A

2010 – MS

Test Paper Code : MS

Time : 3 Hours **Max. Marks : 300**

INSTRUCTIONS

1. The question-cum-answer booklet has **32** pages and has **25** questions. Please ensure that the copy of the question-cum-answer booklet you have received contains all the questions.
2. Write your **Registration Number, Name and the name of the Test Centre** in the appropriate space provided on the right side.
3. Write the answers to the objective questions against each Question No. in the **Answer Table for Objective Questions,** provided on Page No. **7**. Do not write anything else on this page.
4. Each objective question has **4 choices** for its answer: (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used :
   (a) For each correct answer, you will be awarded **6 (Six)** marks.
   (b) For each wrong answer, you will be awarded **–2 (Negative two)** marks.
   (c) Multiple answers to a question will be treated as a wrong answer.
   (d) For each un-attempted question, you will be awarded **0 (Zero)** mark.
   (e) Negative marks for objective part will be carried over to total marks.
5. Answer the subjective question only in the space provided after each question.
6. Do not write more than one answer for the same question. In case you attempt a subjective question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing last only will be evaluated.
7. All answers must be written in blue/black/ blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. No supplementary sheets will be provided to the candidates.
10. **Clip board, log tables, slide rule, calculator, cellular phone or electronic gadgets in any form are NOT allowed.**
11. The question-cum-answer booklet must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this booklet.
12. Refer to special instructions/useful data on the reverse.

Eass.

Eass.

A

2010 – MS

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

REGISTRATION NUMBER						

Name :

Test Centre :

Do not write your Registration Number or Name anywhere else in this question-cum-answer booklet.

I have read all the instructions and shall abide by them.

....

Signature of the Candidate

I have verified the information filled by the Candidate above.

....

Signature of the Invigilator

A

Special Instructions / Useful Data

1. $\ln a$: Natural logarithm of a positive real number a.
2. $\mathbb{R}$: Set of all real numbers
3. $\mathbb{R}^3$: 3-dimensional Euclidean space.
4. $\underset{\sim}{x}^T$: Transpose of a column vector $\underset{\sim}{x}$.
5. $f'(a)$: The first derivative of a function f at the point $x = a$.
6. For an event E, E^C denotes the complement of E.
7. i.i.d. : independent and identically distributed.
8. $N(\mu, \sigma^2)$: Normal distribution with mean μ and variance σ^2.
9. X is $Exp(\lambda)$ random variable means that the probability density function of X is
$$f(x \mid \lambda) = \begin{cases} \lambda e^{-\lambda x}, & x > 0,\ \lambda > 0, \\ 0, & \text{otherwise.} \end{cases}$$
10. X is $G(\alpha, \lambda)$ random variable means that the probability density function of X is
$$f(x \mid \alpha, \lambda) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma\alpha} x^{\alpha-1} e^{-\lambda x}, & x > 0,\ \alpha > 0,\ \lambda > 0, \\ 0, & \text{otherwise.} \end{cases}$$
11. χ_n^2 : Chi-square random variable with n degrees of freedom.
12. t_n : Student's t random variable with n degrees of freedom.
13. $P(\chi_{10}^2 \le 9.15) = 0.48$, $P(\chi_{10}^2 \le 18.30) = 0.95$.
14. $P(t_8 \le 2.3) = 0.975$, $P(t_8 \le 1.9) = 0.95$.

DO NOT WRITE ON THIS PAGE

IMPORTANT NOTE FOR CANDIDATES

- **Questions 1-15 (objective questions) carry *six* marks each and questions 16-25 (subjective questions) carry *twenty one* marks each.**
- **Write the answers to the objective questions in the *Answer Table for Objective Questions* provided on page 7 only.**

Q.1 Let E and F be two events with $P(E)>0$, $P(F\,|\,E)=0.3$ and $P(E\cap F^C)=0.2$. Then $P(E)$ equals

(A) $\frac{1}{7}$ (B) $\frac{2}{7}$ (C) $\frac{4}{7}$ (D) $\frac{5}{7}$

Q.2 Let $X_1,\dots,X_n$ be i.i.d. random variables with the probability density function

$$f(x\,|\,\theta)=\begin{cases}\dfrac{2\theta^2}{x^3}, & x\geq\theta,\\ 0, & \text{otherwise,}\end{cases}$$

where $\theta\,(>0)$ is unknown. Then the maximum likelihood estimator of θ is

(A) $\left(\prod_{i=1}^{n}X_i\right)^{3/2}$ (B) $\frac{1}{n}\sum_{i=1}^{n}\ln X_i$

