Notes for Microeconomics

Julique

gene_introne@163.com

Welcome your discussions and suggestions.

April 28, 2023

Contents

Chapter 1

Choice, Preference, and Utility

1.1 Preparation and An Important Theorem

1. (Definition) A **choice structure** consists of:

   (a) X is a nonempty set of all possible objects that **might** be chosen.

   (b) $\mathcal{A}$ is a set class of **choice sets** $\varnothing \neq A \subseteq X$. $\varnothing \neq A \subseteq X$ is a set of elements that consumer will choose.

   (c) $\varnothing \neq c(A) \subseteq A$ is the **choice rule** by a consumer.

   $$c(A) : \mathcal{A} \to \bigcup \left\{2^{A} : A \in \mathcal{A}\right\} = \bigcup 2^{\mathcal{A}}$$

2. (Definition) $\mathcal{A} := 2^{X} \setminus \{\varnothing\}$.

   (a) A choice function c satisfies **finite nonemptiness** if

   $$\forall A \in \mathcal{A}, \text{with } |A| < \infty, \text{such that}, c(A) \neq \varnothing$$

   (b) A choice function c satisfies **choice coherence** if $\forall x, y \in X$, $\forall A, B \in \mathcal{A}$,

   $$x, y \in A \cap B, x \in c(A), y \notin c(A) \Longrightarrow y \notin c(B)$$

3. (Example) An equivalent (contrapositive) form for "choice coherence" is: $\forall x, y \in X, \forall A, B \in \mathcal{A}$,

   $$x, y \in A \cap B, x \in c(A), y \in c(B) \Longrightarrow y \in c(A), x \in c(B)$$

   Proof. By choice coherence, $\forall x, y \in X, \forall A, B \in \mathcal{A}$,

   $$x, y \in A \cap B, x \in c(A), y \notin c(A) \Longrightarrow y \notin c(B)$$
   $$x, y \in A \cap B, y \in c(B), x \notin c(B) \Longrightarrow x \notin c(A)$$

   then

   $$y \in c(B) \Longrightarrow x, y \notin A \cap B \text{ or } x \notin c(A) \text{ or } y \in c(A)$$
   $$x \in c(A) \Longrightarrow x, y \notin A \cap B \text{ or } y \notin c(B) \text{ or } x \in c(B)$$

and thus

$$x, y \in A \cap B, x \in c(A), y \in c(B) \Longrightarrow y \in c(A), x \in c(B)$$

On the other hand, if $\forall x, y \in X, \forall A, B \in \mathcal{A}$,

$$x, y \in A \cap B, x \in c(A), y \in c(B) \Longrightarrow y \in c(A), x \in c(B)$$

then

$$y \notin c(A), x \notin c(B) \Longrightarrow x, y \notin A \cap B \text{ or} x \notin c(A) \text{ or} y \notin c(B)$$

therefore,

$$x, y \in A \cap B, x \in c(A), y \notin c(A) \Longrightarrow y \notin c(B)$$

$$x, y \in A \cap B, y \in c(B), x \notin c(B) \Longrightarrow x \notin c(A)$$

$\square$

4. (Definition)

   (a) A **preference relation** $\succsim$ is a binary relation on X.

   (b) $x \succsim y$ means that x is weakly preferred to y.

5. (Definition) A preference $\succsim$ is **rational**, if

   (a) (Completeness) $\forall x, y \in X$, either $x \succsim y$ or $y \succsim x$ (or both).

   (b) (Transitivity) $\forall x, y, z \in X, x \succsim y, y \succsim z \Longrightarrow x \succsim z$.

   Note: Completeness implies reflexive.

6. (Definition)

   (a) **Strict preference**, $\succ$ is defined by the asymmetric part of $\succsim$, i.e.,

   $$x \succ y \Longleftrightarrow x \succsim y \text{ but not} y \succsim x$$

   (b) **Indifference**, $\sim$, is defined by the symmetric part of $\succsim$, i.e.,

   $$\succsim \setminus \succ \text{ or } x \sim y \Longleftrightarrow x \succsim y, y \succsim x$$

7. (Remark) Standard Models of Choice in Microeconomics.

   (a) **Choice** behavior for finite X that satisfies finite nonemptiness and choice coherence.

   $$c(A) \subseteq A$$

   (b) **Preference**-maximizing consumer's optimal choices: $\forall A \in \mathcal{A}$,

   $$c_{\succsim}(A) := \{x \in A : \forall y \in A, x \succsim y\}$$

(c) **Utility**-maximizing consumer's optimal choices: $\forall A \in \mathcal{A}$,

$$c_u(A) := \{x \in A : \forall y \in A, u(x) \geqslant u(y)\}$$

Note: If $x \succsim y \Longleftrightarrow u(x) \geqslant u(y)$, we say u represents $\succsim$, and u just gives us a rank of $x \in X$.

8. (**Theorem of Chapter 1**) The three models are equivalent when X is **finite**.

   (a) A choice function c satisfies finite nonemptiness and choice coherence then

      i. There exists $u : X \to \mathbb{R}$ and

      ii. also exists a complete and transitive preference relation $\succsim$ that produces choices according to c.

   (b) If a preference relation $\succsim$ on X is complete and transitive, then

      i. the choice function $c_{\succsim}$ it produces satisfies finite nonemptiness and choice coherence.

      ii. $\exists u : X \mapsto \mathbb{R}$ such that $x \succsim y \Longleftrightarrow u(x) \geqslant u(y)$.

   (c) Given any utility function $u : X \to \mathbb{R}$, then

      i. the choice function $c_u(A)$ it produces satisfies finite nonemptiness and choice coherence.

      ii. the preference relation $\succsim_u$ it produces is complete and transitive.

      iii. the choice function $c_{\succsim_u}$ produced by that preference relation is precisely the choice function c_u produced directly from u.

1.2 Proofs for the Theorem

1. (Lemma 1 for utility to preference and choice) Regardless of the size X, if $u : X \to \mathbb{R}$, then

   (a) The preference relation $\succsim_u$ defined by

   $$x \succsim_u y \Longleftrightarrow u(x) \geqslant u(y)$$

   is complete and transitive.

   (b) The choice function c_u defined by

   $$c_u(A) := \{x \in A : u(x) \geqslant u(y) \ \ \forall y \in A\}$$

   where $A \in \mathcal{A} = 2^X \setminus \{\varnothing\}$, satisfies nonemptiness and choice coherence.

Proof. (a) **Complete**.

$\geqslant$ is a complete relation on $\mathbb{R}$, and thus $\forall x, y \in X$,

$$\text{either } u(x) \geqslant u(y) \text{ or } u(y) \geqslant u(x)$$

then either $x \succsim y$ or $y \succsim x$.

Transitive.

$\forall x, y, z \in X, x \succsim y, y \succsim z \Longrightarrow u(x) \geqslant u(y) \geqslant u(z)$, since $\geqslant$ is transitive on $\mathbb{R}$, then

$$u(x) \geqslant u(z) \Longrightarrow x \succsim z$$

(b) **Finite Nonemptiness**.

If $0 \neq |A| < \infty$, then the image set

$$u(A) := \{r \in \mathbb{R} : r = u(x), \exists x \in A\}$$

is a finite set, and $\max u(A)$ exists, let it be $r^* = u(x^*)$, then

$$\forall r \in u(A), r^* \geqslant r$$

i.e., $\forall x \in A, u(x^*) \geqslant u(x)$, then $x^* \in c_u(A)$, and thus,

$$c_u(A) \neq \varnothing$$

Choice Coherence.

$\forall x, y \in X, \forall A, B \in \mathcal{A}$, with $x, y \in A \cap B$,

$$x \in c_u(A), y \notin c_u(A)$$

then $\forall z \in A, u(x) \geqslant u(z)$, and $\exists z_0 \in A, u(z_0) > u(y)$, therefore,

$$u(x) \geqslant u(z_0) > u(y)$$

then $x \in B, u(x) > u(y) \Longrightarrow y \notin c_u(B)$.

$\square$

2. (Lemma 2.1 for preference to choice) Regardless of the size X, if $\succsim$ is complete and transitive binary relation on X, then the choice function defined by $\forall A \in \mathcal{A} = 2^X \setminus \{\varnothing\}$,

$$c_{\succsim}(A) := \{x \in A : x \succsim y, \forall y \in A\}$$

satisfies finite nonemptiness and choice coherence.

Proof. **Finite Nonemptiness.**

(Mathematical induction) $0 \neq |A| < \infty$.

For $|A| = 1$, $A = \{x\}$ and since $x \succsim x$, $c_{\succsim}(A) = \{x\} \neq \varnothing$.

Suppose for $|A| = n$, $A = \{x_1, \cdots, x_n\}$, $\exists x' \in A$, such that $\forall x \in A, x' \succsim x$.

Then for $|A| = n+1, A = \{x_1, \cdots, x_n, x_{n+1}\}$, and $\exists x' \in \{x_1, \cdots, x_n\}$,

$$\forall x \in \{x_1, \cdots, x_n\}, x' \succsim x$$

by the completeness of $\succsim$,

$$\text{either } x' \succsim x_{n+1} \text{ or } x_{n+1} \succsim x'$$

i.e.,

$$\text{either } x' \in c_{\succsim}(A) \text{ or } x_{n+1} \in c_{\succsim}(A)$$

therefore, A is nonempty.

Choice Coherence.

Suppose $\forall x, y \in X, \forall A, B \in \mathcal{A}$ with $x, y \in A \cap B$, $x \in c(A), y \notin c(A)$, then

$$\forall z \in A, x \succsim z$$

$$\exists z_0 \in A, \text{ not } y \succsim z_0$$

and decompose $\succsim = \succ \cup \sim$, and by $\succsim$ is complete, then

$$\text{not } y \succsim z_0 \Longleftrightarrow z_0 \succ y \Longleftrightarrow z_0 \succsim y \text{ but not } y \succsim z_0$$

Claim that $x \succ y$

> If not, assume that $y \succsim x$, and since $x \succsim z_0$, which yields to
>
> $$y \succsim z_0$$
>
> A contradiction.

Then, $x, y \in B$, then $x \succsim y$ but not $y \succsim x$, therefore, $y \notin c_{\succsim}(B)$. □

3. (Lemma 3.1 for choice to preference and more) Regardless of the size X, suppose the choice function c satisfies nonemptiness and choice coherence.

   (a) Define a binary relation $\succsim_c$ on X by

   $$x \succsim_c y \Longleftrightarrow x \in c(\{x, y\})$$

   Then $\succsim_c$ is complete and transitive.

   (b) Define a new choice function $c_{\succsim_c}(A)$, for $A \in \mathcal{A} = 2^X \setminus \{\varnothing\}$ by

   $$c_{\succsim_c}(A) := \{x \in A : x \succsim_c y, \forall y \in A\}$$

   Then $c_{\succsim_c}$ satisfies finite nonemptiness and choice coherence.

(c) $\forall A \subseteq X$, $c(A) \subseteq c_{\succsim_c}(A)$. Moreover, either $c(A) = \varnothing$ or $c(A) = c_{\succsim_c}(A)$.

Proof. (a) c satisfies finite nonemptiness, then $\forall A \in \mathcal{A}$ with $|A| < \infty$, $c(A) \neq \varnothing$ then $c(\{x, y\}) = \{x, y\}$ or $\{x\}$ or $\{y\}$, i.e.,

$$\text{either } x \in c(\{x, y\}) \text{ or } y \in c(\{x, y\})$$

by definition, either $x \succsim_c y$ or $y \succsim_c x$ and thus $\succsim_c$ is complete. c satisfies choice coherence, then $\forall x, y \in X, \forall A, B \in \mathcal{A}$, with $x, y \in A \cap B, x \in c(A), y \notin c(A)$ then $y \notin c(B)$. For $\forall x, y, z \in X$, and $x \succsim_c y, y \succsim_c z$. Claim that $x \in c(\{x, y, z\})$, if not, suppose $x \notin c(\{x, y, z\})$.

Step 1. Show that $y \notin c(\{x, y, z\})$.

If not, suppose $y \in c(\{x, y, z\})$. Let

$$A = \{x, y, z\}, B = \{x, y\}$$

satisfies $x, y \in A \cap B$, $x \notin c(A), y \in c(A)$ then by choice coherence,

$$x \notin c(B)$$

A contradiction to $x \succsim_c y$. Therefore, $y \notin c(\{x, y, z\})$.

Step 2. Show that $z \notin c(\{x, y, z\})$.

If not, suppose $z \in c(\{x, y, z\})$,

$$A = \{x, y, z\}, B = \{y, z\}$$

satisfies, $y, z \in A \cap B, y \notin c(A), z \in c(A)$ then by choice coherence,

$$y \notin c(B)$$

A contradiction to $y \succsim_c z$.

Therefore,

$$x, y, z \notin c(\{x, y, z\}) \subseteq \{x, y, z\} \Longrightarrow c(\{x, y, z\}) = \varnothing$$

contradicting with finite nonemptiness. And thus, $x \in c(\{x, y, z\})$, since $\{x, z\} \subseteq \{x, y, z\} \Longrightarrow x \in c(\{x, z\})$, then $x \succsim_c z$

(b) Since $\succsim_c$ is complete and transitive, by Lemma 2.1 for preference, we know that $c_{\succsim_c}$ satisfies finite nonemptiness and choice coherence.

