NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

September 19, 2009

Time Allowed: 150 Minutes

Maximum Marks: 30

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 8 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.

- There are three parts to this test: Algebra, Analysis and Geometry. Each part consists of **10** questions adding up to **30** questions in all.

- Answer each question, as directed, in the space provided for it in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so do not answer questions on it.

- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).

- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.

- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol I will denote the identity matrix of appropriate order. All logarithms, unless specified otherwise, are to the base e.

- **Calculators are not allowed.**

Section 1: Algebra

1.1 A polynomial in the variable x leaves a remainder 2 when divided by $(x-3)$ and a remainder 3 when divided by $(x-2)$. Find the remainder when it is divided by $(x-2)(x-3)$.

1.2 Let $a_1, \cdots, a_n \in \mathbb{R}$. Evaluate the determinant:

$$
\begin{vmatrix}
1 + a_1 & a_2 & \cdots & a_n \\
a_1 & 1 + a_2 & \cdots & a_n \\
\cdots & \cdots & \cdots & \cdots \\
a_1 & a_2 & \cdots & 1 + a_n
\end{vmatrix}.
$$

1.3 Find the roots of the equation

$$
27x^3 + 42x^2 - 28x - 8 \;=\; 0
$$

given that they are in geometric progression.

1.4 Which of the following form a group?
a.

$$
G \;=\; \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} \;:\; a \in \mathbb{R}, \; a \neq 0 \right\}
$$

with respect to matrix multiplication.
b. $\mathbb{Z}_4$, the set of all integers modulo 4, with respect to multiplication.
c.

$$
G \;=\; \{ f : [0,1] \rightarrow \mathbb{R} \;;\; f \text{continuous} \}
$$

with respect to the operation defined by $(f.g)(x) = f(x)g(x)$ for all $x \in [0,1]$.

1.5 Let G be a group and let H and K be subgroups of order 8 and 15 respectively. What is the order of the subgroup $H \cap K$?

1.6 Let G be a finite abelian group of odd order. Which of the following define an automorphism of G?
a. The map $x \mapsto x^{-1}$ for all $x \in G$.
b. The map $x \mapsto x^2$ for all $x \in G$.
c. The map $x \mapsto x^{-2}$ for all $x \in G$.

1.7 An algebraic number is one which occurs as the root of a monic polynomial with rational coefficients. Which of the following numbers are algebraic?
a. $5 + \sqrt{3}$
b. $7 + 2^{\frac{1}{3}}$
c. $\cos \frac{2\pi}{n}$, where $n \in \mathbb{N}$

1.8 Given that the matrix
$$\begin{pmatrix} \alpha & 1 \\ 2 & 3 \end{pmatrix}$$
has 1 as an eigenvalue, compute its trace and its determinant.

1.9 Let A be a non-diagonal 2×2 matrix with complex entries such that $A = A^{-1}$. Write down its characteristic and minimal polynomials.

1.10 Pick out the true statements:
a. Let A and B be two arbitrary $n \times n$ matrices. Then

$$(A + B)^2 \;=\; A^2 + 2AB + B^2.$$

b. There exist $n \times n$ matrices A and B such that

$$AB - BA \;=\; I.$$

c. Let A and B be two arbitrary $n \times n$ matrices. If B is invertible, then

$$\mathrm{tr}(A) \;=\; \mathrm{tr}(B^{-1}AB)$$

where $\mathrm{tr}(M)$ denotes the trace of an $n \times n$ matrix M.

Section 2: Analysis

2.1 Evaluate:
$$\lim_{x \to \infty} x \left(\log \left(1 + \frac{x}{2} \right) - \log \frac{x}{2} \right).$$

2.2 Evaluate:
$$\int_0^{\frac{\pi}{2}} \log \tan \theta \; d\theta.$$

2.3 Test the following series for convergence:

a.
$$\sum_{n=1}^{\infty} \left(n^{\frac{1}{n}} - 1 \right)^n$$

b.
$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$$

where $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms.

