MA : MATHEMATICS

Duration : Three Hours   *Maximum Marks* :150

Read the following instructions carefully

1. This question paper contains **20** printed pages including pages for rough work. Please check all pages and report discrepancy, if any.

2. Write your registration number, your name and name of the examination centre at the specified locations on the right half of the ORS.

3. Using HB pencil, darken the appropriate bubble under each digit of your registration number and the letters corresponding to your paper code.

4. All the questions in this question paper are of objective type.

5. Questions must be answered on **O**bjective **R**esponse **S**heet (**ORS**) by darkening the appropriate bubble (marked A, B, C, D) using HB pencil against the question number on the left hand side of the ORS. **Each question has only one correct answer**. In case you wish to change an answer, erase the old answer completely. More than one answer bubbled against a question will be treated as a wrong answer.

6. Questions 1 through 20 are 1-mark questions and questions 21 through 85 are 2-mark questions.

7. Questions 71 through 73 is one set of common data questions, questions 74 and 75 is another pair of common data questions. The question pairs (76, 77), (78, 79), (80, 81), (82, 83) and (84, 85) are questions with linked answers. The answer to the second question of the above pairs will depend on the answer to the first question of the pair. If the first question in the linked pair is wrongly answered or is un-attempted, then the answer to the second question in the pair will not be evaluated.

8. Un-attempted questions will carry zero marks.

9. **NEGATIVE MARKING**: For Q.1 to Q.20, *0.25* mark will be deducted for each wrong answer. For Q.21 to Q.75, *0.5* mark will be deducted for each wrong answer. For the pairs of questions with linked answers, there will be negative marks only for wrong answer to the first question, i.e. for Q.76, Q.78, Q.80, Q.82 and Q.84, *0.5* mark will be deducted for each wrong answer. There is no negative marking for Q.77, Q.79, Q.81, Q.83 and Q.85.

10. Calculator **without data connectivity** is allowed in the examination hall.

11. Charts, graph sheets and tables are NOT allowed in the examination hall.

12. Rough work can be done on the question paper itself. Additional blank pages are given at the end of the question paper for rough work.

Notations and Definitions used in the paper

ϕ	:	The empty set
$X \setminus Y$	:	$\{x \in X : x \notin Y\}$
$\mathbb{N}$	:	The set of all positive integers
$\mathbb{Z}$	:	The set of all integers
$\mathbb{Z}_n$	:	The set of all integers modulo n
$\mathbb{R}$	:	The set of all real numbers
$\mathbb{R}^n$	:	$\{(x_1,...,x_n) : x_i \in \mathbb{R} \text{ for } 1 \le i \le n\}$
$\mathbb{C}$	:	The set of all complex numbers
$\mathbb{Z}[x_1,...,x_n]$	:	The polynomial ring in the variables $x_1,...,x_n$ with coefficients in $\mathbb{Z}$
x'	:	The derivative of the function x
μ	:	The Lebesgue measure on $\mathbb{R}$
$\| \ \|_p$	:	The p-norm for $1 \le p \le \infty$
$C([0,1])$	:	The set of all real-valued continuous functions on $[0,1]$
$C^1([0,1])$	:	The set of all real-valued continuously differentiable functions on $[0,1]$
$C^\infty([0,1])$	:	The set of all real-valued infinitely differentiable functions on $[0,1]$
x^t	:	The transpose of the vector x
M^t	:	The transpose of the matrix M
$N(m,\sigma^2)$	:	The normal distribution with mean m and variance σ^2

Q. 1 – Q. 20 carry one mark each.

Q.1   Consider the subspace $W = \{[a_{ij}] : a_{ij} = 0$ if i is even$\}$ of all 10×10 real matrices. Then the dimension of W is

(A) 25   (B) 50   (C) 75   (D) 100

Q.2   Let S be the open unit disk and $f : S \to \mathbb{C}$ be a real-valued analytic function with $f(0) = 1$. Then the set $\{z \in S : f(z) \neq 1\}$ is

(A) empty   (B) nonempty finite
(C) countably infinite   (D) uncountable

Q.3   Let $E = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq x\}$. Then $\iint\limits_E (x + y)\, dx\, dy$ is equal to

(A) -1 (B) 0 (C) $\dfrac{1}{2}$ (D) 1

Q.4   For $(x, y) \in \mathbb{R}^2$, let

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Then

(A) f_x and f_y exist at $(0, 0)$, and f is continuous at $(0, 0)$

(B) f_x and f_y exist at $(0, 0)$, and f is discontinuous at $(0, 0)$

(C) f_x and f_y do not exist at $(0, 0)$, and f is continuous at $(0, 0)$

(D) f_x and f_y do not exist at $(0, 0)$, and f is discontinuous at $(0, 0)$

Q.5   Let y be a solution of $y' = e^{-y^2} - 1$ on $[0, 1]$ which satisfies $y(0) = 0$. Then

