# List of Symbols, Notations and Data

$B(n, p)$: Binomial distribution with $n$ trials and success probability $p$; $n \in \{1, 2, \dots\}$ and $p \in (0, 1)$

$U(a, b)$: Uniform distribution on the interval $(a, b)$, $-\infty < a < b < \infty$

$N(\mu, \sigma^2)$: Normal distribution with mean $\mu$ and variance $\sigma^2$, $\mu \in (-\infty, \infty)$, $\sigma > 0$

$P(A)$ ꞉ Probability of the event $A$

Poisson$(\lambda)$: Poisson distribution with mean $\lambda, \lambda > 0$

$E(X)$: Expected value (mean) of the random variable $X$

If $Z \sim N(0,1)$, then $P(Z \le 1.96) = 0.975$ and $P(Z \le 0.54) = 0.7054$

$\mathbb{Z}$ ꞉ Set of integers

$\mathbb{Q}$ ꞉ Set of rational numbers

$\mathbb{R}$ ꞉ Set of real numbers

$\mathbb{C}$ ꞉ Set of complex numbers

$\mathbb{Z}_n$: The cyclic group of order $n$

$\mathbb{F}[x]$ : Polynomial ring over the field $\mathbb{F}$

$C[0, 1]$ ꞉ Set of all real valued continuous functions on the interval $[0, 1]$

$C^1[0, 1]$ ꞉ Set of all real valued continuously differentiable functions on the interval $[0, 1]$

$\ell_2$ ꞉ Normed space of all square-summable real sequences

$L^2[0, 1]$: Space of all square-Lebesgue integrable real valued functions on the interval $[0, 1]$

$(C[0, 1], \| \ \|_2)$: The space $C[0, 1]$ with $\|f\|_2 = \left( \int_0^1 |f(x)|^2 \, dx \right)^{\frac{1}{2}}$

$(C[0, 1], \| \ \|_\infty)$: The space $C[0, 1]$ with $\|f\|_\infty = \sup\{ |f(x)| : x \in [0, 1]\}$

$V^\perp$: The orthogonal complement of $V$ in an inner product space

$\mathbb{R}^n$ ꞉ $n$ -dimensional Euclidean space

Usual metric $d$ on $\mathbb{R}^n$ is given by $d\big((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)\big) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}$

$I_n$: The $n \times n$ identity matrix ( $I$ ꞉ the identity matrix when order is NOT specified)

$o(g)$ ꞉ The order of the element $g$ of a group

## Q. 1 – Q. 25 carry one mark each.

Q.1 Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear map defined by
$$T(x, y, z, w) = (x + z, 2x + y + 3z, 2y + 2z, w).$$

Then the rank of $T$ is equal to _____

Q.2 Let $M$ be a $3 \times 3$ matrix and suppose that $1, 2$ and $3$ are the eigenvalues of $M$. If
$$M^{-1} = \frac{M^2}{\alpha} - M + \frac{11}{\alpha} I_3$$
for some scalar $\alpha \neq 0$, then $\alpha$ is equal to _____

Q.3 Let $M$ be a $3 \times 3$ singular matrix and suppose that $2$ and $3$ are eigenvalues of $M$. Then the number of linearly independent eigenvectors of $M^3 + 2M + I_3$ is equal to _____

Q.4 Let $M$ be a $3 \times 3$ matrix such that $M \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix}$ and suppose that $M^3 \begin{pmatrix} 1 \\ -1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ for some $\alpha, \beta, \gamma \in \mathbb{R}$. Then $|\alpha|$ is equal to _____

Q.5 Let $f : [0, \infty) \to \mathbb{R}$ be defined by
$$f(x) = \int_0^x \sin^2(t^2)\, dt.$$

Then the function $f$ is

(A) uniformly continuous on $[0, 1)$ but NOT on $(0, \infty)$
(B) uniformly continuous on $(0, \infty)$ but NOT on $[0, 1)$
(C) uniformly continuous on both $[0, 1)$ and $(0, \infty)$
(D) neither uniformly continuous on $[0, 1)$ nor uniformly continuous on $(0, \infty)$