(c) If $c(A) \neq \varnothing$. $\forall A \in \mathcal{A}$, $\forall x \in c(A)$, and $\forall y \in A$ with $x \neq y$, by finite nonemptiness and choice coherence, $x \in c(\{x, y\})$.

If not, suppose $x \notin c(\{x,y\})$, then by finite nonemptiness, $y \in c(\{x,y\})$, and by choice coherence,

$$x, y \in A \cap \{x,y\}, y \in c(\{x,y\}), x \notin c\{x,y\} \Longrightarrow x \notin c(A)$$

A contradiction!

Then $x \in c(\{x,y\})$ holds for $\forall x \in c(A)$ and $\forall y \in A$ with $x \neq y$.

Therefore, $\forall x \in c(A)$ and $\forall y \in A$ with $x \neq y$, $x \succsim_c y$.

Then

$$x \in c_{\succsim_c}(A)$$

and thus, $c(A) \subseteq c_{\succsim_c}(A)$.

Moreover, since $c_{\succsim_c}(A) \neq \varnothing$, then we consider any $x \in c_{\succsim_c}(A)$.

Suppose $c(A)$ is nonempty, then $\forall y \in c(A) \subseteq c_{\succsim_c}(A)$, by the definition of $c_{\succsim_c}(A)$,

$$x \succsim_c y$$

then $x \in c(\{x,y\})$.

Claim that $x \in c(A)$.

If not,

$$x, y \in A \cap \{x,y\}, y \in c(A), x \notin c(A) \Longrightarrow x \notin c(\{x,y\})$$

A contradiction!

Then $c_{\succsim_c}(A) \subseteq c(A)$, i.e., if $c(A) \neq \varnothing$, $c(A) = c_{\succsim_c}(A)$.

If $|A| = \infty$, then $c(A) = \varnothing$ can hold, since we not assume infinite nonemptiness.

□

Note: We might guess that, $c(A) = c_{\succsim_c}(A)$, but it is not the case. Let $X = [0,1]$, define

$$x \succsim_c y \Longleftrightarrow x \in c(\{x,y\}) = \begin{cases} \{x\}, x \geqslant y \\ \{y\}, y \geqslant x \\ \{x,y\}, x = y \end{cases}$$

then $\mathrm{NBT}_{\succsim_c}(1) = [0,1]$ $c_{\succsim_c}([0,1]) = \{1\}$, but actually, $c([0,1])$ can be $\varnothing$, since we don't assume infinite nonemptiness.

4. (Definition) Let $\succsim$ be a preference relation on X of any size. $\forall x \in X$, **the no-better-than x set** is defined by

$$\mathrm{NBT}(x) := \{y \in X : x \succsim y\}$$

Note: If $\succsim$ is reflexive, i.e., $x \succsim x \Longrightarrow x \in \mathrm{NBT}(x)$, then

$$\mathrm{NBT}(x) \neq \varnothing, \forall x \in X$$

5. (Lemma) If $\succsim$ is complete and transitive, then

$$x \succsim y \Longleftrightarrow \mathrm{NBT}\left(y\right) \subseteq \mathrm{NBT}\left(x\right)$$

and

$$x \succ y \Longrightarrow \mathrm{NBT}\left(y\right) \subsetneqq \mathrm{NBT}\left(x\right)$$

Note: The collection of NBT sets nests.

Proof. If $x \succsim y$, then $\forall z \in \mathrm{NBT}\left(y\right), y \succsim z$, by transitivity, $x \succsim z$, and thus, $z \in \mathrm{NBT}\left(x\right)$, i.e., $\mathrm{NBT}\left(y\right) \subseteq \mathrm{NBT}\left(x\right)$.

If $x \succsim y$ but not $y \succsim x$, then $x \notin \mathrm{NBT}\left(y\right)$ but $x \in \mathrm{NBT}\left(x\right)$ by $x \succsim x$. Therefore, $\mathrm{NBT}\left(y\right) \subsetneqq \mathrm{NBT}\left(x\right)$. □

6. (Lemma 2.2 for preference to utility, finite) If X is a finite set and $\succsim$ is complete and transitive, then the function $u : X \mapsto \mathbb{R}$ defined by

$$u\left(x\right) := \left|\mathrm{NBT}\left(x\right)\right|$$

satisfies

$$u\left(x\right) \geqslant u\left(y\right) \Longleftrightarrow x \succsim y$$

Proof. On the one hand,

$$x \succsim y \Longrightarrow \mathrm{NBT}\left(y\right) \subseteq \mathrm{NBT}\left(x\right) \Longrightarrow \left|\mathrm{NBT}\left(y\right)\right| \leqslant \left|\mathrm{NBT}\left(x\right)\right| \Longrightarrow u\left(x\right) \geqslant u\left(y\right)$$

On the other hand, since $\succsim$ is complete, then either

$$\mathrm{NBT}\left(y\right) \subseteq \mathrm{NBT}\left(x\right) \text{ or } \mathrm{NBT}\left(x\right) \subseteq \mathrm{NBT}\left(y\right)$$

and

$$u\left(x\right) \geqslant u\left(y\right) \Longrightarrow \left|\mathrm{NBT}\left(y\right)\right| \leqslant \left|\mathrm{NBT}\left(x\right)\right|$$

therefore, $\mathrm{NBT}\left(y\right) \subseteq \mathrm{NBT}\left(x\right)$ must holds, then $x \succsim y$.

□

Note: For **infinite** set X, defining $u\left(x\right) := \left|\mathrm{NBT}\left(x\right)\right|$ does **not** work.

7. (Lemma 2.2 for choice to utility, finite) If X is a **finite** set, and choice function $c\left(A\right), \forall A \in \mathcal{A} = 2^X \backslash \{\varnothing\}$ satisfies finite nonemptiness and choice coherence, then there exists a utility function $u : X \mapsto \mathbb{R}$, that produces choices according to c.

Proof. Use c to generate a preference relation $\succsim_c$ as

$$x \succsim_c y \Longleftrightarrow x \in c\left(\{x, y\}\right)$$

then $\succsim_c$ is transitive and complete. Therefore, likewise, we can define $\mathrm{NBT}_{\succsim_c}(x), \forall x \in X$, and

$$u(x) := \left|\mathrm{NBT}_{\succsim_c}(x)\right|$$

then

$$u(x) \geqslant u(y) \Longleftrightarrow x \succsim_c y$$

and since $|A| < \infty$,

$$c(A) = c_{\succsim_c}(A) := \{x \in A : x \succsim_c y, \forall y \in A\} = \{x \in A : u(x) \geqslant u(y), \forall y \in A\}$$

□

8. (Remark)
   (a) Rational preferences relations are **necessary** for utility representation.
      i. For$|X| < \infty$, rational preferences relations are **sufficient** for utility representation.
      ii. For $|X| = \infty$, rational preferences relations are **not sufficient** for utility representation.
   (b) By Lemma 1, 2.1, 2.2, 3.1, 3.2, we showed the important theorem of this chapter.
9. (Example) Lexicographic preference is defined by $\forall x, y \in \mathbb{R}_+^n$,

$$x \succsim_L y \Longleftrightarrow x_1 > y_1 \vee (x_1 = y_1, x_2 > y_2) \vee \cdots \vee (x_1 = y_1, \cdots, x_{n-1} = y_{n-1}, x_n \geqslant y_n)$$

i.e., maximize the former components.

   (a) $\succsim_L$ on $\mathbb{R}_+^2$ satisfies completeness and transitivity.

Proof. **Completeness**.
$(x_1, x_2) \succsim_L (y_1, y_2)$ is defined as

$$x_1 > y_1 \text{ or } (x_1 = y_1 \text{ and } x_2 \geqslant y_2)$$

For any $x, y \in \mathbb{R}_+^2$, since $\geqslant$ is complete in $\mathbb{R}$, we can compare the first element, if $x_1 = y_1$, then we can compare the second element, and thus $\succsim_L$ is complete.
Transitivity.
If $\forall x, y, z \in \mathbb{R}_+^2$, with $x \succsim_L y, y \succsim_L z$, then there're 4 cases

$$\begin{cases} x_1 > y_1 \begin{cases} y_1 > z_1 \Longrightarrow x_1 > z_1 \Longrightarrow x \succsim_L z \\ y_1 = z_1, y_2 \geqslant z_2 \Longrightarrow x_1 > z_1 \Longrightarrow x \succsim_L z \end{cases} \\ x_1 = y_1, x_2 \geqslant y_2 \begin{cases} y_1 > z_1 \Longrightarrow x_1 > z_1 \Longrightarrow x \succsim_L z \\ y_1 = z_1, y_2 \geqslant z_2 \Longrightarrow x_1 = z_1, x_2 \geqslant z_2 \Longrightarrow x \succsim_L z \end{cases} \end{cases}$$

each case is consistent with transitivity. □

(b) $\succsim_L$ on $\mathbb{R}_+^2$ cannot be represented by a utility function.

Proof. Suppose u represents $\succsim_L$. Then $\forall x \in \mathbb{R}_+, (x,1) \succ_L (x,0)$ and thus

$$u(x,1) > u(x,0)$$

And then, $\exists r_x \in \mathbb{Q}$ such that $u(x,1) > r_x > u(x,0)$.

Define $\varphi : \mathbb{R} \to \mathbb{Q}$ as $\varphi : x \mapsto r_x$, now show that φ is an injection, and then generate a contradiction.

$\forall x, y \in \mathbb{R}_+$ with $x \neq y$, W.L.G., assume $x > y$, then

$$(x,0) \succ_L (y,1)$$

and thus

$$u(x,1) > r_x > u(x,0) > u(y,1) > r_y > u(y,0)$$

i.e., $r_x \neq r_y$.

Therefore, $\varphi : \mathbb{R} \to \mathbb{Q}$ is an injection, which is a contradiction, since we cannot construct an injection from $\mathbb{R}$ into $\mathbb{Q}$. □

1.3 Utility on Infinite Sets

1. (Lemma 2.3 preference to utility, countable) Let $\succsim$ be a complete and transitive relation on a countable set X, then $\exists u : X \mapsto \mathbb{R}$,

$$u(x) \geqslant u(y) \Longleftrightarrow x \succsim y$$

Proof. Since X is countable, then let $X = \{x_1, x_2, \cdots\}$, then define $d(x_n) = \frac{1}{2^n}$ and

$$u(x) := \sum_{z \in \mathrm{NBT}(x)} d(z)$$

Check that u represents $\succsim$, firstly note that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} - 1 = 1$$

(a) Suppose $x \succsim y$, then $\mathrm{NBT}(y) \subseteq \mathrm{NBT}(x)$, since $d > 0$, then $u(x) \geqslant u(y)$.

(b) Since NBT sets nest, i.e., by definition,

$$u(x) \geqslant u(y) \Longleftrightarrow \mathrm{NBT}(y) \subseteq \mathrm{NBT}(x)$$

and hence $x \succsim y$.

□

2. (Lemma 2.4 preference to utility, countable and order-dense subset) Let $\succsim$ be a complete and transitive relation on X. Then $\succsim$ can be represented by a utility function u, if and only if, $\exists X^* \subseteq X$, X^* is countable and order-dense in X, that is X^* satisfies $\forall x, y \in X$,

$$x \succ y \Longrightarrow \exists x^* \in X^*, x \succsim x^* \succ y$$

Note: Intuitively, if we can "separate" two elements with strictly preference in the original set X by elements in a countable subset X^*, then we can define a utility function that represents $\succsim$.

Proof. (**If**) Let $X^* = \{x_1^*, x_2^*, \cdots\}$, define $X^* = \{x_1^*, x_2^*, \cdots\}$ and

$$u(x) = \sum_{x^* \in X^* \cap \mathrm{NBT}(x)} d(x^*)$$

Check that u represents $\succsim$.

- $x \succsim y \Longrightarrow \mathrm{NBT}(y) \subseteq \mathrm{NBT}(x) \Longrightarrow (X^* \cap \mathrm{NBT}(y)) \subseteq (X^* \cap \mathrm{NBT}(x))$, then $u(x) \geqslant u(y)$.
- If $x \succ y$, then $\exists x^* \in X^*, x \succsim x^* \succ y$, $\mathrm{NBT}(y) \subsetneqq \mathrm{NBT}(x^*) \subseteq \mathrm{NBT}(x)$, then $u(y) < u(x^*) \leqslant u(x)$, therefore,

$$x \succ y \Longrightarrow u(x) > u(y)$$

by contrapositive,

$$u(y) \geqslant u(x) \Longrightarrow y \succsim x$$

(**Only if**) See Kreps Page.12. ☐

3. (Definition) Let $\succsim$ be a transitive and complete relation on $X = \mathbb{R}_+^k$. We say $\succsim$ is **continuous** if $\forall x, y \in X$, with $x \succ y$, $\exists \varepsilon > 0$, such that

$$\forall x' \in B_\varepsilon(x), \forall y' \in B_\varepsilon(y), x' \succ y'$$

Note:

Specifically, it should be

$$\forall x' \in B_\varepsilon(x) \cap X, \forall y' \in B_\varepsilon(y) \cap X, x' \succ y'$$

Since $\succsim$ is defined on $X \times X$.