(A) $y(x) > 0$ for $x > 0$ (B) $y(x) < 0$ for $x > 0$
(C) y changes sign in $[0, 1]$ (D) $y \equiv 0$ for $x > 0$

Q.6   For the equation $x(x-1)y'' + \sin(x)\, y' + 2x(x-1)y = 0$, consider the following statements
         P: $x = 0$ is a regular singular point.   Q: $x = 1$ is a regular singular point.
Then

(A) both P and Q are true   (B) P is false but Q is true
(C) P is true but Q is false   (D) both P and Q are false

Q.7   Let $G = \mathbb{R} \setminus \{0\}$ and $H = \{-1, 1\}$ be groups under multiplication. Then the map

$\varphi : G \to H$ defined by $\varphi(x) = \dfrac{x}{|x|}$ is

(A) not a homomorphism
(B) a one-one homomorphism, which is not onto
(C) an onto homomorphism, which is not one-one
(D) an isomorphism

Q.8   The number of maximal ideals in $\mathbb{Z}_{27}$ is

(A) 0   (B) 1   (C) 2   (D) 3

Q.9   For $1 \le p \le \infty$, let $\| \ \|_p$ denote the p-norm on $\mathbb{R}^2$. If $\| \ \|_p$ satisfies the parallelogram law, then p

is equal to

(A) 1   (B) 2   (C) 3   (D) ∞

Q.10   Consider the initial value problem $\dfrac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$. The aim is to compute the value of

$y_1 = y(x_1)$, where $x_1 = x_0 + h \, (h > 0)$. At $x = x_1$, if the value of y_1 is equated to the

corresponding value of the straight line passing through (x_0, y_0) and having the slope equal to the

slope of the curve $y(x)$ at $x = x_0$, then the method is called

(A) Euler's method   (B) Improved Euler's method
(C) Backward Euler's method   (D) Taylor series method of order 2

Q.11   The solution of $x u_x + y u_y = 0$ is of the form

(A) $f(y/x)$ (B) $f(x+y)$ (C) $f(x-y)$ (D) $f(xy)$

Q.12   If the partial differential equation $(x-1)^2 u_{xx} - (y-2)^2 u_{yy} + 2 x u_x + 2 y u_y + 2 x y u = 0$ is parabolic

in $S \subseteq \mathbb{R}^2$ but not in $\mathbb{R}^2 \setminus S$, then S is

(A) $\{(x, y) \in \mathbb{R}^2 : x = 1 \text{ or } y = 2\}$ (B) $\{(x, y) \in \mathbb{R}^2 : x = 1 \text{ and } y = 2\}$
(C) $\{(x, y) \in \mathbb{R}^2 : x = 1\}$ (D) $\{(x, y) \in \mathbb{R}^2 : y = 2\}$

Q.13   Let E be a connected subset of $\mathbb{R}$ with at least two elements. Then the number of elements in E is

(A) exactly two   (B) more than two but finite
(C) countably infinite   (D) uncountable

Q.14   Let X be a non-empty set. Let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be two topologies on X such that $\mathfrak{I}_1$ is strictly contained

in $\mathfrak{I}_2$. If $I : (X, \mathfrak{I}_1) \to (X, \mathfrak{I}_2)$ is the identity map, then

(A) both I and I^{-1} are continuous   (B) both I and I^{-1} are not continuous
(C) I is continuous but I^{-1} is not continuous   (D) I is not continuous but I^{-1} is continuous

Q.15   Let $X_1, X_2, \ldots, X_{10}$ be a random sample from $N(80, 3^2)$ distribution. Define

$$S = \sum_{i=1}^{10} U_i \quad \text{and} \quad T = \sum_{i=1}^{10} \left(U_i - \frac{S}{10} \right)^2,$$

where $U_i = \dfrac{X_i - 80}{3}$, $i = 1, 2, \ldots, 10$. Then the value of $E(ST)$ is equal to

(A) 0   (B) 1   (C) 10   (D) $\dfrac{80}{3}$

Q.16 Two (distinguishable) fair coins are tossed simultaneously. Given that ONE of them lands up head, the probability of the OTHER to land up tail is equal to

(A) $\dfrac{1}{3}$ (B) $\dfrac{1}{2}$ (C) $\dfrac{2}{3}$ (D) $\dfrac{3}{4}$

Q.17 Let $c_{ij} \geq 2$ be the cost of the $(i, j)^{th}$ cell of an assignment problem. If a new cost matrix is generated by the elements $c_{ij}^{*} = \dfrac{1}{2} c_{ij} + 1$, then

(A) optimal assignment plan remains unchanged and cost of assignment decreases
(B) optimal assignment plan changes and cost of assignment decreases
(C) optimal assignment plan remains unchanged and cost of assignment increases
(D) optimal assignment plan changes and cost of assignment increases