Q.6 Consider the power series $\sum_{n=0}^{\infty} a_n z^n$, where $a_n = \begin{cases} \frac{1}{3^n} & \text{if } n \text{ is even} \\ \frac{1}{5^n} & \text{if } n \text{ is odd.} \end{cases}$

The radius of convergence of the series is equal to _____

Q.7 Let $C = \{ z \in \mathbb{C} : |z - i| = 2 \}$. Then $\frac{1}{2\pi} \oint_C \frac{z^2 - 4}{z^2 + 4}\, dz$ is equal to _____

Q.8 Let $X \sim B(5, \frac{1}{2})$ and $Y \sim U(0,1)$. Then $\frac{P(X+Y \leq 2)}{P(X+Y \geq 5)}$ is equal to _____

Q.9    Let the random variable $X$ have the distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \dfrac{x}{2} & \text{if } 0 \le x < 1 \\ \dfrac{3}{5} & \text{if } 1 \le x < 2 \\ \dfrac{1}{2} + \dfrac{x}{8} & \text{if } 2 \le x < 3 \\ 1 & \text{if } x \ge 3. \end{cases}$$

Then $P(2 \le X < 4)$ is equal to _____

Q.10    Let $X$ be a random variable having the distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \dfrac{1}{4} & \text{if } 0 \le x < 1 \\ \dfrac{1}{3} & \text{if } 1 \le x < 2 \\ \dfrac{1}{2} & \text{if } 2 \le x < \dfrac{11}{3} \\ 1 & \text{if } x \ge \dfrac{11}{3}. \end{cases}$$

Then $E(X)$ is equal to _____

Q.11    In an experiment, a fair die is rolled until two sixes are obtained in succession. The probability that the experiment will end in the fifth trial is equal to

(A) $\dfrac{125}{6^5}$          (B) $\dfrac{150}{6^5}$          (C) $\dfrac{175}{6^5}$          (D) $\dfrac{200}{6^5}$

Q.12    Let $x_1 = 2.2$, $x_2 = 4.3$, $x_3 = 3.1$, $x_4 = 4.5$, $x_5 = 1.1$ and $x_6 = 5.7$ be the observed values of a random sample of size 6 from a $U(\theta - 1, \ \theta + 4)$ distribution, where $\theta \in (0, \infty)$ is unknown. Then a maximum likelihood estimate of $\theta$ is equal to

(A) 1.8          (B) 2.3          (C) 3.1          (D) 3.6

Q.13    Let $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ be the open unit disc in $\mathbb{R}^2$ with boundary $\partial\Omega$. If $u(x, y)$ is the solution of the Dirichlet problem

$$u_{xx} + u_{yy} = 0 \qquad \text{in } \Omega$$
$$u(x, y) = 1 - 2\, y^2 \qquad \text{on } \partial\Omega,$$

then $u\left(\dfrac{1}{2}, 0\right)$ is equal to

(A) $-1$          (B) $\dfrac{-1}{4}$          (C) $\dfrac{1}{4}$          (D) 1

Q.14    Let $c \in \mathbb{Z}_3$ be such that $\dfrac{\mathbb{Z}_3[X]}{\langle X^3 + c\,X + 1 \rangle}$ is a field. Then $c$ is equal to _____

Q.15    Let $V = C^1[0,1]$, $X = ( C[0,1], \| \quad \|_\infty)$ and $Y = ( C[0,1], \| \quad \|_2)$. Then $V$ is

(A) dense in $X$ but NOT in $Y$
(B) dense in $Y$ but NOT in $X$
(C) dense in both $X$ and $Y$
(D) neither dense in $X$ nor dense in $Y$

Q.16    Let $T : ( C[0,1], \| \quad \|_\infty) \to \mathbb{R}$ be defined by $T(f) = \int_0^1 2x f(x)\, dx$ for all $f \in C[0,1]$. Then $\|T\|$ is equal to _____

