GATE PAPER 2002

Mathematics

Duration : Three hours *Maximum Marks : 150*

Read the following instructions carefully.

1. All answers must be written in ENGLISH.
2. This question paper contains TWO SECTIONS : 'A' and 'B'.
3. Section A consists of two questions of multiple choice type. Question 1 consists of TWENTY FIVE sub-questions of ONE mark each and Question 2 consists of TWENTY FIVE sub-questions of TWO marks each.
4. Answer Section A only on the special machine-gradable OBJECTIVE RESPONSE SHEET (ORS). Questions in Section A will not be graded if answered elsewhere.
5. Write your name, registration number and the name of the Centre at the specified locations on the right half of the ORS for Section A.
6. Using a HB pencil, darken the appropriate bubble under each digit of your registration number.
7. Questions in Section A are to be answered by darkening the appropriate bubble (marked A, B, C or D) using a HB pencil against the question number on the left hand side of the ORS. In case, you wish to change an answer, erase the old answer completely using a good soft eraser.
8. The ORS will be collected after 120 minutes, you may start answering Section B.
9. There will be NEGATIVE marking in Section A. For each wrong answer to 1 - and 2 - mark sub-questions, 0.25 and 0.5 marks will be deducted respectively. More than one answer marked against a question will be deemed as an incorrect response and will be negatively marked.
10. Answer questions in Section B in the answer book. Section B consists of TWENTY questions of FIVE marks each. ANY FIFTEEN out of them have to answered. If more number of questions are attempted, score off the answers not to be evaluated, else only the first fifteen unscored answers will be considered.
11. Answer for each question in Section B should be started on a fresh page. Question numbers must be written legibly and correctly in the answerbook.
12. In all 5 mark questions (Section B), clearly show the important steps in your answers. These intermediate steps will carry partial credit.

NOTE : The symbols Z, R, and C denote the set of all integers, real numbers, and complex numbers respectively. Vector quantities are denoted by bold letters.

SECTION A
(75 marks)

MA-1. This question consists of twenty five sub-questions (1.1-1.25) of one mark each. For each of these sub-questions, four possible answers (A,B,C and D) are given, out of which only one is correct. Answer each sub-question by darkening the appropriate bubble on the objective response sheet (ORS) using an HB pencil. Do not use the ORS for any rough work. Use the Answerbook, if you need to do any rough work.

1.1 *The dimension of the vector space of all 3×3 real symmetric matrices is*

(a) 3 (b) 9
(c) 6 (d) 4

1.2 *Let A be a non-zero upper triangular matrix all of whose eigenvalues are 0. Then I+A is*

(a) invertible (b) singular
(c) idempotent (d) nilpotent

1.3 *The eigenvalues of a skew-symmetric matrix are*

(a) negative (b) real
(c) of absolute value 1 (d) purely imaginary or zero

1.4 *The function $f(z) = z^2$* maps the first quadrant onto

(a) itself (b) upper half plane
(c) third quadrant (d) right half plane

1.5 *The radius of convergence of the power series of the function $f(z) = \dfrac{1}{1-z}$ about $z = \dfrac{1}{4}$ is*

(a) 1 (b) $\frac{1}{4}$
(c) $\frac{3}{4}$ (d) 0

1.6 *Let Γ be any circle enclosing the origin and oriented counter-clockwise. Then the value of the integral $\int_\Gamma \frac{\cos z}{z^2}\,dz$ is*

(a) $2\pi i$ (b) 0
(c) $-2\pi i$ (d) undefined

1.7 *Suppose S_1, S_2 and S_3 are measurable subsets of $[0,1]$, each of measure 3/4, such that the measure of $S_1 \cup S_2 \cup S_3$ is* 1. Then, the measure of $S_1 \cap S_2 \cap S_3$ lies in

