1> Prove $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ & $|\psi\rangle^* = \beta^*|0\rangle - \alpha^*|1\rangle$ are orthogal.

Find $B$ s.t. $B|\psi\rangle = |0\rangle$

& $B|\psi^*\rangle = |1\rangle$

Solution :-

$$\langle\psi^*|\psi\rangle = \begin{bmatrix}\beta & -\alpha\end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

$$= 0$$

~~$\Rightarrow |\psi^*\rangle$ & 1~~

$$\left[\,|\psi\rangle = \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \quad \& \quad |\psi^*\rangle = \begin{pmatrix}\beta^* \\ -\alpha^*\end{pmatrix}\right]$$

$\Rightarrow |\psi^*\rangle$ & $|\psi\rangle$ are orthogonal

Let $M = \begin{pmatrix}|\psi\rangle & |\psi^*\rangle\end{pmatrix}$

$$= \begin{pmatrix}\alpha & \beta^* \\ \beta & -\alpha^*\end{pmatrix}$$

~~is unitary as~~

$$B = M^{-1} = \begin{pmatrix}\alpha^* & \beta^* \\ \beta & -\alpha\end{pmatrix}$$

$$B|\psi\rangle = \begin{pmatrix}\alpha^* & \beta^* \\ \beta & -\alpha\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix}|\alpha|^2+|\beta|^2 \\ \alpha\beta-\alpha\beta\end{pmatrix}$$

$$= \begin{pmatrix}1 \\ 0\end{pmatrix}$$

$$= |0\rangle$$

$$B|\psi^*\rangle = \begin{pmatrix}\alpha^* & \beta^* \\ \beta & -\alpha\end{pmatrix}\begin{pmatrix}\beta^* \\ -\alpha^*\end{pmatrix} = \begin{pmatrix}\alpha^*\beta^* - \alpha^*\beta^* \\ |\beta|^2 + |\alpha|^2\end{pmatrix}$$

$$= \begin{pmatrix}0 \\ 1\end{pmatrix}$$

$$= |1\rangle$$

B is unitary as $B = M^*$

& M is unitary as columns of M are orthonormal.

2) Prove quantum bits can't be cloned.

Let U copies qubits

So, $U|\psi\rangle|\phi\rangle = |\psi\rangle|\psi\rangle$ [ $|\phi\rangle$ is a normalised state]

We know U is unitary i.e. $UU^* = I$

$$U|\psi\rangle|\phi\rangle = |\psi\rangle|\psi\rangle$$
$$U|\psi'\rangle|\phi\rangle = |\psi'\rangle|\psi'\rangle$$

$$\langle\phi|\langle\psi'|U^* U|\psi\rangle|\phi\rangle = \langle\psi'|\langle\psi'||\psi\rangle|\psi\rangle$$

$$\Rightarrow \langle\phi|\langle\psi'||\psi\rangle|\phi\rangle = [\langle\psi'||\psi\rangle]^2$$

$$\Rightarrow \langle\psi'||\psi\rangle\langle\phi||\phi\rangle = [\langle\psi'||\psi\rangle]^2$$

