Notes for Macroeconomics

Julique

gene_introne@163.com

Welcome your discussions and criticisms.

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Contents

Lecture 1

Introduction

1. Macroeconomics can briefly divide into two broad areas.
   (a) Growth: It studies the trend in the economy.
   (b) Business Cycles: It studies the cyclical fluctuations.
2. The growth theory.
   (a) Steady modern growth in GDP per capita.
      i. Exogenous growth theory by Solow (1956);
      ii. Endogenous Growth theory by Romer (1990, 1992), Lucas (1988), Aghion and Howitt (1992);
   (b) The transition to modern growth.
      i. Hansen and Prescott (2002), Galor and Weil (2000) and Galor (2005);
      ii. Before the modern growth, the economy falls into the Malthusian trap (population grows much faster than GDP or food supply);
      iii. Mankind has broken out the trap since industrial revolution.
3. Kaldor's basic stylized facts.
   (a) Labor productivity has grown at a sustained rate.
   (b) Capital per worker has grown at a sustained rate.
   (c) The real interest rate, or return on capital, has been stable.
   (d) The ratio of capital to output has also been stable.
   (e) Capital and labor have captured stable shares of national incomes.
4. Filters
   (a) Nelson and Plosser (1982) concluded that real GDP is best modeled as difference stationary.

(b) Hodrick–Prescott filter (HP filter) is a tool to remove the cyclical component of a time series from raw data.

$$\min_{\tau}\left(\sum_{t=1}^{T}(y_t-\tau_t)^2+\lambda\sum_{t=2}^{T-1}[(\tau_{t+1}-\tau_t)-(\tau_t-\tau_{t-1})]^2\right)$$

- The first term penalizes the cyclical component.
- This second term penalizes variations in the growth rate of the trend component.

HP filter only remove the low frequency fluctuations.

(c) Bandpass filter by Baxter and King's (1994) removes both high and low frequency fluctuations

Lecture 2

Capital Misallocation

2.1 Single-Factor Productivity and TFP

1. Some concepts.

   (a) Single-factor productivity measures reflect units of output produced per unit of a particular input.

   For example, let y denote the firm's output y, and l be the labor input, k be the capital input, then labor productivity is $\frac{y}{l}$, capital productivity is $\frac{y}{k}$.

   (b) TFP (Total Factor Productivity): a productivity concept that invariant to the intensify of use of observable factor inputs.

2. Example 1 for TFP.

   Consider the production function

$$y = Af(k, l, m)$$

   where y is output, $f(\cdot)$ is a function of observable inputs capital k, labor l and intermediate materials m, and A is a factor-neutral shifter, we call it factor-neutral since A does not appear in $f(\cdot)$, for example, $y = f(k, Al, m)$.

   In this setting, TFP is A.

3. Example 2 for TFP.

   Consider an economy with N firms, the output for firm $i, i = 1, \cdots, N$, is

$$y_i = A_i f(k_i, l_i, m_i)$$

   and assume the whole economy produces output by an aggregate production function

$$Y = Af(K, L, M)$$

   where $Y = \sum_{i=1}^{N} y_i, K = \sum_{i=1}^{N} k_i, L = \sum_{i=1}^{N} l_i, M = \sum_{i=1}^{N} m_i$. Then A denotes the TFP of the economy.

Then,

$$Y = \sum_{i=1}^{N} A_i f(k_i, l_i, m_i)$$

We write A as

$$A = \frac{Y}{f(K, L, M)} = \sum_{i=1}^{N} A_i \frac{f(k_i, l_i, m_i)}{f(K, L, M)}$$

define $\phi_i = \frac{f(k_i, l_i, m_i)}{f(K, L, M)}$, then A becomes a weighted average of the TFP of firms,

$$A = \sum_{i=1}^{N} A_i \phi_i$$

Note: We "assume" the aggregate production function here because it is not necessary for the aggregate production function to take the same form as the production functions of firms. We will see in the next lecture.

4. Example for friction.

   Assume the economy has only 1 firm, and the production function for firm is $f(k) = k$. Then for the whole economy, given the total supply of capital K, it can potentially produce $f(K) = K$, but not actually produce, since K is only the supply of capital. If the demand of capital from the firm is $K - \Delta < K$, then

   $$A = \frac{K - \Delta}{K} = 1 - \frac{\Delta}{K} < 1$$

   But potentially, $A = 1$.

2.2 A Model for Competitive Markets

1. Set-ups.

   (a) An economy with a continuum of identical households, a continuum of firms, which is either type 1 or type 2, and a government.

   (b) Households.

      i. Utility function: $U(Y) = Y$.

      ii. Each household is endowed with K units of capital, 1 units of equity share from both firm 1 and firm 2.

      iii. Households are identical, we only consider a representative household.

      iv. There are so many households that every household is a price taker.

   (c) Markets.

      i. Consumption good market: price to buy or sell goods is p.

      ii. Capital market: price rent or lease capital goods is R.

iii. By Walras's law, we can normalize the price of consumption good to 1 so that we only solve for the relative price of capital to the consumption good in equilibrium, i.e., let $p' = 1$, then in relative price form $R' = \frac{R}{p}$, but for simplicity, we still let R denote the relative price.

iv. (Walras's Law) If each individual satisfies his budget constraint, so that the value of the goods sold equals the value of the goods bought, then the total value of all the sales by all individuals equals the total value of all the purchases by all individuals.

Walras's law implies that if all but one market are cleared, then the remaining market must also be cleared.

(d) Government.

i. Revenue: Charge a tax of $\tau_i R_i, i = 1, 2$, for firm i rents from capital market.

ii. Expenditure: Transfer payments T to the households.

iii. The government runs a balanced budget, i.e.,

$$T = \tau_1 R_1 + \tau_2 R_2$$

Therefore, the tax revenue will be transferred to the household as a lump-sum subsidy $T = \tau_1 R_1 + \tau_2 R_2$.

(e) Firms.

i. Firms are representative and price takers.

ii. The output for firm i is

$$y_i = A_i k_i^\alpha, A_i > 0, \alpha \in (0, 1)$$

iii. The cost for firm i is $Rk_i + \tau_i R k_i = (1 + \tau_i)\, k_i$.

Define $R_i := (1 + \tau_i)\, R$, then the cost becomes $R_i k_i$.

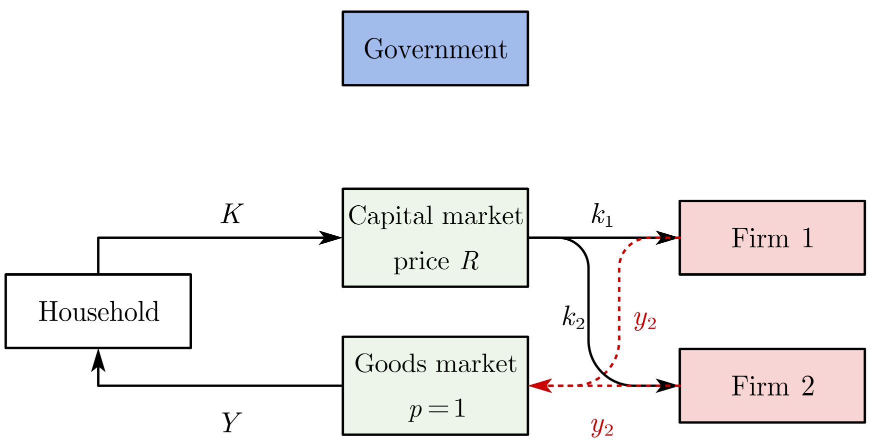

Fig. 2.1. Flow of Goods.

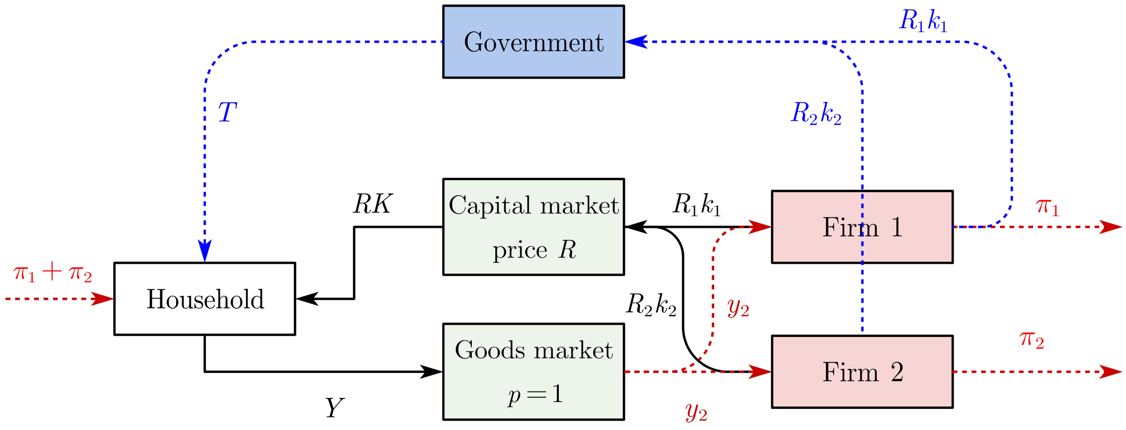

Fig. 2.2. Flow of Income and Expenditure.

2. Solve the representative firms' problem.

$$\pi_i = \max_{k_i} A_i k_i^\alpha - R_i k_i$$

F.O.C. is

$$\alpha A_i k_i^{\alpha-1} - R_i = 0 \Longrightarrow k_i = \left(\frac{R_i}{\alpha A_i}\right)^{\frac{1}{\alpha-1}} = \left(\frac{\alpha A_i}{R_i}\right)^{\frac{1}{1-\alpha}}$$

3. Solve the representative household's problem.

$$\max_{Y} \ U(Y)$$
$$\text{s.t. } Y \leqslant RK + \pi_1 + \pi_2 + T$$

then

$$Y = RK + \pi_1 + \pi_2 + T$$

4. Define the equilibrium.

A competitive equilibrium is$Y, k_1, k_2, \pi_1, \pi_2, R$ such that

(a) Given R_i, k_i solves the firm's problem, i.e., $\alpha A_i k_i^{\alpha-1} - R_i = 0$.

(b) Given R, solves the household's problem, i.e., the household demand is $RK + \pi_1 + \pi_2 + T$.

(c) The capital market clears, $K = k_1 + k_2$.

(d) The consumption good market clears, $Y = RK + \pi_1 + \pi_2 + T$.

5. TFP of the economy,

Let A denote the TFP for the whole economy, suppose the aggregate output takes the form $Y = AK^\alpha$, then

$$A = \frac{Y}{K^\alpha} = \frac{y_1 + y_2}{K^\alpha} = \frac{A_1 k_1^\alpha + A_2 k_2^\alpha}{K^\alpha} = A_1 \left(\frac{k_1}{K}\right)^\alpha + A_2 \left(\frac{k_2}{K}\right)^\alpha$$

Let $\omega_i = \frac{k_i}{K}$, then

$$A = A_1\omega_1^\alpha + A_2\omega_2^\alpha$$

6. Equilibrious TFP of the economy.

   Given the equilibrium, then

$$k_i = \left(\frac{\alpha A_i}{R_i}\right)^{\frac{1}{1-\alpha}} \Longrightarrow \frac{k_1}{k_2} = \left(\frac{A_1/R_1}{A_2/R_2}\right)^{\frac{1}{1-\alpha}}$$

   recall that $R_i = R(1+\tau_i)$, and by the capital market clear, i.e.,

$$K = k_1 + k_2$$

   we have

$$k_i = \frac{\left(\frac{A_i}{1+\tau_i}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{A_1}{1+\tau_1}\right)^{\frac{1}{1-\alpha}} + \left(\frac{A_2}{1+\tau_2}\right)^{\frac{1}{1-\alpha}}} K$$

   then

$$\omega_i^E = \frac{\left(\frac{A_i}{1+\tau_i}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{A_1}{1+\tau_1}\right)^{\frac{1}{1-\alpha}} + \left(\frac{A_2}{1+\tau_2}\right)^{\frac{1}{1-\alpha}}}$$

7. Optimal TFP of the economy.

   Summarize the welfare of this economy by the utility form consumption of the representative household.

$$\max_{y_1,y_2,k_1,k_2} U(Y) \text{ s.t. } Y = y_1 + y_2 y_i = A_i k_i^\alpha, i = 1,2 k_1 + k_2 \leqslant K$$

   Rewrite the problem as

$$\max_{k_1,k_2} A_1 k_1^\alpha + A_2 k_2^\alpha \text{ s.t. } k_1 + k_2 \leqslant K$$

   The equality clearly should be binding, then

$$\max_{k_1} A_1 k_1^\alpha + A_2 (K - k_1)^\alpha$$

   and thus, F.O.C. is

$$A_1 \alpha k_1^{\alpha-1} - A_2 \alpha k_2^{\alpha-1} = 0$$

   then

$$\frac{k_1}{k_2} = \left(\frac{A_2}{A_1}\right)^{\frac{1}{\alpha-1}} = \left(\frac{A_1}{A_2}\right)^{\frac{1}{1-\alpha}}$$

   therefore, consider the welfare of the whole economy, optimal ω should be

$$\omega_i^O = \frac{A_i^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}$$

8. Compare ω_i^O and ω_i^E.

$$\omega_i^O = \frac{A_i^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}} \omega_i^E = \frac{\left(\frac{A_i}{1+\tau_i}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{A_1}{1+\tau_1}\right)^{\frac{1}{1-\alpha}} + \left(\frac{A_2}{1+\tau_2}\right)^{\frac{1}{1-\alpha}}}$$

It is clear that the equilibrium allocation is not socially optimal iff $\tau_1 \neq \tau_2$. Since given A_1, A_2, then ω_i^O is known, take ω_1^E as an example,

$$\omega_1^E = \frac{\left(\frac{A_1}{1+\tau_1}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{A_1}{1+\tau_1}\right)^{\frac{1}{1-\alpha}} + \left(\frac{A_2}{1+\tau_2}\right)^{\frac{1}{1-\alpha}}} = \frac{A_1^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}} + \left(\frac{1+\tau_1}{1+\tau_2}A_2\right)^{\frac{1}{1-\alpha}}}$$

Then $\omega_1^E \neq \omega_1^O$ iff $\tau_1 \neq \tau_2$.

In other words, when there is heterogeneous cost from accessing the capital market, TFP is lower under market allocation than under optimal allocation.

9. (Theorem, First Welfare Theorem) A market in equilibrium under perfect competition will be Pareto optimal in the sense that no further exchange would make one person better off without making another worse off. The requirements for perfect competition are these:

   (a) There are no externalities and no transaction costs, and each actor has perfect information.

   (b) Firms take prices as given.

2.3 A Model for Local Monopoly (Monopolistic Competitive)

1. Modifications and recalls.

   (a) The representative household has the utility $U(Y) = Y$.

   (b) The consumption good is produced by a representative firm. The representative producer uses 2 intermediate good y_1 and y_2 to produce, the output is given by the Constant Elasticity of Substitution (CES) production function

$$Y(y_1, y_2) = \left(y_1^{\frac{\sigma-1}{\sigma}} + y_2^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

   where $\sigma > 1$. And this is where the monopoly comes from, because y_1 and y_2 are not perfectly substitutes, then the intermediate goods producers have some "market power", which allows them to earn a positive profit.

   (c) The two intermediate goods are produced by two representative firms, firm 1 and firm 2, both with output given by

$$Y_i = A_i k_i, i = 1, 2$$

(d) Markets.

i. A competitive market for consumption goods (final goods) market, and price is normalized to 1.

ii. A competitive market for capital, and the rental rate is R.

iii. Two intermediate good market, and the price is $p_i, i = 1, 2$.

- Demand side: p_i is given.
- Supply side: producer faces a demand curve to decide p_i.

(e) Government.

i. Revenue: Charge taxes on capital renting, then the actual rental rate for firm i is $R_i = (1 + \tau_i) R, i = 1, 2$.

ii. Expenditure: A lumpsum transfer payments T to the households.

iii. Runs balanced budget: $T = \tau_1 R k_1 + \tau_2 R k_2$.

2. Solve the representative consumption goods producer's problem.

$$\pi_0 = \max_{y_1, y_2} \left(y_1^{\frac{\sigma-1}{\sigma}} + y_2^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} - p_1 y_1 - p_2 y_2$$

F.O.C.s are

$$\frac{\sigma}{1-\sigma} \left(y_1^{\frac{\sigma-1}{\sigma}} + y_2^{\frac{\sigma-1}{\sigma}} \right)^{\frac{1}{\sigma-1}} \cdot \frac{\sigma - 1}{\sigma} y_i^{-\frac{1}{\sigma}} - p_i = 0, i = 1, 2$$

or

$$p_i = \left(y_1^{\frac{\sigma-1}{\sigma}} + y_2^{\frac{\sigma-1}{\sigma}} \right)^{\frac{1}{\sigma-1}} \cdot y_i^{-\frac{1}{\sigma}}$$

which is the demand curve of y_i, then

$$\pi_0 = \left(y_1^{\frac{\sigma-1}{\sigma}} + y_2^{\frac{\sigma-1}{\sigma}} \right)^{\frac{1}{\sigma-1}} \left[\left(y_1^{\frac{\sigma-1}{\sigma}} + y_2^{\frac{\sigma-1}{\sigma}} \right) - y_1^{1-\frac{1}{\sigma}} - y_2^{1-\frac{1}{\sigma}} \right] = 0$$

Moreover, we write the demand curve as

$$p_i = Y^{\frac{1}{\sigma}} \cdot y_i^{-\frac{1}{\sigma}}$$

because there are many intermediate firms, an intermediate good producer won't affect the aggregate output, then approximately,

$$p_i \propto y_i^{-\frac{1}{\sigma}}$$

3. Discussion on the CES production function.

For $Y(y_1, y_2)$, the elasticity of substitution between good 1 and good 2 is defined as

$$\varepsilon_{12} := -\frac{\mathrm{d} \log \left(\frac{y_1}{y_2} \right)}{\mathrm{d} \log \left(\frac{p_1}{p_2} \right)} = \frac{\mathrm{d} \log \left(\frac{y_1}{y_2} \right)}{\mathrm{d} \log \left(\frac{p_2}{p_1} \right)} = \frac{\mathrm{d}\left(y_1/y_2 \right) p_2/p_1}{\mathrm{d}\left(p_2/p_1 \right) y_1/y_2}$$

and it is a constant for CES production function, let's verify it. Since

$$p_i = \left(y_1^{\frac{\sigma-1}{\sigma}} + y_2^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \cdot y_i^{-\frac{1}{\sigma}}$$

we have

$$\frac{p_1}{p_2} = \left(\frac{y_1}{y_2}\right)^{-\frac{1}{\sigma}} \Longrightarrow \frac{y_1}{y_2} = \left(\frac{p_1}{p_2}\right)^{-\sigma} \Longrightarrow \frac{y_1}{y_2} = \left(\frac{p_2}{p_1}\right)^{\sigma}$$

then

$$\varepsilon_{12} = \sigma \left(\frac{p_2}{p_1}\right)^{\sigma-1} \frac{p_2/p_1}{\left(\frac{p_2}{p_1}\right)^{\sigma}} = \sigma$$

Likewise, for $F(K, L)$, we can define the elasticity of substitution between capital and labor as

$$-\frac{\mathrm{d}\log\left(\frac{K}{L}\right)}{\mathrm{d}\log\left(\frac{r}{w}\right)}$$

Moreover, consider $\left(\omega_1 y_1^{\frac{\sigma-1}{\sigma}} + \omega_2 y_2^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$.

(a) $\sigma \to \infty$, perfectly substitute, $\omega_1 y_1 + \omega_2 y_2$.

(b) $\sigma \to 1$, Cobb-Douglas, $y_1^{\omega_1} y_2^{\omega_2}$.

(c) $\sigma \to 0$, perfectly complement, $\min\{\omega_1 y_1, \omega_2 y_2\}$.

4. Solve the representative intermediate good producers' problem

$$\pi_i = \max_{k_i} p_i y_i - R_i k_i$$

and

$$p_i = Y^{\frac{1}{\sigma}} \cdot y_i^{-\frac{1}{\sigma}} y_i = A_i k_i$$

then the F.O.C. is

$$\frac{\mathrm{d}p_i}{\mathrm{d}k_i} A_i k_i + p_i A_i - R_i = 0$$

and

$$\begin{aligned}
\frac{\mathrm{d}p_i}{\mathrm{d}k_i} &= -\frac{1}{\sigma} Y^{\frac{1}{\sigma}} \cdot (A_i k_i)^{-\frac{1+\sigma}{\sigma}} A_i \\
&= -\frac{1}{\sigma} Y^{\frac{1}{\sigma}} k_i^{-\frac{1}{\sigma}} A_i^{-\frac{1}{\sigma}} \cdot k_i^{-1} \\
&= -\frac{1}{\sigma} \frac{p_i}{k_i}
\end{aligned}$$

Note that, for simplicity, since k_i has little influence on Y, we think Y is not a function of k_i. And thus, F.O.C. becomes

$$-\frac{1}{\sigma} p_i A_i + p_i A_i - R_i = 0 \Longrightarrow \frac{\sigma-1}{\sigma} Y^{\frac{1}{\sigma}} \cdot k_i^{-\frac{1}{\sigma}} A_i^{\frac{\sigma-1}{\sigma}} = R_i$$

or

$$k_i = \left[\frac{\sigma R_i}{\sigma-1} Y^{-\frac{1}{\sigma}} A_i^{-\frac{\sigma-1}{\sigma}}\right]^{-\sigma} = Y A_i^{\sigma-1} \left[\frac{\sigma-1}{\sigma R_i}\right]^{\sigma}$$

Therefore,

$$\pi_i = p_i A_i k_i - R_i k_i = p_i A_i k_i - \frac{\sigma-1}{\sigma} p_i A_i k_i = \frac{1}{\sigma} Y^{\frac{1}{\sigma}} \left(A_i k_i\right)^{1-\frac{1}{\sigma}} > 0$$

It shows that the intermediate good producer, which is competitive monopolistic, earns a positive profit.

5. Solve the representative household's problem.

   It is simple,

$$\max_{Y} \; U(Y) = Y$$
$$\text{s.t. } Y \leqslant \pi_0 + \pi_1 + \pi_2 + T + RK$$

   then

$$Y = \pi_0 + \pi_1 + \pi_2 + T + RK$$

6. Define the equilibrium.

   A competitive equilibrium is $Y, k_1, k_2, \pi_0, \pi_1, \pi_2, R$ such that

   (a) Given R_i, k_i, solves intermediate firms' problem, i.e.,

$$k_i = Y A_i^{\sigma-1} \left[\frac{\sigma-1}{\sigma R_i}\right]^{\sigma}$$

   (b) Given p_i, π_0, y_1, y_2, solves final good producer's problem, i.e.,

$$p_i = \left(y_1^{\frac{\sigma-1}{\sigma}} + y_2^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \cdot y_i^{-\frac{1}{\sigma}}$$

   (c) Given R, solves the household's problem, i.e., the household demand is $\pi_0 + \pi_1 + \pi_2 + T + RK$.

   (d) The capital market clears, $K = k_1 + k_2$.

   (e) The intermediate good markets clear: $y_i = A_i k_i, i = 1, 2$, with $p_i = Y^{\frac{1}{\sigma}} \cdot y_i^{-\frac{1}{\sigma}}$.

   (f) The consumption good market clears, $Y = RK + \pi_1 + \pi_2 + T$.

7. Equilibrious TFP of the economy.

   Likewise, let

$$k_i = \omega_i K_i$$

   then

$$\begin{aligned} A = \frac{Y}{K} &= \frac{1}{K}\left(y_1^{\frac{\sigma-1}{\sigma}} + y_2^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \\ &= \left[\left(A_1\omega_1\right)^{\frac{\sigma-1}{\sigma}} + \left(A_2\omega_2\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \end{aligned}$$

and then

$$k_i = YA_i^{\sigma-1}\left[\frac{\sigma-1}{\sigma R_i}\right]^{\sigma} \Longrightarrow \frac{k_1}{k_2} = \frac{A_1^{\sigma-1}/(1+\tau_1)^{\sigma}}{A_2^{\sigma-1}/(1+\tau_2)^{\sigma}}$$

and capital market clears

$$k_1 + k_2 = K$$

we have

$$\omega_i^E = \frac{A_i^{\sigma-1}(1+\tau_i)^{-\sigma}}{A_1^{\sigma-1}(1+\tau_1)^{-\sigma} + A_2^{\sigma-1}(1+\tau_2)^{-\sigma}}$$

Moreover, since

$$k_i = YA_i^{\sigma-1}\left[\frac{\sigma-1}{\sigma R_i}\right]^{\sigma}$$

we can solve for R as

$$\begin{aligned}
R =& \frac{\sigma-1}{\sigma}\frac{1}{1+\tau_1}\left[\frac{YA_1^{\sigma-1}}{k_1}\right]^{\frac{1}{\sigma}} \\
=& \frac{\sigma-1}{\sigma}\frac{1}{1+\tau_1}\left(A_1^{\frac{\sigma-1}{\sigma}} + \left(\frac{k_2}{k_1}A_2\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} A_1^{\frac{\sigma-1}{\sigma}} \\
=& \frac{\sigma-1}{\sigma}\frac{1}{1+\tau_1}\left[A_1^{\frac{\sigma-1}{\sigma}} + \left(\frac{k_2}{k_1}\frac{A_2}{1+\tau_1}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{1}{\sigma-1}} A_1^{\frac{\sigma-1}{\sigma}} \\
=& \frac{\sigma-1}{\sigma}\frac{1}{1+\tau_1}\left[A_1^{\frac{\sigma-1}{\sigma}} + \left(\frac{A_2^{\sigma-1}(1+\tau_2)^{-\sigma}}{A_1^{\sigma-1}(1+\tau_1)^{-\sigma}}\frac{A_2}{1+\tau_1}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{1}{\sigma-1}} A_1^{\frac{\sigma-1}{\sigma}} \\
=& \frac{\sigma-1}{\sigma}\left[A_1^{\frac{\sigma-1}{\sigma}} + \left(\frac{A_2^{\sigma}(1+\tau_2)^{-\sigma}}{A_1^{\sigma-1}(1+\tau_1)^{1-\sigma}}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{1}{\sigma-1}} A_1^{\frac{\sigma-1}{\sigma}}(1+\tau_1)^{-1} \\
=& \frac{\sigma-1}{\sigma}\left[(1+\tau_1)^{\frac{\sigma-1}{\sigma}}\left(A_1(1+\tau_1)^{-1}\right)^{\frac{\sigma-1}{\sigma}} + \left(A_1(1+\tau_1)^{-1}\right)^{(1-\sigma)\cdot\frac{\sigma-1}{\sigma}}\left(A_2(1+\tau_2)^{-1}\right)^{\sigma-1}\right]^{\frac{1}{\sigma-1}} \\
&\cdot A_1^{\frac{\sigma-1}{\sigma}}(1+\tau_1)^{-1} \\
=& \frac{\sigma-1}{\sigma}\left[\left(A_1(1+\tau_1)^{-1}\right)^{(\sigma-1)\cdot\frac{\sigma-1}{\sigma}+\frac{\sigma-1}{\sigma}} + \left(A_2(1+\tau_2)^{-1}\right)^{\sigma-1}\right]^{\frac{1}{\sigma-1}} \\
&\cdot A_1^{\frac{\sigma-1}{\sigma}}(1+\tau_1)^{\frac{1-\sigma}{\sigma}}\left(A_1(1+\tau_1)^{-1}\right)^{\frac{1-\sigma}{\sigma}} \\
=& \frac{\sigma-1}{\sigma}\left[\left(A_1(1+\tau_1)^{-1}\right)^{\sigma-1} + \left(A_2(1+\tau_2)^{-1}\right)^{\sigma-1}\right]^{\frac{1}{\sigma-1}}
\end{aligned}$$

8. Optimal TFP of the economy.

We can instead consider

$$\max_{\omega_1\in[0,1]}\left[(A_1\omega_1)^{\frac{\sigma-1}{\sigma}} + (A_2(1-\omega_1))^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

F.O.C. is

$$\frac{\sigma}{\sigma-1}\left[(A_1\omega_1)^{\frac{\sigma-1}{\sigma}} + (A_2(1-\omega_1))^{\frac{\sigma-1}{\sigma}}\right]^{\frac{1}{\sigma-1}}\left[\frac{\sigma-1}{\sigma}A_1(A_1\omega_1)^{-\frac{1}{\sigma}} - \frac{\sigma-1}{\sigma}A_2(A_2(1-\omega_1))^{-\frac{1}{\sigma}}\right] = 0$$

then

$$A_1^{\frac{\sigma-1}{\sigma}}\omega_1^{-\frac{1}{\sigma}} - A_2^{\frac{\sigma-1}{\sigma}}\omega_2^{-\frac{1}{\sigma}} = 0$$

and thus

$$\left(\frac{\omega_1}{\omega_2}\right)^{-\frac{1}{\sigma}} = \left(\frac{A_2}{A_1}\right)^{\frac{\sigma-1}{\sigma}} \Longrightarrow \frac{\omega_1}{\omega_2} = \left(\frac{A_1}{A_2}\right)^{\sigma-1}$$

and by

$$\omega_1 + \omega_2 = 1$$

then,

$$\omega_i^O = \frac{A_i^{\sigma-1}}{A_1^{\sigma-1} + A_2^{\sigma-1}}$$

9. Compare ω_i^O and ω_i^E.

$$\omega_i^O = \frac{A_i^{\sigma-1}}{A_1^{\sigma-1} + A_2^{\sigma-1}}$$
$$\omega_i^E = \frac{A_i^{\sigma-1}(1+\tau_i)^{-\sigma}}{A_1^{\sigma-1}(1+\tau_1)^{-\sigma} + A_2^{\sigma-1}(1+\tau_2)^{-\sigma}}$$

Then, $\omega_i^O \neq \omega_i^E$ iff $\tau_1 \neq \tau_2$. Moreover, given the equilibrium capital allocation, we can solve the aggregate TFP as

$$\begin{aligned} A &= \left[\left(A_1\omega_1^E\right)^{\frac{\sigma-1}{\sigma}} + \left(A_2\omega_2^E\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \\ &= \frac{\left\{\left[A_1(1+\tau_1)^{-1}\right]^{\sigma-1} + \left[A_2(1+\tau_2)^{-1}\right]^{\sigma-1}\right\}^{\frac{\sigma}{\sigma-1}}}{A_1^{\sigma-1}(1+\tau_1)^{-\sigma} + A_2^{\sigma-1}(1+\tau_2)^{-\sigma}} \end{aligned}$$

10. Two empirical measure methods of TFP.

    (a) Quantity total factor productivity (TFPQ)

    $$\text{TFPQ}_i = \frac{Y_i}{K_i} = A_i$$

    TFPQ is not directly observable in the data because we do not directly observe the quantity of output.

    (b) Revenue total factor productivity (TFPR)

    $$\text{TFPR}_i = \frac{p_i Y_i}{K_i} = p_i A_i$$

    (c) In the first competitive model, $p = 1$, then TFPR=TFPQ.

    (d) In the second monopolistic model, recall the F.O.C. for intermediate firms

    $$\frac{\sigma-1}{\sigma} p_i A_i = R_i$$

    then

    $$\text{TFPR}_i = \frac{1}{k_i} p_i y_i = A_i p_i = \frac{\sigma}{\sigma-1}(1+\tau_i) R$$

    TFPR tells us firm-level wedges τ_i.