(C) $\max\{X_1,\dots,X_n\}$ (D) $\min\{X_1,\dots,X_n\}$

Q.3 The value of

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{y}e^{-\frac{1}{2}\left(x^2+y^2\right)}\,dx\,dy$$

equals

(A) $\frac{\pi}{4}$ (B) $\frac{1}{2\pi}$ (C) $\frac{1}{4}$ (D) $\frac{1}{2}$

Q.4 The value of

$$\iint_S e^{-(x+y)}\, dx\, dy,$$

where $S = \{(x, y) : 0 < x < 1, y > 0, 1 < x + y < 2\}$, equals

(A) 1 (B) 2 (C) $e^{-1} - e^{-2}$ (D) $e^2 - e$

Q.5 Two coins with probability of heads u and v, respectively, are tossed independently. If P(both coins show up tails) = P(both coins show up heads), then $u + v$ equals

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) 1

Q.6 Let f be a thrice differentiable function on $[-\pi/6, \pi/6]$ such that $f'(x) = 1 + (f(x))^2$. If $f(0) = 1$, then the coefficient of x^2 in Taylor's expansion of f about zero is

(A) 2 (B) 3 (C) 4 (D) 5

Q.7 Let X be a discrete random variable with

$$P(X = k) = \frac{2}{3}\frac{e^{-1}}{k!} + \left(\frac{1}{3}\right)^{k+1}\frac{2}{3}, \quad k = 0, 1, 2, \ldots .$$

Let $E = \{0, 2, 4, \ldots\}$. Then $P(X \in E)$ equals

(A) $\frac{5}{12} + \frac{2}{3}e^{-2}$ (B) $\frac{5}{12} + \frac{1}{3}e^{-2}$ (C) $\frac{7}{12} - \frac{1}{3}e^{-2}$ (D) $\frac{7}{12} + \frac{1}{3}e^{-2}$

Q.8 Let X and Y have the joint probability mass function

$$P(X = n, Y = k) = \left(\frac{1}{2}\right)^{n+2k+1}; \quad n = -k, -k+1, \ldots ; \quad k = 1, 2, \ldots .$$

Then $E(Y)$ equals

(A) 1 (B) 2 (C) 3 (D) 4

Q.9 If X_1 and X_2 are i.i.d. $N(0,1)$ random variables, then $P\left(X_1^2+X_2^2 \le 2\right)$ equals

(A) e^{-1} (B) e^{-2} (C) $1-e^{-1}$ (D) $1-e^{-2}$

Q.10 Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x)=\begin{cases} x^{3/2}, & x \ge 0, \\ -|x|^{3/2}, & x<0. \end{cases}$$

Then f is

(A) continuous everywhere, but not differentiable at $x=0$.
(B) differentiable everywhere except at $x=0$.
(C) differentiable everywhere, but not twice differentiable at $x=0$.
(D) differentiable infinitely many times everywhere.

Q.11 Let $P=\dfrac{\underset{\sim}{x}\,\underset{\sim}{x}^T}{\underset{\sim}{x}^T\underset{\sim}{x}}$ be an $n\times n$ $(n>1)$ matrix, where $\underset{\sim}{x}$ is a nonzero column vector. Then which one of the following statements is FALSE?

(A) P is idempotent (B) P is orthogonal
(C) P is symmetric (D) Rank of P is one

Q.12 Let X and Y be i.i.d. binomial random variables with parameters n and $1/2$ and let Z be another binomial random variable with parameters $2n$ and $1/2$. Then $P(X=Y)$ equals

(A) $P(Z=0)$ (B) $P(Z=n)$ (C) $P(Z=2n-1)$ (D) $P(Z=n+1)$

Q.13 Let X be a random variable with the probability density function

$$f(x\,|\,\theta)=\begin{cases} 2\theta x+1-\theta, & 0<x<1, \quad -1\le\theta\le1, \\ 0, & \text{otherwise.} \end{cases}$$

Based on a sample of size one, the most powerful critical region (rejection region) for testing $H_0:\theta=0$ against $H_1:\theta=1$ at level $\alpha=0.2$ is given by

(A) $X>\dfrac{4}{5}$ (B) $X\le\dfrac{2}{5}$ (C) $X>\dfrac{8}{5}$ (D) $X<\dfrac{4}{5}$

Q.14 Let $P = (p_{ij})$ be a 50×50 matrix, where $p_{ij} = \min(i, j)$; $i, j = 1, \ldots, 50$. Then the rank of P equals

(A) 1 (B) 2 (C) 25 (D) 50

Q.15 If X is a $G(2, 1/2)$ random variable, then $P(X > 4)$ equals

(A) $3e^{-2}$ (B) $2e^{-2}$ (C) e^{-2} (D) $0.5e^{-2}$

Space for rough work

Space for rough work

A

Space for rough work

A

Answer Table for Objective Questions

Write the Code of your chosen answer only in the 'Answer' column against each Question No. Do not write anything else on this page.