(a) $\left[0, \frac{1}{16}\right]$ (b) $\left[\frac{1}{16}, \frac{1}{8}\right]$

(c) $\left[\frac{1}{8}, \frac{1}{4}\right]$ (d) $\left[\frac{1}{4}, 1\right]$

1.8 *Let $f:[0,1] \to \mathbb{R}$ be a bounded Riemann integrable function and let $g:\mathbb{R}\to\mathbb{R}$ be continuous. Then $g \circ f$ is*
(a) Riemann integrable
(b) continuous
(c) Lebesgue integrable, but not Riemann integrable
(d) not necessarily measurable

1.9 *Let V be the volume of a region bounded by a smooth closed surface S. Let r denote the position vector and $\hat{n}$ denote the outward unit normal to S. Then the integral $\iint_S r.\hat{n}\, dS$ equals*
(a) V (b) $\frac{V}{3}$
(c) $3V$ (d) 0

1.10 *Let G be a cyclic group of order 6. Then the number of elements $g \in G$ such that $G = \langle g \rangle$ is*
(a) 5 (b) 3
(c) 4 (d) 2

1.11 *The number of elements of order 5 in the symmetric group S_5 is*
(a) 5 (b) 20
(b) 24 (c) 12

1.12 *The order of the element $(\overline{2}, \overline{2})$ in $Z_4 \times Z_6$ is*
(a) 2 (b) 6
(c) 4 (d) 12

1.13 *Which of the following Banach spaces is not separable?*
(a) $L^1[0,1]$ (b) $L^\infty[0,1]$
(c) $L^2[0,1]$ (d) $C[0,1]$

1.14 *For a subset A of a metric space, which of the following implies the other three?*
(a) A is closed
(b) A is bounded
(c) Closure of B is compact for every $B \subseteq A$
(d) A is compact

1.15 *Let n be a nonnegative integer. The eigenvalues of the Sturm-Liouville problem?*

$$\frac{d^2y}{dx^2} + \lambda y = 0,$$

with boundary conditions $y(0) = y(2\pi),\ \frac{dy}{dx}(0) = \frac{dy}{dx}(2\pi)$ *are*
(a) n (b) $n^2\pi^2$
(c) $n\pi$ (d) n^2

1.16 *The Bessel's function* $\{J_0(\alpha_k x)\}_{k=1}^{\infty}$ *with* α_k *denoting the k-th zero of* $J_0(x)$ *form an orthogonal system on* $[0,1]$ *with respect to weight function*

(a) 1 (b) x^2

(c) x (d) $\sqrt{x}$

1.17 *Linear combinations of solutions of an ordinary differential equation are also solutions if the differential equation is*

(a) Linear nonhomogeneous (b) Linear homogeneous

(c) Nonlinear homogeneous (d) Nonlinear nonhomogeneous

1.18 *Which of the following, concerning the solution of the Neumann problem for Laplace's equation, on a smooth bounded domain, is true ?*

(a) Solution is unique

(b) Solution is unique upto an additive constant

(c) Solution is unique upto a multiplicative constant

(d) No conclusion can be drawn about uniqueness

1.19 *Which of the following satisfies the heat equation (without source terms and with diffusion constant 1) in one space dimension ?*

(a) $\sin\left(\frac{x^2}{4t}\right)$ (b) $e^t \sin x$

(c) $x^2 - t$ (d) $\frac{e^{-x^2/4t}}{\sqrt{t}}$

1.20 *Which of the following is elliptic ?*

(a) Laplace equation (b) Wave equation

(c) Heat equation (d) $u_{xx} + 2u_{xy} - 4u_{yy} = 0$

1.21 *Consider the motion of a three-particle system in the 3-dimensional space. Suppose that all pairs of particles are at invariant distance apart, so that the system constitutes a rigid body. For such a system, the number of degrees of freedom is*

(a) 6 (b) 3

(c) 1 (d) 9

1.22 *If f(x) has an isolated zero of multiplicity 3 at* $x = \xi$, *and the iteration*

$$x_{n+1} = x_n - \frac{3f(x_n)}{f^1(x_n)},\ n = 0, 1, 2, \ldots$$

converges to ξ, *then the rate of convergence is*

(a) linear (b) faster than linear but slower than quadratic

(c) quadratic (d) cubic

1.23 *The best possible error estimate in the Gauss-Hermite formula with 3 points, for calculating the integral* $\int_{-\infty}^{\infty} x^4 e^{-x^2}\,dx$ *is*