$$\Rightarrow \langle\psi'||\psi\rangle = [\langle\psi'||\psi\rangle]^2$$

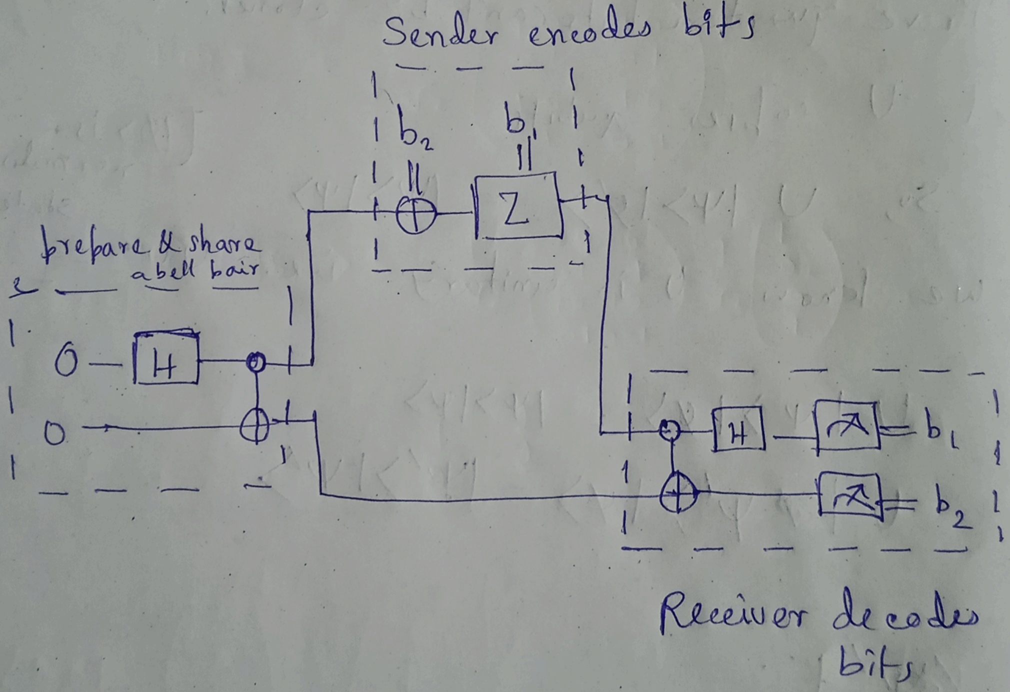
[ $\langle\phi||\phi\rangle = 1$ as $|\phi\rangle$ is normalised]

$$\Rightarrow \langle\psi'|1|\psi\rangle\left(\langle\psi'|1|\psi\rangle - 1\right) = 0$$

$$\Rightarrow \langle\psi'|1|\psi\rangle = 0 \quad \text{or} \quad \langle\psi'|1|\psi\rangle = 1$$

$\Rightarrow$ cloning only possible for pure states from orthonormal basis & impossible for general qubits

3) Dense Coding




$-$ lines carry qubits

$=$ double lines carry classical bits

$b_1, b_2$ are classical boolean bits

When the sender & receiver shares a bell state, two classical bits can be packed into one qubit

Protocol

Suppose,

→ Alice wants to send Bob two classical bits using qubits.

→ The protocol prepares an entangled state which is shared between Bob & Alice.

eg:- $|\Phi^+\rangle = \frac{1}{\sqrt{2}} |0_A 0_B\rangle + |1_A 1_B\rangle$ [bell state]

$[|0_A 0_B\rangle = (|0\rangle_A \otimes |0\rangle_B)]$

→ The qubit denoted by subscript A is sent to Alice & the qubit denoted by subscript B is sent to Bob

~~Encoding~~

## Encoding

Alice applies a quantum gate to her qubit locally & ~~thus~~ thus transforms $|\Phi^+\rangle$ into any of four bell states.

## Encoding

| Alice's message | The gate applied to Alice's ~~message~~ qubit | Resultant entanged state |
|---|---|---|
| 00 | $I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ | $\lvert B_{00}\rangle = \frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$ |
| 01 | $\cancel{\text{CNOT}}\ X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ | $\lvert B_{01}\rangle = \frac{1}{\sqrt{2}}(\lvert 10\rangle + \lvert 01\rangle)$ |
| 10 | $Z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ | $\lvert B_{10}\rangle = \frac{1}{\sqrt{2}}\ \cancel{\lvert 10\rangle +}\ (\lvert 00\rangle - \lvert 11\rangle)$ |
| 11 | $Z * X = iY = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ | $\lvert B_{11}\rangle = \frac{1}{\sqrt{2}}(\lvert 01\rangle - \lvert 10\rangle)$ |

## Decoding

i) Bob performs CNOT on the entangled state using Alice's bit as control & Bob's bit as target

ii) Bob performs $H \otimes I$ on the result ~~and gets the two classical bits~~ and gets the basis qubit state which corresponds to the classical bits sent by Alice

eg:- Bob performs CNOT on $|B_{00}\rangle$ &

then applies $H \otimes I$ on the result to get $|00\rangle$ basis. This denotes that the classical bits sent by Alice is 00