Notes for Macroeconomics II

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Contents

Lecture 1

Ricardian Equivalence (RMT Ch10)

1.1 Settings and Solutions

1. Problem and goals.

   (a) Question: The government is fighting a war, whose cost is exogenous. How would the government tax the household to finance the war?

      i. How?

         - Lump-sum taxes.[1]
         - Government can borrow.

      ii. When?

         - The timing of tax can be varied. E.g., Tax now, or later; in one shot or split it into small installments, etc.

   (b) Objectives.

      i. Study the effect of the timing of taxes.

      ii. Identify conditions under which Ricardian Equivalence holds.

      iii. Review some key tools.

2. Set-ups.

   (a) There are N infinitely lived households in an economy, each trying to maximize a utility over a perishable consumption stream $\{c_t\}$

   $$\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

   where $\beta \in (0,1)$, $u'\left(c\right) > 0, u''\left(c\right) < 0$, and Inada conditions hold ($\lim_{c \to 0} u'\left(c\right) = \infty$, $\lim_{c \to \infty} u'\left(c\right) = 0$).

[1] A lump-sum tax is the same for everyone, which is cannot be evaded, such as head tax (poll tax). It is different from the proportional tax, whose amount depends on other things, such as income or wealth. For example, if the government levy tax on labor (or long hair), people can work less (cut them short) to evade the tax. Moreover, mathematically, the lump-sum tax won't show up in the F.O.C.s of the household.

Also note that there is no uncertainty.

(b) Income is determined by an exogenous endowment process (or a Lucas tree) $\{y_t\}_{t=0}^{\infty}$.

(c) There is a financial market "outside the model".

i. The gross interest rate if exogenously given,[2] which is fixed at R, with $\beta R = 1$.

ii. Perhaps the economy is a "small open economy", note that the economy is so small to have any effect on the interest rate.

(d) Bond:

i. b_t denotes unites of bonds that household bought in period $t-1$.

ii. Each bound is an "IOU" that promises to pay 1 unit of consumption in period t.

(e) The household's period (or iterated, recursive) budget constraint is

$$c_t + R^{-1} b_{t+1} \leqslant y_t + b_t$$

The RHS is the income in period t, and the LHS is the expenditure in period t. Note that $R^{-1} b_{t+1}$ is the price of the bond that pays b_{t+1} units of consumption at period $t+1$, it also can be regarded as the saving of the household.

3. Solution.

$$\max_{\{c_t, b_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } c_t + R^{-1} b_{t+1} \leqslant y_t + b_t$$

Form the Lagrangian,

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + \lambda_t \left(y_t + b_t - c_t - R^{-1} b_{t+1} \right) \right]$$

The F.O.C.s are (Note that $\beta R = 1$)

$$\begin{cases} u'(c_t) - \lambda_t = 0 \\ -\lambda_t R^{-1} + \lambda_{t+1} \beta = 0 \end{cases} \Longrightarrow \begin{cases} u'(c_t) = \lambda_t \\ \lambda_t = \lambda_{t+1} \end{cases} \Longrightarrow c_{t+1} = c_t$$

That is, the optimal consumption path is completely smoothed. Then consider the constraint, which should be binding,

$$\begin{aligned} b_t &= c_t + R^{-1} b_{t+1} - y_t \\ &= c_t - y_t + R^{-1} \left(c_{t+1} + R^{-1} b_{t+2} - y_{t+1} \right) \\ &= c_t + R^{-1} c_{t+1} - \left(y_t + R^{-1} y_{t+1} \right) + R^{-2} b_{t+2} \\ &= \cdots \\ &= \sum_{j=0}^{\infty} R^{-j} c_{t+j} - \sum_{j=0}^{\infty} R^{-j} y_{t+j} + \lim_{j \to \infty} R^{-(j+1)} b_{t+j+1} \end{aligned}$$

[2] Later we will describe general equilibrium analyses of the Ricardian doctrine where the interest rate is determined within the model.

We should impose the no-Ponzi scheme condition as

$$\lim_{j\to\infty} R^{-j} b_{t+j} = 0$$

Therefore, the **permanent budget constraint** is

$$b_t + \sum_{j=0}^{\infty} R^{-j} y_{t+j} = \sum_{j=0}^{\infty} R^{-j} c_{t+j}$$

The LHS is the income and the RHS is the consumption. Then, **the optimal consumption** $\{c_t\}$ satisfies

$$\begin{cases} c_t = c \\ b_t + \sum_{j=0}^{\infty} R^{-j} y_{t+j} = \sum_{j=0}^{\infty} R^{-j} c_{t+j} \end{cases} \Longrightarrow \frac{c}{1-\beta} = b_t + \sum_{j=0}^{\infty} R^{-j} y_{t+j}$$

then

$$c = (1-\beta)\left(b_t + \sum_{j=0}^{\infty} R^{-j} y_{t+j}\right)$$

Note that $b_t + \sum_{j=0}^{\infty} R^{-j} y_{t+j}$ represents the permanent wealth of the household, and they draw $1-\beta$ of it to consume. Moreover, from the permanent budget constraint, if we write

$$\begin{aligned} b_t &= \sum_{j=0}^{\infty} R^{-j} c_{t+j} - \sum_{j=0}^{\infty} R^{-j} y_{t+j} \\ R^{-1} b_{t+1} &= \sum_{j=0}^{\infty} R^{-j+1} c_{t+1+j} - \sum_{j=0}^{\infty} R^{-j+1} y_{t+1+j} \\ &= \sum_{j=0}^{\infty} R^{-j} c_{t+j} - \sum_{j=0}^{\infty} R^{-j} y_{t+j} - c_t + y_t \end{aligned}$$

and make a subtraction,

$$b_t - R^{-1} b_{t+1} = c_t - y_t \Longrightarrow c_t + R^{-1} b_{t+1} = b_t + y_t$$

we will exactly get the period budget constraint, that is, those two budget constraints are equivalent.

4. More set-ups and modifications.

   (a) There is a government, who needs to finance an exogenous stream of expenditure $\{g_t\}$ by

      i. Levying tax τ_t on the households, and/or

      ii. Borrowing B_{t+1} from the financial market.

   (b) B_{t+1} is the debt the government issues in period t, to be paid in period $t+1$.

(c) The budget constraint for the government

$$g_t + B_t = \tau_t + R^{-1} B_{t+1}$$

Note that $R^{-1}B_{t+1}$ is the price of the bond issued by the government. And then, the LHS is the government's expenditure, and the RHS is the income.

5. Solution.

Given the budget constraint of the government's action, we can get

$$\begin{aligned} B_t &= \tau_t - g_t + R^{-1} B_{t+1} \\ &= \tau_t - g_t + R^{-1}\left(\tau_{t+1} - g_{t+1} + R^{-1} B_{t+2}\right) \\ &= \cdots \\ &= \sum_{j=0}^{\infty} R^{-j}\tau_{t+j} - \sum_{j=0}^{\infty} R^{-j} g_{t+j} + \lim_{j\to\infty} R^{-(j+1)} B_{t+j+1} \end{aligned}$$

By imposing

$$\lim_{j\to\infty} R^{-j} B_{t+j} = 0$$

we get the **permanent budget constraint** of the government

$$B_t = \sum_{j=0}^{\infty} R^{-j}\tau_{t+j} - \sum_{j=0}^{\infty} R^{-j} g_{t+j} = \sum_{j=0}^{\infty} R^{-j}\left(\tau_{t+j} - g_{t+j}\right)$$

which implies that the debt issued by the government will be paid by the tax in the end. Now, for the household, since they need to pay taxes, their budget constraint becomes

$$c_t + \tau_t + R^{-1} b_{t+1} \leqslant y_t + b_t$$

We can define $\tilde{y}_t := y_t - \tau_t$, which is the **disposable income** of the household, then the constraint becomes

$$c_t + R^{-1} b_{t+1} \leqslant \tilde{y}_t + b_t$$

Then, **the optimal consumption** $\{c_t\}$ is similarly

$$c = (1-\beta)\left(b_t + \sum_{j=0}^{\infty} R^{-j}\tilde{y}_{t+j}\right) = (1-\beta)\left(b_t + \sum_{j=0}^{\infty} R^{-j}\left(y_{t+j} - \tau_{t+j}\right)\right)$$

1.2 A Simple Example

1. Example.

(a) Assume that $b_0 = B_0 = 0$.

(b) Suppose the government needs to finance an expenditure plan $\{g_0, 0, 0, \cdots\}$, and chooses between two alternative ways to finance the period 0 expenditure.

i. Plan A. Use tax $\tau_0 = g_0$ to pay off the expenditure, in other words,

$$\{\tau_t\}_{t=0}^{\infty} = \{g_0, 0, 0, \cdots\}$$

ii. Plan B. Borrow to finance g_0 in period 0, and pay off the debt in period 1, in other words,

$$\{\tau_t\}_{t=0}^{\infty} = \{\tau_0 = 0, \tau_1 = g_0 R, 0, 0, \cdots\}$$

(c) Assume that households have a stable Lucas tree which is $\forall t \geqslant 1, y_t = y$, where y a constant.

2. Plan A.

If the government tax, then $\{\tau_t\}_{t=0}^{\infty} = \{g_0, 0, 0, \cdots\}$, the **optimal consumption plan** for the household is

$$c = (1-\beta)\left(b_t + \sum_{j=0}^{\infty} R^{-j}\left(y_{t+j} - \tau_{t+j}\right)\right) = (1-\beta)\left(b_t + \sum_{j=0}^{\infty} \beta^j y - \tau_0\right)$$

which is a constant, then we can let $t = 0$, since $b_0 = 0$, $c = (1-\beta)\left(\frac{1}{1-\beta}y - \tau_0\right) = y - (1-\beta)\, g_0$

Then what's **the asset position** ?

Solution 1. Recall the permanent budget constraint,

$$\begin{aligned} b_t &= \sum_{j=0}^{\infty} R^{-j} c_{t+j} - \sum_{j=0}^{\infty} R^{-j} \tilde{y}_{t+j} \\ &= \sum_{j=0}^{\infty} R^{-j} c_{t+j} - \sum_{j=0}^{\infty} R^{-j}\left(y_{t+j} - \tau_{t+j}\right) \end{aligned}$$

Let $t = 1$,

$$\begin{aligned} b_1 &= \sum_{j=0}^{\infty} R^{-j} c_{1+j} - \sum_{j=0}^{\infty} R^{-j}\left(y_{1+j} - \tau_{1+j}\right) \\ &= \frac{c}{1-\beta} - \sum_{j=0}^{\infty} R^{-j} y_{1+j} \\ &= \frac{c-y}{1-\beta} = -g_0 \end{aligned}$$

Likewise,

$$\forall t \geqslant 1, b_t = b_1 = -g_0$$

□

Solution 2. Recall the budget constraint,

$$c_t + R^{-1}b_{t+1} = \tilde{y}_t + b_t \Longrightarrow y - (1-\beta)\,g_0 + R^{-1}b_{t+1} = y - \tau_t + b_t$$

Let $t = 0$, we have

$$b_1 = \frac{1}{\beta}\left[y - g_0 - y + (1-\beta)\,g_0\right] = -g_0$$

Let $t = 1$, we have

$$b_2 = \frac{1}{\beta}\left[(1-\beta)\,g_0 + b_1\right] = -g_0$$

Then by induction, we can conduct that $\forall t \geqslant 1, b_t = b_1 = -g_0$.

So,

$$\{b_t\}_{t=0}^{\infty} = \{0, -g_0, -g_0, \cdots\}$$

□

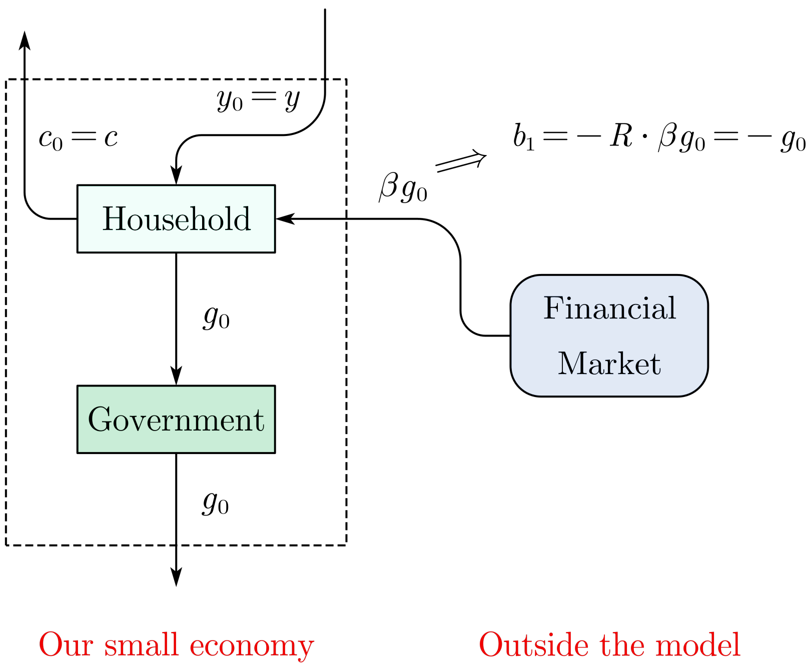

Fig. 1.1. Period 0.

In period 0, household gets y as income, consume $y - (1-\beta)\,g_0$, and give g_0 to the government. Therefore, they need to borrow βg_0 from the financial market, i.e., they short sell a bond [3] with face value g_0 and get the price of this bond βg_0.

In period 1, household need to pay g_0 back to the one bought his or her bond in period 0. And to meet the consumption goal, the household should borrow βg_0 again from the market.

Overall, the household borrows the entire amount βg_0 in period 0 and carry it as a permanent debt. Each period, it pays the interest rate $(R-1)\,\beta g_0 = (1-\beta)\,g_0$ on the debt, but not the principal.

[3] Or "issue" a bond.

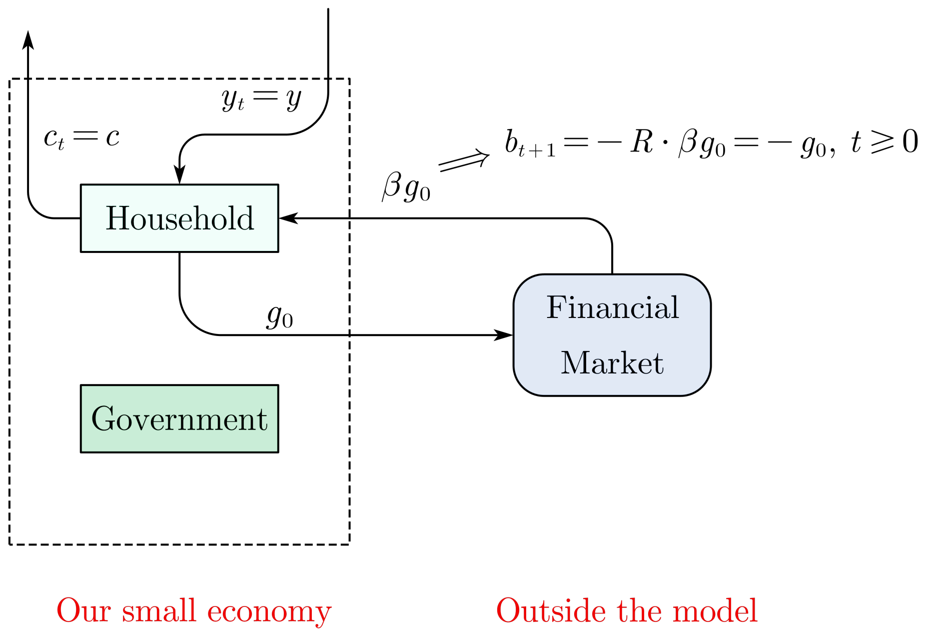

Fig. 1.2. Period 1 and forward.

It may sound incredible, does the model allows the household not to pay the principal? Yes. Recall that our household lives permanently. And the no-Ponzi game condition says

$$\lim_{j\to\infty} R^{-j} b_{t+j} = 0$$

In the example, $\{b_t\}_{t=0}^{\infty} = \{0, -g_0, -g_0, \cdots\}$, it trivially satisfies the condition.

3. Plan B.

$$\{\tau_t\}_{t=0}^{\infty} = \{\tau_0 = 0, \tau_1 = g_0 R, 0, 0, \cdots\}$$

For the household,

$$\begin{aligned} c_t &= (1-\beta)\left[\sum_{j=1}^{\infty} R^{-j}\tilde{y}_{t+j} + b_t\right] \\ &= c_0 = (1-\beta)\left[\sum_{j=1}^{\infty} R^{-j} y_j - \sum_{j=1}^{\infty} R^{-j}\tau_j\right] \\ &= (1-\beta)\left[\frac{y}{1-\beta} - R^{-1}\cdot R\tau_1\right] \\ &= y - (1-\beta)\, g_0 \end{aligned}$$

For the asset position of the household, recall that

$$c_t + R^{-1} b_{t+1} = \tilde{y}_t + b_t$$

Then

$$\begin{aligned} b_1 &= R\left(y - \tau_0 + b_0 - y + (1-\beta)\, g_0\right) = (R-1)\, g_0 \\ b_2 &= R\left(y - \tau_1 + b_1 - y + (1-\beta)\, g_0\right) = -g_0 \\ b_3 &= R\left(y - \tau_2 + b_2 - y + (1-\beta)\, g_0\right) = -g_0 = b_t, t \geqslant 2 \end{aligned}$$

Overall,

$$\{b_t\}_{t=0}^{\infty} = \{0, (R-1)\, g_0, -g_0, -g_0, \cdots\}$$

The household actually postpone to borrow, and they know they will be taxed in period 1, so they save $\beta\,(R-1)\, g_0 = (1-\beta)\, g_0$ in period 0 to meet the optimal consumption.

As for the government,

$$g_t + B_t = \tau_t + R^{-1} B_{t+1}$$

We can get

$$\begin{aligned} B_1 &= g_0 R \\ B_2 &= R\,(B_1 - g_0 R) = 0 \\ B_3 &= 0 = B_t, t \geqslant 2 \end{aligned}$$

then

$$\{B_t\}_{t=0}^{\infty} = \{0, Rg_0, 0, 0, \cdots\}$$

4. Summary.
   (a) The government in Plan A has zero asset position, but in Plan B, the government borrow g_0 in period 0 and pay back $g_0 R$ in period 1 by taxing.
   (b) Plan A to Plan B:
      i. Government's period-1 net saving decreases by Rg_0 (they borrow more or save less in Plan B).
      ii. The household's net position increased by Rg_0 (they borrow less or save more in Plan B).
5. (Definition) Equilibrium. Given initial conditions (b_0, B_0), and equilibrium is a household's plan $\{c_t, b_{t+1}\}_{t=0}^{\infty}$ and the government's policy $\{g_t, \tau_t, B_{t+1}\}_{t=0}^{\infty}$ such that
   (a) The government policy satisfies the government budget constraint.
   (b) Given $\{\tau_t\}_{t=0}^{\infty}$, the household's plan is optimal.
6. (Theorem) The Ricardian Equivalence.

   Given initial conditions $\{b_0, B_0\}$, suppose $\left\{\bar{g}_t, \bar{\tau}_t, \bar{B}_{t+1}\right\}_{t=0}^{\infty}$ and $\left\{\bar{c}_t, \bar{b}_{t+1}\right\}_{t=0}^{\infty}$ is an equilibrium. Then, under an alternative tax plan $\{\hat{\tau}_t\}_{t=0}^{\infty}$ such that

$$\sum_{t=0}^{\infty} R^{-t} \hat{\tau}_t = \sum_{t=0}^{\infty} R^{-t} \bar{\tau}_t$$

$\left\{\bar{c}_t, \hat{b}_{t+1}\right\}_{t=0}^{\infty}$ and $\left\{\bar{g}_t, \hat{\tau}_t, \hat{B}_{t+1}\right\}_{t=0}^{\infty}$ is also an equilibrium, where

$$\hat{b}_t = \sum_{j=0}^{\infty} R^{-j} \left(\bar{c}_{t+j} + \hat{\tau}_{t+j} - y_{t+j}\right)$$

$$\hat{B}_t = \sum_{j=0}^{\infty} R^{-j} \left(\hat{\tau}_{t+j} - \bar{g}_{t+j}\right)$$

Note: The proof is just a restate of the definition of the equilibrium.

(a) $\{\bar{c}_t\}_{t=0}^{\infty}$ solves household's utility maximization problem.

$$\max_{\{c_t,b_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{aligned} s.t. \sum_{j=0}^{\infty} R^{-j} c_{t+j} &= \bar{b}_t + \sum_{j=0}^{\infty} R^{-j} y_{t+j} - \sum_{j=0}^{\infty} R^{-j} \bar{\tau}_{t+j} \\ &= \hat{b}_t + \sum_{j=0}^{\infty} R^{-j} y_{t+j} - \sum_{j=0}^{\infty} R^{-j} \hat{\tau}_{t+j} \end{aligned}$$

Note that, given the no-Ponzi game condition, the recursive budget constraint is equivalent with the permanent budget constraint.

(b) $\left\{\hat{b}_{t+1}, \hat{B}_{t+1}\right\}_{t=1}^{\infty}$ satisfies the budget constraint for household and government respectively.

(c) Note that $\left\{\hat{b}_{t+1}\right\}_{t=0}^{\infty}$ satisfies the transversality condition as well. Firstly, recall the household constraint,

$$R^{-1} b_{t+1} + c_t \leqslant y_t - \tau_t + b_t$$

then we solve b_K backwards as

$$\begin{aligned} b_K &= R(y_{K-1} - \tau_{K-1} + b_{K-1} - c_{K-1}) \\ &= R(y_{K-1} - \tau_{K-1} - c_{K-1}) + R^2 (y_{K-2} - \tau_{K-2} + b_{K-2} - c_{K-2}) \\ &= \cdots \\ &= \sum_{j=0}^{K} R^j (y_{K-j} - \tau_{K-j} - c_{K-j}) + R^K b_0 \end{aligned}$$

therefore,

$$\bar{b}_K - \hat{b}_K = \sum_{j=0}^{K} R^{-j} (\bar{\tau}_{K-j} - \hat{\tau}_{K-j})$$

and thus,

$$R^{-K}\left(\bar{b}_K - \hat{b}_K\right) = \sum_{j=0}^{K} R^{-j-K} (\bar{\tau}_{K-j} - \hat{\tau}_{K-j}) = \sum_{j=0}^{K} R^{-j} (\bar{\tau}_j - \hat{\tau}_j)$$

Let $K \to \infty$, then $\sum_{j=0}^{\infty} R^{-j} (\bar{\tau}_j - \hat{\tau}_j) = 0$, and thus,

$$\lim_{K\to\infty} R^{-K}\hat{b}_K = \lim_{K\to\infty} R^{-K}\bar{b}_K = 0$$

(d) For $\left\{\hat{B}_{t+1}\right\}_{t=1}^{\infty}$, the transversality condition is satisfied similarly.

7. (Remark) What is the necessary condition for Ricardian Equivalence?

(a) Perfect financial market for household and the government.

i. They can borrow at the same interest rate.

ii. The amount of borrowing is unlimited.

(b) Permanently lived households, and thus, they don't care about their descendants.

(c) Government has a lump-sum tax, which is **non-distortionary** and cannot be evaded.

1.3 Borrowing Constraints

1. More modifications.

   Previously, the households are allowed to borrow an unlimited amount of funds, but now we can make some restrictions.

   Consider 2 types of constraints:

   (a) The household can only save, not borrow: $b_t \geqslant 0, \forall t \geqslant 0$.

   (b) The household can borrow up to the present value of its current and expected future endowments: $b_t \geqslant \tilde{b}_t := -\sum_{j=0}^{\infty} R^{-j} y_{t+j}$. The constraint is called "natural debt limit" by Aiyagari (1994).

2. There are 2 ways to see if the borrowing constraint is binding (i.e., if the household is borrowing constrained).

   (a) Method 1: Form a new Lagrangian, to see if the multiplier of the constraint is zero.

   (b) Method 2 (we will use):

   i. Get the optimal path without any borrowing constraint.

   ii. Check if the stream of asset position satisfies the borrowing constraint.

3. (Example) Assume $b_0 = 0$ and $\{y_t\}_{t=0}^{\infty} = \{y_h, y_l, y_h, y_l, \cdots\}$ where $y_h > y_l > 0$, then the optimal consumption is

$$
\begin{aligned}
c &= (1-\beta)\left(b_0 + \sum_{j=0}^{\infty} \beta^j y_j\right) \\
&= (1-\beta)\left(\sum_{j=0}^{\infty} \beta^{2j} y_h + \sum_{j=0}^{\infty} \beta^{2j+1} y_l\right) \\
&= (1-\beta)\left(\frac{y_h}{1-\beta^2} + \frac{\beta y_l}{1-\beta^2}\right) \\
&= (1-\beta)\frac{y_h + \beta y_l}{1-\beta^2} \\
&= \frac{y_h + \beta y_l}{1+\beta} < \frac{y_h + \beta y_h}{1+\beta} \\
&= y_h
\end{aligned}
$$

As for the position of bonds,

$$b_1 = R\left(b_0 + y_0 - c_0\right) = R\left(y_h - \frac{y_h + \beta y_l}{1+\beta}\right) = \frac{\beta\left(y_h - y_l\right)}{\beta\left(1+\beta\right)} = \frac{y_h - y_l}{1+\beta} > 0$$

$$b_2 = R\left(b_1 + y_1 - c_1\right) = R\left(\frac{y_h - y_l}{1+\beta} + y_l - \frac{y_h + \beta y_l}{1+\beta}\right) = 0$$

Likewise,

$$b_t = \begin{cases} \frac{y_h - y_l}{1+\beta}, & t = 2n-1 \\ 0, & t = 2n \end{cases}, n = 0, 1, 2, \cdots$$

Note that $b_t \geqslant 0$, therefore, the "natural debt limit" constraint is never binding.

In this example, the household actually use the income in good states to finance the bad state.

4. (Example) Assume $b_0 = 0$ and $\{y_t\}_{t=0}^{\infty} = \{y_l, y_h, y_l, y_h, \cdots\}$ where $y_h > y_l > 0$.

   Then, regardless of the borrowing constraint, the optimal consumption is

$$c = \frac{y_l + \beta y_h}{1+\beta} > \frac{y_h + \beta y_h}{1+\beta} = y_l$$

   the asset position is

$$b_1 = R\left(b_0 + y_0 - c_0\right) = R\left(y_l - \frac{y_l + \beta y_h}{1+\beta}\right) = \frac{y_l - y_h}{1+\beta} < 0$$

   We now discuss two constraints.

   (a) $b_t \geqslant 0, \forall t \geqslant 0$.

   Then the constraint is binding, i.e., $b_1 = 0$ and $c_0 = y_l$, and from period 1 onward, the solution is $c_t = \frac{y_h + \beta y_l}{1+\beta}, \forall t \geqslant 1$. The asset position is

$$b_t = \begin{cases} \frac{y_h - y_l}{1+\beta}, & t = 2n \\ 0, & t = 2n+1 \end{cases}, n = 1, 2, \cdots$$

   (b) The "natural debt limit".

   Check that $\tilde{b}_1 = -\sum_{j=0}^{\infty} R^{-j} y_{1+j} = -\frac{y_h + \beta y_l}{1-\beta^2}$, and

$$b_1 - \tilde{b}_1 = \frac{y_h - y_l}{1+\beta} - \frac{y_h + \beta y_l}{1-\beta^2} = \frac{y_h - y_l - \beta y_h + \beta y_l - y_h - \beta y_l}{1-\beta^2} = \frac{-\beta y_h - y_l}{1-\beta^2} < 0$$

   therefore, the constraint "natural debt limit" is non-binding.

   Then,

$$c_t = \frac{y_l + \beta y_h}{1+\beta}, t \geqslant 0$$

   and for odd t,

$$b_t = R\left(b_0 + y_0 - c_0\right) = \frac{y_l - y_h}{1+\beta}$$

   for even t,

$$b_t = R\left(b_1 + y_1 - c_1\right) = R\left(\frac{y_l - y_h}{1+\beta} + y_h - \frac{y_l + \beta y_h}{1+\beta}\right) = 0$$

5. (Example) $b_0 = 0$, $y_t = \lambda^t$, and $1 < \lambda < R$. With no borrowing constraint, the optimal consumption is (note that $\beta\lambda < 1$).

$$c = (1-\beta)\sum_{j=0}^{\infty} R^{-j}\lambda^j = \frac{1-\beta}{1-\lambda\beta}$$

and then,

$$\begin{aligned}
b_1 &= R\left(y_0 + b_0 - c_0\right) = R\left(1 - \frac{1-\beta}{1-\lambda\beta}\right) = \frac{1-\lambda}{1-\lambda\beta} < 0 \\
b_2 &= R\left(y_1 + b_1 - c_1\right) = R\left(\lambda + \frac{1-\lambda}{1-\lambda\beta} - \frac{1-\beta}{1-\lambda\beta}\right) \\
&= \frac{1}{\beta}\frac{\lambda - \lambda^2\beta - \lambda + \beta}{1-\lambda\beta} = \frac{1-\lambda^2}{1-\lambda\beta} < 0 \\
b_3 &= R\left(y_2 + b_2 - c_2\right) = R\left(\lambda^2 + \frac{1-\lambda^2}{1-\lambda\beta} - \frac{1-\beta}{1-\lambda\beta}\right) \\
&= \frac{1}{\beta}\frac{\lambda^2 - \lambda^3\beta - \lambda^2 + \beta}{1-\lambda\beta} = \frac{1-\lambda^3}{1-\lambda\beta} < 0 \\
&\vdots \\
b_t &= \frac{1-\lambda^t}{1-\lambda\beta} < 0
\end{aligned}$$

We now discuss two constraints.

(a) $b_t \geqslant 0, \forall t$.

Then $b_1 = 0$, and $c_0 = y_0 + b_0 - b_1 R^{-1} = y_0$.

Likewise, $b_2 = 0$, then $c_2 = y_2$.

Generally, $b_t = 0$, and $c_t = y_t$.

(b) $b_t \geqslant \tilde{b}_t = -\sum_{j=0}^{\infty} R^{-j} y_{t+j} = -\sum_{j=0}^{\infty} (\beta\lambda)^j \cdot \lambda^t = \frac{-\lambda^t}{1-\lambda\beta}, \forall t$.

(c) Note that $b_t - \tilde{b}_t = \frac{1}{1-\lambda\beta} > 0$, then the constraint "natural debt limit" is non-binding.

6. (Example) $b_0 = 0$, $y_t = \lambda^t$, but $\lambda < 1$. Then the household is never borrowing constrained.

$$c_t = (1-\beta)\sum_{j=0}^{\infty} R^{-j}\lambda^j = \frac{1-\beta}{1-\lambda\beta} b_t = \frac{1-\lambda^t}{1-\lambda\beta} > 0$$

7. In what scenario the "natural debt limit" is binding?

Recall the permanent budget constraint,

$$b_t = \sum_{j=0}^{\infty} R^{-j} c_{t+j} - \sum_{j=0}^{\infty} R^{-j} y_{t+j}$$

and since $\tilde{b}_t = -\sum_{j=0}^{\infty} R^{-j} y_{t+j}$, if $b_t = \tilde{b}_t$, then

$$c_{t+j} = 0, \forall j \geqslant 0$$

Intuitively, if the “natural debt limit” is binding, then the agent will spend all her future income to pay the debt.

Therefore, in common cases, households won’t borrow that much, as long as they need to maximize their lifetime utility.

Lecture 2

Fiscal Policy in a Growth Model (RMT Ch11)

2.1 Settings and Solutions

1. Review and Goals.

   (a) Ricardian equivalence is under lump-sum taxes, which doesn't show up in the F.O.C.s of the household.

   (b) We need to enrich the model.

      i. Proportional taxes.

      ii. Endogenous income (from labor and capital).[1]

   (c) In addition, we

      i. endogenize the equilibrium interest rate.

      ii. study capital accumulation.

      iii. use the model to study solution methods.

2. Set-ups.

   (a) No uncertainty.

   (b) Households work and save. They have two sources of income, from labor $w_t n_t$ and capital renting $\eta_t k_t$.

   (c) The total endowment of leisure is 1.

   (d) Firms rent capital form household and hire labors to produce a single consumption good c_t.

   (e) Government needs to finance an exogenous expenditure stream $\{g_t\}_{t=0}^{\infty}$, and can impose taxes $\{\tau_{ct}, \tau_{kt}, \tau_{nt}, \tau_{ht}\}_{t=0}^{\infty}$:

[1] If the income is exogenous, mathematically, the proportional taxes on exogenous income are still the same as lump-sum taxes (won't show up in the F.O.C.s).

i. τ_{ct}: proportional tax rate on consumption

ii. τ_{kt}: proportional tax rate on capital income

iii. τ_{nt}: proportional tax on wage income

iv. τ_{ht}: lump-sum tax ("head" tax or poll tax)

(f) A grand market is open at time 0 to allow households, firms and government to trade claims to consumption from every period.

3. Specific set-ups.

(a) Households: The representative household maximize the utility

$$\max_{\{c_t,n_t,k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) = \max_{\{c_t,n_t,k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(c_t, 1-n_t)$$

where $\beta \in (0,1)$, $U_c > 0, U_{cc} < 0, U_l > 0$.[2] $1 - n_t$ is leisure.

$$\text{s.t.} \quad \sum_{t=0}^{\infty} q_t \left[(1+\tau_{ct}) c_t + k_{t+1} - (1-\delta) k_t\right] \leqslant$$

$$\sum_{t=0}^{\infty} q_t \left[\eta_t k_t - \tau_{kt} (\eta_t - \delta) k_t + (1-\tau_{nt}) w_t n_t - \tau_{ht}\right]$$

The LHS of the constraint is the expenditure, and the RHS is the income.