2.4 Tax Rate with Distribution

1. Set-ups.
   (a) J firms, J is large enough to apply law of large number.
   (b) N is the total supply of the labor market.
   (c) Only one inputs.
   $$y_j = a_j n_j^{1-\theta}, 0 < \theta < 1$$
   (d) Profit maximization given by
   $$\max_{n_j} \; a_j n_j^{1-\theta} - W\tau_j n_j$$
   where W is real wage, a_j is productivity, and τ_j is distortion.
   (e) You may regard τ_j as a measure of tax.
      i. $\tau_j > 1$, the firm suffers an additional tax
      ii. $\tau_j < 1$, the firm receives subsidy.
2. Solve for firms. F.O.C.,
   $$a_j (1-\theta) n_j^{-\theta} - W\tau_j = 0$$
   then
   $$n_j^{-\theta} = \frac{W\tau_j}{a_j (1-\theta)} \Longrightarrow n_j = \left[\frac{(1-\theta) a_j}{W\tau_j}\right]^{\frac{1}{\theta}}$$
   and the output is
   $$y_j = a_j n_j^{1-\theta} = a_j \left[\frac{(1-\theta) a_j}{W\tau_j}\right]^{\frac{1-\theta}{\theta}}$$
3. Solve for labor market and aggregate output.
   $$\sum_{j=1}^{J} n_j = N \Longrightarrow \sum_{j=1}^{J} \left[\frac{(1-\theta) a_j}{W\tau_j}\right]^{\frac{1}{\theta}} = N \Longrightarrow \left(\frac{1-\theta}{W}\right)^{\frac{1}{\theta}} \sum_{j=1}^{J} \left[\frac{a_j}{\tau_j}\right]^{\frac{1}{\theta}} = N$$
   then
   $$W = (1-\theta) \left[\frac{1}{N} \sum_{j=1}^{J} \left(\frac{a_j}{\tau_j}\right)^{\frac{1}{\theta}}\right]^{\theta}$$
   and thus,
   $$n_j = \left[\frac{\frac{a_j}{\tau_j}}{\left[\frac{1}{N}\sum_{j=1}^{J}\left(\frac{a_j}{\tau_j}\right)^{\frac{1}{\theta}}\right]^{\theta}}\right]^{\frac{1}{\theta}} = \frac{\left(\frac{a_j}{\tau_j}\right)^{\frac{1}{\theta}}}{\sum_{j=1}^{J}\left(\frac{a_j}{\tau_j}\right)^{\frac{1}{\theta}}} N$$
   therefore,
   $$y_j = a_j n_j^{1-\theta} = a_j \frac{\left(\frac{a_j}{\tau_j}\right)^{\frac{1-\theta}{\theta}}}{\left[\sum_{j=1}^{J}\left(\frac{a_j}{\tau_j}\right)^{\frac{1}{\theta}}\right]^{1-\theta}} N^{1-\theta}$$

the total aggregate output is

$$Y_t = \sum_{i=1}^{J} y_j = \frac{N^{1-\theta}}{\left[\sum_{j=1}^{J}\left(\frac{a_j}{\tau_j}\right)^{\frac{1}{\theta}}\right]^{1-\theta}} \sum_{i=1}^{J} a_j \left(\frac{a_j}{\tau_j}\right)^{\frac{1-\theta}{\theta}} = \frac{\sum_{i=1}^{J} a_j \left(\frac{a_j}{\tau_j}\right)^{\frac{1-\theta}{\theta}}}{\left[\sum_{j=1}^{J}\left(\frac{a_j}{\tau_j}\right)^{\frac{1}{\theta}}\right]^{1-\theta}} N^{1-\theta}$$

4. If there is no tax, i.e., $\tau_j = 1$.

$$Y_t = \frac{\sum_{i=1}^{J} a_j^{\frac{1}{\theta}}}{\left[\sum_{i=1}^{J} a_j^{\frac{1}{\theta}}\right]^{1-\theta}} N^{1-\theta} = \frac{J \cdot \frac{1}{J}\sum_{i=1}^{J} a_j^{\frac{1}{\theta}}}{J^{1-\theta}\left[\frac{1}{J}\sum_{i=1}^{J} a_j^{\frac{1}{\theta}}\right]^{1-\theta}} N^{1-\theta} \xrightarrow{\mathbb{P}} J^{\theta} N^{1-\theta} \frac{\mathbb{E} a_j^{\frac{1}{\theta}}}{\left[\mathbb{E} a_j^{\frac{1}{\theta}}\right]^{1-\theta}}$$

5. If $\log \tau_j$i.i.d.$\mathcal{N}(\mu, \sigma_\tau^2)$, and assume a_j and τ_j are independent.

$$\begin{aligned}
\tilde{Y}_t &= \frac{\sum_{i=1}^{J} a_j \left(\frac{a_j}{\tau_j}\right)^{\frac{1-\theta}{\theta}}}{\left[\sum_{j=1}^{J}\left(\frac{a_j}{\tau_j}\right)^{\frac{1}{\theta}}\right]^{1-\theta}} N^{1-\theta} = \frac{\sum_{i=1}^{J} a_j^{\frac{1}{\theta}} \tau_j^{-\frac{1-\theta}{\theta}}}{\left[\sum_{j=1}^{J} a_j^{\frac{1}{\theta}} \tau_j^{-\frac{1}{\theta}}\right]^{1-\theta}} N^{1-\theta} \\
&= J^{\theta} N^{1-\theta} \frac{\frac{1}{J}\sum_{i=1}^{J} a_j^{\frac{1}{\theta}} \tau_j^{-\frac{1-\theta}{\theta}}}{\left[\frac{1}{J}\sum_{j=1}^{J} a_j^{\frac{1}{\theta}} \tau_j^{-\frac{1}{\theta}}\right]^{1-\theta}} \xrightarrow{\mathbb{P}} J^{\theta} N^{1-\theta} \frac{\mathbb{E}\left[a_j^{\frac{1}{\theta}} \tau_j^{-\frac{1-\theta}{\theta}}\right]}{\left(\mathbb{E}\left[a_j^{\frac{1}{\theta}} \tau_j^{-\frac{1}{\theta}}\right]\right)^{1-\theta}} \\
&= J^{\theta} N^{1-\theta} \frac{\mathbb{E} a_j^{\frac{1}{\theta}} \cdot \mathbb{E} \tau_j^{-\frac{1-\theta}{\theta}}}{\left[\mathbb{E} a_j^{\frac{1}{\theta}}\right]^{1-\theta} \left(\mathbb{E} \tau_j^{-\frac{1}{\theta}}\right)^{1-\theta}} \\
&= Y_t \cdot \frac{\mathbb{E} \tau_j^{-\frac{1-\theta}{\theta}}}{\left(\mathbb{E} \tau_j^{-\frac{1}{\theta}}\right)^{1-\theta}}
\end{aligned}$$

By Jensen's inequality, $x^{1-\theta}, 0 < \theta < 1$ is a concave function, then it holds that

$$(\mathbb{E}X)^{1-\theta} \geqslant \mathbb{E}X^{1-\theta}$$

therefore,

$$\left(\mathbb{E}\tau_j^{-\frac{1}{\theta}}\right)^{1-\theta} \geqslant \mathbb{E}\tau_j^{-\frac{1-\theta}{\theta}}$$

and thus

$$\tilde{Y}_t \leqslant Y_t$$

Exactly, by $\log \tau_j \sim \mathcal{N}(\mu, \sigma_\tau^2)$,

$$-\frac{1-\theta}{\theta} \log \tau_j \sim \mathcal{N}\left(-\frac{1-\theta}{\theta}\mu, \left(\frac{1-\theta}{\theta}\right)^2 \sigma_\tau^2\right)$$

$$-\frac{1}{\theta} \log \tau_j \sim \mathcal{N}\left(-\frac{1}{\theta}\mu, \frac{\sigma_\tau^2}{\theta^2}\right)$$

then

$$
\begin{aligned}
\frac{\mathbb{E}\tau_j^{-\frac{1-\theta}{\theta}}}{\left(\mathbb{E}\tau_j^{-\frac{1}{\theta}}\right)^{1-\theta}} &= \exp\left\{-\frac{1-\theta}{\theta}\mu + \frac{1}{2}\left(\frac{1-\theta}{\theta}\right)^2 \sigma_\tau^2 - \left[-\frac{1}{\theta}\mu + \frac{1}{2}\frac{\sigma_\tau^2}{\theta^2}\right](1-\theta)\right\} \\
&= \exp\left\{\frac{1}{2}\frac{\theta^2 - 2\theta + 1 - 1 + \theta}{\theta^2}\sigma_\tau^2\right\} \\
&= \exp\left\{\frac{1}{2}\frac{\theta^2 - \theta}{\theta^2}\sigma_\tau^2\right\} \\
&= \exp\left\{\frac{1}{2}\left(1 - \frac{1}{\theta}\right)\sigma_\tau^2\right\} < 1
\end{aligned}
$$

and thus

$$
\tilde{Y}_t \leqslant Y_t
$$

Therefore, tax distortion makes TFP or aggregate output lower.

Lecture 3

Preference and Production

3.1 Preference

1. Log utility and St. Petersburg paradox.

   (a) An infinite gamble: You toss a coin for many times, if you get head on the 1st toss, then you will get \$1 and have the chance to toss it again. If you get head on the 2nd toss, then you will get \$2 and have the chance to toss it again. If you get head on the 3nd toss, then you will get \$4 and have the chance to toss it again. etc.

   (b) The expected payoff:
   $$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n}2^{n-1}=\sum_{n=1}^{\infty}\frac{1}{2}=\infty$$

   (c) The expected utility with log utility:
   $$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n}\log 2^{n-1}=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n}(n-1)\log 2=\left(\frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^{2}}-\frac{1}{1-\frac{1}{2}}+1\right)\log 2=\log 2$$

2. Log utility in macro growth theory.

   (a) Notations.

      i. w_t: wages.

      ii. C_t: consumptions.

      iii. N_t: labor supply or hours of working.

      iv. $L_t = T - N_t$: Leisure or hours of leisure.

   (b) What is balanced growth path?

      i. Kaldor facts: labor productivity grows at a sustained rate, but hours of worked are roughly constant.

      ii. Wage grows at a sustained rate, i.e., $\frac{\dot{w}_t}{w_t}$ is a constant.

      iii. Consumption grows at a sustained rate, i.e., $\frac{\dot{C}_t}{C_t}$ is a constant.

(c) If utility is separable in consumption and leisure.

$$\max_{C_t,N_t} U(C_t)+V(T-N_t)$$
$$\text{s.t. } C_t\leqslant w_tN_t$$

where $\frac{\dot{w}_t}{w_t}=g$.

Solution 1. Assume the constraint is binding, the problem becomes

$$\max_{C_t} U(C_t)+V\left(T-\frac{C_t}{w_t}\right)$$

then

$$U'(C_t)-\frac{C_t}{w_t}V'(T-N_t)=0$$

and thus

$$w_tU'(C_t)=V'(T-N_t)$$

Solution 2. Form the Lagrangian,

$$\mathcal{L}_t=U(C_t)+V(1-N_t)+\lambda_t(w_tN_t-C_t)$$

F.O.C.s are

$$\begin{cases} U'(C_t)-\lambda_t=0\\ -V'(T-N_t)+\lambda_tw_t=0\end{cases}$$

then

$$w_tU'(C_t)=V'(T-N_t)$$

Under balanced growth path, $N_t=N$ is a constant, then by $C_t=w_tN$ (if binding)

$$\frac{\dot{C}_t}{C_t}=\frac{\dot{w}N}{w_tN}=\frac{\dot{w}_t}{w_t}=g$$

Then the first order condition becomes

$$\frac{C_t}{N}U'(C_t)=V'(T-N)\Longrightarrow U'(C_t)=V'(L)\frac{N}{C_t}$$

i.e.,

$$U'(C_t)\propto\frac{1}{C_t}\Longrightarrow U'(C_t)\propto\log C_t$$

(d) If utility is non-separable in consumption and leisure.

$$\max_{C_t,N_t} U(C_t,T-N_t)$$
$$\text{s.t. } C_t\leqslant w_tN_t$$

Likewise, assume the constraint is binding, then

$$w_tU_C(C_t,T-N_t)=U_L(C_t,T-N_t)$$

Under balance growth path, and $C_t=w_tN$,

$$U_C(C_t,T-N)C_t=U_L(C_t,T-N)N$$

(e) (Example 1) Consider the utility form given by King-Plosser-Rebelo (1988)

$$U\left(C_t, L_t\right) = \frac{1}{1-\sigma} C_t^{1-\sigma} V\left(L_t\right)$$

Recall that the F.O.C. under balanced growth path is

$$U_C\left(C_t, T-N\right) C_t = U_L\left(C_t, T-N\right) N$$

then

$$C_t^{-\sigma} V\left(L_t\right) C_t = \frac{1}{1-\sigma} C_t^{1-\sigma} V'\left(L_t\right)$$

C_t cancels on both sides, we call that the utility is **consistent with growing** C_t **under balanced growth path**.

(f) Habit Formation:

$$U = \log\left(C_t - \theta \bar{C}_{t-1}\right) + V\left(T - N_t\right)$$

where $\bar{C}_{t-1}$ is the average consumption of the household, which is also called the habit factor. $\bar{C}_{t-1}$ can be, for example, my neighbor's consumption or my own consumption on $t-1$ period, if $\bar{C}_{t-1}$ is higher, then we will fill much more painful with lower C_t. And we may regard $\bar{C}_{t-1} = w_{t-1} N_{t-1}$, or $C_t \propto \bar{C}_{t-1}$.

This function is consistent with growing C_t.

(g) GHH preference:

$$U = \frac{1}{1-\sigma}\left(C_t - \frac{N_t^{1+\gamma}}{1+\gamma}\right)^{1-\sigma}$$

is not consistent with growing C_t.

3. Time Consistent preferences.

(a) $U\left(c_0, c_1, \cdots, c_T\right) = \sum_{t=0}^{T} \beta^t u\left(c_t\right)$, where $\beta \in (0,1)$ is the discount factor. This is called exponential discounting.

(b) (Example) Log utility is time consistent.

Only consider 3-period problem.

$$\max_{c_0, c_1, c_2} \log c_0 + \beta \log c_1 + \delta \log c_2$$

$$\text{s.t. } c_0 + c_1 + c_2 = y$$

F.O.C.s

$$\frac{1}{c_0} = \frac{\beta}{c_1} = \frac{\delta}{c_2}$$

then

$$c_0 = \frac{1}{1+\beta+\delta} y$$

$$c_1 = \frac{\beta}{1+\beta+\delta} y$$

$$c_2 = \frac{\delta}{1+\beta+\delta} y$$

consider the 2-period problem,

$$\max_{c_1,c_2} \log c_1 + \beta \log c_2$$
$$\text{s.t. } c_1 + c_2 = y - c_0 = \frac{\beta+\delta}{1+\beta+\delta} y$$

likewise,

$$\frac{1}{c_1} = \frac{\beta}{c_2} \Longrightarrow \begin{cases} c_1^* = \frac{1}{1+\beta}\frac{\beta+\delta}{1+\beta+\delta} y \\ c_2^* = \frac{\beta}{1+\beta}\frac{\beta+\delta}{1+\beta+\delta} y \end{cases}$$

let

$$c_1 = c_1^*, c_2 = c_2^*$$

we have

$$\delta = \beta^2$$

(c) For the problem

$$\max_{c_t,c_{t+1},\cdots,c_T} U(c_t, c_{t+1}, \cdots, c_T) = \sum_{\ell=0}^{T-t} d(\ell)\, u(c_{t+\ell})$$
$$\text{s.t. } \sum_{\ell=0}^{T-t} c_{t+\ell} = s_t$$

Strotz (1995) proves that $d(\ell) = \beta^\ell$ is the only way of discounting to make a consistent plan.

3.2 Production

1. Shape of production function.

   (a) Labor share.

      i. For a production function $y = f(k,n)$, one of the first order conditions is$f_n(k,n) = w$.

      ii. Labor share is defined as $\frac{w \cdot n}{y} = \frac{f_n(k,n)\cdot n}{f(k,n)}$.

      iii. For Cobb-Douglas function $f = Ak^\alpha n^{1-\alpha}$, the labor share is

$$\frac{w \cdot n}{f} = \frac{(1-\alpha) Ak^\alpha n^{-\alpha} \cdot n}{Ak^\alpha n^{1-\alpha}} = 1 - \alpha$$

   (b) Alternatives of production function.

$$y = \left(\alpha (A_k k)^\rho + (1-\alpha)(A_n n)^\rho\right)^{\frac{1}{\rho}}$$

      i. $\rho = -\infty, y = \min\{\alpha A_k k + (1-\alpha) A_n n\}$, Leontief.

      ii. $\rho = 0, y = (A_k k)^\alpha (A_n n)^{1-\alpha}$, Cobb-Douglas.

      iii. $\rho = 1$, $y = \alpha A_k k + (1-\alpha) A_n n$, perfect substitute.

iv. Labor share:

$$\begin{aligned}\frac{\frac{\mathrm{d}y}{\mathrm{d}n}n}{y} &= \frac{\frac{1}{\rho}\left(\alpha\left(A_k k\right)^\rho+\left(1-\alpha\right)\left(A_n n\right)^\rho\right)^{\frac{1}{\rho}-1}\cdot\left(1-\alpha\right)A_n^\rho \rho n^{\rho-1}}{\left(\alpha\left(A_k k\right)^\rho+\left(1-\alpha\right)\left(A_n n\right)^\rho\right)^{\frac{1}{\rho}}}\\ &= \frac{\left(1-\alpha\right)\left(A_n n\right)^\rho}{\alpha\left(A_k k\right)^\rho+\left(1-\alpha\right)\left(A_n n\right)^\rho}\end{aligned}$$

2. Micro production and macro production can be inconsistent.

   Consider there is a continuum of J firms, and J is large enough so that we can apply the law of large number. Each firm is endowed with capital K_t, the total labor market supply is N_t.

   The firm solves the problem

$$\max_{N_t^j}\ Y_t^j - w_t N_t^j$$

   where $Y_t^j = \min\left\{K_t, \frac{A_t N_t^j}{z_t^j}\right\}$, z_t^j is a distribution of productivity efficiency among different firms. Then the firm should choose

$$K_t \geqslant \frac{A_t N_t^j}{z_t^j} \geqslant 0$$

   The problem is equivalent with

$$\begin{aligned}&\max_{N_t^j}\ \frac{A_t N_t^j}{z_t^j} - w_t N_t^j\\ &\text{s.t. } 0 \leqslant N_t^j \leqslant \frac{z_t^j}{A_t}K_t\end{aligned}$$

   The firm will produce iff $\frac{A_t}{z_t^j} - w_t \geqslant 0$, define $z_t^* := \frac{A_t}{w_t}$, then

$$N_t = \begin{cases}\frac{z_t^j}{A_t}K_t, & z_t \leqslant z_t^*\\ 0, & z_t > z_t^*\end{cases}$$

   When it is equilibrium, the labor market clears, then

$$\int_0^1 N_t^j \mathrm{d}j = N_t$$

   By the law of large number,

$$N_t = \int_0^1 N_t^j \mathrm{d}j = \frac{1}{J}\sum_{j=1}^J N_t^j \xrightarrow{\mathbb{P}} \mathbb{E}\left[\frac{K_t z}{A_t}\right] = \frac{K_t}{A_t}\int_0^{z_t^*} z\phi\left(z\right)\mathrm{d}z$$

   where ϕ is the PDF of z.

   Let $\phi\left(z\right) = \frac{\theta}{z_{\max}^\theta}z^{\theta-1}, 0 \leqslant z \leqslant z_{\max}$, then

$$N_t = \frac{K_t}{A_t}\int_0^{z_t^*} z\phi\left(z\right)\mathrm{d}z = \frac{K_t}{A_t}\frac{\theta}{z_{\max}^\theta}\int_0^{z_t^*} z^\theta \mathrm{d}z = \frac{K_t}{A_t}\frac{\left(z_t^*\right)^{\theta+1}}{z_{\max}^\theta}\frac{\theta}{\theta+1}$$

and thus,

$$z_t^* \propto \left(\frac{N_t}{K_t}A_t\right)^{\frac{1}{\theta+1}}$$

For the aggregate output, we have

$$Y_t = \int_0^1 Y_t^j \mathrm{d}t = \int_0^1 \frac{A_t N_t^j}{z_t^j}\mathrm{d}t \xrightarrow{\mathbb{P}} \mathbb{E}\left[\frac{A_t N_t^j}{z_t^j}\right] = \mathbb{E}\left[K_t\right] = K_t \int_0^{z_t^*} \frac{\theta}{z_{\max}^\theta} z^{\theta-1}\mathrm{d}z = \frac{K_t}{z_{\max}^\theta} z_t^*$$

then

$$Y_t \propto \left(\frac{N_t}{K_t}A_t\right)^{\frac{1}{\theta+1}} K_t = K_t^{\frac{\theta}{1+\theta}} \left(A_t N_t\right)^{\frac{\theta}{1+\theta}}$$

It takes Cobb-Douglas function form, but firm's production function takes Leontief function form.

Lecture 4

Optimal Control

4.1 Hamiltonian Method

1. Discrete consumption-saving problem.

$$\max_{\{c_t,a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$
$$\text{s.t. } c_t + a_{t+1} = \left(1+r\right) a_t, \forall t \geqslant 0$$
$$c_t, a_{t+1} \geqslant 0, \forall t \geqslant 0$$

2. Continuous consumption-saving problem.

   Let $\mathrm{d}t$ be the duration of every period. Note that $\mathrm{d}t \to 0, \beta \to 1$ (In a very short period of time, we can ignore the discount), and we distribute utility $u\left(c_t\right)$ and c_t uniformly to every small period.

$$\beta \to e^{-\rho \mathrm{d}t}$$
$$r \to r\mathrm{d}t$$
$$u\left(c_t\right) \to u\left(c_t\right)\mathrm{d}t$$
$$c_t \to c_t\mathrm{d}t$$

   Also note that a_t is a state variable if we write the Bellman equation of the problem, so we needn't and cannot approximate it by

$$a_t \approx a_t\mathrm{d}t$$

   We may regard the object function as a summation of the area of infinite rectangles.

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t\right) = \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t u\left(c_t\right) \cdot 1 = \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t u\left(c_t\right) \cdot \left(t+1-t\right)$$

   Partition $[0,T]$ to n subintervals, and $t = \frac{T}{n}$, use the right point of each subinterval to approximate the function value $\beta^t u\left(c_t\right)$.

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t\right) = \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t u\left(c_t\right) \cdot 1 = \lim_{T\to\infty} \sum_{i=1}^{n} e^{-\rho \frac{T}{i}} u\left(c_{T/i}\right) \frac{T}{i}$$

Let $n \to \infty$, we get a Riemann integral (if integrable).

$$\lim_{T\to\infty}\lim_{n\to\infty}\sum_{i=1}^{n} e^{-\rho\frac{T}{i}} u\left(c_{T/i}\right)\frac{T}{i} = \lim_{T\to\infty}\int_0^T e^{-\rho t}u\left(c_t\right)\mathrm{d}t = \int_0^\infty e^{-\rho t}u\left(c_t\right)\mathrm{d}t$$

which is the objective function in continuous case. For the constraint, $c_t + a_{t+1} = (1+r)\,a_t$, we have

$$c_t\mathrm{d}t + a_{t+\mathrm{d}t} = (1+r\mathrm{d}t)\,a_t$$

then

$$a_{t+\mathrm{d}t} - a_t = (r - c_t)\,\mathrm{d}t \Longrightarrow \frac{a_{t+\mathrm{d}t} - a_t}{\mathrm{d}t} = ra_t - c_t$$

Let $\mathrm{d}t \to 0$, then

$$\dot{a}_t = ra_t - c_t$$

Therefore, the problem becomes

$$\begin{aligned} \max_{\{c_t\}_{t\geqslant 0}^{\infty},\{a_t\}_{t>0}^{\infty}} &\int_0^\infty e^{-\rho t}u\left(c_t\right)\mathrm{d}t \\ \text{s.t. } &\dot{a}_t = ra_t - c_t, \forall t\geqslant 0 \\ &c_t, a_t \geqslant 0, \forall t \geqslant 0 \end{aligned}$$

3. Present-value Hamiltonian.

   We can form the Lagrangian,

$$\begin{aligned} \mathcal{L}^{Dis} &= \sum_{t=0}^{\infty}\left[\beta^t u\left(c_t\right) + \lambda_t\left((1+r)\,a_t - c_t - a_{t+1}\right)\right] \\ \mathcal{L}^{Con} &= \int_0^\infty \left[e^{-\rho t}u\left(c_t\right) + \lambda_t\left(ra_t - \dot{a}_t - c_t\right)\right]\mathrm{d}t \end{aligned}$$

   Only consider continuous case.

$$\mathcal{L} = \int_0^\infty \left[e^{-\rho t}u\left(c_t\right) + \lambda_t\left(ra_t - c_t\right)\right]\mathrm{d}t + \int_0^\infty \lambda_t\dot{a}_t\mathrm{d}t$$

   We don't want to deal with $\frac{\partial}{\partial c_t}\dot{a}_t$ or $\frac{\partial}{\partial a_t}\dot{a}_t$, so we first get rid of $\dot{a}_t$ by integration by parts.

$$\int_0^\infty \lambda_t\dot{a}_t\mathrm{d}t = \int_0^\infty \lambda_t\mathrm{d}a_t = a_t\lambda_t\mid_0^\infty - \int_0^\infty a_t\dot{\lambda}_t\mathrm{d}t$$

   Assume $\lim_{t\to\infty}\lambda_t a_t = 0$, which is known as the transversality condition. The interpretation is that the discounted shadow value of the state variable must be zero at infinity. If $\lim_{t\to\infty}\lambda_t a_t > 0$, we can reduce future savings to increase present utility. Moreover, the transversality condition is also crucial for using backward reduction to solve the problem. In our consumption-saving problem, it makes sure that the discounted present value of consumption does not exceed the discounted present value of the income from investment.

   Therefore,

$$\mathcal{L} = \int_0^\infty \left[\underbrace{e^{-\rho t}u\left(c_t\right) + \lambda_t\left(ra_t - c_t\right)}_{\mathcal{H}_t} + \dot{\lambda}_t a_t\right]\mathrm{d}t + a_0\lambda_0$$

F.O.C.s:

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial c_t} &= \int_0^\infty \frac{\partial \mathcal{H}_t}{\partial c_t} \mathrm{d}t = 0 \\
\frac{\partial \mathcal{L}_t}{\partial a_t} &= \int_0^\infty \frac{\partial \mathcal{H}_t}{\partial a_t} + \dot{\lambda}_t \mathrm{d}t = 0
\end{aligned}$$

then

$$\begin{cases} \frac{\partial \mathcal{H}_t}{\partial c_t} = 0 \\ \frac{\partial \mathcal{H}_t}{\partial a_t} = -\dot{\lambda}_t \end{cases} \Longrightarrow \begin{cases} e^{-\rho t} u'(c_t) = \lambda_t \\ r\lambda_t = -\dot{\lambda}_t \end{cases}$$

and thus, to eliminate $\lambda(t)$,

$$\begin{aligned}
r e^{-\rho t} u'(c_t) &= -\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\rho t} u'(c_t)\right) \\
&= -\left[-\rho e^{-\rho t} u'(c_t) + u''(c_t)\dot{c}_t e^{-\rho t}\right] \\
&= \rho e^{-\rho t} u'(c_t) - u''(c_t)\dot{c}_t e^{-\rho t}
\end{aligned}$$

then

$$r u'(c_t) = \rho u'(c_t) - u''(c_t)\dot{c}_t$$

together with the constraint, initial condition, transversality condition,

$$\begin{cases} u''(c_t)\dot{c}_t = (r-\rho) u'(c_t) \\ \dot{a}_t + c_t = r_t a_t \\ a_0 \text{ is given} \\ \lim_{t\to\infty} \lambda_t a_t = 0 \end{cases}$$

we can solve the system.

Note that $\lim_{t\to\infty} \lambda_t a_t = 0$ will pin the system to the saddle path to the steady state, which will be discussed in dynamics of Ramsey model.

4. Current-value Hamiltonian.

$$\begin{aligned}
\mathcal{L}^{Dis} &= \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + q_t\left((1+r)a_t - c_t - a_{t+1}\right)\right] \\
\mathcal{L}^{Con} &= \int_0^\infty e^{-\rho t}\left[u(c_t) + q_t\left(r a_t - \dot{a}_t - c_t\right)\right]\mathrm{d}t
\end{aligned}$$

Likewise,

$$\mathcal{L} = \int_0^\infty e^{-\rho t}\left[u(c_t) + q_t\left(r a_t - c_t\right)\right]\mathrm{d}t - \int_0^\infty e^{-\rho t} q_t \dot{a}_t \mathrm{d}t$$

and

$$\begin{aligned}
\int_0^\infty e^{-\rho t} q_t \dot{a}_t \mathrm{d}t &= e^{-\rho t} q_t a_t \mid_0^\infty - \int_0^\infty a_t \mathrm{d}\left[e^{-\rho t} q_t\right] \\
&= e^{-\rho t} q_t a_t \mid_0^\infty - \int_0^\infty a_t \left(-\rho e^{-\rho t} q_t + e^{-\rho t}\dot{q}_t\right)\mathrm{d}t
\end{aligned}$$

Assume $\lim_{t\to\infty} e^{-\rho t} q_t a_t = 0$, then

$$\mathcal{L} = \int_0^\infty e^{-\rho t}\left[\underbrace{u\left(c_t\right) + q_t\left(ra_t - c_t\right)}_{\mathcal{H}_t^*} + a_t\left(-\rho q_t + \dot{q}_t\right)\right]\mathrm{d}t + q_0 a_0$$

F.O.C.s:

$$\begin{cases} \frac{\partial \mathcal{H}_t^*}{\partial c_t} = 0 \\ \frac{\partial \mathcal{H}_t^*}{\partial a_t} = \rho q_t - \dot{q}_t \end{cases} \implies \begin{cases} u'\left(c_t\right) = q_t \\ r q_t = \rho q_t - \dot{q}_t \end{cases}$$

Moreover,

$$\begin{cases} u'\left(c_t\right) = q_t \\ r q_t = \rho q_t - \dot{q}_t \end{cases} \implies r u'\left(c_t\right) = \rho u'\left(c_t\right) - u''\left(c_t\right)$$

which also yields

$$\begin{cases} u''\left(c_t\right)\dot{c}_t = \left(r-\rho\right)u'\left(c_t\right) \\ \dot{a}_t + c_t = r_t a_t \\ a_0 \text{ is given} \\ \lim_{t\to\infty} e^{-\rho t} q_t a_t = 0 \end{cases}$$

Compare to the present-value Hamiltonian method, the difference is just the formation of Hamiltonian, or definition of **co-state variable**, namely,

$$q\left(t\right) = e^{\rho t}\lambda\left(t\right)$$

5. More about the Euler Equation.

$$u''\left(c_t\right)\dot{c}_t = \left(r-\rho\right)u'\left(c_t\right)$$

rewrite it as

$$\frac{\dot{c}_t}{c_t}\left[\frac{-u''\left(c_t\right)\cdot c_t}{u'\left(c_t\right)}\right] = r - \rho$$

the term $\frac{-u''(c_t)\cdot c_t}{u'(c_t)}$ measures relative risk aversion. And (under the balanced growth path) there is a class of utility with Constant Relative Risk Aversion (CRRA),

$$u\left(c\right) = \frac{c^{1-\gamma} - 1}{1-\gamma}$$

Then

$$\frac{-u''\left(c_t\right)\cdot c_t}{u'\left(c_t\right)} = \frac{\gamma c_t^{-\gamma-1}\cdot c_t}{\frac{(1-\gamma)c_t^{-\gamma}}{1-\gamma}} = \gamma$$

Then the Euler equation of CRRA utility in the problem is

$$\frac{\dot{c}_t}{c_t}\gamma = r - \rho$$

4.2 The General Problem and Hamilton-Jacobi-Bellman Equation

1. The general problem is

$$\max_{\{c_t\}_{t\geqslant 0}} \int_0^\infty e^{-\rho t} \cdot u\left(a_t, c_t\right) \mathrm{d}t$$
$$\text{s.t. } \dot{a}_t = g\left(a_t, c_t\right)$$

   where $\rho > 0, a_0$ is given, u and g are concave and twice differentiable.