Q.17    Let $\tau_1$ be the usual topology on $\mathbb{R}$. Let $\tau_2$ be the topology on $\mathbb{R}$ generated by
$\mathcal{B} = \{ [a,b) \subset \mathbb{R} : -\infty < a < b < \infty \}$. Then the set $\{ x \in \mathbb{R} : 4\,\sin^2 x \le 1 \} \cup \left\{ \dfrac{\pi}{2} \right\}$ is

(A) closed in $(\mathbb{R}, \tau_1)$ but NOT in $(\mathbb{R}, \tau_2)$
(B) closed in $(\mathbb{R}, \tau_2)$ but NOT in $(\mathbb{R}, \tau_1)$
(C) closed in both $(\mathbb{R}, \tau_1)$ and $(\mathbb{R}, \tau_2)$
(D) neither closed in $(\mathbb{R}, \tau_1)$ nor closed in $(\mathbb{R}, \tau_2)$

Q.18    Let $X$ be a connected topological space such that there exists a non-constant continuous function $f : X \to \mathbb{R}$, where $\mathbb{R}$ is equipped with the usual topology. Let $f(X) = \{ f(x) : x \in X \}$. Then

(A) $X$ is countable but $f(X)$ is uncountable
(B) $f(X)$ is countable but $X$ is uncountable
(C) both $f(X)$ and $X$ are countable
(D) both $f(X)$ and $X$ are uncountable

Q.19    Let $d_1$ and $d_2$ denote the usual metric and the discrete metric on $\mathbb{R}$, respectively.
Let $f : (\mathbb{R}, d_1) \to (\mathbb{R}, d_2)$ be defined by $f(x) = x$, $x \in \mathbb{R}$. Then

(A) $f$ is continuous but $f^{-1}$ is NOT continuous
(B) $f^{-1}$ is continuous but $f$ is NOT continuous
(C) both $f$ and $f^{-1}$ are continuous
(D) neither $f$ nor $f^{-1}$ is continuous

Q.20    If the trapezoidal rule with single interval $[0,1]$ is exact for approximating the integral
$\int_0^1 (x^3 - c\,x^2)\,dx$, then the value of $c$ is equal to _____

Q.21    Suppose that the Newton-Raphson method is applied to the equation $2x^2 + 1 - e^{x^2} = 0$ with an initial approximation $x_0$ sufficiently close to zero. Then, for the root $x = 0$, the order of convergence of the method is equal to _____

Q.22    The minimum possible order of a homogeneous linear ordinary differential equation with real constant coefficients having $x^2 \sin(x)$ as a solution is equal to _____

Q.23    The Lagrangian of a system in terms of polar coordinates $(r, \theta)$ is given by
$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) - m g r (1 - \cos(\theta)),$$
where $m$ is the mass, $g$ is the acceleration due to gravity and $\dot{s}$ denotes the derivative of $s$ with respect to time. Then the equations of motion are

(A) $2 \ddot{r} = r \dot{\theta}^2 - g (1 - \cos(\theta))$, $\frac{d}{dt}(r^2 \dot{\theta}) = -g r \sin(\theta)$

(B) $2 \ddot{r} = r \dot{\theta}^2 + g (1 - \cos(\theta))$, $\frac{d}{dt}(r^2 \dot{\theta}) = -g r \sin(\theta)$

(C) $2 \ddot{r} = r \dot{\theta}^2 - g (1 - \cos(\theta))$, $\frac{d}{dt}(r^2 \dot{\theta}) = g r \sin(\theta)$

(D) $2 \ddot{r} = r \dot{\theta}^2 + g (1 - \cos(\theta))$, $\frac{d}{dt}(r^2 \dot{\theta}) = g r \sin(\theta)$

Q.24    If $y(x)$ satisfies the initial value problem
$$(x^2 + y)dx = x \, dy, \qquad y(1) = 2,$$
then $y(2)$ is equal to _____

