1) a) real vectors = $(0,1,0,1)$ & $(0,1,1,1)$

inner product = ~~$[0\,1\,0\,1]\,[0\,1\,1\,1]^T$~~

$$= (0\ 1\ 0\ 1)\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$$

$$= 1+1 = \underline{2} \quad \text{(ans:)}$$

b) matrix representation of $|0101\rangle$:-

$$M_{0101} = \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\\cancel{1}\\0\\0\end{pmatrix}$$

for $|0111\rangle$ matrix representation:-

$$M_{0111} = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\1\\0\\0\\0\\1\end{pmatrix}$$

$\Rightarrow$ inner product of $|0101\rangle$ & $|0111\rangle$ is

$$= M_{0101}(M_{0111})^T$$

$$= 0$$

(ans.)

2> $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$H|0\rangle \otimes H|1\rangle = \frac{1}{2}\left(\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\-1\end{pmatrix}\right)$$

$$= \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$$

$$H \otimes H = \frac{1}{2}\left(\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\right)$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\1 & -1 & 1 & -1\\1 & 1 & -1 & -1\\1 & -1 & -1 & 1\end{pmatrix}$$

$$|0\rangle \otimes |1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$(H \otimes H)(|0\rangle \otimes |1\rangle) = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\1 & -1 & 1 & -1\\1 & 1 & -1 & -1\\1 & -1 & -1 & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$$

$$= H|0\rangle \otimes H|1\rangle$$

(~~brau~~

3) $$H X H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

$$\left[X = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}2 & 0\\0 & -2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$$H \times H |0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$Z |0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \left[Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = H \times H |0\rangle$$

Similarly, we find,

$$H \times H |1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$Z |1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$= H \times H |1\rangle$$

hence, on standard basis $|0\rangle$ & $|1\rangle$,

$$H \times H = Z$$

4) $$M_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$H \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

M

M

$(H\otimes H)\ CNOT\ (H\otimes H)$

$$= \frac{1}{2}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\frac{1}{2}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&-1&-1&1\\1&1&-1&-1\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}$$

In 2qbit, standard bases: $|00\rangle, |01\rangle, |10\rangle$ & $|11\rangle$

$$M|00\rangle = \begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = |00\rangle$$

$$M|01\rangle = \begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = |11\rangle$$

$$M|10\rangle = \begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = |10\rangle$$

$$M|11\rangle = \begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = |01\rangle$$

$\Rightarrow$ first bit flips if second bit is 1

" unchanged " 0

(proved)

5) ~~$\alpha_{0,0}|00\rangle + \alpha_{0,1}|01\rangle + \alpha_{1,0}$~~

5) $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

$$= \alpha_{00}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \alpha_{01}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \alpha_{10}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \alpha_{11}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}\alpha_{00}\\\alpha_{01}\\\alpha_{10}\\\alpha_{11}\end{pmatrix}$$

$$\langle 0| \otimes I = |0\rangle^{*T} \otimes I$$

$$\cancel{= (1 \quad 0) \begin{matrix}1 & 0\\0 & 1\end{matrix}}$$

$$= (1 \quad 0) \otimes \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{pmatrix}$$

$$(\langle 0| \otimes I)(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle)$$

$$= \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\alpha_{00}\\\alpha_{01}\\\alpha_{10}\\\alpha_{11}\end{pmatrix} = \begin{pmatrix}\alpha_{00}\\\alpha_{01}\end{pmatrix} = \alpha_{00}|0\rangle + \alpha_{01}|1\rangle$$

(Ans)

6) $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

P.T. :- $|\psi\rangle$ can't be decomposed as $|\psi_1\rangle \otimes |\psi_2\rangle$

Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ & $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$;

$\boxed{|\alpha_1|^2 + |\beta_1|^2 = |\alpha_2|^2 + |\beta_2|^2 = 1}$ $|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

& $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$

$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$

equating with $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$;

$\alpha_1\alpha_2 = \frac{1}{\sqrt{2}}$; $\beta_1\beta_2 = \frac{1}{\sqrt{2}}$; $\alpha_1\beta_2 = \beta_1\alpha_2 = 0$

~~$\Rightarrow |\alpha_1\alpha_2|^2 + |\beta_1\beta_2|^2 = 1$~~

$|\alpha_1\alpha_2|^2 + |\alpha_1\beta_2|^2 = \frac{1}{2}$

$\Rightarrow |\alpha_1|^2\left(|\alpha_2|^2 + |\beta_2|^2\right) = \frac{1}{2}$

$\Rightarrow |\alpha_1|^2 = \frac{1}{2}$ $\left[|\alpha_2|^2 + |\beta_2|^2 = 1\right.$ as $|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle\left.\right]$

Similarly $|\beta_1\beta_2|^2 + |\beta_1\alpha_2|^2 = \frac{1}{2}$

$\Rightarrow |\beta_1|^2\left(|\alpha_1|^2 + |\beta_1|^2\right) = \frac{1}{2}$ .. (i)

as $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$

hence, $|\alpha_1|^2 + |\beta_1|^2 = 1$

~~$\Rightarrow |\beta_1|^2$~~ from (i); $|\beta_1|^2 = \frac{1}{2}$

Now,

$|\alpha_1\beta_2|^2 = 0$ but $|\alpha_1|^2 = \frac{1}{2}$ $\Rightarrow |\beta_2|^2 = 0$

$|\beta_1\alpha_2|^2 = 0$ but $|\beta_1|^2 = \frac{1}{2}$ $\Rightarrow |\alpha_2|^2 = 0$

$\Rightarrow |\alpha_2|^2 + |\beta_2|^2 = 0$ $\Rightarrow$ can't be possible as $|\alpha_2|^2 + |\beta_2|^2$ must be equal to 1

$\Rightarrow$ contradiction $\quad [|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle]$

$\Rightarrow |\psi\rangle$ can't be written as

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$