GATE – 2003

MATHEMATICS

Time Allowed : 3 Hours *Maximum Marks : 150*

Read the following instructions carefully

1. This question paper contains 90 objective questions. Q. 1—30 carry one mark each and Q. 31—90 carry two marks each.
2. Answer all the questions.
3. Questions must be answered on special machine gradable Objective Response Sheet (**ORS**) by darkening the appropriate bubble (marked A, B, C, D) using HB pencil against the question number on the left hand side of the **ORS**. Each question has only one correct answer. In case you wish to change an answer, erase the old answer completely using a good soft eraser.
4. There will be **Negative** marking. For each wrong answer **0.25** mark from Q. 1—30 and **0.5** mark from Q. 31—90 will be deducted. More than one answer marked against a question will be deemed as an incorrect response and will be negatively marked.
5. Write your registration number, name and name of the Centre at the specified locations on the right half of the **ORS**.
6. Using HB pencil, darken the appropriate bubble under each digit of your registration number.
7. Using HB pencil, darken the appropriate bubble under the letters corresponding to your paper code.
8. No charts or tables are provided in the examination hall.
9. Use the blank pages given at the end of the question paper for rough work.
10. Choose the **Closest** numerical answer among the choices given.
11. This question paper contains 18 pages. Please report if there is any discrepancy.

Q. 1–30 CARRY ONE MARK EACH

The symbols N, Z, Q and R denote the set of natural numbers, integers, rational numbers and real numbers respectively.

1. ***Let T be an arbitrary linear transformation from R^n to R^n which is not one-one. Then***

   (a) Rank T > 0 (b) Rank T = n

   (c) Rank T < n (d) Rank T = $n - 1$

2. ***Let T be a linear transformation from $R^3 \to R^2$ defined by $T(x, y, z) = (x + y, y - z)$. Then the matrix of T with respect to the ordered bases {(1, 1, 1), (1, –1, 0), (0, 1, 0)} and {(1, 1), (1, 0)} is***

   (a) $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

   (c) $\begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}$

3. ***Let the characteristics equation of a matrix M be $\lambda^2 - \lambda - 1 = 0$, then***

   (a) M^{-1} does not exist

   (b) M^{-1} exists but cannot be determined from the data

   (c) $M^{-1} = M + 1$

   (d) $M^{-1} = M - 1$

4. ***Consider a function $f(z) = u + iv$ defined on $|z - i| < 1$ where u, v are real valued functions of x, y. Then f(z) is analytic for u equals to***

   (a) $x^2 + y^2$ (b) $ln\,(x^2 + y^2)$

   (c) e^{xy} (d) $e^{x^2 - y^2}$

5. ***At z = 0, the function $f(z) = z^2 \bar{z}$***

   (a) does not satisfy Cauchy – Reimann equations

   (b) satisfies Cauchy – Reimann equations but is not differentiable

   (c) is differentiable

   (d) is analytic

6. ***The bilinear transformation w, which maps the points 0, 1, ∞ in the z-plane onto the points –i, ∞, 1 in the w-plane is***

   (a) $\dfrac{z-1}{z+i}$ (b) $\dfrac{z-i}{z+1}$

   (c) $\dfrac{z+i}{z-1}$ (d) $\dfrac{z+1}{z-i}$

7. ***The continuous function $f: R \to R$ defined by $f(x) = (x^2 + 1)^{2003}$ is***
   (a) onto but not one-one (b) one-one but not onto
   (c) both one-one and onto (d) neither one-one nor onto

8. ***Diameter of a set S in a metric space with metric d is defined by***
$$diam\ (S) = l.u.b.\ \{d(x, y) \mid x, y \text{ in } S\}$$
Thus, diameter of the cylinder $C = \{(x, y, z) \text{ in } R^3 \mid x^2 + y^2 = 1, -1 < z < 1\}$ in R^3 with standard metric, is
   (a) 2 (b) $2\sqrt{2}$
   (c) $\sqrt{2}$ (d) $\pi + 2$

9. ***Let $X = (0, 1) \cup (2, 3)$ be an open set in R. Let f be a continuous function on X such that the derivative $f'(x) = 0$ for all x. Then the range of f has***
   (a) uncountable number of points
   (b) countably infinite number of points
   (c) at most 2 points
   (d) at most 1 point

10. ***The orthogonal trajectory to the family of circles $x^2 + y^2 = 2\ cx$ (c arbitrary) is described by the differential equation.***
   (a) $(x^2 + y^2)\, y' = 2xy$ (b) $(x^2 - y^2)\, y' = 2xy$
   (c) $(y^2 - x^2)\, y' = xy$ (d) $(y^2 - x^2)\, y' = 2xy$

11. ***Let $y_1(x)$ and $y_2(x)$ be solutions of $y''x^2 + y' + (\sin x)\, y = 0$, which satisfy the boundary conditions $y_1(0) = 0$, $y_1'(1) = 1$ and $y_2(0) = 1$, $y_2'(1) = 0$ respectively. Then***
   (a) y_1 and y_2 do not have common zeroes
   (b) y_1 and y_2 have common zeroes
   (c) either y_1 or y_2 has a zero of order 2
   (d) both y_1 and y_2 have zeroes of order 2