4. (Remark)

(a) If $\succsim$ is not transitive, $\succsim$ can be continuous. Consider a preference $\succsim$ defined on $X = \mathbb{R}_+^2$ as

$$x \succsim y \Longleftrightarrow x_1 \geqslant y_1 \text{ or } x_2 \geqslant y_2$$

(b) If $\succsim$ is not complete, $\succsim$ can be continuous. Consider a preference $\succsim$ defined on $X = \mathbb{R}_+ \cup \{-1\}$ as

$$x \succsim y \Longleftrightarrow x \geqslant y \geqslant 0$$

5. (Theorem) Continuity of preferences on $\mathbb{R}_+^k$ is equivalent to any one of the following

   (a) A sequence $\{x^n\}_{n=1}^{\infty} \subseteq \mathbb{R}_+^k$, $\lim_{n\to\infty} x^n = x$,

      i. $\forall n \geqslant 1, x^n \succsim y \Longrightarrow x \succsim y$.

      ii. $\forall n \geqslant 1, y \succsim x^n \Longrightarrow y \succsim x$.

   (b) A sequence $\{x^n\}_{n=1}^{\infty} \subseteq \mathbb{R}_+^k$, $\lim_{n\to\infty} x^n = x$,

      i. $x \succ y \Longrightarrow \exists N > 0, \forall n \geqslant N, x^n \succ y$.

      ii. $y \succ x \Longrightarrow \exists N > 0, \forall n \geqslant N, y \succ x^n$.

   (c) $\forall x \in \mathbb{R}_+^k, \mathrm{NBT}(x) := \left\{y \in \mathbb{R}_+^k : x \succsim y\right\}, \mathrm{NWT}(x) := \left\{y \in \mathbb{R}_+^k : y \succsim x\right\}$ are closed sets.

   (d) $\forall x \in \mathbb{R}_+^k, \mathrm{SBT}(x) := \left\{y \in \mathbb{R}_+^k : y \succ x\right\}, \mathrm{SWT}(x) := \left\{y \in \mathbb{R}_+^k : x \succ y\right\}$ are (relatively) open sets.

   Note: The set Y is relatively open in another set X if Y is the intersection of X and an open set in the "host space" of X.

   (e) (Definition in MWG) For any sequence of $\{(x^n, y^n)\}_{n=1}^{\infty} \subseteq \mathbb{R}_+^k \times \mathbb{R}_+^k$, $\lim_{n\to\infty} x^n = x, \lim_{n\to\infty} y^n = y$,

$$\forall n \geqslant 1, x^n \succsim y^n \Longrightarrow x \succsim y$$

Proof.

$$\begin{array}{ccccccc}
 & & \text{(e)} & \Longrightarrow & \text{continuity} & & \\
 & & \Uparrow & & & & \\
\text{continuity} & \Longrightarrow & \text{(a)} & \Longrightarrow & \text{(c)} \Longleftrightarrow \text{(d)} & \Longrightarrow & \text{continuity} \\
 & & \Updownarrow & & \text{(complete)} & & \\
 & & \text{(b)} & & & &
\end{array}$$

Step 1. continuity$\Longrightarrow$(a).

Let $x^n \to x$, and $\forall n \geqslant 1, x^n \succsim y$. Suppose $y \succ x$, then $\exists \varepsilon > 0$, such that $\forall y' \in B_\varepsilon(y), \forall x' \in B_\varepsilon(x)$, $y' \succ x'$.

By definition of limit, $\exists N > 0, \forall n \geqslant N, x^n \in B_\varepsilon(x')$, then $x^n \succ y$. A contradiction.

Step 2. (a)$\Longleftrightarrow$(b).

- (a)$\Longrightarrow$(b), let $x^n \to x$, and $x \succ y$, suppose $\forall N > 0, \exists n \geqslant N, y \succsim x^n$, then we can construct $\{x^{n_k}\}_{k=1}^{\infty} \subseteq \{x^n\}$, such that $\forall k \geqslant 1, y \succsim x^{n_k}$, and $x^{n_k} \to x$, then by (a), $y \succsim x$, a contradiction.
- (b)$\Longrightarrow$(a), W.L.G., let $x^n \to x$, $\forall n \geqslant 1, x^n \succsim y$, suppose $y \succ x$, then $\forall N > 0, \exists n \geqslant N, x^n \succ y$, contradiction.

Step 3. (a)$\Longrightarrow$(c).

Let $\{y^n\} \subseteq \mathrm{NBT}(x)$, be an arbitrary convergent sequence, and $y^n \to y$. Then $\forall n \geqslant 1$, $x \succsim y^n$, by (a), we have $x \succsim y$, then $y \in \mathrm{NBT}(x)$. Therefore, $\mathrm{NBT}(x)$ is closed.

Likewise, $\mathrm{NWT}(x)$ is closed.

Step 4. (c)$\Longleftrightarrow$(d).

By assuming that $\succsim$ is complete, it holds that

$$[\mathrm{NBT}(x)]^c = \mathrm{SBT}(x), [\mathrm{NWT}(x)]^c = \mathrm{SWT}(x)$$

by definition of closed and open sets, we know that (c)$\Longleftrightarrow$(d).

Step 5. (c)$\Longrightarrow$continuity.

$\forall x, y \in X$, with$x \succ y$, since$\mathrm{NWT}(x)$ is closed, and the line $\{ax + (1-a)y : a \in [0,1]\}$ is also closed, then

$$S := \mathrm{NWT}(x) \cap \{ax + (1-a)y : a \in [0,1]\}$$

is closed. Let

$$A := \{a \in [0,1] : ax + (1-a)y \in S\} = \{a \in [0,1] : ax + (1-a)y \succsim x\}$$

then $1 \in A, 0 \notin A$. Let $a^* := \inf A$, then $a^* \in A$, i.e.,

$$a^* x + (1-a^*) y \succsim x$$

If not, (by completeness),

$$x \succ a^* x + (1-a^*) y$$

then $a^* x + (1-a^*) y \in [\mathrm{NWT}(x)]^c$, and $[\mathrm{NWT}(x)]^c$ is open, then

$$\exists \varepsilon > 0, N_\varepsilon(a^* x + (1-a^*) y) \subseteq [\mathrm{NWT}(x)]^c$$

therefore, $\exists \tilde{a} > a^*$ such that

$$\tilde{a}x + (1-\tilde{a}) y \in [\mathrm{NWT}(x)]^c \Longleftrightarrow \tilde{a}x + (1-\tilde{a}) y \succ x$$

which is a contradiction to that $a^* = \inf A$.

Moreover, $\forall a \in [0, a^*)$,

$$x \succ ax + (1-a) y$$

Since if not, $ax + (1-a) y \succsim x$, and $a < a^*$, contradicting a^* is the infimum.

And $\text{NBT}(x)$ is closed, then $S' := \text{NBT}(x) \cap \{ax + (1-a)y : a \in [0,1]\}$ is also closed.

We can let $a^n = a^* - \frac{1}{n}, \forall n = 1, 2, \cdots$, and

$$x^n = a^n x + (1 - a^n) y, \forall n \in \mathbb{N}$$

and $x^n = a^n x + (1-a^n) y \to a^* x + (1-a^*) y$, as $n \to \infty$.

Note that $\{x^n\}_{n=1}^{\infty} \subseteq S'$, and S' is closed, then

$$x^n \to a^* x + (1-a^*) y \in S' \subseteq \text{NBT}(x)$$

then

$$x \succsim a^* x + (1-a^*) y$$

Therefore,

$$a^* x + (1-a^*) y \sim x \succ y$$

Let $z := a^* x + (1-a^*) y \succ y$.

Likewise, we can find a b^*, by

$$b^* := \sup \{b \in [0,1] : bz + (1-b) y \succsim y\} \in [0,1)$$

such that

$$z \succ y \sim b^* x + (1-b^*) y$$

Let $\forall b \in (b^*, 1), w := bz + (1-b) y$, then

$$x \succ w \succ y$$

By the openness of $[\text{NWT}(w)]^c = \text{SBT}(w)$ and $[\text{NBT}(w)]^c = \text{SWT}(w)$,

$$\exists \varepsilon_1 > 0, N_{\varepsilon_1}(x) \subseteq \text{SBT}(w) \Longrightarrow \forall x' \in N_{\varepsilon_1}(x), x \succ w$$
$$\exists \varepsilon_2 > 0, N_{\varepsilon_2}(x) \subseteq \text{SWT}(w) \Longrightarrow \forall y' \in N_{\varepsilon_2}(x), w \succ y$$

Let $\varepsilon = \min\{x', y'\}$, then $\forall x' \in N_\varepsilon(x), \forall y' \in N_\varepsilon(x), x' \succ y'$. By definition, $\succsim$ is continuous.

Step 6. (a)+(b)$\Longrightarrow$(e).

Suppose that (e) does not hold, then $\exists \{x^n\}, \{y^n\}$ such that $x^n \succsim y^n$, with

$$\lim_{n\to\infty} x^n = x, \lim_{n\to\infty} y^n = y$$

but

$$y \succ x$$

By (b),

$$\exists N_1 > 0, \forall n \geqslant N_1, y \succ x^n$$
$$\exists N_2 > 0, \forall n \geqslant N_2, y^n \succ x$$

Then, for $N = \max\{N_1, N_2\}, \forall n \geqslant N$,

$$y \succ x^n \succsim y^n \succ x$$

or in particular, $x^n \succ y^N \succ x$, by (a), $x \succsim y^N \succ x$, contradiction.

Step 7. (e)$\Longrightarrow$continuity.

Suppose $\succsim$ is not continuous, then $\exists x, y \in X$, with $x \succ y$, $\forall \varepsilon > 0, \exists x' \in B_\varepsilon(x), \exists y' \in B_\varepsilon(y)$, such that $y' \succsim x'$.

Then we can let $\varepsilon = \frac{1}{n}$, and construct two sequences such that $x^n \to x, y^n \to x$, and

$$y^n \succsim x^n, \forall n \geqslant 1$$

then $y \succsim x$. A contradiction.

$\square$

6. (Example) Lexicographic preference $\succsim_L$ on $\mathbb{R}^2_+$ is not continuous, it's easy to see that its NBT and NWT sets are not closed.

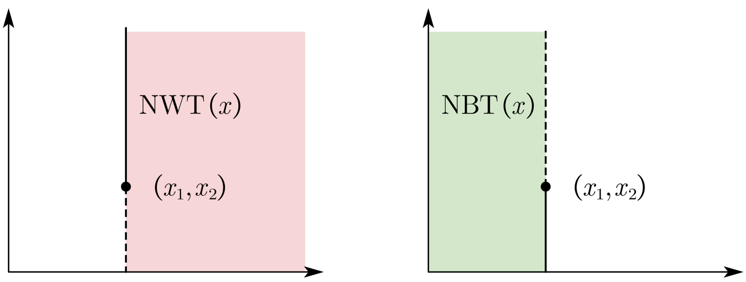

Fig. 1.1. NBT and NWT set.

7. Consider $X = \mathbb{R}^2_+$, define

$$(x_1, x_2) \succsim (y_1, y_2) \Longleftrightarrow x_1 \geqslant x_2 \text{ and } y_1 \geqslant y_2$$

It is not continuous.

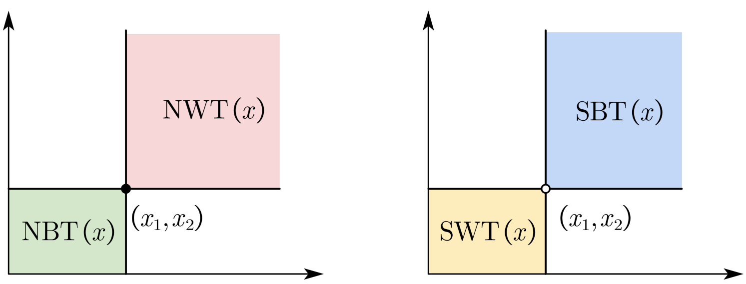

Fig. 1.2. NBT and NWT set.

Since we can let $x = (1,1)\,, y = \left(1, \frac{1}{2}\right)$, then $x \succ y$. But $\forall \varepsilon > 0$, we cannot compare $y' = \left(1 + \frac{\varepsilon}{2}, \frac{1}{2}\right) \in B_\varepsilon(y)$ and $x = (1,1) \in B_\varepsilon(x)$. In other words, $x \succ y'$ doesn't hold, or $\mathrm{SBT}(x)$ is not open.

It may be confusing that $\forall x \in \mathbb{R}^2_+$, $\mathrm{NBT}(x)$ is closed, but the proof of "$\mathrm{NBT}(x)$ is closed $\Longrightarrow$ continuity" used completeness of the preference, so the closedness of $\mathrm{NBT}(x)$ fails to prove the continuity of the preference.

8. (Lemma 2.5 preference to utility, continuous and $\mathbb{R}^k_+$) Let $\succsim$ is a complete, transitive and continuous relation on $X = \mathbb{R}^k_+$, then $\succsim$ can be represented by a utility function u, i.e., $\forall x, y \in X$,

$$u(x) \geqslant u(y) \Longleftrightarrow x \succsim y$$

Proof. We only need to show that there exists a countable subset $X^* \subseteq X$ has the property that if $\forall x, y \in X$ with $x \succ y$, $\exists x^* \in X^*$ such that $x \succsim x^* \succ y$.

