GATE : 2004

MA : MATHEMATICS

Duration : Three Hours *Maximum Marks : 150*

Read the following instructions carefully.

1. This questions paper contains 90 objective questions. Q. 1-30 carry 1 mark each and Q. 31-90 carry 2 marks each.
2. Answer all the questions.
3. Questions must be answered on special machine gradable Objective Response Sheet (**ORS**) by darkening the appropriate bubble (marked A, B, C, D) using HB pencil against the question number on the left hand side of the **ORS**. Each question has only one correct answer. In case you wish to change an answer, erase the old answer completely using a good soft eraser.
4. There will be **NEGATIVE** marking. For each wrong answer, **0.25** mark from Q. 1-30 and **0.5** mark from Q. 31-90 will be deducted. More than one answer marked against a question will be deemed as an incorrect response and will be negatively marked.
5. Write your registration number, name and name of the Centre at the specified locations on the right half of the **ORS**.
6. Using HB pencil, darken the appropriate bubble under each digit of your registration number.
7. Using HB pencil, darken the appropriate bubble under the letters corresponding to your paper code.
8. No charts or tables are provided in the examination hall.
9. Use the blank pages given at the end of the question paper for rough work.
10. Choose the **closest** numerical answer among the choices given.
11. This question paper contains 24 printed pages. Please report, if there is any discrepancy.

Q. 1-30 Carry One Mark Each

The symbols, N, Z, Q, R and C denote the set of natural numbers, integers, rational numbers, real numbers and complex numbers, respectively, throughout the paper.

1. *Let S and T be two subspaces of R^{24} such that dim (S) = 19 and dim (T) = 17. Then, the*
   (a) smallest possible value of dim $(S \cap T)$ is 2
   (b) largest possible value of dim $(S \cap T)$ is 18
   (c) smallest possible value of dim $(S + T)$ is 19
   (d) largest possible value of dim $(S + T)$ is 22

2. *Let $v_1 = (1, 2, 0, 3, 0)$, $v_2 = (1, 2, -1, -1, 0)$, $v_3 = (0, 0, 1, 4, 0)$, $v_4 = (2, 4, 1, 10, 1)$, and $v_5 = (0, 0, 0, 0, 1)$. The dimension of the linear span of $\{v_1, v_2, v_3, v_4, v_5\}$ is*
   (a) 2 (b) 3
   (c) 4 (d) 5

3. *The set $V = \{(x, y) \in R^2 : xy \geq 0\}$ is*
   (a) a vector subspace of R^2
   (b) not a vector subspace of R^2 since every element does not have an inverse in V
   (c) not a vector subspace of R^2 since it is not closed under scalar multiplication
   (d) not a vector subspace of R^2 since it is not closed under vector addition

4. *Let $f : R^4 \to R$ be a linear functional defined by $f(x_1, x_2, x_3, x_4) = -x_2$. If $\langle \cdot, \cdot \rangle$ denotes the standard inner product on R^4, then the unique vector $v \in R^4$ such that $f(w) = \langle v, w \rangle$ for all $w \in R^4$, is*
   (a) (0, – 1, 0, 0) (b) (– 1, 0, – 1, 1)
   (c) (0, 1, 0, 0) (d) (1, 0, 1, – 1)

5. *If D is the open unit disk in C and $f : C \to D$ is analytic with $f(10) = 1/2$, then $f(10 + i)$ is*
   (a) $\frac{1+i}{2}$ (b) $\frac{1-i}{2}$
   (c) $\frac{1}{2}$ (d) $\frac{i}{2}$

6. *The real part of the principal value of 4^{4-i} is*
   (a) 256 cos (ln 4) (b) 64 cos (ln 4)
   (c) 16 cos (ln 4) (d) 4 cos (ln 4)

7. *If $\sin z = \sum_{n=0}^{\infty} a_n (z - \pi/4)^n$, then a_6 equals*
   (a) 0 (b) $\frac{1}{720}$
   (c) $\frac{1}{(720\sqrt{2})}$ (d) $\frac{-1}{(720\sqrt{2})}$

8. ***The equation $x^6 - x - 1 = 0$ has***
   (a) no positive real roots
   (b) exactly one positive real root
   (c) exactly two positive real roots
   (d) all positive real roots

9. ***Let $f, g : (0, 1) \times (0, 1) \to R$ be two continuous functions defined by $f(x, y) = \dfrac{1}{1 + x(1 - y)}$ and $g(x, y) = \dfrac{1}{1 + x(y - 1)}$. Then, on $(0, 1) \times (0, 1)$,***
   (a) f and g are both uniformly continuous
   (b) f is uniformly continuous but g is not
   (c) g is uniformly continuous but f is not
   (d) neither f nor g is uniformly continuous