- c_t is the consumption, $\tau_{ct} c_t$ is the tax on consumption.
- k_t is the capital stock at the beginning of period t, and $k_{t+1} - k_t$ is the expenditure of investment, and δk_t is the depreciation. Hence, $k_{t+1} - (1-\delta) k_t$ is the net investment or expenditure of capital.
- q_t is the time-0 prices of time t consumption good. (We normalize $q_0 = 1$, the price quoted in time-0 consumption good).[3]
- η_t is the rental price of capital in period t, quoted in the time t consumption goods.
- w_t is the wage rate in period t, quoted in the time t consumption goods.
- $\eta_t k_t - \tau_{kt} (\eta_t - \delta) k_t$ is the capital income after depreciation. And note that the government tax the capital after depreciation, this is why capital tax is $\tau_{kt} (\eta_t - \delta) k_t$ rather than $\tau_{kt} \eta_t k_t$.
- $(1-\tau_{nt}) w_t n_t$ is the labor income.
- τ_{ht} is the lump-sum tax.

(b) Firms: Firms have a constant return to scale production technology $F(k,n)$ and solve

$$\max_{\{k_t,n_t\}} F(k_t, n_t) - \eta_t k_t - w_t n_t$$

---

[2] The sign of U_{ll} is not specified, it depends on the utility function.

[3] For example, q_1 is the amount of consumption good at time 0 you should pay to get 1 unit of consumption good at time 1, therefore, $q_1 < q_0 = 1$.

Note that CRS technology in the competitive market yields to that every firm makes zero economic profit.

(c) Government: The government has a time-0 budget constraint.

$$\sum_{t=0}^{\infty} q_t g_t \leqslant \sum_{t=0}^{\infty} q_t \left[\tau_{ct} c_t + \tau_{kt} \left(\eta_t - \delta\right) k_t + \tau_{nt} w_t n_t + \tau_{ht}\right]$$

4. Solution for household.

Form the Lagrangian,

$$\mathcal{L} = \max_{\{c_t, n_t, k_{t+1}\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t U\left(c_t, 1 - n_t\right)$$
$$+ \mu \left(\sum_{t=0}^{\infty} q_t \left[\eta_t k_t - \tau_{kt} \left(\eta_t - \delta\right) k_t + \left(1 - \tau_{nt}\right) w_t n_t - \tau_{ht} - \left(1 + \tau_{ct}\right) c_{nt} - k_{t+1} + \left(1 - \delta\right) k_t\right] \right)$$

Define $U_{c,t} := U_c\left(c_t, 1 - n_t\right), U_{l,t} := U_l\left(c_t, 1 - n_t\right)$. The F.O.C. w.r.t. c_t is

$$\beta^t U_{c,t} = \mu q_t \left(1 + \tau_{ct}\right), \forall t$$

then

$$\mu = \frac{\beta^t U_{c,t}}{q_t \left(1 + \tau_{ct}\right)}, \forall t$$

Let $t = 0$, since $q_0 = 1$,

$$\mu = \frac{U_{c,0}}{1 + \tau_{c0}}$$

It is a rescaled marginal utility, if $\tau_{ct} = 0$, then $\mu = U_{c,0}$ is exactly the marginal utility. Rewrite μ as

$$U_{c,0} = \frac{1}{1 + \tau_{c0}} U_{c,0} + \frac{\tau_{c0}}{1 + \tau_{c0}} U_{c,0}$$

The first term is consumed by the household, the second term is taken by the government.

Generally,

$$\mu = \frac{\beta^t U_{c,t}}{q_t \left(1 + \tau_{ct}\right)} = \frac{\beta^{t+1} U_{c,t+1}}{q_{t+1} \left(1 + \tau_{c,t+1}\right)}$$

then

$$\frac{U_{c,t}}{q_t \left(1 + \tau_{ct}\right)} = \frac{\beta U_{c,t+1}}{q_{t+1} \left(1 + \tau_{c,t+1}\right)}$$

Note that $\frac{1}{q_t} = \frac{q_0}{q_t}$ is the amount of time-t consumption goods that can be bought by 1 unit of time-0 consumption good.

The F.O.C w.r.t labor is

$$-\beta^t U_{l,t} + \mu q_t \left(1 - \tau_{nt}\right) w_t = 0$$

plug $\mu = \frac{\beta^t U_{c,t}}{q_t(1+\tau_{ct})}$ in,

$$U_{l,t} = \frac{U_{c,t}}{1 + \tau_{ct}} \left(1 - \tau_{nt}\right) w_t$$

The LHS is the marginal cost of working (how much utility the household will lose if he works more); the RHS is the marginal benefit of working, i.e., he will get $(1-\tau_{ct})\, w_t$ units of wage income, which is corresponding to $(1-\tau_{nt})\, w_t U_{c,t} \times \frac{1}{1+\tau_{ct}}$ units of utility, rescaled by $\frac{1}{1+\tau_{ct}}$.

The intuition for the F.O.C. of n_t is that marginal cost equals to marginal benefit. And the F.O.C. **actually assumes interior solution**.

(a) $n=1$, then $MC > MB \Longrightarrow U_{l,t} > \frac{U_{c,t}}{1+\tau_{ct}}(1-\tau_{nt})\, w_t$ (Even the marginal benefit is low, the household offers all his time to work).

(b) $n=0$, then $MC < MB \Longrightarrow U_{l,t} < \frac{U_{c,t}}{1+\tau_{ct}}(1-\tau_{nt})\, w_t$ (Even not work, it's painful to work 1 more unit). Therefore, $n_t^* \in (0,1)$.

For example, if the marginal cost is increasing with n_t, and marginal benefit is decreasing with n_t, then we can see the intuition by figure.

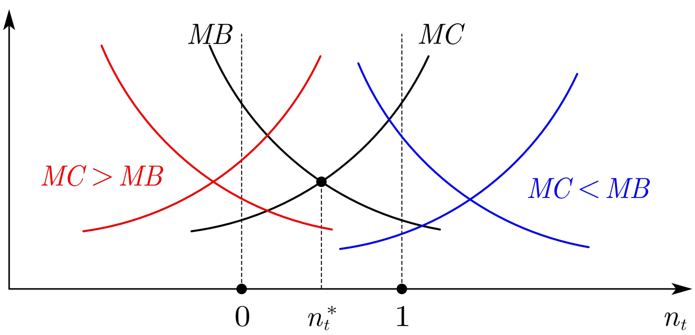

Fig. 2.1. Caption

The F.O.C. w.r.t. k_{t+1} is

$$-\mu q_t + \mu q_{t+1}\left[\eta_{t+1} - \tau_{k,t+1}\left(\eta_{t+1}-\delta\right) + (1-\delta)\right] = 0$$

plug $\mu = \frac{\beta^t U_{c,t}}{q_t(1+\tau_{ct})}$ in,

$$\begin{aligned}\frac{U_{c,t}}{1+\tau_{ct}} &= \frac{\beta U_{c,t+1}}{1+\tau_{c,t+1}}\left[\eta_{t+1} - \tau_{k,t+1}\left(\eta_{t+1}-\delta\right) + (1-\delta)\right] \\ &= \frac{\beta U_{c,t+1}}{1+\tau_{c,t+1}}\left[\left(1-\tau_{k,t+1}\right)\left(\eta_{t+1}-\delta\right) + 1\right]\end{aligned}$$

In order to interpret the equation, we first look at an example in the Ricardian equivalence, suppose the gross interest rate is R_t instead of a constant $R = \frac{1}{\beta}$. Then recall the problem,

$$\max_{\{c_t, b_{t+1}\}_{t=0}^{\infty}} \sum_{t=1}^{\infty} \beta^t u\left(c_t\right) \text{s.t. } c_t + R_t^{-1} b_{t+1} \leqslant y_t + b_t$$

Form the Lagrangian,

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) + \lambda_t\left(y_t + b_t - c_t - R_t^{-1} b_{t+1}\right)\right]$$

then F.O.C.s are

$$\begin{cases} u'(c_t) - \lambda_t = 0 \\ R_t^{-1} = \beta \lambda_{t+1} \end{cases} \Longrightarrow u'(c_t) = \beta R_t u'(c_{t+1})$$

Compare with

$$\frac{U_{c,t}}{1+\tau_{ct}} = \frac{\beta U_{c,t+1}}{1+\tau_{c,t+1}} \left[(1-\tau_{k,t+1})(\eta_{t+1} - \delta) + 1 \right]$$

Then we can think of $\frac{1+\tau_{c,t}}{1+\tau_{c,t+1}} \left[(1-\tau_{k,t+1})(\eta_{t+1} - \delta) + 1 \right]$ as the gross return of saving. or the **gross interest rate**, just like the role of R_t.

Therefore, the LHS is the marginal cost of investment (saving), the RHS is the marginal return of investment.

More specifically, from the F.O.C. w.r.t. k_{t+1},

$$-\mu q_t + \mu q_{t+1} \left[(1-\tau_{k,t+1})(\eta_{t+1} - \delta) + 1 \right] = 0$$

we have

$$\frac{q_t}{q_{t+1}} = (1-\tau_{k,t+1})(\eta_{t+1} - \delta) + 1$$

which is the **implicit interest rate**, or the **market interest rate**. Therefore, we can define

$$R_{t,t+1} := \frac{q_t}{q_{t+1}}, r_{t,t+1} := \frac{q_t}{q_{t+1}} - 1$$

where $r_{t,t+1}$ is called the net interest rate.

5. Solution for firms.

$$\max_{\{k_t, n_t\}} F(k_t, n_t) - \eta_t k_t - w_t n_t$$

The F.O.C.s are

$$\begin{cases} F_k(k_t, n_t) = \eta_t \\ F_n(k_t, n_t) = w_t \end{cases}$$

And note that F is CRS, i.e., homogenous with degree 1, and the market is competitive, every firm makes zero economic profit. For $\alpha > 0$, $F(\alpha k, \alpha n) = \alpha F(k, n)$ differentiate w.r.t. α, $kF_k(\alpha k, \alpha n) + nF_n(\alpha k, \alpha n) = F(k, n)$ and F_k, F_n are homogenous with degree zero, then

$$kF_k(k, n) + nF_n(k, n) = F(k, n)$$

which implies zero profit.

Therefore, there is no accumulation of wealth by firms, and firm's problem can be treated static.

6. Solution for government.

To simplify the model, we assume that the government treats $\{g_t, \tau_{ct}, \tau_{kt}, \tau_{nt}\}_{t=0}^{\infty}$ as **exogenous** (e.g., they are set in law), and uses the lump-sum tax τ_{ht} to balance its budget constraint.

Since the timing of lump-sum tax doesn't changing the optimal consumption plan by the Ricardian equivalence, then we can leave the government's budget problem in the back ground.

7. Combination of the solution.

   To combine those solution, we firstly have the following market clearing condition.

$$F\left(k_t, n_t\right) = g_t + c_t + x_t k_{t+1} = \left(1-\delta\right) k_t + x_t$$

   where x_t is the investment in period t, and we can see it as the amount of investment to reach the level of k_{t+1} in the next period, because k_{t+1} is one of the household's choice variable.

   We can eliminate x_t to get

$$F\left(k_t, n_t\right) = c_t + k_{t+1} - \left(1-\delta\right) k_t + g_t$$

   Basically, the national income is coordinate with our model, i.e., $Y = C + I + G$.

   Overall, now we have the following 6 equations.

$$\begin{cases} \frac{U_{c,t}}{q_t(1+\tau_{ct})} = \frac{\beta U_{c,t+1}}{q_{t+1}(1+\tau_{c,t+1})} \\ U_{l,t} = \frac{U_{c,t}}{1+\tau_{ct}} \left(1-\tau_{nt}\right) w_t \\ \frac{U_{c,t}}{1+\tau_{ct}} = \frac{\beta U_{c,t+1}}{1+\tau_{c,t+1}} \left[\left(1-\tau_{k,t+1}\right)\left(\eta_{t+1}-\delta\right)+1\right] \\ F_k\left(k_t, n_t\right) = \eta_t \\ F_n\left(k_t, n_t\right) = w_t \\ F\left(k_t, n_t\right) = c_t + k_{t+1} - \left(1-\delta\right) k_t + g_t \end{cases}$$

   To simplify, we eliminate η_t and w_t, then

$$\begin{cases} \frac{U_{c,t}}{q_t(1+\tau_{ct})} = \frac{\beta U_{c,t+1}}{q_{t+1}(1+\tau_{c,t+1})} \\ U_{l,t} = \frac{U_{c,t}}{1+\tau_{ct}} \left(1-\tau_{nt}\right) F_n\left(k_t, n_t\right) \\ \frac{U_{c,t}}{1+\tau_{ct}} = \frac{\beta U_{c,t+1}}{1+\tau_{c,t+1}} \left[\left(1-\tau_{k,t+1}\right)\left(F_k\left(k_{t+1}, n_{t+1}\right)-\delta\right)+1\right] \\ F\left(k_t, n_t\right) = c_t + k_{t+1} - \left(1-\delta\right) k_t + g_t \end{cases}$$

   Note that the last 3 equations form a sub-system with variable $\{c_t, n_t, k_{t+1}\}$. When these variables are pinned down, the first equation allows us to solve $\{q_{t+1}\}$.

2.2 Inelastic Labor Supply

1. Simplify the model.

   (a) Assume that leisure does not contribute to utility $U_{lt} = 0$ then $n_t = 1$, which means the household will work all his time.

Then the question is what's the wage? We might think we can pay 0 to the labor, but note that the market is competitive, the labor will finally find a workplace that offers $w_t = F_n(k_t, 1)$, which is the marginal productivity of labor.

Note that $n_t = 1$ is a corner solution, then the F.O.C. w.r.t. n_t

$$U_{l,t} = \frac{U_{c,t}}{1+\tau_{ct}}(1-\tau_{nt}) F_n(k_t, 1)$$

no longer works.

(b) Then the system becomes

$$\begin{cases} \frac{U_{c,t}}{q_t(1+\tau_{ct})} = \frac{\beta U_{c,t+1}}{q_{t+1}(1+\tau_{c,t+1})} \\ \frac{U_{c,t}}{1+\tau_{ct}} = \frac{\beta U_{c,t+1}}{1+\tau_{c,t+1}} \left[(1-\tau_{k,t+1})(F_k(k_{t+1}, 1) - \delta) + 1\right] \\ F(k_t, 1) = c_t + k_{t+1} - (1-\delta) k_t + g_t \end{cases}$$

(c) Define "per worker" production function

$$f\left(\tilde{k}_t\right) := F\left(\frac{k_t}{n_t}, 1\right)$$

where $\tilde{k}_t = \frac{k_t}{n_t}$.

(d) Inelastic labor supply implies $\tilde{k}_t = k_t$.

(e) Since $c_t = F(k_t, 1) - k_{t+1} + (1-\delta) k_t - g_t$ The last 2 equations in the simplified system can be rewritten as

$$\begin{aligned} &\frac{U_c\left(f(k_t) - k_{t+1} + (1-\delta) k_t - g_t\right)}{1+\tau_{ct}} \\ =&\frac{\beta U_c\left(f(k_{t+1}) - k_{t+2} + (1-\delta) k_{t+1} - g_{t+1}\right)}{1+\tau_{c,t+1}} \left[(1-\tau_{k,t+1})(f'(k_{t+1}) - \delta) + 1\right] \end{aligned}$$

2. Effects of taxes.

Look at the last 2 equation in the simplified model,

$$\begin{cases} \frac{U_{c,t}}{1+\tau_{ct}} = \frac{\beta U_{c,t+1}}{1+\tau_{c,t+1}} \left[(1-\tau_{k,t+1})(F_k(k_{t+1}, 1) - \delta) + 1\right] \\ F(k_t, 1) = c_t + k_{t+1} - (1-\delta) k_t + g_t \end{cases}$$

(a) τ_{ct}: it only appears in the first equation, and if τ_{ct} is a constant, then it doesn't matter, otherwise, the variation of τ_{ct} matters.

(b) τ_{nt}: so long as $w > 0$, households will supply $n_t = 1$, then τ_{nt} doesn't matter.

(c) τ_{kt}: it matters and cannot be cancelled.

(d) τ_{ht}: the lump-sum tax doesn't appear in the system.

3. The steady state.

(a) The steady state equilibrium of the system is defined as an equilibrium characterized by **a constant set of values for all endogenous variable**, when the exogenous variables $\{\tau_{ct}, \tau_{kt}, \tau_{nt}, g_t\}$ are constant.

Note that there is no growth in technology, population or anything else, so the steady state is a constant rather than steady state with growth.

(b) Consider the simplified model with $f\left(\tilde{k}_t\right)$,

$$\begin{cases} \frac{U_{c,t}}{q_t(1+\tau_{ct})} = \frac{\beta U_{c,t+1}}{q_{t+1}(1+\tau_{c,t+1})} \\ \frac{U_{c,t}}{1+\tau_{ct}} = \frac{\beta U_{c,t+1}}{1+\tau_{c,t+1}} \left[\left(1-\tau_{k,t+1}\right)\left(f'\left(k_{t+1}\right)-\delta\right)+1\right] \\ f\left(k_t\right) = c_t + k_{t+1} - (1-\delta)\,k_t + g_t \end{cases}$$

On the steady state, $\{c_t, n_t, k_{t+1}\}$ keep constant, and the exogenous variables $\{\tau_{ct}, \tau_{kt}, \tau_{nt}, g_t\}$ also keep the constant (unless we change them by hands), then the first equation implies

$$R_{t+1} = \frac{q_{t+1}}{q_t} = \frac{1+\tau_{ct}}{1+\tau_{c,t+1}}\frac{\beta U_{c,t+1}}{U_{c,t}} = \beta$$

the second equation implies

$$1 = \beta\left[\left(1-\tau_k\right)\left(f'(k)-\delta\right)+1\right]$$

we can define $\rho := \frac{q_t}{q_{t+1}} - 1$ as the steady net interest rate, then

$$\rho = \frac{1}{\beta} - 1 = \left(1-\tau_k\right)\left(f'(k)-\delta\right)$$

and the third equation in the system implies [4]

$$f(k) = c + \delta k + g$$

4. More discussion on steady state.

(a) If $\tau_k = 0$, the economy achieves the "Golden Rule Steady State", which is

$$f'(k) = \rho + \delta$$

(b) If we'd like to maximize the consumption, then

$$c = f(k) - \delta k - g$$

the F.O.C. w.r.t. k is

$$f'(k) = \delta$$

which is the same golden rule in the Solow model.

[4] Firstly, we can simply let $k_{t+1} = k_t = k$ to get it. Secondly, we can regard δk as the amount of investment that we need to replenish to keep the system at the steady state.

   (c) In our model, the household's optimal choice is to share the consumption into future due to the discount effect. But Solow model has no discount effect.

5. The steady state will serve two purposes:
   (a) It is the **destination** of the economy when it is away from the steady state.
   (b) It can be used as the **initial equilibrium** before a shock hits, and conduct comparative static analysis.

6. Methods for getting more from the steady state.
   (a) Log-linearization: it can be used to gain more analytical intuition of the model.
   (b) Numerical method: it can be used to compute the dynamic paths of the economy.
      i. Dynamic programming (The steady state is unknown).
      ii. Shooting algorithm (The steady state is known, so it is faster than dynamic programming).
      iii. Dynare (A software used with MATLAB).

7. Log-linearization: An example.

   Let x be a steady state, and $\hat{x}_t := \ln \frac{x_t}{x}$ be the percentage deviation, then

$$x_t = xe^{\hat{x}_t}$$

   Note that $\hat{x}_t = \ln \frac{x_t}{x} \approx \frac{x_t}{x} - 1$.

   For example, consider the Euler equation,

$$u'(c_t) = \beta R_t u'(c_{t+1})$$

   where $u'(c) = c^{-\gamma}$. In the steady state, assume that $\beta R = 1$, and thus,

$$\left(ce^{\hat{c}_t}\right)^{-\gamma} = \beta R \cdot e^{\hat{R}_t}\left(ce^{\hat{c}_{t+1}}\right)^{-\gamma} \Longrightarrow e^{-\gamma \hat{c}_t} = e^{\hat{R}_t - \gamma \hat{c}_{t+1}}$$

   Then by the first order Taylor's expansion around 0 (since $\hat{c}_t = 0$, $\hat{R}_t - \gamma \hat{c}_{t+1} = 0$ at the steady state),[5] then

$$1 - \gamma \hat{c}_t \approx 1 + \hat{R}_t - \gamma \hat{c}_{t+1}$$

   then

$$\hat{R}_t \approx \gamma\left(\hat{c}_{t+1} - \hat{c}_t\right)$$

   In general, if $\hat{R}_t \uparrow$, then $\hat{c}_{t+1} \uparrow$ and $\hat{c}_t \downarrow$.

   (a) In business cycle model, percentage deviations are small, like 2% to 5%.
   (b) In growth model, we are usually far away from the steady state.

[5] In the example, we can just take log in both hand sides to get the next equation, but in general cases, we may need the Taylor's expansion.

8. Dynamic Programming.

```
# Initialization.
import numpy as np

bet = 0.95
alph = 0.33
delt = 0.2
gamm = 2
g = 0.2
k0 = 0.2

def utility(c):
    return c**(1-gamm)/(1-gamm)
```

```
# Iterations.
import time
time_start = time.time()


k_num = 1000

v_old = np.ones(k_num)
v_new = -1000*np.ones(k_num)
policy = np.zeros(k_num,int)

v_tol = 1e-10
v_ctr = 0

k_min = 0.01
k_max = 4
k_vec = np.linspace(k_min, k_max, k_num)

while np.max(np.abs(v_new-v_old)) > v_tol:
    v_old = np.copy(v_new)
    for w_curr in range(1,k_num):
        k_up = k_vec[w_curr]**alph-g+(1-delt)*k_vec[w_curr]
        c = k_vec[w_curr]**alph - g + (1-delt)*k_vec[w_curr] - k_vec[k_vec<
                                                            =k_up]
        temp = utility(c) + bet * v_old[k_vec<=k_up]
        v_new[w_curr] = np.max(temp)
        policy[w_curr] = np.argmax(temp)
    v_ctr = v_ctr+1

time_end = time.time()
print(v_ctr)
print('Time elapsed::',time_end-time_start)
```

```
# Plot.
```

```
plt.figure(figsize=(6, 4), dpi=300)
plt.plot(k_vec,k_vec[policy])
plt.plot(k_vec,k_vec, '--')
k_argmin = np.argmin(np.abs(k_vec[policy]-k_vec)[1:])
plt.scatter([0,k_vec[k_argmin]],[0,k_vec[k_argmin]], marker='x', alpha= 0.8
                                        )
plt.legend(['Policy Function', '$k_{t+1}=k_t$', 'Fixed Points'])
plt.xlabel('$k_t$')
plt.ylabel('$k_{t+1}$')
plt.show()
```

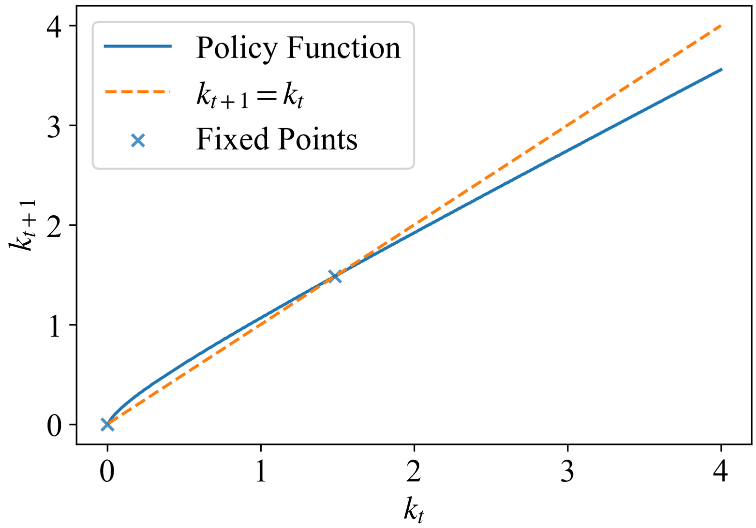

Fig. 2.2. Policy function.

```
T = k_num
k_optimal = np.zeros(T)
k_optimal[0] = 0.1
for i in range(T-1):
    t = np.argmin(abs(k_optimal[i] - k_vec))
    k_optimal[i+1] = k_vec[policy][t]

plt.figure(figsize=(6, 4), dpi=300)
plt.plot(k_optimal)
plt.xlim([0,40])
plt.ylabel('$k_t$')
plt.xlabel('$t$')
plt.show()
```

9. Shooting algorithm.

   Recall the steady state is determined by

$$\begin{cases} 1 - \frac{1}{\beta} = (1 - \tau_k)\left(f'(k) - \delta\right) \\ f(k) = c + \delta k + g \end{cases}$$

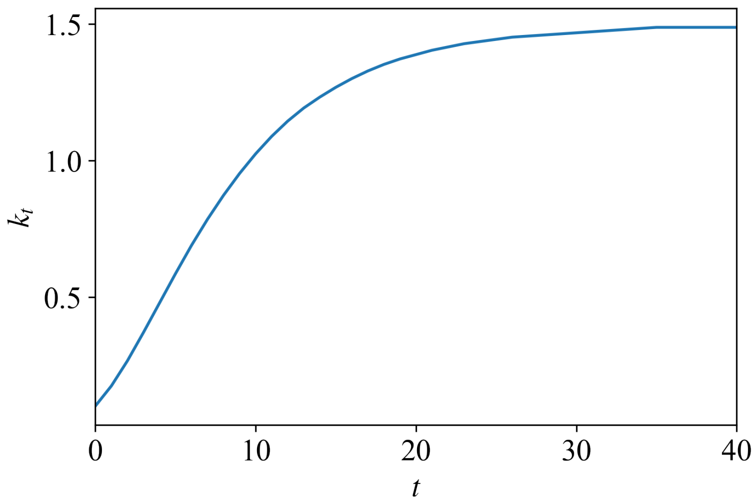

Fig. 2.3. Path of k_t.

The system is

$$\begin{cases} (1)\frac{U_c(c_t,1)}{q_t(1+\tau_{ct})} = \frac{\beta U_c(c_{t+1},1)}{q_{t+1}(1+\tau_{c,t+1})} \\ (2)\frac{U_c(c_t,1)}{1+\tau_{ct}} = \frac{\beta U_c(c_{t+1},1)}{1+\tau_{c,t+1}}\left[(1-\tau_{k,t+1})\left(f'\left(k_{t+1}\right)-\delta\right)+1\right] \\ (3) f\left(k_t\right) = c_t + k_{t+1} - (1-\delta)\, k_t + g_t \end{cases}$$

If k_0 is given, then we can compute $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ by shooting algorithm as following,

$$k_0 \xrightarrow{\text{Guess}} c_0 \xrightarrow{(3)} k_1 \xrightarrow{(2)} c_1 \xrightarrow{(3)} k_2 \to \cdots$$

The path is determined by our initial guess c_0, then a correct c_0 will take use to the steady state (k, c). The only task we need to deal with is to renew our guess, to approach the correct answer.

Note that we have an upper bound of c_0, which is when $k_1 = 0$,

$$c_0 = f\left(k_0\right) + (1-\delta)\, k_0 - g_0$$

Generally, we can define a function to compute the transition path we want.

```
import numpy as np
import matplotlib.pyplot as plt
```

```
def inelastic_labor (T, alph = 1/3, bet = 0.95, gamm = 2, delt = 0.2,
                     g_vec = np.repeat(0.2,T+1),
                     tau_c_vec = np.zeros(T+1), tau_k_vec =  np.zeros(T+1),
                     tol = 1e-5, itr_max = 500):

    c_vec = np.zeros(T+1)
    k_vec = np.zeros(T+1)

    k_ss = ((1/alph)*(1/bet-1+delt))**(1/(alph-1))
```

```
    k0 = k_ss
    c_ss1 = k_ss**alph - delt*k_ss- g_vec[0]
    c_ss2 = k_ss**alph - delt*k_ss- g_vec[-1]

    # initial boundaries for c0
    c0_ub = k0**alph + (1-delt)*k0 - g_vec[0]
    c0_lb = 0
    c0 = (c0_ub+c0_lb)/2


    ii = 0
    while ii < itr_max:
        c_vec[0] = c0
        k_vec[0] = k0

        for t in range(T):
            k_vec[t+1] = k_vec[t]**alph+(1-delt)*k_vec[t]-g_vec[t]-c_vec[t]
            c_vec[t+1] = c_vec[t]*((1+tau_c_vec[t])/(1+tau_c_vec[t+1])*bet*
                         ((1-tau_k_vec[t+1])*
                         (alph*k_vec[t+1]**(alph-1)-delt)+1))**(1/gamm)


        if np.abs(k_vec[T]-k_ss) < tol:
            print("Converged on iteration", ii+1)
            ii = itr_max
        elif k_vec[T] > k_ss:
            c0_lb = c0
        else:
            c0_ub = c0

        ii += 1
        c0 = (c0_ub + c0_lb)/2

    eta_vec = np.zeros(T)
    R_vec = np.zeros(T)

    eta_vec = np.zeros(T)
    R_vec = np.zeros(T)

    for t in range (T):
        eta_vec[t] = alph*k_vec[t]**(alph-1)
        R_vec[t] = c_vec[t]**(-gamm)/(bet*c_vec[t+1]**(-gamm))

    return c_vec, k_vec, R_vec, eta_vec, gamm, k_ss, c_ss1, c_ss2
```

10. Example 1. The system starts with a steady state, and then g grows at $t = 10$. (Financed by lump-sum tax)

(a) Note that k is determined by $1-\frac{1}{\beta}=(1-\tau_k)(f'(k)-\delta)$, therefore, k keeps the same as the old steady state.

(b) But since $c=f(k)-\delta k-g$, then c will decrease $|\Delta g|$.

Take $u(c)=\frac{c^{1-\gamma}}{1-\gamma}$, $f(k)=k^{\alpha}$, assume $\tau_{c,t}=\tau_{k,t}=0$, take $\beta=0.95$, $\alpha=1/3$, $\delta=0.2$, $\gamma=2$, $g_0=g_1\cdots=g_9=0.2$, $g_{10}=g_{11}=\cdots=0.4$.

$$\begin{cases} c_t^{-\gamma}=\beta c_{t+1}^{-\gamma}\left[\alpha k_{t+1}^{\alpha-1}-\delta+1\right] \\ k_t^{\alpha}=c_t+k_{t+1}-(1-\delta)k_t+g_t \end{cases}$$

or

$$\begin{cases} c_{t+1}=c_t\left[\beta\left(\alpha k_{t+1}^{\alpha-1}-\delta+1\right)\right]^{\frac{1}{\gamma}} \\ k_{t+1}=k_t^{\alpha}-c_t+(1-\delta)k_t-g_t \end{cases}$$

We are also concerned about R_t and η_t,

$$R_t=\frac{1+\tau_{c,t}}{1+\tau_{c,t+1}}\left[(1-\tau_{k,t+1})(\eta_{t+1}-\delta)+1\right]=\frac{c_t^{-\gamma}}{\beta c_{t+1}^{-\gamma}}\eta_t=F_k(k_t,n_t)=\alpha k_t^{\alpha-1}$$

Scenario 1. $|\Delta g|$ happens at period 10.

```
T = 60
g_vec = np.array([0.2 if t<=10 else 0.4 for t in range(T+1)])

c_vec, k_vec, R_vec, eta_vec, gamm, k_ss, c_ss1, c_ss2= inelastic_labor(T=T
                                        , g_vec= g_vec, gamm = 2)
```

```
plt.figure(figsize=(12, 10), dpi=300)

axe1 = plt.subplot(2,3,1)
axe1.plot(c_vec)
plt.axhline(c_ss1, ls = '--', alpha = 0.8)
plt.axhline(c_ss2, ls = '--', alpha = 0.8)
axe1.set(xlabel='$t$',ylabel='$c_t$', title = 'Consumption')

axe2 = plt.subplot(2,3,2)
axe2.plot(k_vec)
plt.axhline(k_ss, ls = '--', alpha = 0.8)
axe2.set(xlabel='$t$',ylabel='$k_t$', title = 'Capital')

axe3 = plt.subplot(2,3,3)
axe3.plot(R_vec)
axe3.set(xlabel='$t$',ylabel='$R_t$', title = 'Gross Interest Rate')

axe4 = plt.subplot(2,3,4)
axe4.plot(eta_vec)
axe4.set(xlabel='$t$',ylabel='$\eta_t$', title = 'Rental Rate')

axe5 = plt.subplot(2,3,5)
```

```
axe5.plot(g_vec)
axe5.set(xlabel='$t$',ylabel='$g_t$', title = 'Government Expenditure')

# axe6 = plt.subplot(2,3,6)
# axe6.bar(['$\gamma$'],[gamm], width=0.3)
# axe6.set(title = 'Risk Aversion', ylim = [0,2],xlim = [-0.5,0.5])

plt.tight_layout()
plt.show()
```

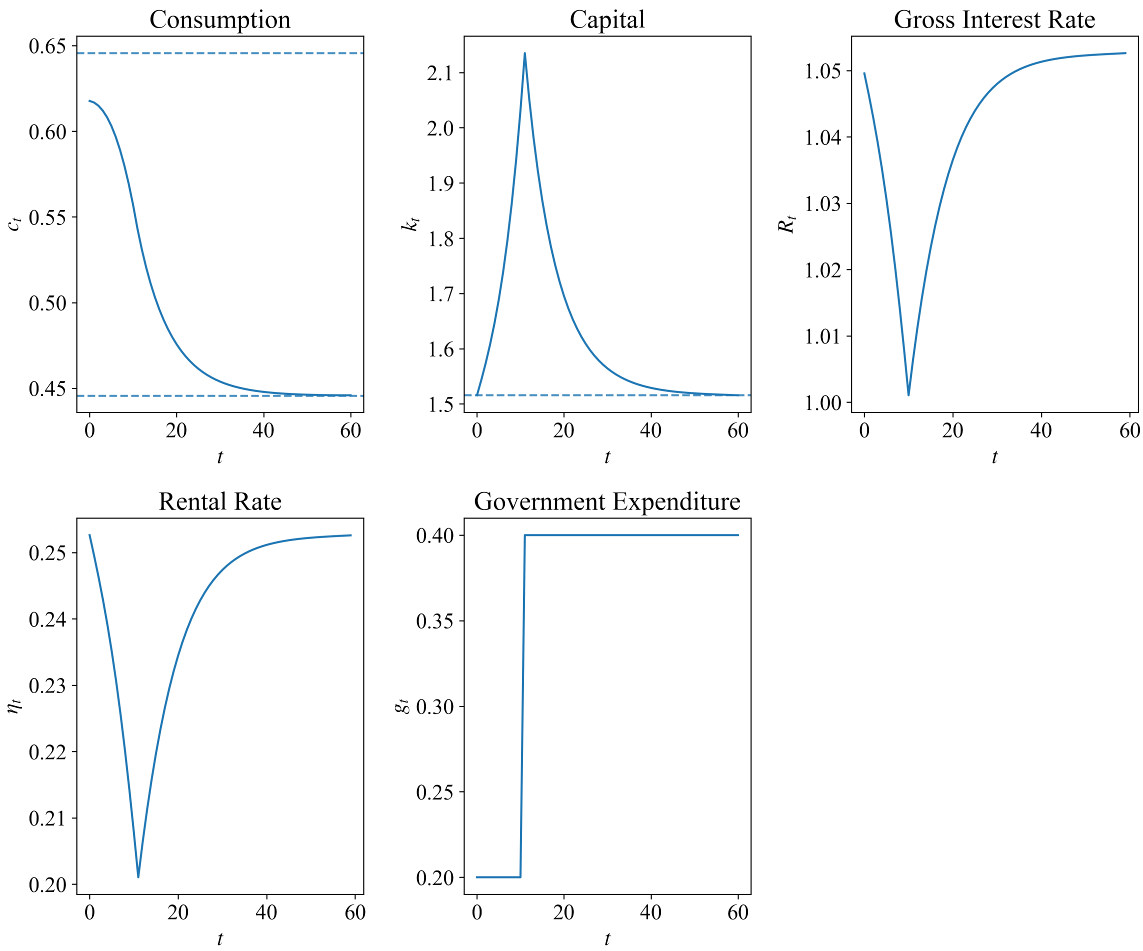

Fig. 2.4. $|\Delta g|$ happens at period 10.