Q.18 Let a primal linear programming problem admit an optimal solution. Then the corresponding dual problem

(A) does not have a feasible solution
(B) has a feasible solution but does not have any optimal solution
(C) does not have a convex feasible region
(D) has an optimal solution

Q.19 In any system of particles, suppose we do not assume that the internal forces come in pairs. Then the fact that the sum of internal forces is zero follows from

(A) Newton's second law
(B) conservation of angular momentum
(C) conservation of energy
(D) principle of virtual displacement

Q.20 Let $q_1, q_2, \cdots, q_n$ be the generalized coordinates and $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$ be the generalized velocities in a conservative force field. If under a transformation φ, the new coordinate system has the generalized coordinates $Q_1, Q_2, \cdots, Q_n$ and velocities $\dot{Q}_1, \dot{Q}_2, \cdots, \dot{Q}_n$. Then the equation

$$\frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$$ takes the form

(A) $\dfrac{\partial L}{\partial Q_k} = \varphi \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{Q}_k}\right)$ (B) $\varphi \dfrac{\partial L}{\partial Q_k} = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{Q}_k}\right)$

(C) $\dfrac{\partial L}{\partial Q_k} = -\varphi \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{Q}_k}\right)$ (D) $\dfrac{\partial L}{\partial Q_k} = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{Q}_k}\right)$

Q. 21 to Q.75 carry two marks each.

Q.21 Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map satisfying

$$T(e_1) = e_2, \ T(e_2) = e_3, \ T(e_3) = 0, \ T(e_4) = e_3,$$

where $\{e_1, e_2, e_3, e_4\}$ is the standard basis of $\mathbb{R}^4$. Then

(A) T is idempotent   (B) T is invertible
(C) Rank $T = 3$ (D) T is nilpotent

Q.22   Let $M = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $V = \{ M x^t : x \in \mathbb{R}^3 \}$. Then an orthonormal basis for V is

(A) $\left\{ (1,0,0)^t, \left(0, \dfrac{2}{\sqrt 5}, \dfrac{1}{\sqrt 5}\right)^t, \left(\dfrac{2}{\sqrt 6}, \dfrac{1}{\sqrt 6}, \dfrac{1}{\sqrt 6}\right)^t \right\}$

(B) $\left\{ (1,0,0)^t, \left(0, \dfrac{1}{\sqrt 2}, \dfrac{1}{\sqrt 2}\right)^t \right\}$

(C) $\left\{ (1,0,0)^t, \left(\dfrac{1}{\sqrt 3}, \dfrac{1}{\sqrt 3}, \dfrac{1}{\sqrt 3}\right)^t, \left(\dfrac{2}{\sqrt 6}, \dfrac{1}{\sqrt 6}, \dfrac{1}{\sqrt 6}\right)^t \right\}$

(D) $\left\{ (1,0,0)^t, (0,0,1)^t \right\}$

Q.23   For any $n \in \mathbb{N}$, let P_n denote the vector space of all polynomials with real coefficients and of degree at most n. Define $T : P_n \to P_{n+1}$ by

$$T(p)(x) = p'(x) - \int_0^x p(t)\, dt .$$

Then the dimension of the null space of T is

(A) 0   (B) 1   (C) n (D) $n+1$

Q.24   Let $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$, where $0 < \theta < \dfrac{\pi}{2}$. Let $V = \{ u \in \mathbb{R}^3 : M u^t = u^t \}$. Then the dimension of V is

(A) 0   (B) 1   (C) 2   (D) 3

Q.25   The number of linearly independent eigenvectors of the matrix $\begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ is

(A) 1   (B) 2   (C) 3   (D) 4

Q.26   Let f be a bilinear transformation that maps -1 to 1, i to ∞ and $-i$ to 0. Then $f(1)$ is equal to

(A) -2 (B) -1 (C) i (D) $-i$

Q.27   Which one of the following does NOT hold for all continuous functions $f : [-\pi, \pi] \to \mathbb{C}$?

(A) If $f(-t) = f(t)$ for each $t \in [-\pi, \pi]$, then $\int\limits_{-\pi}^{\pi} f(t)dt = 2\int\limits_{0}^{\pi} f(t)dt$

(B) If $f(-t) = -f(t)$ for each $t \in [-\pi, \pi]$, then $\int\limits_{-\pi}^{\pi} f(t)dt = 0$

(C) $\int\limits_{-\pi}^{\pi} f(-t)dt = -\int\limits_{-\pi}^{\pi} f(t)dt$

(D) There is an α with $-\pi < \alpha < \pi$ such that $\int\limits_{-\pi}^{\pi} f(t)dt = 2\pi f(\alpha)$

Q.28   Let S be the positively oriented circle given by $|z - 3i| = 2$. Then the value of $\int\limits_{S} \dfrac{dz}{z^2 + 4}$ is

(A) $\dfrac{-\pi}{2}$ (B) $\dfrac{\pi}{2}$ (C) $\dfrac{-i\pi}{2}$ (D) $\dfrac{i\pi}{2}$

Q.29   Let T be the closed unit disk and ∂T be the unit circle. Then which one of the following holds for every analytic function $f : T \to \mathbb{C}$.