10. ***Let S be the surface bounding the region $x^2 + y^2 \le 1, x \ge 0, y \ge 0$ $|z| \le 1$, and $\hat{n}$ be the unit outer normal to S. Then $\iint_S [(\sin^2 x)\,\hat{i} + 2y\,\hat{j} - z(1 + \sin 2x)\hat{k}] \,.\, \hat{n}\, dS$ equals***
   (a) 1
   (b) $\dfrac{\pi}{2}$
   (c) π
   (d) 2π

11. ***Let $f : [0, \infty) \to R$ be defined by***

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}}, & x \ne 0 \\ 0 \;\;, & x = 0 \end{cases}$$

Consider the two improper integrals $I_1 = \int_0^1 f(x)\, dx$ and $I_2 = \int_1^\infty f(x)\, dx$. Then
   (a) both I_1 and I_2 exist
   (b) I_1 exists but I_2 does not
   (c) I_1 does not exist but I_2 does
   (d) neither I_1 nor I_2 exists

12. ***The orthogonal trajectories to the family of straight lines $y = k(x - 1)$, $k \in R$, are given by***
   (a) $(x - 1)^2 + (y - 1)^2 = c^2$
   (b) $x^2 + y^2 = c^2$
   (c) $x^2 + (y - 1)^2 = c^2$
   (d) $(x - 1)^2 + y^2 = c^2$

13. ***If $y = \varphi(x)$ is a particular solution of $y'' + (\sin x)\, y' + 2y = e^x$ and $y = \Psi(x)$ is a particular solution of $y'' + (\sin x)\, y' + 2y = \cos 2x$, then a particular solution of $y'' + (\sin x)\, y' + 2y = e^x + 2\sin^2 x$, is given by***
   (a) $\varphi(x) - \psi(x) + \dfrac{1}{2}$
   (b) $\psi(x) - \varphi(x) + \dfrac{1}{2}$
   (c) $\varphi(x) - \psi(x) + 1$
   (d) $\psi(x) - \varphi(x) + 1$

14. ***Let $P_n(x)$ be the Legendre polynomial of degree $n \le 0$. If $1 + x^{10} = \sum_{n=0}^{10} c_n P_n(x)$, then c_5 equals***
   (a) 0
   (b) $\dfrac{2}{11}$
   (c) 1
   (d) $\dfrac{11}{2}$

15. *Let I be the set of irrational real numbers and let $G = I \cup \{0\}$. Then, under the usual addition of real numbers, G is*

(a) a group, since $\mathbb{R}$ and $\mathbb{Q}$ are groups under addition
(b) a group, since the additive identity is in G
(c) not a group, since addition on G is not a binary operation
(d) not a group, since not all elements in G have an inverse

16. *In the group (Z, +), the subgroup generated by 2 and 7 is*

(a) Z (b) 5Z
(c) 9Z (d) 14Z

17. *The cardinality of the center of Z_{12} is*

(a) 1 (b) 2
(c) 3 (d) 12

18. *Suppose $X = (1, \infty)$ and $T : X \to X$ is such that $d(Tx, Ty) < d(x, y)$ for $x \neq y$. Then*

(a) T has at most one fixed point
(b) T has a unique fixed point, by Banach Contraction Theorem
(c) T has infinitely many fixed points
(d) for every $x \in X$, $\{T^n(x)\}$ converges to a fixed point

19. *Consider R^2 with $\|\cdot\|_1$ norm and $M = \{(x, 0)\} : x \in R\}$. Define $g : M \to R$ by $g(x, y) = x$. Then a Hahn-Banach extension f of g is given by*

(a) $f(x, y) = 2x$ (b) $f(x, y) = x + y$
(c) $f(x, y) = x - 2y$ (d) $f(x, y) = x + 2y$

20. *Let X be an inner product space and $S \subset X$. Then it follows that*

(a) $S\perp$ has nonempty interior
(b) $S\perp = \{0\}$
(c) $S\perp$ is a closed subspace
(d) $(S\perp)\perp = S$

21. *An iterative scheme is given by $x_{n+1} = \frac{1}{5}\left(16 - \frac{12}{x_n}\right)$, $n \in N \cup \{0\}$. Such a scheme, with suitable x_0, will*

(a) not converge (b) converge to 1.6
(c) converge to 1.8 (d) converge to 2

22. *In the (x, t) plane, the characteristics of the initial value problem $u_t + uu_x = 0$, with $u(x, 0) = x$, $0 \le x \le 1$, are*

(a) parallel straight lines
(b) straight lines which intersect at (0, – 1)
(c) non-intersecting parabolas
(d) concentric circles with centre at the origin