(a) 0 (b) 0.30

(c) 0.65 (d) 1.20

1.24 *Let $P(X = n) = \frac{\lambda}{n^2(n+1)}$, where λ is an appropriate constant. Then E(X) is*

(a) $2\lambda + 1$
(b) λ
(c) ∞
(d) 2λ

1.25 *Let x be a non-optimal feasible solution of a linear programming maximization problem and y a dual feasible solution. Then*

(a) The primal objective value at x is greater than the dual objective value at y
(b) The primal objective value at x could equal the dual objective value at y
(c) The primal objective value at x is less than the dual objective value at y
(d) The dual could be unbounded

MA-2. This question consists of TWENTY FIVE sub-questions (2.1 – 2.25) of TWO marks each. For each of these sub-question, four possible answers (A, B, C and D) are given, out of which only one is correct. Answer each sub-question by darkening the appropriate bubble on the OBJECTIVE RESPONSE SHEET (ORS) using a soft HB pencil. Do not use the ORS for any rough work. Use the Answerbook, if you need to do any rough work.

2.1 *Let A be a 3 × 3 matrix with eigenvalues 1, – 1, 0. Then the determinant of $I + A^{100}$ is*

(a) 6
(b) 4
(c) 9
(d) 100

2.2 *Let A be a 2 × 2 orthogonal matrix of trace and determinant 1. Then the angle between Au and u ($u = [1\ 0]^t$) is*

(a) 15°
(b) 30°
(c) 45°
(d) 60°

2.3 *Let $w(z) = \frac{az+b}{cz+d}$ and $f(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ be bilinear (Möbius) transformations. Then the following is also a bilinear transformation*

(a) $f(z)w(z)$
(b) $f(w(z))$
(c) $f(z) + w(z)$
(d) $f(z) + \frac{1}{w(z)}$

2.4 *For the function $f(z) = \sin\frac{1}{z}$, $z = 0$ is a*

(a) removable singularity
(b) simple pole
(c) branch point
(d) essential singularity

2.5 *Pick out the largest of the sets given below on which the sequence of functions $\left\{e^{-n\cos^2 x}\right\}_{n=1}^{\infty}$ converges uniformly*

(a) $[0, \frac{9\pi}{20}) \cup (\frac{11\pi}{20}, \pi]$

(b) $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

(c) $[0, \frac{\pi}{2} - \delta) \cup (\frac{\pi}{2} + \delta, \pi]$, $0 < \delta < \frac{\pi}{100}$

(d) $[0, \pi]$

2.6 *Let $f: R^2 \to R$ be a smooth function with positive definite Hessian at every point. Let $(a, b) \in R^2$ be a critical point of f. Then*

(a) f has a global minimum at (a, b).

(b) f has a local, but not a global minimum at (a, b).

(c) f has a local, but not a global maximum at (a, b).

(d) f has a global maximum at (a, b).

2.7 *Let G be a group of order 30. Let A and B be normal subgroups of orders 2 and 5 respectively. Then the order of the group G/AB is*

(a) 10 (b) 3

(c) 2 (d) 5

2.8 *Let m and n be coprime natural numbers. Then the kernel of the ring homomorphism $\phi : Z \to Z_m \times Z_n$, defined by $\phi(x) = (\bar{x}, \bar{x})$, is*

(a) mZ (b) mnZ

(c) nZ (d) Z

2.9 *Consider the Banach space $C[0, \pi]$ with the supremum norm. The norm of the linear functional $l : C[0, \pi] \to R$, given by $l(f) = \int_0^\pi f(x) \sin^2 x \, dx$, is*

(a) 1 (b) $\frac{\pi}{2}$

(c) π (d) 2π

2.10 *The topology on the real line R generated by left-open right-closed intervals (a, b] is*

(a) strictly coarser than the usual topology

(b) strictly finer than the usual topology

(c) not comparable with the usual topology

(d) same as the usual topology

2.11 *Let X, Y be topological spaces and $f: X \to Y$ be a continuous and bijective map. Then f is a homeomorphism, if*