12. ***For the Sturm Liouville problems : $(1 + x^2)\, y'' + 2xy' + \lambda x^2 y = 0$ with $y'(1) = 0$ and $y'(10) = 0$ the eigen-values, λ, satisfy***
   (a) $\lambda \geq 0$ (b) $\lambda < 0$
   (c) $\lambda \neq 0$ (d) $\lambda \leq 0$

13. ***The number of groups of order n (upto isomorphism) is***
   (a) finite for all values of n
   (b) finite only for finitely many values of n
   (c) finite for infinitely many values of n
   (d) infinite for some values of n

14. ***The set of all real 2×2 invertible matrices acs on R^2 by matrix multiplication. The number of orbits for this action is***
   (a) 1 (b) 2
   (c) 4 (d) infinite

15. *Let l_2 be the set of real sequenc $\{x_n\}$ such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. For x in l_2 define $\|x\|^2 = \sum_{n=1}^{\infty} |x_n|^2$. Consider the set $S = \{x \in l_2$ such that $|x| < 1\}$. Then*

(a) interior of S is compact (b) S is compact

(c) closure of S is compact (d) closure of S is not compact

16. *On $X = C[0, 1]$ define $T : X \to X$ by $T(f)(x) = \int_0^x f(t)\,dt$, for all f in X. Then*

(a) T is one-one and onto (b) T is one-one but not onto

(c) T is not one-one but onto (d) T is neither one-one nor onto

17. *Let M be the length of the initial interval $[a_0, b_0]$ containing a solution of $f(x) = 0$. Let $\{x_0, x_1, x_2, \ldots\}$ represent the successive points generated by the bisection method. Then the minimum number of iterations required to guarantee an approximation to the solution with an accuracy of ε is given by*

(a) $-2 - \dfrac{\log\left(\dfrac{\varepsilon}{M}\right)}{\log 2}$ (b) $-2 + \dfrac{\log\left(\dfrac{\varepsilon}{M}\right)}{\log 2}$

(c) $-2 + \dfrac{\log(M\varepsilon)}{\log 2}$ (d) $-2 - \dfrac{\log\left(\dfrac{\varepsilon}{M}\right)}{(\log 2)^2}$

18. *On evaluating $\int_1^2\int_1^2 \frac{1}{(x+y)}\,dxdy$ numerically by Trapezoidal rule one would get the value*

(a) $\frac{17}{48}$ (b) $\frac{11}{48}$

(c) $\frac{21}{48}$ (d) $\frac{17}{52}$

19. *Complete integral for the partial differential equation $z = px + qy - \sin(pq)$ is*

(a) $z = ax + by + \sin(ab)$ (b) $z = ax + by - \sin(ab)$

(c) $z = ax + y + \sin(b)$ (d) $z = x + by - \sin(a)$

20. *Pick the region in which the following differential equation is hyperbolic*

$$y\,u_{xx} + 2\,xy\,u_{xy} + x\,u_{yy} = u_x + u_y$$

(a) $xy \neq 1$ (b) $xy \neq 0$

(c) $xy > 1$ (d) $xy > 0$

21. ***If the total kinetic energy of a system of particles about the origin is equal to its kinetic energy about the center of mass, then the center of mass is***

(a) at rest
(b) moving along a circle
(c) moving on a straight line
(d) moving along an ellipse

22. ***The number of generalized co-ordinates required to describe motion of a rigid body with one of its points fixed is***

(a) 9
(b) 6
(c) 3
(d) 1

23. ***Let $Q = \{(x, y) \text{ in } R^2 \mid x \geq 0, y \geq 0\}$ be the first quadrant in R^2 with usual topology. Then Q is closed because***

(a) it is compact
(b) it does not contain all its limit points
(c) its complement is open
(d) it is connected

24. ***Let $X = [0, 1] \times [0, 2] \times \ldots \times [0, 10]$ and $f : X \to R$ be a continuous function. Then $f(X)$ is***

(a) $[r_1, r_2] \cup [r_3, r_4]$ for some r_1, r_2, r_3, r_4 in R such that $r_1 \leq r_2 < r_3 \leq r_4$
(b) $(-\infty, r]$ for some r in R
(c) $[r_1, r_2]$ for some r_1, r_2 in R such that $r_1 \leq r_2$
(d) $(-\infty, r_1] \cup [r_2, \infty)$ for some r_1, r_2 in R such that $r_1 \leq r_2$

25. ***Let X_1 and X_2 be independent binomial random variables with $E(X_i) = n_i p$ and $var(X_i) = n_i p(1-p)$ $0 < p < 1$, $i = 1, 2$. Then the distribution of the random variable $Z = n_1 + n_2 - X_1 - X_2$ is***

(a) binomial with mean $(n_1 + n_2)\, p$
(b) binomial with mean $(n_1 + n_2)(1-p)$
(c) Poisson with mean $(n_1 + n_2)\, p$
(d) Poisson with mean $(n_1 + n_2)(1-p)$

26. ***Let –2, 5, –6, 9, –5, –9 be the observed values of a radnom sample of size 6 from a distribution having probability density function,*** $f_\theta(x) = \begin{cases} e^{-(x-\theta)} & \textit{if} \quad x > \theta \\ 0 & \textit{otherwise} \end{cases}$

