GATE PAPER-2001

Mathematics

Duration : Three hours *Maximum Marks : 150*

Read the following instructions carefully

1. This question paper contains TWO SECTIONS : '*A*' and '*B*'
2. Section *A* consists of two questions of the multiple choice type. Question 1 consists of TWENTY FIVE sub-questions of ONE mark each and Question 2 consists of TWENTY FIVE sub-questions of TWO marks each.
3. Answer Section *A* only on the special machine-gradable OBJECTIVE RESPONSE SHEET (ORS). Questions of Section *A* will not be graded if answered anywhere else.
4. Answer problems of Section *B* in the answer-book.
5. Write your name, registration number and the name of the Center at the specified locations on the right half of the *ORS* for Section *A*.
6. Using a soft *HB* pencil darken the appropriate bubble under each digit of your registration number.
7. The Objective Response Sheet will be collected back after 120 minutes have expired from the start of the examination. In case you finish Section A before the expiry of 120 minutes, you may start answering Section *B*.
8. Questions of Section *A* are to be answered by darkening the appropriate bubble (marked *A*, *B*, *C* and *D*) using a soft *HB* pencil against the question on the left-hand side of the Objective Response Sheet.
9. In case you wish to change an answer, erase the old answer completely using a good soft eraser.
10. There is no negative marking.
11. Section *B* consists of TWENTY questions of FIVE marks each. ANY FIFTEEN out of them have to be answered. If more number of questions are attempted, score off the answers not to be evaluated, else only the first fifteen unscored answers will be considered strictly.
12. In all the 5 mark questions, clearly show the steps.

The symbols *R* and *C* denote the set of all real and complex numbers respectively. Vector quantities are denoted by bold letters.

SECTION A
(75 Marks)

MA-1. This question consists of TWENTY FIVE sub-questions (1.1-1.25) of ONE mark each. For each of these sub-questions, four possible answers (*A*, *B*, *C* and *D*) are given, out of which only one is correct. Answer each sub-question by darkening the appropriate bubble on the OBJECTIVE RESPONSE SHEET (ORS) using a *HB* pencil. DO not use the ORS for any rough work. You may like to use the Answer Book for any rough work, if needed.

1.1 *The eigenvalues of a 3×3 real matrix P are 1, – 2, 3. Then*

(*a*) $P^{-1} = \frac{1}{6}(5I + 2P - P^2)$ (*b*) $P^{-1} = \frac{1}{6}(5I - 2P + P^2)$

(*c*) $P^{-1} = \frac{1}{6}(5I - 2P - P^2)$ (*d*) $P^{-1} = \frac{1}{6}(5I + 2P + P^2)$

1.2 *Let $T : C^n \to C^n$ be a linear operator having n distinct eigenvalues. Then*

(*a*) T is invertible (*b*) T is invertible as well as diagonalizable

(*c*) T is not diagonalizable (*d*) T is diagonalizable

1.3 *Let U be a 3×3 complex Hermitian matrix which is Unitary. Then the distinct eigenvalues of U are*

(*a*) $\pm i$ (*b*) $1 \pm i$

(*c*) ± 1 (*d*) $\frac{1}{2}(1 \pm i)$

1.4 *The function sin z is analytic in*

(*a*) $C \cup \{\infty\}$ (*b*) C except on the negative real axis

(*c*) $C - \{0\}$ (*d*) C

1.5 *The series* $\sum_{n=1}^{\infty} \frac{z^n}{n\sqrt{n+1}}$, $|z| \le 1$ *is*

(*a*) uniformly but not absolutely convergent

(*b*) uniformly and absolutely convergent

(*c*) absolutely convergent but not uniformly convergent

(*d*) convergent by not uniformly convergent

1.6 *If $f(z) = z^3$, then it*

(*a*) has an essential singularity at $z = \infty$

(*b*) has a pole of order 3 at $z = \infty$

(*c*) has a pole of order 3 at $z = 0$

(*d*) is analytic at $z = \infty$

1.7 *A uniformly continuous function is*

(*a*) measurable (*b*) not measurable

(*c*) measurable and simple (*d*) integrable and simple

1.8 *Which of the following pair of functions is not a linearly independent pair of solutions of $y'' + 9y = 0$?*