2. Solution 1. Present Value Hamiltonian.

   Form the Lagrangian, and assume $\lim_{t\to\infty} \lambda_t a_t = 0$,

$$\begin{aligned}
\mathcal{L} &= \int_0^\infty e^{-\rho t} u\left(a_t, c_t\right) + \lambda_t\left[g\left(a_t, c_t\right) - \dot{a}_t\right] \mathrm{d}t \\
&= \int_0^\infty \left[e^{-\rho t} u\left(a_t, c_t\right) + \lambda_t g\left(a_t, c_t\right)\right] \mathrm{d}t - \int_0^\infty \lambda_t \dot{a}_t \mathrm{d}t \\
&= \int_0^\infty \left[\underbrace{e^{-\rho t} u\left(a_t, c_t\right) + \lambda_t g\left(a_t, c_t\right)}_{\mathcal{H}_t} + a_t \dot{\lambda}_t\right] \mathrm{d}t + \lambda_0 a_0
\end{aligned}$$

   F.O.C.s

$$\begin{cases} \frac{\partial \mathcal{H}_t}{\partial c_t} = 0 \\ \frac{\partial \mathcal{H}_t}{\partial a_t} = -\dot{\lambda}_t \end{cases} \Longrightarrow \begin{cases} e^{-\rho t} \frac{\partial u}{\partial c_t}\left(a_t, c_t\right) + \lambda_t \frac{\partial g}{\partial c_t}\left(a_t, c_t\right) = 0 \\ e^{-\rho t} \frac{\partial u}{\partial a_t}\left(a_t, c_t\right) + \lambda_t \frac{\partial g}{\partial a_t}\left(a_t, c_t\right) = -\dot{\lambda}_t \end{cases}$$

3. Solution2. Current Value Hamiltonian.

   Form the Lagrangian, and assume $\lim_{t\to\infty} e^{-\rho t} q_t a_t = 0$,

$$\begin{aligned}
\mathcal{L} &= \int_0^\infty e^{-\rho t} \left\{u\left(a_t, c_t\right) + q_t\left[g\left(a_t, c_t\right) - \dot{a}_t\right]\right\} \mathrm{d}t \\
&= \int_0^\infty e^{-\rho t} \left[u\left(a_t, c_t\right) + q_t g\left(a_t, c_t\right)\right] \mathrm{d}t - \int_0^\infty e^{-\rho t} q_t \dot{a}_t \mathrm{d}t \\
&= \int_0^\infty e^{-\rho t} \left[u\left(a_t, c_t\right) + q_t g\left(a_t, c_t\right)\right] \mathrm{d}t - e^{-\rho t} q_t a_t \Big|_0^\infty + \int_0^\infty a_t \mathrm{d}\left(e^{-\rho t} q_t\right) \\
&= \int_0^\infty e^{-\rho t} \left[\underbrace{u\left(a_t, c_t\right) + q_t g\left(a_t, c_t\right)}_{\mathcal{H}_t^*} + \left(-\rho q_t + \dot{q}_t\right) a_t\right] \mathrm{d}t + q_0 a_0
\end{aligned}$$

   F.O.C.s

$$\begin{cases} \frac{\partial \mathcal{H}_t^*}{\partial c_t} = 0 \\ \frac{\partial \mathcal{H}_t^*}{\partial a_t} = \rho q_t - \dot{q}_t \end{cases} \Longrightarrow \begin{cases} \frac{\partial u}{\partial c_t}\left(a_t, c_t\right) + q_t \frac{\partial g}{\partial c_t}\left(a_t, c_t\right) = 0 \\ \frac{\partial u}{\partial a_t}\left(a_t, c_t\right) + q_t \frac{\partial g}{\partial a_t}\left(a_t, c_t\right) = \rho q_t - \dot{q}_t \end{cases}$$

4. Solution 3. Dynamic programming.

Define the value function,

$$V(a_t,t)=\max_{\{c_s\}_{s\geqslant t}}\int_t^{\infty}e^{-\rho(s-t)}\cdot u(a_s,c_s)\,\mathrm{d}s$$
$$\text{s.t. } \dot{a}_s=g(a_s,c_s),s\geqslant t$$

and then form the Bellman equation,

$$V(a_t,t)=\max_{\{c_s\}_{t\leqslant s\leqslant t+\Delta t}}\left\{\int_t^{t+\Delta t}e^{-\rho(s-t)}\cdot u(a_s,c_s)\,\mathrm{d}s+e^{-\rho\Delta t}V(a_{t+\Delta t},t+\Delta t)\right\}$$
$$\text{s.t. } \dot{a}_s=g(a_s,c_s),t\leqslant s\leqslant t+\Delta t$$

Subtract $V(a_t,t)$ on both side and divide by Δt,

$$0=\max_{\{c_s\}_{t\leqslant s\leqslant t+\Delta t}}\left\{\frac{\int_t^{t+\Delta t}e^{-\rho(s-t)}\cdot u(a_s,c_s)\,\mathrm{d}s}{\Delta t}+\frac{e^{-\rho\Delta t}V(a_{t+\Delta t},t+\Delta t)-V(a_t,t)}{\Delta t}\right\}$$

Let $\Delta t\to 0$, by L'Hôpital's rule and Leibniz's rule,

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{\int_t^{t+\Delta t}e^{-\rho(s-t)}\cdot u(a_s,c_s)\,\mathrm{d}s}{\Delta t}&=\lim_{\Delta t\to 0}\left[e^{-\rho(t+\Delta t-t)}\cdot u(a_{t+\Delta t},c_{t+\Delta t})\right]\\&=\lim_{\Delta t\to 0}e^{-\rho\Delta t}u(a_{t+\Delta t},c_{t+\Delta t})\\&=u(a_t,c_t)\end{aligned}$$

and

$$\begin{aligned}&\lim_{\Delta t\to 0}\frac{e^{-\rho\Delta t}V(a_{t+\Delta t},t+\Delta t)-V(a_t,t)}{\Delta t}\\&=\lim_{\Delta t\to 0}e^{-\rho\Delta t}\left[-\rho V(a_{t+\Delta t},t+\Delta t)+V_a(a_{t+\Delta t},t+\Delta t)\,\dot{a}_{t+\Delta t}+V_t(a_{t+\Delta t},t+\Delta t)\right]\\&=-\rho V(a_t,t)+V_a(a_t,t)\,\dot{a}_t+V_t(a_t,t)\\&=-\rho V(a_t,t)+V_a(a_t,t)\,g(a_t,c_t)+V_t(a_t,t)\end{aligned}$$

Then the Bellman equation becomes

$$0=\max_{c_t}\{u(a_t,c_t)+-\rho V(a_t,t)+V_a(a_t,t)\,g(a_t,c_t)+V_t(a_t,t)\}$$

Moreover,

$$\rho V(a_t,t)=\max_{c_t}\left\{u(a_t,c_t)+g(a_t,c_t)\cdot\frac{\partial V}{\partial a_t}(a_t,t)+\frac{\partial V}{\partial t}(a_t,t)\right\}$$

where a_t is given, the equation is called **Hamilton-Jacobi-Bellman equation**. Define

$$\lambda_t:=e^{-\rho t}\cdot\frac{\partial V}{\partial a_t}(a_t,t)$$

then

$$\rho V(a_t,t)=\max_{c_t}\left\{u(a_t,c_t)+e^{\rho t}\lambda_t g(a_t,c_t)+\frac{\partial V}{\partial t}(a_t,t)\right\}$$

F.O.C. is

$$\frac{\partial u}{\partial c_t}(a_t, c_t) + e^{\rho t}\lambda_t \frac{\partial g}{\partial c_t}(a_t, c_t) = 0$$

by the envelope theorem,

$$\rho \frac{\partial V}{\partial a_t}(a_t, t) = \frac{\partial u}{\partial a_t}(a_t, c_t) + e^{\rho t}\lambda_t \frac{\partial g}{\partial a_t}(a_t, c_t) + \frac{\partial^2 V}{\partial a_t \partial t}(a_t, t)$$

Recall what we get in the present value Hamilton,

$$\begin{cases} \frac{\partial \mathcal{H}_t}{\partial c_t} = 0 \\ \frac{\partial \mathcal{H}_t}{\partial a_t} = -\dot{\lambda}_t \end{cases} \Longrightarrow \begin{cases} e^{-\rho t}\frac{\partial u}{\partial c_t}(a_t, c_t) + \lambda_t \frac{\partial g}{\partial c_t}(a_t, c_t) = 0 \\ e^{-\rho t}\frac{\partial u}{\partial a_t}(a_t, c_t) + \lambda_t \frac{\partial g}{\partial a_t}(a_t, c_t) = -\dot{\lambda}_t \end{cases}$$

which is equivalent with this solution. Let's show it.

The first equation is easy to see.

For the second equation, recall the definition of λ_t:

$$\lambda_t := e^{-\rho t} \cdot \frac{\partial V}{\partial a_t}(a_t, t)$$

then

$$\dot{\lambda}_t = -\rho e^{-\rho t}\frac{\partial V}{\partial a_t}(a_t, t) + e^{-\rho t}\frac{\partial^2 V}{\partial t \partial a_t}(a_t, t)$$

plug in,

$$e^{-\rho t}\frac{\partial u}{\partial a_t}(a_t, c_t) + \lambda_t \frac{\partial g}{\partial a_t}(a_t, c_t) = \rho e^{-\rho t}\frac{\partial V}{\partial a_t}(a_t, t) - e^{-\rho t}\frac{\partial^2 V}{\partial t \partial a_t}(a_t, t)$$

which yields,

$$\frac{\partial u}{\partial a_t}(a_t, c_t) + \lambda_t e^{\rho t}\frac{\partial g}{\partial a_t}(a_t, c_t) = \rho \frac{\partial V}{\partial a_t}(a_t, t) - \frac{\partial^2 V}{\partial t \partial a_t}(a_t, t)$$

4.3 Transversality Condition: An Example

1. Consider a classic problem (we don't need to consider nonnegative constraint on c_t by assuming $\lim_{c \to 0} u'(c) = \infty$).

$$\max_{\{c_t\}_{c=0}^{\infty}, \{a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } a_{t+1} = (1+r)a_t - c_t$$
$$a_{t+1} \geqslant 0, a_0 \text{ is given}$$

Form the Lagrangian,

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{u(c_t) - \lambda_t [a_{t+1} - (1+r) \cdot a_t + c_t] + \mu_t a_{t+1}\}$$

F.O.C.s

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial c_t} = \beta^t \left[u' \left(c_t \right) - \lambda_t \right] = 0 \\ \frac{\partial \mathcal{L}}{\partial a_{t+1}} = \beta^t \left[-\lambda_t + \mu_t \right] + \beta^{t+1} \left(1+r \right) \lambda_{t+1} = 0 \\ \mu_t a_{t+1} = 0, \mu_t \geqslant 0, a_{t+1} \geqslant 0 \end{cases}$$

then

$$\begin{cases} u' \left(c_t \right) = \lambda_t \\ \mu_t + \beta \left(1+r \right) \lambda_{t+1} = \lambda_t \end{cases}$$

If $a_T = 0$, then $\forall t \geqslant T, c_t = 0$, which cannot be optimal, therefore,

$$a_{t+1} > 0 \Longrightarrow \mu_t = 0$$

then we get the Euler equation,

$$u' \left(c_t \right) = \beta \left(1+r \right) u' \left(c_{t+1} \right)$$

recall that

$$a_{t+1} = \left(1+r \right) a_t - c_t$$

2. A special case. $u\left(c\right) = \log c$.

The Euler equation and constraint are

$$\begin{cases} \frac{1}{c_t} = \beta \left(1+r \right) \frac{1}{c_{t+1}} \\ a_{t+1} = \left(1+r \right) a_t - c_t \end{cases}$$

and a_0 is given. Then

$$c_T = \beta \left(1+r \right) c_{T-1} = \left[\beta \left(1+r \right) \right]^T c_0$$

and

$$\begin{aligned} a_{T+1} &= \left(1+r \right) a_T - c_T \\ &= \left(1+r \right) \left[\left(1+r \right) a_{T-1} - c_{T-1} \right] - c_T \\ &= \left(1+r \right)^2 a_{T-1} - \left[\left(1+r \right) c_{T-1} + c_T \right] \\ &= \cdots \\ &= \left(1+r \right)^{T+1} a_0 - \sum_{t=0}^{T} \left(1+r \right)^{T-t} c_t \end{aligned}$$

put them together,

$$\begin{aligned} a_{T+1} &= \left(1+r \right)^{T+1} a_0 - \sum_{t=0}^{T} \left(1+r \right)^{T-t} \left[\beta \left(1+r \right) \right]^t c_0 \\ &= \left(1+r \right)^{T+1} \left[a_0 - c_0 \sum_{t=0}^{T} \beta^t \right] \end{aligned}$$

To solve the problem, we first need to figure out what is c_0, and then plug it into the equation to solve a_1, c_1, and so on. Therefore,

$$c_0 = \frac{1}{\sum_{t=0}^{T} \beta^t} \left[a_0 - \frac{a_{T+1}}{(1+r)^{T+1}} \right]$$

In finite-period problem, by completeness and slackness we have $\mu > 0$, then

$$a_{T+1} = 0$$

But for infinite horizon, we cannot have such a condition. If we take $T \to \infty$

$$c_0 = \frac{1}{\sum_{t=0}^{\infty} \beta^t} \left[a_0 - \lim_{T\to\infty} \frac{a_{T+1}}{(1+r)^{T+1}} \right] = (1-\beta) \left[a_0 - \lim_{T\to\infty} \frac{a_{T+1}}{(1+r)^{T+1}} \right]$$

To solve the problem, we may assume that,

$$\lim_{T\to\infty} \frac{a_{T+1}}{(1+r)^{T+1}} = 0$$

which is the transversality condition, and it has the same economic meaning as we explain before.

Lecture 5

Neoclassical Growth Model and Beyond

5.1 Solow Model

1. Set-ups.

   (a) Production function: $Y = F(K, L)$.

      i. Constant return to scale: $F(zK, zL) = zF(K, L)$.

      ii. $k := \frac{K}{L}, y := \frac{Y}{L}$, then $y = \frac{Y}{L} = F\left(\frac{K}{L}, 1\right) = F(k, 1) := f(k)$.

      iii. Concave. $\frac{\mathrm{d}f(k)}{\mathrm{d}k}$ is decreasing.

      iv. $Y = AK^{\alpha}L^{1-\alpha}$ satisfies those properties.

   (b) The saving rate is constant γ.

   (c) Capital stock depreciates at a constant rate δ.

   (d) TFP A is a constant.

2. The model.

$$\dot{k} = \gamma f(k) - \delta k = \gamma A k^{\alpha} - \delta k$$

   (a) The steady state is $k^* = \left(\frac{\gamma A}{\delta}\right)^{\frac{1}{1-\alpha}}$;

   (b) The steady state of output per worker is $y^* = A \cdot \left(\frac{\gamma A}{\delta}\right)^{\frac{\alpha}{1-\alpha}} = A^{\frac{1}{1-\alpha}} \left(\frac{\gamma}{\delta}\right)^{\frac{1}{1-\alpha}}$;

   (c) Growth rate: $\frac{\dot{k}}{k} = \frac{\gamma f(k)}{k} - \delta$, $\frac{\dot{y}}{y} = \alpha\frac{\dot{k}}{k}$.

3. Conclusions. (Ceteris paribus)

   (a) Country with lower income will have higher growth rate (β convergence if they have the same parameter).

   (b) Country with a higher rate of investment will have higher growth.

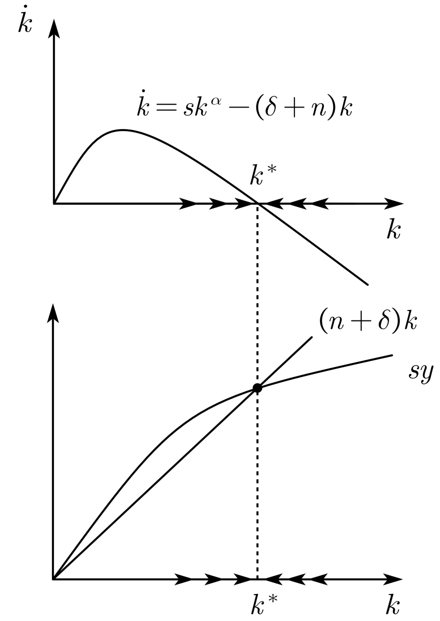

Fig. 5.1. Solow Model.

4. If we let population grows at constant rate $\frac{\dot{L}}{L} = n$, then

$$\dot{K}_t = \gamma F\left(K_t, A_t L_t\right) - \delta K_t$$

then

$$\frac{\dot{K}_t}{L_t} = \gamma f\left(k_t\right) - \delta k_t$$

and

$$\dot{k} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{K}{L}\right) = \frac{\dot{K}L - \dot{L}K}{L^2} = \frac{\dot{K}}{L} - nk$$

then

$$\dot{k} = \gamma f\left(k\right) - \left(\delta + n\right)k$$

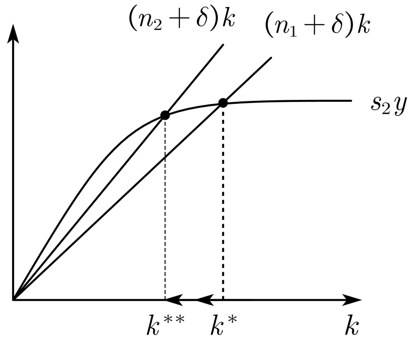

Fig. 5.2. Increase in n.

The population growth rate n implies capital is diluted by population growth, which leads to lower k^* and y^*.

$$k^* = \left(\frac{rA}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$
$$y^* = A^{\frac{1}{1-\alpha}} \left(\frac{r}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

5.2 Endogenous Savings Rate: A One Sector Growth Model

1. Set-up.

   (a) Problem.

   $$\max_{\{C_t, K_{t+1}\}_0^\infty} \sum_{t=0}^{\infty} \beta^t \log C_t$$
   $$\text{s.t. } C_t + K_{t+1} - K_t = -\delta K_t + AK_t^\alpha (X_t N_t)^{1-\alpha}$$

   The constraint shows that the savings rate is endogenous. If it is exogenous, the constraint should be

   $$K_{t+1} - K_t = \gamma AK_t^\alpha (X_t N_t)^{1-\alpha} - \delta K_t$$

   where γ is the savings rate.

   (b) On BGP, $\frac{X_{t+1}}{X_t} = 1 + g$ is the growth of the labor augmented technology.

   (c) C_t is consumption.

   (d) K_t is capital.

   (e) δ is depreciate rate.

   (f) A is a constant.

   (g) α is the capital share.

   (h) $N_t = 1$ keeps a constant, denotes the total labor, so we should not regard N_t as a control variable.

   (i) $Y_t = AK_t^\alpha (X_t N_t)^{1-\alpha}$

2. Solution.

   The Bellman equation is

   $$V(K_t, X_t) = \max_{C_t, K_{t+1}} \log C_t + \beta V(K_{t+1}, X_{t+1})$$
   $$\text{s.t. } C_t + K_{t+1} - K_t = -\delta K_t + AK_t^\alpha X_t^{1-\alpha}$$
   $$K_0 \text{ is given}$$

Note that we plug $N_t = 1$. Form the Lagrangian,

$$\mathcal{L} = \log C_t + \beta V\left(K_{t+1}, X_{t+1}\right) + \lambda_t\left[\left(1-\delta\right)K_t + AK_t^{\alpha}X_t^{1-\alpha} - K_{t+1} - C_t\right]$$

The F.O.C.s are

$$\begin{aligned}
\left[C_t\right] :& \frac{1}{C_t} = \lambda_t \\
\left[K_{t+1}\right] :& \beta\frac{\partial V\left(K_{t+1}, X_{t+1}\right)}{\partial K_{t+1}} = \lambda_t
\end{aligned}$$

then by the Envelop theorem,

$$\frac{\partial V\left(K_t, X_t\right)}{\partial K_t} = \frac{\partial \mathcal{L}}{\partial K_t} = \lambda_t\left[\left(1-\delta\right) + A\alpha K_t^{\alpha-1}X_t^{1-\alpha}\right]$$

and thus,

$$\begin{cases}
\lambda_t = \frac{1}{C_t} \\
\beta\lambda_{t+1}\left[\left(1-\delta\right) + A\alpha K_{t+1}^{\alpha-1}X_{t+1}^{1-\alpha}\right] = \lambda_t
\end{cases}$$

which gives us the Euler equation,

$$\frac{\beta}{C_{t+1}}\left[\left(1-\delta\right) + A\alpha K_{t+1}^{\alpha-1}X_{t+1}^{1-\alpha}\right] = \frac{1}{C_t}$$

With the constraint and impose the transversality condition $\lim_{t\to\infty}\beta^t\lambda^t K_t = 0$, we have a system

$$\begin{cases}
\frac{\beta}{C_{t+1}}\left[\left(1-\delta\right) + A\alpha K_{t+1}^{\alpha-1}X_{t+1}^{1-\alpha}\right] = \frac{1}{C_t} \\
C_t + K_{t+1} - K_t = -\delta K_t + AK_t^{\alpha}X_t^{1-\alpha} \\
\lim_{t\to\infty}\beta^t\frac{K_t}{C_t} = 0 \\
K_0 \text{ is given.}
\end{cases}$$

3. Balanced growth path.

   On the BGP, it holds that

$$\frac{K_{t+1}}{K_t} = \frac{C_{t+1}}{C_t} = \frac{Y_{t+1}}{Y_t}$$

and thus, $\frac{K}{Y}, \frac{C}{Y}$ are constants. We now try to solve these constants, given BGP.

$$Y_t = AK_t^{\alpha}\left(X_tN_t\right)^{1-\alpha} = A\left(\frac{K_t}{Y_t}Y_t\right)^{\alpha}\left(X_tN_t\right)^{1-\alpha} = A\left(\frac{K_t}{Y_t}\right)^{\alpha}Y_t^{\alpha}\left(X_tN_t\right)^{1-\alpha}$$

then plug $N_t = 1$, and we have

$$Y_t = A^{\frac{1}{1-\alpha}}\left(\frac{K}{Y}\right)^{\frac{\alpha}{1-\alpha}}X_t$$

which implies

$$\frac{Y_{t+1}}{Y_t} = \frac{X_{t+1}}{X_t} = 1 + g$$

Recall the Euler equation,

$$\frac{1}{C_t} = \frac{\beta}{C_{t+1}}\left[(1-\delta) + A\alpha K_{t+1}^{\alpha-1} X_{t+1}^{1-\alpha}\right]$$

then

$$\frac{1}{C_t} = \frac{\beta}{C_{t+1}}\left[(1-\delta) + \alpha\frac{Y_{t+1}}{K_{t+1}}\right]$$

given the BGP,

$$\frac{K_{t+1}}{K_t} = \frac{C_{t+1}}{C_t} = \frac{Y_{t+1}}{Y_t} = \frac{X_{t+1}}{X_t} = 1+g$$

then

$$1+g = \beta\left[(1-\delta) + \alpha\frac{Y}{K}\right]$$

which gives us the capital share

$$\frac{K}{Y} = \frac{\alpha}{\frac{1+g}{\beta} - 1 + \delta}$$

As for $\frac{C}{Y}$, by the constraint,

$$C_t + K_{t+1} - K_t = -\delta K_t + AK_t^{\alpha} X_t^{1-\alpha}$$

we get

$$\frac{C_t}{Y_t} + \frac{K_{t+1}}{Y_{t+1}}\frac{Y_{t+1}}{Y_t} = (1-\delta)\frac{K_t}{Y_t} + 1$$

on the BGP,

$$\begin{aligned}
\frac{C}{Y} &= [(1-\delta) - (1+g)]\frac{K}{Y} + 1 \\
&= 1 - (\delta + g)\frac{\alpha}{\frac{1+g}{\beta} - 1 + \delta} \\
&= 1 - \frac{\alpha\beta(\delta+g)}{1+g-\beta+\beta\delta} \\
&> 1 - \frac{\alpha\beta(\delta+g)}{g+\beta\delta} \\
&> 1 - \alpha\beta \\
&> 0
\end{aligned}$$

5.3 A Multisector Model with Endogenous Price

1. Modifications and recalls.

   (a) There are 3 sectors, agriculture, manufacture and services.

   (b) Household's problem.

$$\sum_{t=0}^{\infty} \beta^t \left[\theta_a \log\left(C_{at} - \bar{C}_a\right) + \theta_m \log C_m + \theta_s \log\left(C_{st} + \bar{C}_s\right)\right]$$

$$\text{s.t. } \underbrace{P_{at}C_{at} + C_{mt} + P_{st}C_{st}}_{\text{purchase of consumption}} + \underbrace{K_{t+1} - (1-\delta)K_t}_{\text{purchase of investment}} = \underbrace{W_t + R_t K_t}_{\text{revenue}}$$

(c) C_{at} is consumption for agriculture product, $\bar{C}_a > 0$ captures that agriculture product is essential for living, so $\bar{C}_a$ can be regarded as a necessity, and $C_{at} > \bar{C}_a$.

(d) C_{mt} is consumption for manufactory product.

(e) C_{st} is consumption for services, and $\bar{C}_s > 0$ captures the service is a luxury good, even if $C_{st} = 0$, the utility function is well-defined.

(f) W_t is the wage, and R_t is the rental rate for capital.

(g) We normalized $P_{mt} = 1$.

(h) Production function.

i. Agriculture: $Y_{at} = z_a K_{at}^{\alpha} \left(X_t N_{at}\right)^{1-\alpha}$.

ii. Manufactory: $Y_{mt} = z_m K_{mt}^{\alpha} \left(X_t N_{mt}\right)^{1-\alpha}$.

iii. Service: $Y_{st} = z_s K_{st}^{\alpha} \left(X_t N_{st}\right)^{1-\alpha}$

(i) Remark.

i. First, all production functions have the same capital and labor share, given by α and $1-\alpha$.

ii. Second, the technology progress factors in all sectors are the same, determined by X_t.

iii. The only difference among these three sectors is their sector specific technology level z_a, z_m and z_s.

2. Solve firm's problem. For $j \in \{a, m, s\}$, the problem is

$$\max_{K_{jt}, N_{jt}} P_{jt} z_j K_{jt}^{\alpha} \left(X_t N_{jt}\right)^{1-\alpha} - R_t K_{jt} - W_t N_{jt}$$

The F.O.C.s are

$$\begin{aligned} [K_{jt}] :& P_{jt} z_j \alpha K_{jt}^{\alpha-1} \left(X_t N_{jt}\right)^{1-\alpha} = R_t \\ [N_{jt}] :& P_{jt} z_j K_{jt}^{\alpha} \left(1-\alpha\right) X_t \left(X_t N_{jt}\right)^{-\alpha} = W_t \end{aligned}$$

Because we want to solve household's problem, then we need to solve for P_{jt}. F.O.C.s give us

$$\begin{cases} P_{jt}^{-\alpha} z_j^{-\alpha} \alpha^{-\alpha} \left[K_{jt}^{\alpha-1} \left(X_t N_{jt}\right)^{1-\alpha}\right]^{-\alpha} = R_t^{-\alpha} \\ P_{jt}^{1-\alpha} z_j^{1-\alpha} \left(1-\alpha\right)^{1-\alpha} X_t^{1-\alpha} \left[K_{jt}^{\alpha} \left(X_t N_{jt}\right)^{-\alpha}\right]^{(1-\alpha)} = W_t^{1-\alpha} \end{cases}$$

then

$$P_{jt} z_j \left(1-\alpha\right)^{1-\alpha} \alpha^{\alpha} = W_t^{1-\alpha} R_t^{\alpha} \Longrightarrow P_{jt} = \frac{1}{z_j} \left(\frac{W_t / X_t}{1-\alpha}\right)^{1-\alpha} \left(\frac{R_t}{\alpha}\right)^{\alpha}$$

therefore,

$$P_{at} : P_{mt} : P_{st} = \frac{1}{z_a} : \frac{1}{z_m} : \frac{1}{z_s}$$

given z_m, and we normalize $P_{mt} = 1$, $z_m = 1$, then

$$P_{at} = \frac{z_m}{z_a} = \frac{1}{z_a}, P_{st} = \frac{z_m}{z_s} = \frac{1}{z_s}$$

3. The aggregate production function. The aggregate output is

$$Y_t = \sum_{j\in\{a,m,s\}} P_{jt} z_j K_{jt}^{\alpha} \left(X_t N_{jt}\right)^{1-\alpha}$$

for each sector, recall the F.O.C.s

$$\begin{cases} P_{jt}\alpha \frac{Y_{jt}}{K_{jt}} = R_t \\ P_{jt}\left(1-\alpha\right)\frac{Y_{jt}}{N_{jt}} = W_t \end{cases}$$

If the market clears, then

$$\begin{cases} \alpha \sum_{j\in\{a,m,s\}} P_{jt}Y_{jt} = R_t \sum_{j\in\{a,m,s\}} K_{jt} \\ \left(1-\alpha\right)\sum_{j\in\{a,m,s\}} P_{jt}Y_{jt} = W_t \sum_{j\in\{a,m,s\}} N_{jt} \end{cases} \Longrightarrow \begin{cases} \alpha Y_t = R_t K_t \\ \left(1-\alpha\right) Y_t = W_t N_t \end{cases}$$

therefore,

$$Y_t = R_t K_t + W_t N_t$$

Since $P_m = 1, z_m = 1$, then

$$\left(\frac{W_t/X_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha} = 1$$

where by

$$\begin{cases} \alpha Y_t = R_t K_t \\ \left(1-\alpha\right) Y_t = W_t N_t \end{cases} \Longrightarrow \begin{cases} \frac{R_t}{\alpha} = \frac{Y_t}{K_t} \\ \frac{W_t}{1-\alpha} = \frac{Y_t}{N_t} \end{cases}$$

then

$$\left(\frac{Y_t}{N_t}\frac{1}{X_t}\right)^{1-\alpha}\left(\frac{Y_t}{K_t}\right)^{\alpha} = 1 \Longrightarrow Y_t = K_t^{\alpha}\left(X_t N_t\right)^{1-\alpha}$$

The aggregate production function results come from the assumption all sectors have the same factor share and common technology progress.