23. *Suppose u(x, y) satisfies Laplace's equation : $\nabla^2 u = 0$ in R^2 and $u = x$ on the unit circle. Then, at the origin*

(a) u tends to infinity (b) u attains a finite minimum
(c) u attains a finite maximum (d) u is equal to 0

24. *A circular disk of radius a and mass m is supported on a needle at its centre. The disk is set spinning with initial angular velocity ω_0 about an axis making an angle $\pi/6$ with the normal to the disk. If $\overline{\omega}(t)$ is the angular velocity of the disk at any time t, then its component along the normal equals*

(a) $\frac{\sqrt{3}\omega_0}{2}$ (b) ω_0

(c) $\omega_0 \sin t$ (d) $\left(\frac{\sqrt{3}\omega_0}{2}\right)\cos t$

25. *In R^2 with usual topology, the set $U = \{(x, -y) \in R^2 : x = 0, 1, -1 \text{ and } y \in N\}$ is*

(a) neither closed nor bounded (b) closed but not bounded

(c) bounded but not closed (d) closed and bounded

26. *In R^3 with usual topology, let $V = \{(x, y, z) \in R^3 : x^2 + y^2 + z^2 = 1, y \neq 0\}$ and $W = \{(x, y, z) \in R^3 : y = 0\}$. Then $V \cup W$ is*

(a) connected and compact (b) connected but not compact

(c) compact but not connected (d) neither connected nor compact

27. *Suppose X is a random variable, c is a constant and $a_n = E(X - c)^n$ is finite for all $n \geq 1$. Then $P(X = c) = 1$ if and only if $a_n = 0$ for*

(a) at least one $n \geq 1$ (b) at least one odd n

(c) at least one even n (d) at least two values of n

28. *If the random vector $(X_1, X_2)^T$ has a bivariate normal distribution with mean vector $(\mu, \mu)^T$ and the matrix $((E(X_i X_j)))_{1 \leq i, j \leq 2}$ equals $\begin{pmatrix} \alpha_1 & \mu^2 \\ \mu^2 & \alpha_2 \end{pmatrix}$, where $\mu \in R$ and $\alpha_1, \alpha_2 > \mu^2$, then X_1 and X_2 are*

(a) independent for all α_1 and α_2

(b) independent if and only if $\alpha_1 = \alpha_2$

(c) uncorrelated, but not independent for all α_1, α_2

(d) uncorrelated if and only if $\alpha_1 = \alpha_2$ and in this case they are not independent

29. *If the cost matrix for an assignment problem is given by*

$$\begin{pmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{pmatrix}$$

where a, b, c, d > 0, then the value of the assignment problem is

(a) $a + b + c + d$ (b) $\min\{a, b, c, d\}$

(c) $\max\{a, b, c, d\}$ (d) $4\min\{a, b, c, d\}$

30. Extremals for the variational problem $v[y(x)] = \int_1^2 (y'^2 + x^2 y'^2)\,dx$ *satisfy the differential equation*

(a) $x^2 y'' + 2xy' - y = 0$ (b) $x^2 y'' - 2xy' + y = 0$

(c) $2xy' - y = 0$ (d) $x^2 y'' - y = 0$

Q. 31-90 carry two marks each

31. *Let V be the subspace of R^3 spanned by $u = (1, 1, 1)$ and $v = (1, 1, -1)$. The orthonormal basis of V obtained by the Gram-Schmidt process on the ordered basis (u, v) of V is*

(a) $\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\right)\right\}$

(b) $\{(1, 1, 0), (1, 0, 1)\}$

(c) $\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)\right\}$

(d) $\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)\right\}$

32. *In R^2, $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 x_2 - \alpha(x_2 y_1 + x_1 y_2) + y_1 y_2$ is an inner product*

(a) for all $\alpha \in R$
(b) if and only if $\alpha = 0$
(c) if and only if $\alpha < 1$
(d) if and only if $|\alpha| < 1$

33. *Let $\{v_1, v_2, v_3, v_4\}$ be a basis of R^4 and $v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$ where $a_i \in R$, $i = 1, 2, 3, 4$. Then $\{v_1 - v, v_2 - v, v_3 - v, v_4 - v\}$ is a basis of R^4 if and only if*

(a) $a_1 = a_2 = a_3 = a_4$
(b) $a_1 a_2 a_3 a_4 = -1$
(c) $a_1 + a_2 + a_3 + a_4 \neq 0$
(d) $a_1 + a_2 + a_3 + a_4 \neq 0$

34. *Let $R^{2\times 2}$ be the real vector space of all 2×2 real matrices. For $Q = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$, define a linear transformation T on $R^{2\times 2}$ as $T(P) = QP$. Then the rank of T is*

