thermal expansivity.



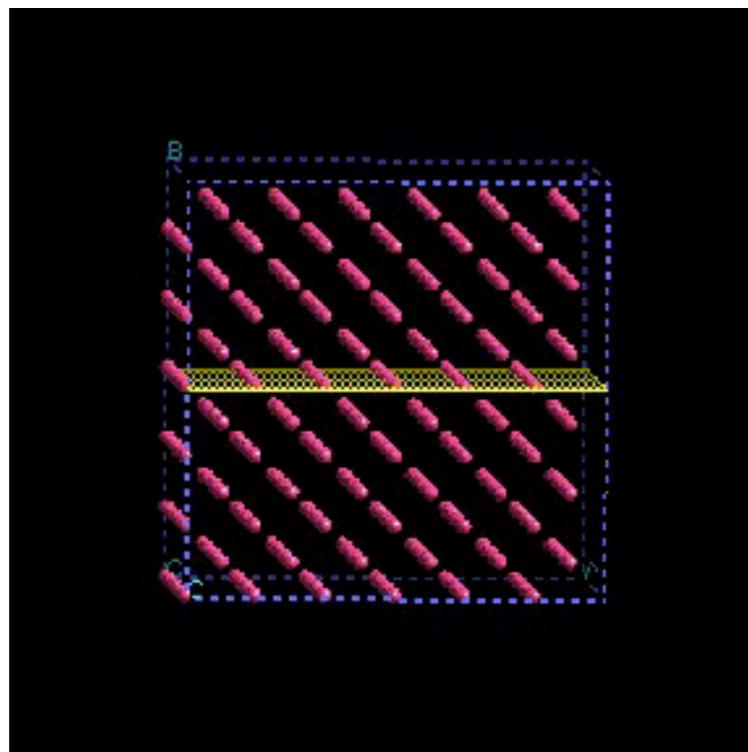
# What is viscosity ?

Viscous deformation is irreversible **flow**

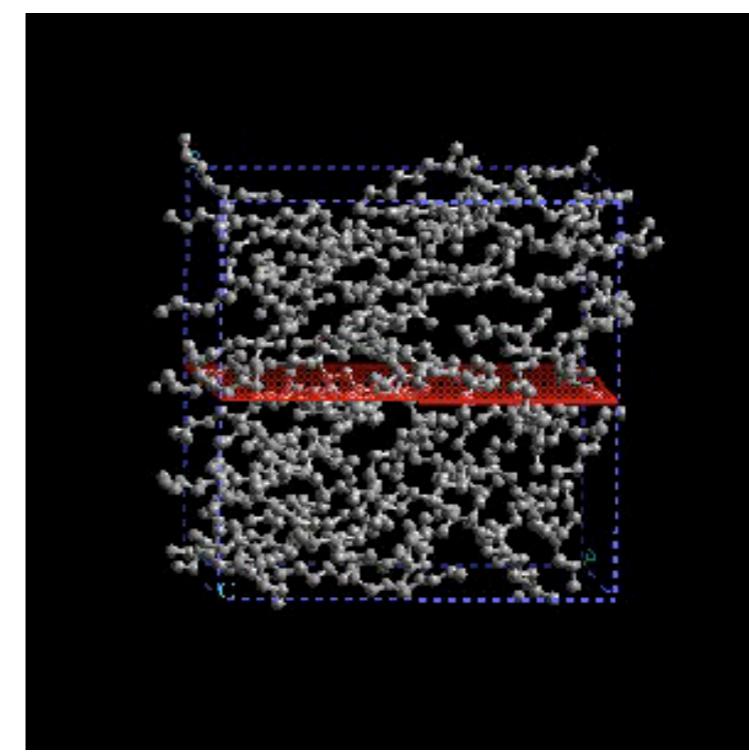
**Viscosity** is a measure of the resistance of a fluid to deform under shear stress. It is commonly perceived as "thickness", or resistance to flow. Viscosity describes a fluid's internal resistance to flow and may be thought of as a measure of fluid friction. Water is runny, having a lower viscosity, while honey is "thick" having a higher viscosity



# Viscous flow mechanism



Liquid sodium - simulation



Liquid octane - simulation

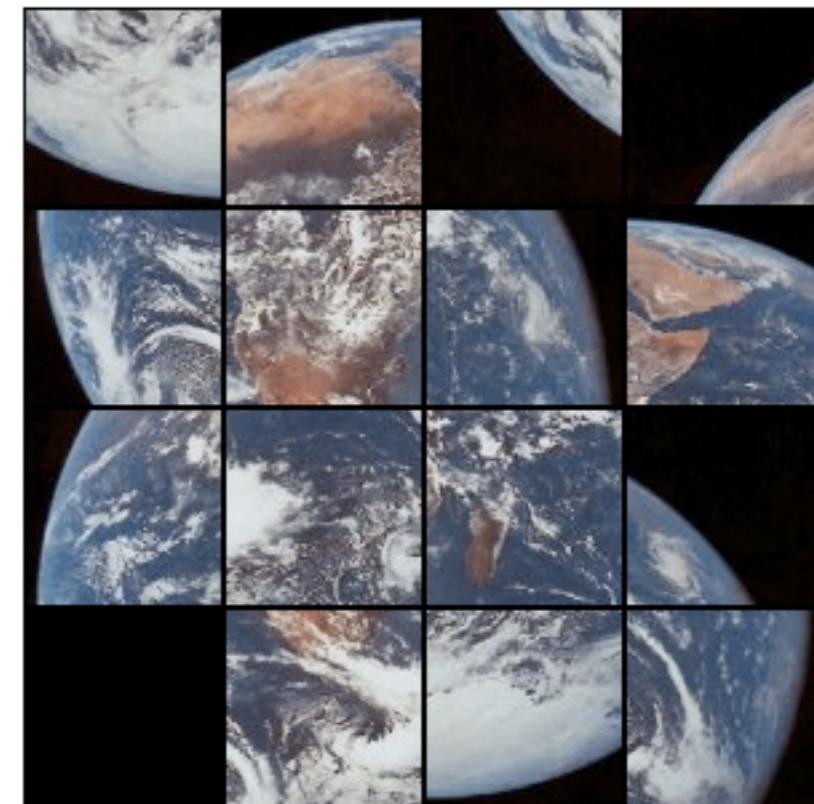
In solids the process is **solid state creep** which occurs through movement of crystal defects:

- Point defects diffusing through the crystal
- Point defects diffusing through grain boundaries
- Dislocation movement

In order to achieve viscous deformation there must be some mechanism for permanently re-organizing atoms or molecules (unlike elastic deformation).

# Diffusion Creep

1	2	3	4	
5	6	7	8	
9	10	11	12	
13	14	15		

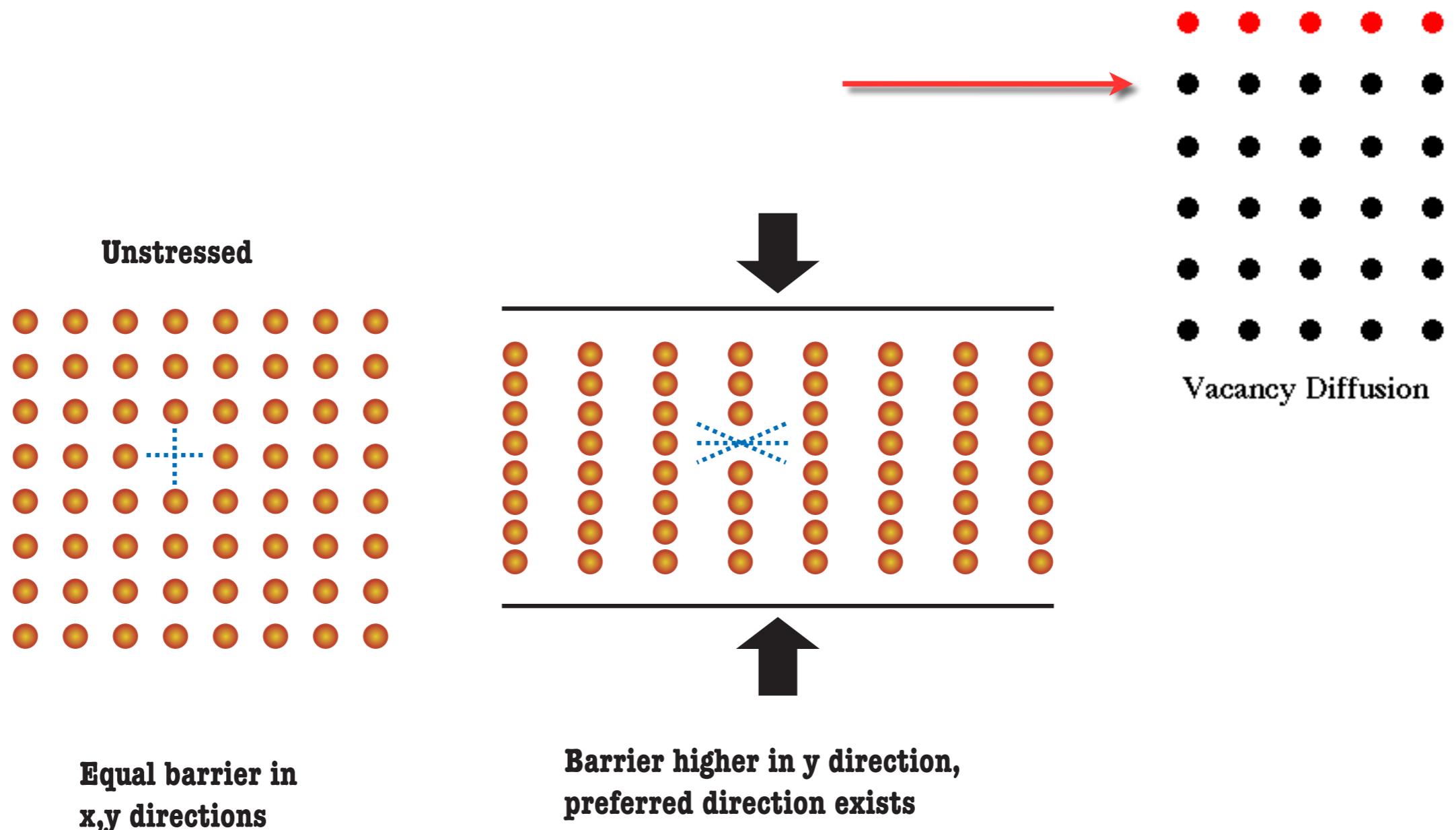


One way is a movement of point defects in the crystal lattice.

If a crystal contains lattice vacancies then diffusion of those vacancies can result in a macroscopic strain.

The vacancies can be considered as real "objects"

# Diffusion Creep



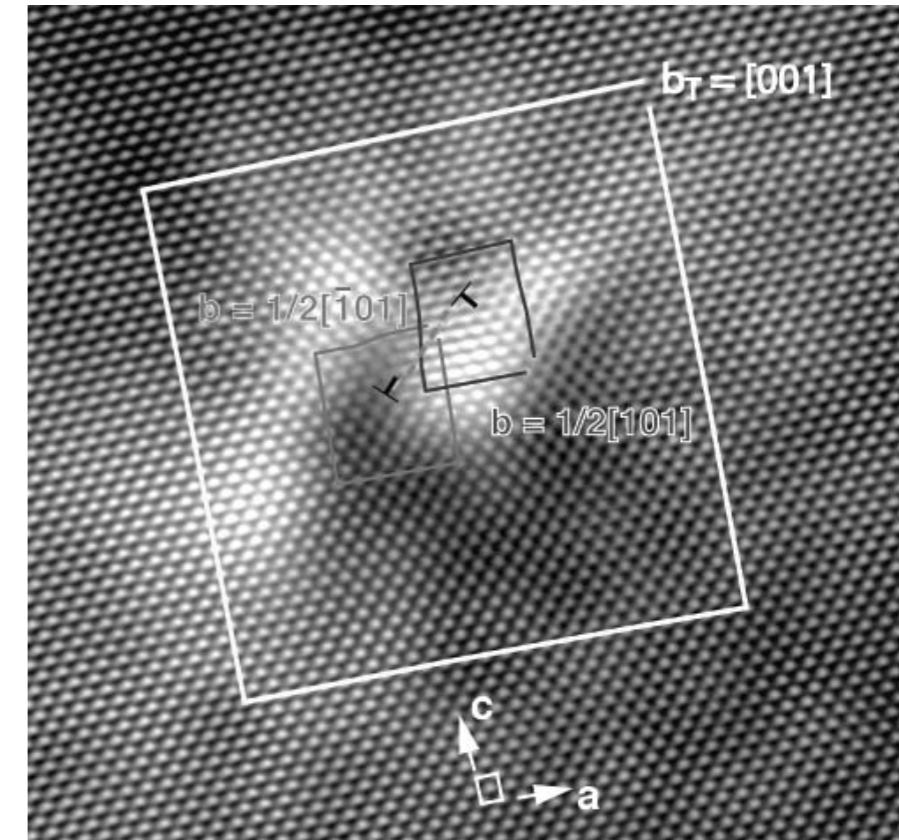
There needs to be a mechanism which "directs" diffusion of the defects to produce the observed macroscopic deformation seen in the movie. This occurs because movement of atoms is slightly easier in the directions with longer bonds

# Dislocation Creep

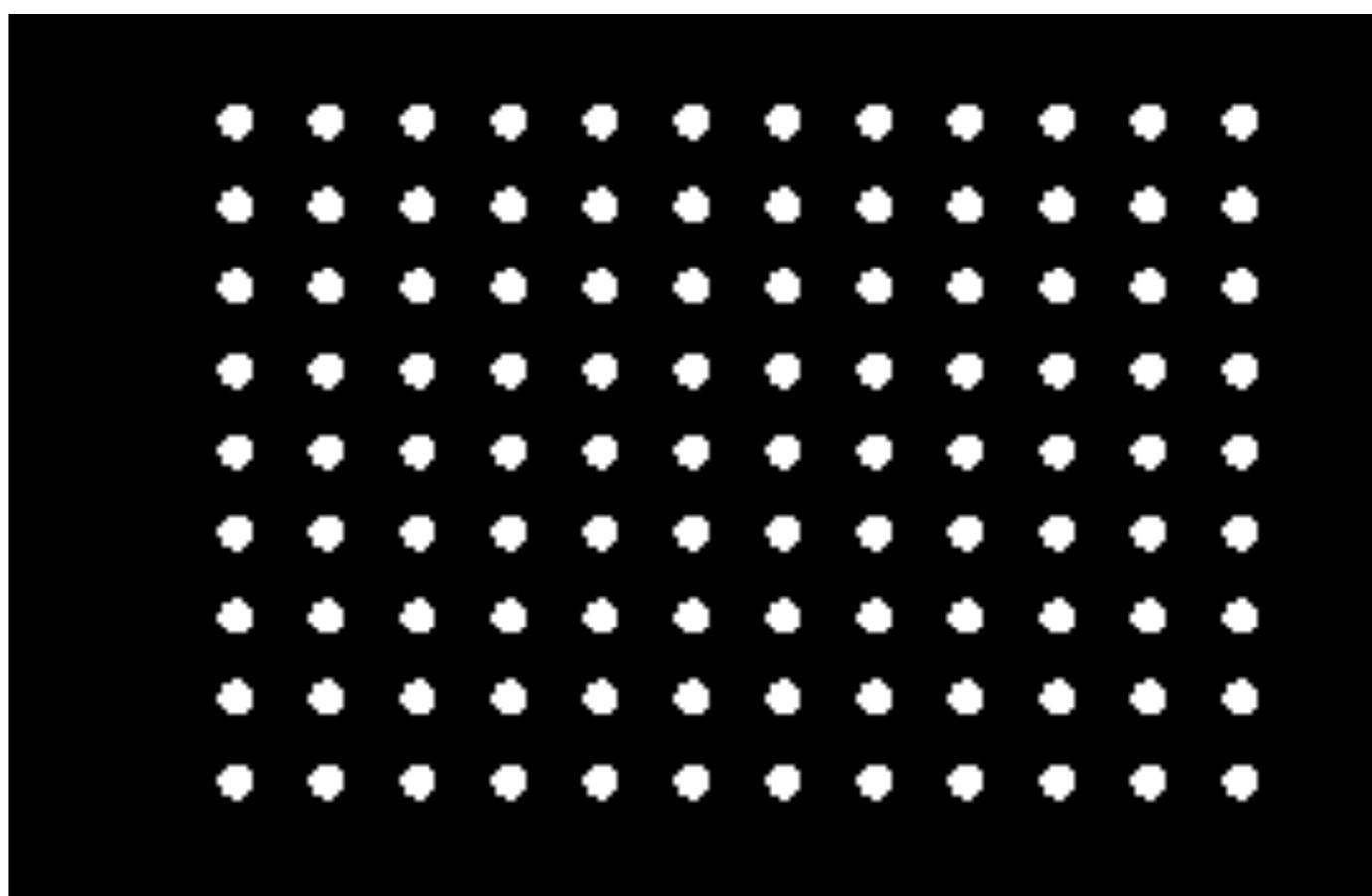
The movement of line defects (**dislocations**) also produces macroscopic deformation.

In each case we can calculate the rate of movement of defects as a function of stress (based on knowing the activation energy required to move atoms).

This movement is a strain rate — if we know strain rate as a function of stress, we know the apparent **viscosity**

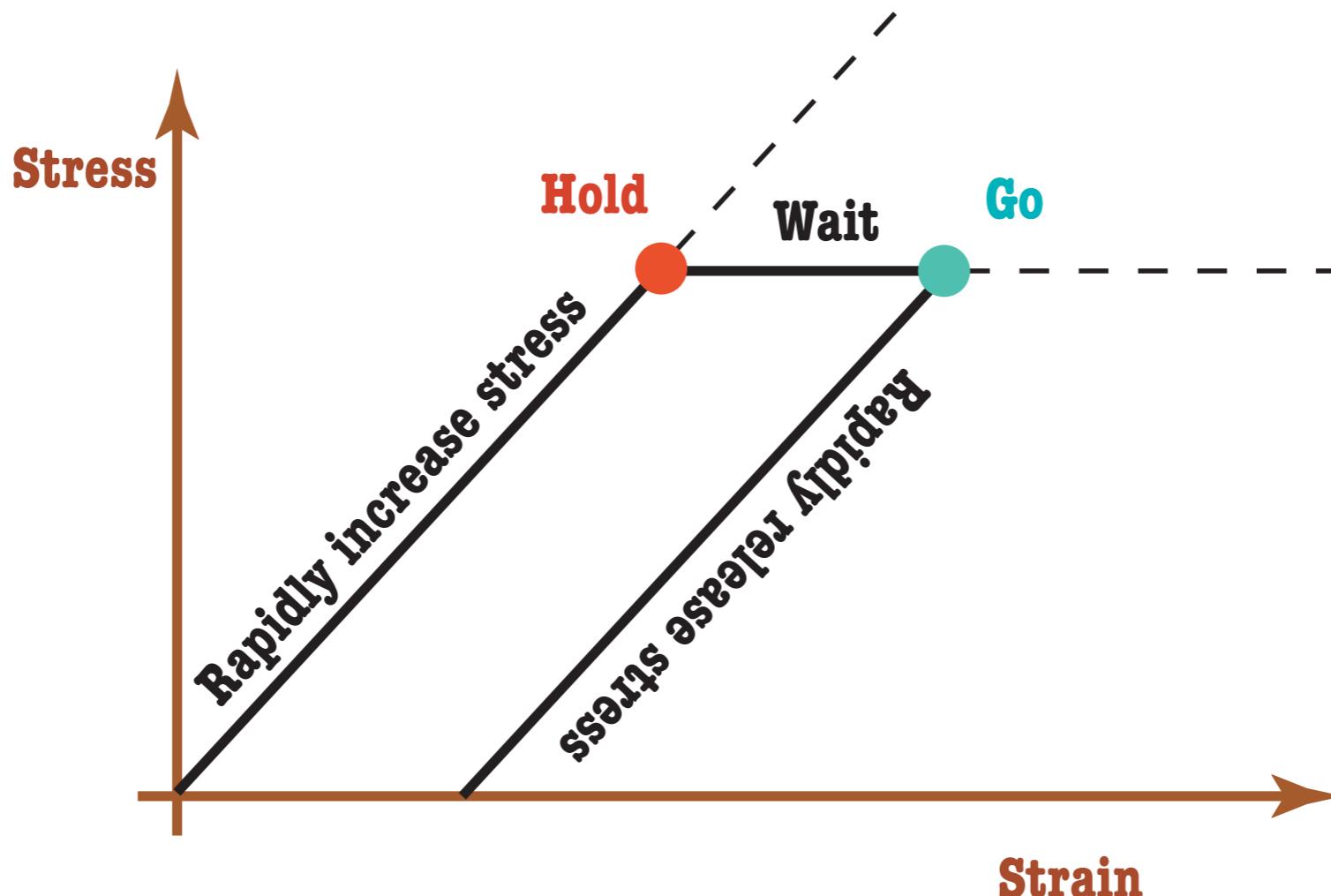


<http://rses.anu.edu.au/research/annrep/ar2003/exp/index.php?p=sharp>



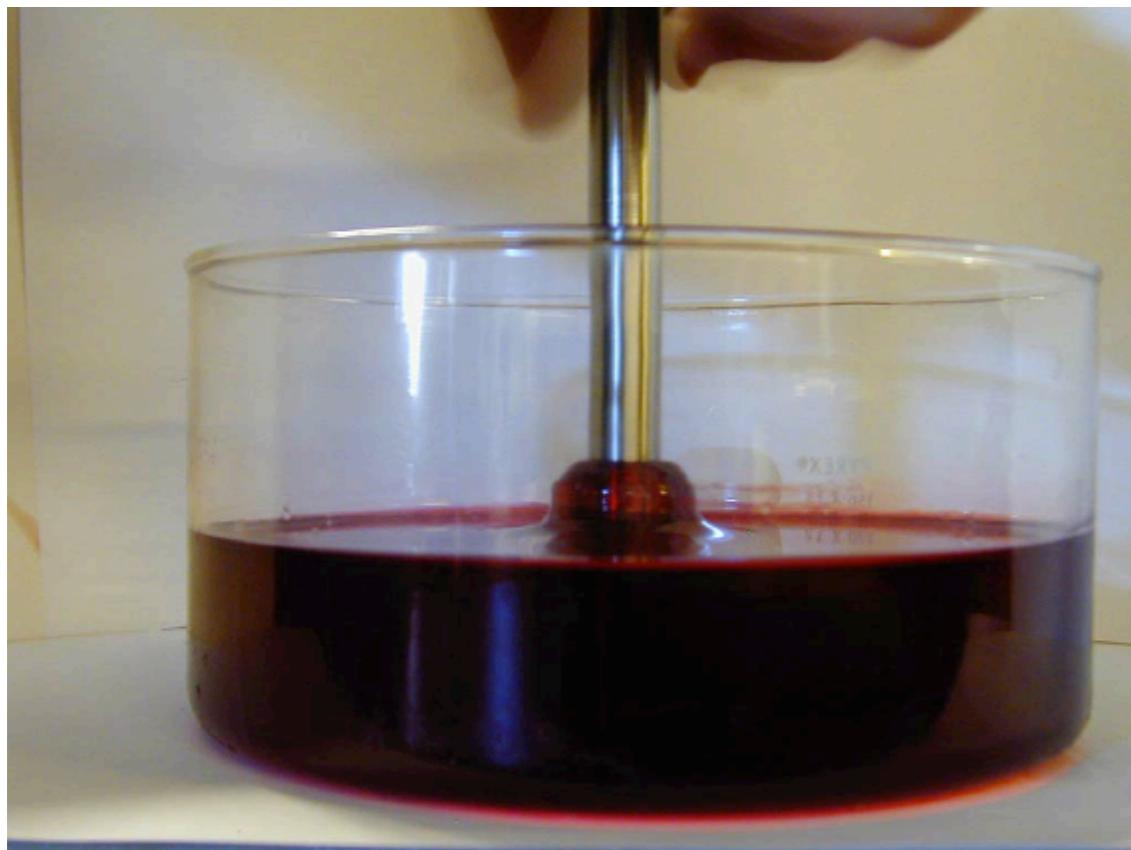
# Viscoelasticity

The reality, which is not surprising given the physical mechanisms involved, is that elastic deformation and viscous deformation occur at the same time.