4. Solve the household's problem.

Since

$$R_t K_t + W_t N_t = Y_t = K_t^{\alpha}\left(X_t N_t\right)^{1-\alpha}$$

the problem becomes

$$\sum_{t=0}^{\infty} \beta^t \left[\theta_a \log\left(C_{at} - \bar{C}_a\right) + \theta_m \log C_m + \theta_s \log\left(C_{st} + \bar{C}_s\right)\right]$$
$$\text{s.t. } P_{at}C_{at} + C_{mt} + P_{st}C_{st} + K_{t+1} = \left(1-\delta\right)K_t + K_t^{\alpha}\left(X_t N_t\right)^{1-\alpha}$$

The Bellman equation is

$$V\left(K_t, X_t\right) = \max_{C_{at}, C_{mt}, C_{st}, K_{t+1}} \theta_a \log\left(C_{at} - \bar{C}_a\right) + \theta_m \log C_m + \theta_s \log\left(C_{st} + \bar{C}_s\right) + \beta V\left(K_{t+1}, X_{t+1}\right)$$
$$\text{s.t. } P_{at}C_{at} + C_{mt} + P_{st}C_{st} + K_{t+1} = \left(1-\delta\right)K_t + K_t^{\alpha}\left(X_t N_t\right)^{1-\alpha}$$

Form the Lagrangian,

$$\begin{aligned}\mathcal{L} =& \theta_a \log\left(C_{at} - \bar{C}_a\right) + \theta_m \log C_m + \theta_s \log\left(C_{st} + \bar{C}_s\right) + \beta V\left(K_{t+1}, X_{t+1}\right) \\ &+ \lambda_t \left[(1-\delta) K_t + K_t^{\alpha} \left(X_t N_t\right)^{1-\alpha} - K_{t+1} - P_{at}C_{at} - C_{mt} - P_{st}C_{st}\right]\end{aligned}$$

The F.O.C.s are

$$\begin{aligned}[C_{at}] &: \frac{\theta_a}{C_{at} - \bar{C}_a} = P_{at}\lambda_t \\ [C_{mt}] &: \frac{\theta_m}{C_{mt}} = \lambda_t \\ [C_{st}] &: \frac{\theta_s}{C_{st} + \bar{C}_s} = P_{st}\lambda_t \\ [K_{t+1}] &: \beta \frac{\partial V\left(K_{t+1}, X_{t+1}\right)}{\partial K_{t+1}} = \lambda_t\end{aligned}$$

and by the Envelope Theorem,

$$\frac{\partial V\left(K_t, X_t\right)}{\partial K_t} = \frac{\partial \mathcal{L}}{\partial K_t} = \lambda_t \left[(1-\delta) + \alpha K_t^{\alpha-1} \left(X_t N_t\right)^{1-\alpha}\right]$$

therefore,

$$\begin{cases} \frac{\theta_a}{C_{at} - \bar{C}_a} \frac{1}{P_{at}} = \frac{\theta_m}{C_{mt}} = \frac{\theta_s}{C_{st} + \bar{C}_s} \frac{1}{P_{st}} \\ \beta \frac{1}{C_{m(t+1)}} \left[(1-\delta) + \alpha \frac{Y_{t+1}}{K_{t+1}}\right] = \frac{1}{C_{mt}} \end{cases}$$

5. Solve the BGP.

Define

$$C_t := C_{mt} + P_{st}C_{st} + P_{at}C_{at}$$

Note that C_t is the value of consumption instead of quantity. By F.O.C.s

$$\begin{cases} P_{at}C_{at} = \frac{\theta_a}{\theta_m} C_{mt} + \bar{C}_a P_{at} \\ P_{st}C_{st} = \frac{\theta_s}{\theta_m} C_{mt} - \bar{C}_s P_{st} \end{cases}$$

then

$$\begin{aligned}C_t &= C_{mt} + P_s C_{st} + P_a C_{at} \\ &= C_{mt} + \frac{\theta_s}{\theta_m} C_{mt} + \frac{\theta_a}{\theta_m} C_{mt} - \bar{C}_s P_{st} + \bar{C}_a P_{at} \\ &= \left(1 + \frac{\theta_s}{\theta_m} + \frac{\theta_a}{\theta_m}\right) C_{mt} - \bar{C}_s P_{st} + \bar{C}_a P_{at}\end{aligned}$$

On the balanced growth path, K_t, Y_t, C_t grows at the same rate. By the Euler equation,

$$\beta \frac{1}{C_{m(t+1)}} \left[(1-\delta) + \alpha \frac{Y_{t+1}}{K_{t+1}}\right] = \frac{1}{C_{mt}} \Longrightarrow \frac{C_{m(t+1)}}{C_{mt}} = \beta \left[(1-\delta) + \alpha \frac{Y_{t+1}}{K_{t+1}}\right]$$

then $\frac{C_{m(t+1)}}{C_{mt}}$ must be a constant on the balanced growth path, since $\frac{Y_{t+1}}{K_{t+1}} = \frac{Y}{K}$ keeps a constant. And since

$$C_t = \left(1 + \frac{\theta_s}{\theta_m} + \frac{\theta_a}{\theta_m}\right) C_{mt} - \bar{C}_s P_{st} + \bar{C}_a P_{at}$$

we should impose

$$\bar{C}_a P_{at} = \bar{C}_s P_{st}$$

recall that

$$P_{at} = \frac{z_m}{z_a} = \frac{1}{z_a}$$
$$P_{st} = \frac{z_m}{z_s} = \frac{1}{z_s}$$

or we impose

$$\frac{\bar{C}_a}{z_a} = \frac{\bar{C}_s}{z_s}$$

to yields

$$C_t = \left(1 + \frac{\theta_s}{\theta_m} + \frac{\theta_a}{\theta_m}\right) C_{mt}$$

Note: The balanced growth then requires the constraint $\frac{\bar{C}_a}{z_a} = \frac{\bar{C}_s}{z_s}$. This is not very satisfactory. As the modern economy often assume that utility and technology are separate.

On the BGP, the reallocation across sectors, let ω_{jt} denote the expenditure share of sector j. Then

$$\omega_{mt} = \frac{C_{mt}}{C_t} = \frac{1}{1 + \frac{\theta_s}{\theta_m} + \frac{\theta_a}{\theta_m}} = \frac{\theta_m}{\theta_m + \theta_s + \theta_a}$$
$$\omega_{at} = \frac{C_{at} P_{at}}{C_t} = \frac{\frac{\theta_a}{\theta_m} C_{mt} + \bar{C}_a P_{at}}{C_{mt}\left(1 + \frac{\theta_s}{\theta_m} + \frac{\theta_a}{\theta_m}\right)} = \frac{\theta_a}{\theta_m + \theta_s + \theta_a} + \frac{P_{at}}{1 + \frac{\theta_s}{\theta_m} + \frac{\theta_a}{\theta_m}} \frac{\bar{C}_a}{C_{mt}}$$
$$\omega_{st} = \frac{C_{st} P_{st}}{C_t} = \frac{\frac{\theta_s}{\theta_m} C_{mt} - \bar{C}_s P_{st}}{C_{mt}\left(1 + \frac{\theta_s}{\theta_m} + \frac{\theta_a}{\theta_m}\right)} = \frac{\theta_s}{\theta_m + \theta_s + \theta_a} - \frac{P_{st}}{1 + \frac{\theta_s}{\theta_m} + \frac{\theta_a}{\theta_m}} \frac{\bar{C}_s}{C_{mt}}$$

as C_{mt} grows at the same rate of C_t, ω_{mt} keeps a constant, ω_{at} declines, and ω_{st} increases overtime.

5.4 Technology Driven Structure Change

1. A simple introduction.

   Consider the discrete time model

   $$\sum_{t=0}^{\infty} \beta^t \log C_t$$

   with the resource constraint

   $$C_t + K_{t+1} = (1 - \delta) K_t + Y_t$$

   where $Y_t = K_t^{\alpha} (X_t N_t)^{1-\alpha}$.

We now consider each period becomes very short. The time interval between two period is Δ,

$$
\begin{aligned}
V(K_t, X_t) &= \max_{\{C_t\}_{t=0}^{\infty},\{K_t\}_{t>0}^{\infty}} \int_0^{\infty} e^{-\rho t} \log C_t \mathrm{d}t \\
&\approx \max_{\{C_t\}_{t=0}^{\infty},\{K_t\}_{t>0}^{\infty}} \Delta \cdot \log C_t + \int_{\Delta}^{\infty} e^{-\rho t} \log C_t \mathrm{d}t \\
&= \max_{\{C_t\}_{t=0}^{\infty},\{K_t\}_{t>0}^{\infty}} \Delta \cdot \log C_t + \int_0^{\infty} e^{-\rho(\tau+\Delta)} \log C_{\tau+\Delta} \mathrm{d}\tau \\
&= \max_{C_t, K_{t+\Delta}} \Delta \cdot \log C_t + e^{-\rho\Delta} V(K_{t+\Delta}, X_{t+\Delta})
\end{aligned}
$$

the constraint becomes

$$
C_t\Delta + K_{t+\Delta} = (1-\delta\Delta) K_t + Y_t\Delta
$$

which implies

$$
\lim_{\Delta\to 0} \frac{K_{t+\Delta} - K_t}{\Delta} = \dot{K}_t = Y_t - C_t - \delta K_t
$$

Plug $Y_t = K_t^{\alpha}(X_t N_t)^{1-\alpha}$, the Bellman equation becomes

$$
\begin{aligned}
V(K_t, X_t) &= \max_{C_t, K_{t+\Delta}} \Delta \cdot \log C_t + e^{-\rho\Delta} V(K_{t+\Delta}, X_{t+\Delta}) \\
\text{s.t. } C_t\Delta + K_{t+\Delta} &= (1-\delta\Delta) K_t + K_t^{\alpha}(X_t N_t)^{1-\alpha}\Delta
\end{aligned}
$$

Form the Lagrangian,

$$
\mathcal{L} = \Delta \cdot \log C_t + e^{-\rho\Delta} V(K_{t+\Delta}, X_{t+\Delta}) + \lambda_t \left[(1-\delta\Delta) K_t + K_t^{\alpha}(X_t N_t)^{1-\alpha}\Delta - C_t\Delta - K_{t+\Delta}\right]
$$

F.O.C.s

$$
\begin{aligned}
[C_t] :& \frac{1}{C_t}\Delta = \lambda_t \Delta \\
[K_{t+\Delta}] :& e^{-\rho\Delta} \frac{\partial V(K_{t+\Delta}, X_{t+\Delta})}{\partial K_{t+\Delta}} = \lambda_t
\end{aligned}
$$

The Envelop Theorem gives us

$$
\frac{\partial V(K_t, X_t)}{\partial K_t} = \lambda_t \left(1 - \delta\Delta + \alpha \frac{Y_t}{K_t}\Delta\right)
$$

then

$$
\begin{cases}
\frac{1}{C_t} = \lambda_t \\
\lambda_{t+\Delta}\left(1 - \delta\Delta + \alpha\frac{Y_{t+\Delta}}{K_{t+\Delta}}\Delta\right) = e^{\rho\Delta}\lambda_t
\end{cases}
$$

By letting $\Delta \to 0$,

$$
e^{\rho\Delta}\lambda_t \approx (1+\rho\Delta)\lambda_t
$$

and thus,

$$
\lambda_{t+\Delta}\left(1 - \delta\Delta + \alpha\frac{Y_{t+\Delta}}{K_{t+\Delta}}\Delta\right) = (1+\rho\Delta)\lambda_t
$$

or

$$\frac{\lambda_{t+\Delta}-\lambda_t}{\Delta}=\rho\lambda_t+\delta\lambda_{t+\Delta}-\alpha\frac{Y_{t+\Delta}}{K_{t+\Delta}}\lambda_{t+\Delta}$$

Let $\Delta\to 0$,

$$\dot{\lambda}_t=\left(\rho+\delta-\alpha\frac{Y_t}{K_t}\right)\lambda_t\Longrightarrow\frac{\dot{\lambda}_t}{\lambda_t}=\rho+\delta-\alpha\frac{Y_t}{K_t}$$

By $\frac{1}{C_t}=\lambda_t\Longrightarrow\frac{\dot{\lambda}_t}{\lambda_t}=-\frac{\dot{C}_t}{C_t}$, we get

$$\frac{\dot{C}_t}{C_t}=\alpha\frac{Y_t}{K_t}-\rho-\delta$$

2. Technology Driven Structure Change.

Now consider there are J sectors, and the problem is

$$V_t(K_t):=V(K_t,X_{1t},\cdots,X_{Jt})=\max_{C_t,K_{t+\Delta}}\log C_t\Delta+e^{-\rho\Delta}V_{t+\Delta}(K_{t+\Delta})$$
$$\text{s.t. } C_t\Delta+K_{t+\Delta}=(1-\delta\Delta)K_t+Y_t\Delta$$

where

$$Y_t=\left[\sum_{j=1}^{J}\omega_j^{\frac{1}{\sigma}}Y_{jt}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},Y_{jt}=AK_{jt}^{\alpha}(X_{jt}N_{jt})^{1-\alpha}$$

and

$$\sum_{j=1}^{J}K_{jt}=K_t,\sum_{j=1}^{J}N_{jt}=N_t$$

We first take a look of each sector's F.O.C.,

$$\frac{Y_{jt}}{K_{jt}}=w_t,\frac{Y_{jt}}{N_{jt}}=r_t\Longrightarrow\frac{K_{jt}}{N_{jt}}=\frac{w_t}{r_t}$$

is a constant for each sector. Then we solve

$$\max Y_t=\left[\sum_{j=1}^{J}\omega_j^{\frac{1}{\sigma}}Y_{jt}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

since $\frac{K_{jt}}{N_{jt}}$ is the same across each sector, then

$$\frac{K_{jt}}{N_{jt}}=\frac{K_t}{N_t}$$

Then,

$$Y_{jt}=AK_{jt}^{\alpha}(X_{jt}N_{jt})^{1-\alpha}=A\left(\frac{K_{jt}}{N_{jt}}\right)^{\alpha}X_{jt}^{1-\alpha}N_{jt}=A\left(\frac{K_t}{N_t}\right)^{\alpha}X_{jt}^{1-\alpha}N_{jt}$$

or

$$\frac{Y_{jt}}{X_{jt}^{1-\alpha}}=A\left(\frac{K_t}{N_t}\right)^{\alpha}N_{jt}$$

Define $Z_{jt} := X_{jt}^{1-\alpha}$, then

$$\sum_{j=1}^{J} \frac{Y_{jt}}{Z_{jt}} = A\left(\frac{K_t}{N_t}\right)^{\alpha} \sum_{j=1}^{J} N_{jt} = AK_t^{\alpha} N_t^{1-\alpha}$$

Regard it as a constraint, then the problem becomes

$$\max \left[\sum_{j=1}^{J} \omega_j^{\frac{1}{\sigma}} Y_{jt}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} + \phi_t \left[AK_t^{\alpha} N_t^{1-\alpha} - \sum_{j=1}^{J} \frac{Y_{jt}}{Z_{jt}}\right]$$

The F.O.C. for Y_{jt} is

$$\omega_j^{\frac{1}{\sigma}} Y_{jt}^{-\frac{1}{\sigma}} \left[\sum_{j=1}^{J} \omega_j^{\frac{1}{\sigma}} Y_{jt}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{1}{\sigma-1}} = \phi_t \frac{1}{Z_{jt}}$$

or

$$\omega_j^{\frac{1}{\sigma}} Y_{jt}^{-\frac{1}{\sigma}} Y_t^{\frac{1}{\sigma}} = \phi_t \frac{1}{Z_{jt}} \Longrightarrow Y_{jt} = Z_{jt}^{\sigma} \phi_t^{-\sigma} \omega_j Y_t \Longrightarrow \frac{Y_{jt}}{Z_{jt}} = Z_{jt}^{\sigma-1} \phi_t^{-\sigma} \omega_j Y_t$$

Then

$$AK_t^{\alpha} N_t^{1-\alpha} = \sum_{j=1}^{J} \frac{Y_{jt}}{Z_{jt}} = \phi_t^{-\sigma} Y_t \sum_{j=1}^{J} Z_{jt}^{\sigma-1} \omega_j \Longrightarrow Y_t = \frac{AK_t^{\alpha} N_t^{1-\alpha}}{\sum_{j=1}^{J} Z_{jt}^{\sigma-1} \omega_j} \phi_t^{\sigma}$$

and then

$$Y_{jt} = Z_{jt}^{\sigma} \phi_t^{-\sigma} \omega_j \frac{AK_t^{\alpha} N_t^{1-\alpha}}{\sum_{j=1}^{J} Z_{jt}^{\sigma-1} \omega_j} \phi_t^{\sigma} = \frac{Z_{jt}^{\sigma} \omega_j}{\sum_{j=1}^{J} Z_{jt}^{\sigma-1} \omega_j} AK_t^{\alpha} N_t^{1-\alpha}$$

Recall that

$$Y_t = \left[\sum_{j=1}^{J} \omega_j^{\frac{1}{\sigma}} Y_{jt}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

plug Y_{jt} into Y_t, we have

$$\begin{aligned}
Y_t &= \left[\sum_{j=1}^{J} \omega_j^{\frac{1}{\sigma}} Z_{jt}^{\sigma-1} \omega_{jt}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \frac{AK_t^{\alpha} N_t^{1-\alpha}}{\sum_{j=1}^{J} Z_{jt}^{\sigma-1} \omega_j} \\
&= \left[\sum_{j=1}^{J} \omega_j Z_{jt}^{\sigma-1}\right]^{\frac{\sigma}{\sigma-1}} \frac{AK_t^{\alpha} N_t^{1-\alpha}}{\sum_{j=1}^{J} Z_{jt}^{\sigma-1} \omega_j} \\
&= AK_t^{\alpha} N_t^{1-\alpha} \left[\sum_{j=1}^{J} \omega_j Z_{jt}^{\sigma-1}\right]^{\frac{1}{\sigma-1}}
\end{aligned}$$

Define

$$Z_t := \left[\sum_{j=1}^{J} \omega_j Z_{jt}^{\sigma-1}\right]^{\frac{1}{\sigma-1}}$$

then $Y_t = AK_t^{\alpha} N_t^{1-\alpha} Z_t$.

(a) $\sigma = 1$, then

$$\begin{aligned}
Z_t &= \lim_{\sigma \to 1} \left[\sum_{j=1}^{J} \omega_j Z_{jt}^{\sigma-1}\right]^{\frac{1}{\sigma-1}} = \lim_{\sigma \to 1} \exp\left[\frac{1}{\sigma-1} \log \sum_{j=1}^{J} \omega_j Z_{jt}^{\sigma-1}\right] \\
&= \exp\left[\lim_{\sigma \to 1} \frac{\sum_{j=1}^{J} \omega_j \log Z_{jt}}{\sum_{j=1}^{J} \omega_j Z_{jt}^{\sigma-1}}\right] = \prod_{j=1}^{J} Z_{jt}^{\omega_j}
\end{aligned}$$

then

$$\frac{\dot{Z}_t}{Z_t} = \sum_{j=1}^{J} \omega_j \frac{\dot{Z}_{jt}}{Z_{jt}}$$

Now we consider the labor share of each sector. Since

$$Y_{jt} = \frac{Z_{jt}^{\sigma}\omega_j}{\sum_{j=1}^{J} Z_{jt}^{\sigma-1}\omega_j} A K_t^{\alpha} N_t^{1-\alpha}$$
$$Y_{jt} = A\left(\frac{K_t}{N_t}\right)^{\alpha} X_{jt}^{1-\alpha} N_{jt} = A\left(\frac{K_t}{N_t}\right)^{\alpha} Z_{jt} N_{jt}$$

then

$$N_{jt} = \frac{Z_{jt}^{\sigma}\omega_j}{\sum_{j=1}^{J} Z_{jt}^{\sigma-1}\omega_j} \frac{A K_t^{\alpha} N_t^{1-\alpha}}{A\left(\frac{K_t}{N_t}\right)^{\alpha} Z_{jt}} = \frac{Z_{jt}^{\sigma-1}\omega_j}{Z_t^{\sigma-1}} N_t$$

Since $N_t = 1$, then

$$\frac{\dot{N}_{jt}}{N_{jt}} = (\sigma - 1)\left(\frac{\dot{Z}_{jt}}{Z_{jt}} - \frac{\dot{Z}_t}{Z_t}\right)$$

is a constant, which implies there are no structure change.

(b) $\sigma > 1$, we have

$$\log Z_t = \frac{1}{\sigma - 1} \log\left[\sum_{j=1}^{J} \omega_j Z_{jt}^{\sigma-1}\right]$$

then

$$\frac{\dot{Z}_t}{Z_t} = \frac{1}{\sigma - 1} \frac{\sum_{j=1}^{J} \omega_j (\sigma - 1) Z_{jt}^{\sigma-2} \dot{Z}_{jt}}{\sum_{j=1}^{J} \omega_j Z_{jt}^{\sigma-1}} = \frac{\sum_{j=1}^{J} \omega_j Z_{jt}^{\sigma-1} \frac{\dot{Z}_{jt}}{Z_{jt}}}{\sum_{j=1}^{J} \omega_j Z_{jt}^{\sigma-1}}$$

Let $\max_{1\leqslant j\leqslant t}\left\{\frac{\dot{Z}_{jt}}{Z_{jt}}\right\} := \frac{\dot{Z}_{\ell t}}{Z_{\ell t}}$ (and assume the maximizer j is unique), then $Z_{\ell t}$ grows the most fast, therefore, as $t \to \infty$,

$$\frac{\dot{Z}_t}{Z_t} \to \frac{\dot{Z}_{\ell t}}{Z_{\ell t}}$$

If $j \neq \ell$, for sufficient large t, we have

$$\frac{\dot{N}_{jt}}{N_{jt}} = (\sigma - 1)\left[\frac{\dot{Z}_{jt}}{Z_{jt}} - \frac{\dot{Z}_t}{Z_t}\right] < 0$$

and hence

$$N_{jt} \to 0$$

(c) $\sigma < 1$, we still have

$$\frac{\dot{Z}_t}{Z_t} \to \frac{\dot{Z}_{\ell t}}{Z_{\ell t}}$$

then

$$\frac{\dot{N}_{jt}}{N_{jt}} = (\sigma - 1)\left[\frac{\dot{Z}_{jt}}{Z_{jt}} - \frac{\dot{Z}_t}{Z_t}\right] > 0$$

and thus

$$\frac{N_{jt}}{N_t} \to 1$$

3. Intuition.

    (a) $\sigma > 1$, the goods from each sector are complementary, then every intermediate good cannot be taken place, and thus we are concerned about the sector with slowest growth, in order to maximize the aggregate output.

    (b) $0 < \sigma < 1$, the goods from each sector are substitute, then we can abandon some sectors with slow growth rate to maximize the aggregate output.

    (c) $\sigma = 1$, $Y_t = \left[\sum_{j=1}^{J} \omega_j^{\frac{1}{\sigma}} Y_{jt}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \to \prod_{J=1}^{J} Y_{jt}^{\omega_j}$ takes Cobb-Douglas function form, there are no structure change.

Lecture 6

Ramsey Model

6.1 The Calculus of Variation

1. General problem.

$$\max_{x(t)\in X} J = \int_a^b f\left(t, x\left(t\right), \dot{x}\left(t\right)\right) \mathrm{d}t$$
$$\text{s.t. } x\left(a\right) = \alpha, x\left(b\right) = \beta$$

   where a, b are constants, $x\left(t\right) : \mathbb{R}_+ \to \mathbb{R}^n$, and $X = C^1\left[a, b\right]$.

2. General solution: derive the Euler equation. Suppose $x^*\left(t\right) \in X$ maximizes (or minimizes) J, and it is unique. Then consider a function $h\left(t\right) \in X$, such that

$$h\left(a\right) = 0, h\left(b\right) = 0$$

   and for $\varepsilon > 0$, let

$$x_\varepsilon\left(t\right) = x^*\left(t\right) + \varepsilon h\left(t\right)$$

   Note that $x_\varepsilon\left(a\right) = \alpha, x_\varepsilon\left(b\right) = \beta$ satisfies the constraint, and $J\left[x_\varepsilon\right]$ attains its maximum (minimum) when $\varepsilon = 0$, therefore

$$\left.\frac{\partial J\left[x_\varepsilon\right]}{\partial \varepsilon}\right|_{\varepsilon=0} = 0$$

   We can regard $h\left(t\right)$ as a perturbation that makes $x_\varepsilon\left(t\right)$ is not a maximizer (or minimizer). And then,

$$\begin{aligned}
\frac{\partial J\left[x_\varepsilon\right]}{\partial \varepsilon} &= \frac{\partial}{\partial \varepsilon}\int_a^b f\left(t, x^*\left(t\right) + \varepsilon h\left(t\right), \dot{x}^*\left(t\right) + \varepsilon \dot{h}\left(t\right)\right) \mathrm{d}t \\
&= \int_a^b \frac{\partial}{\partial \varepsilon} f\left(t, x^*\left(t\right) + \varepsilon h\left(t\right), \dot{x}^*\left(t\right) + \varepsilon \dot{h}\left(t\right)\right) \mathrm{d}t \\
&= \int_a^b f'_x \cdot h\left(t\right) + f'_{\dot{x}} \cdot \dot{h}\left(t\right) \mathrm{d}t
\end{aligned}$$

By integration by parts, and $h(b) = h(a) = 0$, then

$$\begin{aligned}\int_a^b f'_{\dot{x}} \cdot \dot{h}(t)\,\mathrm{d}t &= \int_a^b f'_{\dot{x}} \mathrm{d}h(t) \\ &= f'_{\dot{x}} h(t)\mid_a^b - \int_a^b h(t) \frac{\mathrm{d}f'_{\dot{x}}}{\mathrm{d}t}\mathrm{d}t \\ &= -\int_a^b h(t) \frac{\mathrm{d}f'_{\dot{x}}}{\mathrm{d}t}\mathrm{d}t\end{aligned}$$

then

$$\begin{aligned}\frac{\partial J[x_\varepsilon]}{\partial \varepsilon} &= \int_a^b f'_x \cdot h(t) - h(t)\frac{\mathrm{d}f'_{\dot{x}}}{\mathrm{d}t}\mathrm{d}t \\ &= \int_a^b h(t)\left[f'_x - \frac{\mathrm{d}f'_{\dot{x}}}{\mathrm{d}t}\right]\mathrm{d}t\end{aligned}$$

and thus

$$\left.\frac{\partial J[x_\varepsilon]}{\partial \varepsilon}\right|_{\varepsilon=0} = \left.\int_a^b h(t)\left[f'_x - \frac{\mathrm{d}f'_{\dot{x}}}{\mathrm{d}t}\right]\mathrm{d}t\right|_{\varepsilon=0} = 0$$

which holds for any $h(t) \in X$ with $h(a) = h(b) = 0$, then

$$f'_x - \frac{\mathrm{d}f'_{\dot{x}}}{\mathrm{d}t} = 0, \forall t \in [a, b]$$

Put it in detail,

$$\frac{\partial f(t, x(t), \dot{x}(t))}{\partial x(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial f(t, x(t), \dot{x}(t))}{\partial \dot{x}(t)}\right]$$

This equation is called Euler's equation, which is a necessary condition for the optimization problem.

3. An Example.

Find the curve which joints two points A and B on the plane with the minimum distance.

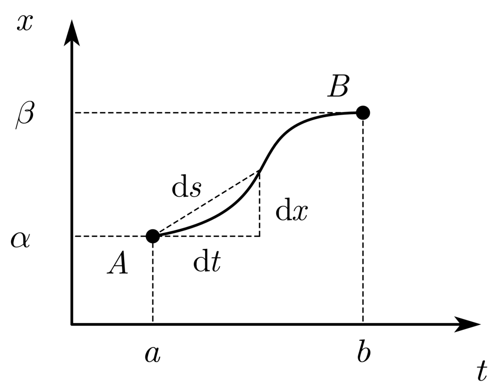

Fig. 6.1. Example

For an infinitesimal movement,

$$\mathrm{d}s = \sqrt{(\mathrm{d}t)^2 + (\mathrm{d}x)^2} = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}\mathrm{d}t = \sqrt{1 + \dot{x}^2(t)}\mathrm{d}t$$

Then the problem becomes

$$\min_{x(t)\in X} J = \int_a^b \sqrt{1 + [\dot{x}(t)]^2}\mathrm{d}t$$
$$\text{s.t. } x(a) = \alpha, x(b) = \beta$$

Let $f = \sqrt{1 + [\dot{x}(t)]^2}$, by Euler equation,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\dot{x}(t)}{\sqrt{1 + [\dot{x}(t)]^2}}\right] \Longrightarrow \frac{\dot{x}(t)}{\sqrt{1 + [\dot{x}(t)]^2}} = const \Longrightarrow \dot{x}(t) = const$$

Then $x(t)$ has the form

$$x(t) = \gamma t + \sigma$$

by $x(a) = \alpha, x(b) = \beta$, we can solve for γ, σ, and get

$$x(t) = \frac{\beta - \alpha}{b - a}x + \frac{a\beta - b\alpha}{a - b}$$

6.2 Ramsey-Cass-Koopmans Model

6.2.1 Set-ups

1. Set-ups.

   (a) The environment: The economy is composed of many identical firms and identical households. The number of firms and households are sufficiently large so that they are all price takers.

   (b) Firms:

      i. Each firm has access to the production technology.

      $$Y = F(K, AL)$$

      and $\forall t > 0, F(tK, tAL) = tF(K, AL)$.

      ii. The profit function of the representative firm in each period is

      $$\Pi_t = Y_t - (r_t + \delta)K_t - w_t L_t$$

   (c) Households:

      i. There are H households, H is a constant, and each household has M_t identical members and $\frac{\dot{M}_t}{M_t} = n$ is a constant.

ii. Each member supplies one unit of labor in one period, then

$$L_t = H \cdot M_t$$

iii. Household maximizes

$$U = \int_{t=0}^{\infty} e^{-\rho t} U(C_t) M_t \mathrm{d}t = \int_{t=0}^{\infty} e^{-\rho t} U(C_t) \frac{L_t}{H} \mathrm{d}t$$

where ρ is the discount rate, C_t is the consumption of a member, and we add up those utility for each person of each period, to get the utility of the household.

iv. Assume $H = 1$ as a normalization.

v. $U(C) = \frac{C^{1-\theta}}{1-\theta}$.

2. Discussion on the utility function $U(C) = \frac{C^{1-\theta}}{1-\theta}$.

(a) $U'(C) = C^{-\theta}, U''(C) = -\theta C^{-\theta-1}$, then U is concave, iff $\theta \geqslant 0$.

(b) The coefficient of relative risk aversion (CRRA) is constant.

$$\frac{-U''(C_t) \cdot C_t}{U'(C_t)} = \theta$$

(c) Given $u(c_1, c_2)$, the elasticity of substitution is defined as

$$\varepsilon_{12} := -\left.\frac{\mathrm{d}\log(c_1/c_2)}{\mathrm{d}\log(p_1/p_2)}\right|_{u_0 = u(c_1, c_2)}$$

which captures how sensitive that the change of p_1 has an influence on c_2, provided that we keep utility as a constant. It is different from marginal rate of substitution, by the F.O.C.s in consumer's problem,

$$\frac{u'_{c_1}}{p_1} = \frac{u'_{c_2}}{p_2} \Longrightarrow \frac{p_1}{p_2} = \frac{u'_{c_1}}{u'_{c_2}} = \mathrm{MRS}_{1,2}$$

Therefore,

$$\varepsilon_{12} = -\frac{\mathrm{d}\log(c_1/c_2)}{\mathrm{d}\log(p_1/p_2)} = -\frac{\mathrm{d}\log(c_1/c_2)}{\mathrm{d}\log(u_{c_1}/u_{c_2})}$$

If we define

$$u(C_t, C_{t+j}) = U(c_t) + e^{-j\rho} U(c_{t+j})$$

We can likewise define the elasticity of intertemporal substitution between consumption at any two points in time as

$$\varepsilon_{t,t+j} := -\left.\frac{\mathrm{d}\log(C_t/C_{t+j})}{\mathrm{d}\log(p_t/p_{t+j})}\right|_{u_0 = u(C_t, C_{t+j})}$$

and thus,

$$\varepsilon_{t,t+j} = -\frac{\mathrm{d}\log(C_t/C_{t+j})}{\mathrm{d}\log\left(u'_{c_t}/u'_{c_{t+j}}\right)} = -\frac{\mathrm{d}\log(C_t/C_{t+j})}{\mathrm{d}\log\left(\frac{e^{-j\rho}C_t^{-\theta}}{C_{t+j}^{-\theta}}\right)}$$

where

$$\mathrm{d}\log\left(\frac{e^{-j\rho}C_t^{-\theta}}{C_{t+j}^{-\theta}}\right)=\mathrm{d}\left[-j\rho-\theta\log\frac{C_t}{C_{t+j}}\right]=-\theta\mathrm{d}\log\frac{C_t}{C_{t+j}}$$

then

$$\varepsilon_{t,t+j}=\frac{1}{\theta}$$

is a constant.