(a) 1
(b) 2
(c) 3
(d) 4

35. *Let P be a $n \times n$ matrix with integral entries and $Q = P + \frac{1}{2} I$, where I denotes the $n \times n$ identity matrix. Then Q is*

(a) idempotent, *i.e.* $Q^2 = Q$
(b) invertible
(c) nilpotent
(d) unipotent, *i.e.*, $Q - I$ is nilpotent

36. *Let M be a square matrix of order, 2 such that rank of M is 1. Then M is*

(a) diagonalizable and nonsingular
(b) diagonalizable and nilpotent
(c) neither diagonalizable nor nilpotent
(d) either diagonalizable or nilpotent but not both

37. *If M is a 7×5 matrix of rank 3 and N is a 5×7 matrix of rank 5, then rank (MN) is*

(a) 5
(b) 3
(c) 2
(d) 1

38. $\int_0^{2\pi} \frac{d\theta}{13 - 5\sin\theta} =$

(a) $-\frac{\pi}{6}$
(b) $-\frac{\pi}{12}$
(c) $\frac{\pi}{12}$
(d) $\frac{\pi}{6}$

39. *In the Laurent series expansion of* $f(z) = \dfrac{1}{z-1} - \dfrac{1}{z-2}$ *valid in the region* $|z| > 2$, *the coefficient of* $\dfrac{1}{z^2}$ *is*

(a) -1 (b) 0
(c) 1 (d) 2

40. *Let* $w = f(z)$ *be the bilinear transformation that maps* $-1, 0$ *and* 1 *to* $-i, 1$ *and* i *respectively. Then* $f(1-i)$ *equals*

(a) $-1+2i$ (b) $2i$
(c) $-2+i$ (d) $-1+i$

41. *For the positively oriented unit circle,* $\oint\limits_{|z|=1} \dfrac{2\,Re(z)}{z+2}\,dz =$

(a) 0 (b) πi
(c) $2\pi i$ (d) $4\pi i$

42. *The number of zeroes, counting multiplicities, of the polynomial* $z^5 + 3z^3 + z^2 + 1$ *inside the circle* $|z| = 2$ *is*

(a) 0 (b) 2
(c) 3 (d) 5

43. *Let* $f = u + iv$ *and* $g = v + iu$ *be non-zero analytic functions on* $|z| < 1$. *Then it follows that*

(a) $f' \equiv 0$ (b) f is conformal on $|z| < 1$
(c) $f \equiv kg$ for some real k (d) f is one to one

44. *If* $f(x, y) = \begin{cases} x^3/(x^2+y^2), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$, *then at* $(0, 0)$

(a) f_x, f_y do not exist
(b) f_x, f_y exist and are equal
(c) the directional derivative exists along any straight line
(d) f is differentiable

45. *Let* $\sigma > 1$ *and* $g(x) = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^x}$, $\sigma \le x < \infty$. *Then* $g(x)$ *is*

(a) not continuous
(b) continuous but not differentiable
(c) differentiable but not continuously differentiable
(d) continuously differentiable

46. *The sequence of functions* (f_n) *on* $[0, 1]$ *with Lebesgue measure, defined by*

$$f_n(x) = \begin{cases} x, & 0 \le x < 1 - 1/n \\ \sqrt{n}, & 1 - 1/n \le x \le 1 \end{cases},\ \text{converges}$$

(*a*) almost everywhere and as well as in L^1
(*b*) almost everywhere but not in L^1
(*c*) in L^1, but not almost everywhere
(*d*) neither almost everywhere nor in L^1

47. ***Consider two sequences $\{f_n\}$ and $\{g_n\}$ of functions where $f_n : [0, 1] \to R$ and $g_n : R \to R$ are defined by***

$$f_n(x) = x^n \text{ and } g_n(x) = \begin{cases} \cos(x - n)\pi/2 & \text{if } x \in [n-1, n+1] \\ 0 & \text{otherwise} \end{cases}. \text{ Then}$$

(*a*) neither $\{f_n\}$ nor $\{g_n\}$ is uniformly convergent
(*b*) $\{f_n\}$ is not uniformly convergent but $\{g_n\}$ is
(*c*) $\{g_n\}$ is not uniformly convergent but $\{f_n\}$ is
(*d*) both $\{f_n\}$ and $\{g_n\}$ are uniformly convergent

48. ***Let $f : [0, 1] \to R$ and $g : [0, 1] \to R$ be two functions defined by***

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n}, n \in N \\ 0 & \text{otherwise} \end{cases} \text{ and } g(x) = \begin{cases} n & \text{if } x = \frac{1}{n}, n \in N \\ 0 & \text{otherwise} \end{cases}. \text{ Then}$$

(*a*) both f and g are Riemann integrable
(*b*) f is Riemann integrable but g is not
(*c*) g is Riemann integrable but f is not
(*d*) neither f nor g is Riemann integrable