Q.25    It is known that Bessel functions $J_n(x)$, for $n \geq 0$, satisfy the identity
$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = J_0(x) + \sum_{n=1}^{\infty} J_n(x) \left( t^n + \frac{(-1)^n}{t^n} \right)$$
for all $t > 0$ and $x \in \mathbb{R}$. The value of $J_0\left(\frac{\pi}{3}\right) + 2 \sum_{n=1}^{\infty} J_{2n}\left(\frac{\pi}{3}\right)$ is equal to _____

## Q. 26 – Q. 55 carry two marks each.

Q.26    Let $X$ and $Y$ be two random variables having the joint probability density function
$$f(x, y) = \begin{cases} 2 & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the conditional probability $P\left(X \leq \frac{2}{3} \mid Y = \frac{3}{4}\right)$ is equal to

(A) $\frac{5}{9}$          (B) $\frac{2}{3}$          (C) $\frac{7}{9}$          (D) $\frac{8}{9}$

Q.27    Let $\Omega = (0,1]$ be the sample space and let $P(\cdot)$ be a probability function defined by
$$P((0,x]) = \begin{cases} \dfrac{x}{2} & \text{if } 0 \leq x < \dfrac{1}{2} \\ x & \text{if } \dfrac{1}{2} \leq x \leq 1. \end{cases}$$

Then $P\left(\left\{\frac{1}{2}\right\}\right)$ is equal to _____

Q.28   Let $X_1, X_2$ and $X_3$ be independent and identically distributed random variables with $E(X_1) = 0$ and $E(X_1^2) = \frac{15}{4}$. If $\psi : (0, \infty) \to (0, \infty)$ is defined through the conditional expectation
$$\psi(t) = E(X_1^2 \mid X_1^2 + X_2^2 + X_3^2 = t), \quad t > 0,$$

then $E(\psi((X_1 + X_2)^2))$ is equal to _____

Q.29   Let $X \sim \text{Poisson}(\lambda)$, where $\lambda > 0$ is unknown. If $\delta(X)$ is the unbiased estimator of $g(\lambda) = e^{-\lambda}(3\lambda^2 + 2\lambda + 1)$, then $\sum_{k=0}^{\infty} \delta(k)$ is equal to _____

Q.30   Let $X_1, \dots, X_n$ be a random sample from $N(\mu, 1)$ distribution, where $\mu \in \{0, \frac{1}{2}\}$. For testing the null hypothesis $H_0 : \mu = 0$ against the alternative hypothesis $H_1 : \mu = \frac{1}{2}$, consider the critical region
$$R = \left\{ (x_1, x_2, \dots, x_n) : \sum_{i=1}^{n} x_i > c \right\},$$
where $c$ is some real constant. If the critical region $R$ has size $0.025$ and power $0.7054$, then the value of the sample size $n$ is equal to _____

Q.31   Let $X$ and $Y$ be independently distributed central chi-squared random variables with degrees of freedom $m \ (\geq 3)$ and $n \ (\geq 3)$, respectively. If $E\left(\frac{X}{Y}\right) = 3$ and $m + n = 14$, then $E\left(\frac{Y}{X}\right)$ is equal to

(A) $\frac{2}{7}$        (B) $\frac{3}{7}$        (C) $\frac{4}{7}$        (D) $\frac{5}{7}$

Q.32   Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed random variables with $P(X_1 = 1) = \frac{1}{4}$ and $P(X_1 = 2) = \frac{3}{4}$. If $\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$, for $n = 1, 2, \dots$, then $\lim_{n \to \infty} P(\overline{X}_n \leq 1.8)$ is equal to _____

Q.33   Let $u(x, y) = 2f(y)\cos(x - 2y)$, $(x, y) \in \mathbb{R}^2$, be a solution of the initial value problem
$$2u_x + u_y = u$$
$$u(x, 0) = \cos(x).$$
Then $f(1)$ is equal to

(A) $\frac{1}{2}$        (B) $\frac{e}{2}$        (C) $e$        (D) $\frac{3e}{2}$

Q.34   Let $u(x, t)$, $x \in \mathbb{R}$, $t \geq 0$, be the solution of the initial value problem
$$u_{tt} = u_{xx}$$
$$u(x, 0) = x$$
$$u_t(x, 0) = 1.$$
Then $u(2,2)$ is equal to _____