3. Firm's problem.

$$\max F\left(K,AL\right)-wL-\left(r+\delta\right)K$$

F.O.C.s

$$\begin{cases}F'_K=r+\delta\\ AF'_L=w\end{cases}$$

by the constant return to scale assumption,

$$F\left(tK,tAL\right)=tF\left(K,AL\right),\forall t>0$$

which yields,

$$F'_K\cdot K+F'_L\cdot AL=F\left(K,AL\right)$$

then,

$$wL+\left(r+\delta\right)K=F\left(K,AL\right)$$

Therefore, the profit for each firm must be zero.

4. Rewrite the production function.

Define $k:=\frac{K}{AL}$, and

$$f\left(k\right):=F\left(k,1\right)=F\left(\frac{K}{AL},1\right)=\frac{1}{AL}F\left(K,AL\right)$$

Recall the F.O.C.s

$$\begin{cases}F'_K=r+\delta\\ AF'_L=w\end{cases}$$

and by

$$F\left(K,AL\right)=AL\cdot f\left(k\right)$$

we have

$$\begin{cases}F'_K=AL\cdot f'\left(k\right)\cdot\frac{1}{AL}=f'\left(k\right)\\ AF'_L=Af\left(k\right)+AL\cdot f'\left(k\right)\cdot\left(-\frac{K}{AL^2}\right)=Af\left(k\right)-\frac{K}{L}f'\left(k\right)\end{cases}$$

then F.O.C.s become

$$\begin{cases}r+\delta=f'\left(k\right)\\ w=A\left[f\left(k\right)-kf'\left(k\right)\right]\end{cases}$$

5. Household's behavior and social planner's problem.

(a) Budget constraint.

i. In discrete time, 1 unit of goods invested at time zero yields

$$\prod_{\tau=0}^{t}(1+r_\tau)$$

In continuous time, firstly, we use $e^{r_\tau} \approx 1 + r_\tau$, then it becomes

$$\prod_{\tau=0}^{t}(1+r_\tau) \approx \prod_{\tau=0}^{t} e^{r_\tau} = e^{\sum_{\tau=0}^{t} r_\tau}$$

where we take that every period persists $(\tau+1)-\tau$, if this becomes infinitesimal, the return becomes

$$e^{R_t} := e^{\int_0^t r_\tau \mathrm{d}\tau}$$

ii. Budget constraint 1: suppose the representative household's initial capital holding is K_0, and K_s at time $t = s$.

$$e^{-R_s}K_s + \int_0^s e^{-R_t}C_tL_t\mathrm{d}t = K_0 + \int_0^s e^{-R_t}w_tL_t\mathrm{d}t$$

The LHS is present value of money to spend or wait to spend, the RHS is how much the household earn.

iii. Budget constraint 2: suppose the representative household's initial capital holding is K_0, and K_s at time $t = s$. Here is another way to write the budget constraint. Recall that

$$e^{R_t} = e^{\int_0^t r_\tau \mathrm{d}\tau} \Longrightarrow \mathrm{d}e^{R_t} = e^{R_t}\mathrm{d}\int_0^t r_\tau \mathrm{d}\tau = r_t e^{R_t}\mathrm{d}t$$

Differentiating both sides of budget constraint w.r.t. s, then

$$-r_s e^{-R_s}K_s + e^{-R_s}\dot{K}_s + e^{-R_s}C_sL_s = e^{-R_s}w_sL_s$$

and thus, for any $s \geqslant 0$,

$$\dot{K}_s + C_sL_s = w_sL_s + r_sK_s$$

change the notation as

$$\dot{K}_t + C_tL_t = w_tL_t + r_tK_t$$

The LSH is the total expenditure, the RHS is the total income. Or in another view, in every period, consumption C_tL_t reduces K_t, and income (labor income and investment income) increases K_t.

iv. The no-Ponzi-game condition:
Recall the original budge constraint,

$$e^{-R_s}K_s + \int_0^s e^{-R_t}C_tL_t\mathrm{d}t = K_0 + \int_0^s e^{-R_t}w_tL_t\mathrm{d}t$$

Let $s \to \infty$,

$$\lim_{s\to\infty} e^{-R_s}K_s + \lim_{s\to\infty}\int_0^s e^{-R_t}C_tL_t\mathrm{d}t = K_0 + \lim_{s\to\infty}\int_0^s e^{-R_t}w_tL_t\mathrm{d}t$$

we need to impose

$$\lim_{s\to\infty} e^{-R_s}K_s \geqslant 0$$

If $\lim_{s\to\infty} e^{-R_s}K_s < 0$, then we can borrow a lot of money to increase the utility without paying back, which is not allowed. And it implies

$$\lim_{s\to\infty}\int_0^s e^{-R_t}C_tL_t\mathrm{d}t \leqslant K_0 + \lim_{s\to\infty}\int_0^s e^{-R_t}w_tL_t\mathrm{d}t$$

which means the lifetime present value of consumption cannot exceed the lifetime income and initial wealth.

(b) Transformation of objective function.

Consider the whole economy, we needn't get H involved.

$$U = \int_{t=0}^{\infty} e^{-\rho t}U(C_t)\,L_t\mathrm{d}t = \int_{t=0}^{\infty} e^{-\rho t}\frac{C_t^{1-\theta}}{1-\theta}L_t\mathrm{d}t$$

Recall that $k_t = \frac{K_t}{A_tL_t}$ and moreover, let $c_t := \frac{C_tL_t}{A_tL_t} = \frac{C_t}{A_t}$, and note that

$$M_t = e^{nt}M_0, L_t = H\cdot M_t \Longrightarrow L_t = e^{nt}L_0$$

Since $A_t = e^{gt}A_0, L_t = e^{nt}L_0$, then the objective function becomes

$$\begin{aligned} U &= \int_{t=0}^{\infty} e^{-\rho t}\frac{A_t^{1-\theta}c_t^{1-\theta}}{1-\theta}L_t\mathrm{d}t \\ &= A_0^{1-\theta}L_0\int_{t=0}^{\infty} e^{-\rho t}\frac{e^{(1-\theta)gt}c_t^{1-\theta}}{1-\theta}e^{nt}\mathrm{d}t \\ &= A_0^{1-\theta}L_0\int_{t=0}^{\infty} e^{(n+(1-\theta)g-\rho)t}\frac{c_t^{1-\theta}}{1-\theta}\mathrm{d}t \end{aligned}$$

Let $\beta = \rho - n - (1-\theta)\,g$, then

$$U = A_0^{1-\theta}L_0\int_{t=0}^{\infty} e^{-\beta t}\frac{c_t^{1-\theta}}{1-\theta}\mathrm{d}t$$

moreover, we only need to maximize

$$\int_{t=0}^{\infty} e^{-\beta t}\frac{c_t^{1-\theta}}{1-\theta}\mathrm{d}t$$

or let $A_0 = L_0 = 1$.

(c) Transformation of budge constraint 1 and No-Ponzi-game condition.

Recall that

$$e^{-R_s}K_s + \int_0^s e^{-R_t}C_tL_t\mathrm{d}t = K_0 + \int_0^s e^{-R_t}w_tL_t\mathrm{d}t$$

since $C_t = A_t c_t = A_0 e^{gt} c_t$, $L_t = L_0 e^{nt}$, $K_t = A_t L_t k_t$, and let

$$\omega_t = \frac{w_t}{A_t} \Longrightarrow w_t = \omega_t A_0 e^{gt}$$

then

$$e^{-R_s} A_0 L_0 e^{(g+n)s} k_s + A_0 L_0 \int_0^s e^{-R_t} e^{(g+n)t} c_t \mathrm{d}t = K_0 + A_0 L_0 \int_0^s e^{-R_t} \omega_t e^{(n+g)t} \mathrm{d}t$$

and thus

$$e^{-R_s} e^{(g+n)s} k_s + \int_0^s e^{-R_t} e^{(g+n)t} c_t \mathrm{d}t = k_0 + \int_0^s e^{-R_t} \omega_t e^{(n+g)t} \mathrm{d}t$$

No-Ponzi-game condition becomes

$$\lim_{s\to\infty} e^{-R_s} K_s = A_0 L_0 \lim_{s\to\infty} e^{-R_s} e^{(g+n)s} k_s \geqslant 0$$

Actually, $\lim_{s\to\infty} e^{-R_s} K_s = 0$, and let $s \to \infty$, the constraint becomes

$$\int_0^\infty e^{-R_t} e^{(g+n)t} c_t \mathrm{d}t = k_0 + \int_0^\infty e^{-R_t} \omega_t e^{(n+g)t} \mathrm{d}t$$

since if $\lim_{s\to\infty} e^{-R_s} K_s > 0$, then we can reduce the savings by consuming more to get a higher utility.

(d) Transformation of budge constraint 2.

Since firms get zero profit, then

$$w_t L_t + (r_t + \delta) K_t = F(K_t, A_t L_t) = Y_t$$

together with the budget constraint

$$\dot{K}_t + C_t L_t = w_t L_t + r_t K_t$$

we have

$$\dot{K}_t + C_t L_t = Y_t - \delta K_t$$

Then divide the equation by $A_t L_t$ on both sides,

$$\frac{\dot{K}_t}{A_t L_t} + c_t = f(k_t) - \delta k_t$$

Now we need to write $\frac{\dot{K}_t}{A_t L_t}$ in terms of k_t,

$$K_t = A_t L_t k_t \Longrightarrow \dot{K}_t = \dot{A}_t L_t k_t + A_t \dot{L}_t k_t + A_t L_t \dot{k}_t$$

then

$$\frac{\dot{K}_t}{A_t L_t} = \frac{\dot{A}_t L_t k_t + A_t \dot{L}_t k_t + A_t L_t \dot{k}_t}{A_t L_t} = (g+n) k_t + \dot{k}_t$$

Therefore,

$$c_t + \dot{k}_t = f(k_t) - (g+n+\delta) k_t$$

Note that we still need to impose $\lim_{s\to\infty} e^{-R_s} e^{(g+n)s} k_s \geqslant 0$.

(e) Now, household solves the decentralized problem

$$\max_{\{c_t\}_{t\geqslant 0}^{\infty}} \int_0^{\infty} e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} \mathrm{d}t$$
$$\text{s.t.} \quad \int_0^{s} e^{-R_t} e^{(g+n)t} \mathrm{d}t = k_0 + \int_0^{s} e^{-R_t} \omega_t e^{(n+g)t} \mathrm{d}t$$

The social planner solves the centralized problem

$$\max_{\{c_t\}_{t\geqslant 0}^{\infty}} \int_0^{\infty} e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} \mathrm{d}t$$
$$\text{s.t.} \ c_t + \dot{k}_t = f(k_t) - (g+n+\delta) k_t$$

Then we will solve these 2 problems respectively and get the same answer.

6.2.2 Solve the Decentralized Problem

1. Solution 1 to decentralized problem by a static way (not rigorous).

$$\max_{\{c_t\}_{t\geqslant 0}^{\infty}} \int_0^{\infty} e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} \mathrm{d}t$$
$$\text{s.t.} \quad \int_0^{\infty} e^{-R_t} e^{(g+n)t} c_t \mathrm{d}t = k_0 + \int_0^{\infty} e^{-R_t} \omega_t e^{(n+g)t} \mathrm{d}t$$

It can be solved by static method, which is not rigorous, but the final result is correct.

Form the Lagrangian,

$$\mathcal{L} = \int_0^{\infty} e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} \mathrm{d}t + \lambda \left[k_0 + \int_0^{\infty} e^{-R_t} \omega_t e^{(n+g)t} \mathrm{d}t - \int_0^{\infty} e^{-R_t} e^{(g+n)t} c_t \mathrm{d}t \right]$$

F.O.C.

$$e^{-\beta t} c_t^{-\theta} = \lambda e^{-R_t} e^{(n+g)t}$$

then

$$c_t^{-\theta} = \lambda e^{-R_t} e^{(n+g+\beta)t}$$

taking log and differentiate w.r.t. t, recall that $R_t = \int_0^t r_\tau \mathrm{d}\tau$ we get

$$-\theta \log c_t = \log \lambda - R_t + (n+g+\beta)\, t \Longrightarrow -\theta \frac{\dot{c}_t}{c_t} = -r_t + (n+g+\beta)$$

and thus

$$\dot{c}_t = \frac{1}{\theta} (r_t - n - g - \beta)\, c_t$$

It's the Euler equation for this problem, it implies the smaller θ, the more sensitive consumption to interest rate change.

Recall that one of the F.O.C.s for the firms

$$r_t + \delta = f'(k_t)$$

Therefore,

$$\dot{c}_t = \frac{1}{\theta} (f'(k_t) - \delta - n - g - \beta)\, c_t$$

which is consistent with what we will deduce in Solution 2.

2. Solution 2 to decentralized problem by calculus of variation.

   The key in decentralized problem is that the household take w_t and r_t as given, therefore, we cannot plug $\begin{cases} r_t = f'(k_t) - \delta \\ w_t = A_t\left[f(k_t) - k_t f'(k_t)\right] \end{cases}$ into the constraint. In other words, we only consider the budget constraint

$$\dot{K}_t + C_t L_t = w_t L_t + r_t K_t$$

   and thus, we have the constraint,

$$c_t + \dot{k}_t = \omega_t - (g+n)\, k_t + r_t k_t$$

   which is equivalent to

$$c_t + \dot{k}_t = f(k_t) - (g+n+\delta)\, k_t$$

   we just take r_t and w_t given.

   Then the problem becomes

$$\max_{\{c_t\}_{t\geqslant 0}^{\infty}} \int_0^{\infty} e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} \mathrm{d}t$$
$$\text{s.t. } c_t + \dot{k}_t = \omega_t - (g+n)\, k_t + r_t k_t$$

   we can solve it by both optimal control and calculus of variation. Here gives a solution by calculus of variation.

   Let

$$f\left(t, k_t, \dot{k}_t\right) = e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} = e^{-\beta t} \frac{\left(\omega_t - (g+n)\, k_t + r_t k_t - \dot{k}_t\right)^{1-\theta}}{1-\theta}$$

   then

$$\begin{aligned} \frac{\partial f}{\partial k_t} &= e^{-\beta t} \cdot (1-\theta)\, c_t^{-\theta} \cdot \left[r_t - (g+n)\right] \\ \frac{\partial f}{\partial \dot{k}_t} &= -e^{-\beta t} \cdot (1-\theta)\, c_t^{-\theta} \\ \frac{\partial^2 f}{\partial t \partial \dot{k}_t} &= \beta e^{-\beta t} (1-\theta)\, c_t^{-\theta} + e^{-\beta t} \cdot (1-\theta)\, \theta c_t^{-\theta-1} \cdot \dot{c}_t \end{aligned}$$

   since the Euler equation is

$$\frac{\partial f}{\partial k_t} = \frac{\partial}{\partial t}\left[\frac{\partial f}{\partial \dot{k}_t}\right]$$

   we have

$$r_t - (g+n) = \beta + \theta c_t^{-1} \dot{c}_t \Longrightarrow \frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - g - n - \beta)$$

   Recall that one of the F.O.C.s for the firms

$$r_t + \delta = f'(k_t)$$

   Therefore,

$$\dot{c}_t = \frac{1}{\theta}\left(f'(k_t) - \delta - n - g - \beta\right) c_t$$

6.2.3 Solve the Centralized Problem

1. Solution 1 to centralized problem by optimal control.

$$\max_{\{c_t\}_{t\geqslant 0}^{\infty}} \int_0^{\infty} e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} \mathrm{d}t$$
$$\text{s.t. } c_t + \dot{k}_t = f(k_t) - (g+n+\delta) k_t$$

We apply the current value Hamilton method by defining

$$\begin{aligned} J &= \int_0^{\infty} e^{-\beta t} \left\{ \frac{c_t^{1-\theta}}{1-\theta} + \lambda_t \left[f(k_t) - (g+n+\delta) k_t - c_t - \dot{k}_t \right] \right\} \mathrm{d}t \\ &= \int_0^{\infty} e^{-\beta t} \left\{ \frac{c_t^{1-\theta}}{1-\theta} + \lambda_t \left[f(k_t) - (g+n+\delta) k_t - c_t \right] \right\} \mathrm{d}t - \int_0^{\infty} e^{-\beta t} \lambda_t \dot{k}_t \mathrm{d}t \end{aligned}$$

and then consider the term

$$\begin{aligned} \int_0^{\infty} e^{-\beta t} \lambda_t \dot{k}_t \mathrm{d}t &= \int_0^{\infty} e^{-\beta t} \lambda_t \mathrm{d}k_t \\ &= e^{-\beta t} \lambda_t k_t \mid_0^{\infty} - \int_0^{\infty} k_t \mathrm{d}\left(e^{-\beta t} \lambda_t\right) \\ &= \lim_{t\to\infty} e^{-\beta t} \lambda_t k_t - \lambda_0 k_0 - \int_0^{\infty} \left(-\beta \lambda_t + \dot{\lambda}_t\right) e^{-\beta t} k_t \mathrm{d}t \end{aligned}$$

by assuming the transversality condition,

$$\lim_{t\to\infty} e^{-\beta t} \lambda_t k_t = 0$$

then

$$\begin{aligned} J = &\int_0^{\infty} e^{-\beta t} \left\{ \frac{c_t^{1-\theta}}{1-\theta} + \lambda_t \left[f(k_t) - (g+n+\delta) k_t - c_t \right] \right\} \mathrm{d}t \\ &+ \lambda_0 k_0 + \int_0^{\infty} \left(-\beta \lambda_t + \dot{\lambda}_t\right) e^{-\beta t} k_t \mathrm{d}t \\ = &\int_0^{\infty} e^{-\beta t} \left\{ \frac{c_t^{1-\theta}}{1-\theta} + \lambda_t \left[f(k_t) - (g+n+\delta+\beta) k_t - c_t + \dot{\lambda}_t k_t \right] \right\} \mathrm{d}t \end{aligned}$$

The F.O.C.s are

$$\begin{aligned} [c_t] &: c_t^{-\theta} - \lambda_t = 0 \\ [k_t] &: \lambda_t \left[f'(k_t) - (g+n+\delta+\beta) \right] + \dot{\lambda}_t = 0 \end{aligned}$$

or by current Hamiltonian method (It has no difference but help us skip the steps above),

$$J = \int_0^{\infty} e^{-\beta t} \left\{ \underbrace{\frac{c_t^{1-\theta}}{1-\theta} + \lambda_t \left[f(k_t) - (g+n+\delta) k_t - c_t \right]}_{\mathcal{H}_t} \right\} \mathrm{d}t - \int_0^{\infty} e^{-\beta t} \lambda_t \dot{k}_t$$

Then by the formula given by current Hamiltonian method,

$$\begin{cases} \frac{\partial \mathcal{H}_t}{\partial c_t} = 0 \\ \frac{\partial \mathcal{H}_t}{\partial k_t} = \beta \lambda_t - \dot{\lambda}_t \end{cases} \Longrightarrow \begin{cases} c_t^{-\theta} - \lambda_t = 0 \\ \lambda_t \left[f'(k_t) - (g+n+\delta) \right] = \beta \lambda_t - \dot{\lambda}_t \end{cases}$$

Therefore,

$$c_t^{-\theta} = \lambda_t \Longrightarrow -\theta \frac{\dot{c}_t}{c_t} = \frac{\dot{\lambda}_t}{\lambda_t}$$

then

$$-\frac{\dot{\lambda}_t}{\lambda_t} = f'(k_t) - (g+n+\delta+\beta) \Longrightarrow f'(k_t) - (g+n+\delta+\beta) = \theta \frac{\dot{c}_t}{c_t}$$

together with the constraint and transversality condition, we get

$$\begin{cases} \dot{c}_t = \frac{c_t}{\theta}\left[f'(k_t) - (g+n+\delta+\beta)\right] \\ \dot{k}_t = f(k_t) - (g+n+\delta)k_t - c_t \\ \lim_{t\to\infty} e^{-\beta t} c_t k_t = 0 \\ k_0 \text{ is given.} \end{cases}$$

2. Solution 2 to centralized problem by calculus of variation.

Recall for the general problem

$$\max_{x(t)\in X} J = \int_a^b f(t, x(t), \dot{x}(t))\, \mathrm{d}t$$
$$\text{s.t. } x(a) = \alpha, x(b) = \beta$$

we deduced Euler equation,

$$\frac{\partial f(t, x(t), \dot{x}(t))}{\partial x(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial f(t, x(t), \dot{x}(t))}{\partial \dot{x}(t)}\right]$$

Likewise, for our problem

$$\max_{\{c_t\}_{t\geqslant 0}^{\infty}} J = \int_0^{\infty} e^{-\beta t}\left\{\frac{c_t^{1-\theta}}{1-\theta} + \lambda_t\left[f(k_t) - (g+n+\delta)k_t - c_t - \dot{k}_t\right]\right\} \mathrm{d}t$$
$$\text{s.t. } \lim_{t\to\infty} e^{-\beta t} c_t k_t = 0, k_0 \text{ is given.}$$

we can apply this method as well.

Firstly, we deduce the Euler equation for c_t.

Let

$$f = e^{-\beta t}\left\{\frac{c_t^{1-\theta}}{1-\theta} + \lambda_t\left[f(k_t) - (g+n+\delta)k_t - c_t - \dot{k}_t\right]\right\}$$

then

$$\frac{\partial f}{\partial c_t} = e^{-\beta t}\left(c_t^{-\theta} - \lambda_t\right) \frac{\partial f}{\partial \dot{c}_t} = 0$$

and thus, the Euler equation for c_t is

$$c_t^{-\theta} = \lambda_t$$

As for k_t, we have

$$\frac{\partial f}{\partial k_t} = e^{-\beta t}\lambda_t\left(f'\left(k_t\right) - \left(g+n+\delta\right)\right)$$
$$\frac{\partial f}{\partial \dot{k}_t} = e^{-\beta t}\lambda_t$$
$$\frac{\partial}{\partial t}\frac{\partial f}{\partial \dot{k}_t} = -\beta e^{-\beta t}\lambda_t + e^{-\beta t}\dot{\lambda}_t$$

and thus, the Euler equation for k_t is

$$\lambda_t\left[f'\left(k_t\right) - \left(g+n+\delta\right)\right] = -\beta\lambda_t + \dot{\lambda}_t$$

or

$$\frac{\dot{\lambda}_t}{\lambda_t} = f'\left(k_t\right) - \left(g+n+\delta+\beta\right)$$

together with $c_t^{-\theta} = \lambda_t$, we still get

$$\begin{cases} \dot{c}_t = \frac{c_t}{\theta}\left[f'\left(k_t\right) - \left(g+n+\delta+\beta\right)\right] \\ \dot{k}_t = f\left(k_t\right) - \left(g+n+\delta\right)k_t - c_t \\ \lim_{t\to\infty} e^{-\beta t}c_t k_t = 0 \\ k_0 \text{ is given} \end{cases}$$

as well.

6.2.4 Steady State and More

1. Steady state: Phase diagram analysis.

   Now consider the system of ODEs.

$$\begin{cases} \dot{c}_t = \frac{c_t}{\theta}\left[f'\left(k_t\right) - \left(g+n+\delta+\beta\right)\right] \\ \dot{k}_t = f\left(k_t\right) - \left(g+n+\delta\right)k_t - c_t \end{cases}$$

   Let $\dot{c}_t = 0, \dot{k}_t = 0$, we solve for c^{ss} and k^{ss},

$$\begin{cases} f'\left(k^{ss}\right) - \left(g+n+\delta+\beta\right) = 0 \\ f\left(k^{ss}\right) - \left(g+n+\delta\right)k_t - c^{ss} = 0 \end{cases}$$

   then

$$\begin{cases} k^{ss} = f'^{-1}\left(g+n+\delta+\beta\right) \\ c^{ss} = f\left(k^{ss}\right) - \left(g+n+\delta\right)k_t \end{cases}$$

   Note that $\lim_{k\to\infty} f'\left(k\right) = 0, \lim_{k\to 0} f'\left(k\right) = \infty$, and f' is strictly decreasing so that the inverse of f' exists.

   Moreover, the maximum of c^{ss} is realized at (the golden rule)

$$\left(g+n+\delta\right) - f'\left(k^{ss}\right) = 0 \Longrightarrow k^{ss} = f'^{-1}\left(g+n+\delta\right)$$

But $\beta = \rho - n - (1-\theta) g > 0$, then

$$f'^{-1}(g+n+\delta+\beta) < f'^{-1}(g+n+\delta)$$

therefore, the vertical line of k^{ss} should locate at the left-hand side of where the peak of c^{ss} is realized.

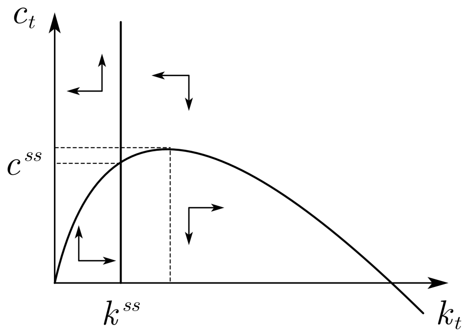

Fig. 6.2. Ramsey Model.

2. Some discussion.

   (a) The steady state satisfies the modified golden rule,

   $$f'(k_t) = \beta + \delta + g + n$$

   (b) The modified-golden-rule saving rate is

   $$\frac{s_t}{y_t} = 1 - \frac{c_t}{y_t} = 1 - \frac{f(k_t) - (g+n+\delta) k_t}{f(k_t)} = (g+n+\delta)\frac{k_t}{f(k_t)}$$

   Define $\alpha := \frac{f'(k_t)k_t}{f(k_t)}$, then

   $$\frac{s_t}{y_t} = (g+n+\delta)\frac{\alpha}{f'(k_t)} = \alpha\frac{g+n+\delta}{\beta+\delta+g+n} < \alpha$$

   (c) Some points about exogenous shock.

      i. Recall the lifetime utility is $\int_0^\infty e^{-\beta t}\frac{c_t^{1-\theta}}{1-\theta}\mathrm{d}t$, if $\beta\uparrow$, then $e^{-\beta t}\downarrow$, which implies consumers become impatient, and will increase their consumption now.

      ii. $\beta = \rho - n - (1-\theta) g$, then

      $$k^{ss} = f'^{-1}(g+n+\delta+\beta) = f'^{-1}(\rho+\delta+\theta g)$$

      and thus, if n and g change, k^{ss} won't change.