Let X^* be all bundles $x \in X$ that all of whose components are rational numbers, i.e., $X^* = \mathbb{Q}^k_+$, then X^* is naturally countable.

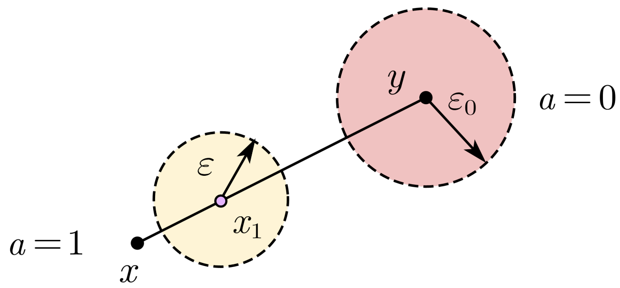

Fig. 1.3. Lemma 2.5.

Then $\forall x, y \in X$ with $x \succ y$, consider the line segment

$$ax + (1-a)\,y, \forall a \in [0,1]$$

Let

$$a_1 := \inf\{a \in [0,1] : ax + (1-a)\,y \succsim x\}$$

$$x_1 := a_1 x + (1-a_1)\,y$$

then $a_1 > 0$, if not, then $y \succsim x$, a contradiction.

Moreover, by continuity $\exists \varepsilon_0 > 0, \forall z \in B_{\varepsilon_0}(y)\,, x \succ z$, therefore, $\forall a < 1 - \frac{\varepsilon_0}{d(x,y)}$, where $d(x,y)$ is the distance between x and y,

$$ax + (1-a)\,y \in B_{\varepsilon_0}(y) \Longleftrightarrow x \succsim ax + (1-a)\,y$$

Claim that $x_1 \sim x$.

> If not, assume $x_1 \succ x$, then $\exists \varepsilon_1 > 0, \forall z \in B_{\varepsilon_1}(x_1), z \succ x$, therefore, $\exists a \in [0,1]$, with $a < a_1$, such that $ax + (1-a)y \succ x$, contradiction to the definition of a_1.
>
> Likewise, if $x \succ x_1$, then $\exists \varepsilon_2 > 0, \forall z \in B_{\varepsilon_2}(x_1), x \succ z$, and thus $\exists a \in [0,1]$, with $a > a_1$ such that $x \succ ax + (1-a)y$. But if $a > a_1$, then $a \in \{a \in [0,1] : ax + (1-a)y \succsim x\}$, contradiction to the definition of a_1.

Since $x_1 \sim x, x \succ y$, then $x_1 \succ y$, and thus $\exists \varepsilon_3 > 0, \forall z \in B_{\varepsilon_3}(x_1), z \succ y$.

$\exists a_2 \in [0,1]$, such that $a_2 < a_1$ and $x_2 := a_2 x + (1-a_2)y \succ y$. And by definition of a_1, we have

$$x \succ x_2 \succ y$$

therefore,

$$\exists \varepsilon_4 > 0, \forall z \in B_{\varepsilon_4}(x_2), x \succ x_2$$
$$\exists \varepsilon_5 > 0, \forall z \in B_{\varepsilon_5}(x_2), x_2 \succ y$$

i.e., $\exists \varepsilon_6 = \frac{1}{2}\min\{\varepsilon_4, \varepsilon_5\}$, $\forall z \in B_{\varepsilon_6}(x_2)$,

$$x \succ z \succ y$$

And since rational numbers are dense in real numbers, $\exists x^* \in \mathbb{Q}_+^k \cap B_{\varepsilon_6}(x_2)$, such that

$$x \succ x^* \succ y$$

which finishes the proof. □

1.4 Choice on Infinite Sets

1. (Remark)

   (a) It is unwise to assume $c(A) \neq \varnothing$ when $|A| = \infty$. For example, $A = [0,1)$, $u(x) = x$, then $c_u(A) = \{x \in A : u(x) \geqslant u(y), \forall y \in A\} = \varnothing$.

   (b) A different approach: define choice only for some subsets $A \subseteq X$.

      i. Restrict the domain of choice function c to the subsets A for which is reasonable to assume that $c(A) = \varnothing$.

      ii. Strengthen finite nonemptiness by dropping the restriction to finite sets.

   (c) Assumption. $x \in A \subseteq \mathrm{NBT}_{\succsim_c}(x) \Longrightarrow c(A) \neq \varnothing$

Notes: $\forall y \in A, y \in \mathrm{NBT}_{\succsim_c}(x) \Longrightarrow x \succsim_c y \Longrightarrow \forall y \in A, x \in c(\{x,y\})$, then $x \in c_{\succsim_c}(A) = \{y \in A : x \succsim_c y\} \neq \varnothing$.

By this assumption, $c(A) \neq \varnothing \Longrightarrow c_{\succsim_c}(A) = c(A) \Longrightarrow x \in c(A)$.

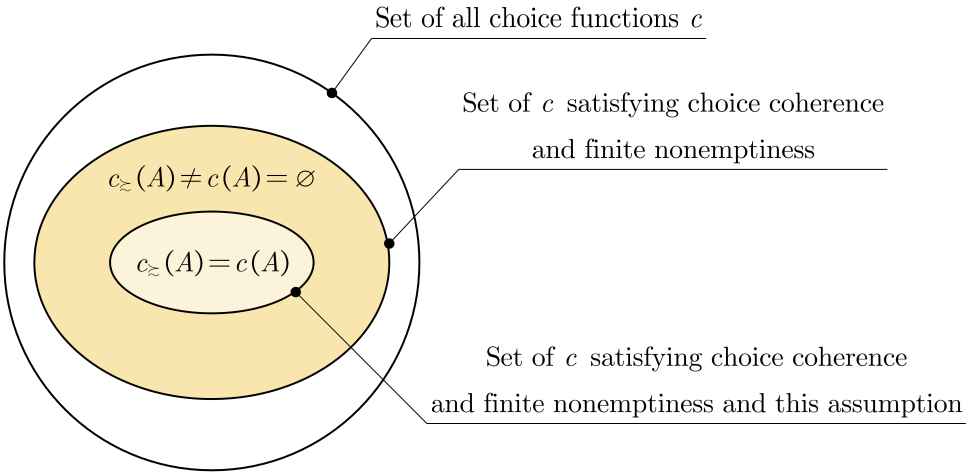

Fig. 1.4. Choice on Infinite Sets

2. (Theorem) A choice function c that satisfies finite nonemptiness and choice coherence is identical to choice generated by the preferences it generates, i.e.,

$$c = c_{\succsim_c}$$

iff

$$x \in A \subseteq \mathrm{NBT}_{\succsim_c}(x) \Longrightarrow c(A) \neq \varnothing$$

Proof. (**Only if**) $c = c_{\succsim_c}$, and $x \in A \subseteq \mathrm{NBT}_{\succsim_c}(x)$, by definition of $c_{\succsim_c}(A)$,

$$c_{\succsim_c}(A) := \{x \in A : x \succsim_c y, \forall y \in A\}$$

then $x \in c_{\succsim_c}(A) = c(A) \Longrightarrow c(A) \neq \varnothing$.

(**If**) Recall that if $c(A) \neq \varnothing$ then $c(A) = c_{\succsim_c}(A)$.

If $c_{\succsim_c}(A) = \varnothing$, since $c(A) \subseteq c_{\succsim_c}(A) \Longrightarrow c(A) = \varnothing$

If $c_{\succsim_c}(A) \neq \varnothing$, let $\forall x \in c_{\succsim_c}(A)$, then $x \in A$.

Claim that $A \subseteq \mathrm{NBT}_{\succsim_c}(x)$, i.e., $\forall y \in A$, then $y \in \mathrm{NBT}_{\succsim_c}(x)$.

If not, suppose $y \notin \mathrm{NBT}_{\succsim_c}(x)$, and $y \in A$, then $y \succ_c x \Longrightarrow x \notin c_{\succsim_c}(A)$, a contradiction with that $x \in c_{\succsim_c}(A)$.

$$x \in A \subseteq \mathrm{NBT}_{\succsim_c}(x) \Longrightarrow c(A) \neq \varnothing$$

therefore, $c(A) = c_{\succsim_c}(A)$. □

3. (Corollary) A choice function c satisfies finite nonemptiness and choice coherence. Then $\forall x \in A, A \nsubseteq \mathrm{NBT}_{\succsim_c}(x) \Longrightarrow c(A) = \varnothing$.

Proof. It's the complementary statement of

$$x \in A \subseteq \mathrm{NBT}_{\succsim_c}(x) \Longrightarrow c(A) \neq \varnothing$$

□

4. (Theorem) Suppose $X = \mathbb{R}_+^k$. Take a choice function c that satisfies finite nonemptiness, choice coherence, and $x \in A \subseteq \mathrm{NBT}_{\succsim_c}(x) \Longrightarrow c(A) \neq \varnothing$. If the preferences $\succsim_c$ generated from c are continuous, then for any nonempty and compact set A, $c(A) \neq \varnothing$.

   Proof. It's a direct corollary of Debreu's Theorem in next chapter or the theorem we proved in previous section. □

Chapter 2

Properties of Preferences and Utility Functions

2.1 Monotonicity

1. (Definition) For a commodity space: $X = \mathbb{R}_{+}^{k}$, $x, y \in X$, and a preference $\succsim$ on X, we say

   (a) $\succsim$ is monotone if $x \geqslant y \Longrightarrow x \succsim y$.

   (b) $\succsim$ is strictly monotone if $x \geqslant y$ and $x \neq y \Longrightarrow x \succ y$.

   (c) $\succsim$ is strictly monotone for strict increases in the bundle if $x \gg y \Longrightarrow x \succ y$.

   Note:

$$x \geqslant y \Longleftrightarrow x_1 \geqslant y_1, x_2 \geqslant y_2, \cdots, x_k \geqslant y_k$$

$$x \gg y \Longleftrightarrow x_1 > y_1, x_2 > y_2, \cdots, x_k > y_k$$

2. (Example)

   (a) $u(x) = \min_{1 \leqslant i \leqslant k} u_i(x_i)$, where $u_i : \mathbb{R}_{+} \rightarrow \mathbb{R}$ is strictly increasing. Then the corresponding preference is strictly monotone for strict increases in x, but not strictly increasing.

   (b) $u(x) = \prod_{i=1}^{k} x_i^{\alpha_i}, \alpha_i > 0$. Then the corresponding preference is strictly monotone for strictly increases in x, but not strictly increasing.

$$x = (0,0), y = (1,0) \Longrightarrow u(x) = u(y) = 0$$

   However, it is strictly increasing away from the axes.

   (c) Lexicographic preference is strictly increasing.

$$(x_1, x_2) \succsim_L (y_1, y_2) \Longrightarrow (x_1 > y_1) \vee (x_1 = y_1, x_2 \geqslant y_2)$$

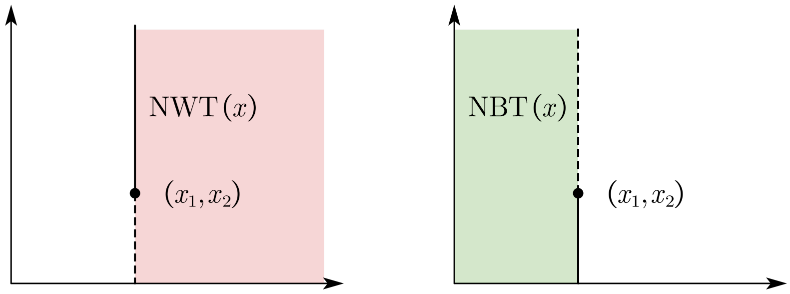

Fig. 2.1. NBT and NWT set.

Moreover, since $\mathrm{NWT}(x) \cap \mathrm{NBT}(x) = \{y \in X : y \sim x\}$, we can see that for Lexicographic preference, $\mathrm{NWT}(x) \cap \mathrm{NBT}(x) = \{x\}$, which implies $\succsim_L$ is antisymmetric.

(d) Define $(x_1, x_2) \succsim (y_1, y_2) \iff x_1 \geqslant y_1$ or $x_2 \geqslant y_2$.

$\succsim$ is not strictly increasing, but strictly increasing for strict in the bundle.

Note: The preference is not transitive.

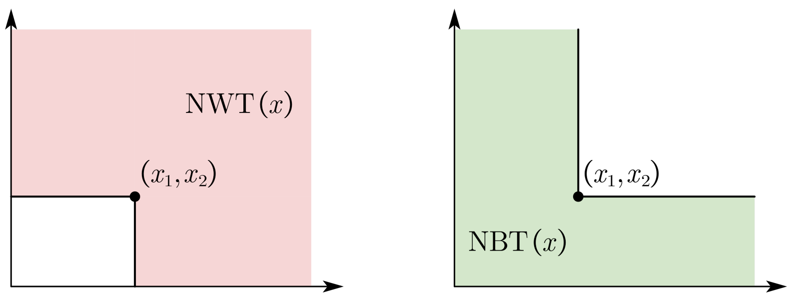

Fig. 2.2. NBT and NWT set.

(e) (Remark) "Strictly monotone for strict increases in the bundle" does not imply "monotone".

Example:

$$u(x) = \begin{cases} 0, & x_1 = x_2 = 0 \\ -1, & x_1 x_2 = 0, \max\{x_1, x_2\} > 0 \\ \min\{x_1, x_2\}, & \min\{x_1, x_2\} > 0 \end{cases}$$

The corresponding $\succsim$ is not monotone since $(0,0) \succ (1,0)$, but it is strictly monotone for strict increases in the bundle.