Note that c_t jumps (or should jump)[6] to a lower value at period 0, and then converges to the new steady state.

Then why c_t cannot jump to the new steady state immediately?

Suppose

$$c_1 = c_2 = \cdots = c_9 = c_{old}^{ss}$$
$$c_{10} = c_{11} = \cdots = c_{new}^{ss}$$

[6]Capital stock cannot jump to a higher value immediately; it can only accumulate or decrease day by day. However, note that capital stock can jump to a lower value if needed, we will see later.

Then consider c_9 and c_{10} in the F.O.C.,

$$u'(c_9) = \beta u'(c_{10})\left[f'(k_{10}) - \delta + 1\right]$$

or

$$u'(c_{old}^{ss}) = \beta u'(c_{new}^{ss})\left[f'(k^{ss}) - \delta + 1\right]$$

Then consider c_8 to c_9, it holds that

$$u'(c_{old}^{ss}) = \beta u'(c_{old}^{ss})\left[f'(k^{ss}) - \delta + 1\right]$$

thus,

$$u'(c_{new}^{ss}) = u'(c_{old}^{ss})$$

which leads to a contradiction. Actually,

$$u'(c_{old}^{ss}) < \beta u'(c_{new}^{ss})\left[f'(k^{ss}) - \delta + 1\right]$$

or

$$u'(c_9) < \beta u'(c_{10})\left[f'(k_{10}) - \delta + 1\right]$$

Then to make the F.O.C. hold, we need let $f'(k_{10}) \downarrow$ or $k_{10} \uparrow$.

Scenario 2. $|\Delta g|$ happens at period 1.

Then (c, k) go to (c^{ss}, k^{ss}) immediately.

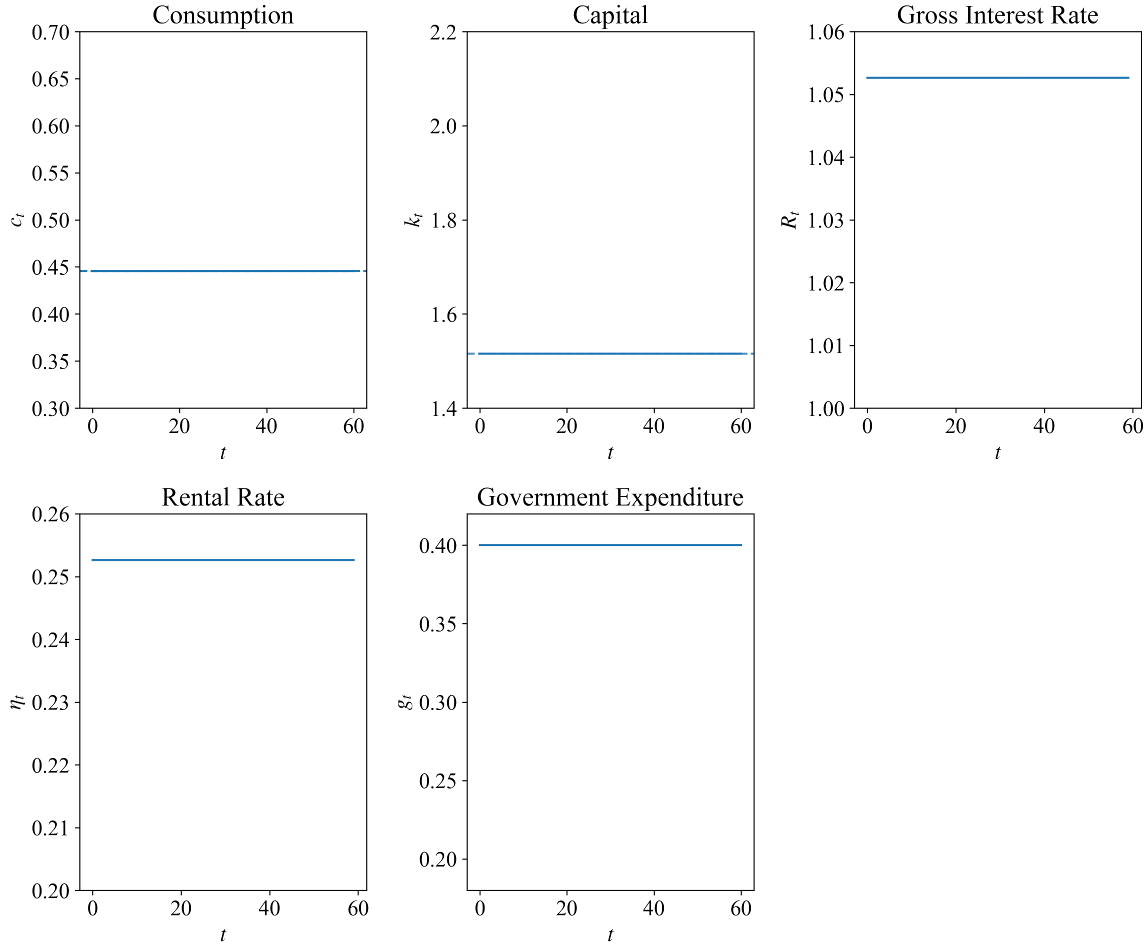

Fig. 2.5. $|\Delta g|$ happens at period 1.

Scenario 3. Change γ.

As γ goes up, the change of c_t becomes more drastic, and the change of k_t becomes flatter.

Consider the F.O.C.,

$$u'(c_t) = \beta R_t u\prime(c_{t+1})$$

then

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta R_t \Longrightarrow \log\frac{c_{t+1}}{c_t} = \frac{1}{\gamma}[\log\beta + \log R_t]$$

As $\gamma\downarrow$, $\frac{1}{\gamma}\uparrow$, then $\frac{c_{t+1}}{c_t}\uparrow$, which means $\{c_t\}$ is backing to the steady state in a faster way, then $\{k_t\}$ needn't change that fast, i.e., $\frac{k_{t+1}}{k_t}\downarrow$.

11. Example 2. The yield curves.

The yield curves show annualized interest rates on bonds of different maturities.

Likewise, let the system start with a steady state, and then g grows at $t=10$.

What will the yield curve in the model be like?

Firstly, recall that $R_{t,t+s} = \frac{q_t}{q_{t+s}}$, and we need to define the annualized interest rate

$$r_{t,t+s} := \frac{1}{s}\ln R_{t,t+s} = \frac{1}{s}\ln\frac{q_t}{q_{t+s}}$$

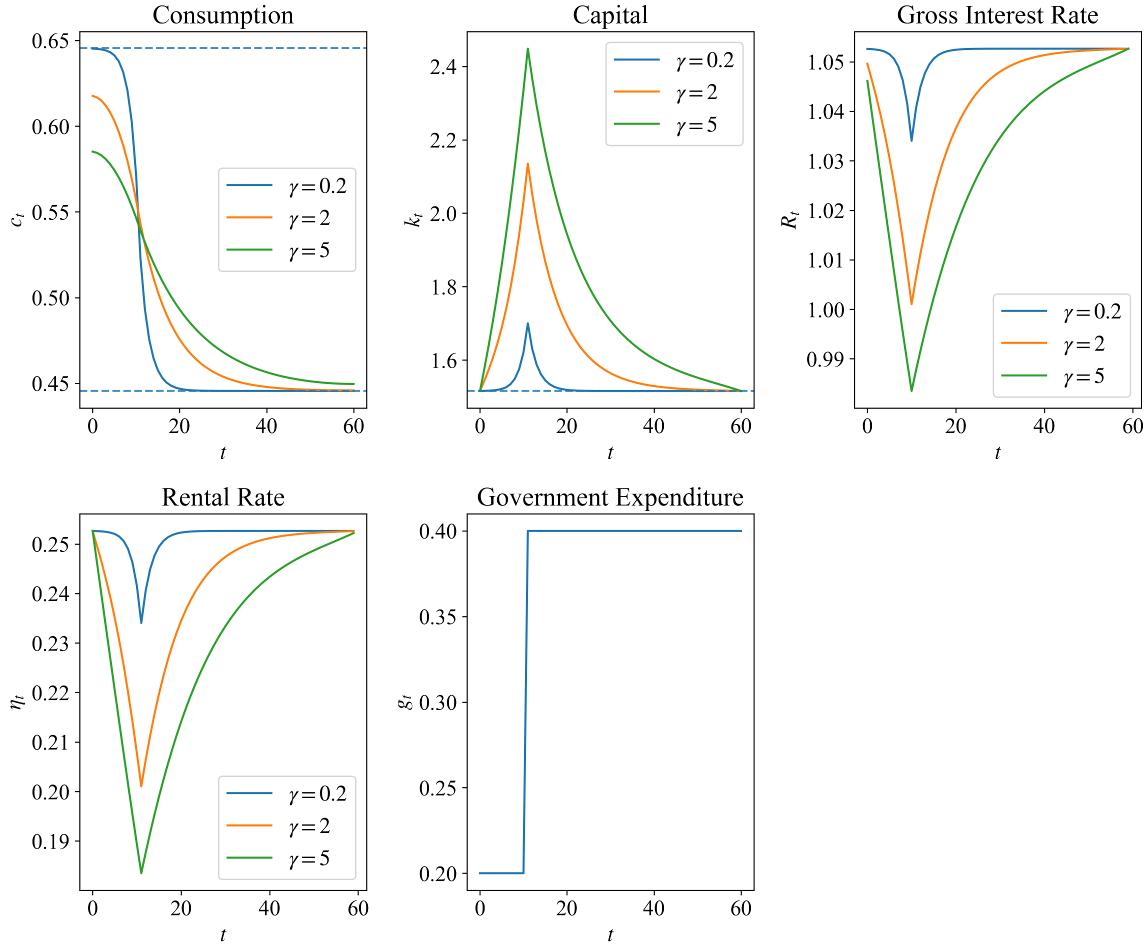

Fig. 2.6. Change γ.

Note that $e^{s \cdot r_{t,t+s}} = R_{t,t+s}$.

Recall that the F.O.C. w.r.t. c_t is

$$\frac{U_{c,t}}{q_t\left(1+\tau_{ct}\right)} = \frac{\beta U_{c,t+1}}{q_{t+1}\left(1+\tau_{c,t+1}\right)}$$

then (set $\tau_{ct} = 0, \forall t$)

$$\frac{q_t}{q_{t+s}} = \frac{c_t^{-\gamma}}{\beta^s c_{t+s}^{-\gamma}} = \beta^{-s}\left(\frac{c_{t+s}}{c_t}\right)^{\gamma}$$

thus,

$$r_{t,t+s} = -\ln\beta + \frac{\gamma}{s}\ln\left(\frac{c_{t+s}}{c_t}\right) = \ln\frac{1}{\beta} + \frac{\gamma}{s}\ln\left(\frac{c_{t+s}}{c_t}\right) \approx \rho + \frac{\gamma}{s}\ln\left(\frac{c_{t+s}}{c_t}\right)$$

Recall that $\rho := \frac{q_t}{q_{t+1}} - 1 = \frac{1}{\beta} - 1$, then $r_{t,t+s}$ equals to a constant part plus a time-varying part.

Also note that in the steady state,

$$r_{t,t+s} = \ln\frac{1}{\beta} + \frac{\gamma}{s}\ln 1 = \ln\frac{1}{\beta}$$

Let's consider q_t, recall the F.O.C.,

$$\frac{q_t}{q_{t+s}} = \beta^{-s}\left(\frac{c_{t+s}}{c_t}\right)^{\gamma}$$

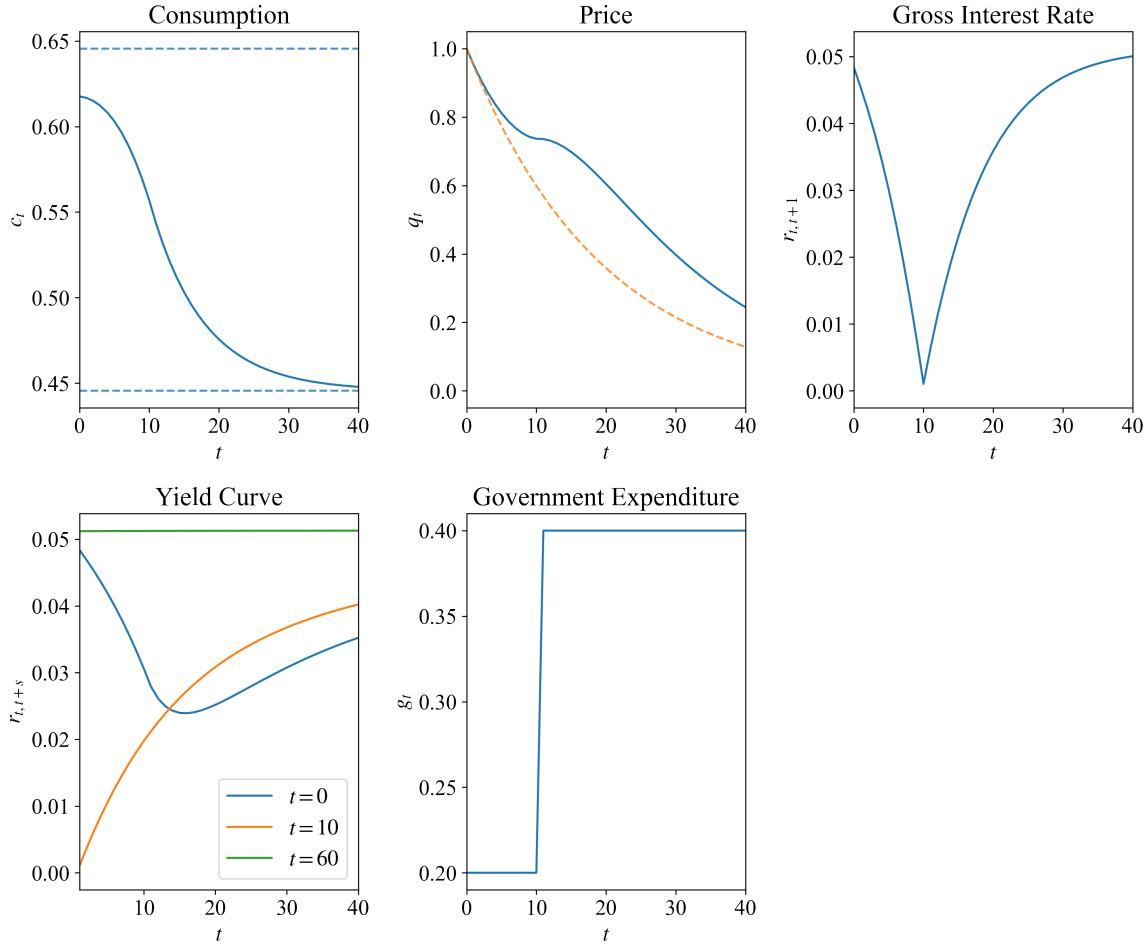

Fig. 2.7. The yield curves.

Let $t = 0$, then

$$\frac{q_0}{q_s} = \beta^{-s} \left(\frac{c_s}{c_0} \right)^{\gamma} \Longrightarrow q_s = \beta^s \left(\frac{c_s}{c_0} \right)^{-\gamma}$$

On the steady state, $q_t^{ss} = \beta^t$.

12. Example 3. The system starts with a steady state, and then τ_c grows at $t = 10$.

We can see that c_t grows in period 0 to 9, this is because the household know they will be taxed in the 10th period thereafter, they will consume more to gain more utility.

If $c_9 = c_{10}$, then by F.O.C.,

$$u'(c_9) = u'(c_{10}) \frac{1 + \tau_{c,9}}{1 + \tau_{c,10}} \left[f'(k) - \delta + 1 \right]$$

actually,

$$\frac{1 + \tau_{c,9}}{1 + \tau_{c,10}} < 1$$

A contradiction.

Therefore, in order to keep the equality, we should let $u'(c_{10}) \uparrow$ then $c_{10} \downarrow$.

In fact, it's a tax evasion behavior to evade the tax in the 10th period.

Look at the R_t, recall that

$$R_t = \frac{1+\tau_{c,t}}{1+\tau_{c,t+1}}\left[f'(k) - \delta + 1\right]$$

in period 10, $\tau_{c,t+1} > \tau_{c,t}$, then R_t drops.

Also note that

$$R_t = \frac{c_t^{-\gamma}}{\beta c_{t+1}^{-\gamma}} \propto \left(\frac{c_{t+1}}{c_t}\right)^{\gamma} \eta_t = F_k(k_t, n_t) = \alpha k_t^{\alpha-1} \propto k_t^{\alpha-1}$$

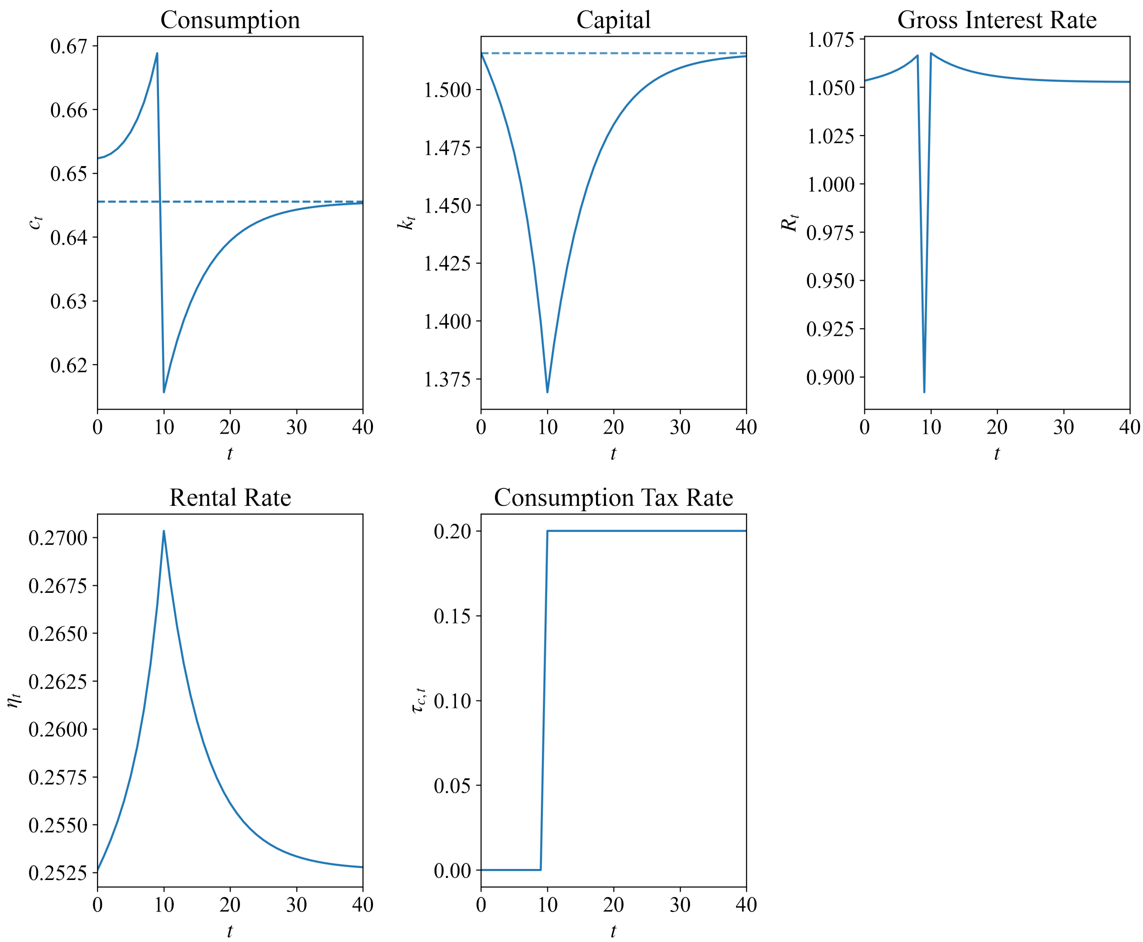

Fig. 2.8. τ_c grows at $t = 10$.

Likewise, we can change γ to see the transition.

As $\gamma \downarrow$, $\frac{c_{t+1}}{c_t} \uparrow$, then why is the case that $\frac{k_{t+1}}{k_t} \uparrow$?

Note that before the shift of the tax policy, $\{c_t\}$ is getting away from the steady state to evade tax and gain more utility, meanwhile, to drag the system back to the steady state, $\{k_t\}$ should change sharply.

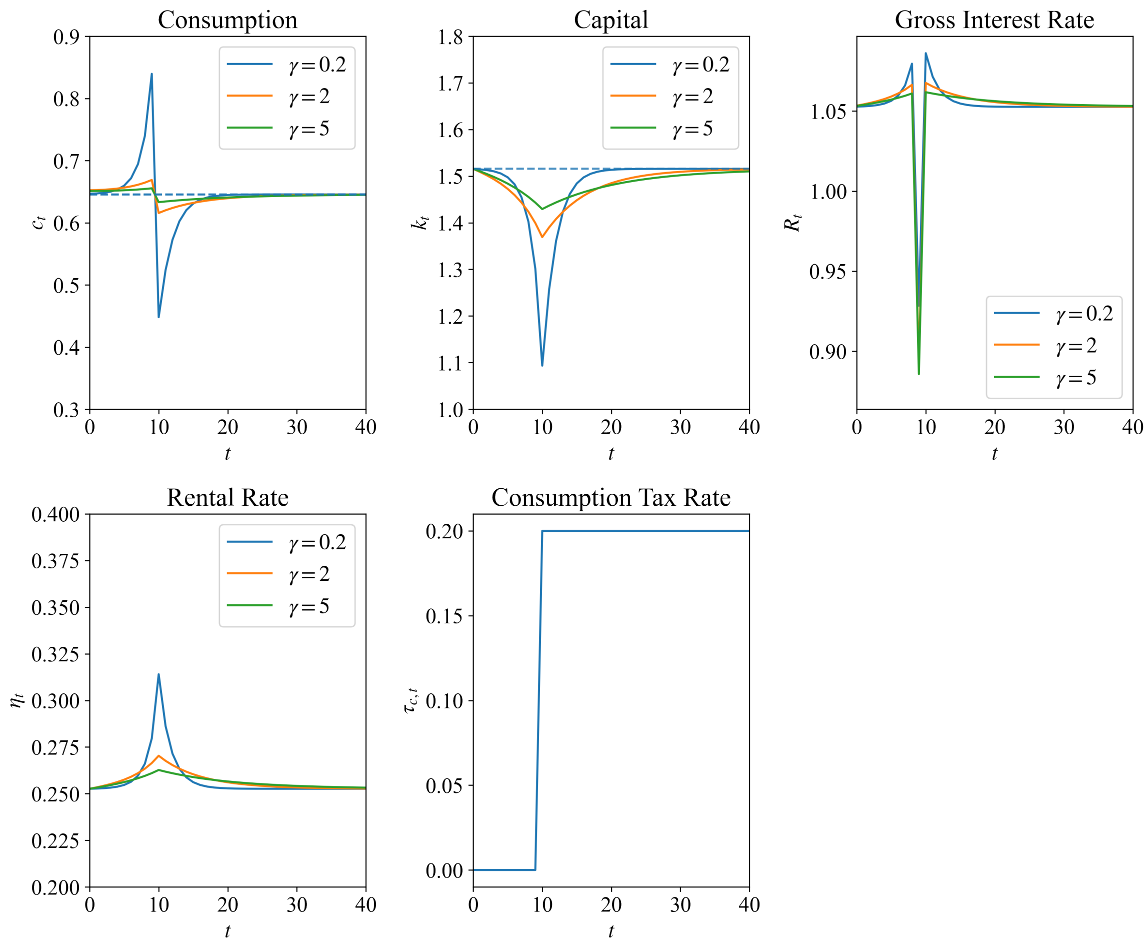

Fig. 2.9. τ_c grows at $t = 10$ (change γ).

13. Example 4. The system starts with a steady state, and then τ_k grows at $t = 10$.

What will the steady state be like?

$$\begin{cases} \rho = (f'(k) - \delta)(1 - \tau_k) \\ c = f(k) - \delta k - g \end{cases}$$

If $\tau_k \uparrow$, then in order to keep ρ a constant, we should make $f'(k) \uparrow$, thus, $k \downarrow$.

Note that if $\tau_k = 0$, $f'(k) = \rho + \delta < \delta$, and $\frac{\mathrm{d}c}{\mathrm{d}k} = f'(k) - \delta = 0$, which implies that $f'(k) = \delta$ maximizes c.

Let $k_{old}^{ss} = f'^{-1}(\rho + \delta), k^* = f'^{-1}(\delta)$, then $k_{old}^{ss} < k^*$, and thus,

$$c_{old}^{ss} = c\mid_{k=k_{old}^{ss}} < c\mid_{k=k^*} = c_{\max}$$

But the new steady state $k_{new}^{ss} < k_{old}^{ss} < k^*$, then

$$c_{new}^{ss} < c_{old}^{ss}$$

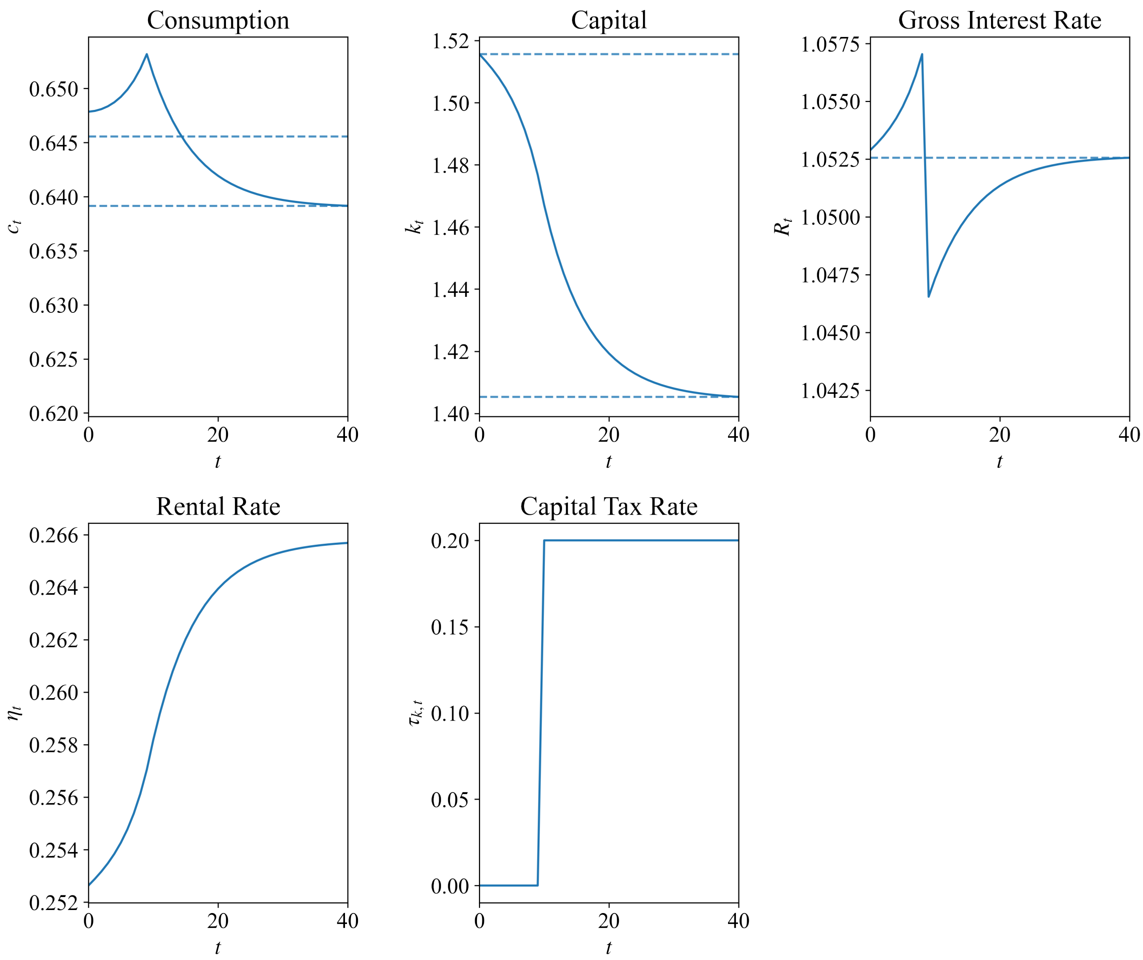

Fig. 2.10. τ_k grows at $t = 10$.

Likewise, we can change γ to see the transition.

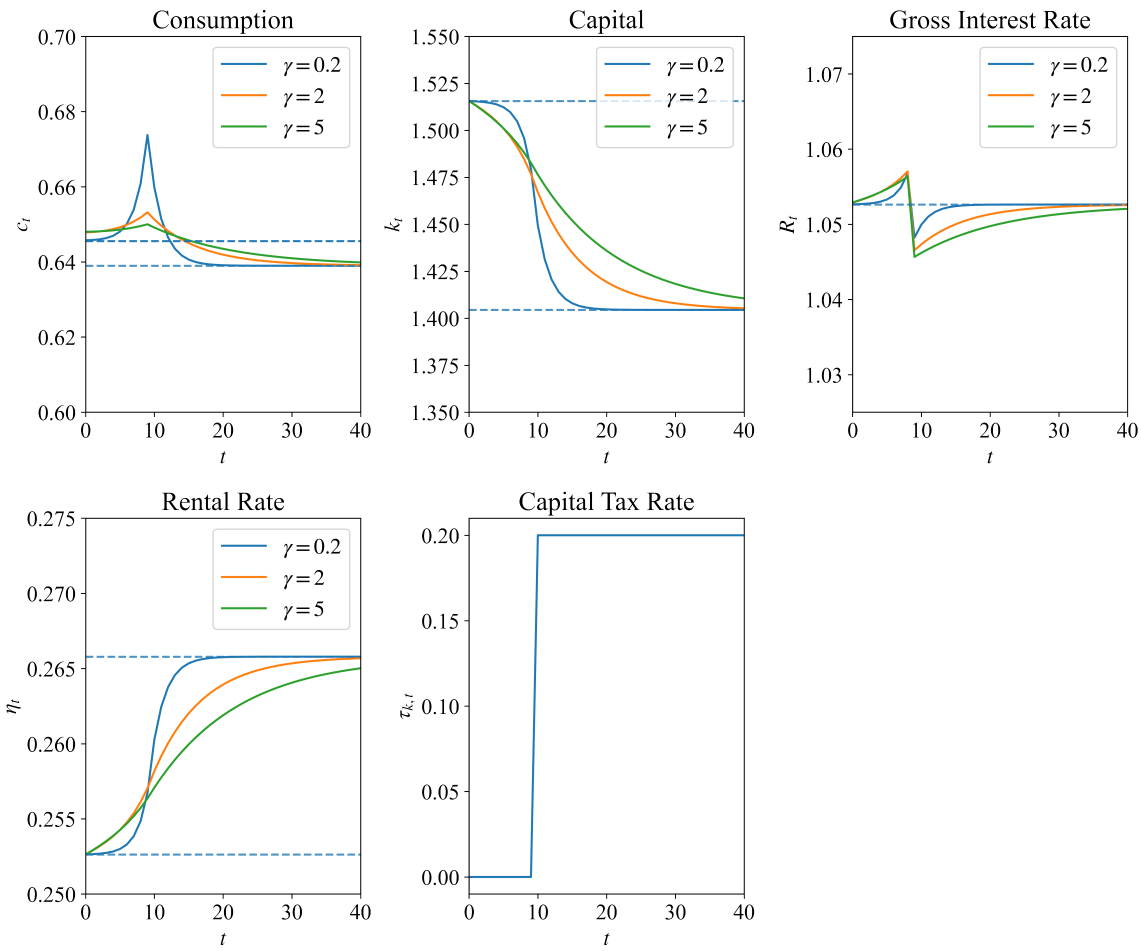

Fig. 2.11. τ_k grows at $t = 10$ (change γ).