(a) X and Y are compact

(b) X is Hausdorff and Y is compact

(c) X is compact and Y is Hausdorff

(d) X and Y are Hausdorff

2.12 *If the integrating factor of*

$$(x^7 y^2 + 3y)\, dx + (3x^8 y - x)\, dy = 0$$

is $x^m y^n$, then

(a) $m = -7, n = 1$ (b) $m = 1, n = -7$

(c) $m = n = 0$ (d) $m = n = 1$

2.13 *The initial value problem*

$$(x^2 - x)\frac{dy}{dx} = (2x - 1)y, \; y(x_0) = y_0$$

has a unique solution if (x_0, y_0) equals

(a) (2, 1) (b) (1, 1)

(c) (0, 0) (d) (0, 1)

2.14 *If u is harmonic on $\{(x,y) \mid x^2+y^2 \le 1\}$, then $\int_0^{2\pi} \frac{\partial u}{\partial n}\, d\theta$ equals*

(where $\frac{\partial u}{\partial n}$ is the normal derivative of u on the boundary of the unit disc)

(a) 2π (b) 1
(c) π (d) 0

2.15 *Let u be a solution of the initial value problem*

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0;\ u(x,0) = x^2,\ \frac{\partial u}{\partial t}(x,0) = 0.$$

Then u(0,1) equals

(a) 1 (b) 0
(c) 2 (d) $\frac{1}{2}$

2.16 *Euler's equation of motion of a rigid body about a fixed point O in the absence of external forces are*

$$A\frac{dw_1}{dt} - (B-C)w_2w_3 = 0,$$

$$B\frac{dw_2}{dt} - (C-A)w_3w_1 = 0,$$

$$C\frac{dw_3}{dt} - (A-B)w_1w_2 = 0,$$

where w_1, w_2, w_3 are components of the angular velocity of the rigid body in the direction of principal axes of inertia at O; A,B,C are principal moments of inertia at O. Let T denote the kinetic energy during the motion and h the magnitude of the angular momentum of the body about O. Then, during the motion

(a) T and h both vary
(b) T varies but h remains constant
(c) T remains constant but h varies
(d) T and h both remain constant

2.17 *The fourth divided difference of the polynomial $3x^3 + 11x^2 + 5x + 11$ over the points x = 0, 1, 4, 6, and 7 is*

(a) 18 (b) 11
(c) 3 (d) 0

2.18 *The polynomial of least degree interpolating the data (0, 4), (1, 5), (2, 8), (3, 13) is*

(a) 4 (b) 3
(c) 2 (d) 1

2.19 *For the matrix*

$$\begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$$

the bound for the eigenvalues predicted by Gershgorin's theorem is

(*a*) 3 (*b*) 1
(*c*) 2 (*d*) 4

2.20 ***An extremal of the functional***

$$I[y(x)] = \int_a^b F\left(x, y, \frac{dy}{dx}\right) dx;\ y(a) = y_1,\ y(b) = y_2$$

satisfies Euler's equation, which in general
(*a*) is a second order linear ordinary differential equation (ODE)
(*b*) is a nonlinear ODE of order greater than two
(*c*) admits a unique solution satisfying the conditions $y(a) = y_1$, $y(b) = y_2$
(*d*) may not admit a solution satisfying the conditions $y(a) = y_1$, $y(b) = y_2$

2.21 ***A lot of 1000 screws contains 1% with major defects and 5% with minor defects. If 50 screws are picked at random and inspected, then the ordered pair (expected number of major defectives, expected number of minor defectives) is***
(*a*) (1, 5) (*b*) (2.5, 0.5)
(*c*) (0.5, 2.5) (*d*) (5, 1)

2.22 ***The sample correlation of the transformed random variables $aX + b$ and $cY + d$ is same as that of X and Y provided***
(*a*) $ac < 0;\ b,d \in (0, \infty)$ (*b*) $ac < 0;\ b,d \in (-\infty, 0)$
(*c*) $ac > 0;\ b,d \in R$ (*d*) $ac < 0;\ b,d \in R$