Then the maximum likelihood estimate of θ is

(a) 9
(b) – 9
(c) $-\frac{4}{3}$
(d) $\frac{4}{3}$

27. ***Suppose that the linear programming problem P : Min $z = c^1 x$ s.t. $Ax \geq b$, $x \geq 0$, where A is an $m \times n$ matrix, c an $n \times 1$ vector and b an $m \times 1$ vector, is being solved by the Dual Simplex Algorithm. Then***

(a) the value of the primal objective function increases at every iteration
(b) the algorithm will always terminate with an optimal solution for the dual
(c) the algorithm will always terminate with an optimal solution to the primal
(d) it is not always possible to obtain a starting basis for this Algorithm

28. *Consider the transportation problem given below. The bracketed elements in the table indicate a feasible solution and the elements on the left hand corner are the costs c_{ij}.*

				a_i
	2 (1)	5	1	1
	1	3 (1)	4 (1)	2
b_j	1	1	1	3

(a) this solution is a basic feasible solution

(b) this solution can be made basic feasible

(c) this is an optimal solution

(d) the problem does not have an optimal solution

29. *Extremals $y = y(x)$ for the variational problem $v[y(x)] = \int_0^1 (y + y')^2\, dx$ satisfy the differential equation*

(a) $y'' + y = 0$ (b) $y'' - y = 0$

(c) $y'' + y' = 0$ (d) $y' + y = 0$

30. *Let $k(x, t) = \begin{cases} x + t & 0 \le t \le x \\ 0 & \text{otherwise} \end{cases}$*

Then, the integral equation $y(x) = 1 + \lambda \int_0^x y(t)\, k(x,t)\, dt$, has

(a) a unique solution for every value of λ

(b) no solution for any value of λ

(c) a unique solution for finitely many values of λ only

(d) infinitely many solutions for finitely many values of λ

Q. 31-90 CARRY TWO MARKS EACH

31. *Consider the matrix $M = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$ and let S_M be the set of 3×3 matrices N such that $MN = 0$. Then the dimension of the real vector space S_M is equal to*

(a) 0 (b) 1

(c) 2 (d) 3

32. *Choose the correct matching from A, B, C and D for the transformation T_1, T_2 and T_3 (mappings from R^2 to R^3) as defined in Group 1 with the statements given in Group 2.*

	Group 1		*Group 2*
P	$T_1(x, y) = (x, x, 0)$	1	Linear transformation of rank 2
Q	$T_2(x, y) = (x, x+y, y)$	2	Not a linear transformation
R	$T_3(x, y) = (x, x+1, y)$	3	Linear transformation of rank 1

(a)	*(b)*	*(c)*	*(d)*
P – 3	P – 1	P – 3	P – 1
Q – 1	Q – 2	Q – 2	Q – 3
R – 2	R – 3	R – 1	R – 2

33. *Let* $M = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & -4 & 0 & 0 \end{pmatrix}$. *Then*

(a) $MM^T = I$ where M^T is the transpose of M amd I is the identify matrix

(b) Column vectors of M form an orthogonal system of vectors

(c) Column vectors of M form an orthonormal system of vectors

(d) (MX, MY) = (X, Y) for all X, Y in R^4 where (,) is the standard inner product on R^4

34. *Let* $M = \begin{pmatrix} 1 & 1+i & 2i & 9 \\ 1-i & 3 & 4 & 7-i \\ -2i & 4 & 5 & i \\ 9 & 7+i & -i & 7 \end{pmatrix}$. Then

(a) M has purely imaginary eigen values

(b) M is not diagonalizable

(c) M has eigen values which are neither real nor purely imaginary

(d) M has only real eigen values

35. *Consider the matrix* $M = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}$ *where a, b and c are non-zero real numbers.*

Then the matrix has

(a) three non-zero real eigen values *(b)* complex eigen values

(c) two non-zero eigen value *(d)* only one non-zero eigen value

36. *The minimal polynomial of* $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ *is*

(a) $(x-1)^2(x-2)$ *(b)* $(x-1)(x-2)^2$

(c) $(x-1)(x-2)$ *(d)* $(x-1)^2(x-2)^2$

37. ***Let γ be the curve : $r = 2 + 4\cos\theta$, $(0 \le \theta \le 2\pi)$. If $I_1 = \int_\gamma \frac{dz}{z-1}$ and $I_2 = \int_\gamma \frac{dz}{z-3}$ then***

(a) $I_1 = 2I_2$ (b) $I_1 = I_2$

(c) $2I_1 = I_2$ (d) $I_1 = 0, I_2 \ne 0$

38. ***Let f(z) be defined on the domain E : $|z - 2i| < 3$ and on its boundary ∂E. Then which of the following statements is always true :***

(a) if $f(z)$ is analytic on E and $f(z) \ne 0$ for any z in E, then $|f|$ attains its maximum on ∂E

(b) if $f(z)$ is analytic on $E \cup \partial E$, then $|f|$ attains its minimum on ∂E

(c) if $f(z)$ is analytic on E and continuous on $E \cup \partial E$, then $|f|$ attains its maximum and minimum on ∂E

(d) if $f(z)$ is analytic on $E \cup \partial E$ and $f(z) \ne 0$ for any z in $E \cup \partial E$, then $|f|$ attains its minimum on ∂E