14. Example 5. The system starts with a steady state, and then g_t grows at $t = 10$ (Financed by lump-sum tax) and back to the original value at $t = 11$.

    The steady state keeps the same (c^{ss}, k^{ss}).

    But the households know they will pay a lump-sum tax $|\Delta g|$ on the period 10, so they actually save the capital, and then pay the tax in period 10, so there is a slump of k_t.

    Note that k_t cannot jump to a higher stage at a moment, but it can jump to a lower stage.

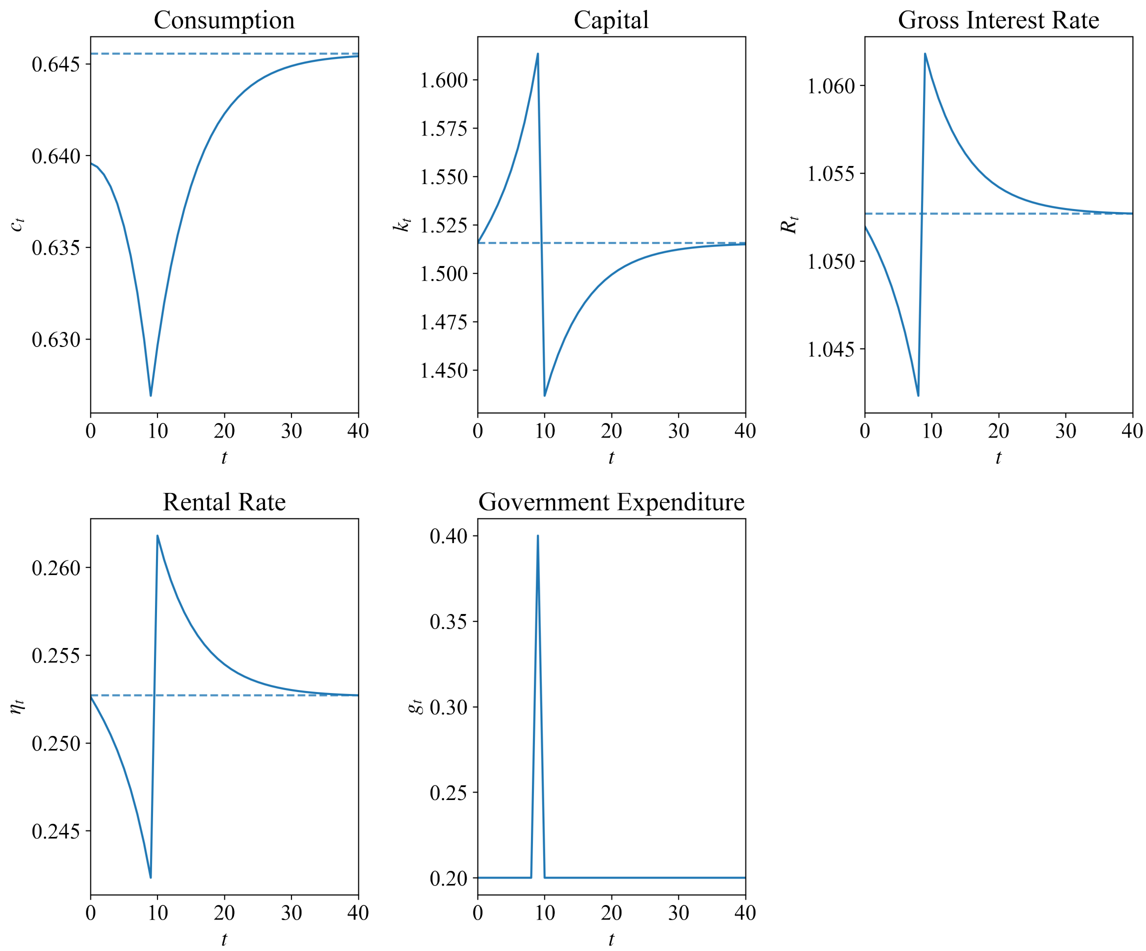

Fig. 2.12. g_t grows at $t = 10$ (Financed by lump-sum tax) and back to the original value at $t = 11$.

2.3 Elastic Labor Supply

1. We will consider elastic labor supply in this section.

2. Steady state.

   Recall the system

$$
\begin{cases}
\frac{U_{c,t}}{q_t(1+\tau_{ct})} = \frac{\beta U_{c,t+1}}{q_{t+1}(1+\tau_{c,t+1})} \\
U_{l,t} = \frac{U_{c,t}}{1+\tau_{ct}} \left(1-\tau_{nt}\right) F_n\left(k_t, n_t\right) \\
\frac{U_{c,t}}{1+\tau_{ct}} = \frac{\beta U_{c,t+1}}{1+\tau_{c,t+1}} \left[\left(1-\tau_{k,t+1}\right)\left(F_k\left(k_{t+1}, n_{t+1}\right)-\delta\right)+1\right] \\
F\left(k_t, n_t\right) = c_t + k_{t+1} - (1-\delta) k_t + g_t
\end{cases}
$$

   and we generally care about the last 3 equations, which decide the steady state as

$$
\begin{cases}
U_l = \frac{U_c}{1+\tau_c}\left(1-\tau_n\right) F_n(k, n) \\
\frac{1}{\beta} - 1 = \left(1-\tau_k\right)\left(F_k(k, n)-\delta\right) \\
F(k, n) = c + \delta k + g
\end{cases}
$$

3. Change the capital into per-work form.

   Define $\tilde{k} := \frac{k}{n}$, then (note that F is CRS)

$$F(k,n) = F\left(n \cdot \tilde{k}, n\right) = nF\left(\tilde{k}, 1\right) = n \cdot f\left(\tilde{k}\right)$$

   thus,

$$F_k(k,n) = nf'\left(\tilde{k}\right) \cdot \frac{1}{n} = f'\left(\tilde{k}\right) F_n(k,n) = f\left(\tilde{k}\right) - n \cdot \frac{k}{n^2} f'\left(\tilde{k}\right) = f\left(\tilde{k}\right) - \tilde{k} f'\left(\tilde{k}\right)$$

   Then the steady state becomes

$$\begin{cases} U_l = \frac{U_c}{1+\tau_c}(1-\tau_n)\left[f\left(\tilde{k}\right) - \tilde{k} f'\left(\tilde{k}\right)\right] \\ \frac{1}{\beta} - 1 = (1-\tau_k)\left(f'\left(\tilde{k}\right) - \delta\right) \\ nf\left(\tilde{k}\right) = c + \delta k + g \end{cases}$$

   The third equation can be rewritten as

$$nf\left(\tilde{k}\right) = c + \delta n \cdot \tilde{k} + g \Longrightarrow n\left[f\left(\tilde{k}\right) - \delta \tilde{k}\right] = c + g$$

4. Specify a utility function with separability.

   Let $u(c, 1-n) = \ln c + B(1-n)$, the utility function has 2 advantages, firstly, c and n are separable, secondly, $u_l = B$ is a constant.

   Then the steady state becomes

$$\begin{cases} (1) Bc = \frac{1}{1+\tau_c}(1-\tau_n)\left[f\left(\tilde{k}\right) - \tilde{k} f'\left(\tilde{k}\right)\right] \\ (2) \frac{1}{\beta} - 1 = (1-\tau_k)\left(f'\left(\tilde{k}\right) - \delta\right) \\ (3) n\left[f\left(\tilde{k}\right) - \delta \tilde{k}\right] = c + g \end{cases}$$

   The second equation specify $\tilde{k}^{ss}$, then we can determine c^{ss} by the first equation, and then solve for n^{ss} by the third one.

5. What are the effects of taxes in the steady state equilibrium?

   (a) $\tau_k \uparrow \overset{(2)}{\Longrightarrow} f'\left(\tilde{k}\right) \uparrow \Longrightarrow \tilde{k} \downarrow$

   (b) $\tau_c \uparrow \overset{(2)}{\Longrightarrow} \tilde{k}^{ss} \rightarrow \overset{(1)}{\Longrightarrow} c^{ss} \downarrow \overset{(3)}{\Longrightarrow} n^{ss} \downarrow$

   (c) $\tau_n \uparrow \overset{(2)}{\Longrightarrow} \tilde{k}^{ss} \rightarrow \overset{(1)}{\Longrightarrow} c^{ss} \downarrow \overset{(3)}{\Longrightarrow} n^{ss} \downarrow$

6. Numerical Example.

   Substitute k into $\tilde{k}$ in the whole system

$$\begin{cases} \frac{\frac{1}{c_t}}{q_t(1+\tau_{ct})} = \frac{\beta \frac{1}{c_{t+1}}}{q_{t+1}(1+\tau_{c,t+1})} \\ Bc_t = \frac{1}{1+\tau_{ct}}(1-\tau_{nt})\left[f\left(\tilde{k}_t\right) - \tilde{k}_t f'\left(\tilde{k}_t\right)\right] \\ \frac{\frac{1}{c_t}}{1+\tau_{ct}} = \frac{\beta \frac{1}{c_{t+1}}}{1+\tau_{c,t+1}}\left[(1-\tau_{k,t+1})\left(f'\left(\tilde{k}_t\right) - \delta\right) + 1\right] \\ n_t f\left(\tilde{k}_t\right) = c_t + n_{t+1}\tilde{k}_{t+1} - n_t(1-\delta)\tilde{k}_t + g_t \end{cases}$$

and

$$R_t = \frac{1+\tau_{ct}}{1+\tau_{c,t+1}} \left[(1-\tau_{k,t+1}) \left(f'\left(\tilde{k}\right) - \delta \right) + 1 \right]$$

Specify $f\left(\tilde{k}_t\right) = \tilde{k}_t^{\alpha}$, then

$$\begin{cases} (1) Bc_t = \frac{1}{1+\tau_{ct}} (1-\tau_{nt})(1-\alpha) \tilde{k}_t^{\alpha} \\ (2) \frac{\frac{1}{c_t}}{1+\tau_{ct}} = \frac{\beta \frac{1}{c_{t+1}}}{1+\tau_{c,t+1}} \left[(1-\tau_{k,t+1}) \left(\alpha \tilde{k}_{t+1}^{\alpha-1} - \delta \right) + 1 \right] \\ (3) n_t \tilde{k}_t^{\alpha} = c_t + n_{t+1} \tilde{k}_{t+1} - n_t (1-\delta) \tilde{k}_t + g_t \end{cases}$$

or

$$\begin{cases} (1) Bc_t = \frac{1}{1+\tau_{ct}} (1-\tau_{nt})(1-\alpha) \left(\frac{k_t}{n_t} \right)^{\alpha} \\ (2) \frac{\frac{1}{c_t}}{1+\tau_{ct}} = \frac{\beta \frac{1}{c_{t+1}}}{1+\tau_{c,t+1}} \left[(1-\tau_{k,t+1}) \left(\alpha \left(\frac{k_{t+1}}{n_{t+1}} \right)^{\alpha-1} - \delta \right) + 1 \right] \\ (3) n_t^{1-\alpha} k_t^{\alpha} = c_t + k_{t+1} - (1-\delta) k_t + g_t \end{cases}$$

Given k_0, we can solve the path by shooting algorithm as

$$k_0 \xrightarrow{\text{Guess}} n_0 \xrightarrow{(1)} c_0 \xrightarrow{(3)} k_1 \xrightarrow{(1)(2)} \begin{cases} c_1 \\ n_1 \end{cases} \xrightarrow{(3)} k_2 \xrightarrow{(1)(2)} \begin{cases} c_2 \\ n_2 \end{cases} \rightarrow \cdots$$

2.4 Two-country Model

1. Set-ups.

   (a) Two countries: home and foreign. Foreign variables are denoted by an asterisk e.g., x^*.

   (b) International trade in goods, capital, and debt, but not labor.

   (c) Inelastic labor supply.

   (d) Utility function is given by $\sum_{t=1}^{\infty} \beta^t u(c_t)$

   (e) Production is CRS with function $F(k_t, n_t)$

   (f) Foreign economy is symmetric to the home economy, and therefore we'll explicitly write down home's equations only/

2. Household's problem.

$$\max_{\left\{c_t, n_t, k_{t+1}, B_t^f\right\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\sum_{t=0}^{\infty} q_t \left[(1+\tau_{ct}) c_t + (k_{t+1} - (1-\delta) k_t) + \left(\tilde{k}_{t+1} - (1-\delta) \tilde{k}_t \right) + R_{t-1,t} B_{t-1}^f \right]$$
$$\leqslant \sum_{t=0}^{\infty} q_t \left[(\eta_t - \tau_{kt} (\eta_t - \delta)) k_t + (\eta_t^* - \tau_{kt}^* (\eta_t^* - \delta)) \tilde{k}_t + (1-\tau_{nt}) w_t n_t + B_t^f - \tau_{ht} \right]$$

where B_t^f is the external debt of the home country (bond issued by home residents that is held by foreign residents); $R_{t-1,t}$ is the gross return on one-period loan from period $t-1$ to period t.

The F.O.C. w.r.t. c_t is

$$\beta^t U_{c,t} = \mu q_t (1+\tau_{ct}) \Longrightarrow \mu = \frac{\beta^t U_{c,t}}{q_t (1+\tau_{ct})}, \forall t$$

or

$$\frac{\beta U_{c,t+1}}{q_{t+1}(1+\tau_{c,t+1})} = \frac{U_{c,t}}{q_t (1+\tau_{ct})}$$

The F.O.C. w.r.t. k_{t+1} is

$$q_t = -q_{t+1}(-(1-\delta)) + (\eta_{t+1} - \tau_{k,t+1}(\eta_{t+1}-\delta))$$

or

$$\frac{q_t}{q_{t+1}} = (1-\tau_{k,t+1})(\eta_{t+1}-\delta)+1$$

The F.O.C. w.r.t. B_t^f is

$$q_t = q_{t+1} R_{t,t+1}$$

Note that the price is the same for two countries,

$$\frac{U_{c,t}/(1+\tau_{ct})}{\beta U_{c,t+1}/(1+\tau_{c,t+1})} = \frac{q_t}{q_{t+1}} = \frac{U_{c,t}^*/(1+\tau_{ct}^*)}{\beta U_{c,t+1}^*/\left(1+\tau_{c,t+1}^*\right)}$$

or

$$\frac{U_{c,t}/(1+\tau_{ct})}{U_{c,t+1}/(1+\tau_{c,t+1})} = \frac{U_{c,t}^*/(1+\tau_{ct}^*)}{U_{c,t+1}^*/\left(1+\tau_{c,t+1}^*\right)}$$

Moreover, $n_t = 1$ by the inelastic labor supply assumption.

3. Firm's Problem.

$$\max_{\{k_t,n_t\}} F(k_t,n_t) - \eta_t k_t - w_t n_t$$

The F.O.C.s are

$$\begin{cases} F_k(k_t,n_t) = \eta_t \\ F_n(k_t,n_t) = w_t \end{cases}$$

Likewise, since the labor $n_t = 1$, then let $f(k) = F(k,1)$, thus,

$$\eta_t = f'(k_t)$$

4. Combine the F.O.C.s and the resource constraint. First consider the home's system

$$\begin{cases} \frac{\beta U_{c,t+1}}{q_{t+1}(1+\tau_{c,t+1})} = \frac{U_{c,t}}{q_t(1+\tau_{ct})} \\ \frac{U_{c,t}}{1+\tau_{ct}} = \frac{\beta U_{c,t+1}}{1+\tau_{c,t+1}}\left[(1-\tau_{k,t+1})(f'(k_{t+1})-\delta)+1\right] \end{cases}$$

We focus on the second one.

Second, the world resource constraint is

$$(c_t + c_t^*) + (g_t + g_t^*) + (k_{t+1} - (1-\delta)\,k_t) + \left(k_{t+1}^* - (1-\delta)\,k_t^*\right) = f(k_t) + f(k_t^*)$$

the world system becomes

$$\begin{cases} \frac{U_{c,t}/(1+\tau_{ct})}{U_{c,t+1}/(1+\tau_{c,t+1})} = \frac{U_{c,t}^*/(1+\tau_{ct}^*)}{U_{c,t+1}^*/\left(1+\tau_{c,t+1}^*\right)} \\ \frac{U_{c,t}}{1+\tau_{ct}} = \frac{\beta U_{c,t+1}}{1+\tau_{c,t+1}}\left[(1-\tau_{k,t+1})\left(f'(k_{t+1})-\delta\right)+1\right] \\ \frac{U_{c,t}^*}{1+\tau_{ct}^*} = \frac{\beta U_{c,t+1}^*}{1+\tau_{c,t+1}^*}\left[\left(1-\tau_{k,t+1}^*\right)\left(f'\left(k_{k+1}^*\right)-\delta\right)+1\right] \\ (c_t + c_t^*) + (g_t + g_t^*) + (k_{t+1} - (1-\delta)\,k_t) + \left(k_{t+1}^* - (1-\delta)\,k_t^*\right) = f(k_t) + f(k_t^*) \end{cases}$$

The system jointly determines the dynamic paths of $\left\{c_t, c_t^*, k_{t+1}, k_{t+1}^*\right\}_{t=0}^{\infty}$.

5. The steady state.

   The steady state solves

$$\begin{cases} \frac{1}{\beta} - 1 = (1-\tau_k)\left(f'(k)-\delta\right) \\ \frac{1}{\beta} - 1 = (1-\tau_k^*)\left(f'(k^*)-\delta\right) \\ (c+c^*) + (g+g^*) + \delta k + \delta k^* = f(k) + f(k^*) \end{cases}$$

6. Simulation.

   (a) Home starts at the steady state.

   (b) Foreign (developing country) has only half initial capital.

   (c) Period utility: $u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \gamma > 0.$,

   (d) Production function: $f(k) = Ak^{\alpha}, \alpha \in (0,1)$

   (e) Parameter values: $\beta = 0.95, \gamma = 2, \delta = 0.2, \alpha = 1/3, A = 1, g = 0.2$, and zero taxes initially.

   See the RMT textbook for results.

Lecture 3

Asset Pricing (RMT Ch13)

3.1 Asset Pricing and Euler Equations

1. Household's choice.

   A consumer or an investor solves

$$\max_{\{C_t, L_t, N_t\}_{t=0}^{\infty}} \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j u\left(c_{t+j}\right) \right]$$

$$\text{s.t. } c_t + R_t^{-1} L_t + P_t N_t \leqslant L_{t-1} + \left(P_t + y_t\right) N_{t-1}$$

   where

   (a) y_t is the stochastic dividend paid by a "Lucas Tree".

   (b) L_t is bond holding.

   (c) N_t is the tree holding.

   (d) R_t is the gross risk-free interest rate.

   (e) P_t is the price of the tree.

   (f) Note that N_{t-1}, L_{t-1} is known.

2. Solution.

   Set up the Lagrangian, at $t = 0$,

$$\mathcal{L} = \max_{\{C_t, L_t, N_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) + \lambda_t \left(L_{t-1} + \left(P_t + y_t\right) N_{t-1} - c_t - R_t^{-1} L_t - P_t N_t \right) \right]$$

   And obtain the F.O.C.s at t: [1]

$$\begin{cases} u'\left(c_t\right) = \lambda_t \\ R_t^{-1} \lambda_t = \beta \mathbb{E}_t \lambda_{t+1} \\ P_t \lambda_t = \beta \mathbb{E}_t \left[\lambda_{t+1} \left(P_{t+1} + y_{t+1}\right) \right] \end{cases} \Longrightarrow \begin{cases} R_t^{-1} u'\left(c_t\right) = \beta \mathbb{E}_t u'\left(c_{t+1}\right) \\ P_t u'\left(c_t\right) = \beta \mathbb{E}_t \left[u'\left(c_{t+1}\right) \left(P_{t+1} + y_{t+1}\right) \right] \end{cases}$$

[1] Note that we can deduce the F.O.C. in a more clear and rigorous way by dynamic programming, i.e., set up the recursive form of the problem and use the envelop theorem.

3. Some alternative assumptions.

   (a) Hall (1987) assumes $R_t = R$ is a constant, so

$$R_t^{-1} u'(c_t) = \beta \mathbb{E}_t u'(c_{t+1}) \Longrightarrow (\beta R)^{-1} u'(c_t) = \mathbb{E}_t u'(c_{t+1})$$

   In this way, we could further assume that

$$u'(c_{t+1}) = (\beta R)^{-1} u'(c_t) + \varepsilon_t$$

   where $\varepsilon_t \sim \mathcal{N}(0,1)$ is independent with c_t, and thus, $u'(c_t) \sim \mathrm{AR}(1)$ is a Markov process. It means only consumption in this period can predict consumption in the next period.

   (b) Efficiency market hypothesis often implies that the present value of stock price or the price of "Lucas Tree" (dividends included) is a martingale.

$$\mathbb{E}_t [P_{t+1} + y_{t+1}] = \frac{1}{\beta} P_t$$

   or to say

$$\mathbb{E}_t [\beta (P_{t+1} + y_{t+1})] = P_t$$

   generally,

$$\mathbb{E}_0 \left[\beta^{t+1} P_{t+1} + \sum_{j=1}^{t+1} \beta^j y_j \right] = P_0$$

   When will this be true?

   Recall the second equation in the F.O.C.s, $P_t u'(c_t) = \beta \mathbb{E}_t [u'(c_{t+1})(P_{t+1} + y_{t+1})]$ $= \beta \mathbb{E}_t [u'(c_{t+1})] \mathbb{E}_t [P_{t+1} + y_{t+1}] + \beta \mathrm{Cov}_t (u'(c_{t+1}), P_{t+1} + y_{t+1})$ thus, note that $u'(c_t)$ is $\sigma(c_t)$-measurable,

$$\frac{P_t}{\beta} = \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right] \mathbb{E}_t [P_{t+1} + y_{t+1}] + \frac{1}{u'(c_t)} \mathrm{Cov}_t (u'(c_{t+1}), P_{t+1} + y_{t+1})$$

   the efficiency market hypothesis holds if

$$\begin{cases} \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right] = 1 \\ \mathrm{Cov}_t (u'(c_{t+1}), P_{t+1} + y_{t+1}) = 0 \end{cases}$$

   Moreover, let's look at two cases.

      i. $\mathrm{Cov}_t (u'(c_{t+1}), P_{t+1} + y_{t+1}) > 0$.
      Intuitively, when $c_{t+1} \downarrow$ then $u'(c_{t+1}) \uparrow$, and thus, $P_{t+1} + y_{t+1} \uparrow$. This kind of asset tends to give us more when we need more, just like an insurance, which is a "good" asset. So, the price P_t is higher.

      ii. $\mathrm{Cov}_t (u'(c_{t+1}), P_{t+1} + y_{t+1}) < 0$.
      Intuitively, this kind of asset tends to give us less when we need more, which is a "bad" asset. So, the price P_t is lower.

3.2 Equilibrium Asset Pricing

1. Lucas Asset pricing model, assumptions.

   (a) One tree per household (total supply of tree is 1 per household).

   (b) Can trade ownership of trees.

   (c) Can issue and buy debts.

   (d) No government.

2. Definition of Equilibrium.

   An equilibrium is this model is the allocation $\{c_t, L_t, N_t\}_{t=0}^{\infty}$, prices $\left\{P_t, R_t^{-1}\right\}_{t=0}^{\infty}$ such that

   (a) Households maximize the permanent expected utility defined as before, subject to the budget constraint.

   $$\max_{\{c_t,L_t,N_t\}_{t=0}^{\infty}} \mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j u\left(c_{t+j}\right)\right]$$
   $$\text{s.t. } c_t + R_t^{-1}L_t + P_tN_t \leqslant L_{t-1} + \left(P_t + y_t\right)N_{t-1}$$

   where $N_{-1} = 1, L_{-1} = 0$.

   (b) Marketing Clearing: $L_t = 0$, $N_t = 1$, $c_t = y_t$.

   Note:

   i. The price systems have no price of consumption goods, since we normalize it to 1.

   ii. The third marketing clearing condition $c_t = y_t$ comes from the Walras's Law, or set $L_t = 0$, $N_t = 1$ in the budget constraint with equality, then

   $$c_t + 0 + P_t = 0 + P_t + y_t \Longrightarrow c_t = y_t$$

   iii. No arbitrage conditions?
   There is no way for household to make themselves better off by trading assets (the equivalent expression of no-arbitrage), which is satisfied in such equilibrium. In equilibrium or any state,

   $$\begin{cases} R_t^{-1}u'\left(c_t\right) = \beta\mathbb{E}_t u'\left(c_{t+1}\right) \\ P_t u'\left(c_t\right) = \beta\mathbb{E}_t\left[u'\left(c_{t+1}\right)\left(P_{t+1} + y_{t+1}\right)\right] \end{cases}$$

   the same kind of assets have the same price.

3. Asset prices.

(a) Debt price.

$$R_t^{-1} = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right]$$

In the equilibrium, $c_t = y_t$, then

$$R_t^{-1} = \beta \mathbb{E}_t \left[\frac{u'(y_{t+1})}{u'(y_t)} \right]$$

(b) Equity price.

$$P_t u'(c_t) = \beta \mathbb{E}_t \left[u'(c_{t+1}) (P_{t+1} + y_{t+1}) \right]$$

and then

$$\begin{aligned}
P_t u'(c_t) =& \beta \mathbb{E}_t \left[u'(c_{t+1}) (P_{t+1} + y_{t+1}) \right] \\
=& \beta \mathbb{E}_t \left[u'(c_{t+1}) P_{t+1} \right] + \beta \mathbb{E}_t \left[u'(c_{t+1}) y_{t+1} \right] \\
=& \beta \mathbb{E}_t \left[\beta \mathbb{E}_{t+1} \left[u'(c_{t+2}) (P_{t+2} + y_{t+2}) \right] \right] + \beta \mathbb{E}_t \left[u'(c_{t+1}) y_{t+1} \right] \\
=& \beta^2 \mathbb{E}_t \left[u'(c_{t+2}) (P_{t+2} + y_{t+2}) \right] \\
& + \beta^2 \mathbb{E}_t \left[u'(c_{t+2}) y_{t+2} \right] + \beta \mathbb{E}_t \left[u'(c_{t+1}) y_{t+1} \right] \\
=& \cdots \\
=& \mathbb{E}_t \left[\sum_{j=1}^{\infty} \beta^j u'(c_{t+j}) y_{t+j} \right] + \lim_{j \to \infty} \beta^j \mathbb{E}_t \left[u'(c_{t+j}) P_{t+j} \right]
\end{aligned}$$

We may impose the transversality condition

$$\lim_{j \to \infty} \beta^j \mathbb{E}_t \left[u'(c_{t+j}) P_{t+j} \right] = 0$$

which implies no asset bubble.

$$\underbrace{P_t u'(c_t)}_{\text{utility cost}} = \underbrace{\mathbb{E}_t \left[\sum_{j=1}^{\infty} \beta^j u'(c_{t+j}) y_{t+j} \right]}_{\text{utility gain of 1 unit tree}} + \lim_{j \to \infty} \beta^j \mathbb{E}_t \left[u'(c_{t+j}) P_{t+j} \right]$$

If, for example, $\lim_{j \to \infty} \beta^j \mathbb{E}_t \left[u'(c_{t+j}) P_{t+j} \right] > 0$, then the cost is greater than gain, so everyone chooses to sell the tree, which cannot lead to an equilibrium.

Therefore,

$$P_t = \mathbb{E}_t \left[\sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} y_{t+j} \right] = \sum_{j=1}^{\infty} \beta^j \mathbb{E}_t \left[\frac{u'(c_{t+j})}{u'(c_t)} y_{t+j} \right]$$

In the equilibrium, $c_t = y_t$, then

$$P_t = \sum_{j=1}^{\infty} \beta^j \mathbb{E}_t \left[\frac{u'(y_{t+j})}{u'(y_t)} y_{t+j} \right]$$

4. The expected hypothesis.

   The expectation hypothesis states that the price or a 2-period bond (no coupon, pays 1 unit of consumption good in 2 periods.) equals to the discounted expected price of a one-period bond, issued in the next period:

$$R_{2,t}^{-1} = R_{1,t}^{-1}\mathbb{E}_t R_{1,t+1}^{-1}$$

   If an economy has 3 assets $\{L_{1,t}, L_{2,t}, N_t\}$, then the problem becomes

$$\max_{\{c_t, L_{1,t}, L_{2,t}, N_t\}_{t=0}^{\infty}} \mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j u\left(c_{t+j}\right)\right]$$
$$\text{s.t. } c_t + R_{1,t}^{-1}L_{1,t} + R_{2,t}^{-1}L_{2,t} + P_t N_t \leqslant L_{1,t-1} + R_{1,t}^{-1}L_{2,t-1} + \left(P_t + y_t\right)N_{t-1}$$

   We now want to examine if the expectation hypothesis holds generally. Let's get started.

   The Lagrangian is

$$\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left[u\left(c_t\right) + \lambda_t\left(L_{1,t-1} + R_{1,t}^{-1}L_{2,t-1} + \left(P_t + y_t\right)N_{t-1} - c_t - R_{1,t}^{-1}L_{1,t} - R_{2,t}^{-1}L_{2,t} - P_t N_t\right)\right]$$

   F.O.C.s are

$$\begin{aligned}
u'\left(c_t\right) &= \lambda_t \\
\lambda_t R_{1,t}^{-1} &= \beta\mathbb{E}_t\lambda_{t+1} \\
\lambda_t R_{2,t}^{-1} &= \beta\mathbb{E}_t\left[\lambda_{t+1}R_{1,t+1}^{-1}\right] \\
\lambda_t P_t &= \beta\mathbb{E}_t\left[\lambda_{t+1}\left(P_{t+1} + y_{t+1}\right)\right]
\end{aligned}$$

   Thus,

$$\begin{aligned}
u'\left(c_t\right)R_{1,t}^{-1} &= \beta\mathbb{E}_t u'\left(c_{t+1}\right) \\
u'\left(c_t\right)R_{2,t}^{-1} &= \beta\mathbb{E}_t\left[u'\left(c_{t+1}\right)R_{1,t+1}^{-1}\right] \\
u'\left(c_t\right)P_t &= \beta\mathbb{E}_t\left[u'\left(c_{t+1}\right)\left(P_{t+1} + y_{t+1}\right)\right]
\end{aligned}$$

   We mainly care about the first and the second equation.

$$\begin{aligned}
R_{2,t}^{-1} &= \beta\mathbb{E}_t\left[\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}R_{1,t+1}^{-1}\right] \\
&= \beta\mathbb{E}_t\left[\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\right]\mathbb{E}_t\left[R_{1,t+1}^{-1}\right] + \beta\mathrm{Cov}_t\left(\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}, R_{1,t+1}^{-1}\right) \\
&= R_{1,t}^{-1}\mathbb{E}_t\left[R_{1,t+1}^{-1}\right] + \beta\mathrm{Cov}_t\left(\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}, R_{1,t+1}^{-1}\right)
\end{aligned}$$

   Moreover, rolling the first one forward 1 period.

$$u'\left(c_{t+1}\right)R_{1,t+1}^{-1} = \beta\mathbb{E}_{t+1}u'\left(c_{t+2}\right) \Longrightarrow R_{1,t+1}^{-1} = \beta\mathbb{E}_{t+1}\left[\frac{u'\left(c_{t+2}\right)}{u'\left(c_{t+1}\right)}\right]$$

then plug $R_{1,t+1}^{-1}$ into the second equation,

$$\begin{aligned} R_{2,t}^{-1} &= \beta^2 \mathbb{E}_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\mathbb{E}_{t+1}\left[\frac{u'(c_{t+2})}{u'(c_{t+1})}\right]\right] \\ &= \beta^2 \mathbb{E}_t\left[\mathbb{E}_{t+1}\left[\frac{u'(c_{t+1})}{u'(c_t)}\cdot\frac{u'(c_{t+2})}{u'(c_{t+1})}\right]\right] \\ &= \beta^2 \mathbb{E}_t\left[\frac{u'(c_{t+2})}{u'(c_t)}\right] \end{aligned}$$

Back to

$$R_{2,t}^{-1} = R_{1,t}^{-1}\mathbb{E}_t\left[R_{1,t+1}^{-1}\right] + \underbrace{\beta \mathrm{Cov}_t\left(\frac{u'(c_{t+1})}{u'(c_t)}, R_{1,t+1}^{-1}\right)}_{\text{Deviation from the expectation hypothesis}}$$

If the deviation is positive, i.e., $\frac{u'(c_{t+1})}{u'(c_t)}$ and $R_{1,t+1}^{-1}$ are positively correlated.

Now if $c_{t+1}\downarrow$, $u'(c_{t+1})\uparrow$, then $R_{1,t+1}^{-1}\uparrow$, meanwhile $R_{2,t}^{-1} > R_{1,t}^{-1}\mathbb{E}_t\left[R_{1,t+1}^{-1}\right]$.

5. Question: What if there are n bonds with maturity varying from 1 to n?

   The expectation hypothesis should be

$$\begin{aligned} R_{n,t}^{-1} &= R_{1,t}^{-1}\mathbb{E}_t\left[R_{n-1,t+1}^{-1}\right] \\ &= R_{1,t}^{-1}\mathbb{E}_t\left[R_{1,t+1}^{-1}R_{1,t+2}^{-1}\cdots R_{1,t+n-1}^{-1}\right] \end{aligned}$$

   The budget constraint should be

$$c_t + \sum_{j=1}^{n} R_{j,t}^{-1}L_{j,t} + P_t N_t \leqslant \sum_{j=1}^{n} R_{j-1,t}^{-1}L_{j,t-1} + (P_t + y_t)N_{t-1}$$

   where we set $R_{0,t}^{-1} = 1$.