49. ***The set of all continuous functions $f : [0, 1] \to R$ satisfying***

$$\int_0^1 t^n f(t)\, dt = 0, \quad n = 0, 1, 2, \ldots$$

(*a*) is empty
(*b*) contains a single element
(*c*) is countably infinite
(*d*) is uncountably infinite

50. ***Let $f : R^3 \to R^3$ be defined by $f(x_1, x_2, x_3) = (x_2 + x_3, x_3 + x_1, x_1 + x_2)$. Then the first derivative of f is***
(*a*) not invertible anywhere
(*b*) invertible only at the origin
(*c*) invertible everywhere except at the origin
(*d*) invertible everywhere

51. ***Let $y = \varphi(x)$ and $y = \psi(x)$ be solutions of $y'' - 2xy' + (\sin x^2)\, y = 0$ such that $\varphi(0) = 1$, $\varphi'(0) = 1$ and $\psi(0) = 1$, $\psi'(0) = 2$. Then the value of the Wronskian $W(\varphi, \psi)$ at $x = 1$ is***
(*a*) 0
(*b*) 1
(*c*) e
(*d*) e^2

52. ***The set of all eigenvalues of the Sturm-Liouville problem***

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'\left(\frac{\pi}{2}\right) = 0,$$

is given by
(*a*) $\lambda = 2n$, $n = 1, 2, 3, \ldots\ldots$
(*b*) $\lambda = 2n$, $n = 0, 1, 2, 3, \ldots\ldots$
(*c*) $\lambda = 4n^2$, $n = 1, 2, 3, \ldots\ldots$
(*d*) $\lambda = 4n^2$, $n = 0, 1, 2, 3, \ldots\ldots$

53. *If Y(p) is the Laplace transform of y(t), which is the solution of the initial value problem* $\frac{d^2y}{dt^2} + y(t) = \begin{cases} 0, & 0 < t < 2\pi \\ \sin t, & t > 2\pi \end{cases}$ *, with* $y(0) = 1$ *and* $y'(0) = 0$*, then Y(p) equals*

(a) $\frac{p}{1+p^2} + \frac{e^{-2\pi p}}{(1+p^2)^2}$ (b) $\frac{p+1}{1+p^2}$

(c) $\frac{p}{1+p^2} + \frac{e^{-2\pi p}}{(1+p^2)}$ (d) $\frac{p(1+p^2)+1}{(1+p^2)^2}$

54. *If* $y = \sum_{m=0}^{\infty} a_m x^m$ *is a solution of* $y'' + xy' + 3y = 0$*, then* $\frac{a_m}{a_{m+2}}$ *equals*

(a) $\frac{(m+1)(m+2)}{m+3}$ (b) $-\frac{(m+1)(m+2)}{m+3}$

(c) $-\frac{m(m-1)}{m+3}$ (d) $\frac{m(m-1)}{m+3}$

55. *The indicial equation for :* $x(1+x^2)\,y'' + (\cos x)y' + (1 - 3x + x^2)\,y = 0$*, is*

(a) $r^2 - r = 0$ (b) $r^2 + r = 0$

(c) $r^2 = 0$ (d) $r^2 - 1 = 0$

56. *The general solution* $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ *of the system*

$x = -x + 2y$

$y = 4x + y$

is given by

(a) $\begin{pmatrix} c_1e^{3t} - c_2e^{-3t} \\ 2c_1e^{3t} + c_2e^{-3t} \end{pmatrix}$ (b) $\begin{pmatrix} c_1e^{3t} \\ c_2e^{-3t} \end{pmatrix}$

(c) $\begin{pmatrix} c_1e^{3t} + c_2e^{-3t} \\ 2c_1e^{3t} + c_2e^{-3t} \end{pmatrix}$ (d) $\begin{pmatrix} c_1e^{3t} - c_2e^{-3t} \\ -2c_1e^{3t} + c_2e^{-3t} \end{pmatrix}$

57. *Let G and H be two groups. The groups* $G \times H$ *and* $H \times G$ *are isomorphic*

(a) for any G and any H (b) only if one of them is cyclic

(c) only if one of them is abelian (d) only if G and H are isomorphic

58. *Let* $H = Z_2 \times Z_6$ *and* $K = Z_2 \times Z_4$*. Then*

(a) H is isomorphic to K since both are cyclic

(b) H is isomorphic to K since 2 divides 6 and g.c.d. (3, 4) = 1

(c) H is not isomorphic to K since K is cyclic whereas H is no

(d) H is not isomorphic to K since there is no homomorphism from H to K

59. *Suppose G denotes the multiplicative group $\{-1, 1\}$ and $S = \{z \in C : |z| = 1\}$. Let G act on S by complex multiplication. Then the cardinality of the orbit of i is*