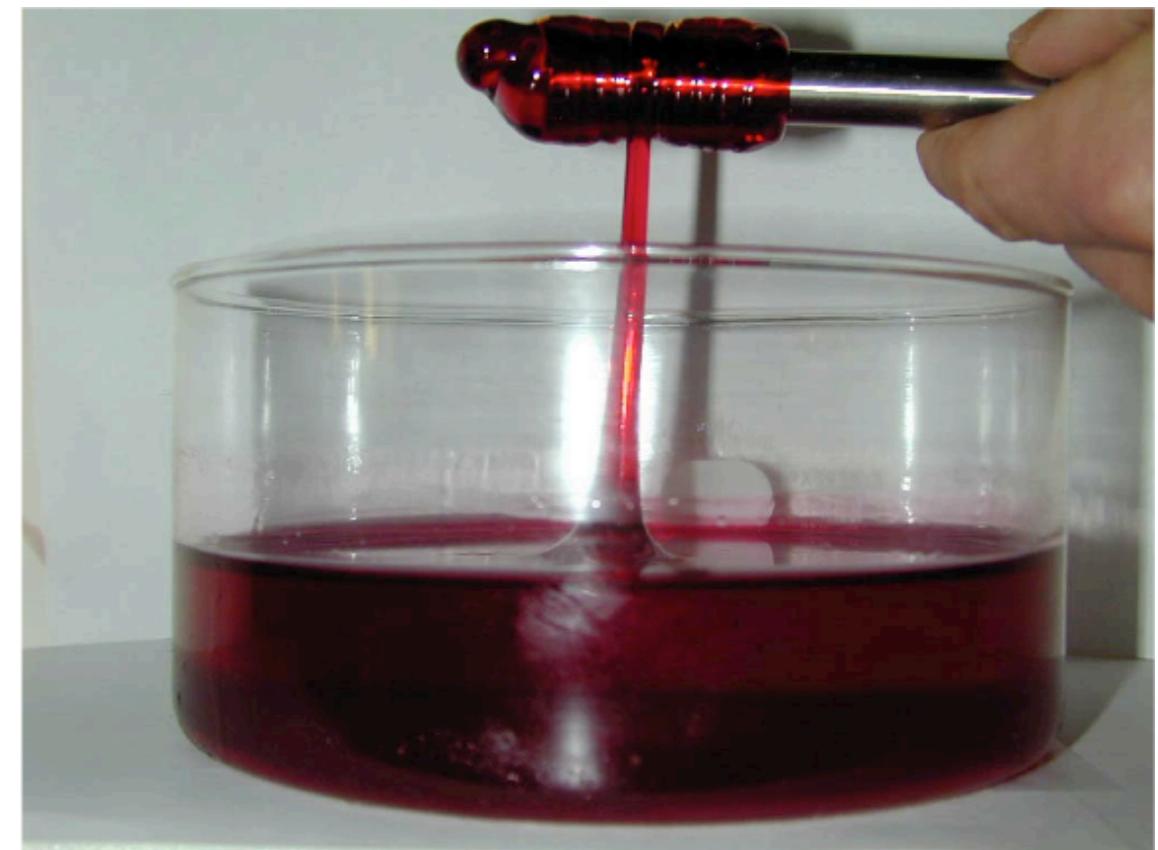


Familiar examples — food and other organic materials which contain long protein molecules in solution

# Viscoelasticity

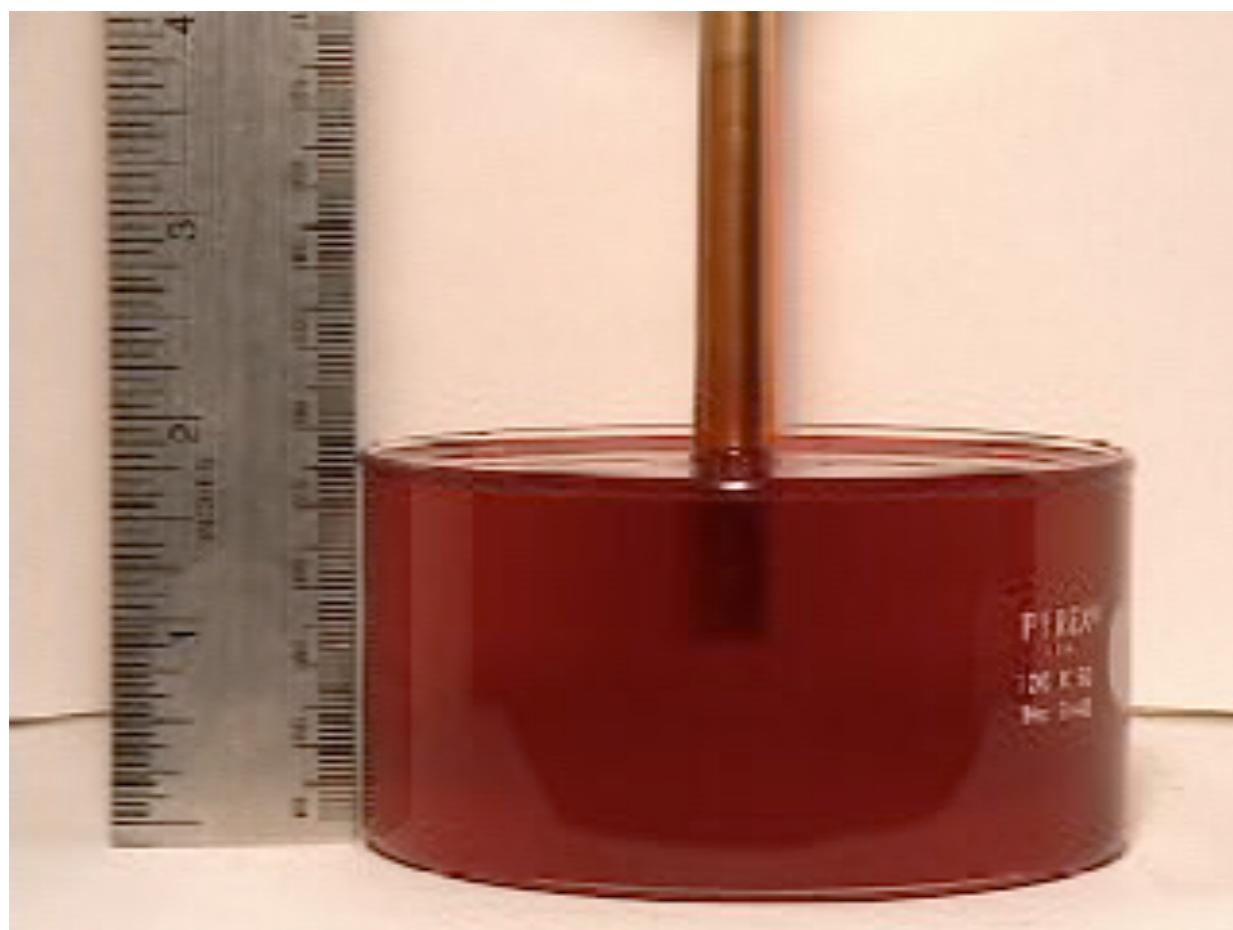


LEFT: Weissenberg effect: A viscoelastic fluid will climb a spinning rod — a viscous fluid is spun away into a hyperbolic shape

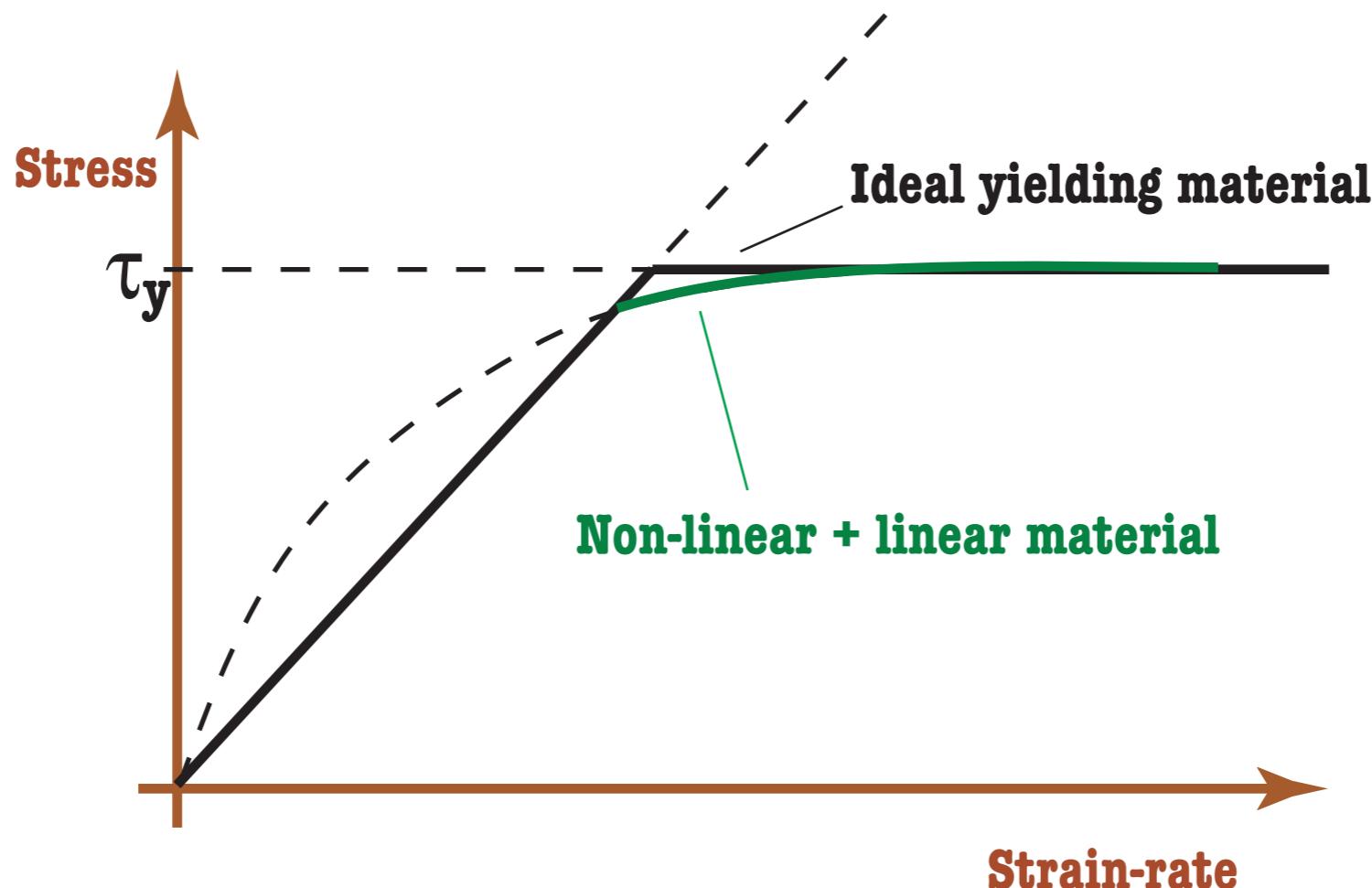


RIGHT: Tubeless siphon: a viscoelastic fluid can be "siphoned" without the use of a hose.

# Viscoelasticity



# Brittle deformation



Our everyday experience is with materials which have a finite strength. Stretched too much or too hard, they will break.

This happens for fluids as well as solids: fluids of this type include toothpaste, gels, sewage sludge (slurries in general) and granular materials.

We can interpret yielding in terms of elastic bond breakage, crack development, frictional bonding in granular materials.

# Application

We now have an idea why / how solid materials can behave as fluids —

- Movement of crystal defects
- Slip on fine-grain faults / cracks

Now we can move on to how the science of fluid dynamics applies to the Earth as a planet

- Convection equations
- How they scale to rocks instead of water or syrup
- Dimensionless numbers
- Flow patterns
- Observables



# Navier Stokes equation

$$\rho_0 \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = \eta \nabla^2 \mathbf{v} - \nabla p - g \rho_0 \alpha (T - T_0) \hat{\mathbf{z}}$$
$$\left( \frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla) T \right) = \kappa \nabla^2 T$$

Depending upon the following physical parameters which vary from experiment to experiment & planet to planet

viscosity	$\eta$	Pa.s
temperature scale	$\Delta T = T - T_0$	K
length scale	$x$	m
density	$\rho_0$	$kg.m^{-3}$
thermal expansivity	$\alpha$	$K^{-1}$
gravity	$g$	$m.s^{-2}$
thermal diffusivity	$\kappa$	$m^2 s^{-1}$

Strength of convection:

$$\text{Activity} = \frac{\text{Parameters increasing convection}}{\text{Processes decreasing convection}}$$

Rather than immediately trying to solve these equations, we can first try to understand the effects of each parameter intuitively ... Consider these 2 examples of heat transport in fluids. How would they be influenced by changing physical parameters ?

Strength of convection:

$$\text{Activity} = \frac{\text{Parameters increasing convection}}{\text{Processes decreasing convection}}$$

$$\text{Activity} = \frac{? \Delta T}{?}$$

## Temperature

Do hot air balloons work better in hot weather or cold ?

Would the lava lamp run faster with a more powerful lamp ?

Rather than immediately trying to solve these equations, we can first try to understand the effects of each parameter intuitively ... Consider these 2 examples of heat transport in fluids. How would they be influenced by changing physical parameters ?

# Scaling

Strength of convection:

$$\text{Activity} = \frac{\text{Parameters increasing convection}}{\text{Processes decreasing convection}}$$

$$\text{Activity} = \frac{?g\Delta T}{?}$$

## Gravity

Would the lava lamp run on the moon, in free fall ?

Which direction would a helium balloon go in the space station ?

Rather than immediately trying to solve these equations, we can first try to understand the effects of each parameter intuitively ... Consider these 2 examples of heat transport in fluids. How would they be influenced by changing physical parameters ?

Strength of convection:

$$\text{Activity} = \frac{\text{Parameters increasing convection}}{\text{Processes decreasing convection}}$$

$$\text{Activity} = \frac{\gamma g \Delta T}{\gamma \kappa}$$

## Thermal Diffusivity

If heat diffuses out of the balloon or the blobs more quickly does their motion become faster or slower ?

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Strength of convection:

$$\text{Activity} = \frac{\text{Parameters increasing convection}}{\text{Processes decreasing convection}}$$

$$\text{Activity} = \frac{\gamma g \alpha \Delta T}{\kappa}$$

## Thermal expansivity

If we used a different fluid in the lava lamp - the same density, but double the thermal expansion, what would happen ?

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Strength of convection:

$$\text{Activity} = \frac{\text{Parameters increasing convection}}{\text{Processes decreasing convection}}$$

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## Density

If we used a different fluid in the lava lamp - double the density, but the same thermal expansion, what would happen ?

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Strength of convection:

$$\text{Activity} = \frac{\text{Parameters increasing convection}}{\text{Processes decreasing convection}}$$

$$\text{Activity} = \frac{\gamma g \rho \alpha \Delta T}{\gamma \kappa \eta}$$

## Viscosity

Would the lava lamp run faster with syrup in place of water ?  
How fast would the balloon rise if the air was very thick & viscous ?

Rather than immediately trying to solve these equations, we can first try to understand the effects of each parameter intuitively ... Consider these 2 examples of heat transport in fluids. How would they be influenced by changing physical parameters ?

Strength of convection:

$$\text{Activity} = \frac{\text{Parameters increasing convection}}{\text{Processes decreasing convection}}$$

$$\text{Activity} = \frac{\gamma g \rho \alpha \Delta T d}{\gamma \kappa \eta}$$

## System size

Is a large lava lamp (same stuff, same temperature difference, same planet) more or less vigorous ?

Rather than immediately trying to solve these equations, we can first try to understand the effects of each parameter intuitively ...

Consider these 2 examples of heat transport in fluids. How would they be influenced by changing physical parameters ?

# Convective activity

**Intuition** about convective systems suggests that the vigour of convection may depend upon the various system parameters according to the following equation:

$$\text{Activity} \approx \frac{g\rho\alpha\Delta T d}{\kappa\eta}$$

This turns out to be roughly correct, but should be

$$\text{Activity} \propto \frac{g\rho\alpha\Delta T d^3}{\kappa\eta}$$

How do we know this is correct, and that we have considered all the relevant effects / parameters ?

# Non-dimensionalization

Consider a prototype planet which is completely flat — the mantle is a layer of fixed depth with fixed top and bottom temperatures and no-shear-stress boundaries.

Rescale with the new variables indicated using a prime (').

$$x = d.x'$$

$$t = (d^2/\kappa)t'$$

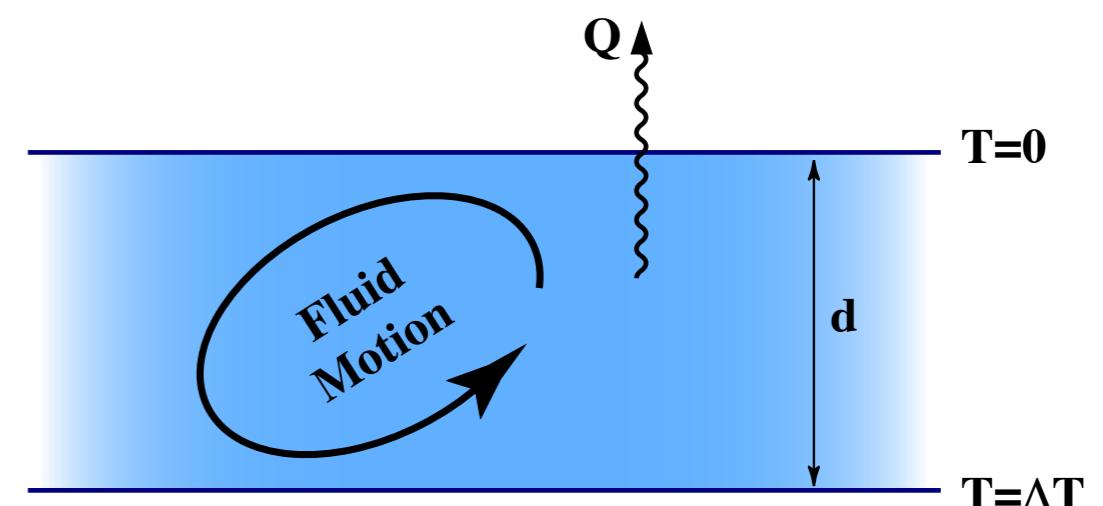
$$T = \Delta T T'$$

$$v = (\kappa/d)v'$$

$$p = p_0 + (\eta\kappa/d^2)p'$$

where

$$\nabla p_0 = -g\rho_0$$



## Non-dimensionalization

Substituting for all the existing terms in the Navier-Stokes equation

$$\frac{\rho_0 \kappa}{d^2} \frac{D}{Dt'} \left( \frac{\kappa}{d} \mathbf{v}' \right) = \frac{\eta}{d^2} \nabla^2 \left( \frac{\kappa}{d} \mathbf{v}' \right) - \frac{\eta \kappa}{d^3} \nabla p' + g \rho_0 \alpha \Delta T T' \hat{\mathbf{z}}$$

Collect up terms:

$$\frac{\rho \kappa}{\eta} \frac{D \mathbf{v}'}{Dt'} = \nabla^2 \mathbf{v}' - \nabla p' + \frac{g \rho_0 \alpha \Delta T d^3}{\kappa \eta} T' \hat{\mathbf{z}}$$

or

$$\frac{1}{\text{Pr}} \frac{D \mathbf{v}'}{Dt'} = \nabla^2 \mathbf{v}' - \nabla p' + \text{Ra} T' \hat{\mathbf{z}}$$

**Pr** is known as the Prandtl number, and **Ra** is known as the Rayleigh number; both are non-dimensional numbers.

## Dimensionless numbers

**Rayleigh number** – measures the intensity of the buoyancy forces

$$\text{Ra} = \frac{g\rho\alpha\Delta T d^3}{\eta\kappa}$$

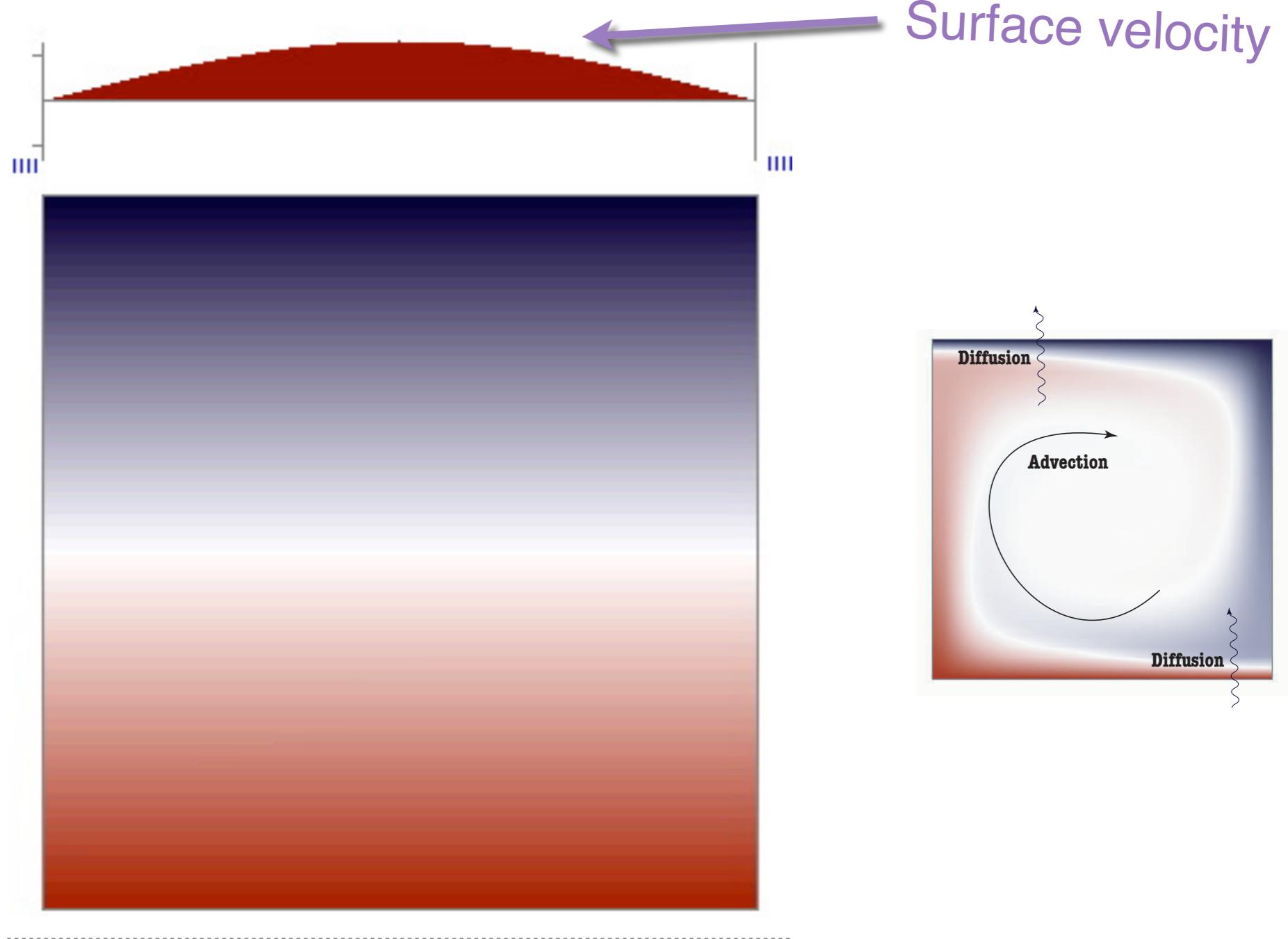
**Reynolds number** – measures the relative importance of inertial to viscous forces

$$\text{Re} = \frac{\rho V_0 d}{\eta} = \frac{V_0 d}{\nu}$$

**Prandtl number** – measures the relative importance of stress diffusion and thermal diffusion

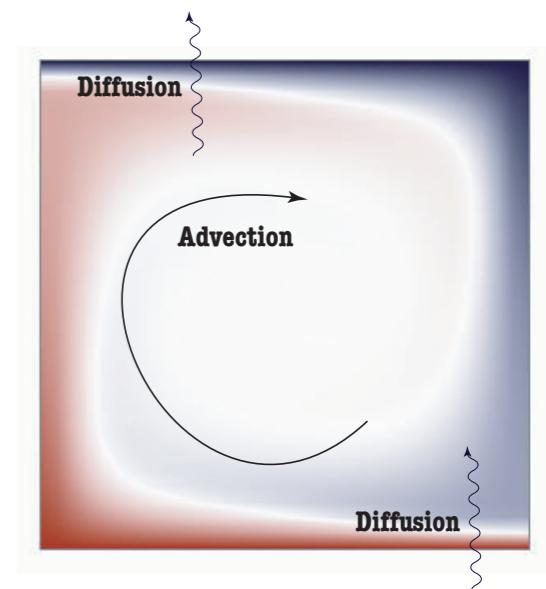
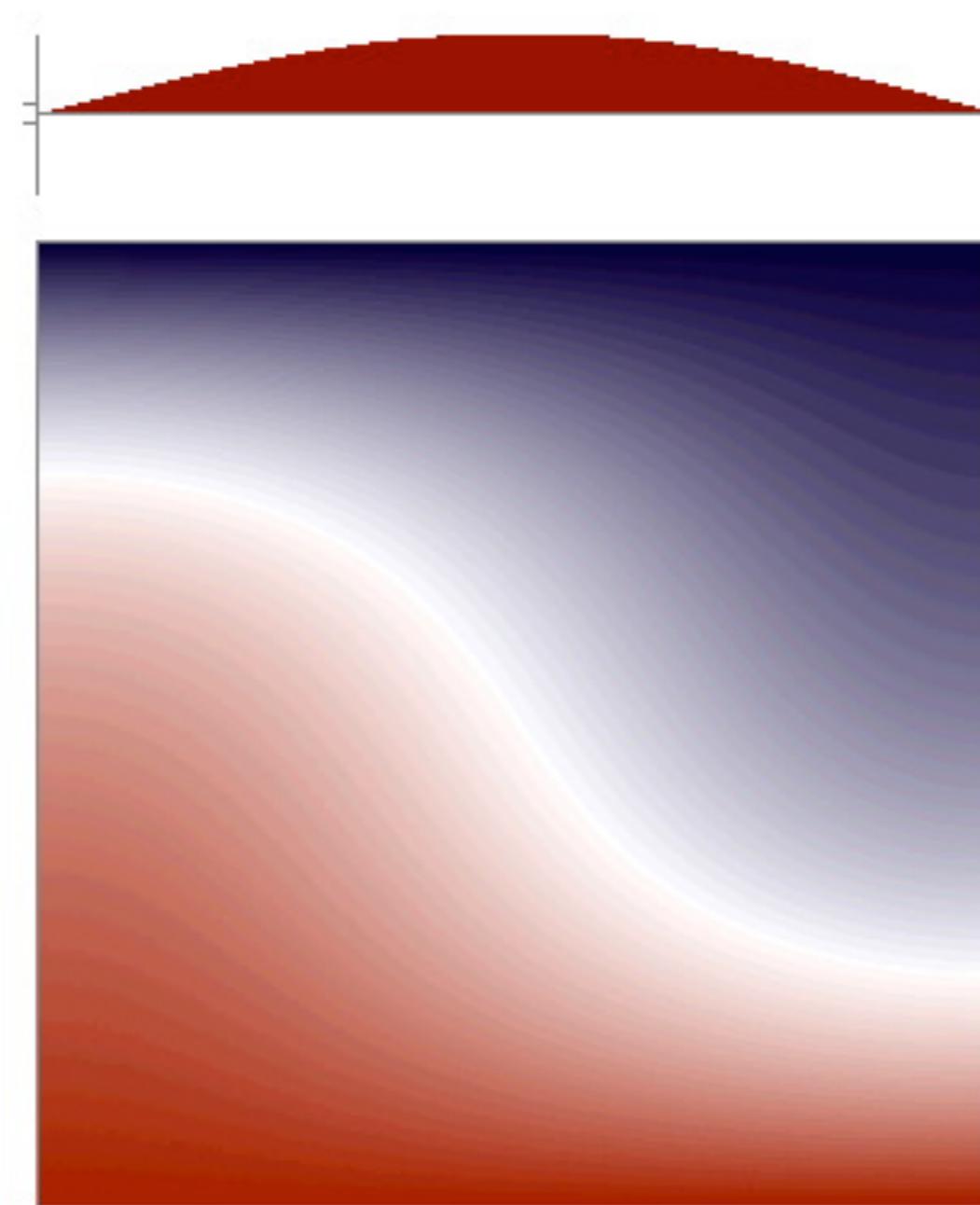
$$\text{Pr} = \frac{\eta}{\rho\kappa} = \frac{\nu}{\kappa}$$