3. (Definition) A preference $\succsim$ on X, we say

(a) $\succsim$ is **globally nonsatiated**, if $\forall x \in X$,

$$\exists y \in X, y \succ x$$

(b) $\succsim$ is **locally nonsatiated**, if $\forall x \in X$,

$$\exists y \in X, \forall \varepsilon > 0, \|y - x\| < \varepsilon, y \succ x$$

Note:

(a) If some good is desirable, then the preference is always locally nonsatiated.

(b) If $\succsim$ is strictly monotone for strict increase in the bundle, then they are locally insatiable, but the converse is not true, e.g., if the indifference curve is like this.

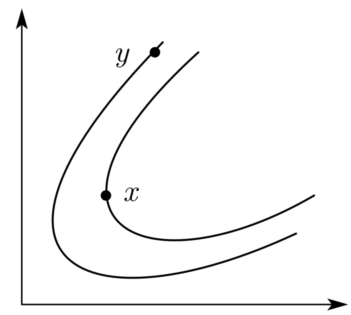

Fig. 2.3. Example

2.2 Convexity

1. (Definition) A preference $\succsim$ on X, we say

   (a) $\succsim$ is **convex**, if $\forall x, y \in X$,with $x \succsim y$, then

   $$\forall a \in [0,1], ax + (1-a) y \succsim y$$

   (b) $\succsim$ is **strictly convex**, if $\forall x, y \in X$, with $x \neq y$, $x \succsim y$, then

   $$\forall a \in (0,1), ax + (1-a) y \succ y$$

   (c) $\succsim$ is **semi-strictly convex**, if $\forall x, y \in X$, with $x \succ y$, then

   $$\forall a \in (0,1), ax + (1-a) y \succ y$$

2. (Theorem) A complete and transitive preference $\succsim$ on X is convex, iff $\forall x \in X$,

   $$\mathrm{NWT}(x) := \{y \in X : y \succsim x\}$$

   is convex.

   Proof. (**If**) Assume that $y \succsim x$, then by reflexivity (which is implied by completeness), $x \succsim x \Longrightarrow \forall a \in (0,1), ay + (1-a) x \succsim x \Longrightarrow \succsim$ is convex.

   (**Only if**) $\succsim$ is convex, then $\forall x, y, z \in X$ with $z \succsim x, y \succsim x$, by completeness,

   $$\text{either } y \succsim z, \text{ or } z \succsim y$$

Without loss of generality, assume $y \succsim z$, then by convexity,

$$\forall a \in [0,1], ay + (1-a) z \succsim z$$

then by transitivity,

$$\forall a \in [0,1], ay + (1-a) z \succsim x \implies ay + (1-a) z \in \mathrm{NWT}(x)$$

□

3. (Theorem) $\succsim$ is complete and transitive.
   (a) If $\succsim$ is convex, then for any convex choice set A, the set $c_{\succsim}(A)$ is convex ($c_{\succsim}(A) = \varnothing$ is not precluded).
   (b) If $\succsim$ is strictly convex, then for any convex choice set A, the set $c_{\succsim}(A)$ contains at most one point.

Proof. (a) $\forall x, y \in c_{\succsim}(A) \subseteq A$, by definition of $c_{\succsim}(A)$, $\forall z \in A, x \succsim z, y \succsim z$. Since $x, y \in A$, then $x \sim y \succsim z$. Of course, $x \succsim y$, then by convexity of $\succsim$, $\forall a \in [0,1]$, $ax + (1-a) y \succsim y \succsim z$, therefore, $ax + (1-a) y \in c_{\succsim}(A)$. And note that $ax + (1-a) y \in A$ since A is convex.

(b) Suppose $\exists x, y \in c_{\succsim}(A)$ with $x \neq y$, then by strict convexity of $\succsim$,

$$\forall a \in (0,1), ax + (1-a) y \succ y$$

And A is convex then $ax + (1-a) y \in A$, then contradicting to $y \in c_{\succsim}(A)$.

□

4. (Definition) A function $f : A \mapsto \mathbb{R}$, and A is convex. We say
   (a) f is **concave** if $\forall x, y \in A, \forall a \in [0,1]$,

$$f(ax + (1-a) y) \geqslant af(x) + (1-a) f(y)$$

   (b) f is **strictly concave** if $\forall x, y \in A, x \neq y, \forall a \in (0,1)$,

$$f(ax + (1-a) y) > af(x) + (1-a) f(y)$$

   (c) f is **quasi-concave** if $\forall x, y \in A, \forall a \in [0,1]$,

$$f(ax + (1-a) y) \geqslant \min \{f(x), f(y)\}$$

   (d) f is **semi-strictly quasi-concave** if f is quasi-concave and $\forall x, y \in A$, with $f(x) > f(y)$, and $\forall a \in (0,1)$,

$$f(ax + (1-a) y) > f(y)$$

(e) f is **strictly quasi-concave** if $\forall x, y \in A, x \neq y, \forall a \in (0,1)$,

$$f(ax + (1-a)y) > \min\{f(x), f(y)\}$$

5. (Theorem)

   (a) Suppose u represents a preference $\succsim$.

      i. If u is concave, then $\succsim$ is convex.

      ii. If u is strictly concave, $\succsim$ is strictly convex.

   (b) Suppose u represents a preference $\succsim$.

      i. u is quasi-concave iff $\succsim$ is convex.

      ii. u is strictly quasi-concave iff $\succsim$ is strictly convex.

      iii. u is semi-strictly quasi-convex iff $\succsim$ is semi-strictly convex.

Proof. Since (strictly) concave implies (strictly) quasi-concave, we only need to show (b).

(**Only if**) If u is quasi-concave, then $\forall x, y \in X$, with $y \succsim x$, then $u(y) \geqslant u(x)$, and thus, $\forall a \in [0,1]$,

$$u(ay + (1-a)x) \geqslant \min\{u(x), u(y)\} = u(x)$$

then $ay + (1-a)x \succsim x \Longrightarrow \succsim$ is convex.

(**If**) If $\succsim$ is convex, likewise, $\forall x, y \in X$, w.l.g., $y \succsim x$ $\forall a \in [0,1]$, then

$$ay + (1-a)x \succsim x$$

which implies

$$u(ay + (1-a)x) \geqslant \min\{u(x), u(y)\}$$

The rest statements are similar to prove. □

6. (Remark) There exists a preference $\succsim$, which is convex, and it admits no concave utility representation.

   E.g., Lexicographic preference has no utility representation, but it is strictly convex (see the NWT set or prove by definition).

$$(x_1, x_2) \succsim (y_1, y_2) \Longrightarrow (x_1 \geqslant y_1) \vee (x_1 = y_1, x_2 \geqslant y_2)$$

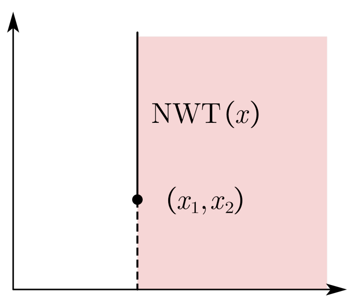

Fig. 2.4. NWT Set.

2.3 Continuity

1. (Theorem, Debreu's Theorem) $\succsim$ is complete and transitive on X.

   (a) If a continuous function u represents $\succsim$, then $\succsim$ is continuous.

   (b) If $\succsim$ is continuous, then it has a continuous utility representation u.

   Proof of part(a). $\forall x, y \in X$ with $x \succ y$, then

$$u(x) > u(y)$$

   Since u is continuous, $\forall \varepsilon > 0$, $\exists \delta_1, \delta_2 > 0$,$\forall x' \in B_{\delta_1}(x), \forall y' \in B_{\delta_2}(y)$,such that

$$|u(x) - u(x')| < \varepsilon, |u(y) - u(y')| < \varepsilon$$

   Let $\varepsilon = \frac{u(x)-u(y)}{3}$, then $\forall x' \in B_{\delta_1}(x), \forall y' \in B_{\delta_2}(y)$,we have

$$\begin{aligned} u(x') - u(y') &= u(x') - u(x) + u(x) - u(y) + u(y) - u(y') \\ &\geqslant u(x) - u(y) - |u(x') - u(x)| - |u(y) - u(y')| \\ &> 3\varepsilon - \varepsilon - \varepsilon \\ &= \varepsilon \end{aligned}$$

   then $x' \succ y'$. Therefore, $\succsim$ is continuous. □

2. (Theorem, A Simple Version of Part(b) Debreu's Theorem) Consider $X = \mathbb{R}_+^k$, If a preference $\succsim$ on X satisfies strict monotonicity and continuity, moreover, assume that

$$0 = (0, \cdots, 0), M = \left(\max_k \{x_k\}, \cdots, \max_k \{x_k\}\right) \in X$$

   then $\succsim$ has a continuous utility representation.

   Proof. 2 steps to prove the theorem.

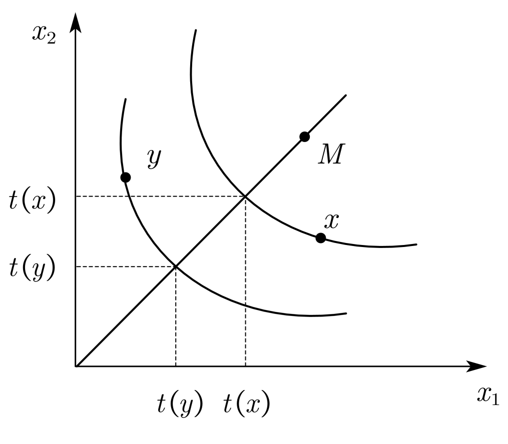

Fig. 2.5. Find a Indifference Alternative on the Diagonal.

Step 1. Define a utility function u and show that u represents $\succsim$.

$\forall x \in \mathbb{R}_+^k$, by continuity, there is a bundle $(t(x), t(x), \cdots, t(x))$ on the main diagonal, such that the consumer is indifferent between $(t(x), t(x), \cdots, t(x))$ and the bundle x (Actually, $\lambda_0 := \inf\{\lambda \in [0,1] : \lambda M \succsim y\}$, $\lambda_0 M$ is such a bundle).

By strict monotonicity, $\forall x \in \mathbb{R}_+^k$, such a bundle $(t(x), t(x), \cdots, t(x))$ is unique, otherwise, contradiction to strict monotonicity.

Moreover, it holds that

$$M = \left(\max_k \{x_k\}, \cdots, \max_k \{x_k\}\right) \succsim x \sim (t(x), \cdots, t(x)) \succsim 0 = (0, \cdots, 0)$$

Let

$$u(x) := t(x)$$

Now show that u represents the references. By transitivity,

$$x \succsim y \Longleftrightarrow (t(x), \cdots, t(x)) \sim x \succsim y \sim (t(y), \cdots, t(y))$$

and by monotonicity,

$$(t(x), \cdots, t(x)) \succsim (t(y), \cdots, t(y)) \Longleftrightarrow t(x) \geqslant t(y)$$

Step 2. Show that u is continuous.

(Proof by contradiction) $\forall x \in \mathbb{R}_+^k$, let (x^n) be a sequence with $x^n \to x$, and assume that $u(x) = t(x)$ is not continuous at x, i.e.,

$$t(x^n) \nrightarrow t(x)$$

By definition of limit, it is easy to show that (x^n) is bounded, i.e., $\exists M^* > 0$, such that

$$(M^*, \cdots, M^*) > x^n \Longleftrightarrow (M^*, \cdots, M^*) \succ x^n$$

In other words, $\forall n \geqslant 1, t(x^n) \in [0, M^*]$, where

$$[0, M^*] = \{t(x_1, \cdots, x_k) : 0 \leqslant x_1, \cdots, x_k \leqslant M\}$$

is a compact interval. And since $t(x^n) \nrightarrow t(x)$, by definition, $\exists \varepsilon > 0, \forall N \geqslant 0, \exists n > N$, such that $|t(x^n) - t(x)| \geqslant \varepsilon \Longleftrightarrow t(x) \notin [t(x) - \varepsilon, t(x) + \varepsilon]$.

Thus, without loss of generality, assume there are infinite elements of (x^n), it forms a subsequence $(x^{n_k}) \subseteq (x^n)$ such that $\forall k \geqslant 1, t(x^{n_k}) \in \left[t(x) + \frac{\varepsilon}{2}, M^*\right]$. Then, by Bolzano-Weierstrass Theorem, $\exists (x^{n_{k_l}}) \subseteq (x^{n_k})$, with

$$t(x^{n_{k_l}}) \to t(x') \in \left[t(x) + \frac{\varepsilon}{2}, M^*\right]$$

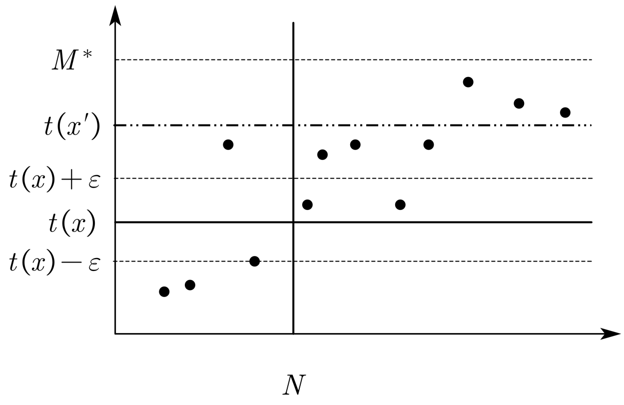

Fig. 2.6. Find a Subsequence.