Q.35   Let $W = \text{Span} \left\{ \frac{1}{\sqrt{2}}(0,0,1,1), \frac{1}{\sqrt{2}}(1,-1,0,0) \right\}$ be a subspace of the Euclidean space $\mathbb{R}^4$. Then the square of the distance from the point $(1,1,1,1)$ to the subspace $W$ is equal to _____

Q.36   Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear map such that the null space of $T$ is
$$\{(x,y,z,w) \in \mathbb{R}^4 : x + y + z + w = 0\}$$
and the rank of $(T - 4\,I_4)$ is 3. If the minimal polynomial of $T$ is $x(x-4)^\alpha$, then $\alpha$ is equal to _____

Q.37   Let $M$ be an invertible Hermitian matrix and let $x, y \in \mathbb{R}$ be such that $x^2 < 4y$. Then

(A) both $M^2 + x\,M + y\,I$ and $M^2 - x\,M + y\,I$ are singular
(B) $M^2 + x\,M + y\,I$ is singular but $M^2 - x\,M + y\,I$ is non-singular
(C) $M^2 + x\,M + y\,I$ is non-singular but $M^2 - x\,M + y\,I$ is singular
(D) both $M^2 + x\,M + y\,I$ and $M^2 - x\,M + y\,I$ are non-singular

Q.38   Let $G = \{ e, x, x^2, x^3, y, xy, x^2y, x^3y \}$ with $o(x) = 4, o(y) = 2$ and $xy = yx^3$. Then the number of elements in the center of the group $G$ is equal to

(A) 1            (B) 2            (C) 4            (D) 8

Q.39   The number of ring homomorphisms from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $\mathbb{Z}_4$ is equal to _____

Q.40   Let $p(x) = 9\,x^5 + 10\,x^3 + 5\,x + 15$ and $q(x) = x^3 - x^2 - x - 2$ be two polynomials in $\mathbb{Q}\,[x]$. Then, over $\mathbb{Q}$,

(A) $p(x)$ and $q(x)$ are both irreducible
(B) $p(x)$ is reducible but $q(x)$ is irreducible
(C) $p(x)$ is irreducible but $q(x)$ is reducible
(D) $p(x)$ and $q(x)$ are both reducible

Q.41   Consider the linear programming problem
$$\text{Maximize } 3\,x + 9\,y,$$
$$\text{subject to} \qquad 2\,y - x \le 2$$
$$3\,y - x \ge 0$$
$$2\,x + 3\,y \le 10$$
$$x, y \ge 0.$$
Then the maximum value of the objective function is equal to _____

Q.42   Let $S = \left\{ (x, \sin\frac{1}{x}) : 0 < x \le 1 \right\}$ and $T = S \cup \{(0,0)\}$. Under the usual metric on $\mathbb{R}^2$,

(A) $S$ is closed but $T$ is NOT closed
(B) $T$ is closed but $S$ is NOT closed
(C) both $S$ and $T$ are closed
(D) neither $S$ nor $T$ is closed

Q.43    Let $H = \left\{ (x_n) \in \ \ell_2 : \sum_{n=1}^{\infty} \frac{x_n}{n} = 1 \right\}$. Then $H$

    (A)  is bounded                                    (B)  is closed

    (C)  is a subspace                             (D)  has an interior point

Q.44    Let $V$ be a closed subspace of $L^2[0, 1]$ and let $f, g \in L^2[0, 1]$ be given by $f(x) = x$ and $g(x) = x^2$. If $V^\perp = \text{Span}\{f\}$ and $Pg$ is the orthogonal projection of $g$ on $V$, then $(g - Pg)(x), \ x \in [0, 1]$, is

    (A)  $\frac{3}{4}x$                 (B)  $\frac{1}{4}x$                 (C)  $\frac{3}{4}x^2$                 (D)  $\frac{1}{4}x^2$