3. Linearization and algebraic analysis near the steady state.

   Recall that

   $$\begin{cases} \dot{c}_t = \frac{c_t}{\theta}\left[f'(k_t) - (g+n+\delta+\beta)\right] \\ \dot{k}_t = f(k_t) - (g+n+\delta) k_t - c_t \end{cases}$$

By the first-order Taylor's expansion near (k^{ss}, c^{ss}),

$$\begin{cases} \dot{c}_t \approx \frac{\partial \dot{c}_t}{\partial c_t}\left(c_t - c^{ss}\right) + \frac{\partial \dot{c}_t}{\partial k_t}\left(k_t - k^{ss}\right) \\ \dot{k}_t \approx \frac{\partial \dot{k}_t}{\partial c_t}\left(c_t - c^{ss}\right) + \frac{\partial \dot{k}_t}{\partial k_t}\left(k_t - k^{ss}\right) \end{cases} \iff \begin{bmatrix} \dot{c}_t \\ \dot{k}_t \end{bmatrix} = \left.\begin{pmatrix} \frac{\partial \dot{c}_t}{\partial c_t} & \frac{\partial \dot{c}_t}{\partial k_t} \\ \frac{\partial \dot{k}_t}{\partial c_t} & \frac{\partial \dot{k}_t}{\partial k_t} \end{pmatrix}\right|_{(k^{ss}, c^{ss})} \begin{bmatrix} c_t - c^{ss} \\ k_t - k^{ss} \end{bmatrix}$$

then the Jacobian matrix is

$$\begin{aligned} \begin{pmatrix} \frac{\partial \dot{c}_t}{\partial c_t} & \frac{\partial \dot{c}_t}{\partial k_t} \\ \frac{\partial \dot{k}_t}{\partial c_t} & \frac{\partial \dot{k}_t}{\partial k_t} \end{pmatrix} &= \begin{pmatrix} \frac{1}{\theta}\left[f'\left(k^{ss}\right) - \left(g+n+\delta+\beta\right)\right] & \frac{c^{ss}}{\theta} f''\left(k^{ss}\right) \\ -1 & f'\left(k^{ss}\right) - \left(g+n+\delta\right) \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{c^{ss}}{\theta} f''\left(k^{ss}\right) \\ -1 & \beta \end{pmatrix} \end{aligned}$$

since $f'' < 0$, let $\tau = -\frac{c^{ss}}{\theta} f''\left(k^{ss}\right) > 0$, then

$$\begin{bmatrix} \dot{c}_t \\ \dot{k}_t \end{bmatrix} = \begin{pmatrix} 0 & -\tau \\ -1 & \beta \end{pmatrix} \begin{bmatrix} c_t - c^{ss} \\ k_t - k^{ss} \end{bmatrix}$$

To decompose the matrix into

$$\begin{pmatrix} 0 & -\tau \\ -1 & \beta \end{pmatrix} = \boldsymbol{V} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \boldsymbol{V}^{-1}$$

we compute

$$\det \begin{pmatrix} \lambda & \tau \\ 1 & \lambda - \beta \end{pmatrix} = 0 \Longrightarrow \lambda^2 - \beta\lambda - \tau = 0 \Longrightarrow \begin{cases} \lambda_1 = \frac{\beta - \sqrt{\beta^2 + 4\tau}}{2} < 0 \\ \lambda_2 = \frac{\beta + \sqrt{\beta^2 + 4\tau}}{2} > 0 \end{cases}$$

since $\beta^2 + 4\tau > 0$, and $\lambda_1, \lambda_2 \in \mathbb{R}$. Then we can solve for eigenvectors, For $\lambda_1 = \frac{\beta - \sqrt{\beta^2 + 4\tau}}{2}$,

$$\begin{bmatrix} \frac{\beta - \sqrt{\beta^2 + 4\tau}}{2} & \tau \\ 1 & \frac{-\beta - \sqrt{\beta^2 + 4\tau}}{2} \end{bmatrix}$$

and

$$\frac{-\beta - \sqrt{\beta^2 + 4\tau}}{2} - \frac{2\tau}{\beta - \sqrt{\beta^2 + 4\tau}} = \frac{\beta^2 + 4\tau - \beta^2 - 4\tau}{2\left(\beta - \sqrt{\beta^2 + 4\tau}\right)} = 0$$

the corresponding eigenvector is

$$\begin{bmatrix} \frac{\beta + \sqrt{\beta^2 + 4\tau}}{2} \\ 1 \end{bmatrix}$$

For $\lambda_2 = \frac{\beta + \sqrt{\beta^2 + 4\tau}}{2}$,

$$\begin{bmatrix} \frac{\beta + \sqrt{\beta^2 + 4\tau}}{2} & \tau \\ 1 & \frac{-\beta + \sqrt{\beta^2 + 4\tau}}{2} \end{bmatrix}$$

likewise, the corresponding eigenvector is

$$\begin{bmatrix} \frac{\beta - \sqrt{\beta^2 + 4\tau}}{2} \\ 1 \end{bmatrix}$$

Therefore,

$$\boldsymbol{V} = \begin{pmatrix} \frac{\beta+\sqrt{\beta^2+4\tau}}{2} & \frac{\beta-\sqrt{\beta^2+4\tau}}{2} \\ 1 & 1 \end{pmatrix}$$

and

$$\boldsymbol{V}^{-1} = \frac{\boldsymbol{V}^*}{\det(\boldsymbol{V})} = \frac{\begin{pmatrix} 1 & -\frac{\beta-\sqrt{\beta^2+4\tau}}{2} \\ -1 & \frac{\beta+\sqrt{\beta^2+4\tau}}{2} \end{pmatrix}}{\sqrt{\beta^2+4\tau}} = \begin{pmatrix} \frac{1}{\sqrt{\beta^2+4\tau}} & -\frac{\beta-\sqrt{\beta^2+4\tau}}{2\sqrt{\beta^2+4\tau}} \\ \frac{-1}{\sqrt{\beta^2+4\tau}} & \frac{\beta+\sqrt{\beta^2+4\tau}}{2\sqrt{\beta^2+4\tau}} \end{pmatrix}$$

then,

$$\boldsymbol{V}^{-1} \begin{bmatrix} \dot{c}_t \\ \dot{k}_t \end{bmatrix} = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \boldsymbol{V}^{-1} \begin{bmatrix} c_t - c^{ss} \\ k_t - k^{ss} \end{bmatrix}$$

let $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \boldsymbol{V}^{-1} \begin{bmatrix} c_t - c^{ss} \\ k_t - k^{ss} \end{bmatrix}$, and then

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \boldsymbol{V}^{-1} \begin{bmatrix} \dot{c}_t \\ \dot{k}_t \end{bmatrix} = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \Longrightarrow \begin{cases} x_1(t) = x_1(0)\, e^{\lambda_1 t} \\ x_2(t) = x_2(0)\, e^{\lambda_2 t} \end{cases}$$

Then by

$$\begin{bmatrix} c_t - c^{ss} \\ k_t - k^{ss} \end{bmatrix} = \boldsymbol{V} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we get the solutions

$$\begin{cases} c_t - c^{ss} = \frac{\beta+\sqrt{\beta^2+4\tau}}{2} x_1(0)\, e^{\lambda_1 t} + \frac{\beta-\sqrt{\beta^2+4\tau}}{2} x_2(0)\, e^{\lambda_2 t} \\ k_t - k^{ss} = x_1(0)\, e^{\lambda_1 t} + x_2(0)\, e^{\lambda_2 t} \end{cases}$$

Since $\lambda_2 > 0$, then we have $\lim_{t\to\infty} x_{2t} = \pm\infty$, to converges to the steady state, we need impose

$$x_2(0) = 0$$

Therefore,

$$\begin{cases} c_t - c^{ss} = \frac{\beta+\sqrt{\beta^2+4\tau}}{2} x_1(0)\, e^{\lambda_1 t} \\ k_t - k^{ss} = x_1(0)\, e^{\lambda_1 t} \end{cases}$$

and since

$$\begin{cases} x_1(t) = \frac{1}{\sqrt{\beta^2+4\tau}} (c_t - c^{ss}) - \frac{\beta-\sqrt{\beta^2+4\tau}}{2\sqrt{\beta^2+4\tau}} (k_t - k^{ss}) \\ x_2(t) = \frac{-1}{\sqrt{\beta^2+4\tau}} (c_t - c^{ss}) + \frac{\beta+\sqrt{\beta^2+4\tau}}{2\sqrt{\beta^2+4\tau}} (k_t - k^{ss}) \end{cases}$$

then $x_2(0) = 0 \Longrightarrow x_2(t) = 0$, which requires for all $t \geqslant 0$,

$$\frac{-1}{\sqrt{\beta^2+4\tau}} (c_t - c^{ss}) + \frac{\beta+\sqrt{\beta^2+4\tau}}{2\sqrt{\beta^2+4\tau}} (k_t - k^{ss}) = 0$$

or

$$c_t - c^{ss} = \frac{\beta+\sqrt{\beta^2+4\tau}}{2} (k_t - k^{ss})$$

therefore,

$$x_1(t) = \left[\frac{\beta + \sqrt{\beta^2 + 4\tau}}{2\sqrt{\beta^2 + 4\tau}} - \frac{\beta - \sqrt{\beta^2 + 4\tau}}{2\sqrt{\beta^2 + 4\tau}}\right](k_t - k^{ss}) = (k_t - k^{ss})$$

then

$$x_1(0) = k_1 - k^{ss}$$

and thus

$$\begin{cases} x_1(t) = e^{\lambda_1 t}(k_t - k^{ss}) \\ x_2(t) = 0 \end{cases}$$

Therefore,

$$\begin{cases} c_t - c^{ss} = \lambda_2 e^{\lambda_1 t}(k_1 - k^{ss}) \\ k_t - k^{ss} = e^{\lambda_1 t}(k_1 - k^{ss}) \end{cases}$$

Lecture 7

Business Cycle Accounting

7.1 An Example without Wedges

1. Set-ups.

   (a) A representative household who consumes, supplies labor and owns the capital stock, and thus the household is the equity holder of the firm.

   (b) Preference: $\sum_{t=0}^{\infty}\beta^t u\left(C_t,N_t\right)=\sum_{t=0}^{\infty}\beta^t\left[\ln C_t-\theta\frac{N_t^{1+\chi}}{1+\chi}\right],0<\beta<1$.

   (c) I_t is the investment at period t, then
   $$K_{t+1}=\left(1-\delta\right)K_t+I_t$$

   (d) Production: $Y_t=A_tK_t^{\alpha}N_t^{1-\alpha}$, with technology shock $\ln A_t\sim\mathrm{AR}\left(1\right)$,
   $$\ln A_{t+1}=\rho\ln A_t+\varepsilon_t,\left|\rho\right|<1,\varepsilon_t\sim N\left(0,\sigma^2\right)$$

   (e) Labor market and capital market are competitive with wage rate w_t, and rental rate R_t (they are relative price, since we normalized the price of consumption good as 1).

2. Household's problem.
   $$\max_{\{C_t,N_t,K_{t+1}\}_0^{\infty}}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left[\ln C_t-\theta\frac{N_t^{1+\chi}}{1+\chi}\right]$$
   $$\text{s.t. }C_t+K_{t+1}-\left(1-\delta\right)K_t\leqslant w_tN_t+R_tK_t+\Pi_t$$

   The bellman equation is
   $$V\left(K_t,A_t\right)=\max_{C_t,N_t,K_{t+1}}\left\{\ln C_t-\theta\frac{N_t^{1+\chi}}{1+\chi}+\beta\mathbb{E}\left[V\left(K_{t+1},A_{t+1}\right)\middle|A_t\right]\right\}$$
   $$\text{s.t. }C_t+K_{t+1}-\left(1-\delta\right)K_t\leqslant w_tN_t+R_tK_t+\Pi_t$$

   Form the Lagrangian,
   $$\begin{aligned}\mathcal{L}=&\ln C_t-\theta\frac{N_t^{1+\chi}}{1+\chi}+\beta\mathbb{E}\left[V\left(K_{t+1},A_{t+1}\right)\middle|A_t\right]\\&+\lambda_t\left[w_tN_t+R_tK_t+\Pi_t-C_t-K_{t+1}+\left(1-\delta\right)K_t\right]\end{aligned}$$

F.O.C.s:

$$\begin{aligned}
[C_t] :& \frac{1}{C_t} = \lambda_t \\
[N_t] :& \lambda_t w_t = \theta N_t^{\chi} \\
[K_{t+1}] :& \beta \mathbb{E}\left[\left. \frac{\partial V\left(K_{t+1}, A_{t+1}\right)}{\partial K_{t+1}} \right| A_t \right] = \lambda_t
\end{aligned}$$

By envelop theorem,

$$\begin{aligned}
\frac{\partial V\left(K_t, A_t\right)}{\partial K_t} &= \frac{\partial \mathcal{L}}{\partial K_t} \\
&= \beta \mathbb{E}\left[\left. \frac{\partial V\left(K_{t+1}, A_{t+1}\right)}{\partial K_{t+1}} \right| A_t \right] \cdot \frac{\partial K_{t+1}}{\partial K_t} + \lambda_t \left(R_t - \frac{\partial K_{t+1}}{\partial K_t} + (1-\delta) \right) \\
&= \lambda_t \left(R_t + (1-\delta) \right)
\end{aligned}$$

Then by rolling 1 more period,

$$\frac{\partial V\left(K_{t+1}, A_{t+1}\right)}{\partial K_{t+1}} = \lambda_{t+1} \left(R_{t+1} + (1-\delta) \right)$$

Therefore, the F.O.C. w.r.t. K_{t+1} becomes

$$\beta \mathbb{E}\left[\left. \lambda_{t+1} \left(R_{t+1} + (1-\delta) \right) \right| A_t \right] = \lambda_t$$

The Euler equation for consumption is

$$\beta \mathbb{E}\left[\left. \frac{1}{C_{t+1}} \left(R_{t+1} + (1-\delta) \right) \right| A_t \right] = \frac{1}{C_t}$$

Therefore,

$$\begin{cases} \frac{1}{C_t} w_t = \theta N_t^{\chi} \\ \beta \mathbb{E}\left[\left. \frac{1}{C_{t+1}} \left(R_{t+1} + (1-\delta) \right) \right| A_t \right] = \frac{1}{C_t} \end{cases}$$

3. The firm's problem.

$$\max_{N_t, K_t} \mathbb{E}_0 \sum_{t=0}^{\infty} M_t \left(A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t - R_t K_t \right)$$

where $M_t = \beta^t \frac{C_0}{C_t}$ is the stochastic discount factor for the equity holder of the firm. F.O.C.s are

$$\begin{aligned}
A_t \alpha \left(\frac{N_t}{K_t} \right)^{1-\alpha} &= R_t \\
A_t (1-\alpha) \left(\frac{K_t}{N_t} \right)^{\alpha} &= w_t
\end{aligned}$$

Note that

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha} = w_t N_t + R_t K_t + \Pi_t$$

4. Combine the F.O.C.s, we get the equilibrium condition for $C_t, N_t, K_{t+1}, Y_t, I_t, w_t, R_t$ and A_t.

$$\begin{aligned}
\theta N_t^{\chi} &= \frac{1}{C_t} w_t \\
\frac{1}{C_t} &= \beta \mathbb{E}\left[\frac{1}{C_{t+1}} \left(R_{t+1} + (1-\delta)\right) \Big| A_t \right] \\
K_{t+1} &= Y_t + (1-\delta) K_t - C_t \\
Y_t &= A_t K_t^{\alpha} N_t^{1-\alpha} \\
I_t &= Y_t - C_t \\
R_t &= A_t \alpha \left(\frac{N_t}{K_t}\right)^{1-\alpha} \\
w_t &= A_t (1-\alpha) \left(\frac{K_t}{N_t}\right)^{\alpha} \\
\ln A_{t+1} &= \rho \ln A_t + \varepsilon_t, \varepsilon_t \sim N\left(0, \sigma^2\right)
\end{aligned}$$

7.2 Introducing Wedges

1. Modifications and recalls.

   (a) Household still considers

   $$\max_{\{C_t, N_t, K_{t+1}\}_0^{\infty}} \mathbb{E}_0 \sum\nolimits_{t=0}^{\infty} \beta^t \left[\ln C_t - \theta \frac{N_t^{1+\chi}}{1+\chi} \right]$$

   but with a different constraint.

   $$C_t + \left(1 + \tau_t^I\right) I_t \leqslant \left(1 - \tau_t^N\right) w_t N_t + R_t K_t + \Pi_t - T_t$$

   (b) $I_t = K_{t+1} - (1-\delta) K_t$.

   (c) T_t is a lumpsum tax.

   (d) τ_t^I is like a tax on investment. It alters the relative price between consumption and investment.

   (e) τ_t^N is a tax on labor income.

   (f) Two wedges.

      i. $1+\tau_t^I$ is the investment wedge, "+" means it takes place of a part of consumption.

      ii. $1-\tau_t^N$ is the labor wedge, "−" means it reduces income.

      iii. Given t, then $\left(1+\tau_t^I\right) I_t \propto I_t$, $\left(1-\tau_t^N\right) w_t \propto w_t$, this is different from a lump-sum tax T_t.

2. Solve household's problem.

The Bellman equation is

$$\begin{aligned} V\left(K_t, A_t\right) = & \max_{C_t, N_t, K_{t+1}} \ln C_t - \theta \frac{N_t^{1+\chi}}{1+\chi} + \beta \mathbb{E}_t\left[V\left(K_{t+1}, A_{t+1}\right)\right] \\ \text{s.t. } & C_t + \left(1+\tau_t^I\right)\left(K_{t+1} - (1-\delta) K_t\right) \leqslant \left(1-\tau_t^N\right) w_t N_t + R_t K_t + \Pi_t - T_t \end{aligned}$$

The Lagrangian is

$$\begin{aligned} \mathcal{L} = & \ln C_t - \theta \frac{N_t^{1+\chi}}{1+\chi} + \beta \mathbb{E}_t\left[V\left(K_{t+1}, A_{t+1}\right)\right] \\ & + \lambda_t\left[\left(1-\tau_t^N\right) w_t N_t + R_t K_t + \Pi_t - T_t - C_t - \left(1+\tau_t^I\right)\left(K_{t+1} - (1-\delta) K_t\right)\right] \end{aligned}$$

F.O.C.s are

$$\begin{aligned} \left[C_t\right] : & \frac{1}{C_t} = \lambda_t \\ \left[N_t\right] : & \theta N_t^\chi = \lambda_t\left(1-\tau_t^N\right) w_t \\ \left[K_{t+1}\right] : & \beta \mathbb{E}_t\left[\frac{\partial V\left(K_{t+1}, A_{t+1}\right)}{\partial K_{t+1}}\right] = \lambda_t\left(1+\tau_t^I\right) \end{aligned}$$

By Envelop theorem,

$$\frac{\partial V\left(K_t, A_t\right)}{\partial K_t} = \lambda_t\left[R_t + \left(1+\tau_t^I\right)(1-\delta)\right]$$

then

$$\begin{cases} \frac{1}{C_t} = \lambda_t \\ \theta N_t^\chi = \lambda_t\left(1-\tau_t^N\right) w_t \\ \beta \mathbb{E}_t\left[\lambda_{t+1}\left[R_{t+1} + \left(1+\tau_{t+1}^I\right)(1-\delta)\right]\right] = \lambda_t\left(1+\tau_t^I\right) \end{cases}$$

and thus

$$\begin{cases} \theta N_t^\chi = \frac{1}{C_t}\left(1-\tau_t^N\right) w_t \\ \beta \mathbb{E}_t\left[\frac{1}{C_{t+1}}\left[R_{t+1} + \left(1+\tau_{t+1}^I\right)(1-\delta)\right]\right] = \frac{1}{C_t}\left(1+\tau_t^I\right) \end{cases}$$

3. Moreover,

   (a) A_t denotes the efficiency wedge.

   (b) G_t denotes the government consumption wedge.

   (c) T_t is chosen by balancing the government's budget:

$$T_t = G_t - \tau_t^I I_t - \tau_t^N w_t N_t$$

4. Combine the F.O.C.s, we get the equilibrium condition for $C_t, N_t, K_{t+1}, Y_t, I_t, w_t, R_t$, then

$$\begin{aligned}
\theta N_t^{\chi} &= \frac{1}{C_t}\left(1-\tau_t^N\right) w_t \\
\frac{1}{C_t}\left(1+\tau_t^I\right) &= \beta \mathbb{E}_t\left[\frac{1}{C_{t+1}}\left[R_{t+1}+\left(1+\tau_{t+1}^I\right)(1-\delta)\right]\right] \\
K_{t+1} &= (1-\delta) K_t + I_t \\
I_t &= Y_t - C_t - G_t \\
R_t &= A_t \alpha \left(\frac{N_t}{K_t}\right)^{1-\alpha} \\
w_t &= A_t (1-\alpha)\left(\frac{K_t}{N_t}\right)^{\alpha}
\end{aligned}$$

Lecture 8

Endogenous Growth

8.1 Romer Model

1. Ideas are non-rival — Paul Romer.

   (a) Rivalrous inputs are usually constant returns to scale in production.

   (b) Ideas are increasing returns to scale in production.

2. Set-ups.

   (a) Production function: $Y = K^{\alpha} (AL_Y)^{1-\alpha}, 0 < \alpha < 1$

   (b) Capital accumulation: $\dot{K} = s_K Y - \delta K$

   (c) Labor: $\frac{\dot{L}}{L} = n$, $L = L_Y + L_A$, where L_Y is the labor that produces consumption good, and L_A is the labor that produces A. And define

   $$s_R = \frac{L_A}{L}$$

   (d) Production function of A: $\dot{A} = \theta L_A^{\lambda} A^{\phi}, 0 < \lambda < 1, \phi < 1$.

   And λ measures the duplicate work of research, ϕ measures the spill-over effect of ideas.

3. BGP.

   Let $y = \frac{Y}{L_Y}, k = \frac{K}{L_Y}$, then

   $$y = k^{\alpha} A^{1-\alpha} \Longrightarrow \frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k} + (1-\alpha) \frac{\dot{A}}{A}$$

   and

   $$k = \frac{K}{L_Y} \Longrightarrow \dot{k} = \frac{\dot{K} L_Y - K \dot{L}_Y}{L_Y^2} = \frac{s_K Y - \delta K}{L_Y} - nk$$

   then

   $$\dot{k} = s_K y - (\delta + n) k$$

and thus, $\frac{\dot{k}}{k} = s_K \frac{y}{k} - (\delta + n)$. Moreover, $\frac{\dot{k}}{k}$ is a constant under BGP, then $\frac{\dot{k}}{k} = \frac{\dot{y}}{y}$, hence

$$\frac{\dot{y}}{y} = \frac{\dot{k}}{k} = \frac{\dot{A}}{A}$$

By $\dot{A} = \theta L_A^{\lambda} A^{\phi}$, then

$$g_A := \frac{\dot{A}}{A} = \frac{vL_A^{\lambda}}{A^{1-\phi}}$$

and g_A is a constant, then

$$0 = \lambda \frac{\dot{L}_A}{L_A} - (1-\phi)\frac{\dot{A}}{A} \Longrightarrow g_A = \frac{\lambda n}{1-\phi}$$

4. An example of comparative statistic.

   Let $\lambda = 1, \phi = 0$,

$$\dot{A} = \theta L_A \Longrightarrow g_A = \frac{\dot{A}}{A} = \theta \frac{L_A}{A} = \theta \frac{s_R L}{A}$$

   Under BGP,

$$g_A = \frac{\dot{L}}{L} = n$$

   If s_R increases to $s_R' > s_R$ at a moment.

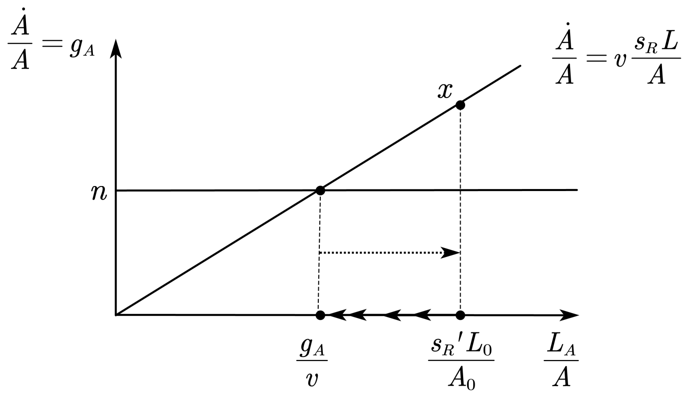

Fig. 8.1. Romer Model, $\lambda = 1, \phi = 0$.

5. Let $\tilde{k} = \frac{K}{AL_Y} = \frac{k}{A}$, $\tilde{y} = \frac{y}{A} \Longrightarrow \tilde{y} = \tilde{k}^{\alpha}$, then

$$\begin{aligned}\dot{\tilde{k}} &= \frac{\dot{k}A - \dot{A}k}{A^2} = \frac{s_K y - (\delta + n)k}{A} - g_A \cdot \tilde{k} \\ &= s_K \tilde{y} - (\delta + n + g_A)\tilde{k}\end{aligned}$$

   To get the steady state, let $\dot{\tilde{k}} = 0$, then

$$s_K \tilde{k}^{\alpha} - (\delta + n + g_A)\tilde{k} = 0 \Longrightarrow \tilde{k} = \left(\frac{s_K}{n + g_A + \delta}\right)^{\frac{1}{1-\alpha}}$$

   then

$$\tilde{y} = \left(\frac{s_K}{n + g_A + \delta}\right)^{\frac{\alpha}{1-\alpha}} \Longrightarrow y = A\left(\frac{s_K}{n + g_A + \delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Since

$$g_A = \theta \frac{s_R L}{A} \Longrightarrow A = \theta \frac{s_R L}{g_A}$$

then

$$y(t) = \left(\frac{s_K}{n + g_A + \delta} \right)^{\frac{\alpha}{1-\alpha}} \theta \frac{s_R}{g_A} L(t)$$

Note that $y = \frac{Y}{L_Y}$, then let $y^* = \frac{Y}{L} = \frac{Y}{L_Y}(1 - s_R)$, then

$$y^*(t) = \left(\frac{s_K}{n + g_A + \delta} \right)^{\frac{\alpha}{1-\alpha}} \theta \frac{s_R}{g_A} (1 - s_R) L(t)$$

6. More setups.

   The Romer economy consists of three sectors: a final-goods sector, an intermediate goods sector, and a research sector.

   (a) The research sector creates new ideas, which take the form of new varieties of capital goods.

   (b) The research sector sells the exclusive right to produce a specific capital good to an intermediate-goods firm.

   (c) The intermediate-goods firm, as a monopolist, manufactures the capital good and sells it to the final-goods sector, which produces output.

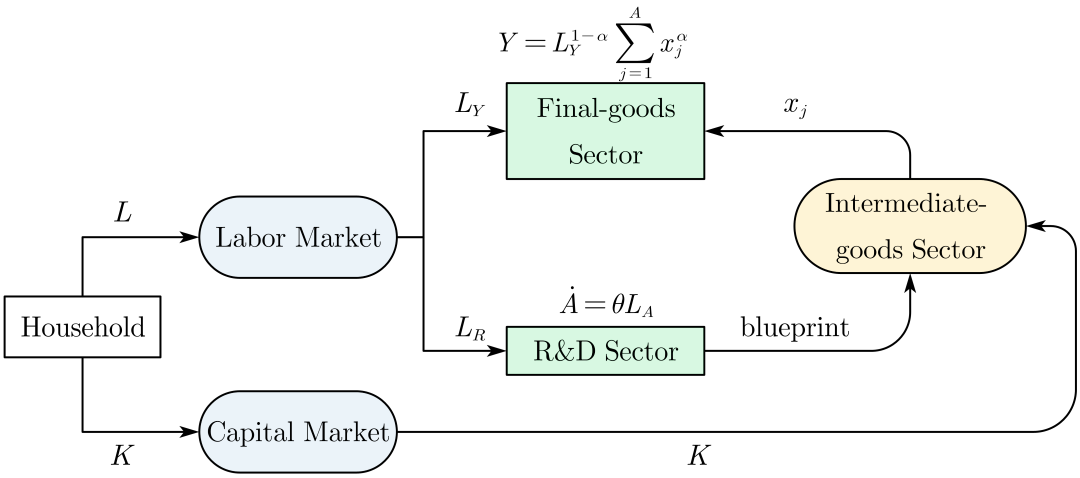

Fig. 8.2. Flow of Production Factors.

7. Final-goods sector.

$$\max_{L_Y, x_j, j=1,2,\cdots,A} L_Y^{1-\alpha} \sum_{j=1}^{A} x_j^\alpha - wL_Y - \sum_{j=1}^{A} p_j x_j$$

   F.O.C.s are

$$w = (1-\alpha) \frac{Y}{L_Y}$$
$$p_j = \alpha L_Y^{1-\alpha} x_j^{\alpha-1}, j = 1, 2, \cdots, A.$$

8. The intermediate-goods producer.

   Assume the production function is $x_j = K_j$, and the capital rental rate is r, then the problem for producer j is

$$\max_{x_j} \ \pi_j = p_j(x_j) x_j - r x_j$$

   where $p_j(x_j) = \alpha L_Y^{1-\alpha} x_j^{\alpha-1}$ is the demand curve that comes from the final-goods sector.

   The F.O.C. is

$$p'(x) x + p(x) = r$$

   or

$$p'(x) \frac{x}{p(x)} + 1 = \frac{r}{p(x)} \Longrightarrow p(x) = \frac{r}{p'(x) \frac{x}{p(x)} + 1}$$

   And

$$p(x) = \alpha L_Y^{1-\alpha} x^{\alpha-1} \Longrightarrow \frac{p'}{p} \cdot x = \alpha - 1$$

   which is the elasticity of demand, then

$$p(x) = \frac{1}{\alpha} r$$

   so that all capital goods sell for the same price. And thus, the profit

$$\pi_j = \pi = (p - r) \cdot x_j = (1 - \alpha) p x_j$$

   The total demand for capital from the intermediate-goods firms must equal the total capital stock in the economy

$$\sum_{j=1}^{A} x_j = \sum_{j=1}^{A} k_j = K$$

   Note that $x_1 = x_2 = \cdots = x_A$, then

$$x_j = x = \frac{K}{A}$$

   therefore, the production function is

$$Y = A L_Y^{1-\alpha} x^\alpha = A L_Y^{1-\alpha} \left(\frac{K}{A}\right)^\alpha = K^\alpha (A L_Y)^{1-\alpha}$$

   And thus,

$$\pi = (1 - \alpha) p x = (1 - \alpha) p \frac{K}{A}$$

   where $pK = \alpha Y$ since

$$\max_{K, L_Y} \ K^\alpha (A L_Y)^{1-\alpha} - pK \Longrightarrow p = \frac{\alpha Y}{K}$$

   therefore,

$$\pi = (1 - \alpha) p x = (1 - \alpha) \alpha \frac{Y}{A}$$

9. An example about asset pricing.

   Consider an asset pays d_{t+1} dividend at $t+1$, and has an ex-dividend price P_{t+1} at $t+1$. Then the ex-dividend price at t should be

$$P_t = \beta \left(P_{t+1} + d_{t+1}\right)$$

   If a duration of a period becomes Δ then we rewrite

$$\begin{aligned} \beta &= e^{-r\Delta} \approx 1 - r\Delta \\ d_{t+1} &\approx d_t \Delta \\ P_{t+1} &\approx P_t + \dot{P}_t \Delta \end{aligned}$$

   plug in,

$$\begin{aligned} P_t &= (1 - r\Delta)\left(P_t + \dot{P}_t\Delta + d_t\Delta\right) \\ &= P_t + \dot{P}_t\Delta + d_t\Delta - rP_t\Delta - r\dot{P}_t\Delta^2 + rd_t\Delta^2 \\ &\approx P_t + \dot{P}_t\Delta + d_t\Delta - rP_t\Delta \end{aligned}$$

   or

$$\dot{P}_t + d_t = rP_t$$

10. The R&D sector.

    Let P_A be the price of a new design.

    The arbitrage equation is

$$rP_A = \pi + \dot{P}_A$$

    The LHS of this equation is the interest earned from investing P_A in the bank; the RHS is the profits plus the capital gain or loss that results from the change in the price of the patent.

$$r = \frac{\pi}{P_A} + \frac{\dot{P}_A}{P_A}$$

    Under BGP, r is constant, then $\frac{\pi}{P_A}$ keeps at a constant, then $\frac{\dot{\pi}}{\pi} = \frac{\dot{P}_A}{P_A}$, Recall that

$$\pi = (1-\alpha)\,\alpha\frac{Y}{A} = (1-\alpha)\,\alpha\frac{yL}{A}$$

    since $g_y = g_A$, we have

$$\frac{\dot{\pi}}{\pi} = \frac{\dot{P}_A}{P_A} = \frac{\dot{L}}{L} = n$$

    therefore,

$$P_A = \frac{\pi}{r-n}$$

    Note that the profits earned by these monopolistic firms are extracted by the inventors, and these profits simply compensate the inventors for the time they spend “prospecting” for new designs.

11. Solve for s_R under BGP.

    The wage rate between R&D sector and final-goods sector should be the same, otherwise, all labor will choose the sector with higher wages. Therefore,

$$w_Y = w_R$$

and since $\dot{A} = \theta L_A$, we let the marginal production by a constant $\theta = \bar{\theta}$, then the wage rate is

$$w_R = \bar{\theta} P_A$$

and

$$w_Y = (1-\alpha)\frac{Y}{L_Y}$$

thus,

$$(1-\alpha)\frac{Y}{L_Y} = \bar{\theta} P_A \Longrightarrow L_Y = (1-\alpha)\frac{Y}{\bar{\theta} P_A}$$

Plug $P_A = \frac{\pi}{r-n}$, $\pi = (1-\alpha)\,\alpha \frac{Y}{A}$, in

$$L_Y = (1-\alpha)\frac{Y}{\bar{\theta}\frac{(1-\alpha)\alpha\frac{Y}{A}}{r-n}} = \frac{1}{\frac{\bar{\theta}}{A}\frac{\alpha}{r-n}} \Longrightarrow \frac{1}{L_Y} = \frac{\bar{\theta}}{A}\frac{\alpha}{r-n}$$

Since

$$\dot{A} = \bar{\theta} L_A \Longrightarrow \frac{\dot{A}}{A} = \frac{\bar{\theta} L_A}{A} \Longrightarrow \frac{g_A}{L_A} = \frac{\bar{\theta}}{A}$$

we get

$$\frac{1}{L_Y} = \frac{g_A}{L_A}\frac{\alpha}{r-n}$$

then

$$\frac{L_A}{L_Y} = \frac{\alpha g_A}{r-n}$$

therefore,

$$s_R = \frac{1}{1+\frac{r-n}{\alpha g_A}}$$

Then s_R is increasing in $g_A = g_k = g_y$ and decreasing in $r-n$. Since if $r-n$ grows, then $P_A = \frac{\pi}{r-n}$ becomes smaller.

8.2 Schumpeterian Model

1. Set-ups and Modifications.

   (a) Final-goods producer uses

$$Y = x_i^{\alpha}\left(A_i L_Y\right)^{1-\alpha}$$

   where x_i has a specific productivity level A_i, i is the version of x_i.

(b) Intermediate-goods producer uses

$$x_i = K_i$$

but the since the final good producer only choose the latest version of x_i, then whenever an intermediate-good producer works, $K_i = K$.

(c) Innovation occurs in steps rather than continuously.