(a) $\sin 3x$, $\sin 3x - \cos 3x$
(b) $\sin 3x + \cos 3x$, $3\sin x - 4\sin^3 x$
(c) $\sin 3x$, $\sin 3x \cos 3x$
(d) $\sin 3x + \cos 3x$, $4\cos^3 x - 3\cos x$

1.9 *Determine the type of the following differential equation*

$$\frac{d^2 y}{dx^2} + \sin(x + y) = \sin x$$

(a) Linear, homogeneous
(b) Nonlinear, homogeneous
(c) Linear, nonhomogeneous
(d) Nonlinear, nonhomogeneous

1.10 *Which of the following is not an integrating factor of $xdy - ydx = 0$?*

(a) $\dfrac{1}{x^2}$
(b) $\dfrac{1}{x^2 + y^2}$
(c) $\dfrac{1}{xy}$
(d) $\dfrac{x}{y}$

1.11 *Let G be a group of order 49. Then*

(a) G is abelian
(b) G is cyclic
(c) G is non-abelian
(d) centre of G has order 7

1.12 *The polynomial $f(x) = x^5 + 5$ is*

(a) irreducible over C
(b) irreducible over R
(c) irreducible over Q
(d) not irreducible over Q

where Q denotes the field of rational number.

1.13 *Given a nontrivial normed linear space, the nontriviality of its dual space is assured by*

(a) the Hahn-Banach Theorem
(b) the Principle of Uniform Boundedness
(c) the Open Mapping Theorem
(d) the Closed Graph Theorem

1.14 *If Δ and ∇ are the forward and the backward difference operators respectively, then $\Delta - \nabla$ is equal to*

(a) $-\Delta\nabla$
(b) $\Delta\nabla$
(c) $\Delta + \nabla$
(d) $\dfrac{\Delta}{\nabla}$

1.15 *One root of the equation $e^x - 3x^2 = 0$ lies in the interval (3, 4). The least number of iterations of the bisection method so that $|error| \le 10^{-3}$ are*

(a) 10
(b) 8
(c) 6
(d) 4

1.16 *If (r, θ, φ) is a harmonic function in a domain D, where (r, θ, φ) are spherical polar co-ordinates, then so is*

(a) $\dfrac{1}{r} f(r, \theta, \varphi)$
(b) $\dfrac{1}{r^2} f\left(\dfrac{1}{r}, \theta, \varphi\right)$
(c) $\dfrac{1}{r^2} f\left(\dfrac{1}{r^2}, \theta, \varphi\right)$
(d) $\dfrac{1}{r} f\left(\dfrac{1}{r}, \theta, \varphi\right)$

1.17 *The solution of the initial value problem*

$$u_{tt} = 4u_{xx}, \quad t > 0, -\infty < x < \infty$$

satisfying the conditions

$$u(x, 0) = x, \quad u_t(x, 0) = 0 \text{ is}$$

(a) x (b) $\frac{x^2}{2}$

(c) $2x$ (d) $2t$

1.18 *In the motion of a two particle system, if two particles are connected by a rigid weightless rod of constant length, then the number of degrees of freedom of the system is*

(a) 2 (b) 3

(c) 5 (d) 6

1.19 *Consider a planet of mass m orbiting around the sun under the inverse square law of attraction $\frac{\mu m}{r^2}$, $\mu > 0$. If the position of the planet at time t is given by the polar co-ordinates (r, θ) then the Lagrangian L of the system is given by*

(a) $\frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu m}{r}$ (b) $\frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu m}{r}$

(c) $\frac{1}{2} m (\dot{r}^2 + \dot{\theta}^2) + \frac{\mu m}{r}$ (d) $\frac{1}{2} m (\dot{r}^2 + \dot{\theta}^2) - \frac{\mu m}{r}$