(A) $|f|$ attains its minimum and its maximum on ∂T

(B) $|f|$ attains its minimum on ∂T but need not attain its maximum on ∂T

(C) $|f|$ attains its maximum on ∂T but need not attain its minimum on ∂T

(D) $|f|$ need not attain its maximum on ∂T and also it need not attain its minimum on ∂T

Q.30   Let S be the disk $|z| < 3$ in the complex plane and let $f : S \to \mathbb{C}$ be an analytic function such that

$f\left(1 + \dfrac{\sqrt{2}}{n}i\right) = -\dfrac{2}{n^2}$ for each natural number n. Then $f(\sqrt{2})$ is equal to

(A) $3 - 2\sqrt{2}$ (B) $3 + 2\sqrt{2}$ (C) $2 - 3\sqrt{2}$ (D) $2 + 3\sqrt{2}$

Q.31   Which one of the following statements holds?

(A) The series $\sum\limits_{n=0}^{\infty} x^n$ converges for each $x \in [-1, 1]$

(B) The series $\sum\limits_{n=0}^{\infty} x^n$ converges uniformly in $(-1, 1)$

(C) The series $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n}$ converges for each $x \in [-1, 1]$

(D) The Deries $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n^2}$ converges uniformly in $(-1, 1)$

Q.32 For $x \in [-\pi, \pi]$, let

$$f(x) = (\pi + x)(\pi - x) \quad \text{and} \quad g(x) = \begin{cases} \cos(1/x) & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Consider the statements

P: The Fourier series of f converges uniformly to f on $[-\pi, \pi]$.

Q: The Fourier series of g converges uniformly to g on $[-\pi, \pi]$.

Then

(A) P and Q are true   (B) P is true but Q is false

(C) P is false but Q is true   (D) both P and Q are false

Q.33 Let $W = \{(x, y, z) \in \mathbb{R}^3 : 1 \leq x^2 + y^2 + z^2 \leq 4\}$ and $F : W \to \mathbb{R}^3$ be defined by

$$F(x, y, z) = \frac{(x, y, z)}{\left[x^2 + y^2 + z^2\right]^{3/2}} \quad \text{for} \quad (x, y, z) \in W. \text{ If } \partial W \text{ denotes the boundary of } W \text{ oriented by}$$

the outward normal n to W, then $\iint\limits_{\partial W} F \cdot n \, dS$ is equal to

(A) 0   (B) 4π (C) 8π (D) 12π

Q.34 For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be a measurable function such that $\left| f_n(t) \right| \leq \dfrac{1}{\sqrt{t}}$ for all

$t \in (0,1]$. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(t) = 1$ if t is irrational and $f(t) = -1$ if t is

rational. Assume that $f_n(t) \to f(t)$ as $n \to \infty$ for all $t \in [0,1]$. Then

(A) f is not measurable

(B) f is measurable and $\displaystyle\int\limits_{[0,1]} f_n \, d\mu \to 1$ as $n \to \infty$

(C) f is measurable and $\displaystyle\int\limits_{[0,1]} f_n \, d\mu \to 0$ as $n \to \infty$

(D) f is measurable and $\displaystyle\int\limits_{[0,1]} f_n \, d\mu \to -1$ as $n \to \infty$

Q.35 Let y_1 and y_2 be two linearly independent solutions of

$$y'' + (\sin x) y, \quad 0 \leq x \leq 1.$$

Let $g(x) = W(y_1, y_2)(x)$ be the Wronskian of y_1 and y_2. Then

(A) $g' > 0$ on $[0,1]$

(B) $g' < 0$ on $[0,1]$

(C) g' vanishes at only one point of $[0,1]$

(D) g' vanishes at all points of $[0,1]$

Q.36 One particular solution of $y''' - y'' - y' + y = -e^x$ is a constant multiple of

(A) xe^{-x} (B) xe^x (C) x^2e^{-x} (D) x^2e^x

Q.37 Let $a, b \in \mathbb{R}$. Let $y = (y_1, y_2)'$ be a solution of the system of equations

$$y_1' = y_2, \quad y_2' = ay_1 + by_2.$$

Every solution $y(x) \to 0$ as $x \to \infty$ if

(A) $a < 0, b < 0$ (B) $a < 0, b > 0$

(C) $a > 0, b > 0$ (D) $a > 0, b < 0$

Q.38 Let G be a group of order 45. Let H be a 3-Sylow subgroup of G and K be a 5-Sylow subgroup of G. Then