39. ***Let f(z) be an analytic function with a simple pole at z = 1 and a double pole at z = 2 with residues 1 and – 2 respectively. Further if f(0) = 0, $f(3) = -\frac{3}{4}$ and f is bounded as $z \to \infty$, then f(z) must be***

(a) $z(z-3) - \frac{1}{4} + \frac{1}{z-1} - \frac{2}{z-1} + \frac{1}{(z-2)^2}$ (b) $-\frac{1}{4} + \frac{1}{z-1} - \frac{2}{z-2} + \frac{1}{(z-2)^2}$

(c) $\frac{1}{z-1} - \frac{2}{z-2} + \frac{5}{(z-2)^2}$ (d) $\frac{15}{4} + \frac{1}{z-1} + \frac{2}{z-2} - \frac{7}{(z-2)^2}$

40. ***An example of a function with a non-isolated essential singularity at z = 2 is***

(a) $\tan\frac{1}{z-2}$ (b) $\sin\frac{1}{z-2}$

(c) $e^{-(z-2)}$ (d) $\tan\frac{z-2}{z}$

41. ***Let f (z) = u (x, y) + iv (x, y) be an entire function having Taylor's series expansion as $\sum_{n=0}^{\infty} a_n z^n$. If f(x) = u (x, 0) and f(iy) = iv(0, y) then***

(a) $a_{2n} = 0$ for all n (b) $a_0 = a_1 = a_2 = a_3 = 0, a_4 \ne 0$

(c) $a_{2n+1} = 0$ for all n (d) $a_0 \ne 0$ but $a_2 = 0$

42. ***Let $I = \int_c \frac{\cot(\pi z)}{(z-i)^2} dz$, where C is the contour $4x^2 + y^2 = 2$ (counter clock-wise). Then I is equal to***

(a) 0 (b) $-2\pi i$

(c) $2\pi i\left(\frac{\pi}{\sin h^2\pi} - \frac{1}{\pi}\right)$ (d) $-\frac{2\pi^2 i}{\sin h^2 \pi}$

43. ***Let $X = \{x \text{ in } Q \mid 0 < x < 1\}$ be the metric space with standard metric from R. The completion of X is***

(a) $\{x \text{ in } Q \mid 0 < x < 1\}$
(b) $\{x \text{ in } R \mid 0 < x < 1\}$
(c) $\{x \text{ in } Q \mid 0 \le x \le 1\}$
(d) $\{x \text{ in } R \mid 0 \le x \le 1\}$

44. ***The function $f(x, y) = (e^x \cos y, e^x \sin y)$ from R^2 to R^2 is***

(a) one-one on all of R^2
(b) one-one on some neighbourhood of any point in R^2
(c) an onto map
(d) such that some neighbourhood of any point surjects onto R^2

45. ***Let E and E_i ($i = 1, 2, \ldots, \infty$) be measurable subsets of the real line such that $E = \bigcup_{i=1}^{\infty} E_i$. Let f be a non-negative function such that f is integrable over E, then***

$$\int_E f\,dx = \sum_{i=1}^{\infty} \int_{E_i} f\,dx \text{ is}$$

(a) true as $\sum_{i=1}^{\infty} \int_{E_i} f\,dx$ is finite
(b) true by dominated convergence theorem
(c) true by Fatou's lemma
(d) not true because $E_i \cap E_j$ may not be empty for some $i \ne j$

46. ***In the inerval [–1, 1], the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^2 + n^2}{n^3}$ is***

(a) uniformly and absolutely convergent
(b) absolutely convergent but not uniformly convergent
(c) neither uniformly nor absolutely convergent
(d) uniformly convergent but not absolutely convergent

47. ***The maximum magnitude of the directional derivative for the surface***

$$x^2 + xy + yz = 9$$

at the point (1, 2, 3) is along the direction

(a) $\hat{i} + \hat{j} + \hat{k}$
(b) $2\hat{i} + 2\hat{j} + \hat{k}$
(c) $\hat{i} + 2\hat{j} + 3\hat{k}$
(d) $\hat{i} - 2\hat{j} + 3\hat{k}$

48. ***Let $B = \{(x, y, z) \mid x, y, z, \in R \text{ and } x^2 + y^2 + z^2 \le 4\}$. Let $v(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$ be a vector-valued function defined on B. If $r^2 = x^2 + y^2 + z^2$, the value of the integral***

$$\iiint_B \nabla \cdot (r^2\, v(x, y, z))\, dV \text{ is}$$

(a) 16π
(b) 32π
(c) 64π
(d) 128π

49. ***For the Initial Value Problem (I.V.P.) : $y' = f(x, y)$ with $y(0) = 0$, which of the following statements is true***

(a) $f(x, y) = \sqrt{xy}$ satisfies Lipschitz's condition and so I.V.P. has unique solution

(b) $f(x, y) = \sqrt{xy}$ does not satisfy Lipschitz's condition and so I.V.P. has no solution

(c) $f(x, y) = |y|$ satisfies Lipschitz's condition and so I.V.P. has unique solution

(d) $f(x, y) = |y|$ does not satisfy Lipschitz's condition still I.V.P. has unique solution

50. ***All real solutions of the differential equation $y'' + 2ay' + by = \cos x$ (where a and b are real constants) are periodic if***