# Thermal convection



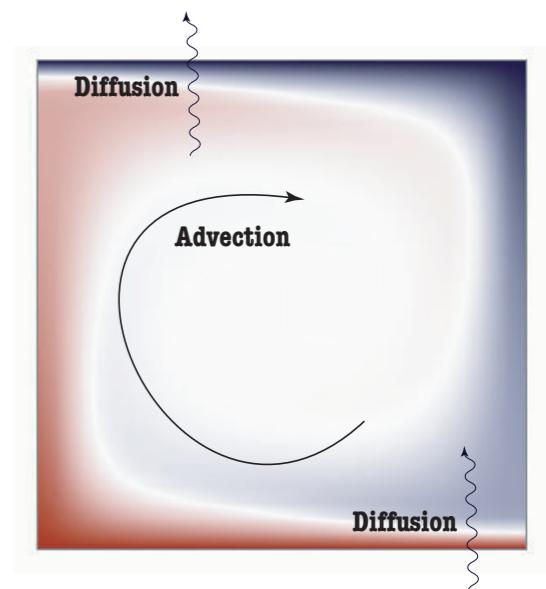
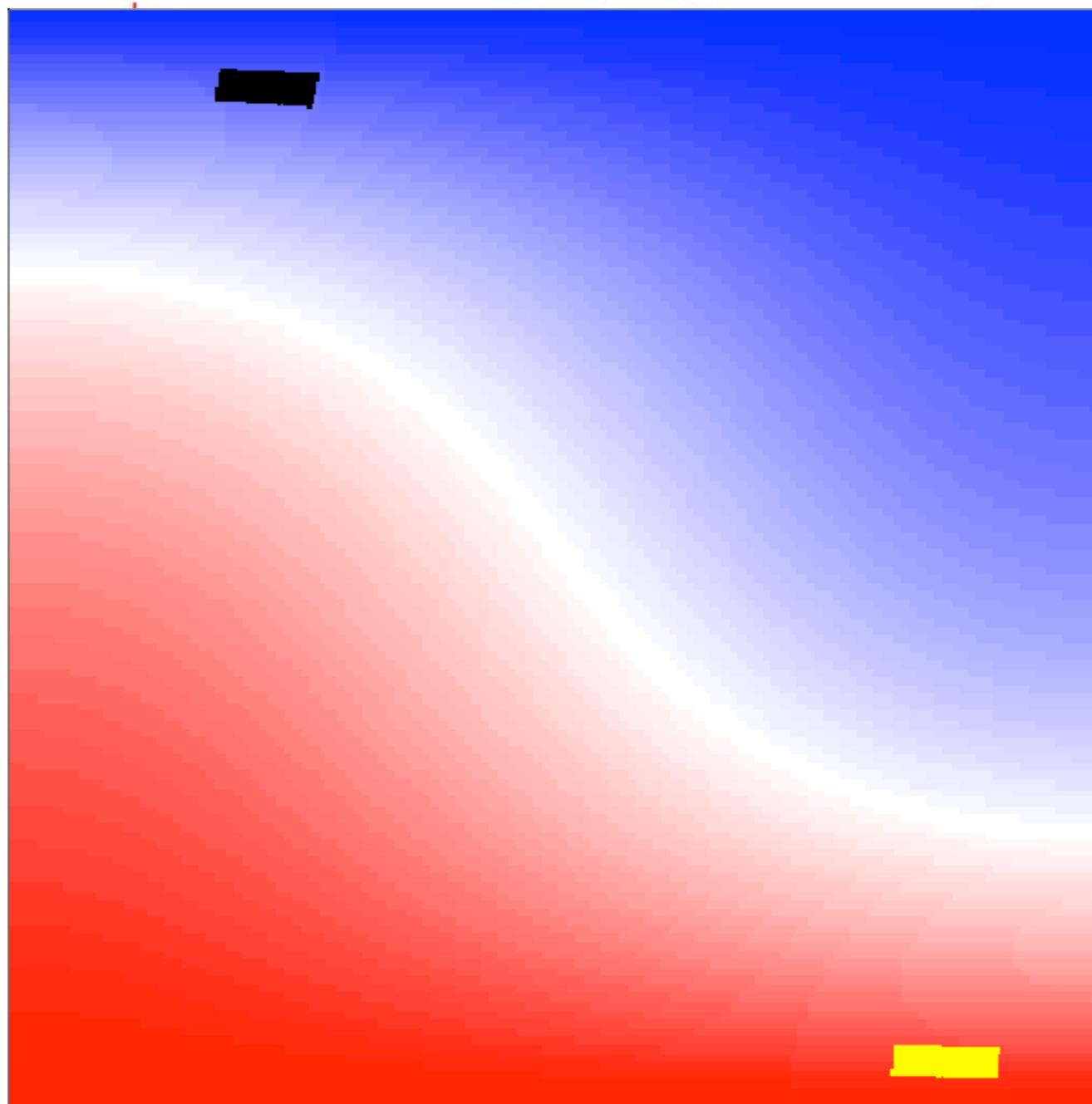
Rayleigh number = 100

# Thermal convection



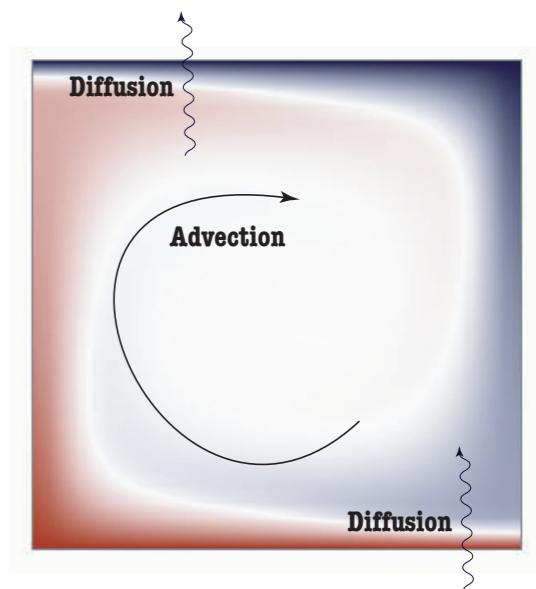
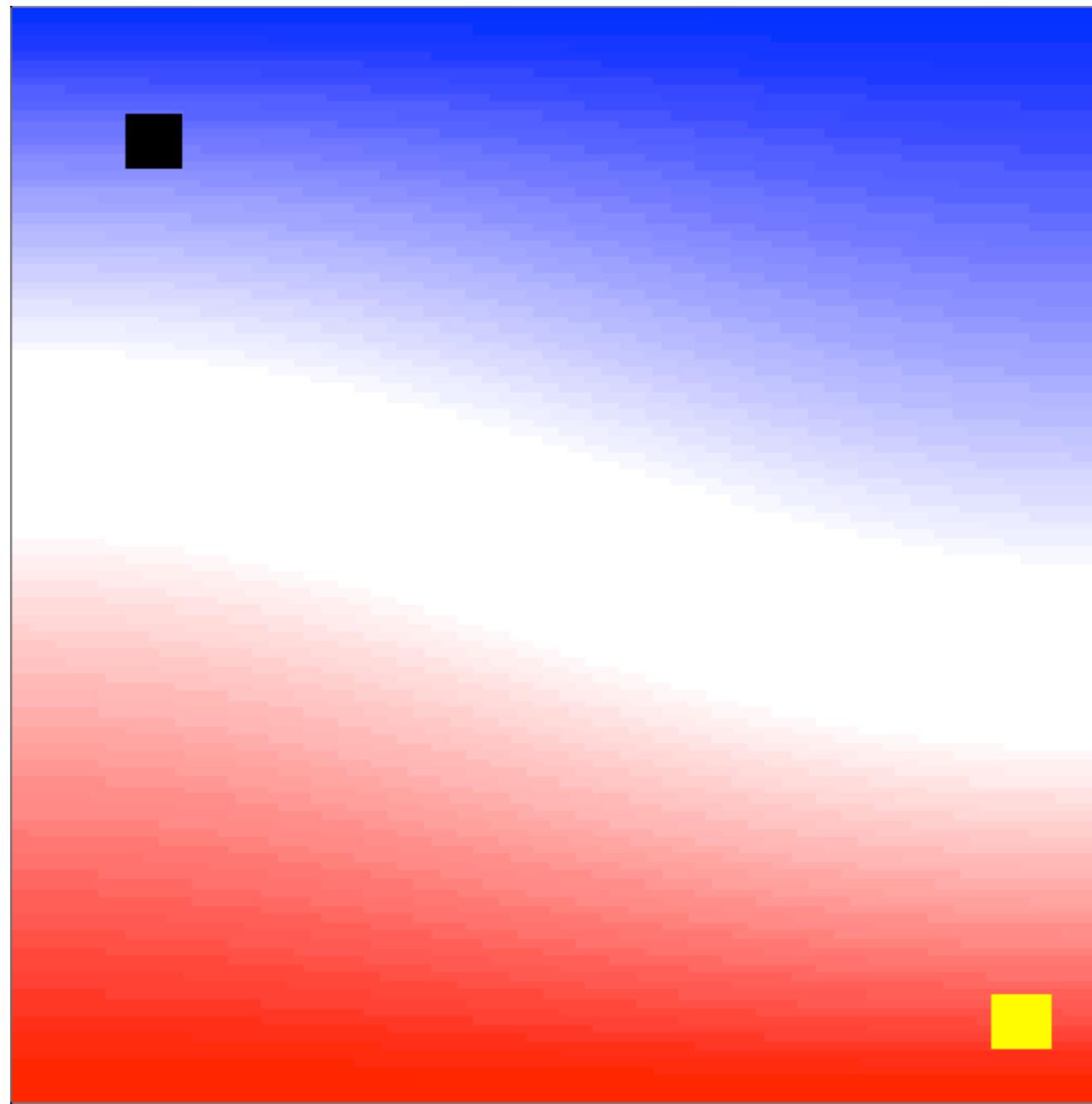
Rayleigh number = 1000

# Thermal convection



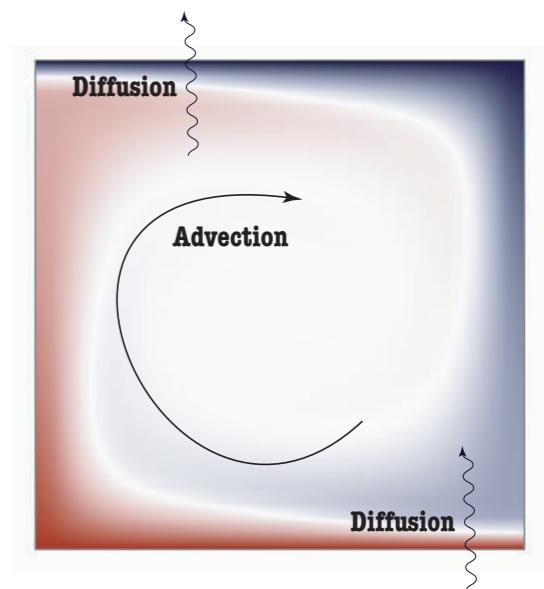
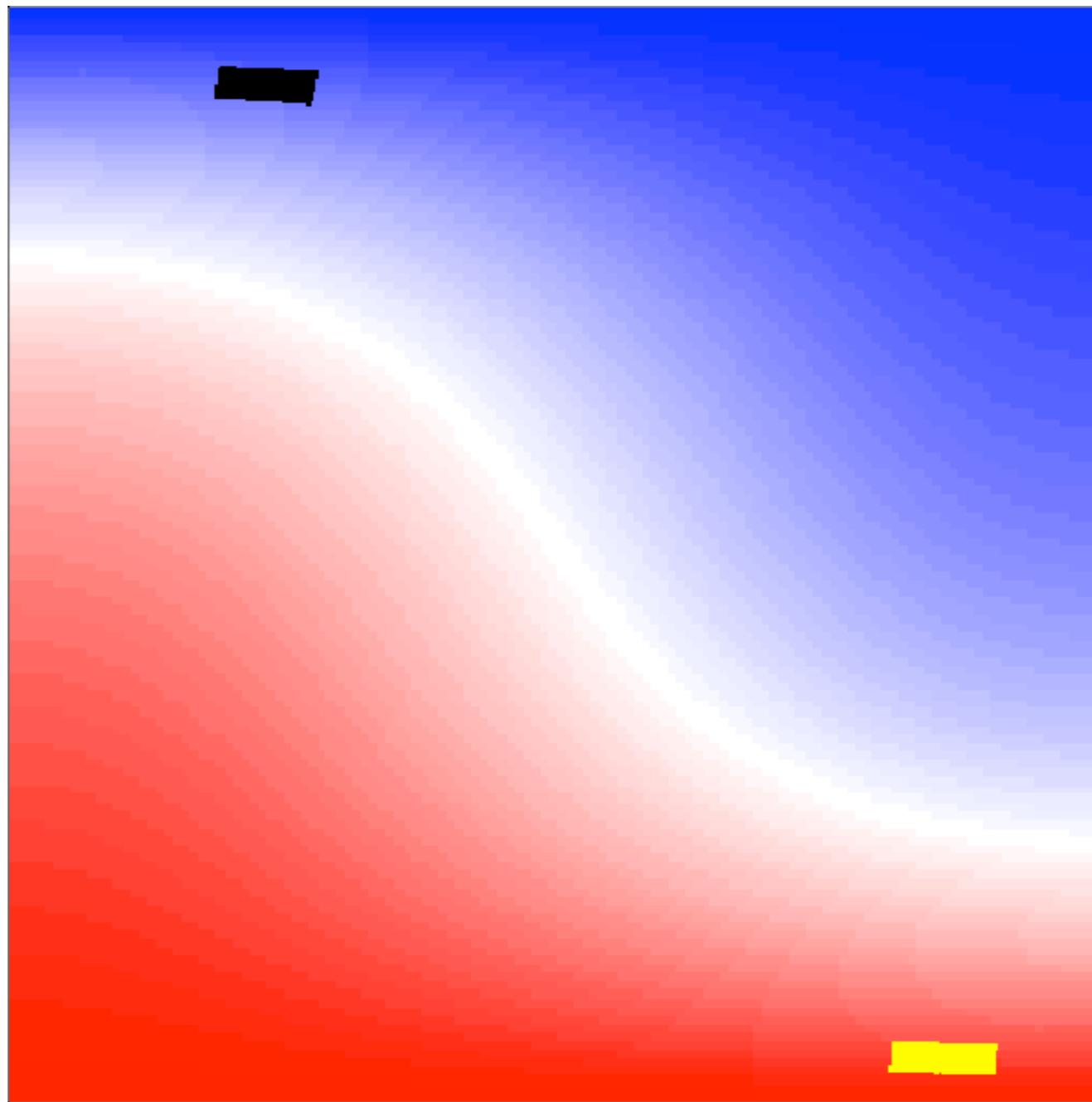
Rayleigh number = 10000

# Thermal convection



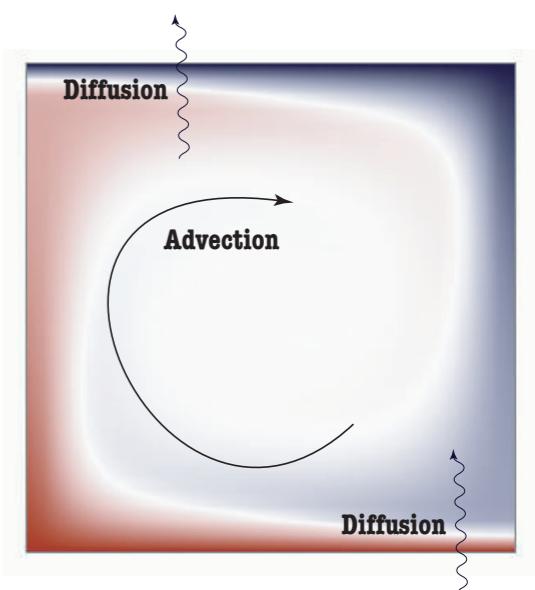
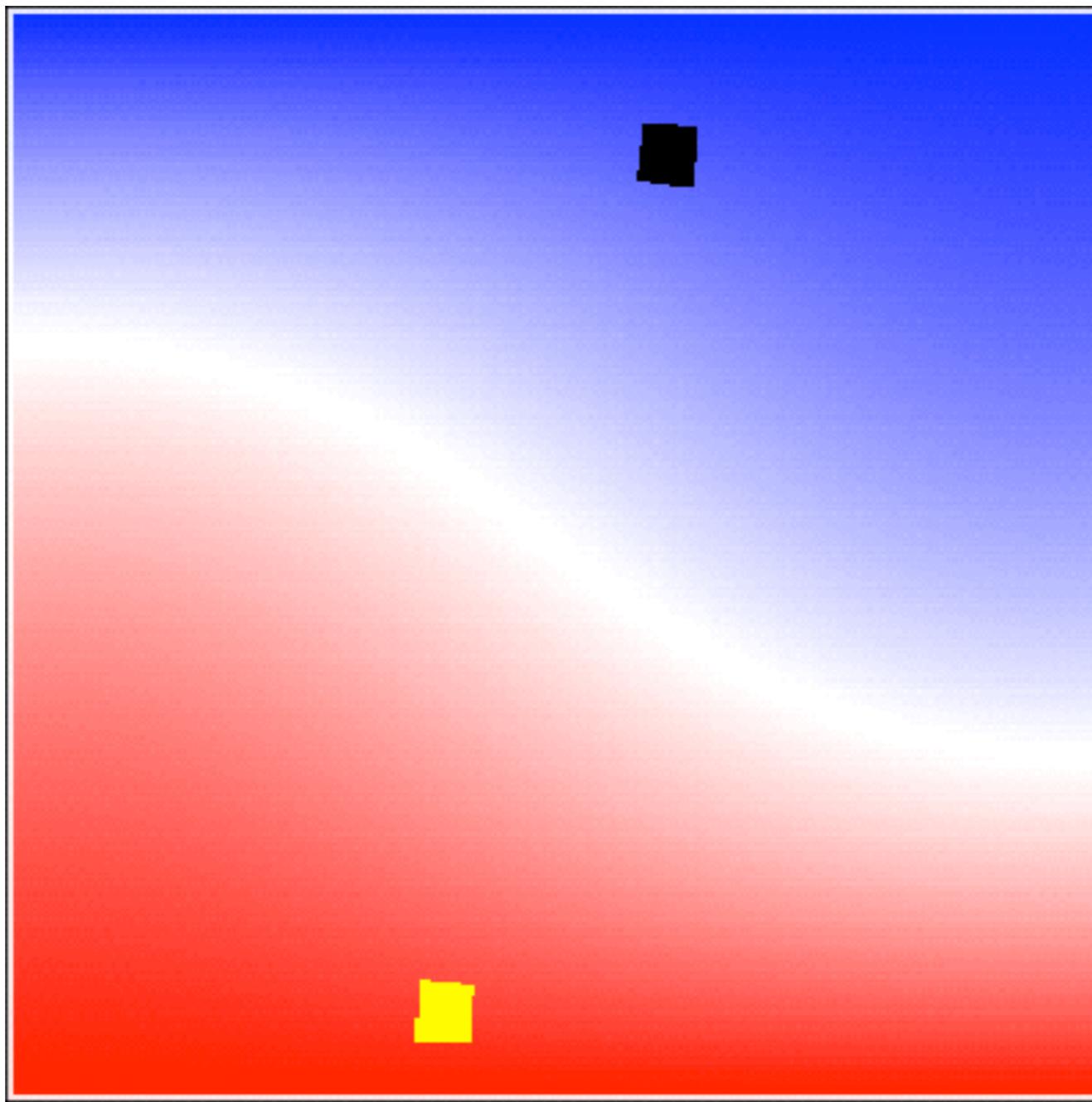
Rayleigh number = 100000

# Thermal convection



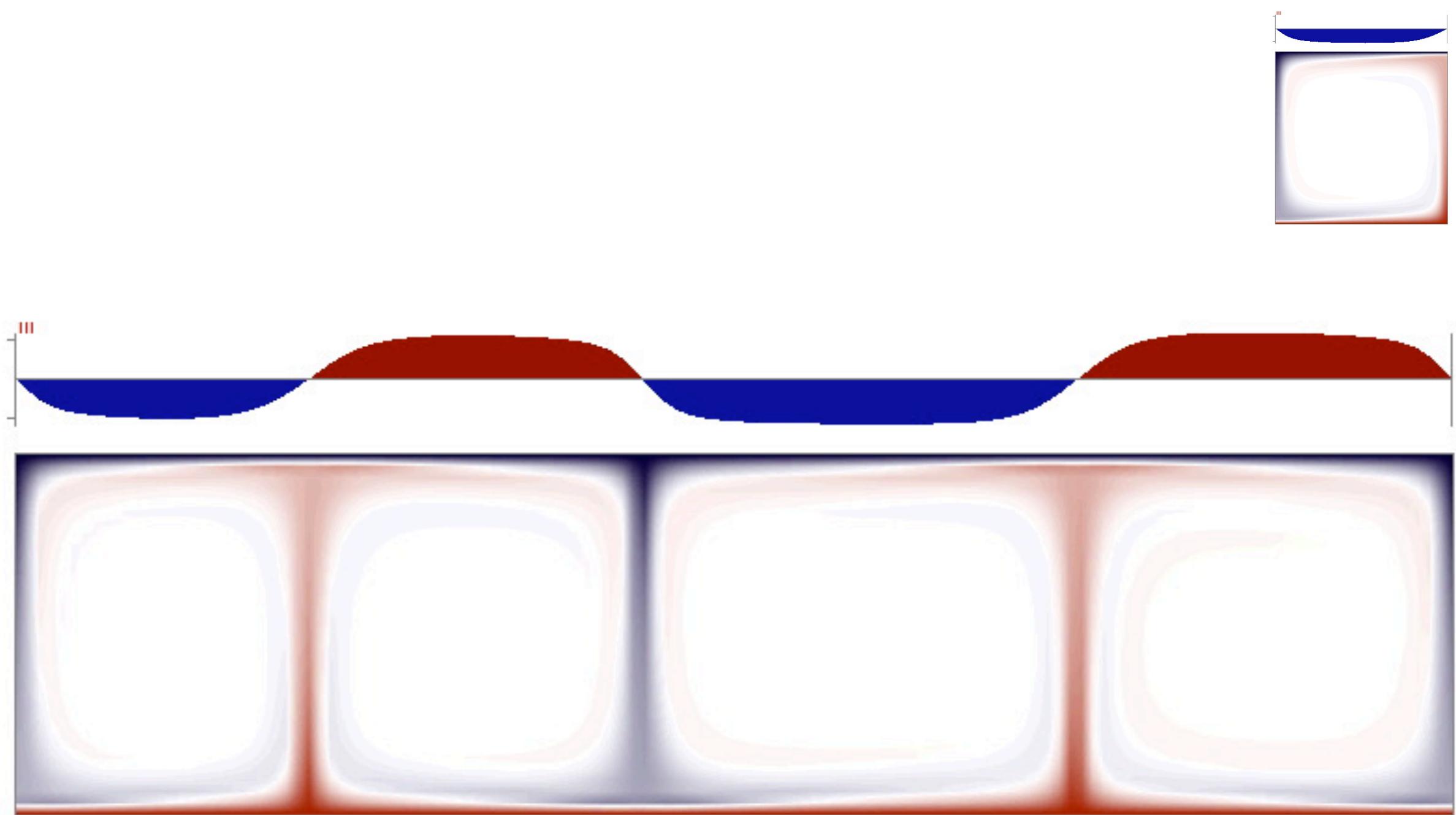
Rayleigh number = 1000000

# Thermal convection



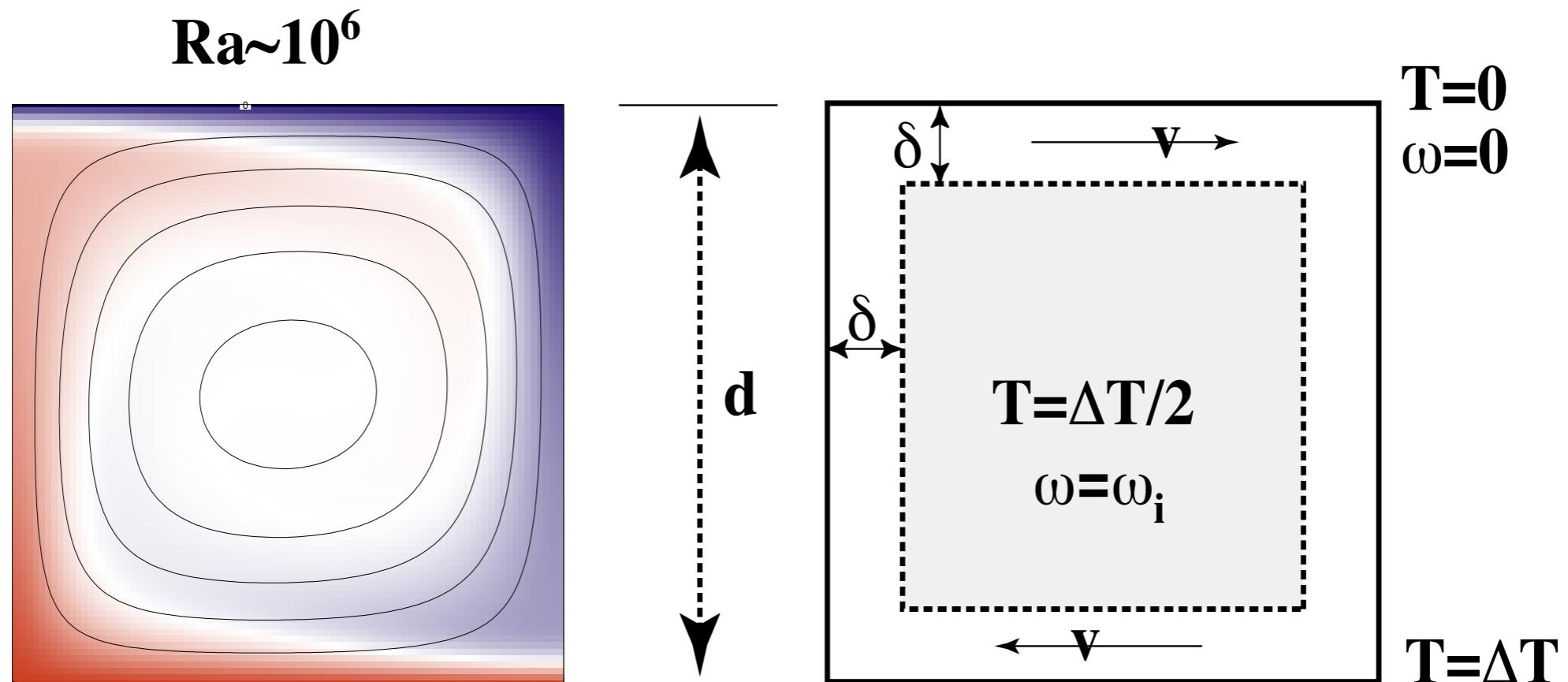
Rayleigh number = 10000000

# Thermal convection



Rayleigh number = 1000000, wide aspect ratio

# Thermal convection

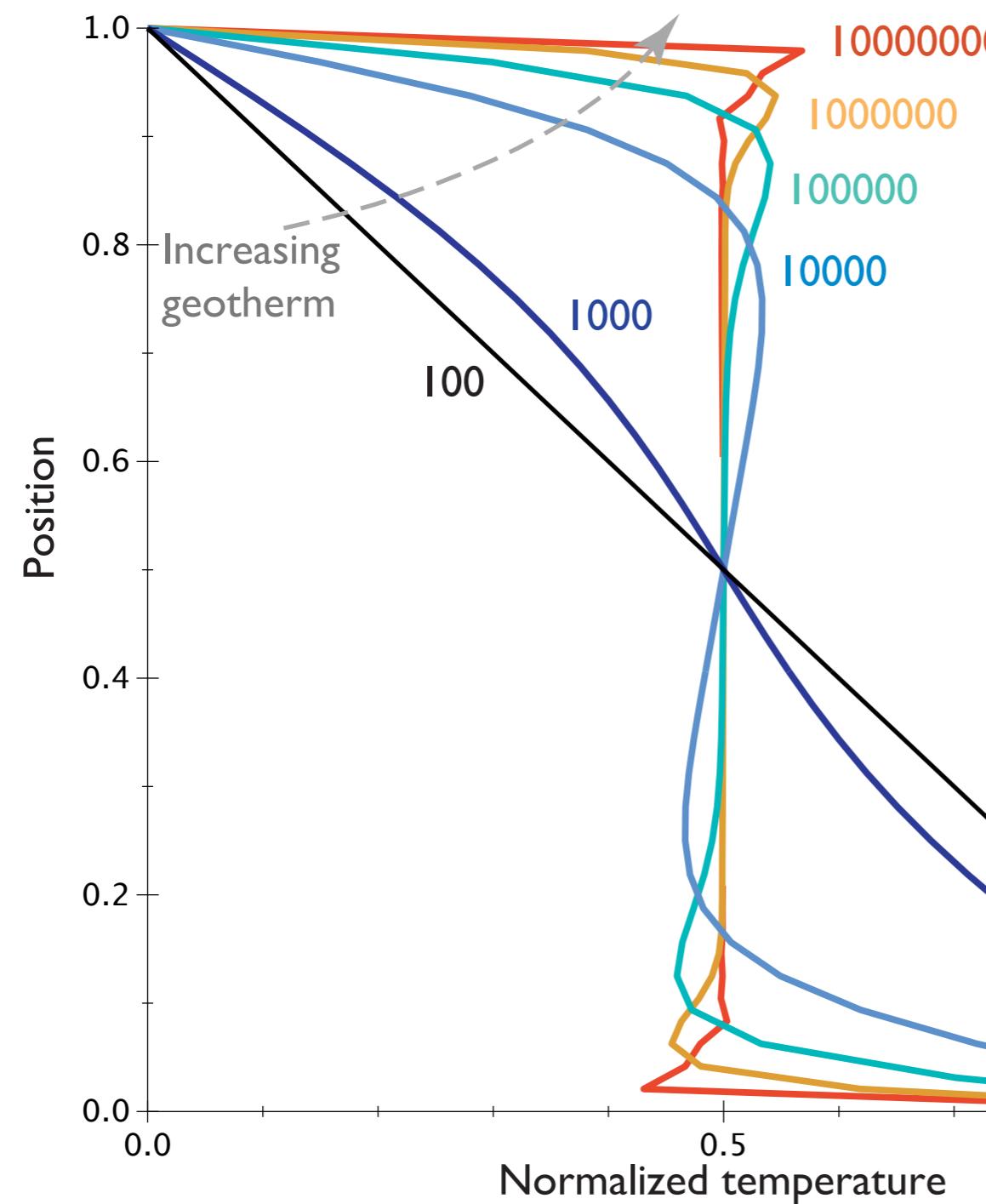


A relatively simple pattern emerges from the competition between advection of heat by fluid flow and diffusion

We will see this picture again when we consider the patterns of heat flow in the Earth's ocean floor.