1.20 *The following statement is false*

(a) Any product of compact spaces is compact

(b) Any product of Hausdorff spaces is Hausdorff

(c) Any product of connected spaces is connected

(d) Any product of metrizable spaces is metrizable

1.21 *The random variable X has a t-distribution with v degrees of freedom. Then the probability distribution of X^2 is*

(a) chi-square distribution with 1 degree of freedom

(b) chi-square distribution with v degrees of freedom

(c) F-distribution with $(1, v)$ degrees of freedom

(d) F-distribution with $(v, 1)$ degrees of freedom

1.22 *Let $S_1 = \{(x, y) \in R^2 : x^2 + y^2 \leq 1\}$ and $S_2 = \{(x, y) \in R^2 : y \leq x^2\}$. Then*

(a) S_1 and S_2 both are convex sets

(b) S_1 is a convex set but S_2 is not a convex set

(c) S_2 is a convex set but S_1 is not a convex set

(d) neither S_1 nor S_2 is a convex set

1.23 *Let T be the matrix (occuring in a typical transportation problem) given by*

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Then

(a) Rank $T = 4$ and T is unimodular
(b) Rank $T = 4$ and T is not unimodular
(c) Rank $T = 3$ and T is unimodular
(d) Rank $T = 3$ and T is not unimodular

1.24 *Consider the primal problem (LP)*

$$\max \quad 4x_1 + 3x_2$$

subject to

$$x_1 + x_2 \le 8$$
$$2x_1 + x_2 \le 10$$
$$x_1 \ge 0, x_2 \ge 10$$

together with its dual (LD). Then

(a) (LP) and (LD) both are infeasible
(b) (LP) and (LD) both are feasible
(c) (LP) is feasible but (LD) is infeasible
(d) (LP) is infeasible but (LD) is feasible

1.25 *The initial value problem corresponding to the integral equation*

$$y(x) = 1 + \int_0^x y(t)\,dt \quad is$$

(a) $y' - y = 0, \quad y(0) = 1$ (b) $y' + y = 0, \quad y(0) = 0$
(c) $y' - y = 0, \quad y(0) = 0$ (d) $y' + y = 0, \quad y(0) = 1$

MA-2. This question consists of TWENTY FIVE sub- questions (2.1-2.25) TWO marks each. For each of these sub-questions, four possible answers (*A*, *B*, *C* and *D*) are given, out of which only one is correct. Answer each sub-question by darkening the appropriate bubble on the OBJECTIVE RESPONSE SHEET (ORS) using a soft *HB* pencil. Do not use the ORS for any rough work. You may like to use the Answer Book for any rough work, if needed.

2.1 *Let A be an $n \times n$ complex matrix whose characteristic polynomial is given by*

$$f(t) = t^n + c_{n-1} t^{n-1} + \ldots + c_1 t + c_0.$$

Then

(a) $\det(A) = c_{n-1}$ (b) $\det(A) = c_0$
(c) $\det(A) = (-1)^n c_{n-1}$ (d) $\det(A) = (-1)^n c_0$

2.2 *Let A be any $n \times n$ non-singular complex matrix and let $B = (\overline{A})^t A$, where $(\overline{A})^t$ is the conjugate transpose of A. If λ is an eigenvalue of B, then*

(a) λ is real and $\lambda < 0$ (b) λ is real and $\lambda \le 0$
(c) λ is real and $\lambda \ge 0$ (d) λ is real and $\lambda > 0$

2.3 *Let $T: C^n \to C^n$ be a linear operator of rank $n-2$. Then*

(a) 0 is not an eigenvalue of T (b) 0 must be an eigenvalue of T

(c) 1 can never be an eigenvalue of T (d) 1 must be an eigenvalue of T

2.4 *The fixed points of $f(z) = \dfrac{2iz+5}{z-2i}$ are*

(a) $1 \pm i$ (b) $1 \pm 2i$

(c) $2i \pm 1$ (d) $i \pm 1$

2.5 *The function $f(z) = |z|^2$ is*

(a) differentiable everywhere (b) differentiable only at the origin

(c) not differentiable anywhere (d) differentiable on real x-axis

2.6 *The connected subsets of the real line with the usual topology are*

(a) all intervals (b) only bounded intervals

(c) only compact intervals (d) only semi-infinite intervals

2.7 *Let $f: [a, b] \to R$ be a bounded function where $-\infty < a < b < \infty$. Then f is Riemann integrable if and only if f is continuous everywhere on $[a, b]$ except on*