Question No.	Answer	Do not write in this column
01		
02		
03		
04		
05		
06		
07		
08		
09		
10		
11		
12		
13		
14		
15		

FOR EVALUATION ONLY

No. of correct answers		Marks	(+)
No. of incorrect answers		Marks	(–)
Total marks in question nos. 1-15			( )

Q.16 (a) Let X_1 and X_2 denote the lifetimes (in months) of bulbs produced at factories F_1 and F_2 respectively. The random variables X_1 and X_2 are Exp (1/8) and Exp (1/2) respectively. A shop procures 80% of its supply of bulbs from factory F_1 and 20% from factory F_2. A randomly selected bulb from the shop is put on test and is found to be working after 4 months. What is the probability that it was procured from factory F_2? (12)

(b) Let X be a $G(4, \lambda)$, $\lambda > 0$, random variable. Find the value of λ that minimizes $E(Y)$, where $Y = X + \frac{3}{4X}$. (9)

A

Q.17 The probability density function of a random variable X is

$$f(x)=\begin{cases} \dfrac{|x|}{2}, & -1 \le x \le 1, \\ \dfrac{3-x}{4}, & 1 < x \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

Find the cumulative distribution function and the probability density function of $Y = |X|$. Also, find the median of the distribution of Y. (21)

A

Q.18 Let the joint probability density function of continuous random variables X and Y be given by

$$f(x, y) = \begin{cases} \dfrac{1}{\alpha \beta} e^{-\left(\frac{x}{\alpha} + \frac{y-x}{\beta}\right)}, & 0 < x \le y < \infty, \quad \alpha > 0, \quad \beta > 0, \quad \alpha \ne \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional probability density function of Y given $X = x \quad (x > 0)$ and the correlation coefficient between X and Y. (21)

A

Q.19 (a) Let X_1, X_2, X_3 be i.i.d. random variables with the probability density function

$$f(x \mid \theta) = \begin{cases} \dfrac{1+\theta x}{2}, & -1 < x < 1; \quad -1 < \theta < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the method of moments estimator (T) of θ. Let $S = X_1 + 2X_2$ be another estimator of θ. Find the efficiency of T relative to S. (12)

(b) Let $X_1, \ldots, X_n$ be independent random variables with X_i having a $N(\theta, 1/i)$ distribution, $\theta \in \mathbb{R}$; $i = 1, \ldots, n$. Find the maximum likelihood estimator of θ. (9)

A

Q.20 (a) Consider the simple linear regression model

$$y_i = \alpha + \beta x_i + \varepsilon_i, \quad i = 1, \ldots, n,$$

where ε_i's are i.i.d. random variables with mean 0 and variance σ^2. Suppose that we have a data set $(x_1, y_1), \ldots, (x_n, y_n)$ with $n = 10$, $\sum_i x_i = 50$, $\sum_i y_i = 40$, $\sum_i x_i^2 = 500$, $\sum_i y_i^2 = 400$ and $\sum_i x_i y_i = 400$. Find the least squares estimates of α and β. Also find an unbiased estimate of σ^2. (12)

(b) Let $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the set of observed values of a random sample from a $N(\mu, \sigma^2)$ distribution, where both μ and σ^2 are unknown. Find a 95% confidence interval for μ.

(You may take the value of $\sqrt{5/6}$ as 0.9.) (9)

A

Q.21 Let $X_1, \ldots, X_5$ be i.i.d. random variables with the probability density function

$$f(x \mid \theta) = \begin{cases} \dfrac{1}{\theta} e^{-x/\theta}, & x > 0,\ \theta > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find the power of the most powerful test for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$ at the level $\alpha = 0.05$. (21)

A

Q.22 (a) Evaluate

$$\lim_{x\to 0}\left[\frac{1}{x^2}-\frac{1}{x\tan x}\right].$$ (12)

(b) Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $a_1=1$ and $\lim_{n\to\infty} a_n=3$. Find the value of $\sum_{n=1}^{\infty}\left[a_{n+1}^2-a_n^2\right]$. (9)

A

Q.23 (a) Test for the convergence of the series

$$\sum_{n=1}^{\infty}\left(1-e^{-1/n}\right)\ln\left(1+\frac{1}{n}\right). \qquad (12)$$

(b) Determine whether the function

$$f(x, y)=\begin{cases}\dfrac{x^4+y^4}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0),\end{cases}$$

is continuous at $(0, 0)$. (9)

A

Q.24 (a) Find the area of the smaller of the two regions enclosed between $\frac{x^2}{9}+\frac{y^2}{2}=1$ and $y^2=x$. (12)

(b) Evaluate

$$\int_1^{\infty}\int_0^{y-1} e^{-\frac{y}{x+1}-x}\,dx\,dy.$$ (9)

A

Q.25 (a) Solve the differential equation

$$(x+y+2)\,dy-(y+2)\,dx=0. \qquad (12)$$

(b) Consider the linear transformation $L : \mathrm{R}^3 \to \mathrm{R}^3$ given by

$$L(x, y, z)=(kx+y+z,\ x+ky+z,\ x+y+kz).$$

Find the matrix P associated with the above transformation with respect to the standard basis. Further, find the values of k for which P has zero as one of its eigen values. (9)

Space for rough work

A

Space for rough work

Space for rough work

A

Space for rough work

Space for rough work

A

A