Convection length and timescales are determined by the Rayleigh number. What is the Ra for the Earth ?

# Thermal convection



$$Ra = \frac{g\rho\alpha\Delta T d^3}{\kappa\eta}$$

The temperature (velocity, stress etc) profiles vary systematically with Rayleigh number.

# Nusselt number

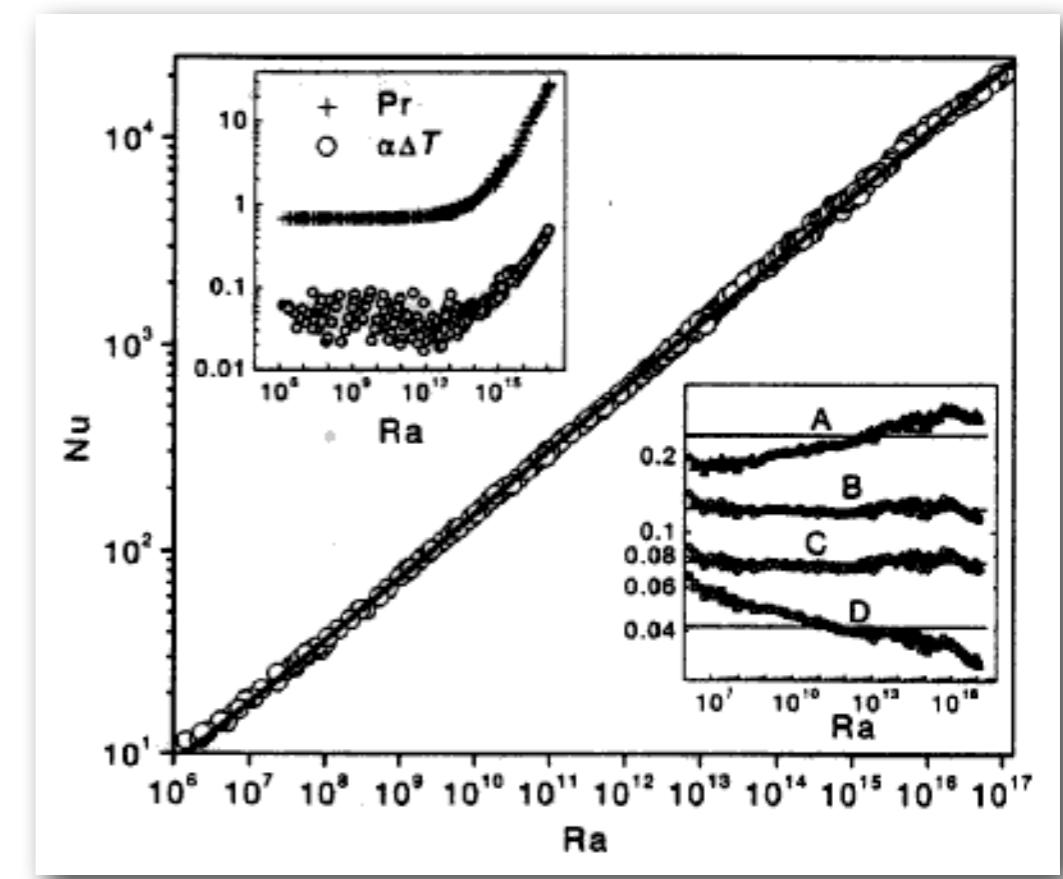
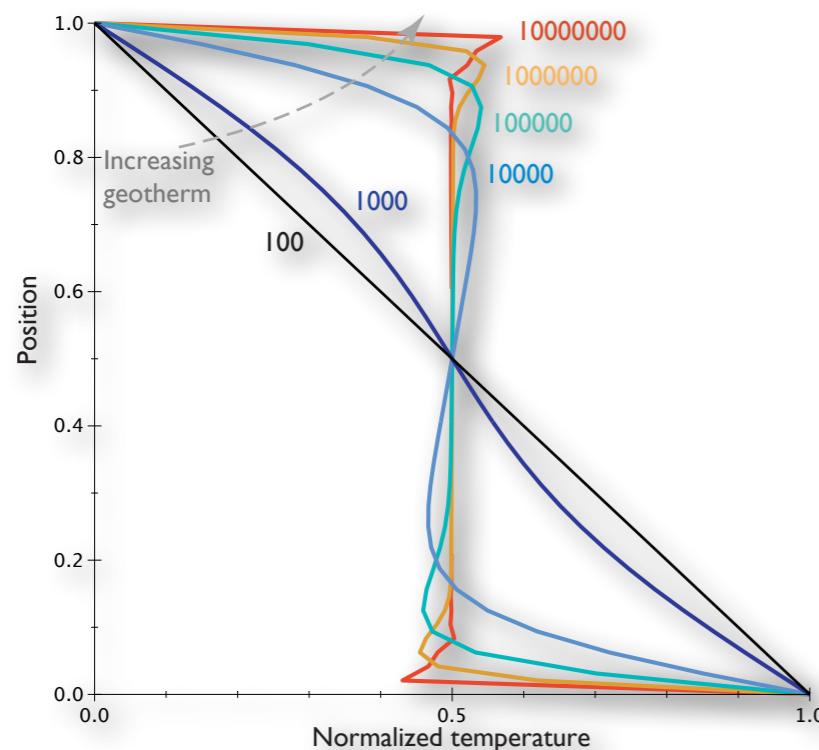
A third, independent, dimensionless number can be derived for the thermally driven flow equations. This is the Nusselt number

$$Nu = \frac{qd}{k\Delta T}$$

It is the ratio of heat transported by fluid motions in the layer ( $q$ ) compared to that transported conductively in the absence of fluid motion.

It can be shown that

$$Nu \propto Ra^{1/3}$$

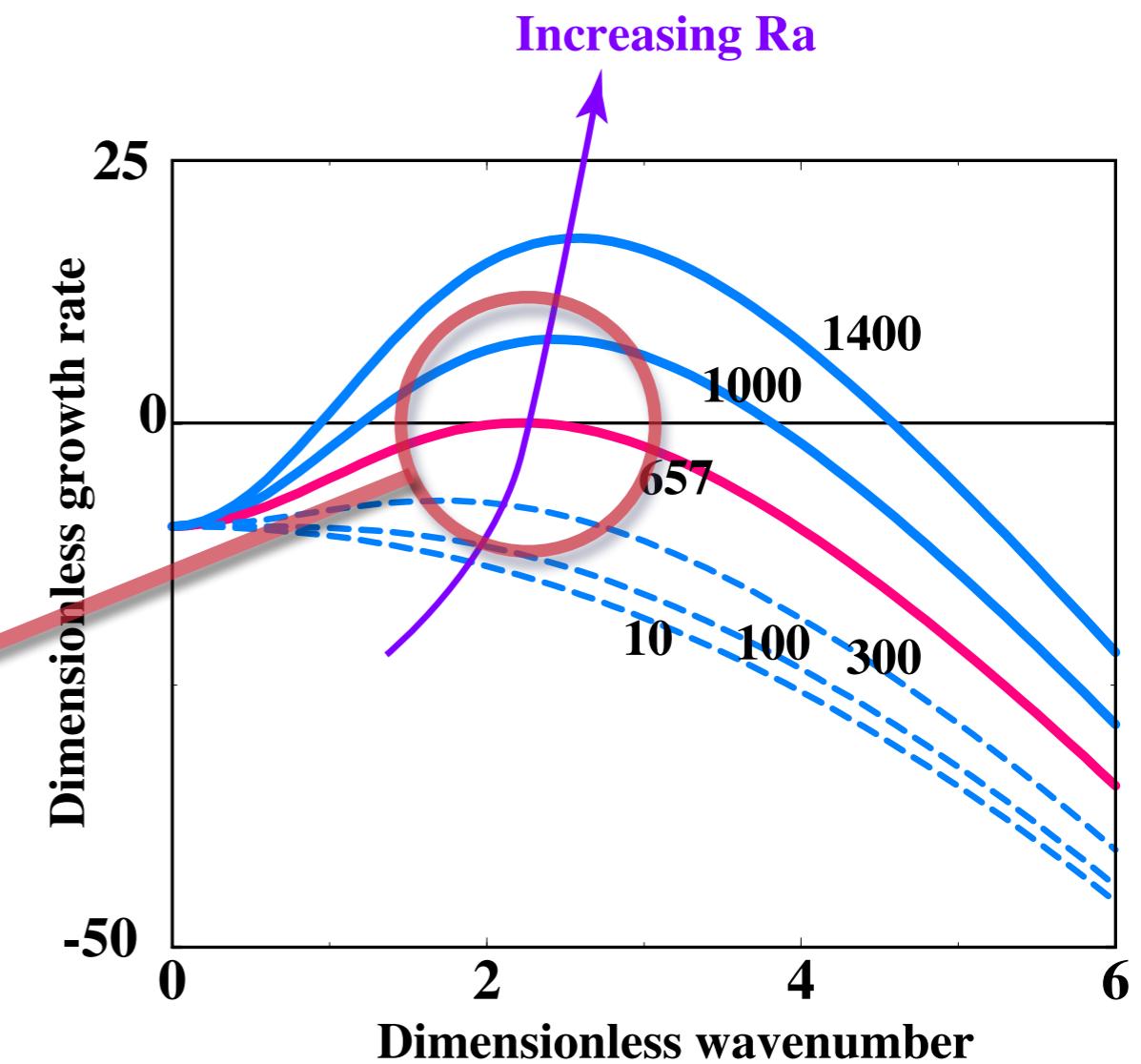


# Critical Rayleigh number

At low Rayleigh number diffusion dominates advection: fluid motion and any lateral temperature variations die away. At high **Ra**, the reverse is true: lateral temperature variations are not damped out, but grow rapidly

$$Ra_c = \frac{27}{4}\pi^4 = 657.51$$

$$k = \frac{\pi}{2^{1/2}} = 2.22$$



# Observables from Convection

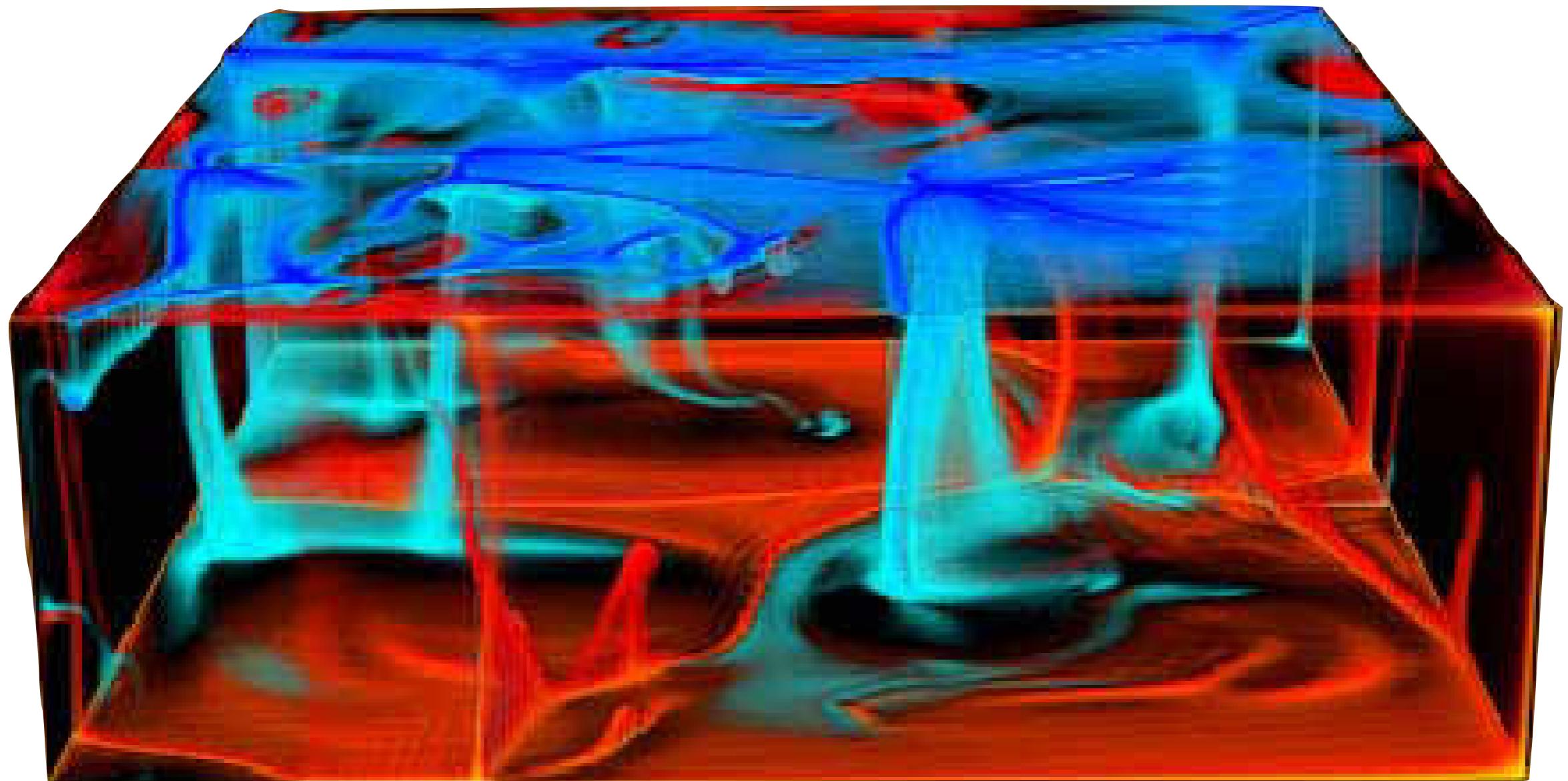
What quantitative information can we obtain about the dynamics of planetary interiors from remote sensing — and from really remote sensing (in orbit) ?

Which remote observations tell us what information ?

- Gravity measurements: density distribution
- Surface topography: surface stress field
- Surface heat flow: near surface temperature gradients
- Seismology: elastic wave speed distribution
- Morphology: deformation styles / relative ages
- E/M: conductivity / magnetic susceptibility

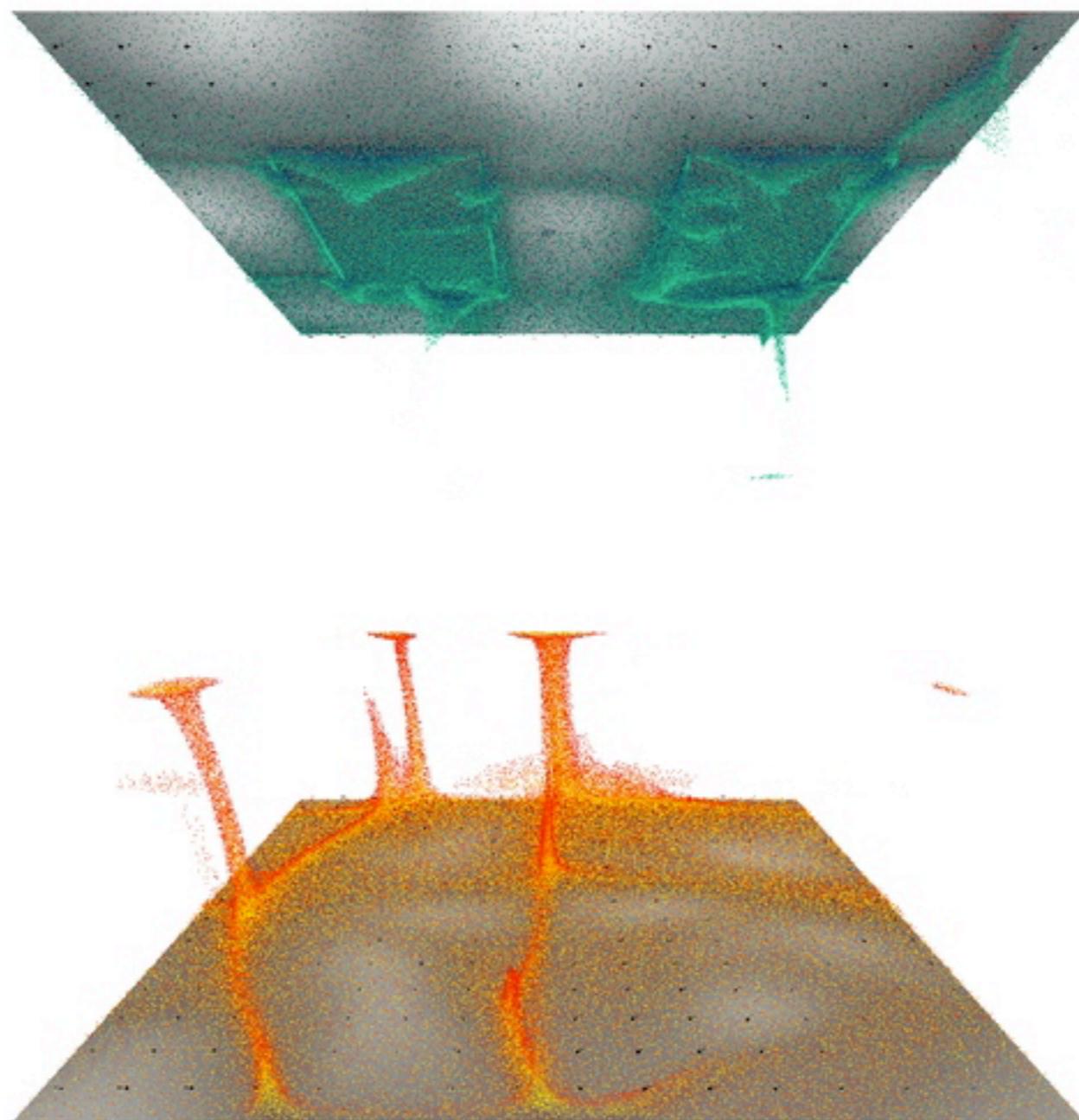
None of these tells us directly what the flow is doing at depth —how to solve this **and** the uncertainty of rheological parameters

# 3d convection model



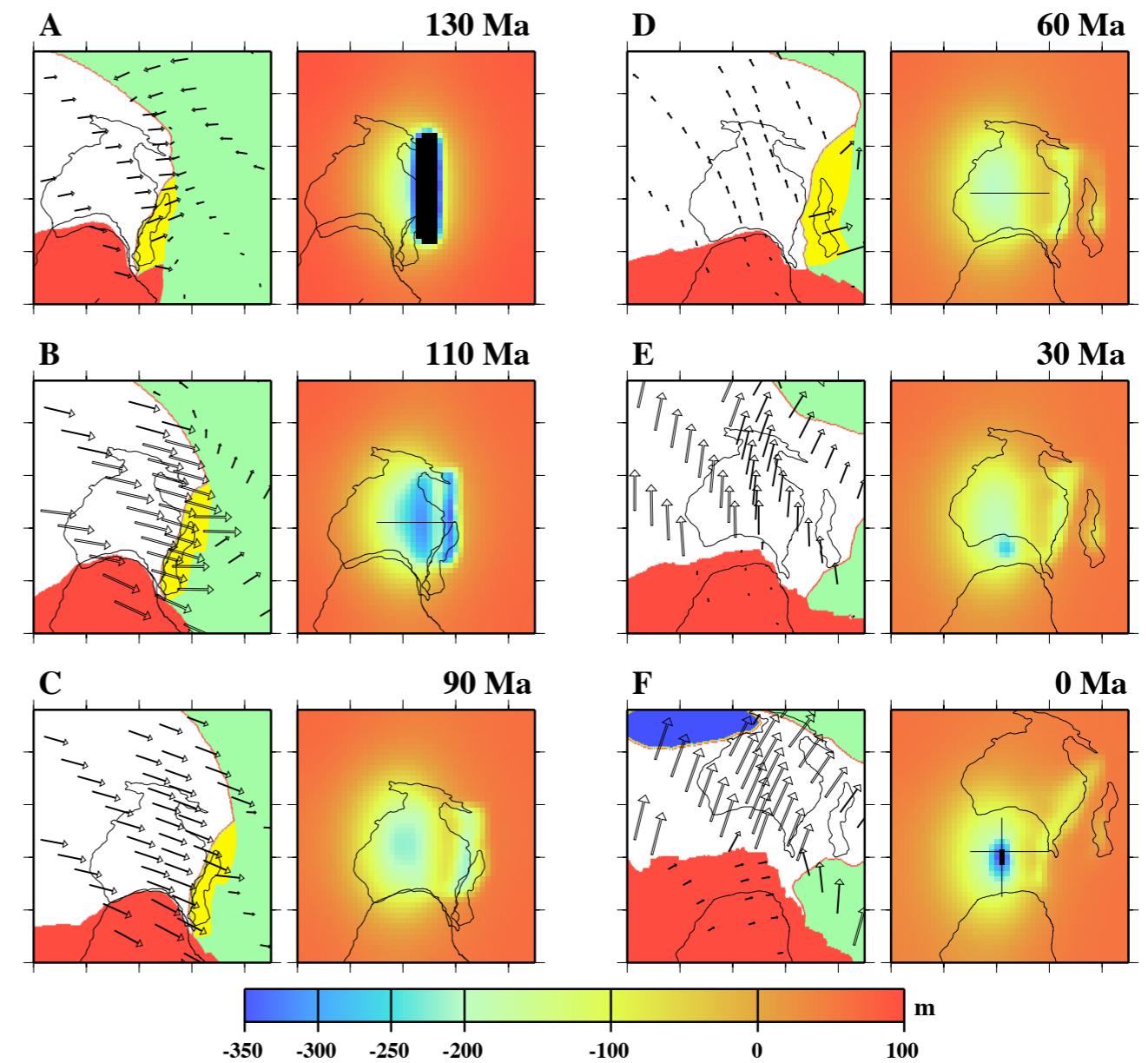
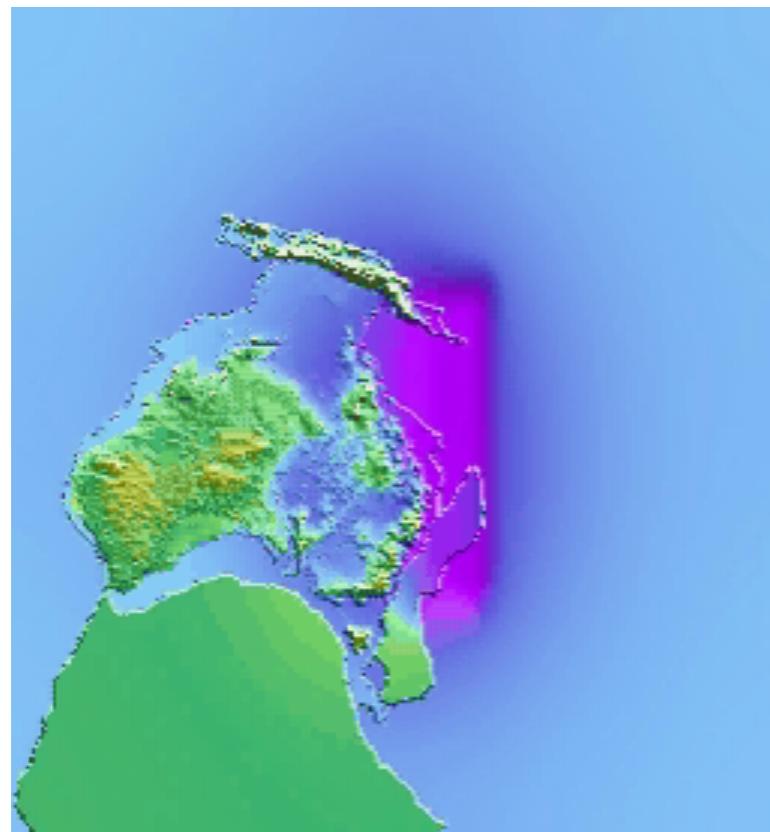
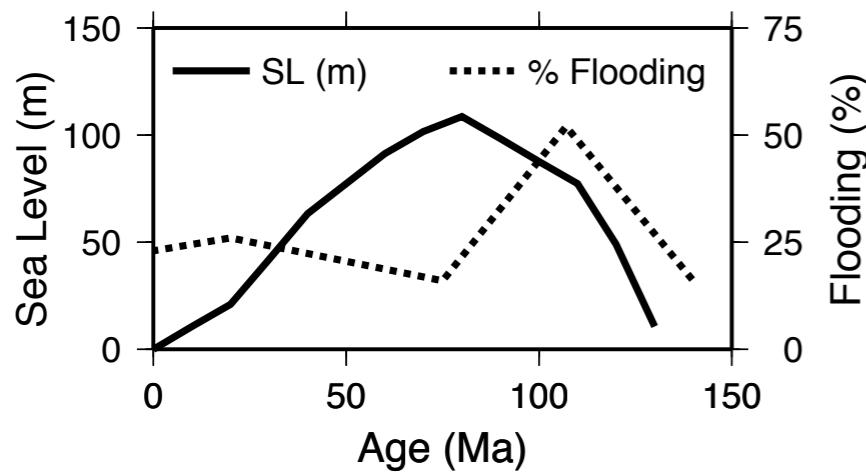
How is the complexity of 3d thermal convection manifest at the surface ? Upwellings, downwellings etc ? How do we see through the lithosphere ?