2.4 Which of the following functions are continuous?

a.
$$f(x) = [x] + (x - [x])^{[x]}, \; x \geq \frac{1}{2},$$

where $[x]$ denotes the largest integer less than, or equal to, x.

b.
$$f(x) = \lim_{n \to \infty} \frac{x^2 e^{nx} + x}{e^{nx} + 1}, \; x \in \mathbb{R}.$$

c.
$$f(x) = \lim_{n \to \infty} \frac{1}{n} \log \left(e^n + x^n \right), \; x \geq 0.$$

2.5 Write down the coefficient of x^7 in the Taylor series expansion of the function
$$f(x) = \log(x + \sqrt{1 + x^2})$$
about the origin.

2.6 Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x = a$. Evaluate:

$$\lim_{n \to \infty} \left\{ f \left(a + \frac{1}{n^2} \right) + f \left(a + \frac{2}{n^2} \right) + \cdots + f \left(a + \frac{n}{n^2} \right) - n f(a) \right\}.$$

2.7 Let
$$f(x, y) = x^4 - 2x^2y^2 + y^4 + x^2 - 6xy + 9y^2.$$
Examine whether f admits a local maximum or minimum at $(0, 0)$.

2.8 Let $z = x + iy \in \mathbb{C}$ and let f be defined by

$$f(z) = y - x - 3x^2 i.$$

If C is the straightline segment joining $z = 0$ to $z = 1 + i$, compute

$$\int_C f(z) \, dz.$$

2.9 Let C be the contour consisting of the lines $x = \pm 2$ and $y = \pm 2$, described counterclockwise in the plane. Compute

$$\int_C \frac{z}{2z + 1} \, dz.$$

2.10 Let
$$f(z) = \frac{5z - 2}{z(z - 1)}.$$
Write down the residues of f at each of its poles.

Section 3: Geometry

3.1 Let M_1 ad M_2 be two points in the plane whose polar coordinates are given as $(12, 4\pi/9)$ and $(12, -2\pi/9)$ respectively. Find the polar coordinates of the midpoint of the line segment joining these points.

3.2 The length of the sides of a rhombus is given as $5\sqrt{2}$. If two of its opposite vertices have coordinates $(3, -4)$ and $(1, 2)$, find the length of the altitude of the rhombus.

3.3 What figure does the equation

$$\sum_{i,j=1}^{2} a_{ij}x_i x_j \ = \ 1$$

represent when $A = (a_{ij})$ is the matrix given by
a.

$$A \ = \ \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}?$$

b.

$$A \ = \ \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}?$$

3.4 Find the locus of a point which moves such that the ratio of its distance from the point $(-5, 0)$ to its distance from the line $5x + 16 = 0$ is $5/4$.

3.5 Express the equations of the curves given below in parametric form in the form $f(x, y) = 0$.
a.

$$x \ = \ \frac{a}{2}\left(t + \frac{1}{t}\right), \ y \ = \ \frac{b}{2}\left(t - \frac{1}{t}\right).$$

b.

$$x \ = \ 2R\cos^2 t, \ y \ = \ R\sin 2t.$$

3.6 Let A_n be the area of the polygon whose vertices are given by the n-th roots of unity in the complex plane. Evaluate:

$$\lim_{n\to\infty} A_n.$$

3.7 Write down the equation of the diameter of the sphere

$$x^2 + y^2 + z^2 + 2x - 6y + z - 11 \ = \ 0$$

which is perpendicular to the plane $5x - y + 2z = 17$.

3.8 Find the values of a for which the plane $x + y + z = a$ is tangent to the sphere $x^2 + y^2 + z^2 = 12$.

3.9 Let $d(P, Q)$ denote the distance between two points P and Q in the plane. Let

$$A \ = \ \{(x, y) \in \mathbb{R}^2 \ : \ xy = 0\}, \ \text{and} \ B \ = \ \{(x, y) \in \mathbb{R}^2 \ : \ xy = 1\}.$$

Compute:

$$\inf_{p \in A, Q \in B} d(P, Q).$$

3.10 A ray, having the origin as its end-point, initially coincides with the x-axis and rotates about the origin in the plane with constant angular velocity ω. A point starts at the origin and moves along the ray with constant velocity v. Write down the parametric equations of the locus of the point in the form $x = \varphi(t), y = \psi(t)$, with time t as the parameter.