(A) both H and K are normal in G
(B) H is normal in G but K is not normal in G
(C) H is not normal in G but K is normal in G
(D) both H and K are not normal in G

Q.39 The ring $\mathbb{Z}\left[\sqrt{-11}\right]$ is

(A) a Euclidean Domain
(B) a Principal Ideal Domain, but not a Euclidean Domain
(C) a Unique Factorization Domain, but not a Principal Ideal Domain
(D) not a Unique Factorization Domain

Q.40 Let R be a Principal Ideal Domain and a, b any two non-unit elements of R. Then the ideal generated by a and b is also generated by

(A) $a + b$ (B) ab (C) $\gcd(a,b)$ (D) $\operatorname{lcm}(a,b)$

Q.41 Consider the action of S_4, the symmetric group of order 4, on $\mathbb{Z}\left[x_1, x_2, x_3, x_4\right]$ given by

$$\sigma \bullet p\left(x_1, x_2, x_3, x_4\right) = p\left(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}\right) \text{ for } \sigma \in S_4.$$

Let $H \subseteq S_4$ denote the cyclic subgroup generated by $(1\,4\,2\,3)$. Then the cardinality of the orbit $O_H\left(x_1 x_3 + x_2 x_4\right)$ of H on the polynomial $x_1 x_3 + x_2 x_4$ is

(A) 1   (B) 2   (C) 3   (D) 4

Q.42 Let $f : \ell^2 \to \mathbb{R}$ be defined by $f(x_1, x_2, \cdots) = \sum_{n=1}^{\infty} \dfrac{x_n}{2^{n/2}}$ for $(x_1, x_2, \cdots) \in \ell^2$. Then $\| f \|$ is equal to

(A) $\dfrac{1}{2}$ (B) 1   (C) 2   (D) $\dfrac{1}{\sqrt{2}-1}$

Q.43 Consider $\mathbb{R}^3$ with norm $\| \ \|_1$, and the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by the 3x3

matrix $\begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 2 \\ 1 & 3 & -3 \end{pmatrix}$. Then the operator norm $\| T \|$ of T is equal to

(A) 6   (B) 7   (C) 8   (D) $\sqrt{42}$

Q.44 Consider $\mathbb{R}^2$ with norm $\| \ \|_\infty$, and let $Y = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 = 0\}$. If $g : Y \to \mathbb{R}$ is defined by $g(y_1, y_2) = y_2$ for $(y_1, y_2) \in Y$, then

(A) g has no Hahn-Banach extension to $\mathbb{R}^2$

(B) g has a unique Hahn-Banach extension to $\mathbb{R}^2$

(C) every linear functional $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $f(-1,1) = 1$ is a Hahn-Banach extension of g to $\mathbb{R}^2$

(D) the functionals $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ given by $f_1(x_1, x_2) = x_2$ and $f_2(x_1, x_2) = -x_1$ are both Hahn-Banach extensions of g to $\mathbb{R}^2$

Q.45 Let X be a Banach space and Y be a normed linear space. Consider a sequence (F_n) of bounded linear maps from X to Y such that for each fixed $x \in X$, the sequence $(F_n(x))$ is bounded in Y. Then

(A) for each fixed $x \in X$, the sequence $(F_n(x))$ is convergent in Y

(B) for each fixed $n \in \mathbb{N}$, the set $\{F_n(x) : x \in X\}$ is bounded in Y

(C) the sequence $(\|F_n\|)$ is bounded in $\mathbb{R}$

(D) the sequence (F_n) is uniformly bounded on X

Q.46 Let $H = L^2([0, \pi])$ with the usual inner product. For $n \in \mathbb{N}$, let

$$u_n(t) = \frac{\sqrt{2}}{\sqrt{\pi}} \sin nt,\ t \in [0, \pi],\ \text{ and }\ E = \{u_n : n \in \mathbb{N}\}.$$

Then

(A) E is not a linearly independent subset of H.

(B) E is a linearly independent subset of H, but is not an orthonormal subset of H.

(C) E is an orthonormal subset of H, but is not an orthonormal basis for H.

(D) E is an orthonormal basis for H.

Q.47 Let $X = \mathbb{R}$ and let $\mathfrak{I} = \{U \subseteq X : X - U \text{ is finite}\} \cup \{\phi, X\}$. The sequence

$$1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ... \text{ in } (X, \mathfrak{I})$$

(A) converges to 0 and not to any other point of X

(B) does not converge to 0

(C) converges to each point of X

(D) is not convergent in X

Q.48 Let $E = \{(x, y) \in \mathbb{R}^2 : |x| \le 1, |y| \le 1\}$. Define $f : E \to \mathbb{R}$ by $f(x, y) = \dfrac{x + y}{1 + x^2 + y^2}$.