# Small-scale instabilities of the (continental) lithosphere



Note how the pattern of small-scale downwellings is highly influenced by steps in the lithosphere

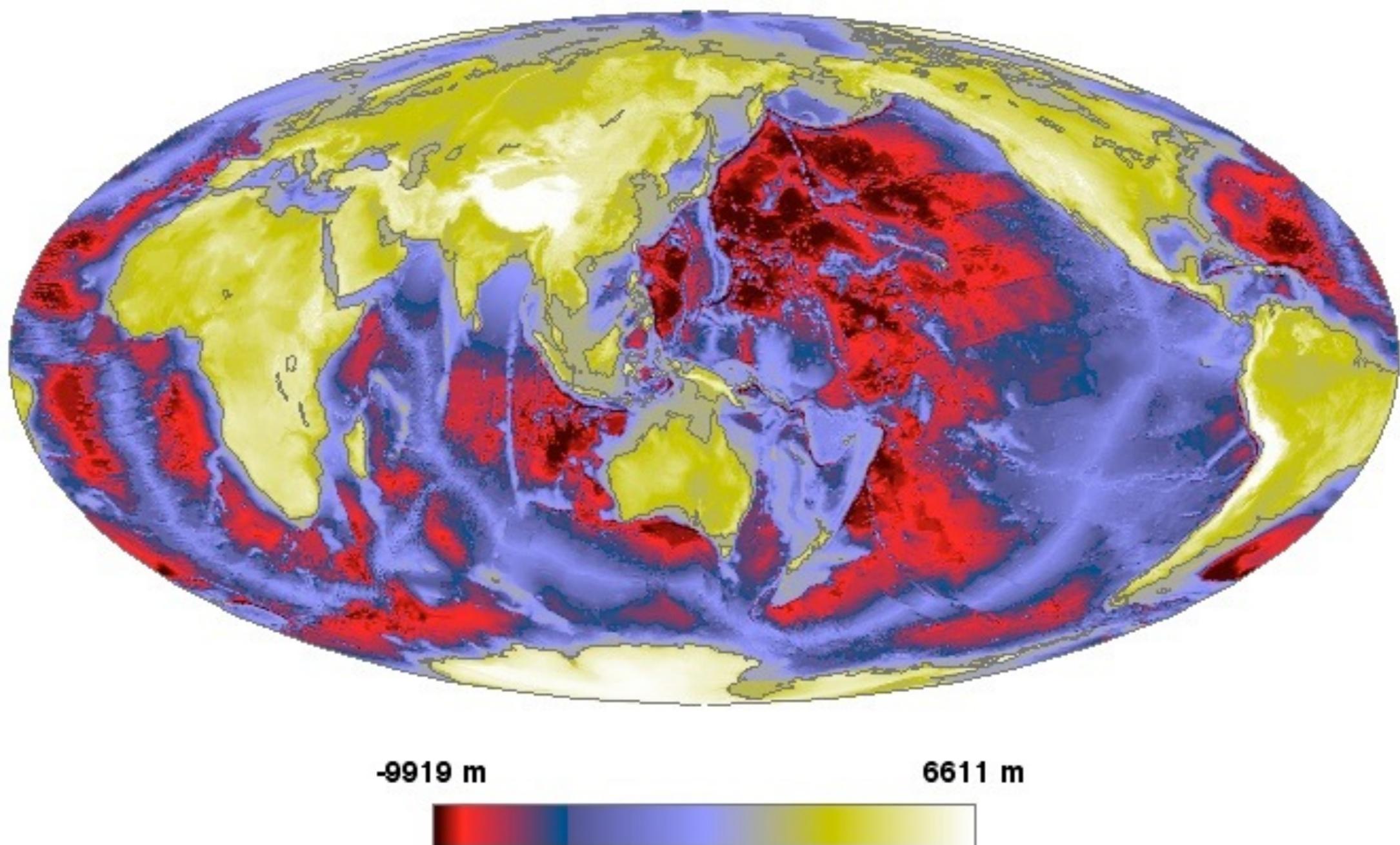
# Messages from the mantle



M. Gurnis, L. Moresi, and R. D. Muller. Models of mantle convection incorporating plate tectonics: the Australian region since the cretaceous. In M. A. Richards, R. Gordon, and R. van der Hilst, editors, AGU Geophysical Monograph 21, pages 211–238. American Geophysical Union, 2000.

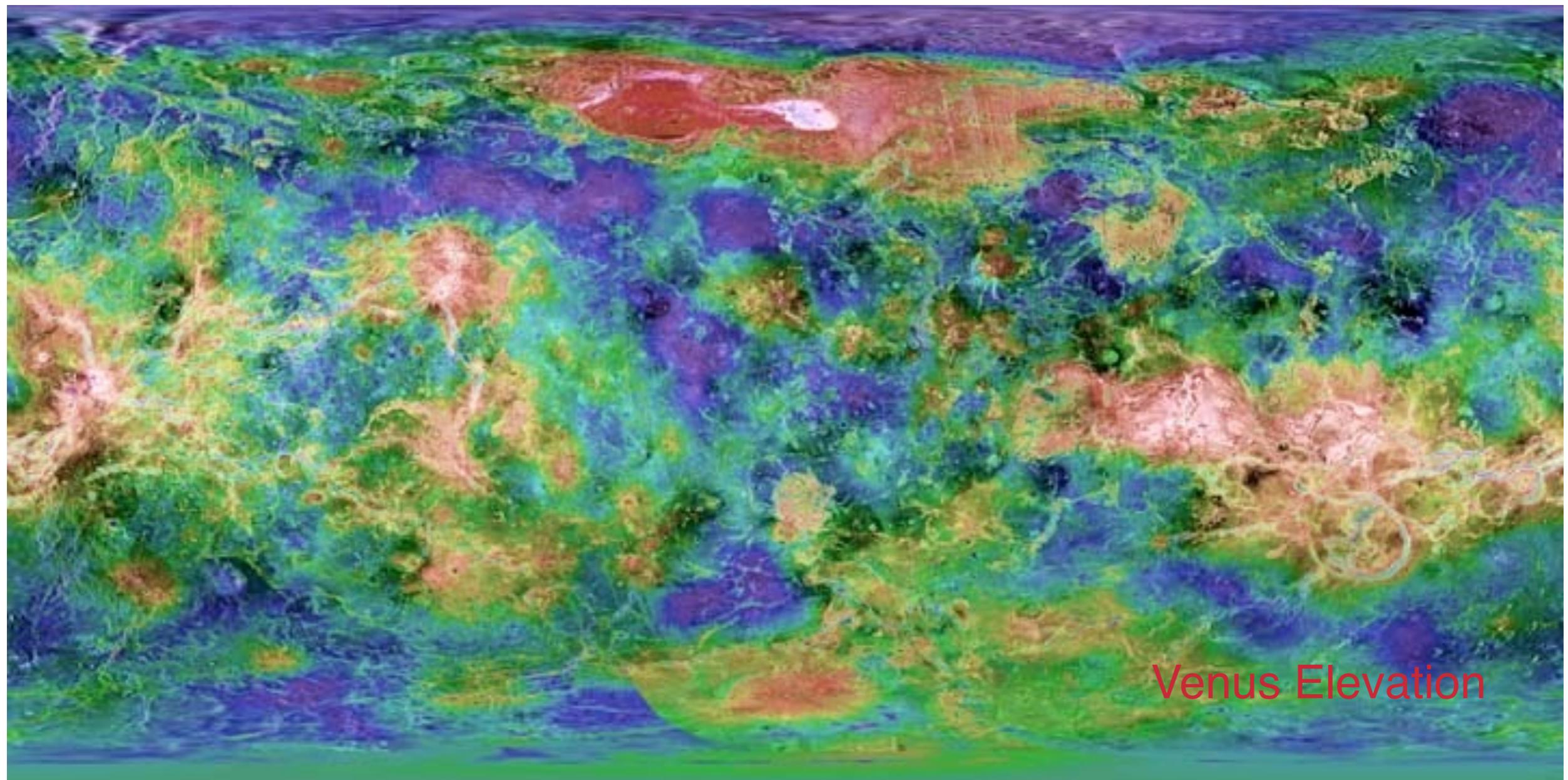
# Surface topography

## ETOPO5 : digital topography



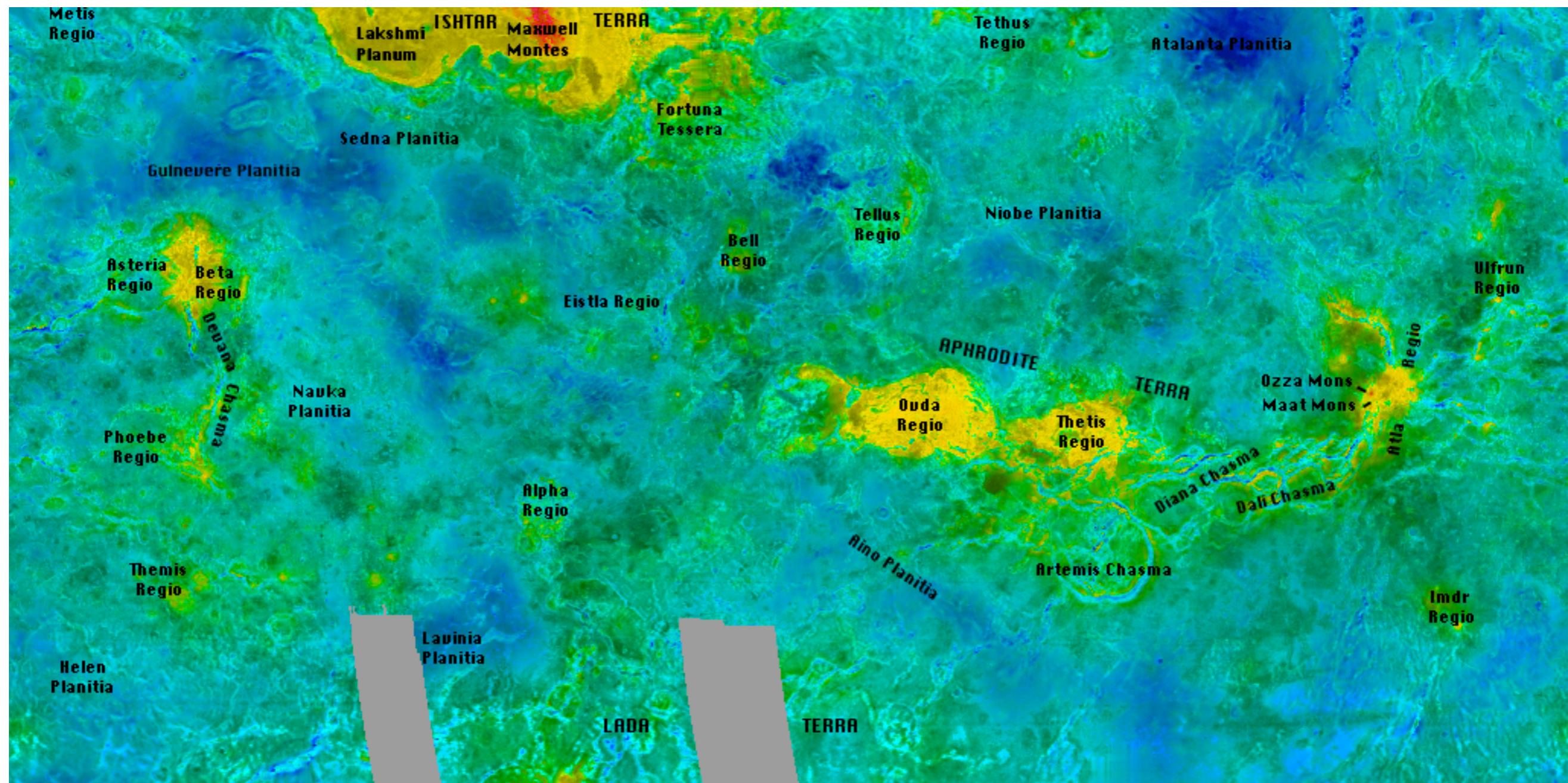
A complex blend of isostasy, strength and mantle dynamics

# Surface topography — Venus

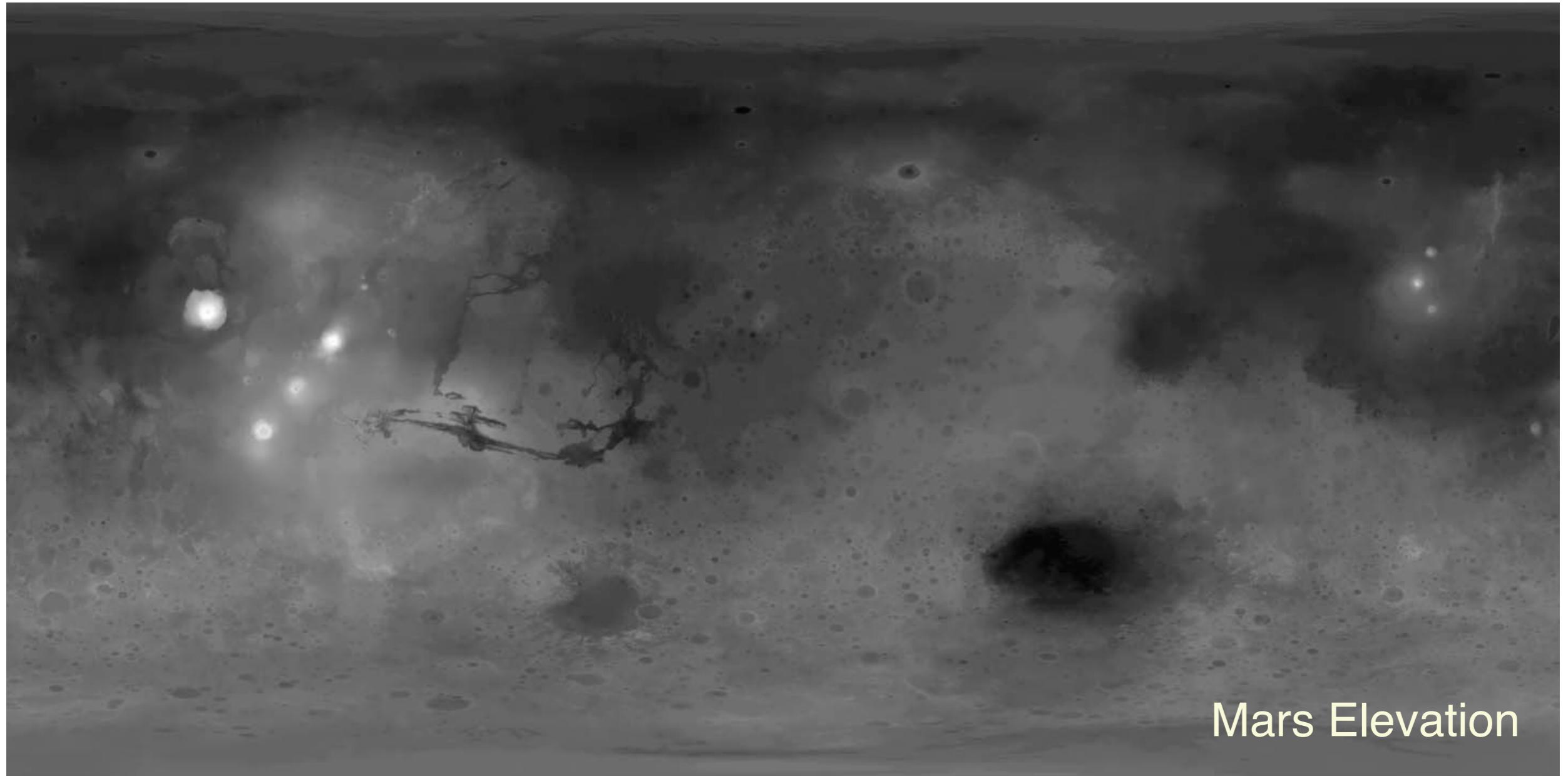


A complex blend of isostasy, strength and mantle dynamics

# Surface topography — Venus

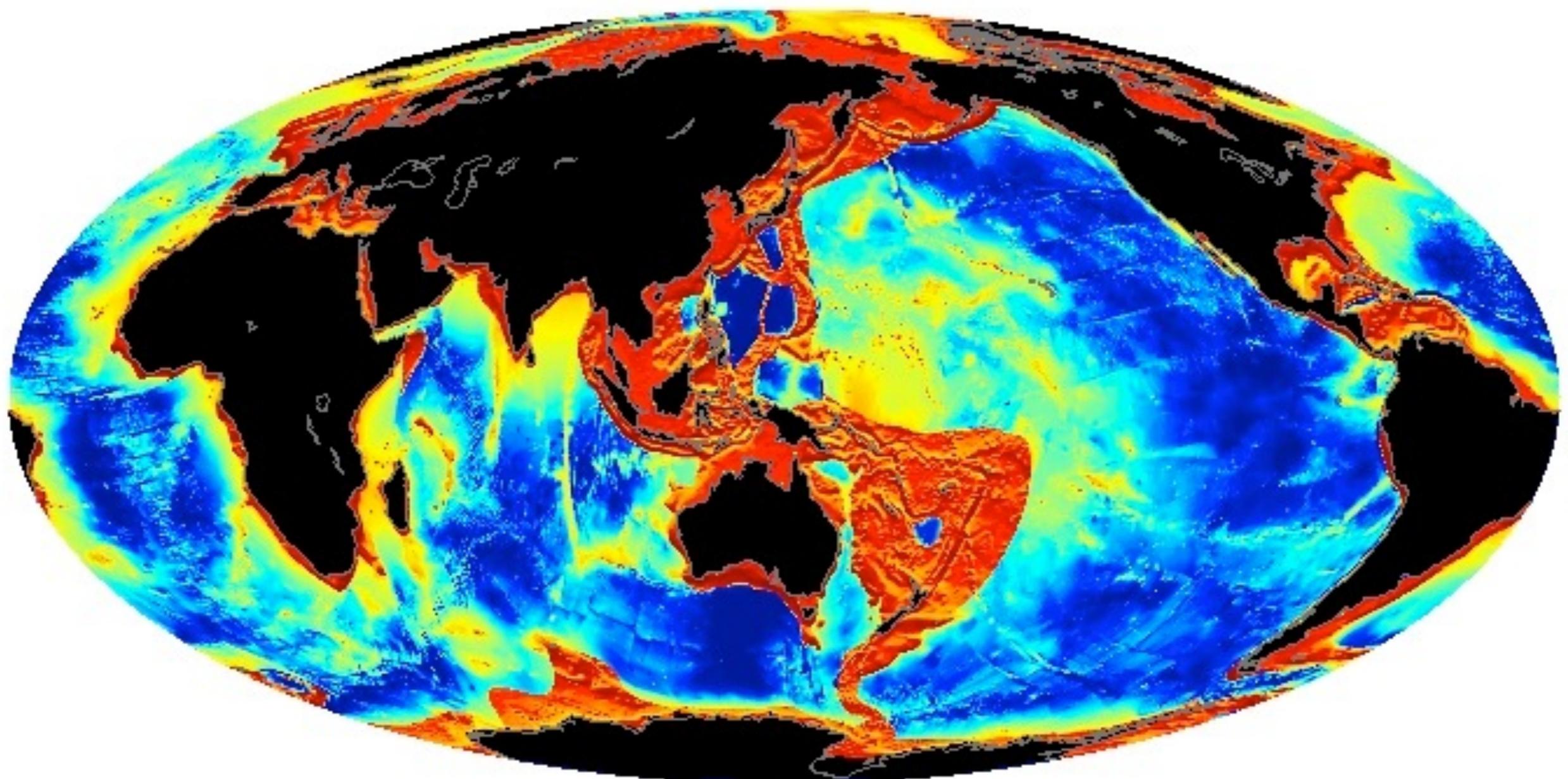


# Surface topography — Mars



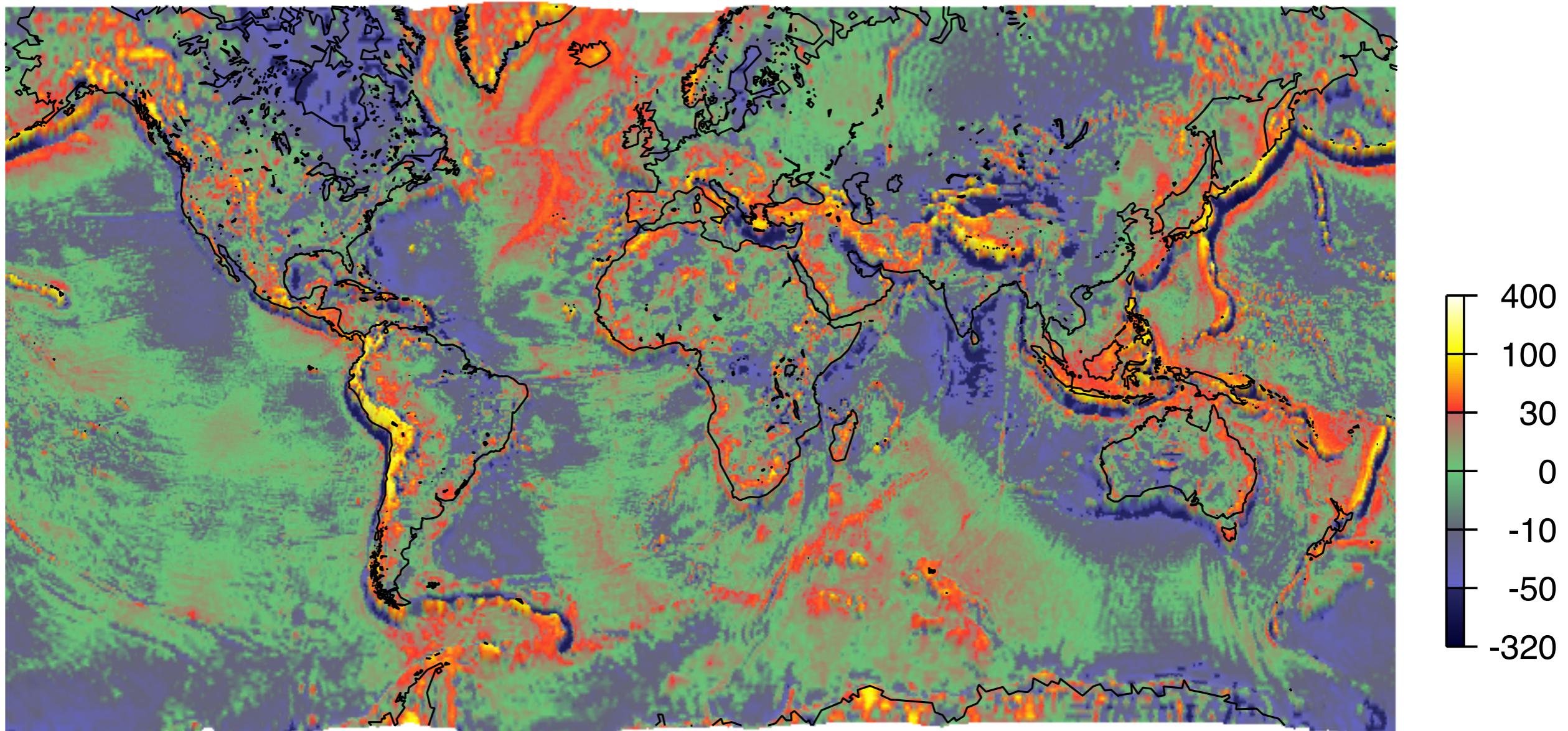
A complex blend of isostasy, strength and mantle dynamics

# Surface topography



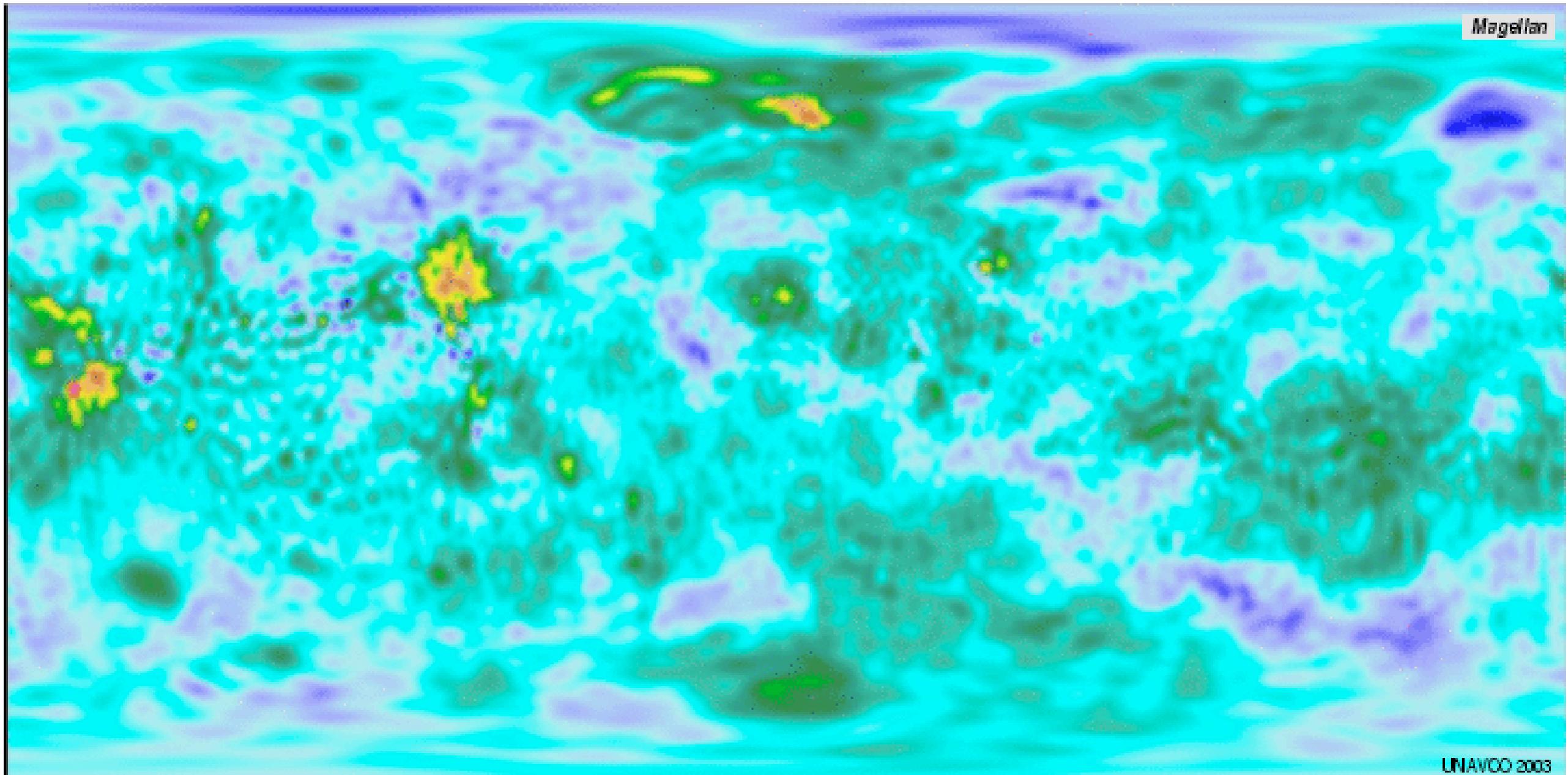
With sediments and depth-age relationships removed