2. Innovation.

$$A_{i+1} = (1+\gamma) A_i \Longleftrightarrow \gamma = \frac{A_{i+1} - A_i}{A_i}$$

where $\gamma > 0$ is the step size, i.e., the amount that productivity rises.

Thus,

$$A_{i+1} > A_i$$

The individual probability of innovation $\bar{\mu}$ given A_i is assumed as

$$\bar{\mu} := \bar{\mu} | \, A_i = \theta \frac{L_A^{\lambda-1}}{A_i^{1-\phi}}$$

where λ measures the duplication of research effort, and ϕ measures the spillover effect of ideas.

Since there are L_A individuals working as researchers, the probability of innovation is

$$\mathbb{P}\left(\text{innovation}\right) = \bar{\mu} L_A = \theta \frac{L_A^{\lambda}}{A_i^{1-\phi}}$$

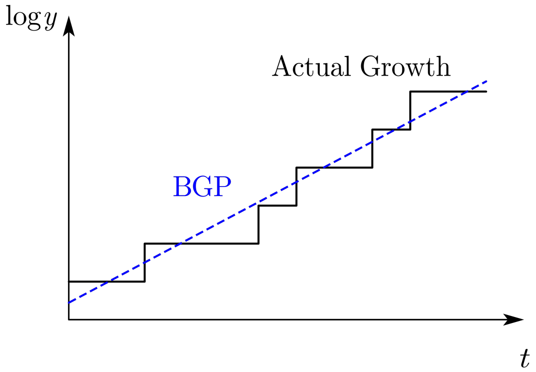

Fig. 8.3. Schumpeterian Model.

Then, we have a probability of innovating $\bar{\mu} L_A$, and the size of innovation $\gamma = \frac{A_{i+1} - A_i}{A_i} \approx \frac{\dot{A}_i}{A_i}$, therefore,

$$\mathbb{E}\left[\left.\frac{\dot{A}_i}{A_i}\right| A_i\right] = \gamma \bar{\mu} L_A = \gamma \theta \frac{L_A^{\lambda}}{A_i^{1-\phi}}$$

or

$$\mathbb{E}\left[\frac{\dot{A}_i}{A_i}\right] = \gamma \bar{\mu} L_A = \gamma \theta \frac{L_A^{\lambda}}{\mathbb{E}\left[A_i\right]^{1-\phi}}$$

Beyond the BGP,

$$\mathbb{E}\left[\frac{\dot{A}}{A}\right]=g_y=g_k=g_A$$

Then by taking log and derivatives,

$$0=\lambda\frac{\dot{L}_A}{L_A}-(1-\phi)\,\mathbb{E}\left[\frac{\dot{A}}{A}\right]\Longrightarrow\mathbb{E}\left[\frac{\dot{A}}{A}\right]=g_A=\frac{\lambda n}{1-\phi}$$

3. Discussion.

   (a) The Schumpeterian model doesn't change the long-run trend growth rate in Romer model.

   (b) γ does not influence the long-run growth.

      i. Larger γ boosts the size of jump in A.

      ii. Larger γ also makes it harder to get to another jump.

      iii. These two effects cancel out.

   (c) Schumpeterian model differs in underlying economics and will differ in the equilibrium value of s_R. We will see it later.

4. Example. Let $\{A_t : t \geqslant 1\}$ be a Poisson process with parameter μ. And the distribution of the time interval T between two innovation is $\exp(\mu)$, i.e., the CDF is

$$F_T(\tau)=1-e^{-\mu\tau}$$

   Then we can compute

$$\mathbb{P}\left(\text{Innovation takes place between } \tau \text{ and } \tau+\mathrm{d}t\right)=\mathbb{P}\left(A_{\tau+\mathrm{d}t}-A_\tau=1\right)=\mu\mathrm{d}t$$

   It just comes from an alternative definition of Poisson process.

   (Definition) A counting process $\{N(t) : t \geqslant 0\}$ is called a Poisson process with parameter $\lambda > 0$, if

   (a) $N(0)=0$.

   (b) The increments are independent.

   (c) As $h \downarrow 0$, $\mathbb{P}\{N(t+h)-N(t)=1\}=\lambda h+o(h)$.

   (d) As $h \downarrow 0$, $\mathbb{P}\{N(t+h)-N(t)\geqslant 1\}=o(h)$.

5. Final-goods producer.

   Final-goods producer need to choose

   (a) The value of x_i.

(b) The version of x_i, i.e., the index i.

But the old and new versions cost the same (we will see later), so they always choose the best one.

$$\max_{L_Y, x_i} x_i^{\alpha} (A_i L_Y)^{1-\alpha} - wL_Y - p_i x_i$$

then

$$w = (1-\alpha) \frac{Y}{L_Y}$$
$$p_i = \alpha x_i^{\alpha-1} (A_i L_Y)^{1-\alpha} = \alpha \frac{Y}{x_i}$$

the elasticity of demand is still

$$\frac{p_i' x_i}{p_i} = \alpha - 1$$

6. Intermediate-goods producer.

$$\max_{x_i} p_i(x_i) x_i - r x_i$$

then

$$p_i'(x_i) x_i + p_i(x_i) = r \Longrightarrow p_i = \frac{1}{1 + \frac{p_i'(x_i) x_i}{p_i(x_i)}} r = \frac{r}{\alpha}$$

The invariant price generates profit (we will see later) that will motivate innovation, and also implies that the price of x_i doesn't depend on the version of x_i.

Therefore, final good firms only buy the best one with highest A_i.

7. Aggregate output and the profit.

If an intermediate producer produces a latest version of x_i, then the economy will only produce x_i, thus,

$$x_i = K$$

The aggregate output becomes

$$Y = K^{\alpha} (A_i L_Y)^{1-\alpha}$$

Therefore,

$$wL_Y = (1-\alpha) Y rK = r x_i = \alpha p_i x_i$$

and the F.O.C. of final-good producer gives us

$$p_i = \alpha \frac{Y}{x_i} \Longrightarrow x_i p_i = \alpha Y$$

then

$$rK = r x_i = \alpha p_i x_i = \alpha^2 Y$$

thus,

$$\pi_i = (p_i - r) x_i = (p_i - \alpha p_i) x_i = (1-\alpha) \frac{rK}{\alpha} = (1-\alpha) \alpha Y > 0$$

8. R&D sector.

   The arbitrage equation of P_A is

$$rP_A = \pi + \dot{P}_A - (\bar{\mu}L_A)\,P_A$$

   (a) rP_A is the return if we put P_A units of money in the bank

   (b) π is the profit coming from the patent.

   (c) $\dot{P}_A - (\bar{\mu}L_A)\,P_A$ is the change of patent's value. Note that the patent will be replaced by the next innovator with probability $\bar{\mu}L_A$.

   Therefore,

$$r = \frac{\pi}{P_A} + \frac{\dot{P}_A}{P_A} - \bar{\mu}L_A$$

   For simplicity,

$$\mu := \bar{\mu}L_A$$

   Given BGP, r is a constant, and

$$\pi = (1-\alpha)\,\alpha Y = (1-\alpha)\,\alpha y L$$

   then

$$\frac{\dot{P}_A}{P_A} = \frac{\dot{\pi}}{\pi} = g_y + n$$

   and

$$g_y = g_A = \mathbb{E}\left[\frac{\dot{A}_i}{A_i}\right] = \gamma\bar{\mu}L_A = \gamma\mu$$

   then

$$r = \frac{\pi}{P_A} + \gamma\mu + n - \mu \Longrightarrow P_A = \frac{\pi}{r - n + (1-\gamma)\,\mu}$$

   Note that no existing monopolist would ever buy the new patent. Why? Because they have to sacrifice their existing profits, meaning they will not pay as much for the new patent. So always new firms coming into existence, which is known as "Arrow Replacement Effect".

9. Solve for s_R.

   Consider the wage rate in different sector,

$$w_R = \bar{\mu}P_A w_Y = (1-\alpha)\,\frac{Y}{L_Y}$$

   and

$$w_R = w_Y$$

   therefore,

$$\bar{\mu}P_A = (1-\alpha)\,\frac{Y}{L_Y}$$

plug $P_A = \frac{\pi}{r-n+(1-\gamma)\mu}$, and $\pi = \alpha(1-\alpha)Y$,

$$\bar{\mu}\frac{\alpha}{r-n+(1-\gamma)\mu} = \frac{1}{L_Y} \Longrightarrow \frac{L_A}{L_Y} = \frac{\alpha\mu}{r-n+(1-\gamma)\mu}$$

then

$$s_R = \frac{1}{1+\frac{r-n+(1-\gamma)\mu}{\alpha\mu}}$$

(a) s_R is decreasing in $r-n$.

(b) μ has two effects.

i. s_R is decreasing in $(1-\gamma)\mu$, it captures the fact that as the probability of innovation goes up, the value of patents declines due to replacement effects.

ii. s_R is increasing in $\alpha\mu$, captures the fact that as the probability of innovation goes up, you are more likely to get a patent in the first place.

On net, the second effect "wins", $s_R = \frac{1}{1+\frac{\frac{r-n}{\mu}+(1-\gamma)}{\alpha}}$, s_R is increasing with μ.

10. Discussion.

The long-run growth rate is

$$g = \frac{\lambda n}{1-\phi}$$

but s_R differs.

(a) Schumpeterian model has higher s_R than Romer model if $g < r-n$. In this case the discount rate is very large, and so people care most about profits in the immediate future and little about the fact that they might be replaced someday.

(b) Schumpeterian model has higher s_R than Romer model if $g > r-n$. In this case the discount rate is low, so people do care about the future replacement a lot.

(c) For two model, it is the same that higher s_R is not necessarily optimal.

Lecture 9

A Premier on Asset Pricing

9.1 Risk Neural Pricing

1. Set-up.

   (a) Two-period, period 2 is uncertain with S exclusive random states $\mathcal{S} = \{1, 2, \cdots, S\}$, each state is with a probability $\pi(s)$, and $\sum_{s=1}^{S} \pi(s) = 1$.

   (b) The households are risk neutral with utility $u(c) = c$.

   (c) Endowment of the household.

      i. q: a unit of risky assets at period 1, q is the price of the asset.

      ii. e_1: endowment consumption goods at period 1.

      iii. e_2: endowment consumption goods at period 2.

   (d) Household take the price of risky asset q as given.

   (e) Households are identical, the representative household solves

$$\max_{a,c_1,c_2} \mathbb{E}_1 [c_1 + \beta c_2(s)]$$
$$\text{s.t. } c_1 + qa = e_1 + q$$
$$c_2(s) = e_2(s) + ay(s), \forall s \in \mathcal{S}$$

2. Solution.

$$\max_a \mathbb{E}_1 [e_1 + q - qa + \beta [e_2(s) + ay(s)]]$$
$$= \max_a \mathbb{E}_1 [(\beta y(s) - q) a + e_1 + q + \beta e_2(s)]$$

   then

$$\mathbb{E}_1 [\beta y(s) - q] = 0$$

   Consider

$$a^* \begin{cases} = \infty, & \mathbb{E}_1[\beta y(s)] - q > 0 \\ \in \mathbb{R}, & \mathbb{E}_1[\beta y(s)] - q = 0 \\ = -\infty, & \mathbb{E}_1[\beta y(s)] - q < 0 \end{cases}$$

Note that market clears iff

$$a^* = 1$$

since the supply of the risky asset never change, if $a^* \neq 1$, then there must be two different household holds different units of risky asset, which is a contradiction with our assumption that households are identical. Then

$$\mathbb{E}_1 \left[\beta y \left(s\right)\right] - q = 0$$

Or from another point, if $\mathbb{E}_1 \left[\beta y \left(s\right)\right] - q \neq 0$, then $a^* = \pm\infty$, which cannot lead to market clear. Therefore,

$$c_1 = e_1 c_2 \left(s\right) = e_2 \left(s\right) + y \left(s\right)$$

and

$$q = \mathbb{E}_1 \left[\beta y \left(s\right)\right] = \sum_{s=1}^{S} \beta \pi \left(s\right) y \left(s\right)$$

3. No compensation for risk when the household is risk neutral.

   (a) The return of a safe bond (that is $y\left(s\right) = 1, \forall s \in \mathcal{S}$) is

   $$r\left(s\right) = \frac{y\left(s\right)}{q} - 1 = \frac{1}{\beta} - 1$$

   (b) The expected return of a risky asset is

   $$\mathbb{E}_1 r\left(s\right) = \frac{\mathbb{E}\left[y\left(s\right)\right]}{q} - 1 = \frac{1}{\beta \mathbb{E}\left[y\left(s\right)\right]} - 1 = \frac{1}{\beta} - 1$$

4. State price.

   Image there is an asset that only pays 1 at state s, then the price is

   $$q_s = \beta \pi \left(s\right)$$

9.2 Pricing of Risk

1. Modification.

   (a) Households are risk averse, so $u\left(c\right)$ is concave.

   (b) $u\left(c\right)$ can be $\log\left(c\right), \frac{c^{1-\gamma}}{1-\gamma}, c - \frac{\sigma}{2}\left(c - c^*\right)^2$.

   (c) Households are identical, the representative household solves

   $$\max_{a, c_1, c_2} \mathbb{E}_1 \left[u\left(c_1\right) + \beta u\left(c_2\left(s\right)\right)\right]$$
   $$\text{s.t. } c_1 + qa = e_1 + q$$
   $$c_2\left(s\right) = e_2\left(s\right) + ay\left(s\right), \forall s \in \mathcal{S}$$

2. Solution.

$$\max_a \mathbb{E}_1 \left[u\left(e_1 + q - qa\right) + \beta u\left(e_2\left(s\right) + ay\left(s\right)\right)\right]$$

F.O.C. is

$$\mathbb{E}_1 \left[-qu'\left(c_1\right) + \beta y\left(s\right) u'\left(c_2\left(s\right)\right)\right] = 0$$

and market clear condition gives us $a = 1$, then

$$c_1 = e_1$$
$$c_2\left(s\right) = e_2\left(s\right) + y\left(s\right)$$

and

$$q = \mathbb{E}_1 \left[\frac{\beta u'\left(e_2\left(s\right) + y\left(s\right)\right)}{u'\left(c_1\right)} y\left(s\right)\right]$$

3. Discussion.

$$q = \mathbb{E}_1 \left[\beta \frac{u'\left(c_2\left(s\right)\right)}{u'\left(c_1\right)} y\left(s\right)\right] \Longrightarrow u'\left(c_1\right) q = \beta \mathbb{E}_1 \left[u'\left(c_2\left(s\right)\right) y\left(s\right)\right]$$

It's like a Euler equation we have seen before. Let $M\left(s\right) = \frac{\beta u'\left(c_2\left(s\right)\right)}{u'\left(c_1\right)}$, which is called the stochastic discount factor, and then

$$q = \mathbb{E}_1 \left[M\left(s\right) y\left(s\right)\right] = \mathbb{E}_1 \left[M\left(s\right)\right] \mathbb{E}_1 \left[y\left(s\right)\right] + \mathrm{Cov}\left(M\left(s\right), y\left(s\right)\right)$$

Note that

$$\mathrm{Cov}\left(M\left(s\right), y\left(s\right)\right) = \mathrm{Cov}\left(\frac{\beta u'\left(c_2\left(s\right)\right)}{u'\left(c_1\right)}, y\left(s\right)\right) = \frac{\beta}{u'\left(c_1\right)} \mathrm{Cov}\left(u'\left(c_2\left(s\right)\right), y\left(s\right)\right)$$

(a) If $c_2\left(s\right) \uparrow \Longrightarrow y\left(s\right) \downarrow$, then $u'\left(c_2\left(s\right)\right) \downarrow \Longrightarrow y\left(s\right) \downarrow$, and thus,

$$\mathrm{Cov}\left(u'\left(c_2\left(s\right)\right), y\left(s\right)\right) > 0$$

then $q > \mathbb{E}_1 \left[M\left(s\right)\right] \mathbb{E}_1 \left[y\left(s\right)\right]$, i.e., price is higher. The intuition is that the risky asset is a "good" asset, and it's like an insurance, which gives us a higher $y\left(s\right)$ at period 2, when $c_2\left(s\right)$ is lower, just like a compensation. Therefore, the price is higher.

(b) If $c_2\left(s\right) \uparrow \Longrightarrow y\left(s\right) \uparrow$, then $u'\left(c_2\left(s\right)\right) \downarrow \Longrightarrow y\left(s\right) \uparrow$, and thus,

$$\mathrm{Cov}\left(u'\left(c_2\left(s\right)\right), y\left(s\right)\right) < 0$$

then $q < \mathbb{E}_1 \left[M\left(s\right)\right] \mathbb{E}_1 \left[y\left(s\right)\right]$, it is a "bad" asset. When $c_2\left(s\right)$ is lower, the asset gives us less.

4. State price.

We normalize the payment at state s to 1, the price of the asset q_s is

$$q_s = \pi\left(s\right) \frac{\beta u'\left(e_2\left(s\right) + 1\right)}{u'\left(e_1\right)}$$

5. Safe asset.
$$q_0 = \mathbb{E}_1\left[\frac{\beta u'(e_2(s)+1)}{u'(e_1)}\cdot 1\right] = \frac{\beta}{u'(e_1)}\mathbb{E}_1\left[u'(e_2(s)+1)\right]$$
Suppose $u''(c)$ is increasing in c or $u'(c)$ is convex, for example,
$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \Longrightarrow u'(c) = c^{-\gamma}, u''(c) = -rc^{-r-1}, u'''(c) = r(r+1)c^{-r-2} > 0$$
then by Jensen's inequality,
$$\mathbb{E}_1\left[u'(e_2(s)+1)\right] \geqslant u'(\mathbb{E}_1[e_2(s)+1])$$
and thus,
$$q_0 = \frac{\beta}{u'(e_1)}\mathbb{E}_1\left[u'(e_2(s)+1)\right] \geqslant \frac{\beta}{u'(e_1)}u'(\mathbb{E}_1[e_2(s)]+1)$$

6. Remark.
   (a) Controlling for the expected income of the household, when the endowment of the household becomes riskier, the value of safe assets goes up.
   (b) Controlling for the riskiness of the income, when the household becomes richer (i.e., $\mathbb{E}_1[e_2(s)]$ is larger), safe assets become less valuable.

7. Equity premium.

   Recall that
$$q = \mathbb{E}_1[M(s)]\,\mathbb{E}_1[y(s)] + \mathrm{Cov}(M(s), y(s))$$
   The price of a safe asset with the same expected payment $\mathbb{E}_1[y(s)]$
$$q_0 = \mathbb{E}_1\left[\frac{\beta u'(e_2(s)+y(s))}{u'(c_1)}\mathbb{E}_1[y(s)]\right] = \mathbb{E}_1[M(s)]\,\mathbb{E}_1[y(s)]$$
   The required compensation for taking on risky assets (risk premium) is
$$q - q_0 = \mathrm{Cov}(M(s), y(s)) = \sum_{s=1}^{S}\pi(s)\left[M(s) - \mathbb{E}_1 M(s)\right]\left[y(s) - \mathbb{E}_1 y(s)\right]$$
   The equity premium is smaller when the stochastic factor $M(s)$, is more negatively correlated with the payment of the asset $y(s)$, or to say, when $M(s)$ is counter-cyclical. Formally, a risk premium is the investment return an asset is expected to yield in excess of the risk-free rate of return. The expected return on the risky asset is
$$\mathbb{E}_1 r(s) = \frac{\mathbb{E}_1 y(s)}{q} - 1$$
   for the safe asset that pays $\mathbb{E}_1 y(s)$,
$$r_0 = \frac{\mathbb{E}_1 y(s)}{q_0} - 1$$
   then the equity premium is defined as
$$\mathbb{E}_1 r(s) - r_0 = \mathbb{E}_1 y(s)\left[\frac{1}{q} - \frac{1}{q_0}\right] = \mathbb{E}_1 y(s)\,\frac{q_0 - q}{q\cdot q_0}$$

9.3 Arbitrage-based pricing

1. Hedge.

   (a) A hedge is an investment that is made with the intention of reducing the risk of adverse price movements in an asset. Normally, a hedge consists of taking an offsetting position in a related security.

   (b) Hedging is analogous to taking out an insurance policy.

   (c) A perfect hedge is one that eliminates all risk in a position or portfolio. In other words, the hedge is 100% inversely correlated to the vulnerable asset.

2. Arbitrage.

   (a) By selling short an asset, the arbitrager faces adverse exposures. When purchasing an asset, the arbitrager receives positive payoffs.

   (b) A risk-free arbitrage is such that the adverse exposure from short-selling securities exactly offsets the payment from purchasing an asset.

3. Arbitrage-based pricing.

   Suppose there are S states, and the price of an insurance that pays 1 only if event s happens is q_s, now we want to price a risky asset that pays $y(s)$ on state s. It must hold that

   $$q(y) = \sum_{s=1}^{S} q_s y(s)$$

   If $q(y) > \sum_{s=1}^{S} q_s y(s)$, then at period 1, we can short sell 1 unit of the risky asset $q(y)$ and buy $y(s)$ units of q_s, then we will gain $q(y) - \sum_{s=1}^{S} q_s y(s) > 0$. And thus, we can make infinite profit by duplicating the portfolio, which cannot lead to market clear.

   It's similar to the case that $q(y) < \sum_{s=1}^{S} q_s y(s)$ by buying 1 unit of risky asset and short selling $y(s)$ units of q_s.

   Moreover, if there are 2 risky assets, one pays $y_1(s)$ and the other pays $y_2(s)$ on state s, then

   $$q(y_1 + y_2) = \sum_{s=1}^{S} q_s [y_1(s) + y_2(s)] = \sum_{s=1}^{S} [q_s y_1(s) + q_s y_2(s)] = q(y_1) + q(y_2)$$

   Recall that

   $$q_s = \pi(s) \frac{\beta u'(e_2(s) + 1)}{u'(e_1)} = \pi(s) \frac{\beta u'(c_2(s))}{u'(c_1)}$$

   then

   $$q(y) = \sum_{s=1}^{S} q_s y(s) = \sum_{s=1}^{S} \pi(s) \frac{\beta u'(c_2(s))}{u'(c_1)} y(s)$$

9.4 Modigliani-Miller Theorem

1. Set-ups.
   (a) The firm's profit (asset): $y(s)$.
   (b) Payment to debt holder: $y_D(s) = \min(D, y(s))$
   (c) Payment to equity holder: $y_E(s) = y(s) - y_D(s) = y(s) + \max(-D, -y(s)) = \max(y(s) - D, 0)$
   (d) Leverage ratio (debt-equity ratio): $\frac{q(y_D)}{q(y_E)}$.
2. The Modigliani-Miller theorem says that the capital structure (debt-equity ratio) of a firm is irrelevant for the total value of the firm.
   (a) $\forall s, y_D(s) + y_E(s) = y(s)$.
   (b) By arbitrage-based pricing, $q(y) = q(y_D + y_E) = q(y_D) + q(y_E)$.
   (c) This is true regardless of the debt face value, D, or equivalently, the leverage of the firm.
3. Modigliani-Miller theorem fails in practice, because of assumptions.
   (a) The firm faces no bankruptcy costs.
   (b) There is no differential taxation of dividends payment to equity holders vs. interest payments to debt holders.
   (c) There is no moral hazard: equity holders seek to maximize the value of the firm.
   (d) There is no adverse selection: information is symmetric.
4. With bankruptcy cost δ.
   (a) Bankruptcy cost is the legal cost associated with default. When the firm's profit is below its promised payment to debt holders, the firm needs to be liquidated or reorganized to service its debt payment.
   (b) The value of the debt contract is

$$y_D(s) = \begin{cases} y(s) - \delta, & y(s) < D \\ D, & y(s) \geqslant D \end{cases}$$

   (c) The total value of the firm is

$$y(s) = y_D(s) + y_E(s) = \begin{cases} y(s) - \delta, & y(s) < D \\ y(s), & y(s) \geqslant D \end{cases}$$

   (d) A higher leverage (a higher D) increases the expected cost from bankruptcy and reduces the firm's value.

5. With tax shield (tax exemption) $\tau(s) = \tau y_D(s)$.

   (a) The total value of the firm is

$$y(s) = y_D(s) + y_E(s) + \tau(s) = \begin{cases} y(s)(1+\tau) - \delta, & y(s) < D \\ y(s) + \tau D, & y(s) \geqslant D \end{cases}$$

   (b) The firm faces a tradeoff in raising debt. While raising debt gives the firm extra tax benefit, it also increases the likelihood of incurring bankruptcy cost.

9.5 Equity Premium and Economic Fluctuations

1. Equity premium puzzle.

   (a) R^f_{t+1} is the return rate of risk-free assets.

$$1 = \mathbb{E}\left[M_{t+1} R^f_{t+1}\right]$$

   (b) $\tilde{R}_{t+1}$ is the return rate of risky asset.

$$1 = \mathbb{E}\left[M_{t+1} \tilde{R}_{t+1}\right]$$

   (c) M_{t+1} is the stochastic discount factor.

   (d) R^e_{t+1} is the excess return, defined as $R^e_{t+1} = \tilde{R}_{t+1} - R^f_{t+1}$.

   Therefore,

$$\mathbb{E}\left[M_{t+1} R^e_{t+1}\right] = \mathbb{E}\left[M_{t+1}\left(\tilde{R}_{t+1} - R^f_{t+1}\right)\right] = \mathbb{E}\left[M_{t+1}\tilde{R}_{t+1}\right] - \mathbb{E}\left[M_{t+1}R^f_{t+1}\right] = 0$$

   and since

$$\mathbb{E}\left[M_{t+1} R^e_{t+1}\right] = \mathbb{E}\left[M_{t+1}\right]\mathbb{E}\left[R^e_{t+1}\right] + \operatorname{Cov}\left(M_{t+1}, R^e_{t+1}\right)$$

   Then

$$\mathbb{E}\left[R^e_{t+1}\right] = -\frac{1}{\mathbb{E}\left[M_{t+1}\right]}\operatorname{Cov}\left(M_{t+1}, R^e_{t+1}\right) = -\operatorname{Cov}\left(\frac{M_{t+1}}{\mathbb{E}\left[M_{t+1}\right]}, R^e_{t+1}\right)$$

   We can normalize $\mathbb{E}M_{t+1} = 1$, then

$$\mathbb{E}\left[R^e_{t+1}\right] = -\operatorname{Cov}\left(M_{t+1}, R^e_{t+1}\right)$$

   Let $u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$, define the consumption growth rate (which we can observe from reality) as

$$\Delta c_{t+1} := \frac{\Delta C_{t+1}}{C_t} = \frac{C_{t+1} - C_t}{C_t} = \frac{C_{t+1}}{C_t} - 1$$

   and then

$$M_{t+1} = \beta\frac{u'(C_{t+1})}{u'(C_t)} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} = \beta\left(1 + \Delta c_{t+1}\right)^{-\gamma} \approx \beta\left(1 - \gamma\Delta c_{t+1}\right)$$

Then

$$\begin{aligned}\mathbb{E}\left[R_{t+1}^{e}\right] &= -\beta\mathrm{Cov}\left(1-\gamma\Delta c_{t+1}, R_{t+1}^{e}\right) = \beta\gamma\mathrm{Cov}\left(\Delta c_{t+1}, R_{t+1}^{e}\right) \\ &= \beta\gamma\mathrm{Corr}\left(\Delta c_{t+1}, R_{t+1}^{e}\right)\sigma\left(\Delta c_{t+1}\right)\cdot\sigma\left(R_{t+1}^{e}\right)\end{aligned}$$

and then the Sharpe ratio is

$$\text{Sharpe Ratio} = \frac{\mathbb{E}\left[R_{r+1}^{e}\right]}{\sigma\left(R_{r+1}^{e}\right)} = \beta\gamma\sigma\left(\Delta c_{t+1}\right)\mathrm{Corr}\left(\Delta c_{t+1}, R_{t+1}^{e}\right) \leqslant \beta\gamma\sigma\left(\Delta c_{t+1}\right)$$

In America's data,

$$\sigma\left(R_{t+1}^{e}\right) = 16\%, \mathbb{E}\left[R_{r+1}^{e}\right] = 4\% \sim 8\%, \sigma\left(\Delta c_{t+1}\right) = 0.01 \sim 0.02$$

then

$$0.25 \leqslant \beta\gamma\sigma\left(\Delta c_{t+1}\right) \leqslant 0.5 \Longrightarrow 1.25 \leqslant \beta\gamma \leqslant 5$$

which will lead to a very high degree of risk aversion γ.

2. The unified ideas behind theories that explains equity premium puzzle.

   Given $u\left(C_t\right) = \frac{C_t^{1-\gamma}}{1-\gamma}$, then the original stochastic discount factor is

   $$M_{t+1} = \beta\frac{u'\left(C_{t+1}\right)}{u'\left(C_t\right)} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$$

   To better calibrate reality, redefine M_{t+1} as

   $$M_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{Y_{t+1}}{Y_t}\right)^{\theta}$$

   where $\left(\frac{Y_{t+1}}{Y_t}\right)^{\theta}$ is the deviation from the original stochastic discount factor, and Y_t usually varies with business cycle.

3. Main theories (We will have a taste of some of them).

   (a) Endogenous risk aversion.

   - Habit formation (Campbell-Cochrane)

   (b) Perceived riskiness.

   - Recursive utility (Epstein-Zin)
   - Long run risks (e.g., Bansal Yaron)
   - Rare Disasters (e.g., Reitz; Barro)
   - Ambiguity aversion, min-max, (Hansen and Scheinkman)
   - Behavioral finance; probability mistakes. (e.g., Shiller, Thaler)

   (c) Risk-bearing capacity:

   - Idiosyncratic risk (e.g., Constantinides and Duffie)

- Leverage; balance-sheet; "institutional" (e.g., Brunnermerier, ...)

4. Habit.

   Let the utility function be

$$u(C) = \frac{(C - X)^{1-\gamma}}{1-\gamma}$$

   where X is the subsistence level of consumption or habit level. Look at the CRRA,

$$r_R = -\frac{Cu''(C)}{u'(C)} = -\frac{-\gamma C (C-X)^{-\gamma-1}}{(C-X)^{-\gamma}} = \gamma \frac{C}{C-X}$$

   As $C \downarrow X$, the risk aversion increase.

   Define $S_t := \frac{C_t - X_t}{C_t}$.

$$M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)} = \beta \left(\frac{C_{t+1} - X_{t+1}}{C_{t+1} - X_t} \right)^{-\gamma} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{S_{t+1}}{S_t} \right)^{-\gamma}$$

5. Recursive utility.

   Let

$$U_t = \left[(1-\beta) C_t^{1-\rho} + \beta \left[\mathbb{E}_t \left(U_{t+1}^{1-\gamma} \right)^{\frac{1-\rho}{1-\gamma}} \right]^{\frac{1-\rho}{1-\gamma}} \right]^{\frac{1}{1-\rho}}$$

   where γ is risk aversion coefficient, $\frac{1}{\rho}$ is the elasticity of intertemporal substitution.

   Note that when $\rho = \gamma$, the function reduces to time separable power utility.

$$U_t = \left[(1-\beta) C_t^{1-\gamma} + \beta \left[\mathbb{E}_t \left(U_{t+1}^{1-\gamma} \right) \right] \right]^{\frac{1}{1-\gamma}}$$

   A problem with the time separable utility is that it cannot tell apart risk aversion from the elasticity of intertemporal substitution.