(a) $a = 1$ and $b = 0$ (b) $a = 0$ and $b = 1$

(c) $a = 1$ and $b \neq 0$ (d) $a = 0$ and $b \neq 1$

51. ***Let $y = \Psi(x)$ be a bounded solution of the equation : $(1 - x^2)y'' - 2xy' + 30y = 0$. Then***

(a) $\int_{-1}^{1} x^3 \Psi(x)\, dx \neq 0$ (b) $\int_{-1}^{1} (1 + x^3 + x^4)\psi(x)\, dx \neq 0$

(c) $\int_{-1}^{1} x^5 \Psi(x)\, dx = 0$ (d) $\int_{-1}^{1} x^{2m} \Psi(x)\, dx = 0$ for all $n \in N$

52. ***$\int x^3 J_0(x)\, dx$ is equal to (upto a constant)***

(a) $x J_0(x) - x^3 J_1(x)$ (b) $x^2 J_0(x) + J_1(x)$

(c) $x^3 J_1(x) - 2x^2 J_2(x)$ (d) $2x^2 J_1(x) + x J_2(x)$

53. ***Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of $xy'' + y' + x^2 y = 0$, in the neighbourhood of $x = 0$. If $y_1(x)$ is a power series around $x = 0$, then***

(a) $y_2(x)$ is bounded around $x = 0$

(b) $y_2(x)$ is unbounded around $x = 0$

(c) $y_2(x)$ has power series solution

(d) $y_2(x)$ has solution of the form $\sum_{n=1}^{\infty} b_n x^{n+r}$, where $r \neq 0$, and $b_0 \neq 0$

54. ***Consider the following system of differential equations in $x(t)$, $y(t)$ and $z(t)$***

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Then there exists a choice of 3 linearly independent vectors u, v, w in R^3 such that vectors, forming a fundamental set of solutions of the above system, are given by

(a) $e^{-t}u, e^t v, te^t w$ (b) $e^t u, te^t v, t^2 e^t w$

(c) $e^{-t}u, te^{-t}v, e^t w$ (d) $u, tv, e^t w$

55. ***Any subgroup of Q (the group of rational numbers under addition) is***

(a) cyclic and finitely generated but not abelian and normal

(b) cyclic and abelian but not finitely generated and normal

(c) abelian and normal but not cyclic and finitely generated

(d) finitely generated and normal but not cyclic and abelian

56. ***Let σ and τ be the permutations defined by***

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 3 & 5 & 7 & 9 & 6 & 4 & 8 & 2 \end{pmatrix} \text{ and } \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 3 & 4 & 9 & 6 & 5 & 2 & 1 \end{pmatrix}$$

Then

(a) σ and τ generate the group of permutations on {1, 2, 3, 4, 5, 6, 7, 8, 9}

(b) σ is contained in the group generated by τ

(c) τ is contained in the group generated by σ

(d) σ and τ are in the same conjugacy class

57. ***Upto isomorphism, the number of abelian groups of order 10^5 is***

(a) 2 (b) 5

(c) 7 (d) 49

58. ***Set of multiples of 4 forms an ideal in Z, the ring of integers under usual addition and multiplication. This ideal is***

(a) a prime ideal but not a maximal ideal

(b) a maximal ideal but not a prime ideal

(c) both a prime ideal and a maximal ideal

(d) neither a prime ideal nor a maximal ideal

59. ***Let C [0, 1] be the set of all continuous functions defined on the interval [0, 1]. On this set, define addition and multiplication pointwise. Then C [0, 1] is***

(a) a group but not a ring

(b) a ring but not an integral domain

(c) a field

(d) an integral domain but not a field

60. ***Let $X = C^1 [0,1]$ and $Y = C [0, 1]$ both having the norm $|f| = \sup\{ |f(x)|, 0 \le x \le 1\}$. Define $T : X \to Y$ by $T (f) = f'$, where f' denotes the derivative of f. Then T is***

(a) linear and continuous (b) not linear but continuous

(c) is linear and not continuous (d) is not linear and not continuous

61. ***Let B a Banach space (not finite dimensional) and $T : B \to B$ be a continuous operator such that the range of T is B and $T (x) = 0 \Rightarrow x = 0$. Then***

(a) T maps bounded sets to compact sets

(b) T^{-1} maps bounded sets to compact sets

(c) T^{-1} maps bounded sets to bounded sets

(d) T maps compact sets to opoen sets

62. ***Let the sequence $\{e_n\}$ be a complete orthonormal set in a Hilbert space H. Then***

(a) for all bounded linear operators T on H, the sequence $\{Te_n\}$ is convergent in H

(b) for the identity operator I on H the sequence $\{Ie_n\}$ is convergent in H

(c) for all bounded linear functionals f on H the sequence $\{fe_n\}$ is convergent in R

(d) none of the above

63. ***Let $A : H \to H$ by any bounded linear operator on a complex Hilbert space H such that $\|Ax\| = \|A^*x\|$ for all x in H, where A* is the adjoint of A. If there is a non-zero x in H such that $A^*(x) = (2+3i)x$, then A is***

(a) an unitary operator on H
(b) a self-adjoint operator on H but not unitary
(c) a self-adjoint operator on H but not normal
(d) a normal operator