## World Gravity Anomalies (OSU91)



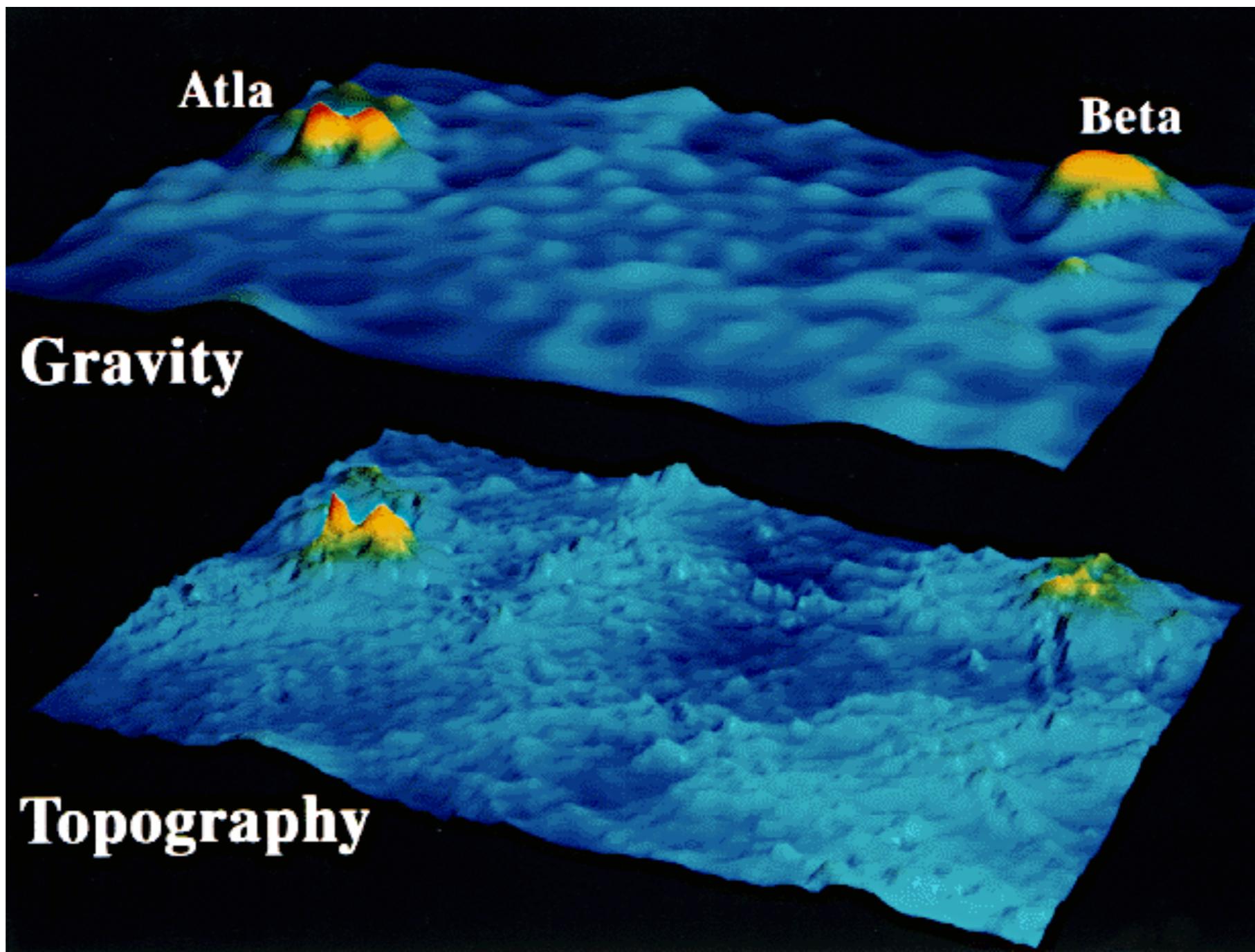
What we see (or don't see ) is mainly isostasy

# Gravity field



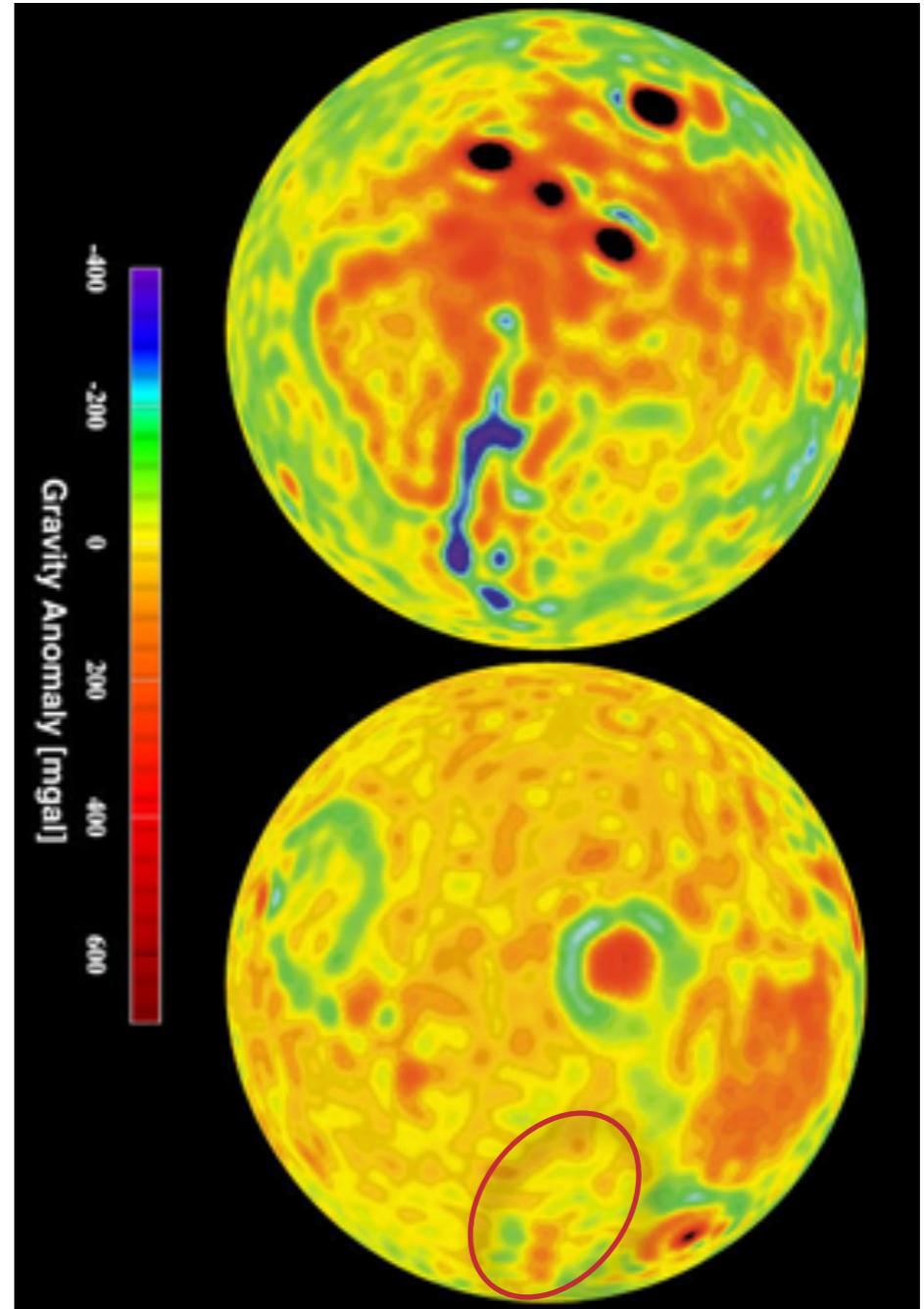
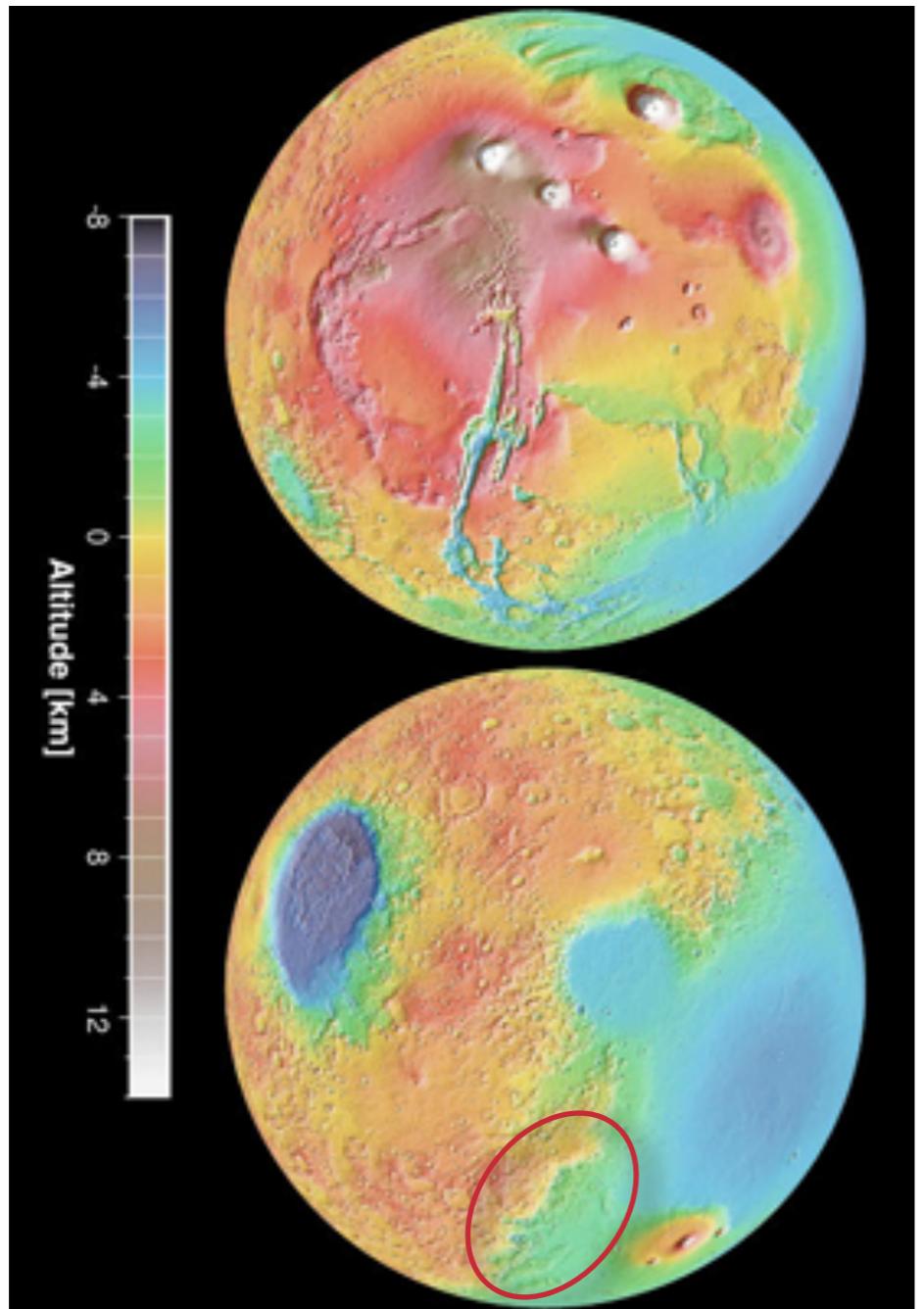
Venus is similar in some ways — whole regions of topography are invisible. But some still stand out strongly.

# Gravity & topography



Relative magnitude of gravity and topography is generally more useful as each has different sensitivity to deep mass, crustal strength, isostatic compensation etc.

# Gravity & topography

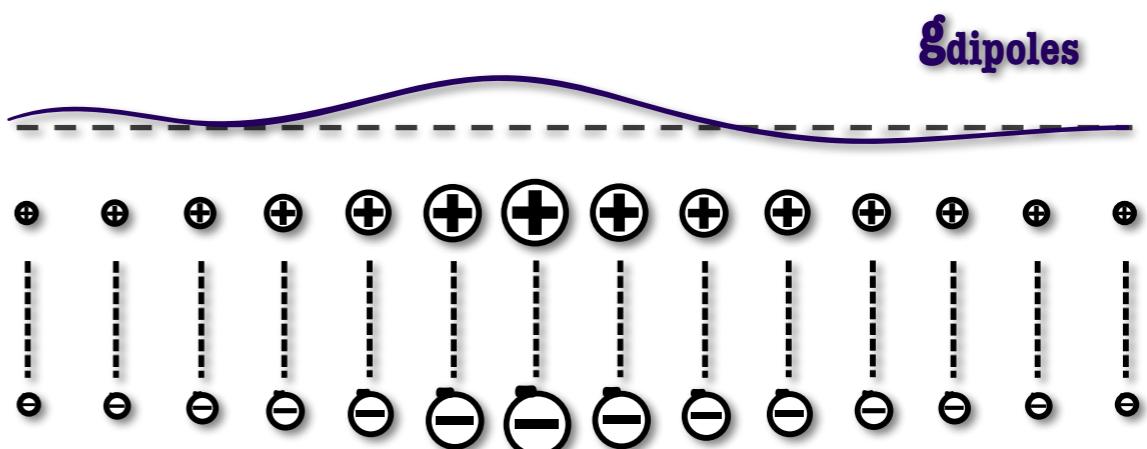
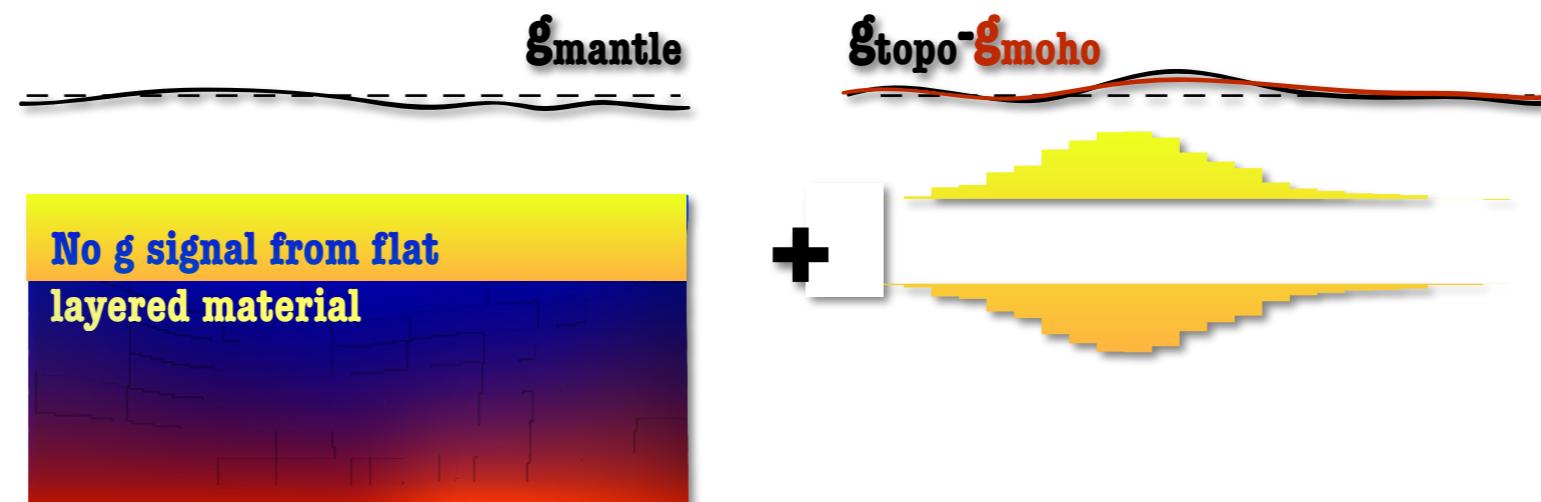
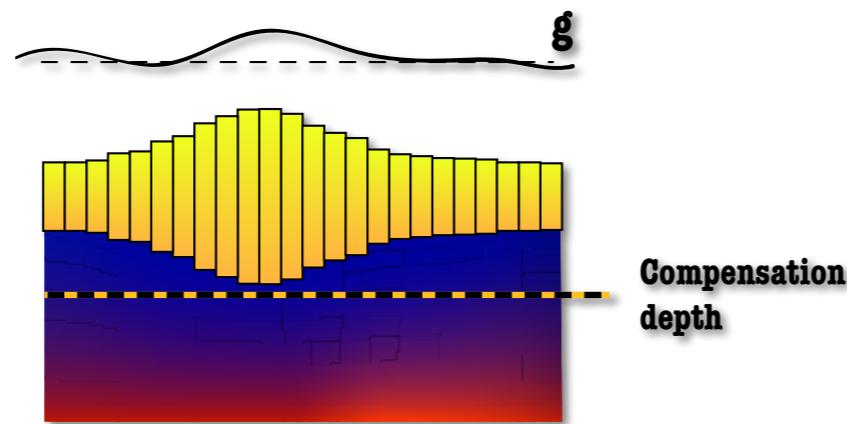


- crustal thickening,
- impact craters

- tharsis rise
- volcanic edifices.

# Gravity anomaly and Isostasy

The lack of gravity (anomaly) signature associated with continental topography on Earth is explained simply by ISOSTASY — wherever there is a topographic high, there is a corresponding crustal root which produces a gravitational dipole



# Geoid

Gravity and topography both reveal the goings on in the interior directly and indirectly. They have very different sensitivities to deep mass.

It is common to use **geoid in place of gravity** since this is relatively easy to obtain over the oceans. It is basically equivalent information to knowledge of the gravity field.

**Geoid:** *The gravitational equipotential which best coincides (in a least squares sense) with mean sea level.*

For other planets some other reference level is required.

Also, consider what happens if mean sea level changes over time.

# Equipotentials

**gravity** – the force of nature which manifests itself as an attraction between objects with mass.

The gravitational force between two objects is

$$F = \frac{Gm_1m_2}{r^2}$$

If one of the bodies is planet sized and spherical then the acceleration experienced by a nearby object is

$$g = \frac{GM}{r^2}$$

It takes energy to lift an object against the force of gravity This produces a gravitational potential energy:

$$\Delta E = GMm \left( \frac{1}{r_2} - \frac{1}{r_1} \right)$$

# Equipotentials

Energy required to take a test mass from some point above a spherical planet out to an infinite distance:

$$U = \frac{GM}{r}$$

Surfaces where  $U$  is constant are spherical. These surfaces are perpendicular to the direction of the gravitational force.

In the realistic, non-spherical planet, the **equipotentials** are not spherical and the field lines are not radial.

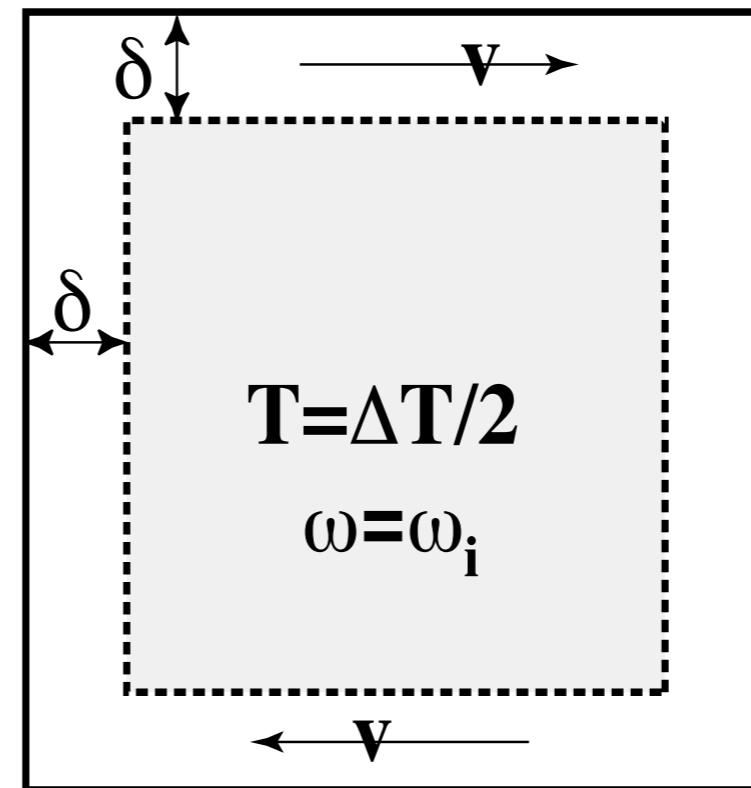
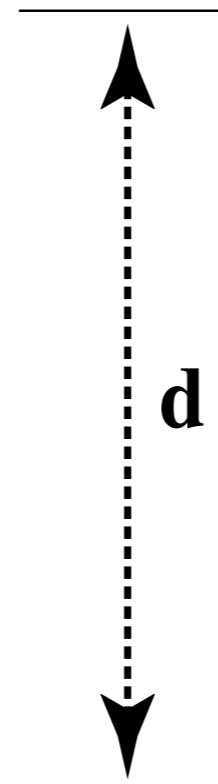
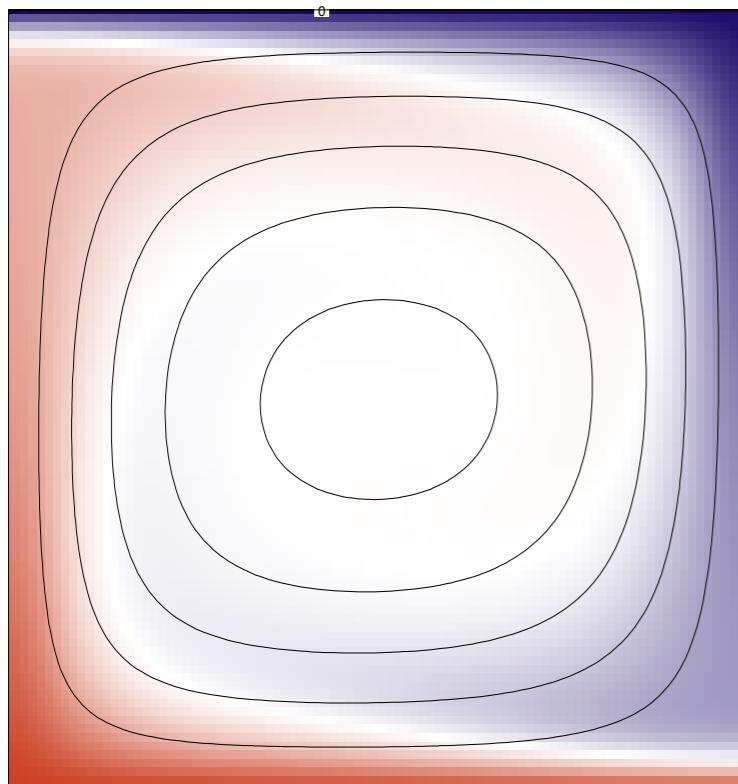
$$g = -\nabla U$$

gradient vector

$U$  contains all the information about the gravitational field  $g$  plus a constant of integration.

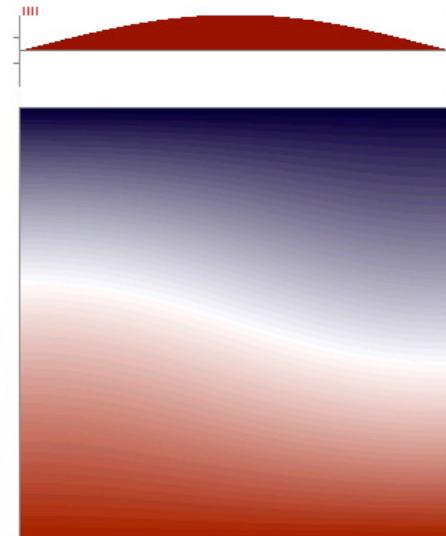
# Boundary layer theory

$\text{Ra} \sim 10^6$



$$\begin{aligned} T &= 0 \\ \omega &= 0 \end{aligned}$$

$$T = \Delta T$$

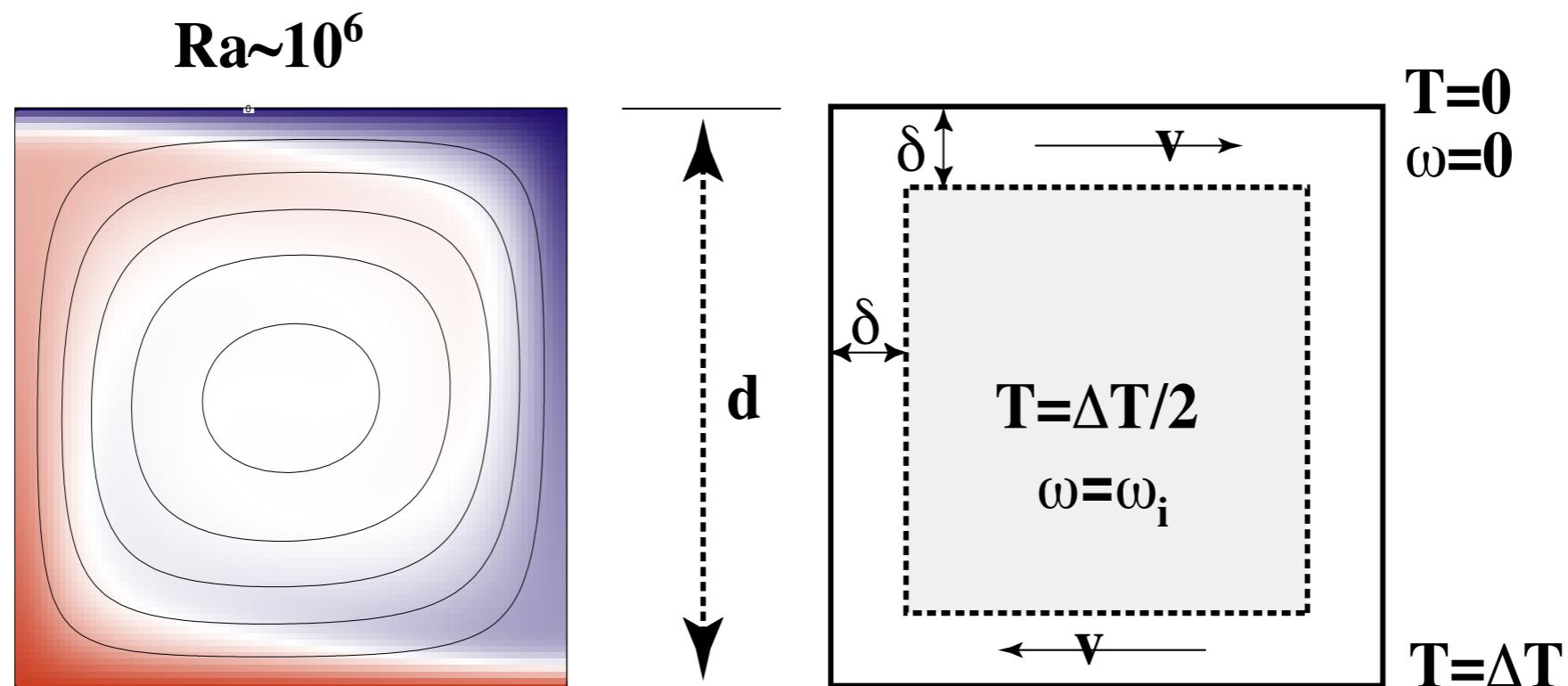


We can analyze simple, constant viscosity convection using boundary layer theory — provided we already have some idea of the geometry of fully developed convection.