Since $(x^{n_{k_l}}) \subseteq (x^n)$, then $x^{n_{k_l}} \to x$, but

$$\left(t\left(x^{n_{k_l}}\right), \cdots, t\left(x^{n_{k_l}}\right)\right) \to \left(t\left(x'\right), \cdots, t\left(x'\right)\right)$$

Then by continuity of preferences, and

$$x^{n_{k_l}} \sim \left(t\left(x^{n_{k_l}}\right), \cdots, t\left(x^{n_{k_l}}\right)\right) \Longrightarrow \begin{cases} x^{n_{k_l}} \succsim \left(t\left(x^{n_{k_l}}\right), \cdots, t\left(x^{n_{k_l}}\right)\right) \\ \left(t\left(x^{n_{k_l}}\right), \cdots, t\left(x^{n_{k_l}}\right)\right) \succsim x^{n_{k_l}} \end{cases}$$

then

$$\begin{cases} \lim_{l\to\infty} x^{n_{k_l}} \succsim \lim_{l\to\infty} \left(t\left(x^{n_{k_l}}\right), \cdots, t\left(x^{n_{k_l}}\right)\right) \\ \lim_{l\to\infty} \left(t\left(x^{n_{k_l}}\right), \cdots, t\left(x^{n_{k_l}}\right)\right) \succsim \lim_{l\to\infty} x^{n_{k_l}} \end{cases} \Longrightarrow \begin{cases} x' \succsim \left(t\left(x'\right), \cdots, t\left(x'\right)\right) \\ \left(t\left(x'\right), \cdots, t\left(x'\right)\right) \succsim x' \end{cases}$$

i.e.,

$$x \sim \left(t\left(x'\right), \cdots, t\left(x'\right)\right)$$

but

$$x \sim \left(t\left(x\right), \cdots, t\left(x\right)\right)$$

and $t\left(x'\right) > t\left(x\right)$, by strict monotonicity, we deduced a contradiction. □

3. (Remark) If $\succsim$ is continuous, and u represents $\succsim$, then we cannot conclude that u is continuous. Since $v = f \circ u$ where f is strictly monotone, still represents $\succsim$, but f can be discontinuous, and thus v can be discontinuous.

2.4 Separability

1. (Remark) Notations.

   (a) K is the list of commodity names.

   (b) $J \subseteq K$.

(c) The consumption bundle $x = x_K$, x_J is a sub-consumption bundle that contains the component of x on the list J.

(d) (Example) $K = \{1, 2, 3\}, J = \{1, 3\}, x = (x_1, x_2, x_3), x_J = (x_1, x_3)$.

2. (Definition) Let $J_1, J_2, \cdots, J_N$ be a list of N mutually exclusive subsets of K, i.e., $\forall n \neq m, J_n \cap J_m = \varnothing$. A preference $\succsim$ is **weakly separable** into $J_1, J_2, \cdots, J_N$ if $\forall n = 1, 2, \cdots, N$, $\forall x_{J_n}, x'_{J_n} \in \mathbb{R}_+^{|J_n|}, \forall x_{J_n^c}, x'_{J_n^c} \in \mathbb{R}_+^{|J_n^c|}$,

$$\left(x_{J_n}, x_{J_n^c}\right) \succsim \left(x'_{J_n}, x_{J_n^c}\right) \Longleftrightarrow \left(x_{J_n}, x'_{J_n^c}\right) \succsim \left(x'_{J_n}, x'_{J_n^c}\right)$$

Note:

(a) No matter how we change the sub-consumptions on J_n^c, the preference relation of sub-consumptions on J_n is preserved.

(b) Lexicographic preference is weakly separable.

3. (Theorem) Suppose u represents $\succsim$. Preference $\succsim$ is weakly separable into , iff $\exists u_n : \mathbb{R}_+^{|J_n|} \to \mathbb{R}, n = 1, 2, \cdots, N$ and for the image set $u_n\left(\mathbb{R}_+^{J_n}\right)$, $v : u_1\left(\mathbb{R}_+^{|J_1|}\right) \times \cdots \times u_N\left(\mathbb{R}_+^{|J_N|}\right) \times \mathbb{R}_+^{|(J_1 \cup \cdots \cup J_N)^c|} \to \mathbb{R}$ such that v is strictly increasing in its first N arguments, and

$$u(x) = v\left(u_1\left(x_{J_1}\right), u_2\left(x_{J_2}\right), \cdots, u_N\left(x_{J_N}\right), x_{(J_1 \cup \cdots \cup J_N)^c}\right)$$

Proof. (**If**) We have such a utility represents $\succsim$, then, $\forall n = 1, 2, \cdots, N$, $\forall x_{J_n}, x'_{J_n} \in \mathbb{R}_+^{|J_n|}, \forall x_{J_n^c}, x'_{J_n^c} \in \mathbb{R}_+^{|J_n^c|}$, and since v is strictly increasing in the n-th argument, then

$$\left(x_{J_n}, x_{J_n^c}\right) \succsim \left(x'_{J_n}, x_{J_n^c}\right) \Longleftrightarrow u_n\left(x_{J_n}\right) \geqslant u_n\left(x'_{J_n}\right) \Longleftrightarrow \left(x_{J_n}, x'_{J_n^c}\right) \succsim \left(x'_{J_n}, x'_{J_n^c}\right)$$

(**Only if**) Suppose u represents $\succsim$, and $\succsim$ is weakly separable, $\forall n \geqslant 1$, fix $x^*_{J_n^c}$, define

$$u_n\left(x_{J_n}\right) := u\left(x_{J_n}, x^*_{J_n^c}\right), n = 1, 2, \cdots, N$$

and

$$v\left(u_1\left(x_{J_1}\right), \cdots, u_N\left(x_{J_N}\right), x_{(J_1 \cup \cdots \cup J_N)^c}\right) := u\left(x_{J_1}, \cdots, x_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right)$$

Now, show that the definition does not depend on the particular x_{J_n}, i.e., rigorously, we need to show that our utility functions are well defined.

In other words, for any two consumption bundles x and x', with

$$u_n\left(x_{J_n}\right) = u_n\left(x'_{J_n}\right), n = 1, 2, \cdots, N$$

$$x_{(J_1 \cup \cdots \cup J_N)^c} = x'_{(J_1 \cup \cdots \cup J_N)^c}$$

then by definition of v, we have

$$v\left(u_1\left(x_{J_1}\right), \cdots, u_N\left(x_{J_N}\right), x_{(J_1 \cup \cdots \cup J_N)^c}\right) = v\left(u_1\left(x'_{J_1}\right), \cdots, u_N\left(x'_{J_N}\right), x_{(J_1 \cup \cdots \cup J_N)^c}\right)$$

we want to show that $x \sim x'$.

Suppose $\forall n \geqslant 1$, $x'_{J_n} \in \mathbb{R}_+^{|J_n|}$ such that

$$u_n\left(x_{J_n}\right) = u_n\left(x'_{J_n}\right), n = 1, 2, \cdots, N$$

then

$$u\left(x_{J_1}, \cdots, x_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right) = u\left(x'_{J_1}, \cdots, x'_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right)$$

and,

$$u\left(\left(x_{J_n}, x^*_{J_n^C}\right)\right) = u_n\left(x_{J_n}\right) = u_n\left(x'_{J_n}\right) = u\left(\left(x'_{J_n}, x^*_{J_n^C}\right)\right)$$

which implies

$$\left(x_{J_n}, x^*_{J_n^C}\right) \sim \left(x'_{J_n}, x^*_{J_n^C}\right)$$

By weak separability, we can replace $x^*_{J_n^c}$ by

$$u\left(x_{J_1}, \cdots, x_{J_{n-1}}, x'_{J_{n+1}}, \cdots, x'_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right)$$

then

$$\begin{aligned}
&\left(x_{J_1}, \cdots, x_{J_{n-1}}, x_{J_n}, x'_{J_{n+1}}, \cdots, x'_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right) \\
&\sim \left(x_{J_1}, \cdots, x_{J_{n-1}}, x'_{J_n}, x'_{J_{n+1}}, \cdots, x'_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right)
\end{aligned}$$

which holds for $\forall n = 1, 2, \cdots, N$. In detail, we have

$$\begin{aligned}
\left(x_{J_1}, x'_{J_2}, \cdots, x'_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right) &\sim \left(x'_{J_1}, x'_{J_2}, \cdots, x'_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right) \\
\left(x_{J_1}, x_{J_2}, x'_{J_3}, \cdots, x'_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right) &\sim \left(x_{J_1}, x'_{J_2}, x'_{J_3}, \cdots, x'_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right) \\
&\vdots \\
\left(x_{J_1}, \cdots, x_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right) &\sim \left(x_{J_1}, \cdots, x_{J_{N-1}}, x'_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right)
\end{aligned}$$

Then by transitivity of $\sim$,

$$\left(x'_{J_1}, x'_{J_2}, \cdots, x'_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right) \sim \left(x_{J_1}, \cdots, x_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right)$$

Now show that v is strictly increasing in the first N arguments. $\forall n = 1, 2, \cdots, N$,if $u_n\left(x'_{J_n}\right) > u_n\left(x_{J_n}\right)$, then

$$\left(x'_{J_n}, x^*_{J_n^c}\right) \succ \left(x_{J_n}, x^*_{J_n^c}\right)$$

Likewise, by weak separability,

$$\begin{aligned}
&\left(x_{J_1}, \cdots, x_{J_{n-1}}, x_{J_n}, x_{J_{n+1}}, \cdots, x_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right) \\
&\qquad \succ \left(x_{J_1}, \cdots, x_{J_{n-1}}, x'_{J_n}, x_{J_{n+1}}, \cdots, x_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right)
\end{aligned}$$

Then

$$\begin{aligned}
&v\left(u_1\left(x_{J_1}\right), \cdots, u\left(x_{J_{n-1}}\right), u\left(x_{J_n}\right), u\left(x_{J_{n+1}}\right), \cdots, u_N\left(x_{J_N}\right), x_{(J_1 \cup \cdots \cup J_N)^c}\right) \\
&= u\left(x_{J_1}, \cdots, x_{J_{n-1}}, x_{J_n}, x_{J_{n+1}}, \cdots, x_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right) \\
&> u\left(x_{J_1}, \cdots, x_{J_{n-1}}, x'_{J_n}, x_{J_{n+1}}, \cdots, x_{J_N}, x_{(J_1 \cup \cdots \cup J_N)^c}\right) \\
&= v\left(u_1\left(x_{J_1}\right), \cdots, u\left(x_{J_{n-1}}\right), u\left(x'_{J_n}\right), u\left(x_{J_{n+1}}\right), \cdots, u_N\left(x_{J_N}\right), x_{(J_1 \cup \cdots \cup J_N)^c}\right)
\end{aligned}$$

which holds for $\forall n = 1, 2, \cdots, N$. □

4. (Definition) Let $J_1, J_2, \cdots, J_N$ be a partition of K, i.e., $\forall n \neq m$, $J_n \cap J_m = \varnothing$, $\bigcup_{i=1}^{N} J_i = K$. A preference $\succsim$ is **strongly separable** into $J_1, J_2, \cdots, J_N$ if $\forall \ell \in \mathbb{N}, 1 \leqslant \ell \leqslant N$, for the indices $\{n_1, \cdots, n_\ell\} \subseteq \{1, \cdots, N\}$,

$$L := J_{n_1} \cup J_{n_2} \cup \cdots \cup J_{n_\ell}$$

then $\forall x_L, x'_L \in \mathbb{R}_+^{|L|}, \forall x_{L^c}, x'_{L^c} \in \mathbb{R}_+^{|L^c|}$,

$$(x_L, x_{L^c}) \succsim (x'_L, x_{L^c}) \Longleftrightarrow (x_L, x'_{L^c}) \succsim (x'_L, x'_{L^c})$$

5. (Theorem) Suppose a preference $\succsim$ is continuous and strongly separable in J_1 through J_N. Suppose that $\succsim$ is nontrivial on at least 3 of the commodity indices sets J_1 through J_N. Then $\exists u_n : \mathbb{R}_+^{|J_n|} \to \mathbb{R}$, such that

$$u(x) = \sum_{n=1}^{N} u_n (x_{J_n})$$

represents $\succsim$.

Conversely, if preference $\succsim$ is represents by a utility function u taking the form $\sum_{n=1}^{N} u_n (x_{J_n})$ where $u_n : \mathbb{R}_+^{|J_n|} \to \mathbb{R}$ then $\succsim$ is strongly separable.

Note:

(a) For $X = \mathbb{R}_+^k, k \geqslant 3$, lexicographic preference is strongly separable.

(b) Strong separability of preference implies addictively separable utility function.

(c) We say $\succsim$ is nontrivial on J_n, if $\exists x_{J_n}, x'_{J_n} \in \mathbb{R}_+^{|J_n|}$, $\exists x^*_{J_n^c} \in \mathbb{R}_+^{|J_n^c|}$ such that $\left(x_{J_n}, x^*_{J_n^c}\right) \succ \left(x'_{J_n}, x^*_{J_n^c}\right)$.