(a) 1 (b) 2

(c) 5 (d) infinite

60. *The number of 5-Sylow subgroups of Z_{20} is*

(a) 1 (b) 4

(c) 5 (d) 6

61. *Let $S = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in R\right\}$ be the ring under matrix addition and multiplication.*

Then the subset $\left\{\begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} : p \in R\right\}$ is

(a) not an ideal of S

(b) an ideal but not a prime ideal of S

(c) is a prime ideal but not a maximal ideal of S

(d) is a maximal ideal of S

62. *Consider $S = C[x^5]$, complex polynomials is x^5, as a subset of $T = C[x]$, the ring of all complex polynomials. Then*

(a) S is neither an ideal nor a subring of T

(b) S is an ideal, but not a subring of T

(c) S is a subring but not an ideal of T

(d) S is both a subring and an ideal of T

63. *Which of the following statements is true about $S = Z[x]$?*

(a) S is an Euclidean domain since all its ideals are principal

(b) S is an Euclidean domain since Z is an Euclidean domain

(c) S is not an Euclidean domain since S is not even an integral domain

(d) S is not an Euclidean domain since it has non-principal ideals

64. *Let X be the space of bounded real sequences with sup norm. Define a linear operator $T : X \to X$ by*

$$T(x) = \left(\frac{x_1}{1}, \frac{x_2}{2}, \cdots\right) \text{ for } x = (x_1, x_2, \ldots) \in X. \text{ Then}$$

(a) T is bounded but not one to one

(b) T is one to one but not bounded

(c) T is bounded and its inverse (from range of T) exists but is not bounded

(d) T is bounded and its inverse (from range of T) exists and is bounded

65. *Let X be the space of real sequences having finitely many non-zero terms with $\|\cdot\|_p$, $1 \le p \le \infty$. Then*

(a) f is continuous only for $p = 1$ (b) f is continuous only for $p = 2$

(c) f is continuous only for $p = \infty$ (d) f is not continuous for any p, $1 \le p \le \infty$

66. ***Let $X = C^1\,[0, 1]$ with the norm $\| x \| = \| x \|_\infty + \| x' \|_\infty$ (where x' is the derivative of x) and $Y = C^1[0, 1]$ with sup norm. If T is the identity operator from X into Y, then***

(*a*) T and T^{-1} are continuous
(*b*) T is continuous but T^{-1} is not
(*c*) T^{-1} is continuous but T is not
(*d*) neither T nor T^{-1} is continuous

67. ***Let $X = C[-1, 1]$ with the inner product defined by***

$$\langle x, y\rangle = \int_{-1}^{1} x(t)\, y(t)\, dt\,.$$

Let Y be the set of all odd functions in X. Then

(*a*) $Y \perp$ is the set of all even functions in X
(*b*) $Y \perp$ is the set of odd functions in X
(*c*) $Y \perp = \{0\}$
(*d*) $Y \perp$ is the set of all constant functions in X

68. ***Let $X = l^2$, the space of all square-summable sequences with***

$$\| x \| = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}\,,\ \text{for } x = (x_i) \in X.$$

Define a sequence $\{T_n\}$ of linear operators on X by $T_n(x) = (x_1, x_2, ..., x_n, 0, 0, ...)$. Then

(*a*) T_n is an unbounded operator for sufficiently large n
(*b*) T_n is bounded but not compact for all n
(*c*) T_n is compact for all n but $\lim_{n\to\infty} T_n$ is not compact
(*d*) T_n is compact for all n and so is $\lim_{n\to\infty} T_n$

69. ***To find the positive square root of $a > 0$ by solving $x^2 - a = 0$ by the Newton-Raphson method, if x_n denotes the n^{th} iterate with $x_0 > 0$, $x_0 \neq \sqrt{a}$, then the sequence $\{x_n, n \geq 1\}$ is***

(*a*) strictly decreasing
(*b*) strictly increasing
(*c*) constant
(*d*) not convergent

70. ***In solving the ordinary differential equation $y' = 2x$, $y(0) = 0$ using Euler's method, the iterates y_n, $n \in N$ satisfy***

(*a*) $y_n = x_n^{\,2}$
(*b*) $y_n = 2x_n$
(*c*) $y_n = x_n\, x_{n-1}$
(*d*) $y_n = x_{n-1} + x_n$

71. ***The characteristic curves of the partial differential equation***

$$(2x + u)u_x + (2y + u)\, u_y = u,$$

passing through (1, 1) for any arbitrary initial values prescribed on a non-characteristic curve are given by