# Boundary layer theory

We assume the geometry of fully developed convection is as follows

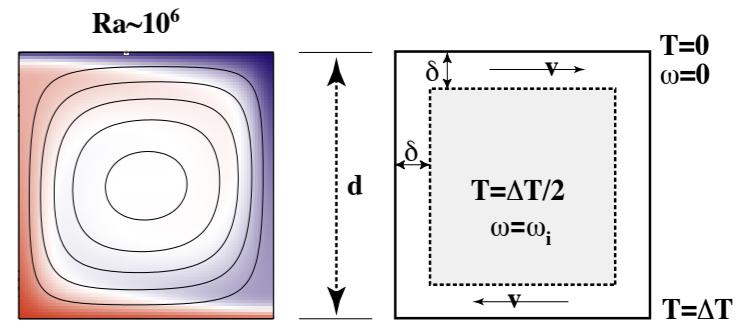
- Narrow thermal boundary layers
- Approximately isothermal core



Thin horizontal boundary layers where heat diffuses into the layer and is carried away by horizontal advection. These are connected by vertical boundary layers of similar dimension.

Gradients along boundary layers are typically on a scale of the layer depth (large so leads to small gradients). Gradients across the boundary layers are typically large because the boundary layers are thin

# Boundary layer theory



For the vertical thermal boundary layer, assume that:

$$\frac{\partial v_2}{\partial x_2} \approx \frac{V}{\delta}; \quad \frac{\partial v_2}{\partial x_1} \approx \frac{V}{d}; \quad v_1 = 0; \quad \frac{\partial T}{\partial x_1} \approx \frac{\Delta T}{\delta}$$

So that

*Stokes equation*

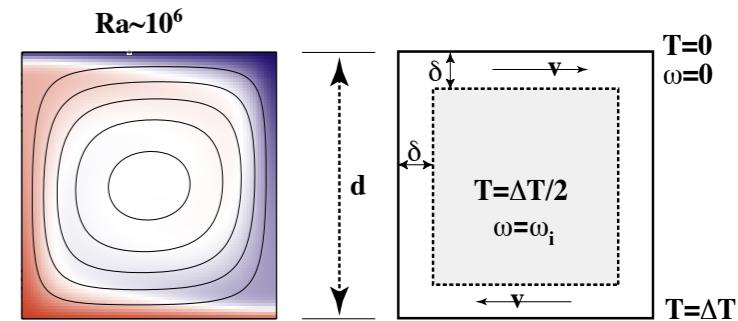
$$\frac{\partial^3 v_2}{\partial x_1^3} - \frac{\partial^3 v_1}{\partial x_2^3} + \frac{\partial^3 v_2}{\partial x_1 \partial x_2^2} - \frac{\partial^3 v_1}{\partial x_2 \partial x_1^2} = \frac{g \rho \alpha}{\eta} \frac{\partial T}{\partial x_1}$$

becomes

$$\cancel{\frac{V}{d^3}} + \frac{V}{d \delta^2} \approx \frac{g \rho \alpha \Delta T}{\eta} \frac{1}{\delta}$$

$$\rightarrow \frac{V d^3}{d \delta^2} \approx \frac{g \rho \alpha d^3 \Delta T}{\eta} \frac{1}{\delta}$$

# Boundary layer theory



Now assume that the system is approximately steady so

$$\kappa \nabla^2 T = (\mathbf{v} \cdot \nabla) T$$

and in the horizontal thermal boundary layer, assume that:

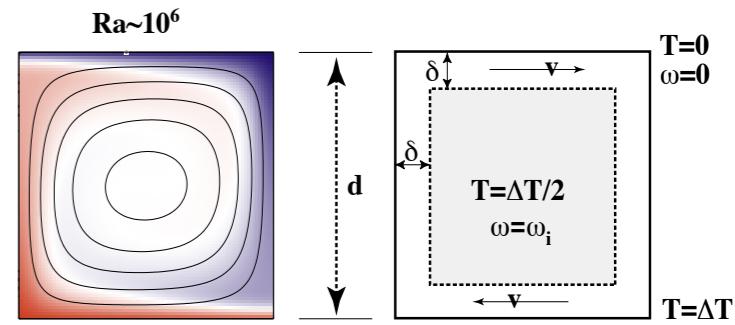
$$v_1 \approx V; \quad v_2 \approx 0; \quad \frac{\partial T}{\partial x_1} \approx \frac{\Delta T}{d}; \quad \frac{\partial T}{\partial x_2} \approx \frac{\Delta T}{\delta};$$

Which produces

$$\kappa \nabla^2 T = (\mathbf{v} \cdot \nabla) T \quad \longrightarrow \quad \frac{v \Delta T}{d} \sim \frac{\Delta T \kappa}{\delta^2}$$

# Boundary layer theory

Eliminating  $v/d \sim \kappa/\delta^2$  gives



$$\frac{\delta}{d} \sim Ra^{-1/3}$$

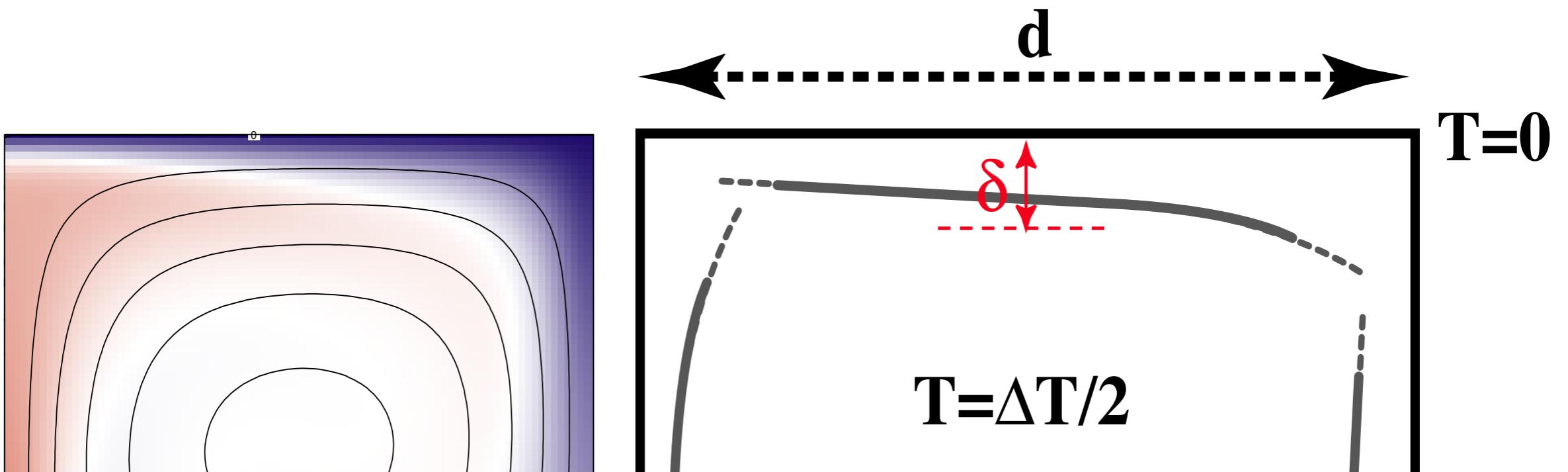
and

$$v \sim \frac{\kappa}{d} Ra^{2/3}$$

This theory balances transfer of momentum and temperature across and out of the boundary layer with advection of each quantity along the boundary layer to maintain a steady state.

$$Nu \sim \frac{\rho C_p v \Delta T \delta}{(k \Delta T / d) d} \sim Ra^{1/3}$$

## Boundary layer theory - version II

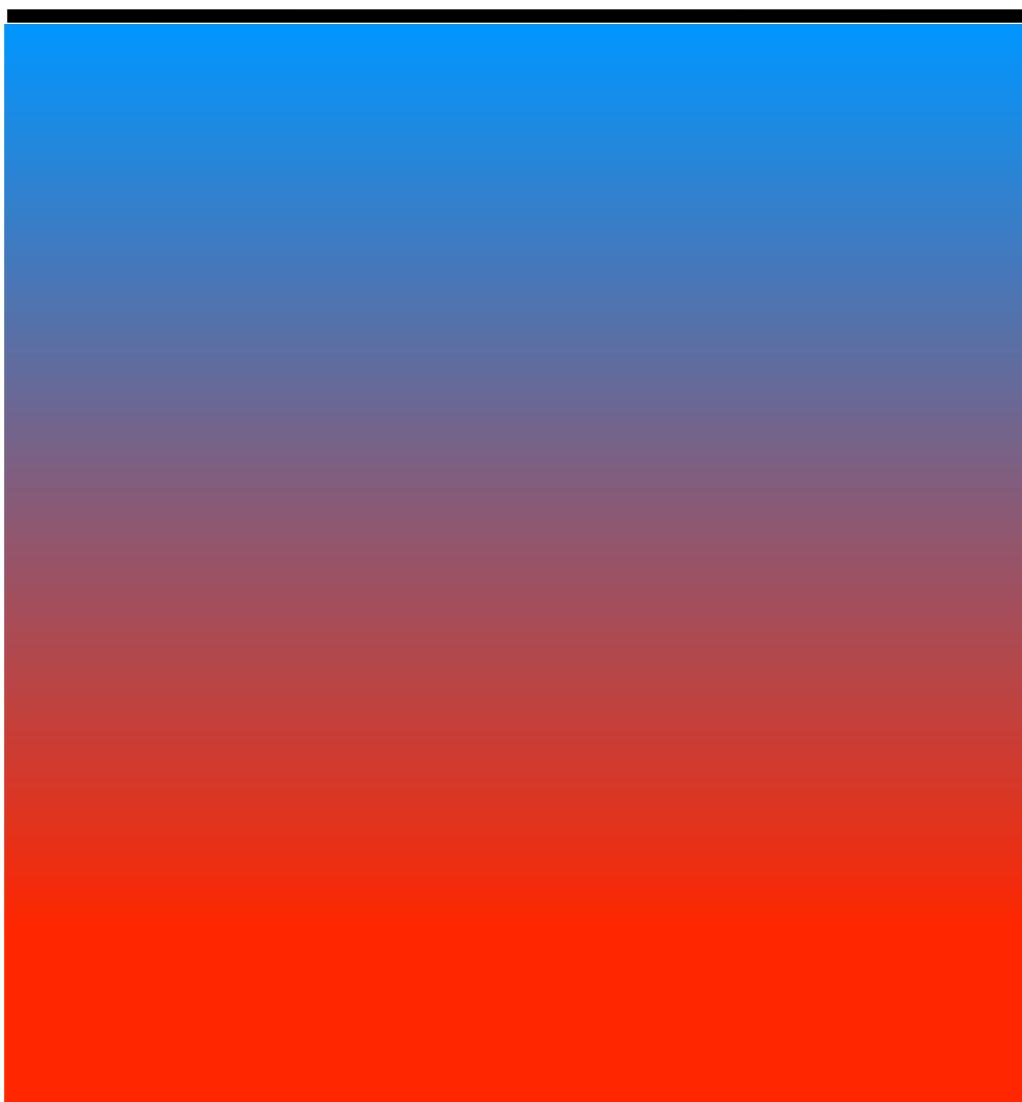


We can also try to account for the variation in the boundary layer thickness as it moves along the horizontal boundary.

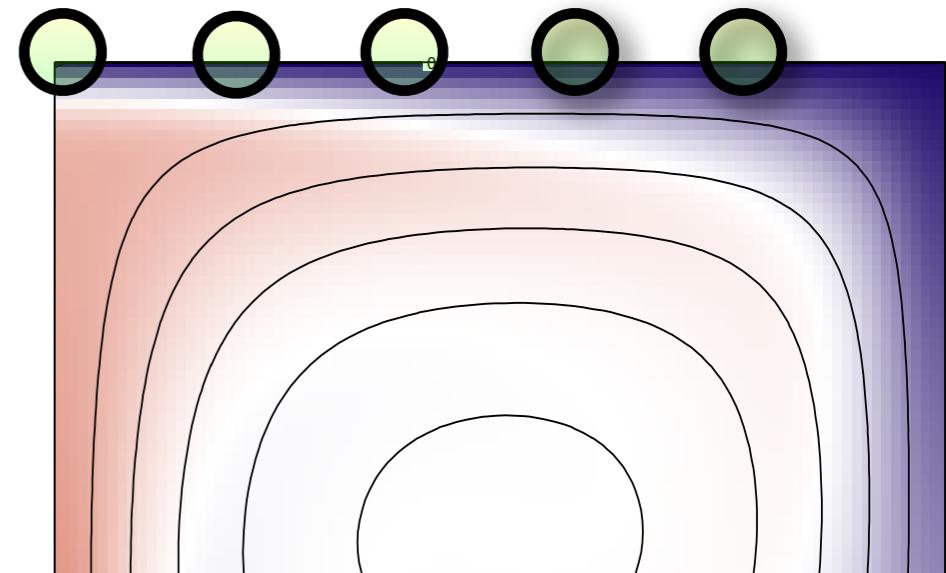
Consider the boundary layer to be very thin above the upwelling plume (left side), thickening as it cools to the right. In a Lagrangian frame this is a 1D diffusion equation.

# Boundary layer model & Cooling 1/2 space

Sudden change of surface temperature

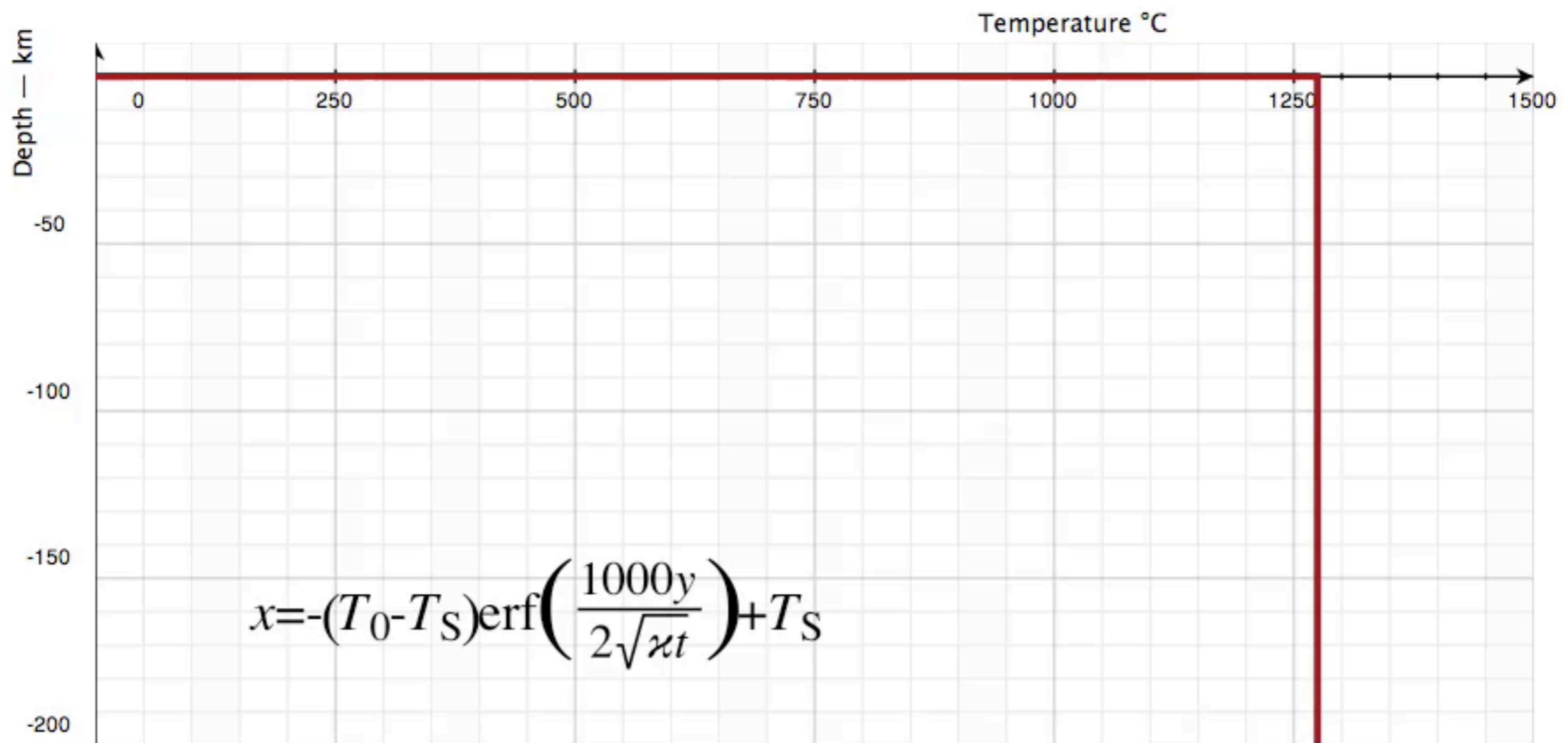


Uniformly warm/cool



# Cooling lithosphere

At any given point riding along on the plate



## Heating/cooling 1/2 space

The solution for the temperature in an infinite half space with a suddenly-imposed change in surface temperature at time  $t = 0$ :

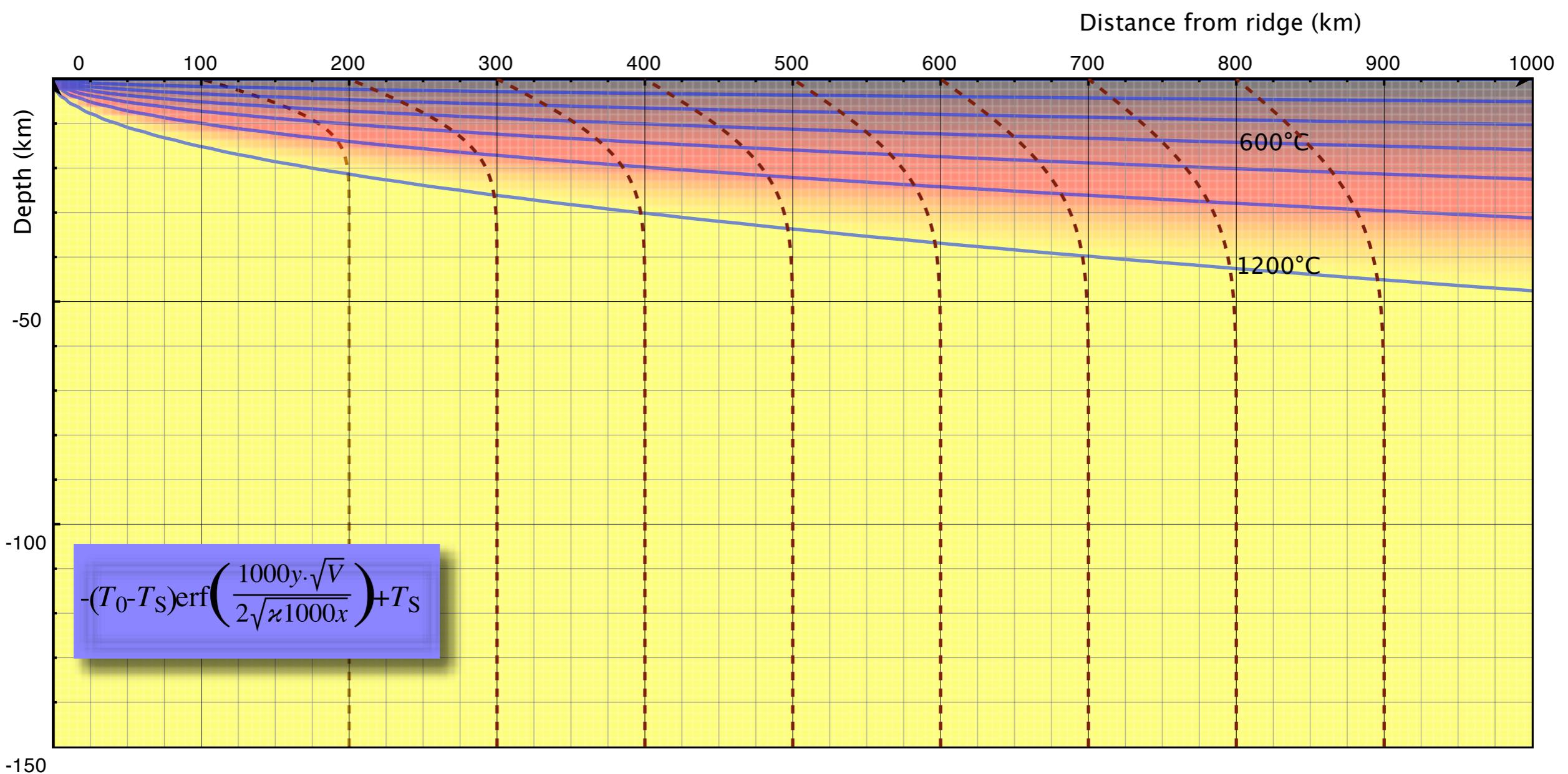
$$\frac{T - T_0}{T_s - T_0} = \operatorname{erfc} \left( \frac{x_2}{2\sqrt{\kappa t}} \right)$$

Heating

$$\frac{T - T_s}{T_0 - T_s} = \operatorname{erf} \left( \frac{x_2}{2\sqrt{\kappa t}} \right)$$