2.23 ***There are two identical locks, with two identical keys, and the keys are among the six different ones which a person carries in his pocket. In a hurry he drops one key somewhere. Then the probability that the locks can still be opened by drawing one key at random is equal to***
(*a*) $\frac{1}{3}$ (*b*) $\frac{5}{6}$
(*c*) $\frac{1}{30}$ (*d*) $\frac{1}{12}$

2.24 ***The system $Ax \le 0$, where A is an $n \times n$ matrix,***
(*a*) may not have a nonzero solution
(*b*) always has a nonzero solution
(*c*) always has at least 2 linearly independent solutions
(*d*) always has at least n linearly independent solutions

2.25 ***Consider the following linear program P : Max $\sum_{j=1}^{n} c_j x_j$ subject to***

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i,\ 1 \le i \le m \quad \text{and} \quad x_j \ge 0,\ 1 \le j \le n.$$

Suppose that we are keeping the c_j's and a_{ij}'s fixed and varying the b_i's. Suppose that P is unbounded for some set of the parameter values b_i. Then, for every choice of b_i's,
(*a*) *P* is unbounded or infeasible
(*b*) *P* is unbounded
(*c*) The dual problem to *P* has a finite optimum
(*d*) The dual problem to *P* is unbounded

SECTION B
(75 Marks)

This section consists of TWENTY questions of FIVE marks each. ANY FIFTEEN out of them have to be answered on the Answer Book provided.
$(15 \times 5 = 75)$

3. *Let J_n be the $n \times n$ matrix each of whose entries equals 1. Find the nullity and the characteristic polynomial of J_n.*

4. *Evaluate the integral*

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}, \; a > 0,$$

by the method of residue calculus.

5. *Let a,b be real numbers with $0 < a < b$. Define sequences $\{a_n\}$ and $\{b_n\}$ recursively by*

$$a_{n+1} = \sqrt{a_n b_n} \text{ and } b_{n+1} = \frac{a_n + b_n}{2}, \quad \text{where } a_1 = a,\ b_1 = b.$$

Show that $\{a_n\}$ is an increasing sequence, $\{b_n\}$ is a decreasing sequence, and both converge to the same limit.

6. *Let $f(t)$ be a real-valued continuous function on [0, 1] such that*

$$\int_0^1 f(t)t^n dt = 0, \text{ for all } n = 0, 1, 2, \ldots .$$

Prove that $f(t)$ vanishes identically.

7. *Let p be a prime and q be a prime divisor of $2^p - 1$. Find the order of $\overline{2}$ (the residue class of 2) in the multiplicative group G of non zero residue classes of integers modulo q. Conclude that $q > p$.*

8. *Let X be an infinite dimensional Banach space. Prove that X can not have countable dimension as a vector space.*

9. *Show that X is Hausdorff topological space if and only if the diagonal Δ defined by*

$$\Delta = \{(x, y) \in X \times X \mid x = y\}$$

is a closed subset of $X \times X$ (with product topology).

10. *Find the general solution of the differential equation*

$$\frac{d^4y}{dx^4} - y = x \sin x.$$

11. *Find the general solution of the differential equation*

$$(x-1)^2 \frac{d^2y}{dx^2} + (x - 1)\frac{dy}{dx} \ldots$$

in powers of $(x-1)$ using the Fröbenius method.

12. *Solve the initial value problem*

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0, x \in (-\infty, \infty)$$

$$u(x, 0) = x, \quad \frac{\partial u}{\partial t}(x, 0) = \frac{x}{1+x^2}$$

13. *Consider the initial value problem*

$$\frac{\partial u}{\partial y} + u\frac{\partial u}{\partial x} = 0, \quad y > 0, x \in R,$$

$$u(x, 0) = f(x).$$

Show that the solution is constant along the characteristics. Hence deduce that if f is decreasing monotonically, the solution cannot exist as a single valued function for all $y > 0$.