(a) the empty set (b) a set of measure zero

(c) a finite number of points (d) a countably infinite number of points

X 2.8 *The general solution of the differential equation*

$$\frac{dy}{dx} + \tan y \tan x = \cos x \sec y \text{ is}$$

(a) $2 \sin y = (x + c - \sin x \cos x) \sec x$ (b) $\sin y = (x+c) \cos x$

(c) $\cos y = (x+c) \sin x$ (d) $\sec y = (x+c) \cos x$

2.9 *The eigenvalues of the Sturm Liouville system*

$$y'' + \lambda y = 0, \quad 0 \le x \le \pi$$
$$y(0) = 0,\ y'(\pi) = 0 \quad \text{are}$$

(a) $\dfrac{n^2}{4}$ (b) $\dfrac{(2n-1)^2 \pi^2}{4}$

(c) $\dfrac{(2n-1)^2}{4}$ (d) $\dfrac{n^2 \pi^2}{4}$

2.10 *The differential equation whose linearly independent solutions are $\cos 2x$, $\sin 2x$ and e^{-x} is*

(a) $(D^3 + D^2 + 4D + 4)\, y = 0$ (b) $(D^3 - D^2 + 4D - 4)\, y = 0$

(c) $(D^3 + D^2 - 4D - 4)\, y = 0$ (d) $(D^3 - D^2 - 4D + 4)\, y = 0$

where $D = \dfrac{d}{dx}$.

2.11 *Let $(Z, +)$ denote the group of all integers under addition. Then the number of all automorphisms of $(Z, +)$ is*

(a) 1 (b) 2

(c) 3 (d) 4

2.12 *Let G be a finite group of order 200. Then the number of subgroups of G of order 25 is*

(a) 1 (b) 4

(c) 5 (d) 10

2.13 *If p is prime, and Z_{p^4} denote the ring of integers modulo p^4, then the number of maximal ideals in Z_{p^4} is*

(a) 4 (b) 2
(c) 3 (d) 1

2.14 *All norms on a normed vector space X are equivalent provided*

(a) X is reflexive (b) X is complete
(c) X is finite dimensional (d) X is an inner product space

2.15 *The space l_p is a Hilbert space if and only if*

(a) $p > 1$ (b) $p =$ even
(c) $p = \infty$ (d) $p = 2$

2.16 *The least squares approximation of first degree to the function $f(x) = \sin x$ over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is*

(a) $\frac{24x}{\pi^3}$ (b) $\frac{24x}{\pi^2}$

(c) $\frac{24x}{\pi}$ (d) $24x$

2.17 *The order of the numerical differentiation formula*

$$f''(x_0) = \frac{1}{12h^2}[-\{f(x_0 - 2h) + f(x_0 + 2h)\} + 16\{f(x_0 - h) + f(x_0 + h)\} - 30 f(x_0)] \text{ is}$$

(a) 2 (b) 3
(c) 4 (d) 1

2.18 *The method*

$$y_n + 1 = y_n + \frac{1}{4}(k_1 + 3k_2),\ n = 0, 1, \ldots\ldots$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{2h}{3}, y_n + \frac{2}{3}k_1\right)$$

is used to solve the initial value problem

$$y' = f(x, y) = -10y, \quad y(0) = 1$$

The method will produce stable results if the step size h satisfies

(a) $0.2 < h < 0.5$ (b) $0 < h < 0.5$
(c) $0 < h < 1$ (d) $0 < h < 0.2$

2.19 *The general integral of the partial differential equation*

$$(y + zx)z_x - (x + yz)z_y = x^2 - y^2 \text{ is}$$

(a) $F(x^2 + y^2 + z^2, xy + z) = 0$ (b) $F(x^2 + y^2 - z^2, xy + z) = 0$
(c) $F(x^2 - y^2 - z^2, xy + z) = 0$ (d) $F(x^2 + y^2 + z^2, xy - z) = 0$

where F is an arbitrary function.