(*a*) $x = y$
(*b*) $x^2 + y^2 = 2$
(*c*) $x + y = 2$
(*d*) $x^2 - xy + y^2 = 1$

72. ***The solution of Laplace's equation***

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

in the unit disk with boundary conditions $u(1, \theta) = 2\cos^2\theta$ is given by

(a) $1 + r^2 \cos 2\theta$ (b) $1 + \ln r + r\cos 2\theta$

(c) $2r^3\cos^2\theta$ (d) $1 - r^2 + 2r^2\cos^2\theta$

73. *For the heat equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ on } R \times [0, T], \quad \text{with } u(x, 0) = u_0(x),\ u_0 \in L^2(R),$$

(a) the solution is reversible in time

(b) if $u_0(x)$ has compact support, so does $u(x, t)$ for any given t

(c) if $u_0(x)$ is discontinuous at a point, so is $u(x, t)$ for any given t

(d) if $u_0(x) \geq 0$ for all x, then $u(x, t) \geq 0$ for all x and $t > 0$

74. *If $u(x, t)$ satisfies the wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad x \in R,\ t > 0, \text{ with initial conditions}$$

$$u(x, 0) = \begin{cases} \sin \dfrac{\pi x}{c}, & 0 \leq x \leq c \\ 0 & \text{elsewhere} \end{cases}, \text{ and } u_t(x, 0) = 0 \text{ for all } x, \text{ then for a given } t > 0,$$

(a) there are values of x at which $u(x, t)$ is discontinuous

(b) $u(x, t)$ is continuous, but $u_x(x, t)$ is not continuous

(c) $u(x, t)$, $u_x(x, t)$ are continuous, but $u_{xx}(x, t)$ is not continuous

(d) $u(x, t)$ is smooth for all x

75. *A rigid body is acted on by two forces, $F_1 = a\,\hat{i} + b\,\hat{j} - 3\hat{k}$ at the point $(1, 2, -1)$ and $F_2 = \hat{i} + a\hat{j} + b\hat{k}$ at the point $(-1, 0, 1)$. If the force system is equipollent to the force F and the couple G, which have no components along $\hat{k}$, then F equals*

(a) $2\hat{i} + 4\hat{j}$ (b) $2\hat{i} - 4\hat{j}$

(c) $4\hat{i} + 2\hat{j}$ (d) $4\hat{i} - 2\hat{j}$

76. *A frictionless wire, fixed at R, rotates with constant angular velocity ω about a vertical axis RO (O is the origin and R is above O), making a constant angle α with it. A particle P of unit mass is constrained to move on the wire. If the mass of the wire is negligible, distance OR is h and RP is $r(t)$ at any time t, then the Lagrangian of the motion is*

(a) $\frac{1}{2}\ \dot{r}^2 - g(h - r\cos\alpha)$ (b) $\frac{1}{2}\ (\dot{r}^2 + \omega^2 r^2) + g\, r\cos\alpha$

(c) $\frac{1}{2}\ (\dot{r}^2 + \omega^2 r^2 \sin^2\alpha) - g(h - r\cos\alpha)$ (d) $\frac{1}{2}\ (\dot{r}^2 + r^2\sin^2\alpha) - gh$

77. *In R with the usual topology, the set $U = \{x \in R : -1 \leq x \leq 1, x \neq 0\}$ is*

(a) neither Hausdorff nor first countable

(b) Hausdorff but not first countable

(c) first countable but not Hausdorff

(d) both Hausdorff and first countable

78. Suppose $U = \{x \in Q : 0 \le x \le 1\}$ and $V = \{x \in Q : 0 < x < 2\}$. Let n and m be the number of connected components of U and V respectively. Then

(a) $m = n = 1$ (b) $m = n \neq 1$

(c) $m = 2n$, m, n finite (d) $m > 2n$

79. Let $f : [0, 1] \to R$ be the continuous function defined by

$$f(x) = \frac{(x-1)(x-2)}{(x-3)(x-4)}.$$

Then the maximal subset of R on which f has a continuous extension is

(a) $(-\infty, 3)$ (b) $(-\infty, 3) \cup (4, \infty)$

(c) $R \backslash \{3, 4\}$ (d) R

80. Suppose $U = (0, 1/2) \times (0, 1/2)$, $V = (-1/2, 0) \times (-1/2, 0)$ and D be the open unit disk with centre at origin of R^2. Let f be a real valued continuous function on D such that $f(U) = 0$. Then it follows that

(a) $f(v) = 0$ for every v in V (b) $f(v) \neq 0$ for every v in V

(c) $f(v) = 0$ for some v in V (d) f can assume any real value on V

81. Suppose X is a random variable and $f, g : R \to R$ are measurable functions such that $f(X)$ and $g(X)$ are independent, then