Q.45    Let $p(x)$ be the polynomial of degree at most $3$ that passes through the points $(-2, 12), (-1, 1)$, $(0,2)$ and $(2, -8)$. Then the coefficient of $x^3$ in $p(x)$ is equal to _____

Q.46    If, for some $\alpha, \beta \in \mathbb{R}$, the integration formula
$$\int_0^2 p(x)dx = p(\alpha) + p(\beta)$$
holds for all polynomials $p(x)$ of degree at most $3$, then the value of $3(\alpha - \beta)^2$ is equal to _____

Q.47    Let $y(t)$ be a continuous function on $[0, \infty)$ whose Laplace transform exists. If $y(t)$ satisfies
$$\int_0^t (1 - \cos(t - \tau)) \, y(\tau) \, d\tau = t^4,$$
then $y(1)$ is equal to _____

Q.48    Consider the initial value problem
$$x^2 y'' - 6y = 0, \qquad y(1) = \alpha, \ y'(1) = 6.$$
If $y(x) \to 0$ as $x \to 0^+$, then $\alpha$ is equal to _____

Q.49    Define $f_1, f_2 : [0,1] \to \mathbb{R}$ by
$$f_1(x) = \sum_{n=1}^{\infty} \frac{x \sin(n^2 x)}{n^2} \quad \text{and} \quad f_2(x) = \sum_{n=1}^{\infty} x^2 (1 - x^2)^{n-1}.$$
Then

    (A)  $f_1$ is continuous but $f_2$ is NOT continuous

    (B)  $f_2$ is continuous but $f_1$ is NOT continuous

    (C)  both $f_1$ and $f_2$ are continuous

    (D)  neither $f_1$ nor $f_2$ is continuous

Q.50    Consider the unit sphere $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and the unit normal vector $\hat{n} = (x, y, z)$ at each point $(x, y, z)$ on $S$. The value of the surface integral
$$\iint_S \left\{ \left( \frac{2x}{\pi} + \sin(y^2) \right) x + \left( e^z - \frac{y}{\pi} \right) y + \left( \frac{2z}{\pi} + \sin^2 y \right) z \right\} \, d\sigma$$
is equal to _____

Q.51    Let $D = \{(x, y) \in \mathbb{R}^2 : 1 \le x \le 1000, \ 1 \le y \le 1000\}$. Define

$$f(x, y) = \frac{x\,y}{2} + \frac{500}{x} + \frac{500}{y}.$$

Then the minimum value of $f$ on $D$ is equal to _____

Q.52    Let $\mathbb{D} = \{\, z \in \mathbb{C} : |z| < 1 \,\}$. Then there exists a non-constant analytic function $f$ on $\mathbb{D}$ such that for all $n = 2, 3, 4, \ldots$

(A) $f\left(\frac{\sqrt{-1}}{n}\right) = 0$                     (B) $f\left(\frac{1}{n}\right) = 0$

(C) $f\left(1 - \frac{1}{n}\right) = 0$                         (D) $f\left(\frac{1}{2} - \frac{1}{n}\right) = 0$

Q.53    Let $\sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurent series expansion of $f(z) = \dfrac{1}{2\,z^2 - 13\,z + 15}$ in the annulus $\frac{3}{2} < |z| < 5$. Then $\dfrac{a_1}{a_2}$ is equal to _____

Q.54    The value of $\dfrac{i}{4 - \pi} \displaystyle\int_{|z|=4} \dfrac{dz}{z\,\cos(z)}$ is equal to _____

Q.55    Suppose that among all continuously differentiable functions $y(x)$, $x \in \mathbb{R}$, with $y(0) = 0$ and $y(1) = \frac{1}{2}$, the function $y_0(x)$ minimizes the functional

$$\int_0^1 \left(e^{-(y'-x)} + (1 + y)y'\right)dx.$$

Then $y_0\left(\frac{1}{2}\right)$ is equal to

(A) 0                  (B) $\frac{1}{8}$                  (C) $\frac{1}{4}$                  (D) $\frac{1}{2}$

# END OF THE QUESTION PAPER