2.20 *The differential equation governing the damped motion of a certain coil spring of unit mass under the action of an external force is given by*

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 24x = 30\cos\omega t,$$

where x(t) is the displacement from the equilibrium position at time t and ω is a constant. The resonant frequency when the forcing function is in resonance with the system is

(a) $\sqrt{\frac{2}{\pi}}$ (b) $\frac{\pi}{2}$

(c) 2π (d) $\frac{2}{\pi}$

2.21 *A metric space is always*

(a) first countable (b) second countable

(c) Lindelof (d) separable

2.22 *Let X be the indiscrete space and Y a T_0 space. If $f : X \to Y$ is continuous, then*

(a) X must be a one-point space (b) Y must be discrete

(c) f must be a constant (d) Y must be a one-point space

2.23 *Let (X, Y) be the co-ordinates of a point chosen at random inside the disc $x^2 + y^2 \le r^2$ where $r > 0$. The probability that $Y > mX$ is*

(a) $\frac{1}{2^r}$ (b) $\frac{1}{2}$

(c) $\frac{1}{2^m}$ (d) $\frac{1}{2^{m+r}}$

2.24 *Let (X, Y) be a two-dimensional random variable such that*

$$E(X) = E(Y) = 3,\ Var(X) = Var(Y) = 1 \text{ and } Cov(X, Y) = \frac{1}{2}.$$

Then $P(|X - Y| > 6)$ is

(a) less than $\frac{1}{6}$ (b) equal to $\frac{1}{2}$

(c) equal to $\frac{1}{3}$ (d) greater than $\frac{1}{2}$

2.25 *Let Z* denote the optimal value of LPP*

$\max Z = 4x_1 + 6x_2 + 2x_3$

such that

$3x_1 + 2x_2 + x_3 = 12$

$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$

Then

(a) $10 \le Z^* \le 20$ (b) $20 < Z^* \le 30$

(c) $30 < Z^* \le 40$ (d) $Z^* > 40$

SECTION B
(75 Marks)

This section consists of TWENTY questions of FIVE marks each. ANY FIFTEEN out of them have to be answered on the Answer Book provided. $(15 \times 5 = 75)$

3. *Let $T : V \to V$ be a linear transformation on a vector space V over a field K satisfying the property $Tx = 0 \Rightarrow x = 0$. If $x_1, x_2, \ldots, x_n$ are linearly independent elements in V, show that $Tx_1, Tx_2, \ldots, Tx_n$ are also linearly independent.*

4. *Let $T : V \to V$ be a linear operator on a finite dimensional vector space V over a field K and let $p(t)$ be the minimal polynomial of T. If T is diagonalizable, show that*

$$p(t) = (t - \lambda_1)(t - \lambda_2)\ldots(t - \lambda_r)$$

for some distinct scalars $\lambda_1, \lambda_2, \ldots, \lambda_r$.

5. *Suppose $z = a$ is an isolated singularity of $f(z)$. Prove that $f(z)$ cannot be bounded in a neighbourhood of $z = a$.*

6. *Evaluate*

$$\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5}\, dx$$

using the method of residues.

7. *The function f is defined on $[0, 1]$ as follows*

$$f(x) = x \sin \frac{1}{x},\ x \neq 0$$

$$f(0) = 0.$$

Find the (Lebesgue) measure of the set $\{x \mid f(x) \geq 0\}$.