3.3 Review: Complete Market Model

1. Set-up.

   (a) Household $i = 1, 2, \cdots, I$, they have the same standard preference

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

   (b) Exogenous endowment y_t^i in period t. The sup-script denotes for different households.

   (c) No initial wealth, $y_0^i = 0, \forall i$.

   (d) State:

      i. $s_t = \left\{y_t^1, y_t^2, \cdots, y_t^I\right\}$.

ii. $\pi\left(s^t\right)$ is the probability of history if we are in period $t-j$, we may denote $s^t = \{s_{t-j}, s_{t-j+1}, \cdots, s_t\}$.

iii. $\pi\left(s_{t+1} \middle| s^t\right) = \mathbb{P}\left(s_{t+1} \middle| s^t\right) = \mathbb{P}\left(s_{t+1} \middle| \mathcal{F}_t\right)$

2. Two ways to form a competitive market.

   (a) Arrow-Debreu securities.

      i. Time 0 trading: Contingent claims traded in period 0, exercised every period.

      ii. $q^0\left(s^t\right)$: the time-0 price of asset that pays 1unit of consumption if history s^t realized.

      iii. $a^i\left(s^t\right)$: agent i's holding of the asset contingent on state s^t.

   (b) Arrow securities

      i. Sequential trading.

      ii. One-period ahead contingent claims.

3. Price Arrow-Debreu securities.

   Household i's problem.

$$\max_{\{c^i(s^t), a^i(s^t)\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t^i\right) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi\left(s^t\right) u\left(c^i\left(s^t\right)\right)$$
$$\text{s.t.} \quad \begin{cases} c^i\left(s^t\right) = y^i\left(s^t\right) + a^i\left(s^t\right) \\ \sum_{t=0}^{\infty} \sum_{s^t} q^0\left(s^t\right) a^i\left(s^t\right) = 0 \end{cases}$$

The first constraint implies if the household wants to borrow, then she buys security issued by other households, otherwise, she will issue a security and sell it to others in need. And $c^i\left(s^t\right) = y^i\left(s^t\right) + a^i\left(s^t\right)$ holds state by state, and it has no uncertainty within a state.

The second constraint is the constraint for time-0 trading, which means the selling revenue equals to the buying expenditure. Again, if the household i wants to borrow in state $s^t = \tilde{s}$, then she must buy some securities, say $a^i\left(\tilde{s}\right)$. Note that we assume so initial wealth, then, household i should issue some securities $a^i\left(\bar{s}\right)$ for some other state $\bar{s}$, to sell at time 0 to get the wealth. Overall, the non-initial-wealth condition gives us the second condition.

Merge two constraint into

$$\max_{\{a^i(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi\left(s^t\right) u\left(c^i\left(s^t\right)\right)$$
$$\text{s.t.} \quad \sum_{t=0}^{\infty} \sum_{s^t} q^0\left(s^t\right) \left[c^i\left(s^t\right) - y^i\left(s^t\right)\right] = 0$$

Let μ^i be the Lagrangian multiplier on the constraint, then

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \left[\beta^t \pi \left(s^t \right) u \left(c^i \left(s^t \right) \right) - \mu^i q^0 \left(s^t \right) \left[c^i \left(s^t \right) - y^i \left(s^t \right) \right] \right]$$

F.O.C.s are

$$\beta^t \pi \left(s^t \right) u' \left(c^i \left(s^t \right) \right) = \mu^i q^0 \left(s^t \right), \forall i, s^t$$

For another household j:

$$\beta^t \pi \left(s^t \right) u' \left(c^j \left(s^t \right) \right) = \mu^j q^0 \left(s^t \right)$$

then

$$\frac{u' \left(c^i \left(s^t \right) \right)}{u' \left(c^j \left(s^t \right) \right)} = \frac{\mu^i}{\mu^j}, \forall i, j, s^t$$

If the initial wealth $\sum_{t=0}^{\infty} \sum_{s^t} q^0 \left(s^t \right) y^i \left(s^t \right) = \sum_{t=0}^{\infty} \sum_{s^t} q^0 \left(s^t \right) y^j \left(s^t \right)$, then household i and j have the same Lagrangian function, thus, $\mu^i = \mu^j$, then they will have identical consumption path.

For another period $t + j$:

$$\beta^{t+j} \pi \left(s^{t+j} \right) u' \left(c^i \left(s^{t+j} \right) \right) = \mu^i q^0 \left(s^{t+j} \right)$$

then

$$\beta^j \frac{\pi \left(s^{t+j} \right)}{\pi \left(s^t \right)} \frac{u' \left(c^i \left(s^t \right) \right)}{u' \left(c^i \left(s^{t+j} \right) \right)} = \frac{q^0 \left(s^{t+j} \right)}{q^0 \left(s^t \right)}$$

Assuming that the market for Arrow-Debreu securities **opens at time** t. We get the pricing equation conditional on time t information:

$$q^0 \left(s^{t+j} \middle| s_t \right) = \beta^j \frac{\pi \left(s^{t+j} \right)}{\pi \left(s^t \right)} \frac{u' \left(c^i \left(s^t \right) \right)}{u' \left(c^i \left(s^{t+j} \right) \right)}$$

We may assume $q^t \left(s^t \right) = q^t = 1$, and it doesn't depend on s^t anymore, since 1 unit of goods in period t exactly costs 1 unit of goods in period t, it has no discount effect or risk.

Moreover,

$$\frac{\pi \left(s^{t+j} \right)}{\pi \left(s^t \right)} = \frac{\pi \left(s^{t+j} \right)}{\pi \left(s_t \right)} = \frac{\pi \left(s^{t+j} \right)}{1} = \mathbb{P} \left(s^{t+j} \middle| s_t \right) = \pi \left(s^{t+j}, s_t \right)$$

Thus,

$$q^0 \left(s^{t+j} \middle| s_t \right) = \beta^j \pi \left(s^{t+j}, s_t \right) \frac{u' \left(c^i \left(s_t \right) \right)}{u' \left(c^i \left(s^{t+j} \right) \right)}$$

3.4 Prices of State-contingent Bonds

1. Previously, we learnt the price of the state-contingent bonds.

   Note that we assume the evolvement of states to be a **Markov Chain**, thus,

$$\mathbb{P} \left(s_{t+1} \middle| s^t \right) = \mathbb{P} \left(s_{t+1} \middle| s_t \right)$$

holds by Markov property.

Here, the conclusion is generalized to a j-period state contingent claim that promises 1 unite of consumption payments in state s_j, given that the state of today is s, the price of the claim is

$$Q_j\left(s_{t+j}\middle|\, s_t=s\right)=\beta^j\frac{u'\left(s_j\right)}{u'\left(s\right)}f^j\left(s_j,s\right)$$

The s_j is called contingent state, and $f^j\left(s_j,s\right)$ is the j-periods ahead conditional probability density. $f^j\left(s_j,s\right)$ satisfies

$$f^j\left(s_j,s\right)=\int_{\mathcal{S}}f\left(s_j,s_{j-1}\right)f^{j-1}\left(s_{j-1},s\right)\mathrm{d}s_{j-1}$$

where $\mathcal{S}$ is the set all possible states. Moreover,

$$\int_{-\infty}^{s'}f^j\left(w,s\right)\mathrm{d}w=\mathbb{P}\left\{s_{t+j}\leqslant s'\middle|\, s_t=s\right\}$$

For example, $Q_2\left(s_j=12\middle|\, s_t=10\right)$ is the price of the state-contingent bond that pays 1 unite of consumption in 2 periods if the state turns out to be $s_j=12$.

Rewrite the pricing equation as

$$Q_j\left(s_{t+j}\middle|\, s_t=s\right)u'\left(s\right)=\beta^ju'\left(s_j\right)f^j\left(s_j,s\right)$$

The LHS is the unitality cost of buying the bond, the RHS is the utility gain or pay-off.

2. Two examples for pricing with state-contingent bonds: If a state-contingent bonds are available, then all other assets are redundant.

   (a) A risk-free bond that promises to pay 1unit of consumption.

   We can buy one unite of bond for all states.

$$\begin{aligned}R_t^{-1}&=\int_{\mathcal{S}}Q\left(s_{t+1}\middle|\, s_t=s\right)\mathrm{d}s_{t+1}=\int_{\mathcal{S}}\beta\frac{u'\left(s_{t+1}\right)}{u'\left(s\right)}f\left(s_{t+1},s\right)\mathrm{d}s_{t+1}\\&=\beta\mathbb{E}_t\frac{u'\left(s_{t+1}\right)}{u'\left(s\right)}\end{aligned}$$

   where $Q\left(s_{t+1}\middle|\, s_t=s\right)=Q_1\left(s_{t+1}\middle|\, s_t=s\right)$.

   (b) Price a stock.

   Let $y_{t+1}=y\left(s_{t+1}\right)$ be the dividend payment in $t+1$ period, $P_{t+1}=P\left(s_{t+1}\right)$ be the price in $t+1$ period. To price the stock, we can buy $y_{t+1}+P_{t+1}$ units of bonds for every state s_{t+1}.

   Then

$$\begin{aligned}P_t&=\int_{\mathcal{S}}\left[y\left(s_{t+1}\right)+P\left(s_{t+1}\right)\right]Q\left(s_{t+1}\middle|\, s_t=s\right)\mathrm{d}s_{t+1}\\&=\int_{\mathcal{S}}\left[y\left(s_{t+1}\right)+P\left(s_{t+1}\right)\right]\beta\frac{u'\left(s_{t+1}\right)}{u'\left(s\right)}f\left(s_{t+1},s\right)\mathrm{d}s_{t+1}\\&=\beta\mathbb{E}_t\left[\frac{u'\left(s_{t+1}\right)}{u'\left(s_t\right)}\left(y_{t+1}+P_{t+1}\right)\right]\end{aligned}$$

3. Insurance premium.

   Consider an insurance against a Lucas tree "crop failures". Let $s_t \geqslant 0$ denotes the output from the tress. The insurance promises to pay one unit of consumption in all states when $s' = s_{t+1} \leqslant \alpha$.

   Then the price of such an insurance should be (assume that $\mathcal{S} \subseteq \mathbb{R}$)

$$q_\alpha(s) = \int_{[0,\alpha]} Q(s'|\, s)\, \mathrm{d}s' = \int_{[0,\alpha]} \beta \frac{u'(s')}{u'(s)} f(s', s)\, \mathrm{d}s'$$

   Moreover, it can be rewritten as

$$\begin{aligned}
q_\alpha(s) &= \frac{\beta}{u'(s)} \int_{[0,\alpha]} u'(s') f(s', s)\, \mathrm{d}s' \\
&= \frac{\beta}{u'(s)} \mathbb{P}(s' \leqslant \alpha|\, s) \int_{[0,\alpha]} u'(s') \frac{f(s', s)}{\mathbb{P}(s' \leqslant \alpha|\, s)} \mathrm{d}s' \\
&= \frac{\beta}{u'(s)} \mathbb{P}(s' \leqslant \alpha|\, s) \int_{[0,\alpha]} u'(s') \frac{\mathbb{P}(s'|\, s)}{\mathbb{P}(s' \leqslant \alpha|\, s)} \mathrm{d}s' \\
&= \frac{\beta}{u'(s)} \mathbb{P}(s' \leqslant \alpha|\, s) \int_{\mathcal{S}} \mathbb{I}_{[0,\alpha]}(s')\, u'(s') \frac{\mathbb{P}(s'|\, s)}{\mathbb{P}(s' \leqslant \alpha|\, s)} \mathrm{d}s' \\
&= \frac{\beta}{u'(s)} \mathbb{P}(s' \leqslant \alpha|\, s)\, \mathbb{E}\left[\mathbb{I}_{[0,\alpha]}(s')\, u'(s') \middle|\, \{s' \leqslant \alpha\}, s \right] \\
&= \frac{\beta}{u'(s)} \mathbb{P}(s' \leqslant \alpha|\, s)\, \mathbb{E}\left[u'(s') |\, \{s' \leqslant \alpha\}, s \right]
\end{aligned}$$

   If the agent is risk neutral, then she will have constant $u'(\cdot)$, then

$$\bar{q}_\alpha(s) = \beta \mathbb{P}(s' \leqslant \alpha|\, s)$$

   which is the **actuarily fair** price of the insurance, i.e., the discounted expected payoff of the insurance.

   If $\mathbb{E}[u'(s')|\, \{s' \leqslant \alpha\}, s] > u'(s)$, then $q_\alpha(s) > \bar{q}_\alpha(s)$, we call the difference $q_\alpha(s) - \bar{q}_\alpha(s)$ the insurance premium, which implies that the agent is willing to pay more above the expected payoff to insure against the adverse income shock.

   What kinds of people tend to pay more for insurance?

   If $u'(s)$ is more less than $\mathbb{E}[u'(s')|\, \{s' \leqslant \alpha\}, s]$, then the agent will pay more premium. Now, low $u'(s)$ implies high s, or high today's consumption. Then, it seems that richer people tend to pay more.

   Moreover, high $\mathbb{E}[u'(s')|\, \{s' \leqslant \alpha\}, s]$, which means the tail expectation of low future income (or consumption) is high, will make people pay more.

4. Man-made uncertainty.

   (a) We can apply lottery/gamble with state-contingent bond.

   (b) Consider a one-period lottery that pays a stochastic payoff ω, the probability density of the payoff, conditional on s' and s is $h(\omega, s', s)$.

(c) The price of the lottery is
$$\beta \int_{\mathcal{S}} \int_{\Omega} \frac{u'(s')}{u'(s)} \omega h(\omega, s', s) f(s', s) \, \mathrm{d}\omega \mathrm{d}s'$$

(d) If the payoff ω doesn't depend on s', then $h(\omega, s', s) = h(\omega, s)$, then the price will be
$$\beta \int_{\mathcal{S}} \frac{u'(s')}{u'(s)} f(s', s) \, \mathrm{d}s' \int_{\Omega} \omega h(\omega, s) \, \mathrm{d}\omega = R_t^{-1} \mathbb{E}[\omega | s]$$

5. Equivalent martingale measure and European option pricing.

(a) Assume that state s_t evolves according to a Markov chain with transition probabilities $\pi(s_{t+1} | s_t)$.

(b) Let an asset pay a steam of dividends $\{d(s_t)\}_{t \geqslant 0}$.

(c) Cum-dividend.

i. Cum-dividend means that the person who owns the asset at the end of time t is entitled to the time t dividend.

ii. Ex-dividend means that the person who owns the asset at the end of the period does not receive the time t dividend.

iii. The cum-dividend time t price of this asset $a(s_t)$ can be expressed recursively as

$$\begin{aligned} a(s_t) &= d(s_t) + \beta \sum_{s_{t+1} \in \mathcal{S}} \frac{u'\left[c_{t+1}^i(s_{t+1})\right]}{u'\left[c_t^i(s_t)\right]} a(s_{t+1}) \pi(s_{t+1} | s_t) \\ &= d(s_t) + \beta \mathbb{E}_t \left[\frac{u'\left[c_{t+1}^i(s_{t+1})\right]}{u'\left[c_t^i(s_t)\right]} a(s_{t+1}) \right] \end{aligned}$$

where c_t^i is the consumption of agent i at date t in state s_t. This equation holds for every agent i.

(d) Change the probability into $\tilde{\pi}(s_{t+1} | s_t)$. Let
$$\tilde{\pi}(s_{t+1} | s_t) = R_t \beta \frac{u'\left[c_{t+1}^i(s_{t+1})\right]}{u'\left[c_t^i(s_t)\right]} \pi(s_{t+1} | s_t)$$
where
$$R_t^{-1} = R_t^{-1}(s_t) = \beta \mathbb{E}_t \left(\frac{u'\left[c_{t+1}^i(s_{t+1})\right]}{u'\left[c_t^i(s_t)\right]} \right)$$
then
$$\begin{aligned} a(s_t) &= d(s_t) + \beta \sum_{s_{t+1} \in \mathcal{S}} \frac{u'\left[c_{t+1}^i(s_{t+1})\right]}{u'\left[c_t^i(s_t)\right]} a(s_{t+1}) \pi(s_{t+1} | s_t) \\ &= d(s_t) + \beta \sum_{s_{t+1} \in \mathcal{S}} \frac{u'\left[c_{t+1}^i(s_{t+1})\right]}{u'\left[c_t^i(s_t)\right]} a(s_{t+1}) \tilde{\pi}(s_{t+1} | s_t) \frac{1}{R_t \beta} \frac{u'\left[c_t^i(s_t)\right]}{u'\left[c_{t+1}^i(s_{t+1})\right]} \\ &= d(s_t) + R_t^{-1} \sum_{s_{t+1} \in \mathcal{S}} a(s_{t+1}) \tilde{\pi}(s_{t+1} | s_t) \\ &= d(s_t) + R_t^{-1} \tilde{\mathbb{E}}_t \left[a(s_{t+1}) | s_t \right] \end{aligned}$$

Note that $\tilde{\pi}\left(s_{t+1} \mid s_t\right)$ is also a probability measure with the assumption that $u' > 0$, then $\tilde{\pi} \in [0,1]$, and

$$\sum_{s_{t+1}\in\mathcal{S}} \tilde{\pi}\left(s_{t+1} \mid s_t\right) = R_t\beta \sum_{s_{t+1}\in\mathcal{S}} \frac{u'\left[c_{t+1}^i\left(s_{t+1}\right)\right]}{u'\left[c_t^i\left(s_t\right)\right]}\pi\left(s_{t+1} \mid s_t\right) = R_t^{-1}R_t = 1$$

(e) The transformed or "twisted" transition density $\tilde{\pi}\left(s_{t+1} \mid s_t\right)$ can be used to define a twisted joint density $\tilde{\pi}_t\left(s^t\right) = \tilde{\pi}\left(s_t \mid s_{t-1}\right)\cdots\tilde{\pi}\left(s_1 \mid s_0\right)\tilde{\pi}\left(s_0\right)$ For example,

$$\begin{aligned}\tilde{\pi}\left(s_{t+1}, s_{t+2} \mid s_t\right) &= \tilde{\pi}\left(s_{t+1} \mid s_t\right)\cdot\tilde{\pi}\left(s_{t+2} \mid s_{t+1}\right)\\ &= R_t\left(s_t\right)R_{t+1}\left(s_{t+1}\right)\beta^2\frac{u'\left[c_{t+1}^i\left(s_{t+1}\right)\right]}{u'\left[c_t^i\left(s_t\right)\right]}\pi\left(s_{t+1} \mid s_t\right)\pi\left(s_{t+2} \mid s_{t+1}\right)\end{aligned}$$

(f) Equivalent martingale measure $\tilde{\pi}$.

i. "Equivalent": Under the assumption that $u' > 0$, $\tilde{\pi}$ assigns positive probability to any event that is assigned positive probability by π, and vice versa.

ii. "Martingale":

Insight 1:

Consider a special $\{d_t\}$ such that $d_T = d\left(s_T\right), d_t = 0$ for $t \neq T$.

Then

$$\begin{aligned}a_T\left(s_T\right) &= d\left(s_T\right)\\ a_{T-1}\left(s_{T-1}\right) &= R_{T-1}^{-1}\tilde{\mathbb{E}}_{T-1}\left[a_T\left(s_T\right)\right]\\ a_{T-2}\left(s_{T-2}\right) &= R_{T-2}^{-1}\tilde{\mathbb{E}}_{T-2}\left[R_{t-2}^{-1}R_{T-1}^{-1}a_T\left(s_T\right)\right]\\ &\vdots\\ a_t\left(s_t\right) &= R_t^{-1}\tilde{\mathbb{E}}_t\left[R_{t+1}^{-1}R_{t+2}^{-1}\cdots R_{T-1}^{-1}a_T\left(s_T\right)\right]\end{aligned}$$

Now fix $t < T$ and define the "deflated" or "interest-adjusted" asset price process

$$\tilde{a}_{t,t+j} = \frac{a_{t+j}}{R_tR_{t+1}\cdots R_{t+j-1}}$$

then

$$\tilde{\mathbb{E}}_t\left[\tilde{a}_{t,t+j}\right] = a_t\left(s_t\right) = \tilde{a}_{t,t}, \forall j = 1,2\cdots,T-t$$

Then the interest adjusted asset price is a martingale: using the twisted measure.

Insight 2:

Define the likelihood ratio as

$$L\left(s^t\right) := \frac{\tilde{\pi}_t\left(s^t\right)}{\pi_t\left(s^t\right)}$$

Then

$$L\left(s^t\right) = \frac{\tilde{\pi}_t\left(s^t\right)}{\pi_t\left(s^t\right)} = \frac{\tilde{\pi}_t\left(s_t \mid s_{t-1}\right)\tilde{\pi}_t\left(s^{t-1}\right)}{\pi_t\left(s_t \mid s_{t-1}\right)\pi_t\left(s^{t-1}\right)} = \frac{\tilde{\pi}_t\left(s_t \mid s_{t-1}\right)}{\pi_t\left(s_t \mid s_{t-1}\right)}L\left(s^{t-1}\right)$$

thus,

$$\begin{aligned}\mathbb{E}_{t-1}\left[L\left(s^{t}\right)\right] &= L\left(s^{t-1}\right)\mathbb{E}_{t}\left[\frac{\tilde{\pi}_{t}\left(s_{t}|\,s_{t-1}\right)}{\pi_{t}\left(s_{t}|\,s_{t-1}\right)}\right] \\ &= L\left(s^{t-1}\right)\sum_{s_{t+1}\in\mathcal{S}}\frac{\tilde{\pi}_{t}\left(s_{t}|\,s_{t-1}\right)}{\pi_{t}\left(s_{t}|\,s_{t-1}\right)}\cdot\pi_{t}\left(s_{t}|\,s_{t-1}\right) = L\left(s^{t-1}\right)\end{aligned}$$

then $L\left(s^{t}\right)$ is a martingale.

(g) Using the twisted measure, the best prediction of the future interest-adjusted asset price is its current value. Thus, when the equivalent martingale measure is used to price assets, we have so-called risk-neutral pricing. Recall that

$$a\left(s_{t}\right) = d\left(s_{t}\right) + \beta\mathbb{E}_{t}\left[\frac{u'\left[c_{t+1}^{i}\left(s_{t+1}\right)\right]}{u'\left[c_{t}^{i}\left(s_{t}\right)\right]}a\left(s_{t+1}\right)\right]$$

and

$$a\left(s_{t}\right) = d\left(s_{t}\right) + R_{t}^{-1}\tilde{\mathbb{E}}_{t}\left[a\left(s_{t+1}\right)|\,s_{t}\right]$$

then

$$\left[a\left(s_{t}\right) - d\left(s_{t}\right)\right]R_{t} = \tilde{\mathbb{E}}_{t}\left[a\left(s_{t+1}\right)|\,s_{t}\right]$$

Under the equivalent martingale measure, asset pricing reduces to calculating the conditional expectation of the stream of dividends that defines the asset.

(h) European option pricing

The payoff is $\max\left(0, a_{T} - K\right) \equiv \left(a_{T} - K\right)^{+}$, let Y_{t} be the price of the option at $t < T$, then $Y_{T} = \left(a_{T} - K\right)^{+}$, and

$$Y_{t} = R_{t}^{-1}\tilde{\mathbb{E}}_{t}\left[R_{t+1}^{-1}\cdots R_{T-1}^{-1}\left(a_{T} - K\right)^{+}\right] = \tilde{\mathbb{E}}_{t}\left[\frac{\left(a_{T} - K\right)^{+}}{R_{t}R_{t+1}\cdots R_{T-1}}\right]$$

Black and Scholes (1973) used a particular continuous-time specification of $\tilde{\pi}$ that made it possible to solve Y_{t} analytically.

3.5 The Modigliani-Miller Theorem

1. Set-up.

   (a) A firm issues both bonds (gives fixed income) and equities (gives proportional income).

   (b) Question:

      i. What's the optimal finance structure of a firm?

      ii. What's the relative return of two assets compared to the value of the firm?

   (c) Consider a firm whose owner has one Lucas tree; the tree entitles its owner to the stream of dividends $\{y_{t} > 0\}$.

(d) The owner sells two types of assets on the market:

i. B units of bonds:

- Each promises to pay a fixed interest r per period.
- No explicit maturity, the bonds can be traded in all the periods.
- The owner makes sure that $rB < y(s), \forall s$, so that the firm is always liquid.

ii. N shares of stock: divide the title to the rest of the income to N shares.

- Each pays $\frac{y_t - rB}{N}$ dividends every period.

2. The valuation of the firm.

(a) Bond:

$$P_t^B = \sum_{j=1}^{\infty} \int_{\mathcal{S}} rQ_j\left(s_{t+j} \middle| s_t\right) \mathrm{d}s_{t+j}$$

(b) Equity:

$$P_t^N = \sum_{j=1}^{\infty} \int_{\mathcal{S}} \frac{y_{t+j} - rB}{N} Q_j\left(s_{t+j} \middle| s_t\right) \mathrm{d}s_{t+j}$$

Given the state s_{t+j}, $y(s_{t+j}) = y_{t+j}$ has no uncertainty.

(c) Total value:

$$\begin{aligned} BP_t^B + NP_t^N &= \sum_{j=1}^{\infty} \int_{\mathcal{S}} rBQ_j\left(s_{t+j} \middle| s_t\right) \mathrm{d}s_{t+j} + \sum_{j=1}^{\infty} \int_{\mathcal{S}} \left[y_{t+j} - rB\right] Q_j\left(s_{t+j} \middle| s_t\right) \mathrm{d}s_{t+j} \\ &= \sum_{j=1}^{\infty} \int_{\mathcal{S}} y_{t+j} \beta^j \frac{u'(s_{t+j})}{u'(s_t)} f\left(s_{t+j} \middle| s_t\right) \mathrm{d}s_{t+j} \\ &= \sum_{j=1}^{\infty} \beta^j \mathbb{E}_t \left[y_{t+j} \frac{u'(s_{t+j})}{u'(s_t)}\right] \\ &= P_t \end{aligned}$$

which is equal to the value of the tree, and it is independent of r, N or B.

(d) Therefore, the owner cannot gain by selling fragments of the tree at greater total value.

Conclusion: The division of the output from the tree into debts and equity does change the return structure of the assets.

E.g., we can create a risk-free asset from the intrinsically risky tree.

3. Return of the asset.

(a) For each asset, the return consists of **capital gain** (value of the asset) and **coupon, dividend or interest return** (interest rate or dividends payments).

(b) Assume that $u(c) = \ln c$, and y_t is i.i.d.

Why i.i.d.? It's a benchmark model, in which we cannot predict the future, the idiosyncratic return only depends on the luck today.

(c) Rewrite the pricing formula,

$$\begin{aligned}
P_t^B &= \sum_{j=1}^{\infty} \int_{\mathcal{S}} rQ_j\left(s_{t+j}\middle| s_j\right) \mathrm{d}s_{t+j} = r\sum_{j=1}^{\infty} \beta^j \mathbb{E}_t\left[\frac{u'\left(y_{t+j}\right)}{u'\left(y_t\right)}\right] = r\sum_{j=1}^{\infty} \beta^j \mathbb{E}_t\left[\frac{y_t}{y_{t+j}}\right] \\
&= r\sum_{j=1}^{\infty} \beta^j y_t \mathbb{E}_t\left[y^{-1}\right] = \frac{\beta}{1-\beta} r\mathbb{E}\left[y^{-1}\right] y_t \\
P_t^N &= \sum_{j=1}^{\infty} \int_{\mathcal{S}} \frac{y_{t+j}-rB}{N} Q_j\left(s_{t+j}\middle| s_j\right) \mathrm{d}s_{t+j} = \sum_{j=1}^{\infty} \beta^j \mathbb{E}_t\left[\frac{y_{t+j}-rB}{N}\frac{u'\left(y_{t+j}\right)}{u'\left(y_t\right)}\right] \\
&= \sum_{j=1}^{\infty} \beta^j \mathbb{E}_t\left[\frac{y_{t+j}-rB}{N}\frac{y_t}{y_{t+j}}\right] = \frac{1}{N}\frac{\beta}{1-\beta}\mathbb{E}_t\left[1-rBy^{-1}\right] y_t \\
&= \frac{\beta}{1-\beta}\frac{1-rB\mathbb{E}\left[y^{-1}\right]}{N} y_t \\
P_t &= \sum_{j=1}^{\infty} \int_{\mathcal{S}} y_{t+j} Q_j\left(s_{t+j}\middle| s_j\right) \mathrm{d}s_{t+j} = \sum_{j=1}^{\infty} \beta^j y_t = \frac{\beta}{1-\beta} y_t
\end{aligned}$$

Note that $P_t^B, P_t^N, P_t \propto y_t$, i.e., the prices are all proportional to the current output and the proportion is not time variant.

(d) The capital gains of each asset are the same:

$$\frac{P_{t+1}^B}{P_t^B} = \frac{P_{t+1}^N}{P_t^N} = \frac{P_{t+1}}{P_t} = \frac{y_{t+1}}{y_t}$$

which implies that the capital gains are not only the same in expectation, but also in each realization.

(e) What about the interest return?

Benchmark:

$$I^0 = \mathbb{E}_t\left[\frac{y_{t+1}}{P_t}\right] = \frac{1-\beta}{\beta}\mathbb{E}_t\left[\frac{y_{t+1}}{y_t}\right] = \frac{1-\beta}{\beta}\frac{\mathbb{E}\left[y\right]}{y_t}$$

The bond:

$$I^B = \frac{r}{P_t^B} = \frac{1-\beta}{\beta}\frac{1}{\mathbb{E}\left[y^{-1}\right] y_t}$$

Note that $\mathbb{E}\left[y^{-1}\right] > \left[\mathbb{E}y\right]^{-1}$, therefore,

$$I^B = \frac{1-\beta}{\beta}\frac{1}{\mathbb{E}\left[y^{-1}\right] y_t} < \frac{1-\beta}{\beta}\frac{\mathbb{E}y}{y_t} = I^0$$

The equity:

$$\begin{aligned}
I^N &= \frac{1}{P_t^N}\mathbb{E}_t\left[\frac{y_{t+1}-rB}{N}\right] = \frac{\mathbb{E}_t\left[\frac{y_{t+1}-rB}{N}\right]}{\frac{\beta}{1-\beta}\frac{1-rB\mathbb{E}\left[y^{-1}\right]}{N} y_t} = \frac{1-\beta}{\beta}\frac{1}{y_t}\frac{\mathbb{E}_t\left[y_{t+1}-rB\right]}{1-rB\mathbb{E}\left[y^{-1}\right]} \\
&= \frac{1-\beta}{\beta}\frac{1}{y_t}\frac{\mathbb{E}\left[y\right]-rB}{1-rB\mathbb{E}\left[y^{-1}\right]} = \frac{1-\beta}{\beta}\frac{1}{y_t}\frac{1-rB\left(\mathbb{E}y\right)^{-1}}{1-rB\mathbb{E}\left[y^{-1}\right]}\mathbb{E}y \\
&> \frac{1-\beta}{\beta}\frac{1}{y_t}\mathbb{E}y = I^0
\end{aligned}$$

since

$$\frac{1-rB\left(\mathbb{E}y\right)^{-1}}{1-rB\mathbb{E}\left[y^{-1}\right]}>\frac{1-rB\mathbb{E}\left[y^{-1}\right]}{1-rB\mathbb{E}\left[y^{-1}\right]}=1$$

Thus, the debt offers lower equilibrium return than the tree and the equity offers higher return.

(f) Moreover, the interest return of equity is proportional to

$$\begin{aligned}\frac{1-rB\left(\mathbb{E}y\right)^{-1}}{1-rB\mathbb{E}\left[y^{-1}\right]}&=1+\frac{rB\mathbb{E}\left[y^{-1}\right]-rB\left(\mathbb{E}y\right)^{-1}}{1-rB\mathbb{E}\left[y^{-1}\right]}=1+\frac{rB\left(\mathbb{E}\left[y^{-1}\right]-\left(\mathbb{E}y\right)^{-1}\right)}{1-rB\mathbb{E}\left[y^{-1}\right]}\\&=1+\frac{\mathbb{E}\left[y^{-1}\right]-\left(\mathbb{E}y\right)^{-1}}{\frac{1}{rB}-\mathbb{E}\left[y^{-1}\right]}\end{aligned}$$

Note that $\mathbb{E}\left[y^{-1}\right]-\left(\mathbb{E}y\right)^{-1}>0$, and by $y\left(s\right)>rB,\forall s$,

$$\frac{1}{rB}-\mathbb{E}\left[y^{-1}\right]>\frac{1}{rB}-\frac{1}{\mathbb{E}y}>0$$

If rB increases, then I^N will increase. However, the interest return from the debt is independent of r or B.

Lecture 4

Asset Pricing Empirics (RMT Ch14)

4.1 Equity Premium Puzzle

1. What is Equity Premium Puzzle?

   Hansen and Singleton (1983) took the CRRA utility function, made assumption about the consumption and asset return processes (based on Lucas' model of asset pricing), found that

   (a) Equity premium puzzle: an unrealistic high risk aversion parameter is necessary to justify the equity return premium over bond.

   (b) risk-free rate puzzle: If risk aversion was indeed that high, risk-free rate should be much higher than observed.

2. How the puzzle is raised?

   (a) Rewrite the asset pricing equation.