64. ***If the scheme corresponding to the Newton-Raphson method for solving the system of nonlinear equation : $x^2 + y^2 - 10 = 0$, $x^2 y - 3 = 0$ is***

$$x^{k+1} = x^k + \{f(x,y)\}_{(x^k, y^k)} \text{ and } y^{k+1} = y^k + \{g(x,y)\}_{(x^k, y^k)}$$

then f(x, y) and g(x, y) are respectively given by

(a) $-\dfrac{(x^2+y^2-10)}{2x}$ and $-\dfrac{(x^2y-3)}{x^2}$

(b) $\dfrac{x^2(y^2-x^2+10)-6y}{2x(x^2-y^2)}$ and $\dfrac{y^3-10y+3}{x^2-y^2}$

(c) $\dfrac{x^2(3y^2-x^2-10p+6y}{2x(x^2-y^2)}$ and $\dfrac{y^3-10y+3}{x^2-y^2}$

(d) $-(x^2+y^2-10)$ and $-(x^2y-3)$

65. ***A lower bound on the polynomial interpolation error $e_2(\bar{x})$ for $f(x) = \ln(x)$, with $x_0 = 2$, $x_1 = 2$, $x_2 = 4$ and $\bar{x} = \dfrac{5}{4}$ is given by***

(a) $\dfrac{1}{256}$ (b) $\dfrac{1}{64}$

(c) $\dfrac{1}{512}$ (d) 0

66. ***Consider the Quadrature formula***

$$\int_0^h f(x)dx = \left\{\alpha f(0) + \beta f\left(\frac{3h}{4}\right) + \gamma f(h)\right\} h$$

The values of α, β, γ for which this is exact for polynomials of as high degree as possible, are

(a) $\alpha = \dfrac{5}{18}, \beta = \dfrac{8}{9}, \gamma = -\dfrac{1}{6}$ (b) $\alpha = \dfrac{1}{2}, \beta = -\dfrac{1}{4}, \gamma = \dfrac{3}{4}$

(c) $\alpha = 0, \beta = 1, \gamma = -\dfrac{1}{4}$ (d) $\alpha = 1, \beta = 2, \gamma = 3$

67. ***Consider the (Cholesky's) algorithm given below for LL^T decomposition of a Symmetric Positive Definite matrix A :***

Compute $L_{11} = A_{11}^{1/2}$

For $i = 2$ to N
Compute $L_{i,1} = A_{i,1} / L_{i,1}$

For $i = 2$ to N
Compute $L_{j,j} = \left(A_{j,j} - \sum_{m=1}^{j-1} L_{j,m}^2 \right)^{1/2}$

Right alternative for filling the shaded box to complete the above algorithm is

(a) For $i = j+1$ to N
Compute $L_{i,j} = \frac{1}{L_{j,j}} \left(A_{i,j} - \sum_{m=1}^{j-1} L_{i,m} L_{j,m} \right)$

(b) For $i = j$ to N
Compute $L_{i,j} = \frac{1}{L_{j,j}} \left(A_{i,j} - \sum_{m=1}^{j-1} L_{i,m} L_{j,m} \right)$

(c) For $i = j$ to N
Compute $L_{i,j} = \frac{1}{L_{j,j}} \left(A_{i,j} - \sum_{m=1}^{j-1} L_{i,m} L_{j,m} \right)$

(d) For $i = j+1$ to N
Compute $L_{i,j} = \frac{1}{L_{j,j}} \left(A_{i,j} - \sum_{m=1}^{j-1} L_{i,m}^2 \right)$

68. ***Let $u = \psi(x, t)$ be the solution to the initial value problem***

$$u_{tt} = u_{xx} \text{ for } -\infty < x < \infty,\ t > 0$$

with $u(x, 0) = \sin(x)$, $u_t(x, 0) = \cos(x)$ then the value of $\psi(\pi/2, \pi/6)$ is

(a) $\sqrt{3}/2$ *(b)* $1/2$

(c) $1/\sqrt{2}$ *(d)* 1

69. ***Consider the boundary value problem :***

$u_{xx} + u_{yy} = 0$ in $\Omega = \{(x, y);\ x^2 + y^2 < 1\}$ with $\frac{\partial u}{\partial n} = x^2 + y^2$ on the boundary of Ω

$\left(\frac{\partial u}{\partial n} \text{ denotes the normal derivative of } u \right)$. Then its solution $u(x, y)$

(a) is unique and is identically zero *(b)* is unique upto a constant

(c) does not exist *(d)* is unique and non-zero

70. ***The Cauchy problem $u_x - u_y = 2$ with the Cauchy data on Γ: $(s, -s, 2s)$ has***

(a) one solution
(b) two solutions
(c) no solution
(d) infinite solutions

71. ***Let $u(r, \theta, z, t)$ be the solution to the heat conduction problem $u_t = u_{xx}$ in $\Omega \times [0, T]$, where $\Omega = \{(r, \theta, z) \mid 0 < r_1 \le r \le r_2, 0 \le \theta \le 2\pi, 0 \le z \le L\}$ with compatible initial and boundary conditions :***

$$u(r, \theta, z, 0) = \begin{cases} 0 & \text{in the interior of } \Omega \\ g(\theta) & \text{on } \partial\Omega \end{cases}, \text{ and } u(r, \theta, z, t) = f(\theta) \text{ on } \partial\Omega \text{ for } t > 0$$