Proof. Rigorous proof is difficult, and thus, omitted. □

2.5 Quasi-linearity and Homotheticity

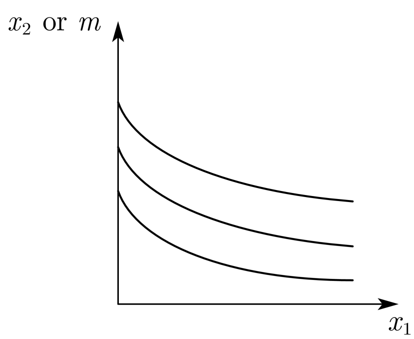

Fig. 2.7. Quasi-linear in x_2 or m.

1. (Definition) A preference $\succsim$ on $\mathbb{R}_+^k$ is **quasi-linear in the k-th** commodity if they can be represented by a utility function of the form

$$U(x,m) = u(x) + m$$

for some sub-utility function $u : \mathbb{R}_+^{k-1} \to \mathbb{R}$, where $x \in \mathbb{R}_+^{k-1}, m \in \mathbb{R}_+$.

Meanwhile, we also say U has a **quasi-linear form**.

2. (Theorem) Preference $\succsim$ on has a quasi-linear representation in the k-th commodity iff the following properties hold.

   (a) $\forall x \in \mathbb{R}_+^{k-1}, \forall m, m' \in \mathbb{R}_+$ with $(x,m) \succsim (x,m')$ iff $m \geqslant m'$.

   (b) $\forall x, x' \in \mathbb{R}_+^{k-1}, \forall m, m' \in \mathbb{R}_+$, with $(x,m) \succsim (x',m')$ iff $\forall m'' \in \mathbb{R}_+$, $(x, m+m'') \succsim (x', m'+m'')$.

   (c) $\forall x, x' \in \mathbb{R}_+^{k-1}$, such that $\exists m, m' \in \mathbb{R}_+$ $(x,m) \sim (x',m')$.

   Notes: The intuition.

   (a) More money is better than less.

   (b) No wealth effect: Money won't change the preference between two sub-bundles.

   (c) Money can be used to calibrate the difference in worth of any two sub-bundles. If U has a quasi-linear form, for any two bundles (x,m), (x',m'), define $m^* := U(x,m) - U(x',m')$, then

   $$(x,m) \sim (x, m' + m^*)$$

   Actually, it's a proof of (c).

   More Note: Lexicographic preferences satisfy properties (a) and (b). But it fails to satisfy (c), since if $x_1 > y_1$, we cannot make $x \sim y$ by changing x_2, y_2.

   Proof for the only if part. (a) $U(x,m) = u(x) + m \geqslant u(x) + m' = U(x,m') \iff (x,m) \succsim (x,m')$

   (b) $U(x,m) \geqslant U(x',m') \iff u(x) + m + m'' \geqslant u(x') + m' + m'' \iff (x, m+m'') \succsim (x', m'+m'')$.

   (c) See note above.

   □

3. (Example) Consider $X = \mathbb{R}_+^2$, if $\succsim$ on X is quasi-linear in 2-nd component, continuous and strictly monotonic, then the indifference curve has a cross point with the 2-nd (vertical) axis.

Proof. Take $\forall\,(x,m) \in \mathbb{R}^2_+$, and $0 \in \mathbb{R}_+$ then by property (c), $\exists m', m''$, such that

$$(x, m') \sim (0, m'')$$

and by strictly monotonicity and $x > 0$, we must have $m'' > m'$, otherwise

$$m'' \leqslant m', 0 < x \Longrightarrow (x, m') \succ (0, m'')$$

a contradiction. Therefore, let $m'' - m' > 0$, then

$$(x, m' - m' + m) \sim (0, m'' - m' + m) \Longrightarrow (x, m) \sim (0, m'' - m' + m)$$

which finishes the proof. □

Moreover, if $u(0) = 0$, $\lim_{x\to\infty} u(x) = \infty$, then every indifference curve will hit the 1-st (horizontal) axis.

If $\lim_{x\to\infty} u(x) = k < \infty$, then the indifference curve through $(0, k)$ where $U(0, k) = k$, will asymptote the 1-st axis.

And thus, indifference curve through $(0, m), m < k$ will hit the 1st axis, but indifference curve through $(0, m), m \geqslant k$ will asymptote the 1-st axis.

4. (Definition) A preference $\succsim$ is **homothetic** if $\forall x, y \in X, \forall \lambda \geqslant 0$,

$$x \succsim y \Longrightarrow \lambda x \succsim \lambda y$$

Note: It's not a natural assumption, for example, we only need to take 2 pills of a kind of drug every day, then $2 \succ 1, 2 \succ 4$.

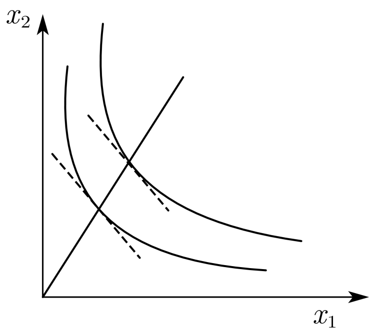

Fig. 2.8. Homotheticity.

5. (Definition) If $A \subseteq \mathbb{R}^k$ is a cone, that is $x \in A \Longrightarrow \lambda x \in A, \lambda \geqslant 0$, a function $f : A \to \mathbb{R}$ is **homogenous of degree** $\alpha \geqslant 0$, if $\forall \lambda \geqslant 0$, $x \in A$,

$$f(\lambda x) = \lambda^\alpha f(x)$$

If $\alpha = 1$, we simply say that f is **homogenous**.

6. (Theorem) A preference $\succsim$ is homothetic iff it can be represented by a continuous and homogenous utility function.

   Proof of a simple version. (**If**) Let u be a homogenous function and represent $\succsim$, then $\forall \alpha > 0$,

$$x \succsim y \Longrightarrow u(x) \geqslant u(y) \Longrightarrow u(\alpha x) = \alpha u(x) \geqslant \alpha u(y) = u(\alpha y) \Longrightarrow \alpha x \succsim \alpha y$$

   (**Only if**) $\succsim$ is homogenous, and add some assumptions of $\succsim$ to meet the conditions of the simple version of Debreu's Theorem, then we get a utility representation of $\succsim$, which is

$$u(x) := t(x)$$

   where $(t(x), \cdots, t(x)) \sim x$. Let $e = (1, \cdots, 1)$, then by homotheticity,

$$x \sim t(x) \cdot e \Longrightarrow \alpha x \sim \alpha t(x) \cdot e$$

   and since $\alpha x \sim t(\alpha x) \cdot e \Longrightarrow \alpha t(x) \cdot e \sim t(\alpha x) \cdot e$. By the continuity and strictly monotonicity of u,

$$u(\alpha x) = \alpha u(x)$$

   Therefore, under these strong assumptions, we construct a homogenous utility function.

   □

7. (Example) Consider $X = \mathbb{R}^2_+$. If $\succsim$ on X is strictly monotonic, continuous and homothetic, then the indifference curve has a cross point with each axis.

   Proof. Take $\forall x = (x_1, x_2) \in \mathbb{R}^2_+$, by continuity of $\succsim$, we have constructed a utility function $u(x) := t(x)$, where $(t(x), \cdots, t(x)) \sim x$.

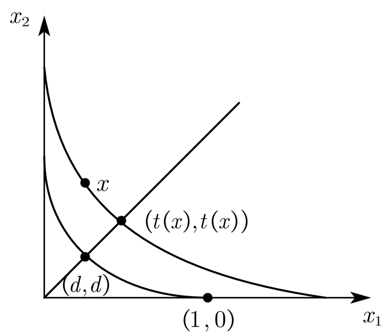

Fig. 2.9. Homotheticity.

   Therefore, by strict monotonicity,

$$(1, 1) \succ (0, 1) \succ (0, 0)$$

   and then $\exists d \in (0, 1)$, such that

$$(1, 0) \sim (d, d)$$

If $(x_1, x_2) \in \mathbb{R}^2_{++}$, then $t(x) > 0$. Then, by homotheticity, let $\lambda = \frac{t(x)}{d}$, then

$$(\lambda, 0) \sim (t(x), t(x)) \sim x$$

and likewise, we also can show that $\exists \lambda' > 0$ such that

$$(0, \lambda') \sim x$$

If $(x_1, x_2) = (0, 0)$, then itself is a cross point with the axis.

$\square$

Chapter 3

Basics of Consumer Demand

3.1 Set-ups and Facts

1. (Remark) Set-ups and notations.

   (a) The space of consumption bundles X is $\mathbb{R}_+^k$.

   (b) The income is y.

   (c) The consumer takes price $p = (p_1, \cdots, p_k) \in \mathbb{R}_{++}^k$ as given.

   (d) Budget set or budget-feasible consumption bundles is

   $$\mathbf{B}(p, y) = \left\{x \in \mathbb{R}_+^k : p \cdot x \leqslant y\right\}$$

   (e) Assume that consumer's preference $\succsim$ is complete, transitive and continuous, the there exists a continuous u that represents $\succsim$.

   (f) Consumer's Problem (CP):

   $$\begin{aligned} \max u(x) \\ \text{s.t. } p \cdot x \leqslant y \\ x \geqslant 0 \end{aligned}$$

2. (Theorem) Fix a continuous utility function u, and consider the CP with $p \gg 0, y \geqslant 0$, then

   (a) Fix $p \gg 0, y \geqslant 0$, if x is a solution of the CP, $\forall \lambda > 0$, x is also a solution for $(\lambda p, \lambda y)$.

   (b) $\forall p \gg 0, \forall y \geqslant 0$, the CP has at least one solution, i.e., $\exists x \in \mathbf{B}(p, y)$, x maximize $u(x)$ over $\mathbf{B}(p, y)$.

   (c) If u is quasi-concave (i.e., the preference $\succsim$ is convex), then $\forall p \gg 0, \forall y \geqslant 0$, the set of solutions to the CP is convex. Moreover, if u is strictly quasi-concave (i.e., the preference $\succsim$ is strictly convex), the CP has a unique solution.

   (d) Given $p \gg 0, y \geqslant 0$, if the preference $\succsim$ is locally insatiable and x is a solution to CP, then $p \cdot x = y$.

Proof. (a) $\mathbf{B}(\lambda p, \lambda y) = \mathbf{B}(p, y)$.

(b) If $\mathbf{B}(p, y) \subseteq \mathbb{R}^k_+$ is a nonempty compact set, then the continuous function has a solution on $\mathbf{B}(p, y)$.

- $\forall (p, y) \in \mathbb{R}^k_{++} \times \mathbb{R}_+$, $0 \in \mathbf{B}(p, y)$, therefore, $\mathbf{B}(p, y)$ is nonempty.
- $\forall \{x^n\} \subseteq \mathbf{B}(p, y)$ with $x^n \to x$, it holds that $x^n \cdot p \leqslant y$, by taking the limit, $x \cdot p = \lim_{n\to\infty} x^n \cdot p \leqslant y \Longrightarrow x \in \mathbf{B}(p, y)$. Therefore, $\mathbf{B}(p, y)$ is closed.
- $\forall (p, y) \in \mathbb{R}^k_{++} \times \mathbb{R}_+$.
  If $y = 0$, then $\mathbf{B}(p, y) = \{0\}$ is bounded.
  If $y > 0$, let $\delta = \frac{y}{\frac{1}{2}\min_{1\leqslant i\leqslant k} p_i}$, then $\forall x \in \mathbf{B}(p, y)$,

$$p \cdot x = p_1 x_1 + \cdots + p_k x_k \leqslant y \Longrightarrow \forall i, p_i x_i \leqslant y \Longrightarrow \forall i, x_i \leqslant \frac{y}{p_i} < \delta$$

  then

$$\mathbf{B}(p, y) \subseteq B_\delta(0)$$

  where $B_\delta(0)$ is bounded, therefore, $\mathbf{B}(p, y)$ is bounded.

(c) Firstly, $\mathbf{B}(p, y)$ is a convex set, since $\forall x_1, x_2 \in \mathbf{B}(p, y), \forall a \in [0, 1]$,

$$[ax_1 + (1-a)x_2] \cdot p \leqslant ay + (1-a)y = y$$

then $ax_1 + (1-a)x_2 \in \mathbf{B}(p, y)$.

Secondly, u is quasi-concave then $\succsim$ is quasi-convex, then the choice set $c_\succsim$ is convex. u is strictly quasi-concave then $\succsim$ is strictly quasi-convex, then the choice set $c_\succsim$ contains at most one point. By (b), CP always has a solution, then $c_\succsim$ contains only 1 element.

Moreover, we can show (c) without taking the convexity of $\succsim$ into account.

If $x_1, x_2 \in \mathbf{B}(p, y)$ both maximize u, let $u(x_1) = u(x_2) = u^*$ then $\forall a \in [0, 1]$

$$u^* \geqslant u(ax_1 + (1-a)x_2) \geqslant \min\{u(x_1), u(x_2)\} = u^*$$

then $ax_1 + (1-a)x_2 \in \mathbf{B}(p, y)$ also maximize u.