8. *Consider two metric spaces (R, d_1), (R, d_2) where*

$$d_1(y, z) = |y - z| \text{ and } d_2(y, z) = \left| \frac{y}{1+|y|} - \frac{z}{1+|z|} \right|.$$

Let the functions $f, f_n : [0, \infty] \to R$ be defined by

$$f(x) = x, \quad f_n(x) = x\left(1 + \frac{1}{n}\right), \quad \text{for } 0 \leq x < \infty,$$

where $[0, \infty]$ is the subspace of (R, d_1). Show that f_n converges to f uniformly on $[0, \infty]$ when R has metric d_2 but f_n does not converge uniformly to f on $[0, \infty]$ when R has metric d_1.

9. *(a) Find the general solution of the differential equation*

$$\frac{d^2 y}{dx^2} + 4y = \sec^2 2x$$

using the method of variation of parameters.

(b) *Construct Green's function for the boundary value problem,*

$$y'' + y = -x, \qquad y(0) = y(\pi) = 0$$

if it exists.

10. *Show that the alternating group An, $n \geq 3$ is generated by all cycles of length 3.*
11. *Let R be a commutative principal ideal domain with identity $1 \neq 0$ and let P be a non-zero prime ideal fo R. Show that P is a maximal ideal of R.*
12. *In an inner product space X, fix $b \in X$ and define $f(x) = \langle x, b \rangle$ where $\langle x, b \rangle$ is the inner product of x with b. Show that f is a continuous linear functional and $\| f \| = \| b \|$.*
13. *Find the value of p such that the integration method*

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}\,[f(x_0) + f(x_1)] + ph^3\,[f''(x_0) + f''(x_1)]$$

where $x_1 = x_0 + h$, provides exact result for highest degree polynomial. Find also the order of the method and the error term.

14. *Set up the Gauss-Siedel iteration scheme in matrix form to solve the system of equations*

$$\begin{pmatrix} 4 & 1 & 2 \\ 1 & 5 & 1 \\ 2 & 1 & 4 \end{pmatrix} x = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$$

Is this method convergent ? If yes, find its rate of convergence.

15. *Find the region in which the partial differential equation*

$$u_{xx} - yu_{xy} + xu_x + yu_y + u = 0$$

is hyperbolic and reduce it to a canonical form.

16. *Determine the curve joining two points, which generates a surface of revolution of minimum area, when revolved about the x-axis.*
17. *Let X be a group with identity e and let $p : X \to R$ be a function satisfying*
    (i) $p(x) \geq 0$ *for all* $x \in X$, $p(x) = 0$ *if* $x = e$
    (ii) $p(xy) \leq p(x) + p(y)$ *for all* $x, y \in X$
    (iii) $p(x^{-1}) = p(x)$, *for all* $x \in X$.
    Define $d(x, y) = p(x^{-1}y)$. Show that d is a metric on X.
18. *The number N of the persons getting injured in a bomb blast at a busy market place is a random variable having a Poisson distribution with parameter $\lambda\,(\geq 1)$. A person injured in the explosion may either suffer a minor injury requiring first aid or suffer a major injury requiring hospitalisation. Let the number of persons with minor injury be N_1 and the conditional distribution of N_1 given N be*

$$P\left(N_1 = \frac{i}{N}\right) = \frac{1}{N}, \quad t = 1, 2 \ldots, N$$

Find the expected number of persons requiring hospitalisation in a bomb blast.

19. *Let $X_1, X_2, \ldots, X_{200}$ be identically and independently distributed random variables each with mean 0 and variance 1. Show that*

$$\sqrt{\frac{\pi}{2}}\ P(|X_1X_2 + X_3X_4 + \ldots + X_{199}X_{200}| < 10)$$

is approximately equal to $\int_0^1 e^{\frac{-x^2}{2}}\,dx.$

20. *The following table gives the scores on some scale of four ability groups taught by three different teaching methods*

	Teaching method		
Ability group	*A*	*B*	*C*
1	5	9	4
2	8	7	2
3	12	15	7
4	7	11	9

Test whether or not the teaching methods are equally effective. You may use appropriate values from the following upper percentile values of F-distribution for your test

$F_{2,6,.01} = 10.92\quad F_{3,6,.01} = 9.78,\quad F_{2,6,.05} = 5.14\quad F_{3,6,.05} = 4.76.$