   For stock or the tree,

$$P_t u'(C_t) = \beta \mathbb{E}_t \left[u'(C_{t+1}) (P_{t+1} + y_{t+1}) \right]$$

   then

$$1 = \beta \mathbb{E}_t \left[\frac{u'(C_{t+1})}{u'(C_t)} \frac{P_{t+1} + y_{t+1}}{P_t} \right]$$

   define

$$1 + r^s_{t+1} := \frac{P_{t+1} + y_{t+1}}{P_t}$$

   then

$$1 = \beta \mathbb{E}_t \left[\frac{u'(C_{t+1})}{u'(C_t)} \left(1 + r^s_{t+1}\right) \right]$$

   For the 1-period bond,

$$R_t^{-1} u'(C_t) = \beta \mathbb{E}_t \left[u'(C_{t+1}) \right] \Longrightarrow 1 = \beta \mathbb{E}_t \left[\frac{u'(C_{t+1})}{u'(C_t)} R_t \right]$$

Note that the bond in real life only pays nominal return instead of real return, and we don't know the inflation rate in advance, so we may define

$$1 + r_{t+1}^{b} := R_t$$

Overall,

$$1 = \beta \mathbb{E}_t \left[\frac{u'(C_{t+1})}{u'(C_t)} \left(1 + r_{t+1}^{i}\right) \right], i = s, b$$

(b) Assume the consumption and return processes.

i. Utility function:

$$u(C) = \frac{C^{1-\gamma}}{1-\gamma}$$

ii. Consumption process:

$$\frac{C_{t+1}}{C_t} = \bar{C} \exp\left\{ \varepsilon_{c,t+1} - \frac{\sigma_c^2}{2} \right\}$$

where $\bar{C} > 1$ is the mean growth rate in data.

iii. Return process:

$$1 + r_{t+1}^{i} = \left(1 + \bar{r}^{i}\right) \exp\left\{ \varepsilon_{i,t+1} - \frac{\sigma_i^2}{2} \right\}, i = s, b$$

where $1 + \bar{r}^{i}$ is the mean return.

Moreover, assume $\{\varepsilon_{c,t+1}, \varepsilon_{s,t+1}, \varepsilon_{b,t+1}\}$ are zero mean, jointly normal, and with variance $\{\sigma_c^2, \sigma_b^2, \sigma_s^2\}$.

(c) Find the puzzles.

$$\begin{aligned} 1 &= \beta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(1 + r_{t+1}^{i}\right) \right] \\ &= \beta \bar{C}^{-\gamma} \left(1 + \bar{r}^{i}\right) \mathbb{E}_t \exp\left\{ -\gamma \varepsilon_{c,t+1} + \varepsilon_{i,t+1} + \gamma \frac{\sigma_c^2}{2} - \frac{\sigma_i^2}{2} \right\} \end{aligned}$$

Note that

$$-\gamma \varepsilon_{c,t+1} + \varepsilon_{i,t+1} + \gamma \frac{\sigma_c^2}{2} - \frac{\sigma_i^2}{2} \sim \mathcal{N}\left(\gamma \frac{\sigma_c^2}{2} - \frac{\sigma_i^2}{2}, \gamma^2 \sigma_c^2 + \sigma_i^2 - 2\gamma \mathrm{Cov}\left(\varepsilon_c, \varepsilon_i\right) \right)$$

thus,

$$\begin{aligned} 1 &= \beta \bar{C}^{-\gamma} \left(1 + \bar{r}^{i}\right) \exp\left\{ \gamma \frac{\sigma_c^2}{2} - \frac{\sigma_i^2}{2} + \frac{1}{2}\gamma^2 \sigma_c^2 + \frac{1}{2}\sigma_i^2 - \frac{1}{2}\gamma \mathrm{Cov}\left(\varepsilon_c, \varepsilon_i\right) \right\} \\ &= \beta \bar{C}^{-\gamma} \left(1 + \bar{r}^{i}\right) \exp\left\{ \gamma (1 + \gamma) \frac{\sigma_c^2}{2} - \gamma \mathrm{Cov}\left(\varepsilon_c, \varepsilon_i\right) \right\}, i = s, b \end{aligned}$$

then

$$\left(1 + \bar{r}^{s}\right) \exp\left\{ -\gamma \mathrm{Cov}\left(\varepsilon_c, \varepsilon_s\right) \right\} = \left(1 + \bar{r}^{b}\right) \exp\left\{ -\gamma \mathrm{Cov}\left(\varepsilon_c, \varepsilon_b\right) \right\}$$

or

$$\frac{1+\bar{r}^s}{1+\bar{r}^b} = \exp\left\{-\gamma \text{Cov}\left(\varepsilon_c, \varepsilon_b\right) + \gamma \text{Cov}\left(\varepsilon_c, \varepsilon_s\right)\right\}$$

Taking log

$$\begin{aligned}\bar{r}^s - \bar{r}^b &\approx \log\left(1+\bar{r}^s\right) - \log\left(1+\bar{r}^b\right) \\ &= \gamma\left[\text{Cov}\left(\varepsilon_c, \varepsilon_s\right) - \text{Cov}\left(\varepsilon_c, \varepsilon_b\right)\right]\end{aligned}$$

where $\bar{r}^s - \bar{r}^b$ is so called **equity premium**.

We need to find the moments of data to see if the theory is true.

See the result from Kocherlakota (1996a), take the U.S. annual data, 1889–1978,

Table 4.1: The U.S. annual data, 1889–1978.

		$1+r^s_{t+1}$	$1+r^b_{t+1}$	C_{t+1}/C_t
	Mean		Covariance	
$1+r^s_{t+1}$	1.070	0.0274	0.00104	**0.00219**
$1+r^b_{t+1}$	1.010		0.00308	**-0.000193**
C_{t+1}/C_t	1.018			0.00127

We can see that the equity premium is about 6Then $\text{Cov}\left(\varepsilon_s, \varepsilon_b\right)$ should be very small.

$$\bar{r}^s - \bar{r}^b \approx \gamma \text{Cov}\left(\varepsilon_c, \varepsilon_s\right)$$

Approximating $\text{Cov}\left(\varepsilon_s, \varepsilon_c\right) \approx \text{Cov}\left(1+r^s_{t+1}, \frac{C_{t+1}}{C_t}\right)$,[1] then

$$\gamma = \frac{0.06}{0.00219} = 27.3973$$

This is in stark contrast with micro-level evidence, which would suggest that $\gamma < 5$. This is known as "equity premium puzzle".

[1]Note that $i = s, b$, $\left(1+\bar{r}^i\right)\bar{C} \approx 1$,

$$\begin{aligned}\text{Cov}\left(1+r^i_{t+1}, \frac{C_{t+1}}{C_t}\right) &= \text{Cov}\left(\left(1+\bar{r}^i\right)\exp\left\{\varepsilon_{i,t+1} - \frac{\sigma_i^2}{2}\right\}, \bar{C}\exp\left\{\varepsilon_{c,t+1} - \frac{\sigma_c^2}{2}\right\}\right) \\ &= \left(1+\bar{r}^i\right)\bar{C}e^{-\frac{\sigma_i^2}{2}-\frac{\sigma_c^2}{2}}\text{Cov}\left(e^{\varepsilon_{i,t+1}}, e^{\varepsilon_{c,t+1}}\right) \\ &\approx e^{-\frac{\sigma_i^2}{2}-\frac{\sigma_c^2}{2}}\left(\mathbb{E}\left[e^{\varepsilon_{i,t+1}+\varepsilon_{c,t+1}}\right] - \mathbb{E}\left[e^{\varepsilon_{i,t+1}}\right]\mathbb{E}\left[e^{\varepsilon_{c,t+1}}\right]\right) \\ &= e^{-\frac{\sigma_i^2}{2}-\frac{\sigma_c^2}{2}}\left(\exp\left\{\frac{\sigma_i^2}{2} + \frac{\sigma_c^2}{2} + \frac{1}{2}\cdot 2\text{Cov}\left(\varepsilon_s, \varepsilon_i\right)\right\} - e^{\frac{\sigma_i^2}{2}+\frac{\sigma_c^2}{2}}\right) \\ &= e^{\text{Cov}\left(\varepsilon_s, \varepsilon_i\right)} - 1 \\ &\approx \text{Cov}\left(\varepsilon_s, \varepsilon_i\right)\end{aligned}$$

If we accept $\gamma = 27.4$, then

$$\begin{aligned}1 &= \beta R_t^f \mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right] = \beta R_t^f \bar{C}^{-\gamma}\mathbb{E}_t \exp\left\{-\gamma\varepsilon_{c,t+1} + \gamma\frac{\sigma_c^2}{2}\right\}\\ &= \beta R_t^f \bar{C}^{-\gamma}\exp\left\{\gamma(\gamma+1)\frac{\sigma_c^2}{2}\right\}\end{aligned}$$

then

$$r^f \approx \log R_t^f = \log\frac{1}{\beta} + \gamma\log\bar{C} - \gamma(1+\gamma)\frac{\sigma_c^2}{2}$$

Let $\beta = 0.995$, $\gamma = 27.4$, and note that

$$\begin{aligned}\mathbb{E}\left[\frac{C_{t+1}}{C_t}\right] &= \bar{C}\exp\left\{-\frac{1}{2}\sigma_c^2 + \frac{1}{2}\sigma_c^2\right\} = \bar{C}\\ \mathrm{Var}\left[\frac{C_{t+1}}{C_t}\right] &= \bar{C}^2\left[\exp\left\{-\frac{1}{2}\sigma_c^2 + \frac{1}{2}\sigma_c^2\right\}\sqrt{e^{\sigma_c^2}-1}\right]^2 \approx \sigma_c^2\end{aligned}$$

then

$$r^f \approx \log\frac{1}{\beta} + \gamma\log\bar{C} - \gamma(1+\gamma)\frac{\sigma_c^2}{2} = 21.3838\%$$

which is way too high.

3. Another way to present "risk premium puzzle".

   Gallant, Hansen, and Tauchen (1990) and Hansen and Jagannathan (1991) interpret the equity premium puzzle in terms of the high "market price of risk" implied by time series data on asset returns.

   Given the preference $\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t u(C_t)$, recall the price of an asset paying p_{t+1} on period $t+1$ is

$$q_t = \beta\mathbb{E}_t\left[\frac{u'(C_{t+1})}{u'(C_t)}p_{t+1}\right] = \mathbb{E}_t[m_{t+1}p_{t+1}]$$

   or

$$q_t = \mathbb{E}_t[m_{t+1}p_{t+1}] = \mathbb{E}_t[m_{t+1}]\mathbb{E}_t[p_{t+1}] + \mathrm{Cov}_t(m_{t+1}, p_{t+1})$$

   Applying the Cauchy-Schwarz inequality, or the correlation coefficient

$$\frac{\mathrm{Cov}_t(m_{t+1}, p_{t+1})}{\sigma_t(m_{t+1})\sigma_t(p_{t+1})} \in (0,1)$$

   thus,

$$\begin{aligned}\frac{q_t}{\mathbb{E}_t[m_{t+1}]} &= \mathbb{E}_t[p_{t+1}] + \frac{\mathrm{Cov}_t(m_{t+1}, p_{t+1})}{\sigma_t(m_{t+1})\sigma_t(p_{t+1})}\frac{\sigma_t(m_{t+1})\sigma_t(p_{t+1})}{\mathbb{E}_t[m_{t+1}]}\\ &\geqslant \mathbb{E}_t[p_{t+1}] - \frac{\sigma_t(m_{t+1})\sigma_t(p_{t+1})}{\mathbb{E}_t[m_{t+1}]}\end{aligned}$$

   Let p_{t+1} be the return R_{t+1} on the asset, then $q_t = 1$, then

$$\frac{1}{\mathbb{E}_t[m_{t+1}]} \geqslant \mathbb{E}_t[R_{t+1}] - \frac{\sigma_t(m_{t+1})}{\mathbb{E}_t[m_{t+1}]}\sigma_t(R_{t+1})$$

Note that $\mathbb{E}_t\left[m_{t+1}\right] = R_{f,t+1}^{-1}$, thus,

$$\mathbb{E}_t\left[R_{t+1}\right] \leqslant R_{f,t+1} + \frac{\sigma_t\left(m_{t+1}\right)}{\mathbb{E}_t\left[m_{t+1}\right]}\sigma_t\left(R_{t+1}\right)$$

The return on any asset is bounded by the sum of the risk-free rate $R_{f,t}^{-1}$ and **the market price of risk** $\frac{\sigma_t(m_{t+1})}{\mathbb{E}_t[m_{t+1}]}$ times the conditional standard deviation of the return $\sigma_t\left(R_{t+1}\right)$.

Or

$$\frac{\mathbb{E}_t\left[R_{t+1}\right] - R_{f,t+1}}{\sigma_t\left(R_{t+1}\right)} \leqslant \frac{\sigma_t\left(m_{t+1}\right)}{\mathbb{E}_t\left[m_{t+1}\right]}$$

The Sharpe ratio of asset is bounded by the market price of risk.

4. What's wrong?

   (a) The utility function is wrong.

   (b) The assumptions of consumption and return processes are wrong.

4.2 Hansen-Jaganathan Bound

1. A Drawback.

   Previously,

$$\mathbb{E}_t\left[\beta\frac{u'\left(C_{t+1}\right)}{u'\left(C_t\right)}R_{j,t+1}\right] = \mathbb{E}_t\left[m_{t+1}R_{j,t+1}\right] = 1$$

   where $m_t = \beta\frac{u'(C_{t+1})}{u'(C_t)}$ is called the stochastic discount factor (SDF), which depends on the pricing model and the utility function.

2. General method.

   Hansen and Jaganathan (1991) proposed a method, which is not model dependent and has no utility function, we only need to

   (a) Prove that there exists an SDF.

   (b) Infer properties of the SDF: a relationship between $\sigma\left(m\right)$ and $\mathbb{E}\left(m\right)$ that can be recovered from data.

   (c) The relationship set a benchmark for different pricing models. For a specific model, we can examine whether the SDF it produces satisfies the Hansen-Jaganathan Bound.

   (d) It's a less structured way of stating an equity premium puzzle.

3. Set-up.

   (a) Suppose there are J primitive assets, and none of them has a payoff structure that is a linear combination of the others. Let $x_{J\times 1}$ be the vector of random payoffs.

For example, if there are S states, then

$$x_{J\times 1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_J \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1S} \\ x_{21} & x_{22} & \cdots & x_{1S} \\ \vdots & \vdots & \ddots & \vdots \\ x_{J1} & x_{J2} & \cdots & x_{JS} \end{bmatrix}$$

and $x_1, x_2, \cdots, x_J$ are linearly independent. We can regard $x_1, \cdots, x_J$ as random variables with a discrete distribution.

Generally, $x_1, \cdots, x_J$ are random variables, and $x_{J\times 1}$ is a random vector.

(b) Assume that $\mathbb{E}\left[xx^{\mathrm{T}}\right]$ and $\left(\mathbb{E}\left[xx^{\mathrm{T}}\right]\right)^{-1}$ exist.

(c) Let $q_{J\times 1}$ denote the prices of the J assets.

(d) Consider a portfolio formed as a combination of these primitive assets, let $c \in \mathbb{R}^J$ represent the portfolio weights. Then the payoff of the portfolio is $c \cdot x$.

(e) The **space of the attainable payoffs** is defined by

$$P := \left\{p : p = c \cdot x, \text{ for some } c \in \mathbb{R}^J\right\}$$

Then price of a portfolio ϕ is a mapping $P \to \mathbb{R}$, and $\phi(P) \subseteq \mathbb{R}$.

4. Properties of ϕ.

(a) If the Law of One Price holds, then

$$\phi(c \cdot x) = \phi(c_1 x_1 + c_2 x_2 + \cdots + c_J x_J) = c_1 \phi(x_1) + c_2 \phi(x_2) + \cdots + c_J \phi(x_J)$$

thus, ϕ is a linear functional on P.

(b) If $x_1, x_2, \cdots, x_J$ are returns, and since return = payoff/price = x_j/q_j, then $q = \mathbf{1}$, the unit vector, and

$$\phi(c \cdot x) = c \cdot \mathbf{1}$$

(c) According to the Riesz representation theorem,[2] a linear functional ϕ can be represented as the inner product of the random payoff x with some scalar random variable y:

$$\phi(p) = \mathbb{E}[yp], \forall p \in P$$

that we call y the stochastic discount factor.

(d) In particular,

$$\phi(1) = \mathbb{E}[y]$$

which is the inverse of the interest rate of a risk-free bond.

[2] Let X be a Hilbert space (i.e., a complete inner product space) of random variable with inner product $\langle x_1, x_2\rangle = \mathbb{E}[x_1 x_2]$, Riesz representation theorem says that, if ϕ is a continuous linear functional $\phi : X \to \mathbb{R}$. There exists a unique element $y \in X$, such that $\phi(x) = \mathbb{E}[yx]$.

5. Empirical practice.

   In the data, we can observe the price q and payoff x, and we know that

$$\mathbb{E}[yx] = q$$

   Consider the population regression

$$y = a + x^{\mathrm{T}} b + e$$

   with estimated coefficients

$$\begin{aligned} b &= [\operatorname{Cov}(x,x)]^{-1} \operatorname{Cov}(x,y) \\ a &= \mathbb{E}y - \mathbb{E}\left[x^{\mathrm{T}} b\right] \end{aligned}$$

   where $\operatorname{Cov}(x,x) = \mathbb{E}\left[xx^{\mathrm{T}}\right] - \mathbb{E}[x](\mathbb{E}[x])^{\mathrm{T}}$ and $\mathbb{E}[ex] = 0$, in other words, e is orthogonal to x. Then

$$\operatorname{Var}(y) = \operatorname{Var}\left(x^{\mathrm{T}} b\right) + \operatorname{Var}(e) \Longrightarrow \operatorname{Var}(y) \geqslant \operatorname{Var}\left(x^{\mathrm{T}} b\right)$$

   thus, $\operatorname{Var}\left(x^{\mathrm{T}} b\right)$ is a lower bound of $\operatorname{Var}(y)$. Now, the question is how to know $\operatorname{Var}\left(x^{\mathrm{T}} b\right)$. In particular, we do not know b. Given $\mathbb{E}[yx] = q$, then

$$\operatorname{Cov}(x,y) = \mathbb{E}[xy] - \mathbb{E}[x]\,\mathbb{E}[y] = q - \mathbb{E}[x]\,\mathbb{E}[y]$$

   and since $b = [\operatorname{Cov}(x,x)]^{-1} \operatorname{Cov}(x,y)$,

$$b = [\operatorname{Cov}(x,x)]^{-1} (q - \mathbb{E}[x]\,\mathbb{E}[y])$$

   Here are the steps to find the Hansen-Jaganathan bound.

   (a) Pick an $\mathbb{E}y$. Recall that $\phi(1) = \mathbb{E}[y] = r_f^{-1}$, so using the risk-free rate is a good idea.

   (b) Calculate $b = [\operatorname{Cov}(x,x)]^{-1} (q - \mathbb{E}[x]\,\mathbb{E}[y])$.[3]

   (c) Calculate $\operatorname{Var}\left(x^{\mathrm{T}} b\right)$.

   (d) Change $\mathbb{E}y$, and plot $\left(\mathbb{E}y, \sqrt{\operatorname{Var}(x^{\mathrm{T}} b)}\right)$.

   Also note that $[\operatorname{Cov}(x,x)]^{\mathrm{T}} = \operatorname{Cov}(x,x)$, then

$$\begin{aligned} \sigma(y) \geqslant \sqrt{\operatorname{Var}(x^{\mathrm{T}} b)} &= \sqrt{b^{\mathrm{T}} \operatorname{Cov}(x,x)\, b} \\ &= \sqrt{(q - \mathbb{E}[x]\,\mathbb{E}[y])^{\mathrm{T}} [\operatorname{Cov}(x,x)]^{-1} (q - \mathbb{E}[x]\,\mathbb{E}[y])} \end{aligned}$$

   The RHS determines a parabola as a function of $\mathbb{E}y$.

[3]If x contains return rates, then $b = [\operatorname{Cov}(x,x)]^{-1} (\mathbf{1} - \mathbb{E}[x]\,\mathbb{E}[y])$.

6. The Hansen-Jaganathan and CRRA.

Under CRRA utility function, the stochastic discount factor is

$$m_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

then

$$\ln m_{t+1} = \ln \beta - \gamma \ln \frac{C_{t+1}}{C_t}$$

assume again,

$$\ln C_{t+1} = \mu + \ln C_t + \sigma_c \varepsilon_{t+1}, \varepsilon_{t+1} \sim \mathcal{N}(0,1), \text{ i.i.d.}$$

then

$$\ln m_{t+1} = \ln \beta - \gamma (\mu + \sigma_c \varepsilon_{t+1})$$

therefore,

$$\ln m_{t+1} \sim \mathcal{N}\left(\ln \beta - \gamma\mu, \gamma^2 \sigma_c^2\right), \text{ i.i.d.}$$

Moreover, [4]

$$\begin{aligned} \mathbb{E}(m) &= e^{\ln \beta - \gamma\mu + \frac{1}{2}\gamma^2\sigma_c^2} = \beta \exp\left\{ -\gamma\mu + \frac{1}{2}\gamma^2\sigma_c^2 \right\} \\ \sigma(m) &= \mathbb{E}(m) \cdot \sqrt{e^{\gamma^2\sigma_c^2} - 1} \approx \mathbb{E}(m)\gamma\sigma_c \end{aligned}$$

quarterly U.S. data 1948:2 to 2005:4 tells us

$$\mu = 0.004952, \sigma_c = 0.005050$$

When γ changes, since σ_c and μ are the same the same order of magnitude and both are small, change in $\mathbb{E}(m)$ mainly dominated by the $e^{-\gamma\mu}$ term. Therefore, as γ increase, $\mathbb{E}(m)$ will decrease. Let $\beta = 0.995$, changing the value of γ, we can plot $(\mathbb{E}(m), \sigma(m))$ for CRRA preferences.

The solid line on the northeast corner is the Hansen-Jaganathan lower bound for $\sigma(m)$. Any theory that is in line with the data in real life should produce a stochastic discount factor y such that, given $\mathbb{E}y$, $\sigma(y)$ should be larger than the Hansen-Jaganathan bound.

However, under CRRA, the cross dots are far away from the H-J bound, and we can see the puzzles from the graph.

(a) The higher γ, the higher $\sigma(m)$ and the lower $\mathbb{E}m$.

(b) Focusing on the southeast region gives us the equity premium puzzle. $\mathbb{E}(m) \in (0.95, 1) \Longleftrightarrow r_f \in (0, 5.26\%)$ is the reasonable interval for risk-free rate. Under the reasonable risk-free rate, to satisfy the high equity premium, we need a very high γ.

(c) Focusing on the northeast region gives us the risk-free rate puzzle. Even if we accept the high γ, the risk-free rate is extremely high. $\mathbb{E}(m) \in (0.80, 0.85) \Longleftrightarrow r_f \in (17.64\%, 25\%)$.

[4] $\ln X \sim \mathcal{N}(\mu, \sigma^2) \Longrightarrow \mathbb{E}X = \exp\left\{\mu + \frac{1}{2}\sigma^2\right\}$, $\mathrm{Var}(X) = \exp\left\{2\mu + \sigma^2\right\}\left(e^{\sigma^2} - 1\right) = (\mathbb{E}X)^2\left(e^{\sigma^2} - 1\right)$.

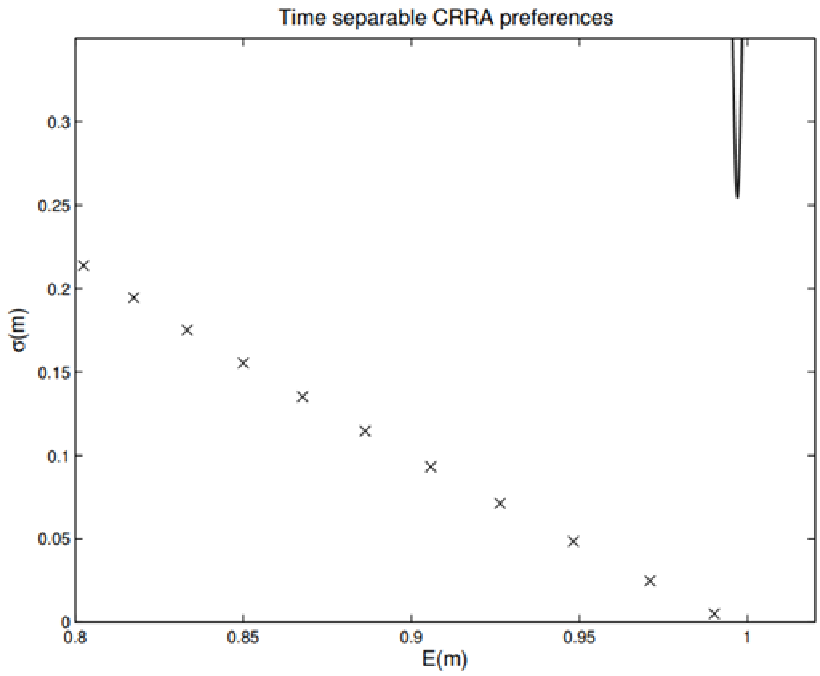

Fig. 4.1. CRRA and H-J bound.

7. Continue on CRRA.

Let $\beta = \exp(-\rho)$, $\mathbb{E}(m) = \exp\{-r_t\}$, where ρ is the discount rate, r_t is the continuous risk-free rate (or the short rate), then

$$\mathbb{E}(m) = \exp\left\{-\rho - \gamma\mu + \frac{1}{2}\gamma^2\sigma_c^2\right\} \Longrightarrow r_t = \rho + \gamma\mu - \frac{1}{2}\gamma^2\sigma_c^2$$

We can see that r_t consists of three terms

(a) discounting in preferences ρ

(b) consumption growth μ, reflecting the growth or deviations of consumption from a smooth path across time.

(c) taste for smooth consumption γ

(d) and a **precautionary saving motive** $\frac{1}{2}\gamma^2\sigma_c^2$

Recall

$$\mathbb{E}(m) = \exp\left\{-\rho - \gamma\mu + \frac{1}{2}\gamma^2\sigma_c^2\right\}$$
$$\sigma(m) \approx \mathbb{E}(m)\gamma\sigma_c$$

The time separable CRRA specification causes two conceptually distinct consumer attitudes to find expression through γ.

(a) If $\gamma \uparrow$, the price of risk $\frac{\sigma(m)}{\mathbb{E}(m)}$ increase, it is because higher values of γ indicate more hatred of risk and a higher price of risk.

Here γ represents the consumer's distaste for **atemporal** gambles.

We can see that

$$\gamma = -\frac{Cu''}{u'}$$

is the relative risk aversion (RRA): It governs the willingness to **substitute across states**.

(b) $\mathbb{E}(m)$ is dominated by $\exp\{-\gamma\mu\}$, then $\mathbb{E}(m)\downarrow$ is decreasing as $\gamma\uparrow$.

Here γ expresses the representative consumer's distaste for deviations of consumption from a smooth path **across time**.

We also can see that

$$\frac{1}{\gamma} = -\frac{\mathrm{d}\ln\left(\frac{C_{t+1}}{C_t}\right)}{\mathrm{d}\ln\left(\frac{u'(C_{t+1})}{u'(C_t)}\right)} = \frac{\mathrm{d}\ln\left(\frac{C_{t+1}}{C_t}\right)}{\mathrm{d}\ln\left(\frac{u'(C_t)}{u'(C_{t+1})}\right)} = \frac{\mathrm{d}\ln\left(\frac{C_{t+1}}{C_t}\right)}{\mathrm{d}\ln\left[\frac{1}{\beta}(1+r_f)\right]} = \frac{\mathrm{d}\ln\left(\frac{C_{t+1}}{C_t}\right)}{\mathrm{d}\ln(1+r_f)}$$

is the elasticity of intertemporal substitution (EIS): It governs the willingness to **substitute across time**.

(c) γ also affects $\mathbb{E}(m)$ through the precautionary saving motive term $\exp\left(\frac{1}{2}\gamma^2\sigma_c^2\right)$, which comes from the consumer's dislike of risky consumption streams. Therefore, $\gamma\uparrow$, the dislike of risky should be compensated by a larger $\mathbb{E}(m)$, indicating $r_f\downarrow$.

8. Why CRRA is wrong?

   (a) We need a high γ to explain the high equity premium, but this leads to low EIS, which gives us a high r_f.

   (b) The assumption for consumption growth says consumption always grow at a positive rate, then everyone would like to borrow, driving up the interest rate.

   (c) Comparing to CRRA, H-J bound gives us a higher $\sigma(m)$. Recall that

$$m_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \Longrightarrow \ln m_{t+1} = \ln\beta - \gamma\ln\frac{C_{t+1}}{C_t}$$

   we have a low $\sigma\left(\ln\left(\frac{C_{t+1}}{C_t}\right)\right)$ in reality, thus, we need a high γ to attain the high $\sigma(m_{t+1})$.

4.3 Recursive Utility

1. Key idea. Let marginal utility $u'(C)$ depend on things besides current consumption, which allows $u'(C)$ to be much more volatile than C.

   (a) e.g., habit persistent (Campbell and Cochrane, 1999).

   (b) Recursive preferences: make MU depend on expected future consumption.

2. A general framework.

   Recursive preference is defined (by Kreps and Proteus 1978)

$$V_t = w\left(C_t, \xi\left(V_{t+1}\right)\right)$$

   where

$$\xi = f^{-1}\left(\mathbb{E}f\left(V_{t+1}\right)\right)$$

   (a) $\xi\left(V_{t+1}\right)$ is the certainty equivalent operator that translates random future utility unto consumption (or current utility) units,

$$f\left(\xi\right) = \mathbb{E}f\left(V_{t+1}\right)$$

   making it comparable to current consumption.

   It captures the risk aversion.

   (b) $W\left(\cdot,\cdot\right)$ is the time aggregator that combines current consumption and the certainty equivalent of future utility into a measure of current utility.

   It captures the intertemporal substitution.

   (c) Example. A familiar utility function.

$$U_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) = u\left(c_0\right) + \xi\left(U_1\right) = w\left(c_0 + \xi\left(U_1\right)\right)$$

   where $U_1 = u\left(c_1\right) + \beta u\left(c_2\right) + \cdots$, $\xi\left(U_1\right) = \beta\mathbb{E}_0\left(U_1\right)$, $w\left(x,y\right) = u\left(x\right) + y$.

3. Example.

   Epstein and Zin (1989) developed the widely used example

$$w\left(C_t, \xi\right) = \left[\left(1-\delta\right)C_t^{1-\eta} + \delta\xi^{1-\eta}\right]^{\frac{1}{1-\eta}}$$

$$\xi_t\left(V_{t+1}\right) = \left[\mathbb{E}_t\left(V_t^{1-\gamma}\right)\right]^{\frac{1}{1-\gamma}}$$

   If $\eta = 1, w\left(C,\xi\right) = C^{1-\delta}\xi^{\delta}$, where $\frac{1}{\eta}$ is the elasticity of intertemporal substitution, that is the elasticity of substitution between consumption today and the certainty equivalent ξ of continuation utility tomorrow.

   In this example, both time aggregator and certainty equivalent are CES functions.

4. SDF for the general recursive utility.

   Consider an asset that promises an uncertain payoff p_{t+1} units of consumption goods in period $t+1$. Let q_t denote the price of this asset in today's consumption good, and $V_t\left(\left\{C_{t+j}\right\}_{j=0}^{\infty}\right)$ denote the utility of a consumer.

   The F.O.C.s imply

$$\frac{\partial V_t}{\partial C_t} q_t = \sum_{s_{t+1}\in\mathcal{S}} \frac{\partial V_t}{\partial C_{t+1}\left(s_{t+1}\right)} p_{t+1}\left(s_{t+1}\right)$$

Note that the discount factor β and probability of each state do not explicitly show- up in the F.O.C.s, it depends on the utility function form, and may implicitly embedded in the term $\frac{\partial V_t}{\partial C_{t+1}(s_{t+1})}$.

Proof. We can see it by a simplified case. Consider the problem

$$\max_{C_t, N_{t-1}} V_t = u\left(C_t\right) + \beta \mathbb{E}_t V_{t+1} \text{s.t. } C_t + p_t N_t \leqslant q_t N_{t-1}$$

The F.O.C.s are

$$\begin{cases} \frac{\partial V_t}{\partial C_t} = \lambda_t \\ \lambda_t q_t = \beta \mathbb{E}_t\left[\lambda_{t+1} p_{t+1}\right] \end{cases} \Longrightarrow \frac{\partial V_t}{\partial C_t} q_t = \beta \mathbb{E}_t\left[\frac{\partial V_{t+1}}{\partial C_{t+1}} p_{t+1}\right]$$

Moreover,

$$\frac{\partial V_t}{\partial C_t} q_t = \beta \mathbb{E}_t\left[\frac{\partial V_{t+1}}{\partial C_{t+1}} p_{t+1}\right] = \beta \int \frac{\partial V_{t+1}}{\partial C_{t+1}\left(s_{t+1}\right)} p_{t+1}\left(s_{t+1}\right) f\left(\left.s_{t+1}\right| s_t\right) \mathrm{d} s_{t+1}$$

Note that

$$V_t = u\left(C_t\right) + \beta \mathbb{E}_t V_{t+1} = u\left(C_t\right) + \beta \int V_{t+1} f\left(\left.s_{t+1}\right| s_t\right) \mathrm{d} s_{t+1}$$

thus,

$$\frac{\partial V_t}{\partial C_{t+1}\left(s_{t+1}\right)} = \beta \frac{\partial V_{t+1}}{\partial C_{t+1}\left(s_{t+1}\right)} f\left(\left.s_{t+1}\right| s_t\right)$$

plug into the F.O.C.,

$$\frac{\partial V_t}{\partial C_t} q_t = \int \frac{\partial V_t}{\partial C_{t+1}\left(s_{t+1}\right)} p_{t+1}\left(s_{t+1}\right) \mathrm{d} s_{t+1} = \sum_{s_{t+1}} \frac{\partial V_t}{\partial C_{t+1}\left(s_{t+1}\right)} p_{t+1}\left(s_{t+1}\right)$$

□

Then

$$q_t = \sum_{s_{t+1} \in \mathcal{S}} \frac{\partial V_t / \partial C_{t+1}\left(s_{t+1}\right)}{\partial V_t / \partial C_t} p_{t+1}\left(s_{t+1}\right)$$

Compare to the Hansen-Jagannathan's definition of the SDF,

$$q_t = \mathbb{E}_t\left[m_{t+1} p_{t+1}\right] = \sum_{s_{t+1} \in \mathcal{S}} m\left(s_{t+1}\right) p\left(s_{t+1}\right) f\left(\left.s_{t+1}\right| s_t\right)$$

we get

$$m\left(s_{t+1}\right) = \frac{\partial V_t / \partial C_{t+1}\left(s_{t+1}\right)}{\partial V_t / \partial C_t} \frac{1}{f\left(\left.s_{t+1}\right| s_t\right)}$$

Generally, the SDF is normalized by $f\left(\left.s_{t+1}\right| s_t\right)$.