Cooling

# Heating/cooling 1/2 space

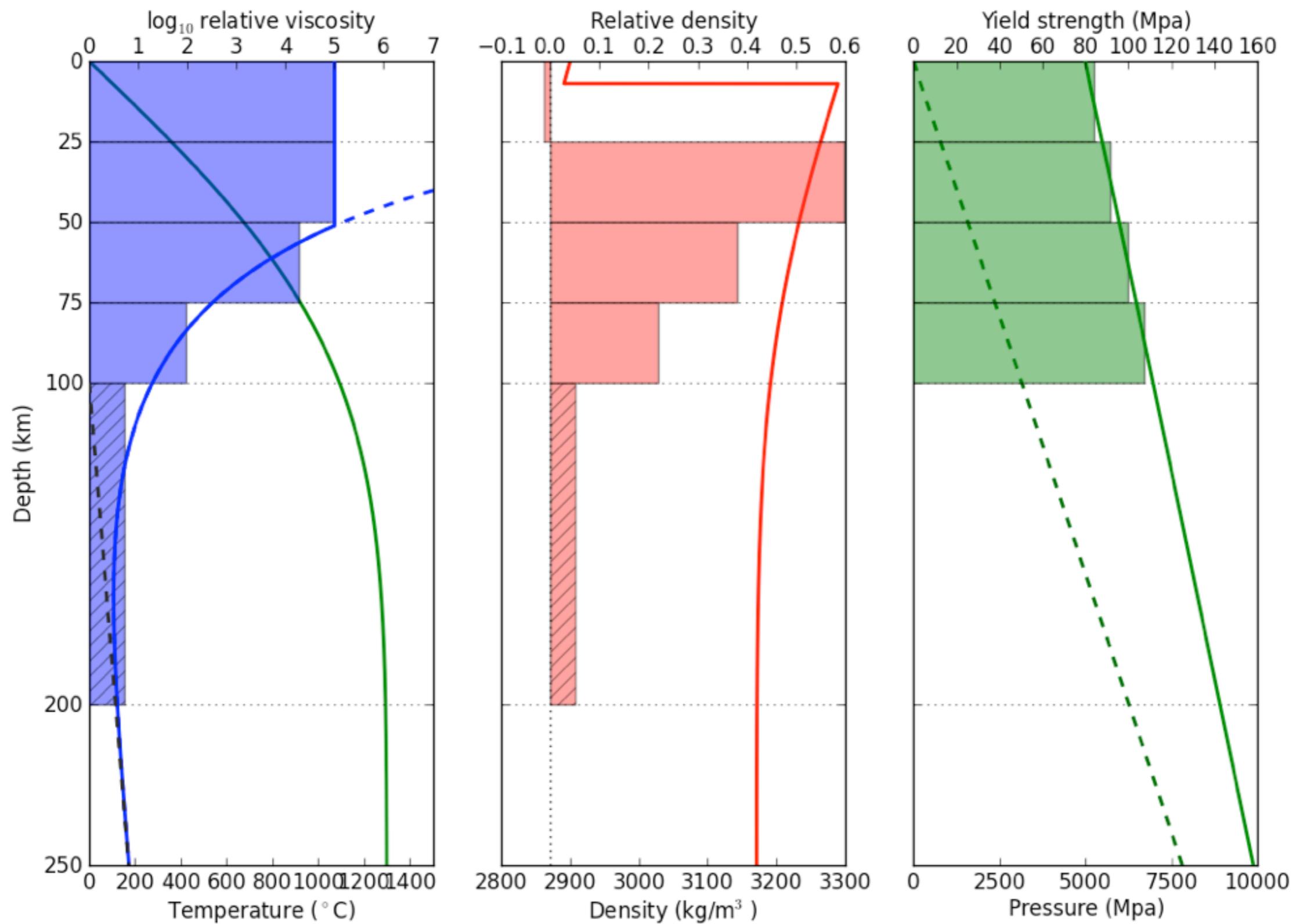


Lithosphere / boundary layer thickness – choose an isotherm

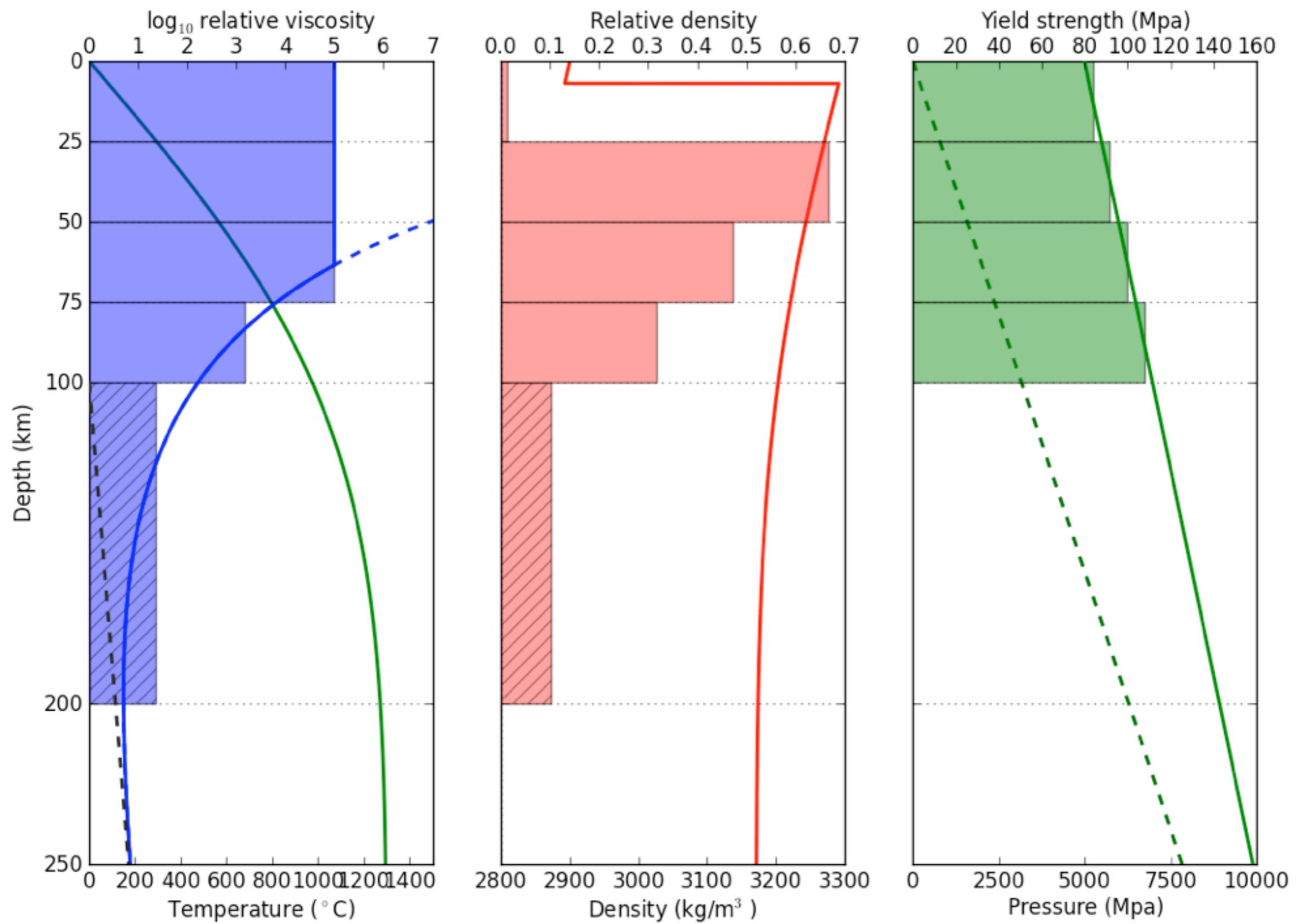
$$\delta \propto \sqrt{\kappa t}$$

$$\delta \propto \sqrt{\kappa x/v}$$

# 80 Myr old lithosphere

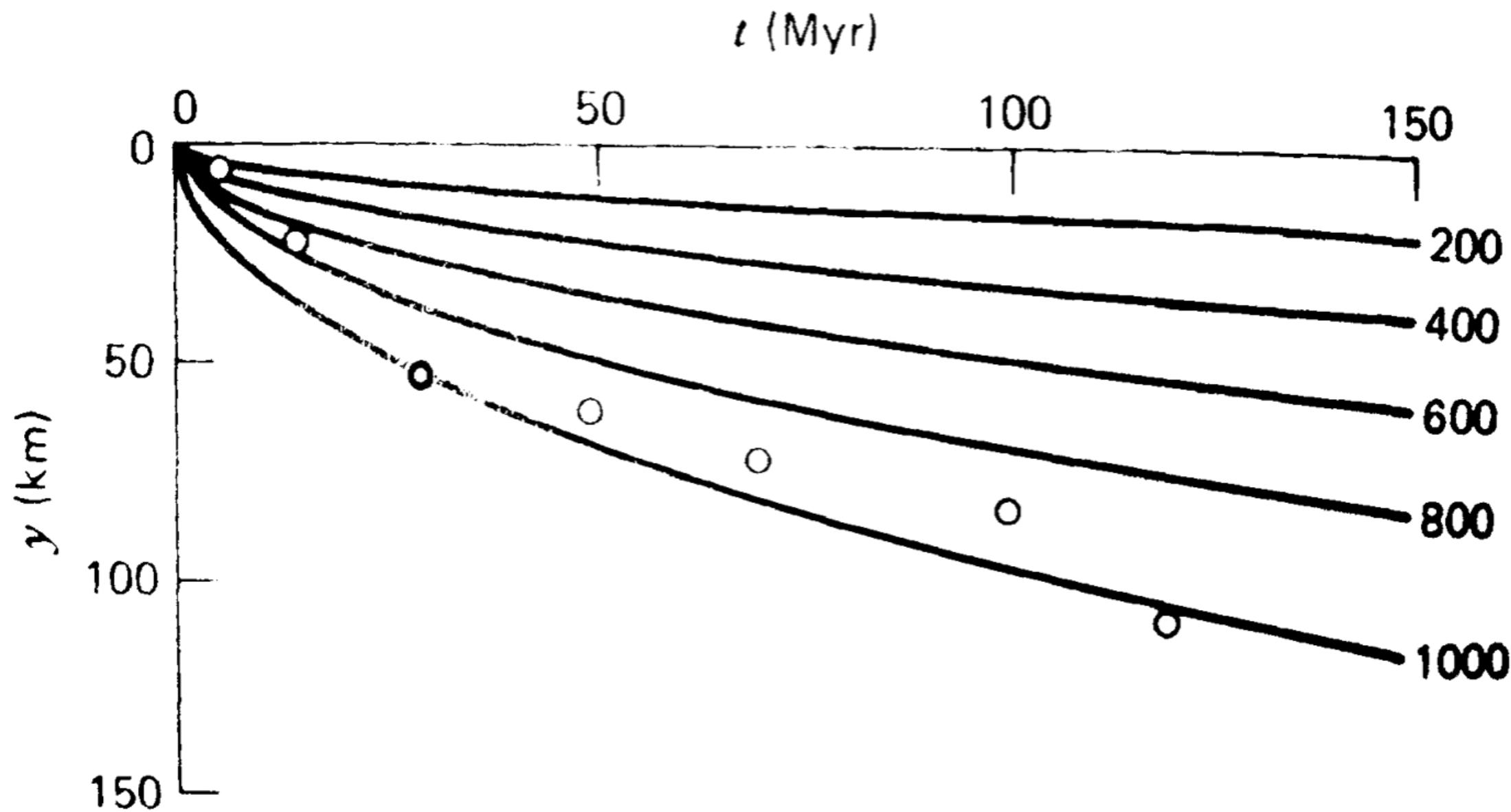


# 120 Myr old lithosphere



# Agreement with observation

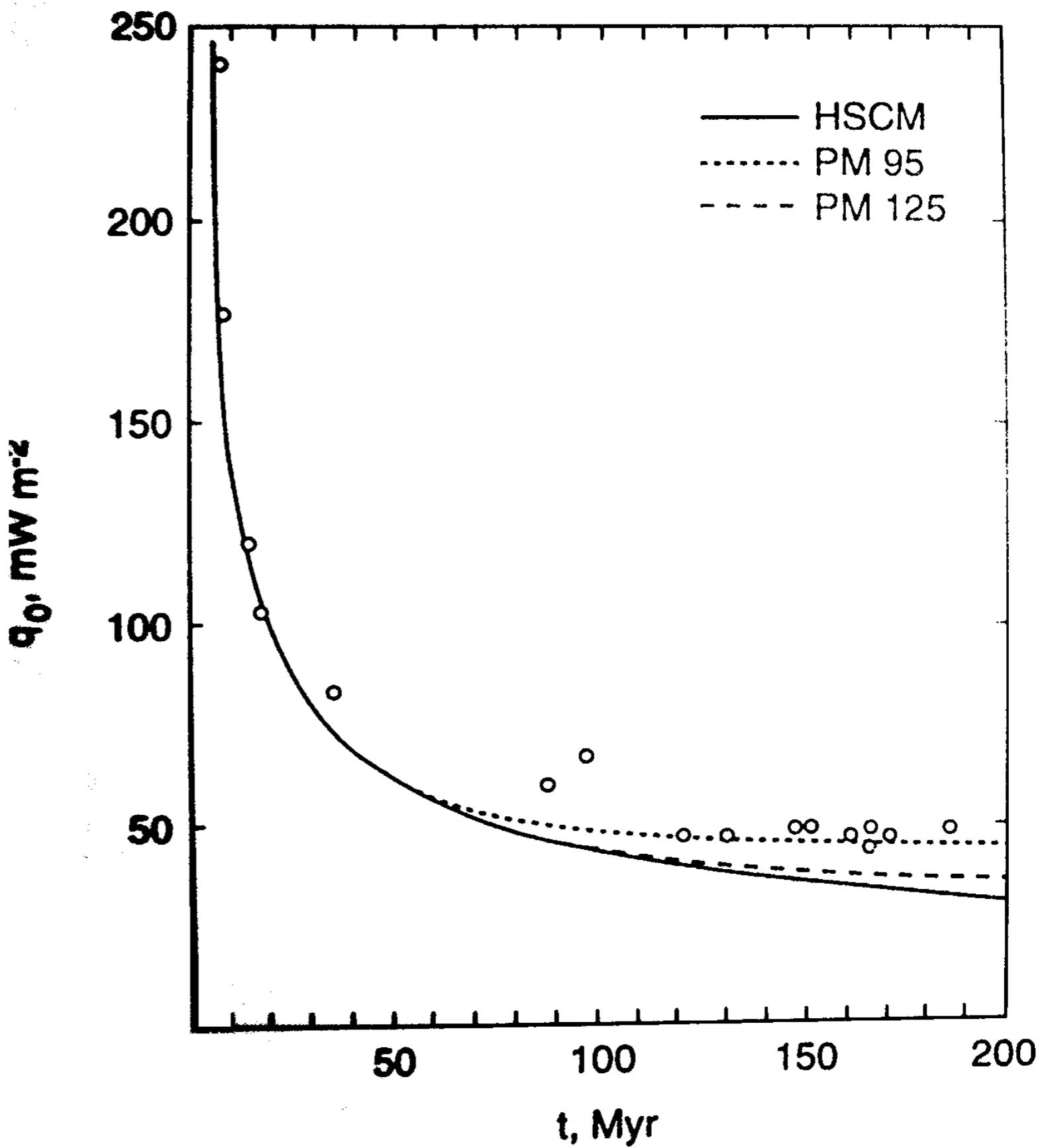
This predicts the thickness of the thermal boundary layer which can be measured by seismology.



# Agreement with observation

We can differentiate the cooling half-space model to give the temperature gradient at the surface, and hence the heat flux.

The oceanic heat flux is hard to measure at very young ages, but heat flux agrees quite well (at least until about 80-100 Myr)

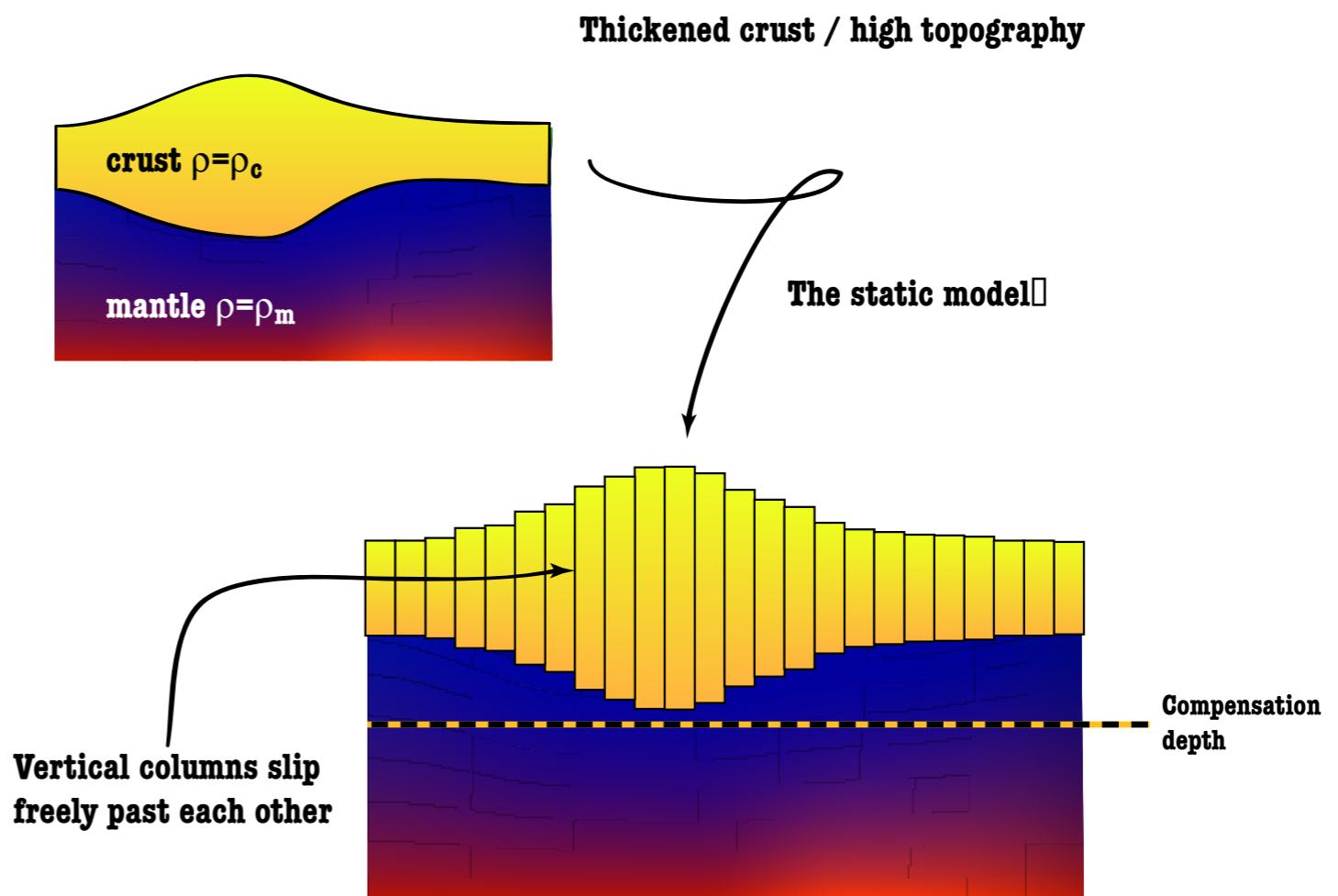


# Sea floor depth calculation

Given a few more assumptions, it is possible to turn thermal cooling into a prediction for seafloor depth.

As the thermal boundary layer cools it contracts which produces a small density change. Isostasy maps this density variation into a surface topography.

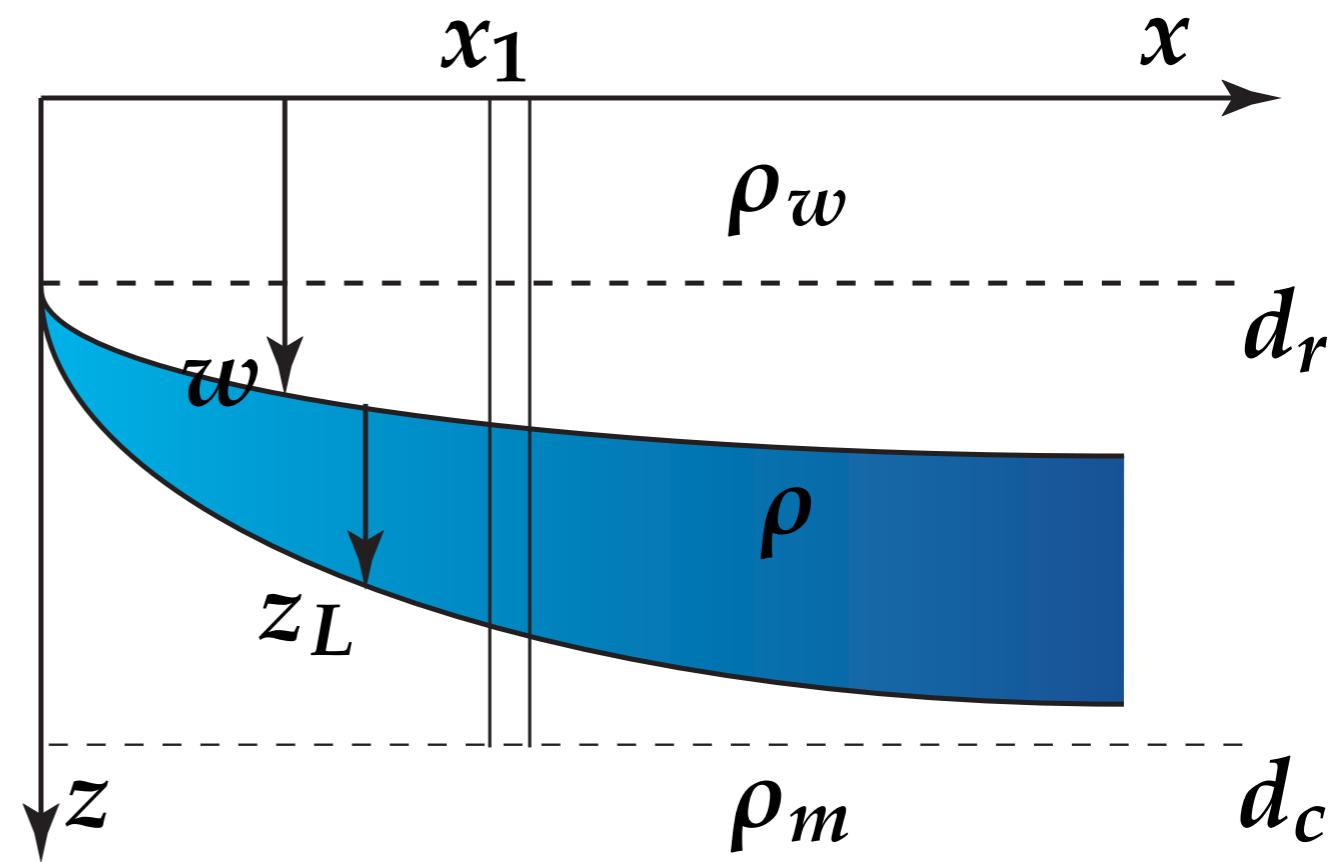
$$\int_0^{d_c} \rho(z) dz = c$$



## Sea floor depth calculation

In the case of the Earth's ocean floor, we integrate downwards from sea level:

$$\int_{d_r}^{d_c} \rho(z) dz = \rho_m(d_c - d_r)$$



Assume no density variations beneath the boundary layer and compare the density column at any  $x$  with the column at  $x = 0$ .

$$w\rho_w + \int_w^{z_L} \rho(z) dz = \rho_m z_L$$

## Sea floor depth calculation

Rearranging, and changing the limits of the integral

$$w(\rho_w - \rho_m) + \int_{d_r}^{z_L} (\rho(z) - \rho_m) dz = 0$$

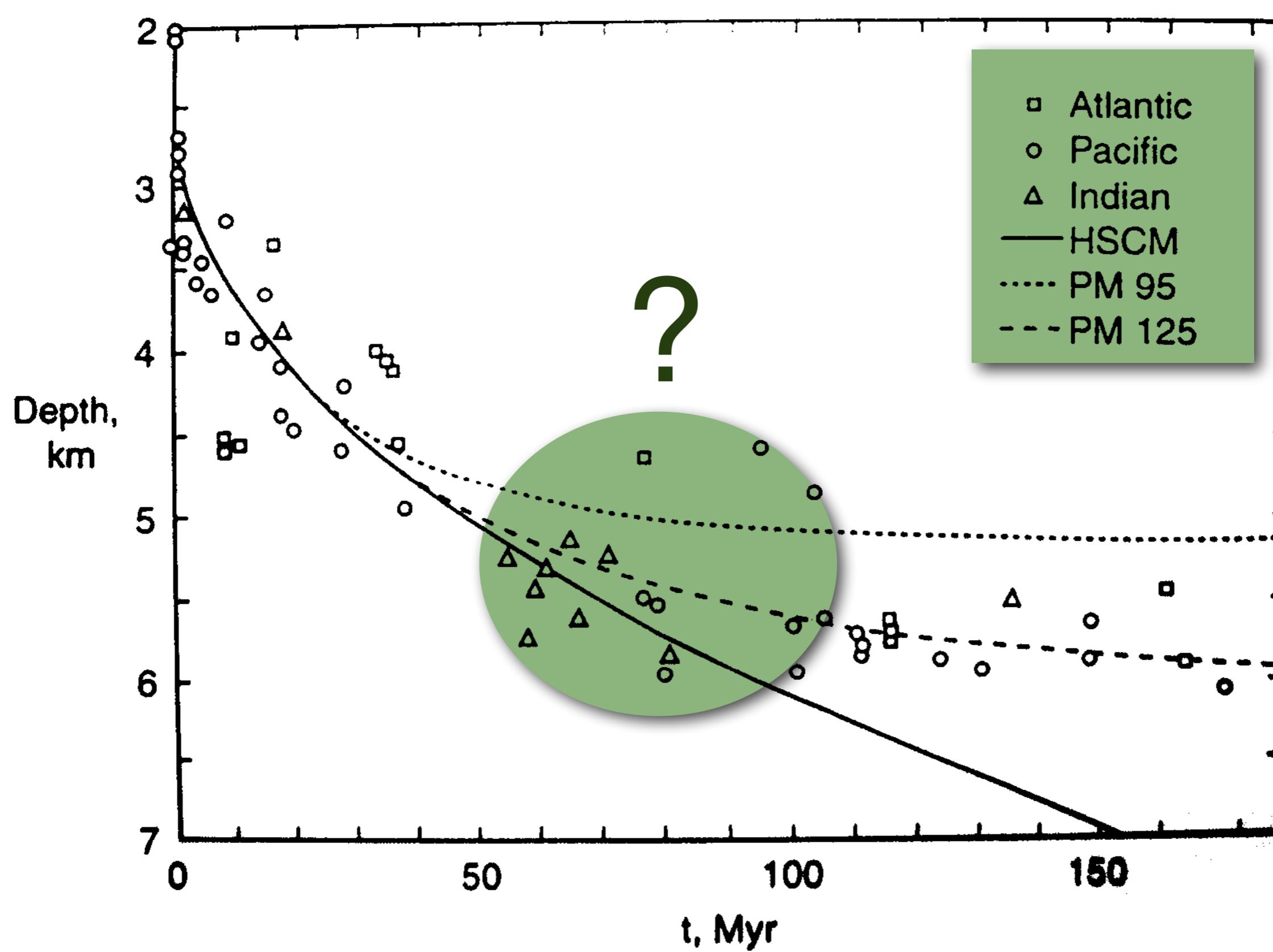
But  $(\rho - \rho_m) = \rho_m \alpha (T_1 - T)$  and  $T(z)$  was calculated earlier:

$$w(\rho_w - \rho_m) = \rho_m \alpha (T_1 - T_0) \int_0^\infty \operatorname{erfc} \left[ \frac{z}{2} \left( \frac{v}{\kappa x} \right)^{1/2} \right]$$

The ocean depth as a function of seafloor age is

$$w = \frac{2\rho_m \alpha (T_1 - T_0)}{\rho_m - \rho_w} \left[ \frac{\kappa x}{\pi v} \right]^{1/2}$$

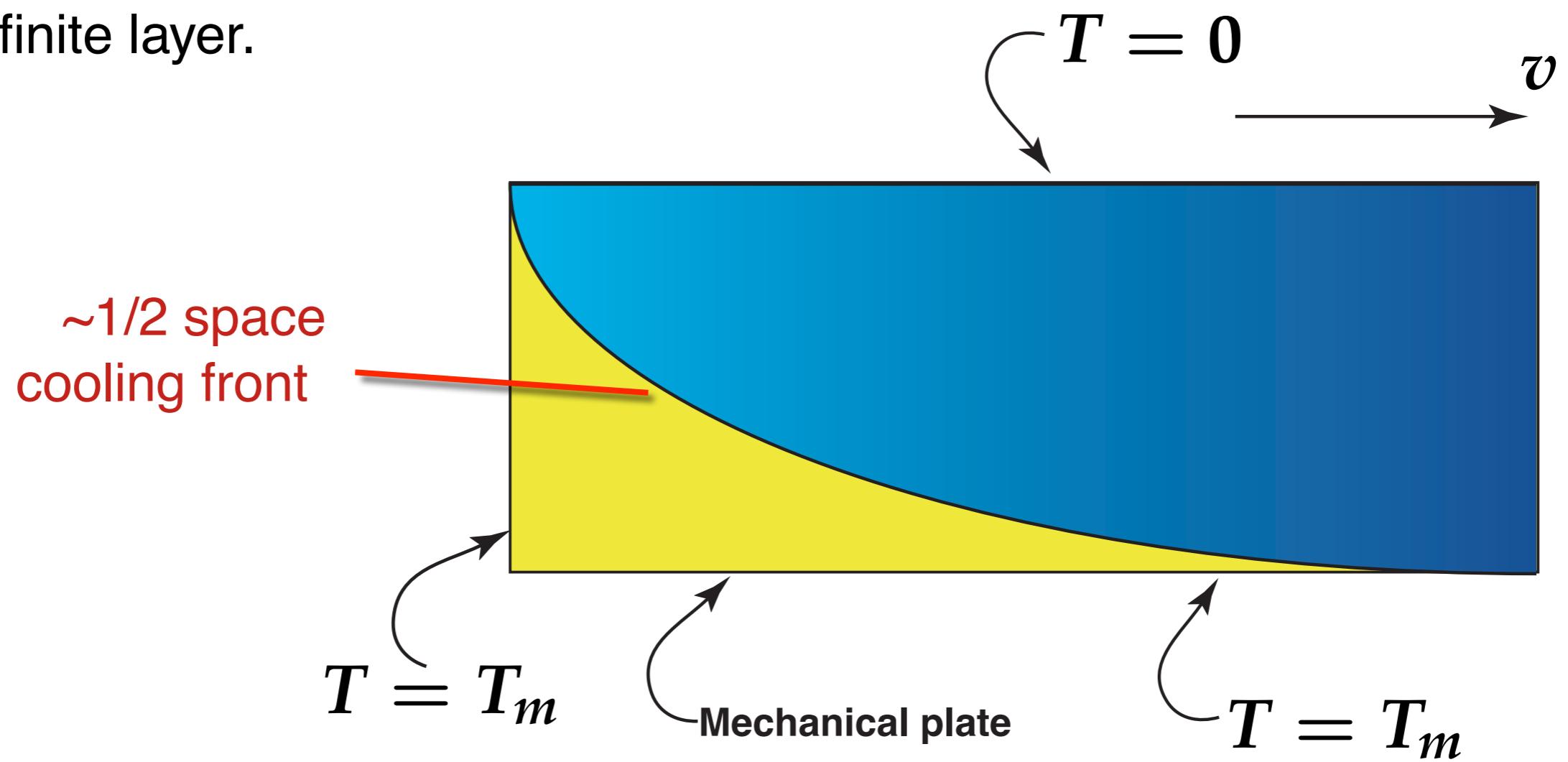
## Agreement with observations



## The plate model

The half-space cooling model holds until about 80Myr. After that the lithosphere thickness appears to plateau. This requires additional heat into the lithosphere at depth.

One suggestion was that the plates are the **fundamental** objects in the system which only cool to a certain thickness. They are modeled as mechanically distinct — this is not a half space but a finite layer.



## The plate model

Solve for a static plate initially:

$$\begin{aligned} T &= T_m \quad \text{at} \quad t = 0 & 0 \leq z \leq z_{\text{plate}} \\ T &= T_0 \quad \text{at} \quad z = 0 & t > 0 \\ T &= T_m \quad \text{at} \quad z = z_{\text{plate}} & t > 0 \end{aligned}$$

Solution has the form

$$T = T_0 + (T_m - T_0) \left[ \frac{z}{z_{\text{plate}}} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \exp \left( -\frac{\kappa n^2 \pi^2 t}{z_{\text{plate}}^2} \right) \sin \left( \frac{n \pi z}{z_{\text{plate}}} \right) \right]$$

## The plate model

At small times this reverts to the cooling half space solution, and at large times to an equilibrium profile:

$$T = T_0 + (T_m - T_0) \frac{z}{z_{\text{plate}}}$$

The surface heat flow at large times is

$$q_{0\infty} = \frac{k(T_m - T_0)}{z_{\text{plate}}}$$

i.e. the heat flux tends to a non-zero limit.

# Boundary layer theory: summary

- Relates quantities such as Nusselt number, typical velocity to Rayleigh number — scaling laws.
- Gives a 1D picture of the convection cell — ability to transfer heat or momentum as a "black-box"
- Leads to the possibility of parameterized convection — thermal history models for planets which do not require full solution of convection equations.
- Extension to compute surface topography and heat flow for plates.