(a) X is degenerate

(b) both $f(X)$ and $g(X)$ is degenerate

(c) either $f(X)$ or $g(X)$ is degenerate

(d) X, $f(X)$ and $g(X)$ could all be non-degenerate

82. Suppose $X_1, X_2, \ldots X_n$ is a random sample from a $N(\mu, \sigma^2)$ distribution, where μ is known, but σ^2 is not. If $\overline{X} = \frac{1}{n}\sum_{i=0}^{n} X_i$ and $S = \sqrt{\frac{1}{n}\sum_{i=0}^{n}(X_i - \mu)^2}$, then the pair $(\overline{X}, S)$ is

(a) complete and sufficient (b) complete but not sufficient

(c) sufficient but not complete (d) neither sufficient nor complete

83. If X and Y are random variables with $0 < Var(X), Var(Y) < \infty$, consider the statements : (I) $Var(E(Y/X)) = Var(Y)$ and (II) the correlation co-efficient between X and Y is ± 1. Then

(a) (I) implies (II) and (II) implies (I)

(b) (I) implies (II) but (II) does not imply (I)

(c) (II) implies (I) but (I) does not imply (II)

(d) neither does (I) imply (II) nor does (II) imply (I)

84. If the random variable X has a Poisson distribution with parameter λ and the parametric space has three elements 3, 4 and k, then to test the null hypothesis $H_0 : \lambda = 3$ vs. the alternative hypothesis $H_1 : \lambda \neq 3$, a uniformly most powerful test at any level $\alpha \in (0.1)$ exists for any sample size

(a) for all $k \neq 3, 4$ (b) if and only if $k > 4$

(c) if and only if $k < 3$ (d) if and only if $k > 3$

85. Suppose the random variable X has a uniform distribution P_θ in the interval $[\theta - 1, \theta + 1]$, where $\theta \in Z$. If a random sample of size n is drawn from this distribution, then P_θ almost surely for all $\theta \in Z$, a maximum likelihood estimator (MLE) for θ

(a) exists and is unique (b) exists but may or may not be unique

(c) exists but cannot be unique (d) does not exist

86. *A χ^2 (chi-squared) test for independence between two attributes X and Y is carried out at 2.5% level of significance on the following 2 × 2 contingency table showing frequencies*

Y \ X	X_1	X_2
Y_1	1	0
Y_2	1	d

If the upper 2.5% point of the χ_1^2 distribution is given as 5.0, then the hypothesis of independence is to be rejected if and only if

(a) $d > 1$ (b) $d > 3$ (c) $d > 5$ (d) $d > 9$

87. *Consider the Linear Programming Problem (LPP) :*
Maximize x_1
subject to : $3x_1 + 4x_2 \le 10$, $5x_1 - x_2 \le 9$, $3x_1 - 2x_2 \ge -2$, $x_1 - 3x_2 \le 3$, $x_1, x_2 \ge 0$.
The value of the LPP is

(a) $\frac{9}{5}$ (b) 2 (c) 3 (d) $\frac{10}{3}$

88. *Given that the eigenvalues of the integral equation*

$$y(x) = l \int_0^{2\pi} \cos(x+t)\, y(t)\, dt$$

are $\frac{1}{\pi}$ and $-\frac{1}{\pi}$ with respective eigenfunctions cos x and sin x. Then the integral equation

$$y(x) = \sin x + \cos x + \lambda \int_0^{2\pi} \cos(x+t)\, y(t)\, dt \text{ has}$$

(a) unique solution for $\lambda = 1/\pi$ (b) unique solution for $\lambda = -1/\pi$
(c) unique solution for $\lambda = \pi$ (d) no solution for $\lambda = -\pi$

89. *The values of λ for which the integral equation*

$$y(x) = \lambda \int_0^1 (6x - t)\, y(t)\, dt$$

has a nontrivial solution, are given by the roots of the equation

(a) $(3\lambda - 1)(2 + \lambda) - \lambda^2 = 0$ (b) $(3\lambda - 1)(2 + \lambda) + 2 = 0$
(c) $(3\lambda - 1)(2 + \lambda) - 4\lambda^2 = 0$ (d) $(3\lambda - 1)(2 + \lambda) + \lambda^3 = 0$

90. *The extremals for the functional*

$$v[y(x)] = \int_{x_0}^{x_1} (xy' + y'2)\, dx$$

are given by the following family of curves :

(a) $y = c_1 + c_2 x + \left(\frac{x^2}{4}\right)$ (b) $y = 1 + c_1 x + c_2 \left(\frac{x^2}{4}\right)$

(c) $y = c_1 + x + c_2 \left(\frac{x^2}{4}\right)$ (d) $y = c_1 + c_2 x - \left(\frac{x^2}{4}\right)$