   In this theory, the stochastic discount factor is

$$M_{t+1} = \beta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left[\frac{U_{t+1}}{\left(\mathbb{E}_t U_{t+1}^{1-\gamma} \right)^{\frac{1}{1-\gamma}}} \right]^{\rho - \gamma} \right]$$

   By the concavity of $x^{1-\gamma}$,

$$\mathbb{E}_t U_{t+1}^{1-\gamma} \leqslant \left[\mathbb{E}_t U_{t+1} \right]^{1-\gamma}$$

   then $\left(\mathbb{E}_t U_{t+1}^{1-\gamma} \right)^{\frac{1}{1-\gamma}} \leqslant \mathbb{E}_t U_{t+1}$.

   If $\rho = \gamma$, $M_{t+1} = \beta \mathbb{E}_t \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$, becomes the original one.

   If $\rho \neq \gamma$, the wedge is $\mathbb{E}_t \left[\frac{U_{t+1}}{\left(\mathbb{E}_t U_{t+1}^{1-\gamma} \right)^{\frac{1}{1-\gamma}}} \right]^{\rho - \gamma}$.

6. Idiosyncratic risk.

   Suppose we observe the equilibrium consumption of individual agent i, then

$$C_{i,t} = \delta_{i,t} C_t$$

   Then the stochastic discount factor for agent i is

$$M_{i,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{\delta_{i,t+1}}{\delta_{i,t}} \right)^{-\gamma}$$

7. A brief summary.

   (a) Empirical: Asset prices are driven by a large, time-varying, business-cycle correlated risk premium.

   (b) Recessions are phenomena of risk aversion.

Lecture 10

Equilibrium with Complete Markets

10.1 Efficient Allocation

1. Set-ups.

   (a) Two periods, $t = 1, 2$.

   (b) Two types of agents, $i = 1, 2$.

   (c) Pay off relevant events (states), $s \in \{1, 2, \cdots, S\}$.

   (d) Probability that state s takes place $\pi(s)$.

   (e) Endowment of agent i: e_1^i at period 1, $e_2^i(s)$ at period 2 on state s. And $e_1 = e_1^1 + e_1^2, e_2(s) = e_2^1(s) + e_2^2(s)$.

   (f) Consumption of agent i: c_1^i at period 1, $c_2^i(s)$ at period 2 on state s.

   (g) Payoff of agent i.

   $$U_i = u\left(c_1^i\right) + \beta \sum_{s=1}^{S} \pi(s)\, u\left(c_2^i(s)\right)$$

   where u is increasing and concave, $\lim_{c \to 0} u'(c) = \infty$.

2. Agent i's problem.

   $$\max_{c_1^i, c_2^i(s), s=1,\cdots,S} \quad U_i = u\left(c_1^i\right) + \beta \sum_{s=1}^{S} \pi(s)\, u\left(c_2^i(s)\right)$$
   $$\text{s.t. } c_1^i + \sum_{s=1}^{S} q_s c_2^i(s) \leqslant e_1^i + \sum_{s=1}^{S} q_s e_2^i(s)$$

   The LHS of the budget constraint is the market value of her income and the RHS is the market value of her expenditure.

3. Equilibrium.

   A competitive equilibrium is consumption allocations for agent $i = 1, 2$, $c_1^i, c_2^i(s)$ for all $s \in \{1, 2, \cdots, S\}$, and prices for insurance contracts q_s for all $s \in \{1, 2, \cdots, S\}$ such that

   (a) given prices q_s, consumption allocations for agent i maximize her expected utility subject to her budget constraint.

   (b) markets for all insurance contracts clear.

4. Efficient allocation.

   (a) Pareto efficient allocation: any reallocation that makes one agent strictly better off also makes one or more other agents worse off.

   (b) Welfare theorems.

      i. First welfare theorem: any competitive equilibrium is efficient.

$$\forall \text{competitive equilibrium} \Longrightarrow \text{efficient}$$

      ii. Second welfare theorem: any efficient allocation corresponds to a competitive equilibrium with certain endowment.

$$\forall \text{efficient}, \exists \text{endowment} + \text{efficient} \Longrightarrow \text{competitive equilibrium}$$

      Therefore, finding all Pareto efficient allocations helps solving competitive equilibrium.

5. Social planner's problem.

$$\max_{c_1^i, c_2^i(s)} W = \sum_{i=1}^{2} \theta_i U_i$$

   where θ_i is the Pareto weight on agent i.

6. Solve the one-period problem.

   (a) $U_i = u\left(c_1^i\right)$.

   (b) Normalize $\theta_1 + \theta_2 = 1$, let $\theta_1 = \theta$, then $\theta_2 = 1 - \theta$.

   (c) Normalize $e_1^1 + e_1^2 = 1$.

   The Problem becomes

$$\max_{c_1^1} \theta u\left(c_1^1\right) + (1-\theta) u\left(1 - c_1^1\right)$$

   then

$$\theta u'\left(c_1^1\right) = (1-\theta) u'\left(1 - c_1^1\right) \Longrightarrow \frac{u'\left(1 - c_1^1\right)}{u'\left(c_1^1\right)} = \frac{\theta}{1-\theta}$$

7. Solve the two-period problem.

$$\max_{c_1^1, c_1^2, c_2^1(s), c_2^2(s)} \theta \mathbb{E}\left[u\left(c_1^1\right) + \beta u\left(c_2^1(s)\right)\right] + (1-\theta) \mathbb{E}\left[u\left(c_1^2\right) + \beta u\left(c_2^2(s)\right)\right]$$

$$\text{s.t. } c_1^1 + c_1^2 = e_1$$

$$c_2^1(s) + c_2^2(s) = e_2(s), s = 1, 2, \cdots, S$$

or

$$\begin{cases} \max_{c_1^1} \; \theta u\left(c_1^1\right) + (1-\theta)\, u\left(e_1 - c_1^1\right) \\ \max_{c_2^1(s)} \; \theta \mathbb{E}\left[u\left(c_2^1(s)\right)\right] + (1-\theta)\, \mathbb{E}\left[u\left(e_2(s) - c_2^1(s)\right)\right] \end{cases}$$

F.O.C.s: $\forall s \in \{1, 2, \cdots, S\}$,

$$\begin{cases} \theta u'\left(c_1^1\right) = (1-\theta)\, u'\left(e_1 - c_1^1\right) \\ \theta u'\left(c_2^1(s)\right) = (1-\theta)\, u'\left(e_2(s) - c_2^1(s)\right) \end{cases} \quad \Longrightarrow \frac{u'\left(c_1^2\right)}{u'\left(c_1^1\right)} = \frac{u'\left(c_2^2(s)\right)}{u'\left(c_2^1(s)\right)} = \frac{\theta}{1-\theta}$$

If $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, then $\forall s \in \{1, 2, \cdots, S\}$,

$$\left(\frac{e_1 - c_1^1}{c_1^1}\right)^{-\gamma} = \left(\frac{e_2(s) - c_2^1(s)}{c_2^1(s)}\right)^{-\gamma} = \frac{\theta}{1-\theta}$$

or

$$\frac{c_1^1}{e_1} = \frac{c_2^1(s)}{e_2(s)} = \rho := \frac{\theta^{\frac{1}{\gamma}}}{\theta^{\frac{1}{\gamma}} + (1-\theta)^{\frac{1}{\gamma}}}$$

which implies that the consumption for an agent fluctuates in proportion to the aggregate fluctuation, and the social planner distributes consumption and risks "smoothly" across agents.

This is because the planner insures agents against idiosyncratic risks independent from the aggregate endowment fluctuation.

8. (Remark)

   (a) Our goal is to solve for equilibrium consumption and asset prices.

   (b) Allocations and prices as functions of Pareto weights is summarized in ρ.

   (c) Now we need to solve for ρ by individual agents' budget constraint.

   (d) In other words, we have found the efficient equilibrium, now we need to find what endowment with a competitive equilibrium corresponds to the efficient equilibrium.

9. Agent's problem.

$$\max_{c_1^i, c_2^i(s), s=1,\cdots,S} \; U_i = u\left(c_1^i\right) + \beta \sum_{s=1}^{S} \pi(s)\, u\left(c_2^i(s)\right)$$
$$\text{s.t.} \;\; c_1^i + \sum_{s=1}^{S} q_s c_2^i(s) \leqslant e_1^i + \sum_{s=1}^{S} q_s e_2^i(s)$$

Since the model only has 2 period, the constraint must be binding.

$$\max_{c_2^i(s), s=1,\cdots,S} \; U_i = u\left(e_1^i + \sum_{s=1}^{S} q_s e_2^i(s) - \sum_{s=1}^{S} q_s c_2^i(s)\right) + \beta \sum_{s=1}^{S} \pi(s)\, u\left(c_2^i(s)\right)$$

The F.O.C. w.r.t. $c_2^i(s)$ is

$$\beta \pi(s)\, u'\left(c_2^i(s)\right) = q_s u'\left(c_1^i\right)$$

then

$$q_s = \beta \frac{u'\left(c_2^i\left(s\right)\right)}{u'\left(c_1^i\right)} \pi\left(s\right)$$

Recall that

$$\frac{c_1^1}{e_1} = \frac{c_2^1\left(s\right)}{e_2\left(s\right)} = \rho := \frac{\theta^{\frac{1}{\gamma}}}{\theta^{\frac{1}{\gamma}} + \left(1-\theta\right)^{\frac{1}{\gamma}}}$$

If $u\left(c\right) = \frac{c^{1-\gamma}}{1-\gamma} \Longrightarrow u'\left(c\right) = c^{-\gamma}$, then

$$q_s = \beta\pi\left(s\right) \frac{u'\left(\rho e_2\left(s\right)\right)}{u'\left(\rho e_1\right)} = \beta\pi\left(s\right)\left[\frac{e_2\left(s\right)}{e_1}\right]^{-\gamma}$$

And by the budget constraint of agent 1,

$$c_1^1 + \sum_{s=1}^{S} q_s c_2^1\left(s\right) = \rho\left[e_1 + \sum_{s=1}^{S} q_s e_2\left(s\right)\right] = e_1^1 + \sum_{s=1}^{S} q_s e_2^1\left(s\right)$$

then

$$\rho = \frac{e_1^1 + \sum_{s=1}^{S} q_s e_2^1\left(s\right)}{e_1 + \sum_{s=1}^{S} q_s e_2\left(s\right)}$$

which implies that if agent 1 is richer or luckier, i.e., the endowment is larger, ρ is larger.

Recall that

$$\rho = \frac{\theta^{\frac{1}{\gamma}}}{\theta^{\frac{1}{\gamma}} + \left(1-\theta\right)^{\frac{1}{\gamma}}} = \frac{1}{1 + \left(\frac{1}{\theta} - 1\right)^{\frac{1}{\gamma}}}$$

is increasing in θ.

Therefore, if agent 1 is richer or luckier, the Pareto weight for agent 1 is larger.

10.2 Equilibrium Characterization and Implication

1. (Remark)

$$\begin{aligned} c_1^1 &= \rho e_1, c_2^1\left(s\right) = \rho e_2\left(s\right) \\ c_1^2 &= \left(1-\rho\right) e_1, c_2^2\left(s\right) = \left(1-\rho\right) e_2\left(s\right) \end{aligned}$$

   (a) The insurance is only against idiosyncratic risk, not the aggregate risk.

   (b) Why does the lucky agent pay unlucky agent?

   It is because the lucky agent can increase the consumption in other states by reducing the consumption in lucky states, to smooth the consumption among states. Therefore, luck or not is just an idiosyncratic event.

2. Implication on asset pricing.

   Consider an asset with payments $y\left(s\right)$,

$$y\left(s\right) = y_e\left(s\right) + \varepsilon\left(s\right)$$

where y_e is correlated with e_2, while $\varepsilon(s)$ is independent from $e_2(s)$, and $\mathbb{E}[\varepsilon(s)|\, e_2(s)] = \mathbb{E}[\varepsilon(s)] = 0$.

Then the price of the asset is

$$\begin{aligned}
q(y) &= \mathbb{E}_1\left[\beta\frac{u'(e_2(s))}{u'(e_1)}y(s)\right] \\
&= \mathbb{E}_1\left[\beta\frac{u'(e_2(s))}{u'(e_1)}[y_e(s)+\varepsilon(s)]\right] \\
&= \mathbb{E}_1\left[\beta\frac{u'(e_2(s))}{u'(e_1)}y_e(s)\right] + \mathbb{E}_1\left[\beta\frac{u'(e_2(s))}{u'(e_1)}\varepsilon(s)\right] \\
&= \mathbb{E}_1\left[\beta\frac{u'(e_2(s))}{u'(e_1)}y_e(s)\right] + \mathbb{E}_1\left[\beta\frac{u'(e_2(s))}{u'(e_1)}\mathbb{E}[\varepsilon(s)|\, e_2(s)]\right] \\
&= \sum_{s=1}^{S}\pi(s)\,\beta\frac{u'(e_2(s))}{u'(e_1)}y_e(s)
\end{aligned}$$

where we may regard $\mathbb{E}[\varepsilon(s)|\, e_2(s)]$ as $\mathbb{E}_2[\varepsilon(s)]$. Recall that the general asset pricing equation is

$$q(y) = \mathbb{E}[M(s)]\,\mathbb{E}[y(s)] + \mathrm{Cov}(M(s), y(s))$$

where $M(s) = \beta\frac{u'(e_2(s))}{u'(e_1)}$. For this asset,

$$\begin{aligned}
q(y) &= \mathbb{E}[M(s)]\,\mathbb{E}[y_e(s)+\varepsilon(s)] + \mathrm{Cov}(M(s), y_e(s)+\varepsilon(s)) \\
&= \mathbb{E}[M(s)]\,\mathbb{E}[y_e(s)] + \mathrm{Cov}(M(s), y_e(s))
\end{aligned}$$

And the price of a safe asset with the same expected payment $\mathbb{E}_1[y(s)] = \mathbb{E}_1[y_e(s)]$ is

$$q_0 = \mathbb{E}_1\left[\frac{\beta u'(e_2(s))}{u'(e_1)}\mathbb{E}_1[y_e(s)]\right] = \mathbb{E}_1[M(s)]\,\mathbb{E}_1[y_e(s)]$$

The risk premium for the asset is

$$q - q_0 = \mathrm{Cov}(M(s), y_e(s))$$

Therefore, the risk premium on the risky asset depends not on idiosyncratic fluctuations in the asset payment. Instead, it depends on the correlation between the asset payment and the aggregate output e_2 reflected in the stochastic discount factor $M(s) = \beta\frac{u'(e_2(s))}{u'(e_1)}$.

Lecture 11

Overlapping Generations Model

11.1 Time 0 Trading

1. Set-ups.

   (a) Time is discrete, the economy starts at 1, so $t = 1, 2, \cdots$.

   (b) Infinite types of agents, $i = 0, 1, 2 \cdots$, indexed by the period of birth, they only live 1 period.

   (c) A single nonstorable good at each date.

   (d) The utility of agent i is

      i. $U^i(c^i) = u(c_i^i) + u\left(c_{i+1}^i\right)$ for $i \geqslant 1$.

      ii. $U^0(c^0) = u(c_1^0)$.

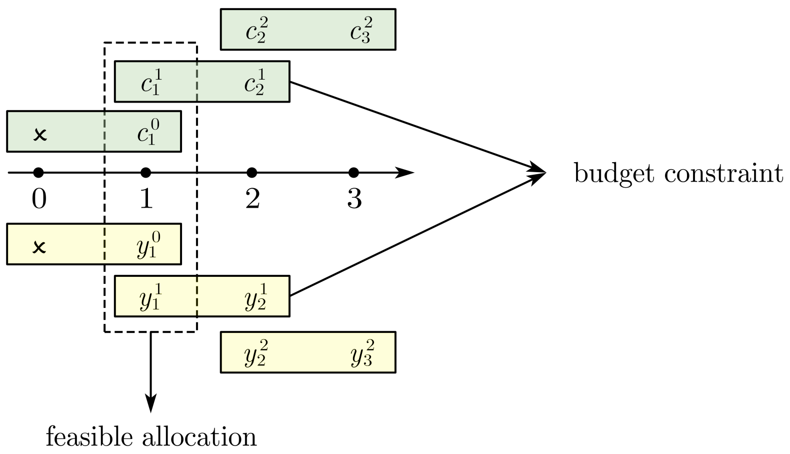

Fig. 11.1. Time 0 Trading.

   (e) Endowment of agent i: $y_i^i \geqslant 0, y_{i+1}^i \geqslant 0$, and $y_t^i = 0$ otherwise.

2. Time 0 trading: A clearinghouse at time 0.

At date 0, there are complete market in time t consumption goods with date 0 price q_t^0. A household's budget constraint is,

$$\sum_{t=1}^{\infty} q_t^0 c_t^i \leqslant \sum_{t=1}^{\infty} q_t^0 y_t^i$$

or, since $c_i^i, c_{i+1}^i \neq 0, y_i^i, y_{i+1}^i \neq 0$, the constraint for agent i becomes

$$q_i^0 c_i^i + q_{i+1}^0 c_{i+1}^i \leqslant q_i^0 y_i^i + q_{i+1}^0 y_{i+1}^i$$

For $i \neq 0$, consider

$$\begin{aligned} \max_{c_i^i, c_{i+1}^i} \ & U^i\left(c^i\right) = u\left(c_i^i\right) + u\left(c_{i+1}^i\right) \\ \text{s.t. } & q_i^0 c_i^i + q_{i+1}^0 c_{i+1}^i \leqslant q_i^0 y_i^i + q_{i+1}^0 y_{i+1}^i \end{aligned}$$

Form the Lagrangian,

$$\mathcal{L} = u\left(c_i^i\right) + u\left(c_{i+1}^i\right) + \mu^i\left(q_i^0 y_i^i + q_{i+1}^0 y_{i+1}^i - q_i^0 c_i^i - q_{i+1}^0 c_{i+1}^i\right)$$

F.O.C.s are

$$\begin{aligned} u'\left(c_i^i\right) &= \mu^i q_i^0 \\ u'\left(c_{i+1}^i\right) &= \mu^i q_{i+1}^0 \\ c_t^i &= 0, t \notin \{i, i+1\} \end{aligned}$$

And an allocation is feasible if

$$c_t^t + c_t^{t-1} \leqslant y_t^t + y_t^{t-1}, \forall t \geqslant 1$$

3. Stationary Equilibrium.

   An allocation is stationary if $\forall i \geqslant 1, c_{i+1}^i = c_o, c_i^i = c_y$, where c_o and c_y are constant.

   Here the subscript o denotes old and y denotes young.

4. Example 1.

   Let $\varepsilon \in (0, 0.5)$, the endowments are

$$\begin{aligned} y_i^i &= 1 - \varepsilon, \forall i \geqslant 1 \\ y_{i+1}^i &= \varepsilon, \forall i \geqslant 0 \\ y_t^i &= 0, \ \text{o.w.} \end{aligned}$$

   Then $y_i^i > 0.5 > y_{i+1}^i$.

   This economy has many equilibria. We describe two stationary equilibria by the guess-and-verify method.

(a) Equilibria.

- **Equilibrium H**: a high-interest-rate equilibrium.
  Set

$$\begin{aligned} q_t^0 &= 1, \forall t \geqslant 1 \\ c_i^i &= c_{i+1}^i = 0.5, \forall i \geqslant 1 \\ c_1^0 &= \varepsilon < 0.5 \end{aligned}$$

  To verify it is an equilibrium, we can see that each household's first-order conditions are satisfied and that the allocation is feasible.
  Note that for $t = 1$,

$$c_1^1 + c_1^0 = \varepsilon + 0.5 < 1 = y_1^1 + y_1^0$$

- **Equilibrium L**: a low-interest-rate equilibrium.
  Set

$$\begin{aligned} q_1^0 &= 1 \\ \frac{q_{t+1}^0}{q_t^0} &= \frac{u'(\varepsilon)}{u'(1-\varepsilon)} := \alpha > 1 \\ c_t^i &= y_t^i, \forall i \geqslant 0, t \geqslant 1 \end{aligned}$$

  Then, as the economy start, there is an equilibrium. All agent needn't to have an access to the financial market, we say it is autarkic, with prices being set to eradicate all trade.

(b) Note that the equilibrium H is not autarkic, since

$$\begin{aligned} c_i^i &= 0.5 < y_i^i = 1 - \varepsilon \\ c_{i+1}^i &= 0.5 > y_{i+1}^i = \varepsilon \end{aligned}$$

then the agents must trade in the financial market to get to the equilibrium.

(c) Why low interest rate? Since if we let $R_i := \frac{q_i^0}{q_{i+1}^0} = \frac{1}{\alpha}$, it is less than 1.

(d) Equilibrium H Pareto dominates equilibrium L. In equilibrium H every generation after the initial old one is better off and no generation is worse off than in equilibrium L.

$$u(1-\varepsilon) + u(\varepsilon) \leqslant 2u(0.5)$$

The condition of the first fundamental theorem of welfare that is violated by equilibrium L is the assumption that the value of the aggregate endowment at the equilibrium prices is finite.

5. To find equilibria, we define the offer curve.

The household's offer curve $\psi\left(c_i^i, c_{i+1}^i\right) = 0$ is the locus of $\left(c_i^i, c_{i+1}^i\right)$ that solves

$$\max_{c_i^i, c_{i+1}^i} U\left(c^i\right) = u\left(c_i^i\right) + u\left(c_{i+1}^i\right)$$
$$\text{s.t. } c_i^i + \alpha_i c_{i+1}^i \leqslant y_i^i + \alpha_i y_{i+1}^i$$

where $\alpha_i := \frac{q_{i+1}^0}{q_i^0} > 0$, is the reciprocal of the one-period gross rate of return from period i to $i+1$ and is treated as a parameter.

Evidently, the offer curve $\psi\left(c_i^i, c_{i+1}^i\right) = 0$ is the curve that solves

$$\begin{cases} c_i^i + \alpha_i c_{i+1}^i = y_i^i + \alpha_i y_{i+1}^i \\ \frac{u'\left(c_{i+1}^i\right)}{u'\left(c_i^i\right)} = \frac{q_{i+1}^0}{q_i^0} = \alpha_i \end{cases}$$

Thus, the equilibrium allocation is given by

$$\begin{cases} \psi\left(c_i^i, c_{i+1}^i\right) = 0 \\ c_i^i + c_i^{i-1} = y_i^i + y_i^{i-1} \end{cases}$$

Given c_1^0, we can compute $c_i^i, c_i^{i+1}, \forall i \geqslant 1$.

$$c_1^0 \xrightarrow{c_1^1 = y_1^0 + y_1^1 - c_1^0} c_1^1 \xrightarrow{\psi\left(c_i^i, c_{i+1}^i\right) = 0} c_2^1 \xrightarrow{c_2^2 = y_2^2 + y_2^1 - c_2^1} c_2^2 \xrightarrow{\psi\left(c_i^i, c_{i+1}^i\right) = 0} c_3^2 \to \cdots$$

and

$$q_i^0 = u'\left(c_i^i\right), \forall i \geqslant 1$$

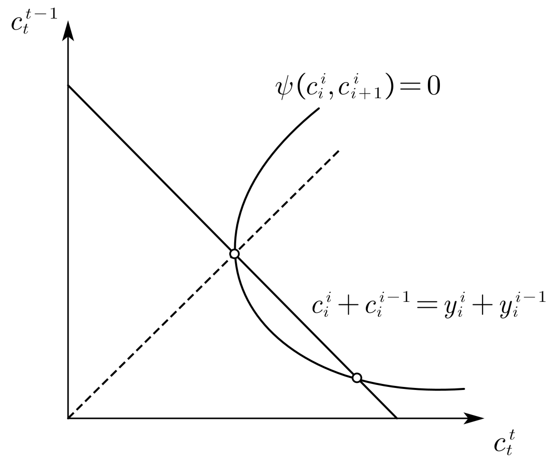

Fig. 11.2. Find Equilibria.

6. Example 2.

If we change the endowment of the initial old to be $y_1^0 = \varepsilon > 0$, and $\delta = 1 - \varepsilon > 0$ units of consumption at $t = \infty$.

$$\sum_{t=1}^{\infty} q_t^0 y_t^0 = q_1^0 \varepsilon + \delta \lim_{t \to \infty} q_t^0$$

For equilibrium H, $q_t^0 = 1, \forall t \geqslant 1$, and thus, $\alpha_t = 1, \forall t \geqslant 1$.

$$\lim_{t\to\infty} q_t^0 = 1$$

For equilibrium L, $R < 1$, then $q_t^0 = q_{t-1}^0 \frac{1}{R} = q_1^0 \frac{1}{R^{t-1}}$,

If we normalize $q_1^0 = 1$, then

$$\lim_{t\to\infty} q_t^0 = \lim_{t\to\infty} \frac{1}{R^{t-1}} = \infty$$

Confronted with such prices, the initial old would demand unbounded consumption. That is not feasible. Therefore, such a price system cannot be an equilibrium.

7. Example 3. Lucas Tree.

A Lucas tree pays d at every period, so the value of a tree is

$$V = \sum_{t=1}^{\infty} d \cdot q_t^0 = d \sum_{t=1}^{\infty} q_t^0$$

Let the initial old endowment be y_1^0, plus a "Lucas tree", then the budget constraint for the initial old person becomes

$$q_1^0 c_1^0 = d \sum_{t=1}^{\infty} q_t^0 + q_1^0 y_1^0$$

The offer curve stays the same, but the feasible allocation condition becomes

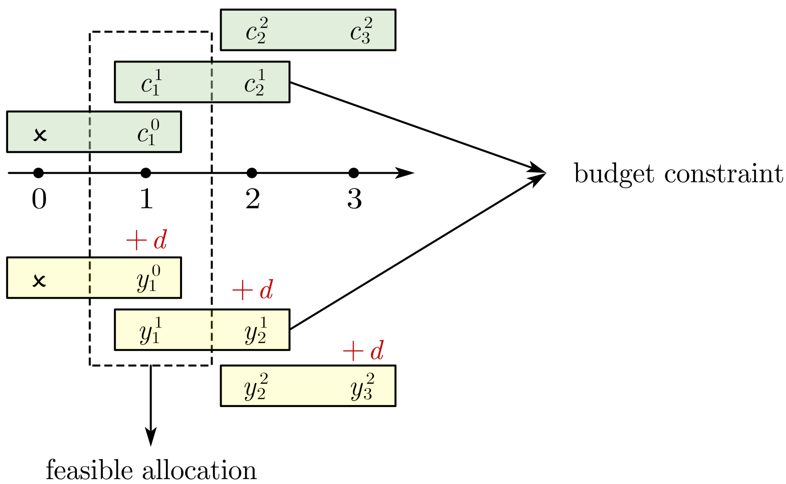

Fig. 11.3. Lucas Tree.

$$c_i^i + c_i^{i-1} = y_i^i + y_i^{i-1} + d$$

To find an equilibrium,

$$\sum_{t=1}^{\infty} q_t^0 < \infty \Longrightarrow \lim_{t\to\infty} q_t^0 = 0$$

And

$$q_t^0 = \frac{1}{R^{t-1}} \to 0 \Longrightarrow R > 1 \Longrightarrow \alpha < 1$$

Thus, the only equilibrium has $\alpha < 1$.

11.2 Sequential Trading

1. Set-up.

   (a) We abandon the time 0, complete markets trading arrangement and replace it with sequential trading in which a durable asset.

   (b) At date $t = 1$, old agents are endowed in the aggregate with $M > 0$ units of intrinsically worthless currency.

   (c) No one has promised to redeem the currency for goods (inconvertible or fiat).

   Note: Samuelson (1958) showed, there exists a system of expectations that makes the unbacked currency be valued.

   (d) For each date $t \geqslant 1$, young agents purchase m_t^i units of currency at a price of $\frac{1}{p_t}$ units of the time t consumption good, where $p_t > 0$ is the time t price level.

   Thus, in total, the agent spends $\frac{m_i^i}{p_i}$ units of consumption good to get the currency.

   (e) For each date $t \geqslant 1$, each old agent exchanges his holdings of currency for time t consumption good.

2. Budget constraint.

   The budget constraints of a young agent born in $i \geqslant 1$ are

$$
\begin{aligned}
c_i^i + \frac{m_i^i}{p_i} &\leqslant y_i^i \\
c_{i+1}^i &\leqslant y_{i+1}^i + \frac{m_i^i}{p_{i+1}} \\
m_i^i &\geqslant 0
\end{aligned}
$$

   Given $m_i^i \geqslant 0$,

$$\left(c_{i+1}^i - y_{i+1}^i\right) p_{i+1} \leqslant m_i^i \leqslant \left(y_i^i - c_i^i\right) p_i \Longrightarrow y_i^i + y_{i+1}^i \frac{p_{i+1}}{p_i} \geqslant c_i^i + c_{i+1}^i \frac{p_{i+1}}{p_i}$$

   We set

$$\frac{p_{i+1}}{p_i} = \alpha_i = \frac{q_{i+1}^0}{q_i^0}$$

3. Equilibrium.

   (a) A nominal price sequence is a positive sequence $\{p_i\}_{i \geqslant 1}$.

   (b) An equilibrium with valued fiat money is a feasible allocation and a nominal price sequence with $p_t < \infty, \forall t \geqslant 1$ such that given the price sequence $\{p_i\}_{i \geqslant 1}$, the allocation solves the household's problem for each $i \geqslant 1$.

   (c) We call $p_t < \infty$, i.e., the fiat money is valued, a "monetary equilibrium". And if $\frac{1}{p_t} = \infty$, we call it a "nonmonetary equilibrium".

4. Computing the equilibria.

   Define a saving function s as

$$s\left(\alpha_i; y_i^i, y_{i+1}^i\right) = y_i^i - c_i^i$$

   Whenever is an equilibrium, at time i, the net saving of generation i should equal to the net dissaving of generation $i-1$, then

$$s\left(\alpha_i; y_i^i, y_{i+1}^i\right) = \frac{M}{p_i} \Longleftrightarrow y_i^i - c_i^i = \frac{M}{p_i}$$

   where $\alpha_i = \frac{q_{i+1}^0}{q_i^0} = \frac{p_{i+1}}{p_i}$.

5. Example.

   If $u(c) = \ln c$, and $\left(y_i^i, y_{i+1}^i\right) = (w_1, w_2)$, with $w_1 > w_2$.

   The saving function is

$$s(\alpha_i) = \frac{1}{2}(w_1 - \alpha_i w_2)$$

   Therefore,

$$\frac{1}{2}\left(w_1 - \frac{p_{t+1}}{p_t}w_2\right) = \frac{M}{p_t} \Longrightarrow p_t = \frac{2M}{w_1} + \frac{w_2}{w_1}p_{t+1}$$

   the solution for this difference equation is

$$p_t = \frac{2M}{w_1\left(1 - \frac{w_2}{w_1}\right)} + c\left(\frac{w_2}{w_1}\right)^t$$

   where $c \geqslant 0$ is a constant.

   If $c = 0$, there is a unique stationary solution.

   If $c > 0$, the solutions have uniformly higher price levels than the $c = 0$ solution, since $\frac{w_2}{w_1} < 1$, $\lim_{t\to\infty} p_t = p_t|_{c=0}$.

6. Equivalence.

   Let $\bar{c}^i$ denote a competitive equilibrium allocation with time 0 trading (trade by Arrow-Debreu security) and suppose it satisfies $\bar{c}_1^1 < y_1^1$. Then there exists an equilibrium with sequential trading (trade by fiat money) of the monetary economy with allocation that satisfies

$$c_i^i = \bar{c}_i^i, i \geqslant 1$$
$$c_{i+1}^i = \bar{c}_{i+1}^i, i \geqslant 1$$

   Note: $\bar{c}_1^1 < y_1^1$ implies that the initial endowment of young generation is higher than the initial consumption, thus, the agent has some left goods to trade.