Further, if

$$M = \max\{u, (r, \theta, z, T) \mid (r, \theta, z) \in \Omega\},$$
$$M_1 = \max\{u(r, \theta, z, T) \mid r = r_1, 0 \le \theta \le 2\pi, 0 < z < L\}$$
$$M_2 = \max\{u(r, \theta, z, T) \mid r = r_2, 0 \le \theta \le 2\pi, 0 < z < L\}$$

then

(a) $M_1 \le M \le M_2$
(b) $M_2 \le M \le M_1$
(c) $M \le \max(M_1, M_2)$
(d) $M \ge \max(M_1, M_2)$

72. ***A particle of unit mass is moving under gravitational field, along the cycloid $x = \phi - \sin\phi$, $y = 1 + \cos\phi$. Then the Lagrangian for the motion is***

(a) $\dot{\phi}^2(1 + \cos\phi) - g(1 - \cos\phi)$
(b) $\dot{\phi}^2(1 - \cos\phi) + g(1 + \cos\phi)$
(c) $\dot{\phi}^2(1 - \cos\phi) - g(1 + \cos\phi)$
(d) $2\dot{\phi}^2(1 - \cos\phi) - g(1 + \cos\phi)$

Data for Q. 73 – 74 is given below. Solve the problems and choose correct answers.

Three particles of masses 1, 2, and 4 move under a force field such that their position vectors at any time t are respectively given by

$$\bar{r}_1 = 2\hat{i} + 4t^2\hat{k}, \quad \bar{r}_2 = 4t\hat{i} - \hat{k}, \quad \bar{r}_1 = (\cos\pi t)\hat{i} + (\sin\pi t)\hat{j}$$

73. ***For the above motion which of the following is true***

(a) the total momentum is zero
(b) the total momentum has constant magnitude
(c) the force acting on the system is constant
(d) the force acting on the system has constant magnitude

74. ***The angular momentum of the system about the origin at $t = 1/2$ is given by***

(a) zero vector
(b) $4(-4\hat{j} + \pi\hat{k})$
(c) $-4(-4\hat{j} + \pi\hat{k})$
(d) $-4(\pi\hat{j} + \hat{k})$

75. ***A cube of unit mass is suspended vertically from one of its edges. If the length of its edge is $\sqrt{2}$, then the length of the equivalent simple pendulum is***

(a) $\frac{4}{3}$
(b) $\frac{2}{3}$
(c) $\frac{2\sqrt{2}}{3}$
(d) $2\sqrt{2}$

76. ***Consider the following statements concerning topological spaces :***
(P) Continuous image of a non-compact space is non-compact
(Q) Every metrizable space is normal
Then
(a) both P and Q are true
(b) P is true and Q is false
(c) P is false and Q is true
(d) both P and Q are false.

77. ***In R^3, with the usual topology, let B be the unit closed ball with center at the origin, and T be the closure of an inscribed tetrahedron. Let $f : T \to R$ be any continuous function. Then***
(a) f has an extension to B $\Rightarrow f$ is a constant function
(b) not every f has an extension to B
(c) f always has an extension to B
(d) if f has an extension to B then $f(T) \subseteq [0, 1]$

78. ***Let PQR be a triangle in R^2 with the usual topology. Define***
$X = Int\,(PQR) \cup \{P, Q, R\}$.
Then the number of connected components of X is
(a) 1
(b) 2
(c) 3
(d) 4

79. ***In R^2 with the usual topology, let $X = \{(x, |x|)$ such that $-1 \le x \le 1\}$. Let $p : X \to [-1, 1]$ be the map defined by $p(x, y) = x$ for all (x, y) in X. Then p is***
(a) a homoomorphism as p is one-one, onto and continuous
(b) a homoomorphism as both p and p^{-1} are one-one, onto and continuous
(c) not a homoomorphism as p is not continous
(d) not a homoomorphism as p^{-1} is not continuous

80. ***E_1, E_2 are independent events such that***

$$P(E_1) = \frac{1}{4},\ P(E_2/E_1) = \frac{1}{2} \text{ and } P(E_1/E_2) = \frac{1}{4}.$$

Define random variables X and Y by

$$X = \begin{cases} 1 & \text{if } E_1 \text{ occurs} \\ 0 & \text{if } E_1 \text{ does not occur} \end{cases},\quad Y = \begin{cases} 1 & \text{if } E_2 \text{ occurs} \\ 0 & \text{if } E_2 \text{ does not occur} \end{cases}$$

Consider the following statements
α : ***X is uniformly distributed on the set {0, 1}***
β : ***X and Y are identically distributed***
$\gamma : P\{X^2 + Y^2 = 1\} = 1/2$
$\delta : P\{X\,Y = X^2 Y^2\} = 1$
Choose the correct combination
(a) (α, β)
(b) (α, γ)
(c) (β, γ)
(d) (γ, δ)

81. *Let X and Y be the time (in hours) taken by Saurabh and Sachin to solve a problem. Suppose that each of X and Y are uniformly distributed over the interval [0, 1]. Assume that Saurabh and Sachin start to solve the problem independently. Then, the probability that the problem will be solved in less than 20 minutes is*