14. *Consider a pendulum consisting of a bob of mass m at the end of a rod of length a. If the bob is pulled to one side through an angle α to the downward vertical and released, show that the time required for one complete oscillation is given by*

$$T = 4\sqrt{\frac{a}{g}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \theta}}$$

where $k = \sin\left(\frac{\alpha}{2}\right)$ and g is the acceleration due to gravity.

Assume that the mass of the rod and the air resistance are negligible.

15. *Determine the LU decomposition of the matrix*

$$A = \begin{bmatrix} 5 & -2 & -3 \\ 20 & -5 & -13 \\ 35 & -5 & -17 \end{bmatrix}$$

with L having all its diagonal entries 1; and hence solve the system $AX = [0\ 2\ 13]^t$.

16. *Using the Runge-Kutta method of order 4 and taking the step size $h = 0.1$, determine $y(0.1)$, where $y(x)$ is the solution of*

$$\frac{dy}{dx} + 2xy^2 = 0, \quad y(0) = 1.$$

17. *Solve the integral equation*

$$\phi(x) = x + \int_0^1 (x-\xi)\phi(\xi)d\xi.$$

18. Let $X_1, X_2, \ldots, X_{100}$ be independent and identical Poisson random variables with parameter $\lambda = 0.03$. Let $S = \sum_{i=1}^{100} X_i$. Use the Central Limit Theorem to evaluate $P(\{S \geq 3\})$ and compare the result with the exact probability of the event $\{S \geq 3\}$.

19. Let $X_1, X_2, \ldots, X_n$ be a random sample from exponential density $f_\theta(x) = \theta e^{-\theta x}$, $x \geq 0$, $\theta > 0$. Find the maximum likelihood estimate (MLE) of θ. Also, find the MLE of $P_\theta(X_1 \geq 1)$. Further show that both the estimators are consistent.

20. (a) Let $X \sim B(n_1, p_1)$ and $Y \sim B(n_2, p_2)$, where p_1 and p_2 are unknowns. Further, let X and Y be statistically independent. Construct an approximate $100(1-\alpha)\%$ confidence interval for $(p_1 - p_2)$. [3]

(b) An antibiotic for pneumonia was injected into 100 patients with kidney malfunctions (uremic patients) and into 100 patients with no kidney malfunctions (normal patients). Some allergic reaction developed in 38 of the uremic patients and in 21 of the normal patients. Use the result in part (a) to construct a 95% confidence interval for the difference between the two population proportions (Use the appropriate table value $Z_{0.025} = 1.960$, $Z_{0.05} = 1.645$). [2]

21. Consider the Linear Program

$$\text{Max} \sum_{i=1}^{4} c_i x_i,$$

subject to

$$\sum_{i=1}^{4} a_i x_i \leq a_0,$$

$$0 \leq x_1, x_2, x_3, x_4 \leq 1.$$

where $a_i > 0$, $c_i > 0$ for $i = 1, 2, 3, 4$ and $a_0 > 0$

(i) Write the dual of this Linear Programming Problem. [3]

(ii) Assuming

$$\frac{c_1}{a_1} \geq \frac{c_2}{a_2} \geq \frac{c_3}{a_3} \geq \frac{c_4}{a_4},$$

$$a_1 + a_2 \leq a_0, \quad \text{and} \quad a_1 + a_2 + a_3 > a_0,$$

show that the feasible solution

$$x_1 = x_2 = 1,\ x_3 = \frac{a_0 - a_1 - a_2}{a_3},\ x_4 = 0,$$

is an optimal solution. [2]

22. Consider the optimal assignment problem, in which n persons $P_1, P_2, \ldots, P_n$ are to be assigned n jobs $J_1, J_2, \ldots, J_n$ and where the effectiveness rating of the person P_i for the job J_j is $a_{ij} > 0$. The objective is to find an assignment of persons to jobs, that is, a permutation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ which assigns person P_i to job $J_{\sigma(i)}$, so as to maximize the total effectiveness $\sum_{i=1}^{n} \alpha_{i\sigma(i)}$. Show that in any optimal assignment, at least one person is assigned a job at which he is best.