Then the range of f is a

(A) connected open set   (B) connected closed set

(C) bounded open set   (D) closed and unbounded set

Q.49 Let $X = \{1,2,3\}$ and $\Im = \{\phi, \{1\}, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$. The topological space $(X, \Im)$ is said to have the property P if for any two proper disjoint closed subsets Y and Z of X, there exist disjoint open sets U, V such that $Y \subseteq U$ and $Z \subseteq V$. Then the topological space $(X, \Im)$

(A) is T_1 and satisfies P

(B) is T_1 and does not satisfy P

(C) is not T_1 and satisfies P

(D) is not T_1 and does not satisfy P

Q.50 Which one of the following subsets of $\mathbb{R}$ (with the usual metric) is NOT complete?

(A) $[1, 2] \cup [3, 4]$ (B) $[0, \infty)$

(C) $[0, 1)$ (D) $\{0\} \cup \left\{ \dfrac{1}{n} : n \in \mathbb{N} \right\}$

Q.51 Consider the function

$$f(x) = \begin{cases} k(x - [x]), & 0 \le x < 2 \\ 0, & \text{otherwise,} \end{cases}$$

where $[x]$ is the integral part of x. The value of k for which the above function is a probability density function of some random variable is

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{2}$ (C) 1 (D) 2

Q.52 For two random variables X and Y, the regression lines are given by $Y = 5X - 15$ and $Y = 10X - 35$. Then the regression coefficient of X on Y is

(A) 0.1 (B) 0.2 (C) 5 (D) 10

Q.53 In an examination there are 80 questions each having four choices. Exactly one of these four choices is correct and the other three are wrong. A student is awarded 1 mark for each correct answer, and -0.25 for each wrong answer. If a student ticks the answer of each question randomly, then the expected value of his/her total marks in the examination is

(A) -15 (B) 0 (C) 5 (D) 20

Q.54 Let $X_1, X_2, ..., X_n$ be a random sample from uniform distribution on $[0, \theta]$. Then the maximum likelihood estimator (MLE) of θ based on the above random sample is

(A) $\dfrac{2}{n} \sum_{i=1}^{n} X_i$ (B) $\dfrac{1}{n} \sum_{i=1}^{n} X_i$

(C) $\text{Min}\{X_1, X_2, ..., X_n\}$ (D) $\text{Max}\{X_1, X_2, ..., X_n\}$

Q.55 The cost matrix of a transportation problem is given by

1	2	3	4
4	3	2	0
0	2	2	1

The following are the values of variables in a feasible solution.

$$x_{12} = 6, \ x_{23} = 2, \ x_{24} = 6, \ x_{31} = 4, \ x_{33} = 6$$

Then which of the following is correct?

(A) The solution is degenerate and basic   (B) The solution is non-degenerate and basic
(C) The solution is degenerate and non-basic   (D) The solution is non-degenerate and non-basic

Q.56 The maximum value of $z = 3x_1 - x_2$ subject to $2x_1 - x_2 \le 1$, $x_1 \le 3$ and $x_1, x_2 \ge 0$ is

(A) 0   (B) 4   (C) 6   (D) 9

Q.57 Consider the problem of maximizing $z = 2x_1 + 3x_2 - 4x_3 + x_4$ subject to

$$x_1 + x_2 + x_3 \qquad\quad = \ 2,$$
$$x_1 - x_2 + x_3 \qquad\quad = \ 2,$$
$$2x_1 + 3x_2 + 2x_3 - x_4 \ = \ 0,$$
$$x_1, x_2, x_3, x_4 \qquad\quad \ge \ 0.$$

Then

(A) $(1,0,1,4)$ is a basic feasible solution but $(2,0,0,4)$ is not
(B) $(1,0,1,4)$ is not a basic feasible solution but $(2,0,0,4)$ is
(C) neither $(1,0,1,4)$ nor $(2,0,0,4)$ is a basic feasible solution
(D) both of $(1,0,1,4)$ and $(2,0,0,4)$ are basic feasible solutions

Q.58 In the closed system of a simple harmonic motion of a pendulum, let H denote the Hamiltonian and E be the total energy. Then

(A) H is a constant and $H = E$ (B) H is a constant but $H \ne E$
(C) H is not constant but $H = E$ (D) H is not constant and $H \ne E$

Q.59 The possible values of α for which the variational problem

$$J[y(x)] = \int_0^1 \left(3y^2 + 2x^3 y'\right) dx, \ y(\alpha) = 1$$

has extremals are

(A) $-1, 0$ (B) $0, 1$ (C) $-1, 1$ (D) $-1, 0, 1$

Q.60 The functional $\int_0^1 \left(y'^2 + x^3\right) dx$, given $y(1) = 1$, achieves its

(A) weak maximum on all its extremals
(B) weak minimum on all its extremals
(C) weak maximum on some, but not on all of its extremals
(D) weak minimum on some, but not on all of its extremals

