2016

BOOKLET No. TEST CODE : PMB

Afternoon

Duration of test : 2 hours

Write your *registration number*, *test code*, *booklet no.*, etc. in the appropriate places on your ANSWER BOOKLET.

This test has questions arranged in two groups.

Each group consists of 6 questions.

You need to answer 4 questions FROM EACH GROUP.

Each question carries 10 marks. Total marks = 80.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET
AND/OR ON YOUR ANSWER BOOKLET.
CALCULATORS ARE NOT ALLOWED.

STOP ! WAIT FOR THE SIGNAL TO START

INSTRUCTIONS FOR CANDIDATES

- Please answer FOUR questions from EACH group.
- Each question carries 10 marks. Total marks : 80.
- $\mathbb{R}, \mathbb{C}, \mathbb{Q}$ and $\mathbb{N}$ denote respectively the set of all real numbers, the set of all complex numbers, the set of all rational numbers and the set of all positive integers.

Group A

1. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers defined as follows: $x_1 = 1$ and for all $n \in \mathbb{N}$, $x_{n+1} = (3 + 2x_n)/(3 + x_n)$.

   (a) Show that there exists $\lambda \in (0, 1)$ such that for all $n \geq 2$,

   $$|x_{n+1} - x_n| \leq \lambda |x_n - x_{n-1}|.$$

   (b) Prove that $\lim_{n\to\infty} x_n$ exists and find its value.

2. Examine, with justification, whether the following limit exists:

   $$\lim_{N\to\infty} \int_N^{e^N} x e^{-x^{2016}} dx.$$

   If the limit exists, then find its value.

3. Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ that takes every real value exactly twice? Justify your answer.

4. Suppose $f : [0, 1] \to \mathbb{R}$ is a bounded function such that f is Riemann integrable on $[a, 1]$ for every $a \in (0, 1)$. Is f Riemann integrable on $[0, 1]$? Justify your answer. **[P. T. O.]**

5. Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a surjective function such that

$$\|h(\mathbf{x}) - h(\mathbf{y})\| \geq 3\|\mathbf{x} - \mathbf{y}\|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$. Show that the image of every open set (in $\mathbb{R}^2$) under the map h is an open set (in $\mathbb{R}^2$).

6. Suppose that $g : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function and

$$D = \{(x,y) \in \mathbb{R}^2 : 0 < x < y < 1\}.$$

Define a new function $G : [0,1] \times [0,1] \to \mathbb{R}$ by

$$G(x,v) = \int_0^x g(u,v)du, \quad (x,v) \in [0,1] \times [0,1].$$

Now define another function $\psi : D \to \mathbb{R}$ by

$$\psi(x,y) = \int_x^y G(x,v)dv, \quad (x,y) \in D.$$

Does $\frac{\partial \psi}{\partial y}$ exist at every point in D? Justify your answer.

Group B

7. Let A be a 2×2 matrix with complex entries. Suppose that $\det(A) = 0$ and $\operatorname{trace}(A) \neq 0$. Show the followimg:

   (a) $\operatorname{Kernel}(A) \cap \operatorname{Range}(A) = \{\mathbf{0}\}$.

   (b) $\mathbb{C}^2 = \operatorname{span}\big(\operatorname{Kernel}(A) \cup \operatorname{Range}(A)\big)$.

8. Suppose that B is a nonzero 2×2 matrix with complex entries. Prove that $B^2 = 0$ if and only if the B and the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are similar.

9. Let S_{17} be group of all permutations of 17 distinct symbols. How many subgroups of order 17 does S_{17} have? Justify your answer.

10. Suppose that H and K are two subgroups of a group G. Assume that $[G : H] = 2$ and K is not a subgroup of H. Show that $HK = G$.

11. For any ring R, let $R[X]$ denote the ring of all polynomials with indeterminate X and coefficients from R. Examine, with justification, whether the following pairs of rings are isomorphic:

    (a) $\mathbb{R}[X]$ and $\mathbb{C}[X]$.

    (b) $\mathbb{Q}[X]/(X^2 - X)$ and $\mathbb{Q} \times \mathbb{Q}$.

12. For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let $\mathbb{Q}(\alpha)$ be the smallest subfield of $\mathbb{R}$ containing $\mathbb{Q} \cup \{\alpha\}$. Find a basis for the vector space $\mathbb{Q}(\sqrt{3} + \sqrt{5})$ over $\mathbb{Q}(\sqrt{15})$.