5. Example.

Tallarini (2000)'s utility specification is

$$V_t = C_t^{1-\beta}\left[\left(\mathbb{E}_t V_{1+t}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}\right]^\beta$$

Question: Does $(\mathbb{E}m, \sigma(m))$ implied by this utility function satisfy the Hansen-Jagannathan bound?

To compute the stochastic discount factor, we consider to rewrite the utility function.

Taking log,

$$\ln V_t = (1-\beta)\ln C_t + \frac{\beta}{1-\gamma}\ln \mathbb{E}_t V_{1+t}^{1-\gamma} = (1-\beta)\ln C_t + \frac{\beta}{1-\gamma}\ln \mathbb{E}_t\left[e^{(1-\gamma)\ln V_{1+t}}\right]$$

thus,

$$\frac{1}{1-\beta}\ln V_t = \ln C_t + \frac{\beta}{(1-\gamma)(1-\beta)}\ln \mathbb{E}_t\left[\exp\left\{(1-\gamma)\ln V_{1+t}\right\}\right]$$

Let $\theta := \frac{-1}{(1-\gamma)(1-\beta)}, U_t := \frac{\ln V_t}{1-\beta}$, we can rewrite the utility function as

$$U_t = \ln C_t - \beta\theta \ln \mathbb{E}_t\left[\exp\left\{-\frac{U_{t+1}}{\theta}\right\}\right]$$

then U_t is now linear in $\ln C_t$. Now we compute the SDF for U_t.

$$\begin{aligned} U_t &= \ln C_t - \beta\theta \ln\left[\int_{\mathcal{S}} \exp\left\{-\frac{1}{\theta}U_{t+1}(C_{t+1}, s_{t+1})\right\} f(s_{t+1}|\, s_t)\, \mathrm{d}s_{t+1}\right] \\ &= \ln C_t - \beta\theta \ln\left[\sum_{s_{t+1}\in\mathcal{S}} \exp\left\{-\frac{1}{\theta}U_{t+1}(C_{t+1}, s_{t+1})\right\} f(s_{t+1}|\, s_t)\right] \end{aligned}$$

and

$$\frac{\partial U_t}{\partial C_t} = \frac{1}{C_t}\frac{\partial U_t}{\partial C_{t+1}(s_{t+1})} = \frac{\partial U_t}{\partial U_{t+1}}\frac{\partial U_{t+1}}{\partial C_{t+1}(s_{t+1})} = \beta\frac{\exp\left\{-\frac{1}{\theta}U_{t+1}\right\} f(s_{t+1}|\, s_t)}{\mathbb{E}_t\left[\exp\left\{-\frac{1}{\theta}U_{t+1}\right\}\right]} \cdot \frac{1}{C_{t+1}(s_{t+1})}$$

Then the SDF is

$$m(s_{t+1}) = \frac{\partial U_t/\partial C_{t+1}(s_{t+1})}{\partial U_t/\partial C_t}\frac{1}{f(s_{t+1}|\, s_t)} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-1}\frac{\exp\left\{-\frac{1}{\theta}U_{t+1}\right\}}{\mathbb{E}_t\left[\exp\left\{-\frac{1}{\theta}U_{t+1}\right\}\right]}$$

Recall the SDF for CRRA,

$$m_{\text{CRRA}}(s_{t+1}) = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$$

the extra term for Tallarini (2000) is the red term.

The SDF for Tallarini (2000) consists of two components

(a) The standard term $\beta\left(\frac{C_{t+1}}{C_t}\right)^{-1}$ reflects the natural discount rate and the marginal utility ratio of current consumption.

(b) An additional "wedge" term, $g_{t+1} := \frac{\exp\left\{-\frac{1}{\theta}U_{t+1}\right\}}{\mathbb{E}_t\left[\exp\left\{-\frac{1}{\theta}U_{t+1}\right\}\right]}$, that comes from the recursive nature of the utility function.

To simplify the problem and solve the explicit form of U_t, assume that

$$\ln C_{t+1} = \mu + \ln C_t + \sigma_c \varepsilon_{t+1}, \varepsilon_{t+1} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$$

Now, to compute $\mathbb{E}m$ and $\sigma(m)$, we should go through several steps.

Step 1. we use guess-and-verify method solve for U_t. Guess

$$U_t = k_0 + k_1 \ln C_t$$

Plug it into

$$U_t = \ln C_t - \beta\theta \ln \mathbb{E}_t \left[\exp\left\{-\frac{U_{t+1}}{\theta}\right\}\right]$$

we get

$$\begin{aligned}
k_0 + k_1 \ln C_t &= \ln C_t - \beta\theta \ln \mathbb{E}_t \left[\exp\left\{-\frac{k_0 + k_1 \ln C_{t+1}}{\theta}\right\}\right] \\
&= \ln C_t - \beta\theta \ln \mathbb{E}_t \left[\exp\left\{-\frac{k_0}{\theta} - \frac{k_1}{\theta}\left(\mu + \ln C_t + \sigma_c \varepsilon_{t+1}\right)\right\}\right] \\
&= \ln C_t - \beta\theta \ln \left(\exp\left\{-\frac{k_0}{\theta} - \frac{k_1}{\theta}\left(\mu + \ln C_t\right)\right\} \mathbb{E}_t \left[\exp\left(-\frac{k_1 \sigma_c}{\theta}\varepsilon_{t+1}\right)\right]\right) \\
&= \ln C_t - \beta\theta \left(-\frac{k_0}{\theta} - \frac{k_1}{\theta}\left(\mu + \ln C_t\right) + \frac{1}{2}\frac{k_1^2 \sigma_c^2}{\theta^2}\right) \\
&= \left(1 + \beta k_1\right) \ln C_t + \beta k_0 + \beta k_1 \mu - \frac{1}{2}\frac{\beta k_1^2 \sigma_c^2}{\theta}
\end{aligned}$$

Comparing the coefficients, we get

$$k_1 = \frac{1}{1-\beta} k_0 = \frac{\beta}{1-\beta}\left[\frac{\mu}{1-\beta} - \frac{1}{2}\frac{\sigma_c^2}{\theta\left(1-\beta\right)^2}\right]$$

therefore,

$$U_t = \frac{\beta}{\left(1-\beta\right)^2}\left[\mu - \frac{\sigma_c^2}{2\theta\left(1-\beta\right)}\right] + \frac{1}{1-\beta}\ln C_t$$

Step 2. Solve g_{t+1}. Recall that

$$g_{t+1} = \frac{\exp\left\{-\frac{1}{\theta}U_{t+1}\right\}}{\mathbb{E}_t\left[\exp\left\{-\frac{1}{\theta}U_{t+1}\right\}\right]}$$

Consider

$$\begin{aligned}
-\frac{1}{\theta}U_{t+1} &= -\frac{k_0}{\theta} - \frac{k_1}{\theta}\ln C_{t+1} = -\frac{k_0}{\theta} - \frac{k_1}{\theta}\left[\mu + \ln C_t + \sigma_c \varepsilon_{t+1}\right] \\
&= -\frac{k_0 + k_1 \ln C_t}{\theta} - \frac{k_1}{\theta}\mu - \frac{\sigma_c}{\theta\left(1-\beta\right)}\varepsilon_{t+1} \\
&= -\frac{U_t + k_1 \mu}{\theta} - \frac{\sigma_c}{\theta\left(1-\beta\right)}\varepsilon_{t+1}
\end{aligned}$$

Let $\tilde{U}_t = -\frac{U_t + k_1\mu}{\theta}$then

$$\mathbb{E}_t\left[\exp\left\{-\frac{1}{\theta}U_{t+1}\right\}\right] = \exp\left\{\tilde{U}_t + \frac{\sigma_c^2}{2\theta^2\left(1-\beta\right)^2}\right\}$$

then

$$g_{t+1} = \exp\left\{\tilde{U}_t - \frac{\sigma_c}{\theta\left(1-\beta\right)}\varepsilon_{t+1} - \tilde{U}_t - \frac{\sigma_c^2}{2\theta^2\left(1-\beta\right)^2}\right\} = \exp\left\{\omega\varepsilon_{t+1} - \frac{\omega^2}{2}\right\}$$

where $\omega = \frac{-\sigma_c}{\theta(1-\beta)} = \sigma_c\left(1-\gamma\right)$.

Step 3. Compute m_t.

Recall that

$$m\left(s_{t+1}\right) = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-1} g_{t+1}$$

Take log,

$$\ln m_{t+1} = \ln\beta - \ln\frac{C_{t+1}}{C_t} + \omega\varepsilon_{t+1} - \frac{\omega^2}{2}$$

Since $\ln C_{t+1} - \ln C_t = \mu + \sigma_c\varepsilon_{t+1}$,

$$\begin{aligned}
\ln m_{t+1} &= \ln\beta - \mu + \left(\omega - \sigma_c\right)\varepsilon_{t+1} - \frac{\omega^2}{2} \\
&= \ln\beta - \mu - \frac{\sigma_c^2\left(1-\gamma\right)^2}{2} - \gamma\sigma_c\varepsilon_{t+1} \\
&\sim \mathcal{N}\left(\ln\beta - \mu - \frac{\sigma_c^2\left(1-\gamma\right)^2}{2}, \gamma^2\sigma_c^2\right)
\end{aligned}$$

Step 4. Compute $\mathbb{E}m$ and $\sigma\left(m\right)$.

$$\begin{aligned}
\mathbb{E}m &= \exp\left(\ln\beta - \mu - \frac{\sigma_c^2\left(1-\gamma\right)^2}{2} + \frac{\gamma^2\sigma_c^2}{2}\right) \\
&= \beta\exp\left(-\mu + \frac{2\gamma-1}{2}\sigma_c^2\right) \\
\sigma\left(m\right) &= \mathbb{E}\left[m\right]\cdot\sqrt{e^{\gamma^2\sigma_c^2}-1} \approx \mathbb{E}\left[m\right]\gamma\sigma_c
\end{aligned}$$

Note that σ_c^2 is very small, thus, $\mathbb{E}m$ is dominated by $\exp\left(-\mu\right)$. Therefore, when γ increases, $\mathbb{E}\left[m\right]$ nearly keeps the same.

Recall for CRRA, when $\mathbb{E}m$ varies, $\sigma\left(m\right)$ changes too little to attain the H-J bound.

$$\begin{aligned}
\mathbb{E}\left(m_{\text{CRRA}}\right) &= \beta\exp\left\{-\gamma\mu + \frac{1}{2}\gamma^2\sigma_c^2\right\} \\
\sigma\left(m_{\text{CRRA}}\right) &\approx \mathbb{E}\left(m\right)\gamma\sigma_c^2
\end{aligned}$$

$\mathbb{E}\left(m_{\text{CRRA}}\right)$ is dominated by $\exp\left(-\gamma\mu\right)$ instead. Therefore, $\mathbb{E}\left(m_{\text{CRRA}}\right)$ drops when γ gets larger.

4.4 Ambiguity Aversion

1. Intro.

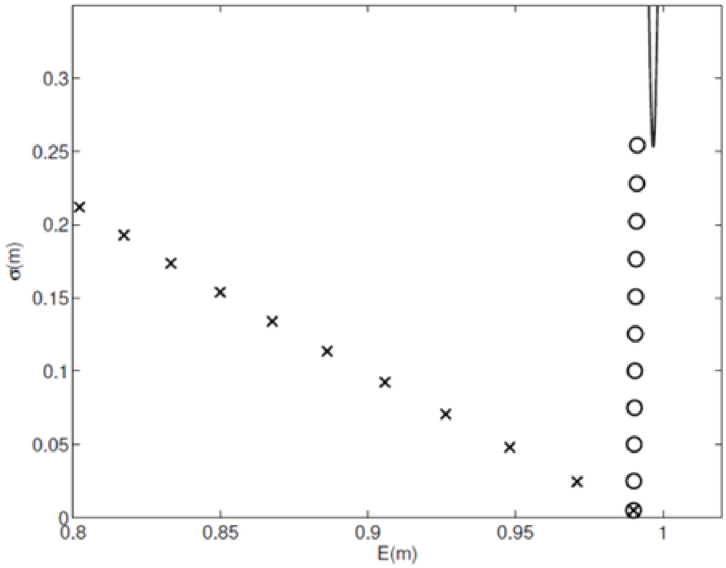

Fig. 4.2. Tallarini (2000) and H-J bound.

(a) The ambiguity aversion theory is a theory as an alternative explanation to the high manifested risk aversion.

(b) Hansen and Sargent's value function obeys the iteration

$$W_t = \ln C_t + \beta T_t \left(W_{t+1} \right)$$

where T_t is a certainty equivalent function that evaluates the uncertain W_{t+1} and translate it into units of certain time-t equivalent utils.

In particular, if $T_t = \mathbb{E}_t$, then the value of the uncertain W_{t+1} is equivalent to a certain value that is given by its expectation.

Now, we need to derive a specification for T that captures aversion of model uncertainty, whose meaning is yet to be illuminated.

2. Set-up.

(a) The agent has a value function $z(\varepsilon)$ which depends on realization of ε.

(b) The agent has a benchmark distribution of ε given by $\phi(\varepsilon)$.

(c) The agent dose not completely trust his own model. In other words, model may be wrong: for example, the actual distribution may be an unknown alternative $\tilde{\phi}(\varepsilon)$.

(d) We focus on one way to evaluate the value function— the agent will have an incentive to guard against the worst possibility.

3. Mathematical expression to realize the idea.

Let $T(z)$ be the worst outcome, and $g(\varepsilon) := \frac{\tilde{\phi}(\varepsilon)}{\phi(\varepsilon)}$ be the likelihood ratio that captures the distortion to the probability distribution, then

$$T(z) = \min_{g(\varepsilon)} \int z(\varepsilon) \tilde{\phi}(\varepsilon) \mathrm{d}\varepsilon = \min_{g(\varepsilon)} \int z(\varepsilon) \phi(\varepsilon) g(\varepsilon) \mathrm{d}\varepsilon \text{s.t.} \int \tilde{\phi}(\varepsilon) \mathrm{d}\varepsilon = 1$$

Note that the worst outcome is the agent's benchmark distribution $\phi(\varepsilon)$ is totally wrong, and is replaced with the alternative distribution $\tilde{\phi}(\varepsilon)$.

However, we want something between totally wrong and totally right. Therefore, we need to revise $T(z)$ to restrict attention in the neighborhood of the model.

Consider the K-L distance or relative entropy between $\tilde{\phi}(\varepsilon)$ and $\tilde{\phi}(\varepsilon)$, defined as

$$
\begin{aligned}
D\left(\phi, \tilde{\phi}\right) &:= \int \tilde{\phi}(\varepsilon) \ln \frac{\tilde{\phi}(\varepsilon)}{\phi(\varepsilon)} \mathrm{d}\varepsilon = \int \tilde{\phi}(\varepsilon) \ln g(\varepsilon) \, \mathrm{d}\varepsilon \\
&= \int \phi(\varepsilon) g(\varepsilon) \ln g(\varepsilon) \, \mathrm{d}\varepsilon
\end{aligned}
$$

By Jensen's inequality,[5] we can find that $D\left(\phi, \tilde{\phi}\right) \geqslant 0$. Moreover, $D\left(\phi, \tilde{\phi}\right) = 0$ iff $\phi = \tilde{\phi}$.

Also let $\mathrm{ent}(g) := D\left(\phi, \tilde{\phi}\right)$, which captures the distance or the "twist" of the benchmark model. With this definition, we may alter the $T(z)$ as

$$
\begin{aligned}
T(z) = \min_{g(\varepsilon)} \left\{ \int z(\varepsilon) \tilde{\phi}(\varepsilon) \, \mathrm{d}\varepsilon + \theta \mathrm{ent}(g) \right\} &= \min_{g(\varepsilon)} \int \left[z(\varepsilon) + \theta \ln g(\varepsilon) \right] \phi(\varepsilon) g(\varepsilon) \, \mathrm{d}\varepsilon \\
\text{s.t.} \quad \int \tilde{\phi}(\varepsilon) \, \mathrm{d}\varepsilon &= \int \phi(\varepsilon) g(\varepsilon) \, \mathrm{d}\varepsilon = 1
\end{aligned}
$$

where θ is the penalty parameter, controlling the deviation.

If $\theta = 0$, then there is no penalty and thus no trust in the bench mark, backing to the previous $T(z)$.

If $\theta \to \infty$, to minimize the object, $\mathrm{ent}(g) = 0$ must hold, then $\phi = \tilde{\phi}$, which means the agent completely trust the benchmark distribution.

Set up the Lagrangian,

$$
\mathcal{L} = \min_{g(\varepsilon)} \int \left[z(\varepsilon) + \theta \ln g(\varepsilon) \right] \phi(\varepsilon) g(\varepsilon) \, \mathrm{d}\varepsilon + \lambda \left[\int \phi(\varepsilon) g(\varepsilon) \, \mathrm{d}\varepsilon - 1 \right]
$$

The F.O.C. is

$$
\theta \phi(\varepsilon) + \left[z(\varepsilon) + \theta \ln g(\varepsilon) \right] \phi(\varepsilon) + \lambda \phi(\varepsilon) = 0
$$

or

$$
\theta + z(\varepsilon) + \theta \ln g(\varepsilon) + \lambda = 0
$$

then

$$
g(\varepsilon) = \exp\left\{ \frac{-\lambda - \theta - z(\varepsilon)}{\theta} \right\}
$$

The constraint gives us

$$
\begin{aligned}
1 = \int \phi(\varepsilon) g(\varepsilon) \, \mathrm{d}\varepsilon &= \int \phi(\varepsilon) \exp\left\{ \frac{1 - \lambda - \theta - z(\varepsilon)}{\theta} \right\} \mathrm{d}\varepsilon \\
&= \exp\left\{ -\frac{\lambda}{\theta} - 1 \right\} \int \phi(\varepsilon) \exp\left\{ -\frac{z(\varepsilon)}{\theta} \right\} \mathrm{d}\varepsilon
\end{aligned}
$$

[5] $D\left(\tilde{\phi}, \phi\right) = \int \tilde{\phi}(x) \ln \frac{\tilde{\phi}(x)}{\phi(x)} \mathrm{d}x = \mathbb{E}_{\tilde{\phi}}\left[\ln \frac{\tilde{\phi}(x)}{\phi(x)}\right] = -\mathbb{E}_{\tilde{\phi}}\left[\ln \frac{\phi(x)}{\tilde{\phi}(x)}\right] \geqslant -\ln \mathbb{E}_{\tilde{\phi}}\left[\frac{\phi(x)}{\tilde{\phi}(x)}\right] = -\ln \int \phi(x) \, \mathrm{d}x = 0.$

then

$$\exp\left\{-\frac{\lambda}{\theta}-1\right\}=\frac{1}{\mathbb{E}_{\phi}\left[\exp\left\{-\frac{z(\varepsilon)}{\theta}\right\}\right]}$$

Therefore,

$$g(\varepsilon)=\exp\left\{\frac{-\lambda-\theta-z(\varepsilon)}{\theta}\right\}=\frac{\exp\left\{-\frac{z(\varepsilon)}{\theta}\right\}}{\mathbb{E}_{\phi}\left[\exp\left\{-\frac{z(\varepsilon)}{\theta}\right\}\right]}$$

then

$$\begin{aligned}
T(z) &= \min_{g(\varepsilon)}\int\left[z(\varepsilon)+\theta\ln g(\varepsilon)\right]\phi(\varepsilon)\,g(\varepsilon)\,\mathrm{d}\varepsilon \\
&= \int\left[z(\varepsilon)+\theta\left(-\frac{z(\varepsilon)}{\theta}-\ln\mathbb{E}_{\phi}\left[\exp\left\{-\frac{z(\varepsilon)}{\theta}\right\}\right]\right)\right]\phi(\varepsilon)\,g(\varepsilon)\,\mathrm{d}\varepsilon \\
&= -\theta\ln\mathbb{E}_{\phi}\left[\exp\left\{-\frac{z(\varepsilon)}{\theta}\right\}\right]\int\tilde{\phi}(\varepsilon)\,\mathrm{d}\varepsilon \\
&= -\theta\ln\mathbb{E}_{\phi}\left[\exp\left\{-\frac{z(\varepsilon)}{\theta}\right\}\right]
\end{aligned}$$

Therefore, the value function will be

$$W_t=\ln C_t+\beta T_t\left(W_{t+1}\right)\;=\ln C_t-\beta\theta\ln\mathbb{E}_{\phi}\left[\exp\left\{-\frac{W_{t+1}}{\theta}\right\}\right]$$

It takes the form $f^{-1}\left(\mathbb{E}f\left(z\right)\right)$ as the recursive term.

Recall the Tallarini (2000)'s utility function,

$$U_t=\ln C_t-\beta\theta\ln\mathbb{E}_t\left[\exp\left\{-\frac{U_{t+1}}{\theta}\right\}\right]$$

Consequently, they are identical! Therefore, the SDF m_{t+1} and $\left(\mathbb{E}m,\sigma\left(m\right)\right)$ will also be identical.

4. Two ways to meet the H-J bound.

   (a) Tallarini: increase γ.

   (b) Hansen and Sargent: Decrease θ.

   (c) As a result, a moderate amount of model uncertainty will take us to the H-J bound.

5. Risk aversion and ambiguity aversion (Jargon).

   (a) Risk: unknown draw from a certain distribution.

   (b) Ambiguity: uncertainty of the distribution.

4.5 Cost of Business Cycle

1. This section studies why γ is important, main results are from Lucas.

   Recall Tallarini (2000)'s utility function as

   $$U_t = \ln C_t - \beta\theta \ln \mathbb{E}_t \exp\left\{-\frac{U_{t+1}}{\theta}\right\}$$

   Under the assumption,

   $$\ln C_{t+1} = \mu + \ln C_t + \sigma_c \varepsilon_{t+1}, \varepsilon_{t+1} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$$

   we have

   $$\mathbb{E}_t C_{t+1} = C_t \exp\left\{\mu + \frac{1}{2}\sigma_c^2\right\}$$

   We ask: if consumption instead follows the deterministic process

   $$C_{t+1}^d = C_t^d \exp\left\{\mu + \frac{\sigma_c^2}{2}\right\}$$

   which share the same trend with the stochastic consumption process but contains no uncertainty, how much would agent benefit from the elimination of consumption uncertainty (volatility of business cycle)?

   Instead, Lucas asks: what percentage of initial (and therefore permanent) consumption the agent is willing to give up to eliminate business cycle uncertainty?

2. Think up a way to fit data.

   By the assumption,

   $$\ln C_{t+1} = \mu + \ln C_t + \sigma_c \varepsilon_{t+1}$$

   we have deduced the utility for the consumption with uncertainty before,

   $$U_t = \frac{\beta}{(1-\beta)^2}\left[\mu - \frac{\sigma_c^2}{2\theta(1-\beta)}\right] + \frac{1}{1-\beta}\ln C_t$$

   Note that $\theta = \frac{-1}{(1-\gamma)(1-\beta)}$, then

   $$U_t = \frac{\beta}{(1-\beta)^2}\left[\mu + \frac{(1-\gamma)\sigma_c^2}{2}\right] + \frac{1}{1-\beta}\ln C_t$$

   Under the deterministic process, the utility becomes

   $$U_t^d = \ln C_t^d - \beta\theta \ln \mathbb{E}_t \exp\left\{-\frac{U_{t+1}^d}{\theta}\right\} = \ln C_t^d + \beta U_{t+1}^d$$

   Guess $U_t^d = k_0 + k_1 \ln C_t^d$, and then

   $$\begin{aligned} k_0 + k_1 \ln C_t^d &= \ln C_t^d + \beta\left[k_0 + k_1 \ln C_{t+1}^d\right] \\ &= \ln C_t^d + \beta\left[k_0 + k_1\left(\mu + \frac{\sigma_c^2}{2} + \ln C_{t+1}^d\right)\right] \\ &= (1+\beta k_1)\ln C_t^d + \beta\left[k_0 + k_1\left(\mu + \frac{\sigma_c^2}{2}\right)\right] \end{aligned}$$

thus,

$$k_1 = \frac{1}{1-\beta} k_0 = \frac{\beta}{(1-\beta)^2}\left(\mu + \frac{\sigma_c^2}{2}\right)$$

Combine the utilities with the stochastic and the deterministic consumption,

$$\begin{cases} U_t = \frac{\beta}{(1-\beta)^2}\left[\mu + \frac{(1-\gamma)\sigma_c^2}{2}\right] + \frac{1}{1-\beta}\ln C_t \\ U_t^d = \frac{\beta}{(1-\beta)^2}\left(\mu + \frac{\sigma_c^2}{2}\right) + \frac{1}{1-\beta}\ln C_t^d \end{cases}$$

and equate $U_t = U_t^d$,

$$\ln \frac{C_t}{C_t^d} = \frac{\beta}{1-\beta}\left[-\mu - \frac{(1-\gamma)\,\sigma_c^2}{2\theta} + \mu + \frac{\sigma_c^2}{2}\right] = \frac{\beta}{1-\beta}\frac{\gamma\sigma_c^2}{2}$$

which is the percentage permanent consumption value of business cycle uncertainty. All we need to do now is to estimate $\frac{\beta}{1-\beta}\frac{\gamma\sigma_c^2}{2}$ by data.

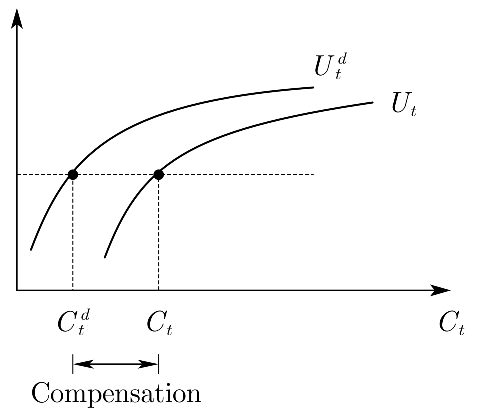

Fig. 4.3. Cost of business cycle.

3. Answer the question by data.

   (a) Take $\beta = 0.99, \sigma_c = 0.005, \gamma = 3$, we solve $\ln \frac{C_t}{C_t^d} \approx 0.3713\%$.

   (b) Lucas did it with the CRRA utility function, in which $\ln \frac{C_t}{C_t^d} \approx 0.13\%$.

   And Lucas argues that we should focus on how to boost economic growth instead of avoid business cycles, since if the growth rate is 0.13% or more, people don't care about the volatility anymore.

   (c) If change γ to 50, the welfare cost becomes $\ln \frac{C_t}{C_t^d} \approx 6.19\%$.

   (d) Therefore, γ matters for the importance of business cycle and stabilization policies.

4.6 Consumption Heterogeneity

1. This section discusses the consumption heterogeneity under some assumptions, justifying the reasonability of high γ.

2. Under recursive utility and ambiguity aversion,

$$m_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)} g(\varepsilon_{t+1})$$

where $g(\varepsilon_{t+1})$ accounts for volatility that $\{C_t\}$ cannot generate.

Constantinides and Duffie (1996) instead relied on CRRA utility, but assumes that **households are heterogenous**.

Let $R_{j,t+1}, j = 1, \cdots, J$ be a list of return rates on assets, for example, the return rate of a stock with prices $\{p_t\}$ and dividends $\{d_t\}$, $R_{j,t+1} = \frac{d_{j,t+1}+p_{j,t+1}}{p_{j,t}}$.

For individual household $i, i \in I$, the F.O.C. is

$$\mathbb{E}_t \left[m^i_{t+1} R_{j,t+1} \right] = 1, j = 1, 2, \cdots, J$$

where $m^i_{t+1} = \beta \left(\frac{C^i_{t+1}}{C^i_t} \right)^{-\gamma}$.

Let C_t denote the aggregate consumption. Assume that individual consumption is given by

$$C^i_t = \delta^i_t C_t, \forall i \in I$$

where $\{\delta^i_t\}$ is the consumption share process, given by

$$\delta^i_{t+1} = \delta^i_t \exp \left\{ \eta^i_{t+1} y_{t+1} - \frac{y^2_{t+1}}{2} \right\}$$

where $\eta^i_{t+1} \sim \mathcal{N}(0, 1)$ are i.i.d. across households, y_{t+1} governs the volatility of δ^i_{t+1}. Note that $\eta^i_{t+1}, \delta^i_{t+1}$ are both time-varying, and $\mathbb{E}\left[\delta^i_{t+1} \middle| \delta^i_t, y_{t+1} \right] = \delta^i_t$, so that $\{\delta^i_t\}$ given y_t is a geometric random walk process.

Further assume that

$$y^2_{t+1} = a + b \ln \left(\frac{C_{t+1}}{C_t} \right), b < 0$$

Note that $b < 0$ ensures that when consumption growth rate decreases, y^2_{t+1} will increase (It's kind of arbitrary, but helps conclude).

For individual household i, the consumption growth is

$$\frac{C^i_{t+1}}{C^i_t} = \frac{\delta^i_{t+1}}{\delta^i_t} \frac{C_{t+1}}{C_t} = \exp \left\{ \eta^i_{t+1} y_{t+1} - \frac{y^2_{t+1}}{2} \right\} \frac{C_{t+1}}{C_t}$$

By $\mathbb{E}_t \left[m^i_{t+1} R_{j,t+1} \right] = 1, \forall i \in I$,

$$\mathbb{E}_t \left[\beta \exp \left\{ -\gamma \eta^i_{t+1} y_{t+1} + \gamma \frac{y^2_{t+1}}{2} \right\} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{j,t+1} \right] = 1, \forall i \in I$$

by the Law of iterated expectations, we can aggregate across household i as

$$\mathbb{E}_t \left[\beta \mathbb{E} \left[\exp \left\{ -\gamma \eta^i_{t+1} y_{t+1} + \gamma \frac{y^2_{t+1}}{2} \right\} \middle| y_{t+1} \right] \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{j,t+1} \right] = 1$$

thus,

$$\mathbb{E}_t\left[\beta\exp\left\{\gamma\frac{y_{t+1}^2}{2}+\frac{1}{2}\gamma^2y_{t+1}^2\right\}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}R_{j,t+1}\right]=1$$

or

$$\mathbb{E}_t\left[\beta\exp\left\{\frac{y_{t+1}^2\left(\gamma+1\right)\gamma}{2}\right\}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}R_{j,t+1}\right]=1$$

Plug our assumption $y_{t+1}^2=a+b\ln\left(\frac{C_{t+1}}{C_t}\right)$ into it,

$$\mathbb{E}_t\left[\beta\exp\left\{\frac{a\gamma\left(\gamma+1\right)}{2}+\frac{b\gamma\left(\gamma+1\right)}{2}\ln\left(\frac{C_{t+1}}{C_t}\right)\right\}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}R_{j,t+1}\right]=1$$

or

$$\mathbb{E}_t\left[\beta\exp\left\{\frac{a\gamma\left(\gamma+1\right)}{2}\right\}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma+\frac{b\gamma(\gamma+1)}{2}}R_{j,t+1}\right]=1$$

Let

$$\hat{\beta}=\beta\exp\left\{\frac{a\gamma\left(\gamma+1\right)}{2}\right\}\hat{\gamma}=\gamma-\frac{b\gamma\left(\gamma+1\right)}{2}>\gamma$$

then

$$\mathbb{E}_t\left[\hat{\beta}\left(\frac{C_{t+1}}{C_t}\right)^{-\hat{\gamma}}R_{j,t+1}\right]=1$$

Once we fit the equation with data in reality, we get $\hat{\gamma}$, which is apparently greater than the "theoretical" γ implied by the model. So, in this case, estimates of γ constructed in usual ways from aggregate consumption data are distorted upwards.

Lecture 5

Incomplete Markets Models (RMT Ch18)

1. Introduction

   (a) Many interesting questions can only be answered with heterogenous agents.

      i. Income and wealth distributions

      ii. Inequality and growth

   (b) When play with heterogeneity

      i. We need to track **distribution** of agents. Usually, there are no closed-form solutions. Therefore, it demands computation (thanks to the availability of micro-level data and improvement of computing power).

      ii. We need

         - idiosyncratic risk
         - incomplete market

2. Overview.

   (a) No aggregate uncertainty: GE heterogenous household models with different assumptions on asset supply.

      i. Aiyagari model: growth model with incomplete market, **capital in positive net supply**

      ii. Huggett model: **private IOUs in zero net supply**

      iii. Bewley models: **money or bonds in positive net supply**

   (b) With aggregate uncertainty: Krusell-Smith model.

5.1 Precautionary Savings

1. A review of complete market.

Suppose there is one unit of households. Individual households with idiosyncratic income uncertainty $\{y_{h,t}\}$.

Moreover, there is a complete set of state-contingent bonds whose prices are given by $\{q(s_{t+1}|\, s_t)\}_{s_{t+1}\in\mathcal{S}}$.

F.O.C. implies

$$q(s_{t+1}|\, s_t) = \beta\frac{u'(c_{h,t+1}(s_{t+1}))}{u'(c_{h,t})}f(s_{t+1}|\, s_t)$$

where the subscript h of $c_{h,t+1}$ denotes the index of household, showing the heterogeneity.