Q.61 The integral equation

$$x(t) = \sin t + \lambda \int_0^t \left(s^2 t^3 + e^{s^2 + t^3} \right) x(s)\, ds, \ 0 \le t \le 1, \ \lambda \in \mathbb{R}, \ \lambda \ne 0$$

has a solution for

(A) all non-zero values of λ

(B) no value of λ

(C) only countably many positive values of λ

(D) only countably many negative values of λ

Q.62 The integral equation $x(t) - \int_0^1 [\cos t \ \sec s \ x(s)]\, ds = \sinh t, \ 0 \le t \le 1$, has

(A) no solution

(B) a unique solution

(C) more than one but finitely many solution

(D) infinitely many solutions

Q.63 If $y_{i+1} = y_i + h\,\varphi(f, x_i, y_i, h), \ i = 1, 2, \cdots$, where

$\varphi(f, x, y, h) = af(x, y) + bf(x + h, y + hf(x, y))$, is a second order accurate scheme to solve the

initial value problem $\dfrac{dy}{dx} = f(x, y), \ y(x_0) = y_0$, then a and b, respectively, are

(A) $\dfrac{h}{2}, \ \dfrac{h}{2}$ (B) $1, \ -1$ (C) $\dfrac{1}{2}, \ \dfrac{1}{2}$ (D) $h, \ -h$

Q.64 If a quadrature formula $\dfrac{3}{2} f\left(-\dfrac{1}{3}\right) + K f\left(\dfrac{1}{3}\right) + \dfrac{1}{2} f(1)$, that approximates $\displaystyle\int_{-1}^{1} f(x)\, dx$, is found to

be exact for quadratic polynomials, then the value of K is

(A) 2 (B) 1 (C) 0 (D) -1

Q.65 If $\begin{pmatrix} 1 & 4 & 3 \\ 2 & 7 & 9 \\ 5 & 8 & a \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & -53 \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$, then the value of a is

(A) -2 (B) -1 (C) 1 (D) 2

Q.66 Using the least squares method, if a curve $y = ax^2 + bx + c$ is fitted to the collinear data points $(-1, -3)$, $(1, 1)$, $(3, 5)$ and $(7, 13)$, then the triplet (a, b, c) is equal to

(A) $(-1, 2, 0)$ (B) $(0, 2, -1)$ (C) $(2, -1, 0)$ (D) $(0, -1, 2)$

Q.67 A quadratic polynomial $p(x)$ is constructed by interpolating the data points $(0,1)$, $(1, e)$ and $(2, e^2)$. If $\sqrt{e}$ is approximated by using $p(x)$, then its approximate value is

(A) $\dfrac{1}{8}\left(3 + 6e - e^2\right)$ (B) $\dfrac{1}{8}\left(3 - 6e + 2e^2\right)$

(C) $\dfrac{1}{8}\left(3 - 6e - e^2\right)$ (D) $\dfrac{1}{8}\left(3 + 6e - 2e^2\right)$

Q.68 The characteristic curve of $2yu_x + (2x+y^2)u_y = 0$ passing through $(0, 0)$ is

(A) $y^2 = 2(e^x + x - 1)$ (B) $y^2 = 2(e^x - x + 1)$

(C) $y^2 = 2(e^x - x - 1)$ (D) $y^2 = 2(e^x + x + 1)$

Q.69 The initial value problem $u_x + u_y = 1$, $u(s,s) = \sin s$, $0 \le s \le 1$, has

(A) two solutions   (B) a unique solution

(C) no solution   (D) infinitely many solutions

Q.70 Let $u(x,t)$ be the solution of $u_{tt} - u_{xx} = 1$, $x \in \mathbb{R}, t > 0$, with

$u(x,0) = 0$, $u_t(x,0) = 0$, $x \in \mathbb{R}$. Then $u(1/2, \, 1/2)$ is equal to

(A) $\dfrac{1}{8}$ (B) $-\dfrac{1}{8}$ (C) $\dfrac{1}{4}$ (D) $-\dfrac{1}{4}$

Common Data Questions

Common Data for Questions 71, 72, 73:

Let $X = C([0,1])$ with sup norm $\| \ \|_\infty$.

Q.71 Let $S = \left\{ x \in X : \|x\|_\infty \le 1 \right\}$. Then

(A) S is convex and compact
(B) S is not convex but compact
(C) S is convex but not compact
(D) S is neither convex nor compact

Q.72 Which one of the following is true?