(a) $\frac{1}{3}$ (b) $\frac{5}{9}$

(c) $\frac{8}{9}$ (d) $\frac{4}{9}$

82. *The value of the limit*

$$\lim_{n\to\infty}\sum_{j=n}^{4n}\binom{4n}{j}\left(\frac{1}{4}\right)^{j}\left(\frac{3}{4}\right)^{4n-j} \quad \text{equals}$$

(a) 0 (b) $\frac{1}{4}$

(c) $\frac{1}{2}$ (d) $\frac{3}{4}$

83. *Let f(x) and g(x) be two probability mass functions (p.m.f) defined by*

$$f(x) = \frac{1}{6},\ x = 1, 2, 3, 4, 5, 6 \text{ and}$$

$$g(x) = \begin{cases} 1/12, & \text{if } x = 1, 2 \\ 1/2, & \text{if } x = 3 \\ 1/9, & \text{if } x = 4, 5, 6 \end{cases}$$

Let X be a random sample of size one from a distribution having p.m.f. $h(x) \in \{f(x), g(x)\}$. To test the hypothesis $H_0 : h = f$ vs $H_1 : h = g$, a most powerful test of size $\alpha = 1/6$ rejects H_0 iff

(a) $x = 1$ (b) $x = 2$

(c) $x = 3$ (d) $x \in \{4, 5, 6\}$

84. *Consider the linear programming problem P1 : Min $z = c'x$ s.t. $Ax = b$, $x \geq 0$, where A is an $m \times n$ matrix, $m \leq n$, c and x are $n \times 1$ vectors and b an $m \times 1$ vector. Let K denote the set of feasible solutions for P1. Then,*

(a) the number of positive x's in any feasible solution of P1 can never exceed m, and if it is less than m, the feasible solution is a degenerate basic feasible solution

(b) every feasible solution of P1 in which m variables are positive is a basic feasible solution and ${}^{n}C_{m}$ is the total number of basic feasible solutions

(c) in solving P1 by the simplex algorithm a new basis and a new extreme point of the constraint set are generated after every pivot step

(d) K is a convex set and if the value of the objective function at an extreme point x^* of K is better than its values at all the neighbouring extreme points, then x^* is an optimal solution of P1

85. ***Consider the linear programming formulation (P2) of optimally assigning n men to n jobs with respect to some costs $\{c_{ij}\}_{i,j=1}^{n}$. Let A denote the coefficient matrix of the constraint set. Then,***

(a) rank of A is $2n-1$ and every basic feasible solution of P2 is integer valued

(b) rank of A is $2n-1$ and every basic feasible solution of P2 is integer valued

(c) rank of A is $2n$ and every basic feasible solution of P2 is integer valued

(d) rank of A is $2n$ and every basic feasible solution of P2 is not integer valued

36. ***Simplex tableau for phase I of the simplex algorithm for a linear programming problem is given below (x_3, x_4, x_5 are artificial variables) :***

Basis	x_1	x_2	x_3	x_4	x_5	RHS
$Z_j - C_J$	0	0	–2	–2	0	0
x_1	1	0	3/5	1/5	0	2
x_2	0	1	–2/5	1/5	0	0
x_3	0	0	–1	–1	1	0

Choose the correct statement

(a) the tableau does not show the end of phase I, since the artificial variable x_5 is in the basis

(b) the tableau does show the end of phase I since the value of the phase I objective function is zero

(c) the constraints for the original linear programming problem are not redundant

(d) the original linear programming problem does not have a feasible solution

87. ***Given below is the final tableau of a linear programming problem (x_4 and x_5 are slack variables) :***

Basis	x_1	x_2	x_3	x_4	x_5	RHS
$Z_j - C_J$	0	0	3	5	1	8
x_1	1	0	1	4	–1	2
x_2	0	1	2	–1	1	3

If the right hand side vector $\begin{pmatrix}1\\3\end{pmatrix}$ of the problem gets changed to $\begin{pmatrix}1+\theta\\3\end{pmatrix}$, then the current basic feasible solution is optimal for

(a) all $\theta \le 2$

(b) all $\theta \ge -\frac{1}{4}$

(c) all $\theta \in \left[-\frac{1}{2}, 2\right]$

(d) no non-zero value of θ

88. *The functional* $v[y(x)] = \int_0^2 [(y')^2 + 6xy + x^3]\, dx$, $y(0) = 0$, $y(2) = 2$ *can be extremized on the curve*

(a) $y = x$ (b) $2y = x^3$

(c) $y = x^3 - 6x$ (d) $2y = x^3 - 2x$

89. *The integral equation*

$$y(x) = \int_0^x (x-t)\, y(t)\, dt - x\int_0^1 (1-t)\, y(t)\, dt$$

is equivalent to

(a) $y'' - y = 0,\ y(0) = 0,\ y(1) = 0$

(b) $y'' - y = 0,\ y(0) = 0,\ y'(0) = 0$

(c) $y'' + y = 0,\ y(0) = 0,\ y(1) = 0$

(d) $y'' + y = 0,\ y(0) = 0,\ y'(0) = 0$

90. *The integral equation* $y(x) = \lambda \int_0^{2\pi} \sin(x+t)\, y(t)\, dt$ *has*

(a) no solution for any value of λ

(b) unique solution for every value of λ

(c) infinitely many solutions for only one value of λ

(d) infinitely many solutions for two values of λ