Rewrite the F.O.C. as

$$\frac{q(s_{t+1}|\, s_t)}{\beta f(s_{t+1}|\, s_t)} = \frac{u'(c_{h,t+1}(s_{t+1}))}{u'(c_{h,t})}, \forall s_{t+1}$$

The LHS doesn't depend on the index h, i.e., the LHS is identical for all households.

Therefore, for example, for $h = 1, 2$,

$$\frac{u'(c_{1,t+1})}{u'(c_{1,t})} = \frac{u'(c_{2,t+1})}{u'(c_{2,t})}$$

or

$$\frac{u'(c_{1,0})}{u'(c_{2,0})} = \frac{u'(c_{1,1})}{u'(c_{2,1})} = \cdots = \frac{u'(c_{1,t})}{u'(c_{2,t})}$$

If initially, $u'(c_{1,0}) > u'(c_{2,0})$, then the agent 2 consumes more than agent 1 forever, in other words,

$$u'(c_{1,t}) > u'(c_{2,t}), \forall t \Longleftrightarrow c_{1,t} < c_{2,t}, \forall t$$

If we further assume that households start equal with identical c_0, then all household will have equal consumption in very state. Therefore, idiosyncratic income shock cease to matter, and only aggregate shock matters. Thus, we only need to care about aggregate shocks if complete market is present.

Moreover, the individual consumer's state-s consumption is determined by average output only:

$$c(s_{t+1}) = \int_0^1 y_{h,t+1}\mathrm{d}h := \bar{y}_{t+1}$$

Asset prices reflect the aggregate output as well:

$$q(s_{t+1}|\, s_t) = \frac{u'(\bar{y}_{t+1})}{u'(\bar{y}_t)}\beta f(s_{t+1}|\, s_t)$$

If without aggregate output uncertainty, every household is fully insured and enjoys constant consumption over time despite income uncertainty at individual level.

2. Household' s problem in partial equilibrium. To specify how the market is incomplete, we make following assumptions.

   (a) Assume the household has an unpredictable tree with i.i.d. endowment process $\{y_t\}$.

(b) Exogenous interest rate $R = \frac{1}{\beta}$.

(c) A fundamental problem with asset exchanges: the borrower may not pay back (contract default).

(d) There is a limit how much the household can borrow. (In other words, we impose a borrowing constraint in the model. [1])

The individual's problem is

$$\begin{aligned} \max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } & R^{-1} a_{t+1} + c_t \leqslant a_t + y_t \\ & a_{t+1} \geqslant 0 \end{aligned}$$

where we impose the (ad-hoc) no-borrow constraint $a_{t+1} \geqslant 0$, a_{t+1} denotes the units of bond the units of bond the individual holds into period $t+1$.

$$\mathcal{L} = \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \lambda_t \left[a_t + y_t - R^{-1} a_{t+1} - c_t \right] - \mu_t a_{t+1} \right\}$$

Note that $R^{-1} a_{t+1} + c_t \leqslant a_t + y_t$ is always binding, then

$$\begin{aligned} u'(c_t) &= \lambda_t \\ \lambda_t R^{-1} - \mu_t &= \mathbb{E}_t \beta \lambda_{t+1} \\ \mu_t a_{t+1} &= 0 \\ \mu_t &\geqslant 0 \end{aligned}$$

If $a_{t+1} > 0$, then $\mu_t = 0$, note that $\beta R = 1$,

$$u'(c_t) = \mathbb{E}_t u'(c_{t+1})$$

If $a_{t+1} = 0$, then $\mu_t \geqslant 0$ holds trivially,

$$\beta \mathbb{E}_t u'(c_{t+1}) \leqslant \lambda_t R = R u'(c_t) \iff u'(c_t) \geqslant \mathbb{E}_t u'(c_{t+1})$$

Therefore,

$$\begin{aligned} u'(c_t) &= \lambda_t \\ \lambda_t R^{-1} - \mu_t &= \mathbb{E}_t \beta \lambda_{t+1} \\ \mu_t a_{t+1} &= 0 \\ \mu_t &\geqslant 0 \end{aligned}$$

When borrowing constraint is binding, then the household is willing to borrow more but cannot. If the household can borrow more regardless of the constraint, then $c_t \uparrow$, $u'(c_t) \downarrow$to meet $u'(c_t) = \mathbb{E}_t u'(c_{t+1})$.

[1] It can be exogenous or endogenous.

We can solve it by dynamic programming as well. Consider

$$\begin{aligned} v(a_t) = & \max_{\{c_t\}_{t=0}^{\infty}} u(c_t) + \beta \mathbb{E}_t v(a_{t+1}) \\ & \text{s.t. } R^{-1} a_{t+1} + c_t = a_t + y_t \\ & a_{t+1} \geqslant 0 \end{aligned}$$

or

$$v(a_t) = \max_{\{c_t\}_{t=0}^{\infty}} u\left(a_t + y_t - R^{-1} a_{t+1}\right) + \beta \mathbb{E}_t v(a_{t+1}) - \mu_t a_{t+1}$$

F.O.C.s are

$$\begin{aligned} -R^{-1} u'(c_t) + \beta \mathbb{E}_t v'(a_{t+1}) - \mu_t &= 0 \\ \mu_t a_{t+1} &= 0 \\ \mu_t &\geqslant 0 \end{aligned}$$

by Envelop theorem,

$$v'(a_t) = u'(c_t)$$

then

$$\begin{aligned} \beta \mathbb{E}_t v'(c_{t+1}) - \mu_t &= R^{-1} u'(c_t) \\ \mu_t a_{t+1} &= 0 \\ \mu_t &\geqslant 0 \end{aligned}$$

we can as well have

$$\begin{aligned} & u'(c_t) \geqslant \mathbb{E}_t u'(c_{t+1}) \\ & u'(c_t) = \mathbb{E}_t u'(c_{t+1}) \text{ if } a_{t+1} > 0 \end{aligned}$$

Note that $u' > 0$, the inequality $u'(c_t) \geqslant \mathbb{E}_t u'(c_{t+1})$ implies that $\{u'(c_t)\}$ is a nonnegative sup-martingale.

By the Doob's martingale convergence theorem, there exists a random variable c, such that $\lim_{t\to\infty} u'(c_t) = u'(c)$, a.s.

Claim that the value of $u'(c) = 0$, then $c_t \to \infty$, and $a_t \to \infty$.

Intuitive Proof. Suppose the marginal utility $u'(c_t)$ converges to $u'(c)$, but $u'(c) = 0$a.s. does not hold, then the consumption c can only take finite positive values.

However, no matter how much asset the household accumulated, it will never choose a constant consumption path.

Suppose the contrary that $\exists T, \forall t \geqslant T, c_t = c$. Let a_{T+1} be the asset position at the end of time T, then the only feasible constant consumption is $(R-1) a_{T+1} + y_{\min}$, where $y_{\min}$ is the minimum realization of income y.

However, if $y > y_{\min}$, the household will find it optimal to deviate, otherwise, its asset is going to accumulate when output goes above $y_{\min}$, contradicting to the envelop theorem $u'(c) = v'(a)$.

The intuition of the result depends on the shape of utility function. The result requires the mild assumption that the marginal utility is convex.

The optimal consumption path is captured by

$$u'(c_t) \geqslant \mathbb{E}_t u'(c_{t+1})$$

If $u''' > 0$, by Jensen's inequality,

$$u'(c_t) \geqslant \mathbb{E}_t u'(c_{t+1}) > u'(\mathbb{E}_t c_{t+1})$$

therefore,

$$c_t < \mathbb{E}_t c_{t+1}$$

which implies $\{c_t\}$ is a sub-martingale.

Intuitively, the household wants to make her cross-states average marginal utility in period $t+1$ to equal to the marginal utility in period t, i.e., to attain $u'(c_t) = \mathbb{E}_t u'(c_{t+1})$. Precautionary saving is the excess saving due to uncertainty, sometimes households need such precautionary saving to maximize their utility.

5.2 Aiyagari and Huggett

1. Aiyagari's set-up.

   (a) Stochastic growth model with incomplete market.

   (b) Intrinsically homogenous agents are subject to idiosyncratic shocks. They become heterogenous ex-post due to market incompleteness.

   (c) Firms in the model have the CRS production function $Y_t = F(K_t, N_t)$.

   Firm's problem is

   $$\max_{K_t, N_t} F(K_t, N_t) - \tilde{r}_t K_t - w_t N_t$$

   The production factor markets are competitive, so the firms take the prices $\tilde{r}_t, w_t$ as given.

   (d) A household's objective is to solve

   $$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \text{s.t. } c_t + k_{t+1} - (1-\delta) k_t \leqslant \tilde{r}_t k_t + w_t s_t$$

   where s_t is a random variable capturing labor income risks (e.g., the employment rates). Assume $\{s_t\}$ is governed by a Markov chain with transition matrix P, note that s_t is idiosyncratic across households.

   Note that we assume inelastic labor supply here, so $u(c_t, n_t)$ degenerates to $u(c_t)$.

(e) There is no aggregate shock. $\{s_t\}$ is all uncertainty in this model.

(f) Households save via owing capital. Define the asset position $a_t := k_t$ and **the net interest rate** $r_t := \tilde{r}_t - \delta$. The budget constraint can be written as

$$c_t + a_{t+1} \leqslant (1 + r_t)\, a_t + w_t s_t$$

We assume the borrowing constraint

$$a_{t+1} \geqslant -\phi$$

where ϕ captures the looseness of the borrowing constraint, assume that

$$\phi \in \left[0, \frac{ws_1}{r}\right]$$

where s_1 is the minimum value that s can take, then $\frac{ws_1}{r}$ is the minimum labor income ws_1 normalized by the steady state interest rate r.

Such constraint is also known as the natural borrowing constraint.

2. Solution to Aiyagari's model. Assume that $a_t \in \mathcal{A}$, where $\mathcal{A}$ is a discrete set of values.

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } c_t + a_{t+1} \leqslant (1 + r_t)\, a_t + w_t s_t$$

The Bellman equation is

$$v(a, s) = \max_{a' \in \mathcal{A}} u(ws + (1 + r)\, a - a') + \beta \sum_{s' \in \mathcal{S}} v(a', s')\, \mathbb{P}\{s' | s\}$$

We cannot directly solve the stochastic model. We can only calibrate the probability of a household or proportion of households with certain (a_t, s_t). There are only two heterogeneity sources in our model.

(a) A household comes into period t with wealth a_t.

(b) The household then receives an exogenous labor income shock s_t.

Therefore, the pair (a_t, s_t) summarized the environment of the household, and thus, we can regard each pair is a stamp for a household. All subsequent decisions about consumption and saving are based on (a_t, s_t).

Define the distribution of (a_t, s_t) pairs (or households):

$$\lambda_t(a, s) = \mathbb{P}\{a_t = a, s_t = s\}$$

Let the optimal policy function be g, thus $a_{t+1} = g(a_t, s_t)$, it is the choice of capital or asset held by the households.

$$\begin{aligned}
\lambda_{t+1}\left(a', s'\right) &= \mathbb{P}\left\{a_{t+1} = a', s_{t+1} = s'\right\} \\
&= \sum_{a\in\mathcal{A}}\sum_{s\in\mathcal{S}} \mathbb{P}\left\{\left. a_{t+1} = a', s_{t+1} = s' \right| a_t = a, s_t = s\right\} \cdot \mathbb{P}\left\{a_t = a, s_t = s\right\} \\
&= \sum_{a\in\mathcal{A}}\sum_{s\in\mathcal{S}} \mathbb{P}\left\{\left. a_{t+1} = a', s_{t+1} = s' \right| a_t = a, s_t = s\right\} \cdot \lambda_t\left(a, s\right) \\
&= \sum_{a\in\mathcal{A}}\sum_{s\in\mathcal{A}} \mathbb{P}\left\{\left. a_{t+1} = a' \right| a_t = a, s_t = s\right\} \cdot \mathbb{P}\left\{\left. s_{t+1} = s' \right| s_t = s\right\} \cdot \lambda_t\left(a, s\right) \\
&= \sum_{a\in\mathcal{A}}\sum_{s\in\mathcal{S}} \mathbb{I}_{\{a'=g(a,s)\}} \cdot \mathbb{P}\left\{\left. s_{t+1} = s' \right| s_t = s\right\} \cdot \lambda_t\left(a, s\right) \\
&= \sum_{s\in\mathcal{S}}\sum_{a\in\{a:a'=g(a,s)\}} \mathbb{P}\left\{\left. s_{t+1} = s \right| s_t = s\right\} \cdot \lambda_t\left(a, s\right)
\end{aligned}$$

or we can write

$$\lambda_{t+1}\left(a', s'\right) = \sum_{a\in\mathcal{A}}\sum_{s\in\mathcal{S}} I\left(a', s, a\right) \cdot P\left(\left. s' \right| s\right) \cdot \lambda_t\left(a, s\right)$$

where

$$I\left(a', s, a\right) = \begin{cases} 1, & \text{if } a' = g\left(s, a\right) \\ 0, & \text{o.w.} \end{cases}, P\left(\left. s' \right| s\right) = \mathbb{P}\left\{\left. s' = s \right| s_t = s\right\}$$

A stationary distribution λ of $\{\lambda_t\}$ is a time invariant distribution that solves

$$\lambda_{t+1}\left(a', s'\right) = \sum_{s\in\mathcal{S}}\sum_{a\in\{a:a'=g(a,s)\}} P\left(\left. s' \right| s\right) \cdot \lambda_t\left(a, s\right)$$

3. Stationary equilibrium for Aiyagari.[2]

We previously assumed the labor supply is inelastic, then the labor market always clears. The aggregate labor supply in the limit is

$$N = \xi_\infty^{\mathrm{T}} \bar{s}$$

where ξ_∞ is the invariant distribution associated with P, and $\bar{s}$ is the exogenous specified vector of individual employment rates. We can regard N as an expectation or weighted average of households' labor supply. A **stationary equilibrium** of the model is defined as: A policy function $g\left(a, s\right)$, a probability distribution $\lambda\left(a, s\right)$, allocation K and price system $\left(r, w\right)$, such that

(a) (Firm's optimality) $w = F_N\left(K, N\right)$, $r = F_K\left(K, N\right) - \delta$.

(b) (Household's optimality) Given prices w, r, $g\left(a, s\right)$ solves household's optimization problem.

(c) (Stationary) $\lambda\left(a, s\right)$ is a stationary distribution of $\{\lambda_t\left(a, s\right)\}$, solving

$$\lambda_{t+1}\left(a', s'\right) = \sum_{s\in\mathcal{S}}\sum_{a\in\{a:a'=g(a,s)\}} P\left(\left. s' \right| s\right) \cdot \lambda_t\left(a, s\right)$$

[2]We study the long-run equilibrium of the model by examine its stationary equilibrium.

(d) (Feasibility) Asset market clears:

$$K = \sum_{a \in \mathcal{A}} \sum_{s \in \mathcal{S}} \lambda(a, s) g(a, s)$$

where the RHS is the total household savings, and the LHS is the total demand of capital by firms.

4. Remarks on Stationary equilibrium.

(a) By Walras' Law, consumption good market clears, and thus aggregate consumption is given by

$$C := \sum_{a \in \mathcal{A}} \sum_{s \in \mathcal{S}} \lambda(a, s) c(a, s)$$

where $c(a, s) = (1 + r) a - a' + ws = (1 + r) a - g(a, s) + ws$.

(b) In a stationary equilibrium, aggregate variables (w, r, N, K) are constant and same as the steady state.

(c) However, individual-level variables are moving "up and down" in the stationary distributions as their luck change.

(d) Assets are in positive net supply, as capital comes from positive asset creation.

5. Compute the Aiyagari's equilibrium.

Note that $N = \xi_\infty^{\mathrm{T}} \bar{s}$ is given exogenously.

(a) Pick K, compute $(w, r) = (F_N(N, K), F_K(N, K))$.

(b) Given r, solve household's problem for policy function $g(a, s)$ and compute the stationary distribution $\lambda(a, s)$.

(c) Check if market clearing condition is satisfies. Let capital supply be

$$A := \mathbb{E}[a](r) = \sum_{a \in \mathcal{A}} \sum_{s \in \mathcal{S}} \lambda(a, s) g(a, s)$$

If $A \neq K$, then pick $\tilde{K} = \frac{K+A}{2}$.

6. Analysis by graph. We now aggregate capital as a function of r, and use the classic demand-supply graph analyze the differences between the incomplete and complete market.

(a) The demand side is from firm's optimal capital input, i.e., the capital demand K solves

$$F_K(K, N) = \tilde{r} = r + \delta$$

(b) The supply side is derived from household's saving problem. The capital supply A is

$$A = \mathbb{E}[a](r) = \sum_{a \in \mathcal{A}} \sum_{s \in \mathcal{S}} \lambda(a, s) g(a, s)$$

Moreover, when $\beta(1+r)=1$, household's consumption and saving both diverge to ∞.

However, we should get $A<\infty$, which means the equilibrium interest rate has an upper bond $\rho := \frac{1}{\beta}-1$.

Also note that when the market is complete, the Euler equation

$$u'(c_t)=\beta(1+r)\,u'(c_{t+1})$$

implies that $\beta(1+r)=1$ in the steady state. (ρ is such an interest rate)

(c) As r drops, total saving A decreases, but average asset is bounded by $-\phi$.

Now, assume $\phi=\min\left\{b, \frac{ws_1}{r}\right\}$, where b represents the bound in the ad hoc borrowing constraint.

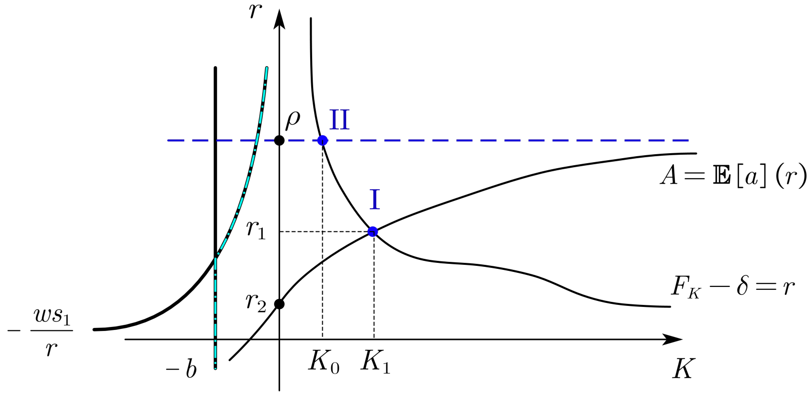

Fig. 5.1. Aiyagari.

The green dashed line is the borrowing constraint. Equilibrium I with (K_1, r_1) is for the Aiyagari's incomplete market model.

Equilibrium II with (K_0,ρ) is for the complete market. From Equilibrium II to Equilibrium I, households save more with lower utility but higher capital due to precautionary savings, and thus, lower interest rate.

7. Effect of borrowing constraint.

Recall the original constraints

$$c_t+a_{t+1}=(1+r)\,a_t+w_ts_ta_{t+1}\geqslant -\phi$$

Define $\hat{a}_t=a_t+\phi$, then

$$c_t+\hat{a}_{t+1}=(1+r)\,\hat{a}_t+w_ts_t+r\phi\hat{a}_{t+1}\geqslant 0$$

If $r=0$, then the rewritten budget constraints are identical with original constraints, $\hat{a}_{t+1}$ is as good as a_t. Therefore, the change of ϕ can be seen by the change of the crossing point of K-axis and $A=\mathbb{E}[a](r)$.

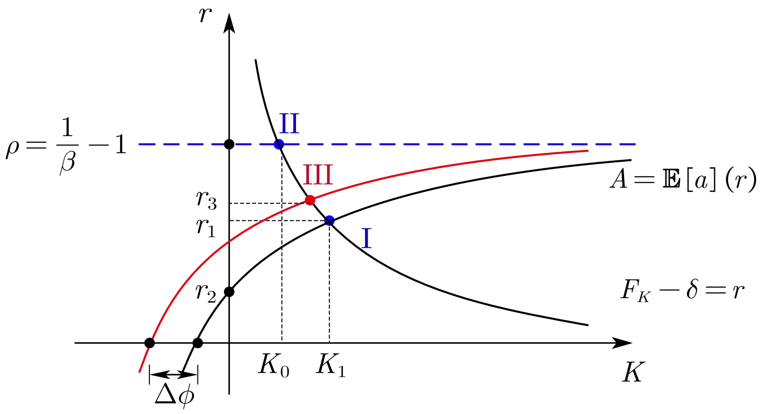

Fig. 5.2. Effect of borrowing constraint.

If we relax the borrowing constraint, i.e., let $\phi \uparrow$, the consumers can borrow more, which reduces the supply of capital.

Equilibrium II is our first best point, and if $\phi \uparrow$, the new equilibrium III with $r_3 > r_1$ is closer to the first best equilibrium.

8. Set-up modifications (Huggett's model):

   (a) The pure endowment economy with no production sector.

   (b) Household's endowment is governed by the Markov chain $(P, \bar{s})$, and can borrow or lend at constant rate r, subject to $a_{t+1} \geqslant -\phi$.

   (c) The household's problem is

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } c_t + a_{t+1} = (1+r) a_t + s_t$$
$$a_{t+1} \geqslant -\phi$$

9. Stationary equilibrium for Huggett's model.

   Equilibrium in Huggett's economy is defined as a market rate r and a policy function $g(a,s)$ such that

   (a) (Optimality) Given r, $g(a,s)$ solves household's optimization problem.

   (b) (Stationary) Distribution $\lambda(a,s)$ induced by $(P, \bar{s})$ and the policy function $g(a,s)$ is the stationary distribution of the controlled chain.

   (c) (Feasibility) Market clears:

$$\mathbb{E}a = \sum_{a,s} \lambda(a,s)\, g(a,s) = 0$$

   Note: In a model without heterogeneity, we study the representative agent model, and the market clearing condition there also states the bond demand is zero, which means no one borrow or save in the equilibrium. However, because of heterogeneity, consumers here are allowed to borrow or save so long as the total supply of capital is zero.

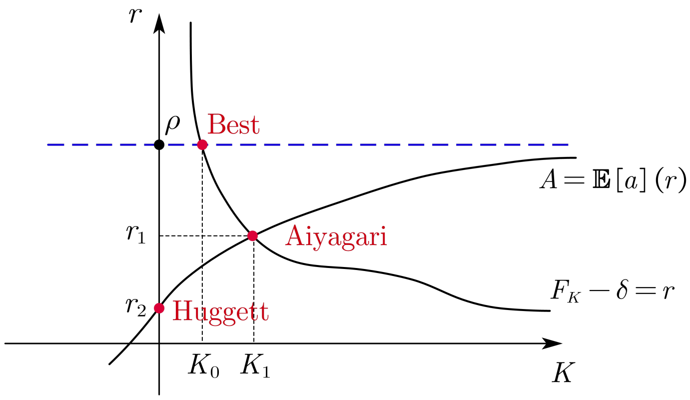

Fig. 5.3. Huggett.

10. Compute the equilibrium for Huggett.

    (a) Pick an r, compute household's optimal saving function $g(a, s)$ and the associated stationary distribution $\lambda(a, s)$.

    (b) Then compute $\mathbb{E}a(r)$, if $\mathbb{E}a(r) > 0$ the lower r, otherwise increase r, and recompute until $\mathbb{E}a(r) = 0$.

11. Remark on the Huggett's model.

    (a) Huggett shows that:

       i. Market equilibrium is suboptimal;

       ii. But under the natural borrowing constraint, r is close to ρ. Households can achieve substantial self-insurance (meaning save by yourself).

    (b) The model can be thought of as a "free banking" (or "inside money") model of circulating private IOUs.

5.3 Bewley Models

1. Set-up for fiat money in Bewley Model.

    (a) Two points about fiat money.

       i. No intrinsic value: be made of some papers.

       ii. No officially guaranteed exchange of money into good, otherwise, it is commodity money (such as gold, silver). For example, we cannot give some fiat money to the central bank, asking for consumption goods.

(b) No borrowing constraint: $\phi = 0$, therefore, no saving.

(c) Government issues a fixed nominal amount M units of money. "Money" is intended to be used as an official method of payment, and thus universally accepted across households in every period of time.

(d) Individuals can hold, but not issue money.

2. Household's problem.

Household's asset position is now

$$a_{t+1} = \frac{m_{t+1}}{P}$$

where P is the price of goods in the stationary equilibrium.

Their saving problem is then

$$c_t + a_{t+1} \leqslant (1+r)\, a_t + s_t$$

Then

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } c_t + a_{t+1} = (1+r)\, a_t + s_t$$
$$a_{t+1} \geqslant 0, a_t = \frac{m_t}{P} \ \forall t$$

3. Remarks on equilibrium.

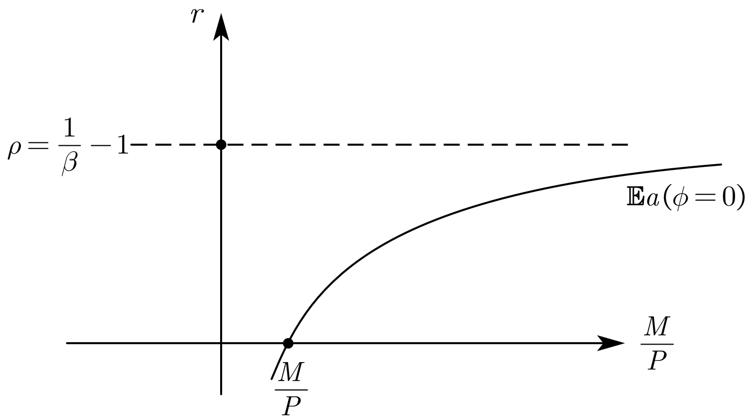

Fig. 5.4. Bewley.

(a) The equilibrium now depends on two prices $\{r, P\}$.

(b) In an equilibrium when money is valued ($P > 0$), $r = 0$, because the nominal interest rate is obviously zero.

i. Due to price invariability, so is the real interest rate.

ii. The equilibrium price of funds is therefore zero.

(c) In equilibrium, asset demand still equals to asset supply, which requires

$$\mathbb{E}a\,(0) = \frac{M}{P}$$

This relation determines the long-run price level in the economy.

(d) Given $\phi = 0$, the long-run price is positive and finite, as $\mathbb{E}a\,(r)$ is bounded below by zero (i.e., the no-borrowing constraint), it must cross the $r = 0$ axis at a positive value.

(e) Let $\bar{m} := \mathbb{E}a\,(0; \phi = 0)$ denote the real money/asset balance under a no-borrowing constraint.

(f) The quantity theory of money obviously holds in the long run: As an increase in M results in an equi-proportional increase in P. It only affects P, not $\mathbb{E}a\,(0)$.

4. The value of money: static comparative.

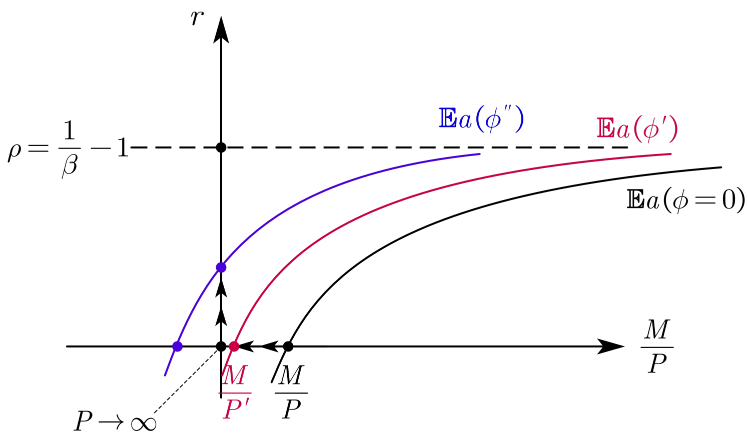

Fig. 5.5. The value of money.

(a) Private credit will not change the equilibrium interest rate, but it does change the valuation of fiat money.

(b) $\phi' > \phi$, the new equilibrium interest rate will still be zero, because on net, there are more lenders than borrowers. Then $\frac{M}{P'} < \frac{M}{P} \Longrightarrow P' > P$, so, money is less valuable.

(c) $\phi'' > \phi' > \phi$, it is the case that $P \to \infty \Longrightarrow \frac{M}{P} \to 0$, implying money is no longer valuable. Money is out of the system so that the model is back to Huggett's economy.

i. So long as $\phi \in (0, \bar{m})$, money is still valued, and in circulation.

ii. A qualitative change happens when ϕ hits $\bar{m}$: interest rate rises above zero, and money stops circulating in the system (money is out).

(d) Thus, in an equilibrium in which bonds pay positive interest rate, money should not be circulating.

i. Keynes circumvented the problem by postulating that money provides additional benefits in terms of liquidity value.

ii. What purpose does money serve in this economy?
It is a storage of value.

5.4 Aggregate Uncertainty

1. Set-up.

   (a) Krusell and Smith (1998) ask a particular (and important) question: Are representative agent models good approximations of heterogeneous agent's ones?

   (b) Households:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \text{s.t. } c + a' = (1+r)\, a + ws$$

   where s is an idiosyncratic employment shock.

   (c) Firms: Production technology is given by

$$F(K, N) = ZK^{\alpha} N^{1-\alpha}$$

   where Z is an aggregate productivity shock governed by a Markov chain.

2. Households' problem.

   The households need to take in the distribution $\lambda(a, s)$ as a state variable. The Bellman equation for the optimization problem is

$$V(a, s, \lambda, Z) = \max_{c,a'} \{u(c) + \beta \mathbb{E}_t [V(a', s', \lambda', Z')]\}$$
$$\text{s.t. } c + a' = w(Z, \lambda)\, s + (1 + r(Z, \lambda))\, a$$
$$\lambda' = H(\lambda, Z, Z')$$

   where function H represents the transition of distributions.

$$V(a, s, \lambda, Z) = \max_{a'} \{u(w(Z, \lambda)\, s + (1 + r(Z, \lambda))\, a - a') + \beta \mathbb{E}_t [V(a', s', \lambda', Z')]\}$$

   The F.O.C. is

$$u'(c) = \beta \mathbb{E}_t [V_2(a', s', \lambda', Z')]$$

   By the Envelop theorem,

$$V_2(a, s, \lambda, Z) = (1 + r(Z, \lambda))\, u'(c)$$

   then

$$u'(c) = \beta \mathbb{E}_t [(1 + r(Z', \lambda'))\, u'(c')]$$

   or

$$u'(c_t) = \beta \mathbb{E}_t [(1 + r(Z_{t+1}, \lambda_{t+1}))\, u'(c_{t+1})] = \beta \mathbb{E}_t [(1 + r_{t+1})\, u'(c_{t+1})]$$

   To solve the problem, we need to forecast the net interest rate r:

(a) On the capital demand side

$$r+\delta=\frac{\partial F\left(K,N\right)}{\partial K}\Longrightarrow r=Z\alpha\left(\frac{K}{N}\right)^{\alpha-1}-\delta$$

where Z serves as a demand shifter.

(b) On the supply side, desired saving depends on the distribution of K, a random function that is history dependent and time-varying

$$K'=\sum_{a,s}\lambda\left(a,s\right)a'\left(a,s,Z,\lambda\right)$$

households need to keep track of λ in order to forecast future capital stock, and with it, future prices.

3. Estimate the H function.

(a) We cannot analytically prove the existence or uniqueness of a recursive equilibrium.

(b) The key challenge is to track the wealth distribution $\lambda\left(a,s\right)$, or namely, the function $H\left(\lambda,Z,Z'\right)$.

(c) Krusell-Smith's idea is to only track a small number of moments of the distribution in a numerical computation. Make the households boundedly rational, in that they only estimate an approximate H function

$$m'=H_n\left(m,Z,Z'\right)$$

where m denotes the first n moments of asset distribution.

(d) Then we need to choose the number of moments and the functional form of the function H_n.

(e) Krusell and Smith pick $n=1$ and assume that

$$\ln K'=\beta_0\left(Z\right)+\beta_1\left(Z\right)\ln K$$

where β_0,β_1 are parameters to be estimated.

(f) Effectively, use K to replace λ in the state space.

4. Compute the equilibrium.

(a) Guess the $\beta=\left[\beta_0,\beta_1\right]$, and then get the function H.

(b) Solve households' problem to obtain the policy function $a'=g\left(a,s,Z,K\right)$.

(c) Simulate for large number of T periods, large number N of households:

i. Initial conditions for economy $\left(Z_0,K_0\right)$, for each household $\left(a_0^i,s_0^i\right)$.

ii. Draw random sequences $\{Z_t\}_{t=1}^{T}$ and $\{s_t^i\}_{t=1,i=1}^{T,N}$. Use decision rule $a' = g(a, s, Z, K)$ and

$$K' = \sum_{a,s} K(a, s)\, a'(a, s, Z, K)$$

to generate $\{a_t^i\}_{t=1,i=1}^{T,N}$.

iii. Aggregate $K_t = \frac{1}{N}\sum_{i=1}^{N} a_t^i$.

(d) Run the regression to estimate β:

$$\ln K' = \hat{\beta}_0(Z) + \hat{\beta}_1(Z)\ln K$$

(e) If the R^2 is high, and $\hat{\beta}$ are close to β, then stop. Otherwise update guess for β.

(f) If guesses for β converge, but R^2 is still low, then we should add higher moments to the aggregate law of motion or use different functional form.

5. Key insights and what's the next?

(a) Key finding: A real business cycle model with complete markets does a good job of approximating the prices and the aggregate allocation of a model with identical preferences and technology but in which only a single asset, physical capital, can be traded.

(b) A side note: realistic income distribution fails to produce realistic wealth distribution, especially the high concentration of wealth at the upper end. (Model Gini coefficient is much lower than the real one.)

(c) New developments that utilize parallelization and more efficient algorithms have been developed. We leave the discussion to a formal computational economics course.