(A) $C^\infty([0,1])$ is dense in X

(B) X is dense in $L^\infty([0,1])$

(C) X has a countable basis

(D) There is a sequence in X which is uniformly Cauchy on $[0, 1]$ but does not converge uniformly on $[0, 1]$

Q.73 Let $I = \left\{ x \in X : x(0) = 0 \right\}$. Then

(A) I is not an ideal of X
(B) I is an ideal, but not a prime ideal of X
(C) I is a prime ideal, but not a maximal ideal of X
(D) I is a maximal ideal of X

Common Data for Questions 74, 75:

Let $X = C^1([0,1])$ and $Y = C([0,1])$, both with the sup norm. Define $F : X \to Y$ by $F(x) = x + x'$

and $f(x) = x(1) + x'(1)$ for $x \in X$.

Q.74 Then
(A) F and f are continuous   (B) F is continuous and f is discontinuous
(C) F is discontinuous and f is continuous   (D) F and f are discontinuous

Q.75   Then
- (A) F and f are closed maps
- (B) F is a closed map and f is not a closed map
- (C) F is not a closed map and f is a closed map
- (D) neither F nor f is a closed map

Linked Answer Questions: Q.76 to Q.85 carry two marks each.

Statement for Linked Answer Questions 76 & 77:

Let $N = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Q.76   Then N is
- (A) non-invertible
- (C) symmetric
- (B) skew-symmetric
- (D) orthogonal

Q.77   If M is any 3×3 real matrix, then trace$\left(NMN'\right)$ is equal to

- (A) $[\text{trace}(N)]^2 \text{trace}(M)$
- (C) $\text{trace}(M)$
- (B) $2\,\text{trace}(N) + \text{trace}(M)$
- (D) $[\text{trace}(N)]^2 + \text{trace}(M)$

Statement for Linked Answer Questions 78 & 79:

Let $f(z) = \cos z - \dfrac{\sin z}{z}$ for non-zero $z \in \mathbb{C}$ and $f(0) = 0$. Also, let $g(z) = \sinh z$ for $z \in \mathbb{C}$.

Q.78   Then $f(z)$ has a zero at $z = 0$ of order
- (A) 0
- (B) 1
- (C) 2
- (D) greater than 2

Q.79   Then $\dfrac{g(z)}{zf(z)}$ has a pole at $z = 0$ of order

- (A) 1
- (B) 2
- (C) 3
- (D) greater than 3

Statement for Linked Answer Questions 80 & 81:

Let $n \geq 3$ be an integer. Let y be the polynomial solution of $\left(1 - x^2\right)y'' - 2xy' + n(n-1)y = 0$ satisfying $y(1) = 1$.

Q.80   Then the degree of y is
- (A) n
- (B) $n-1$
- (C) less than $n-1$
- (D) greater than $n+1$

Q.81   If $I = \displaystyle\int_{-1}^{1} y(x)x^{n-3}\,dx$ and $J = \displaystyle\int_{-1}^{1} y(x)x^n\,dx$, then

- (A) $I \neq 0,\ J \neq 0$
- (C) $I = 0,\ J \neq 0$
- (B) $I \neq 0,\ J = 0$
- (D) $I = 0,\ J = 0$

Statement for Linked Answer Questions 82 & 83:
Consider the boundary value problem

$$u_{xx} + u_{yy} = 0, \ x \in (0, \pi), \ y \in (0, \pi),$$

$$u(x, 0) = u(x, \pi) = u(0, y) = 0.$$

Q.82 Any solution of this boundary value problem is of the form

(A) $\displaystyle\sum_{n=1}^{\infty} a_n \sinh nx \sin ny$

(B) $\displaystyle\sum_{n=1}^{\infty} a_n \cosh nx \sin ny$

(C) $\displaystyle\sum_{n=1}^{\infty} a_n \sinh nx \cos ny$

(D) $\displaystyle\sum_{n=1}^{\infty} a_n \cosh nx \cos ny$

Q.83 If an additional boundary condition $u_x(\pi, y) = \sin y$ is satisfied, then $u(x, \pi/2)$ is equal to

(A) $\dfrac{\pi}{2}\left(e^{\pi} - e^{-\pi}\right)\left(e^x + e^{-x}\right)$

(B) $\dfrac{\pi\left(e^x + e^{-x}\right)}{4\left(e^{\pi} - e^{-\pi}\right)}$

(C) $\dfrac{\pi\left(e^x - e^{-x}\right)}{\left(e^{\pi} + e^{-\pi}\right)}$

(D) $\dfrac{\pi}{2}\left(e^{\pi} + e^{-\pi}\right)\left(e^x + e^{-x}\right)$

Statement for Linked Answer Questions 84 & 85:
Let a random variable X follow the exponential distribution with mean 2. Define
$Y = [X - 2 \mid X > 2]$.

Q.84 The value of $P(Y \geq t)$ is

(A) $e^{-t/2}$ (B) e^{-2t} (C) $\dfrac{1}{2}e^{-t/2}$ (D) $\dfrac{1}{2}e^{-t}$

Q.85 The value of $E(Y)$ is equal to

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{2}$ (C) 1 (D) 2

END OF THE QUESTION PAPER