When u is strictly quasi-concave, then suppose $x_1, x_2 \in \mathbf{B}(p, y)$ with $x_1 \neq x_2$ both maximize u, then $\forall a \in (0, 1)$,

$$u(ax_1 + (1-a)x_2) > \min\{u(x_1), u(x_2)\} = u(x_1) = u(x_2)$$

contradicting with x_1, x_2 maximize u.

(d) Suppose $\succsim$ is locally insatiable and $x \in \underset{x\in\mathbf{B}(p,y)}{\arg\max}\ u(x)$ but $p \cdot x < y$.

Let $\varepsilon = \frac{y - p\cdot x}{\max_{1\leqslant i\leqslant k} p_i}$, by locally insatiability, $\exists x' \in B_\varepsilon(x)$ such that

$$x' \succ x$$

which implies $u(x') > u(x)$.

And $x' \in \mathbf{B}(p, y)$ since

$$\begin{aligned} x' \cdot p = \sum_{i=1}^{k} x_i' p_i < \sum_{i=1}^{k} (x_i + \varepsilon) p_i = p \cdot x + \varepsilon \sum_{i=1}^{k} p_i \\ \leqslant p \cdot x + \varepsilon \max_{1 \leqslant i \leqslant k} p_i = p \cdot x + y - p \cdot x = y \end{aligned}$$

which contradicts that x maximize u on $\mathbf{B}(p, y)$.

□

3.2 Marshallian Demand Correspondence and Indirect Utility Function

1. (Definition)

   (a) A mapping ϕ is called **correspondence** from X to Y, denoted by $\phi : X \rightrightarrows Y$ is defined as a function $\phi : X \to 2^Y$, $x \mapsto \phi(x)$,where $x \in X$,$\phi(x) \subseteq Y$.

   (b) The **graph** of the correspondence $\phi : X \rightrightarrows Y$, is defined as

   $$\{(x, y) \in X \times Y : y \in \phi(x)\}$$

   (c) Let $\phi : X \rightrightarrows Y$, we say

   - ϕ is **nonempty valued**, if $\forall x \in X, \phi(x) \neq \varnothing$.
   - ϕ is **convex valued**, if $\forall x \in X, \phi(x)$ is convex.
   - ϕ is **singleton valued**, if $\forall x \in X, |\phi(x)| = 1$.
   - ϕ is **closed valued**, if $\forall x \in X, \phi(x)$ is closed.
   - ϕ is **locally bounded** if $\forall x \in X$, $\exists \varepsilon > 0$, $\forall x' \in B_\varepsilon(x)$ there is a bounded subset $Y(x) \subseteq Y$ such that $\phi(x') \subseteq Y(x)$.

2. (Definition) Let $\phi : X \rightrightarrows Y$, we say

   (a) ϕ is **upper semi-continuous**, if $\forall \{x_n\}_{n=1}^{\infty} \in X^{\mathbb{N}}$ with $x_n \to x \in X$, and $\forall \{y_n\}_{n=1}^{\infty} \in Y^{\mathbb{N}}$, with $\forall n \geqslant 1, y_n \in \phi(x_n)$, and $\lim_{n\to\infty} y_n$ exists, then $\lim_{n\to\infty} y_n = y \in \phi(x)$.

   (b) ϕ is **lower semi-continuous**, if $\forall x \in X, \forall \{x_n\}_{n=1}^{\infty} \in X^{\mathbb{N}}$ with $x_n \to x \in X$, and $\forall y \in \phi(x)$, $\exists \{y_n\}_{n=1}^{\infty} \in Y^{\mathbb{N}}$ with $\forall n \geqslant 1, y_n \in \phi(x_n)$, such that $\lim_{n\to\infty} y_n = y$.

   (c) ϕ is **continuous**, if ϕ is both upper and lower semi-continuous.

3. (Theorem) The correspondence $\phi : X \rightrightarrows Y$ upper semi-continuous iff the graph of ϕ is (relatively) closed in $X \times Y$.

   Note: We say "relatively closed" since X itself maybe an open set.

4. (Example)

   Note that for a statement,

   $$\text{if } p, \text{then } q$$

   If p is always false, then the statement is always true.

   (a) $f(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$

      i. $f(x)$ is upper semi-continuous.
      Note that $\lim_{x \to 0} \frac{1}{x}$ does not exist in $\mathbb{R}$.

      ii. $f(x)$ is not lower semi-continuous, since

      $$x_n = \frac{1}{n} \to 0, y_n = f(x_n) \to \infty \neq f(0) = 0$$

   (b) $\phi(x) = \begin{cases} \{3\}, & x < 2 \\ [2,3], & 2 \leqslant x \leqslant 3 \\ \{2\}, & x > 3 \end{cases}$.

      i. $\phi(x)$ is upper semi-continuous.

      ii. $\phi(x)$ is not lower semi continuous, since $x_n = 2 - \frac{1}{n} \to 2$, $y_n = f(x_n) \to 3$, for $y \in \phi(2) \setminus \{3\}$, we cannot approach, for example $y = \frac{5}{2}$ by the sequence $\{x_n\}$.

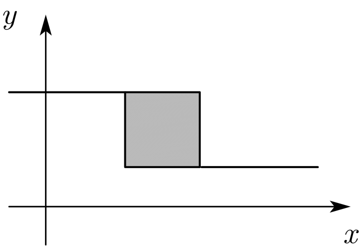

Fig. 3.1. Example (b).

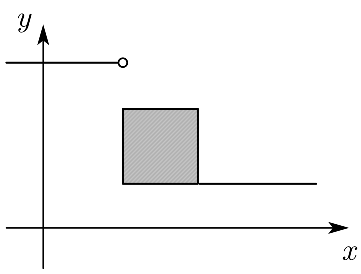

Fig. 3.2. Example (c).

   (c) $\phi(x) = \begin{cases} \{4\}, & x < 2 \\ [2,3], & 2 \leqslant x < 3 \\ \{2\}, & x > 3 \end{cases}$ is neither upper semi-continuous nor lower semi-continuous.

(d) $\phi(x) = \begin{cases} \{4\}, & x < 2 \\ \varnothing, & x = 2 \\ [2,3] & 2 < x < 3 \\ \{2\}, & x \geqslant 3 \end{cases}$

i. $\phi(x)$ is not upper semi-continuous.

ii. $\phi(x)$ is lower semi-continuous.

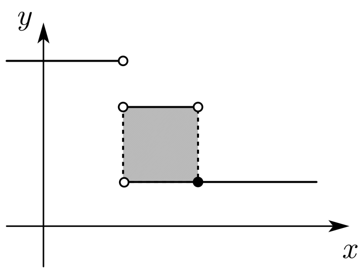

Fig. 3.3. Example (d).

5. (Theorem)

(a) A singleton-valued correspondence ϕ is lower semi-continuous iff it describes a continuous function, in which case it is also upper semi-continuous.

(b) A singleton-valued correspondence ϕ that is locally bounded is upper semi-continuous$\Longleftrightarrow\phi$ is lower semi-continuous $\Longleftrightarrow$ the function described by ϕ is continuous.

6. (Theorem, Berge's Theorem, the Theorem of the Maximum) Consider

$$\max_z \; F(z,\theta)$$
$$\text{s.t. } z \in A(\theta)$$

Let $Z(\theta)$ be the set of solution of this problem for the parameter θ, and let $f(\theta) = \sup\{F(z,\theta) : z \in A(\theta)\}$.

If

(a) F is continuous in (z,θ).

(b) $\theta \rightrightarrows A(\theta)$ is lower semi-continuous.

(c) $\forall\theta, \exists B(\theta) \subseteq A(\theta)$ such that $Z(\theta) \subseteq B(\theta)$, and

$$\sup_{z\in B(\theta)} F(z,\theta) = \sup_{z\in Z(\theta)} F(z,\theta)$$

and $\theta \rightrightarrows B(\theta)$ is upper semi-continuous and locally bounded.

Then

(a) $\forall\theta, Z(\theta) \neq \varnothing$ and $\theta \rightrightarrows Z(\theta)$ is upper semi-continuous and locally bounded.

(b) $\theta \to f(\theta)$ is continuous.

7. (Definition) Fixing u,

   (a) The set $\mathbf{D}(p,y) = \underset{x\in\mathbf{B}(p,y)}{\arg\max}\ u(x)$ is called **Marshallian demand** at price p and income y. And the correspondence $(p,y) \rightrightarrows \mathbf{D}(p,y)$ is called **Marshallian demand correspondence**.

   (b) The number $v(p,y) = \max_{x\in\mathbf{B}(p,y)} u(x)$ is called the **indirect utility** at price p and income y. And the function $(p,y) \mapsto v(p,y)$ is called indirect utility function.

8. (Remark) To apply Berge's Theorem, we may regard $\mathbf{B}(p,y)$ in CP as$B(\theta) = A(\theta) \subseteq A(\theta)$ in original Berge's Theorem.

   And let's restate the theorem in CP.

$$\max_x \ u(x)$$
$$\text{s.t. } x \in \mathbf{B}(p,y)$$

   Let $\mathbf{D}(p,y)$ be the set of solution of this problem for the parameter (p,y), and let $v(p,y) = \sup\{u(x) : x \in \mathbf{B}(p,y)\}$.

   If

   (a) u is continuous in x.

   (b) $(p,y) \rightrightarrows \mathbf{B}(p,y)$ is lower semi-continuous.

   (c) $(p,y) \rightrightarrows \mathbf{B}(p,y)$ is upper semi-continuous and locally bounded.

   Then

   (a) $\forall (p,y), \mathbf{D}(p,y) \neq \varnothing$ and $(p,y) \rightrightarrows \mathbf{D}(p,y)$ is upper semi-continuous and locally bounded.

   (b) $(p,y) \mapsto v(p,y)$ is continuous.

   Therefore, we only need to show $(p,y) \rightrightarrows \mathbf{B}(p,y)$ is a locally bounded correspondence, and a continuous correspondence.

9. (Theorem, Berge's Theorem applied to the consumer's problem)

   (a) $\mathbf{D}(p,y)$ is upper semi-continuous. And if for some open set in $\mathbb{R}^k_{++} \times \mathbb{R}_+$, $\mathbf{D}(p,y)$ is singleton valued, then $\mathbf{D}(p,y)$ is continuous.

   (b) $v(p,y)$ is continuous.

   Proof. 3 steps to prove it.

Step 1. Show that $(p, y) \rightrightarrows \mathbf{B}(p, y)$ is locally bounded.

$\forall (p, y) \in \mathbb{R}^k_{++} \times \mathbb{R}_+$, let $\varepsilon = \frac{1}{3}\min_{1 \leqslant i \leqslant k} p_i$, then $\forall (p', y') \in B_\varepsilon(p, y)$, for $x \in \mathbf{B}(p', y')$, it holds that $p' \cdot x \leqslant y' < y + \varepsilon$. Since $\forall i, p'_i x_i \geqslant 0$, then

$$p'_i x_i < y + \varepsilon$$

then $x_i < \frac{y+\varepsilon}{p'_i}$. Recall that

$$p'_i > p_i - \varepsilon \geqslant 3\varepsilon - \varepsilon = 2\varepsilon$$

then

$$0 \leqslant x_i < \frac{y+\varepsilon}{2\varepsilon}$$

Therefore, $\mathbf{B}(p', x') \subseteq \left[0, \frac{y+\varepsilon}{2\varepsilon}\right]^k \subseteq \mathbb{R}^k_+$, where $\left[0, \frac{y+\varepsilon}{2\varepsilon}\right]^k$ is a bounded set, and thus $(p, y) \rightrightarrows \mathbf{B}(p, y)$ is locally bounded.

Step 2. Show that $(p, y) \rightrightarrows \mathbf{B}(p, y)$ is continuous.

Step 2.1. Show upper-continuity.

We take any convergent sequence $\{(x^n, p^n, y^n)\}$ with $x^n \in \mathbf{B}(p^n, y^n)$ and with limit $(x^n, p^n, y^n) \to (x, p, y)$, since

$$x^n \cdot p^n \leqslant y^n$$

then

$$\lim_{n\to\infty}(x^n \cdot p^n) \leqslant \lim_{n\to\infty} y^n$$

and thus $x \cdot p \leqslant y \Longrightarrow x \in \mathbf{B}(p, y)$. Therefore, $(p, y) \rightrightarrows \mathbf{B}(p, y)$ is upper continuous.

Step 2.2. Show lower-continuity.

$\forall \{(p^n, y^n)\}$ with $(p^n, y^n) \to (p, y)$, and $\forall x \in \mathbf{B}(p, y)$, we need to construct a sequence $\{x^n\}$ such that

$$x^n \in \mathbf{B}(p^n, y^n), x^n \to x$$

If $y = 0$, since $p \gg 0$ then $x = 0$, therefore $\{x^n\} = \{0, 0, \cdots\}$ is such a sequence. If $y > 0$, and we can assume $x > 0$, let

$$x^n = \frac{y^n}{y}\frac{p \cdot x}{p^n \cdot x}x$$

then $x^n \cdot p^n = \frac{y^n}{y}\frac{p \cdot x}{p^n \cdot x}x \cdot p^n = \frac{y^n}{y}p \cdot x \leqslant \frac{y^n}{y}y = y^n$, then $x^n \in \mathbf{B}(p^n, y^n)$. And

$$\lim_{n\to\infty} x^n = \lim_{n\to\infty}\frac{y^n}{y}\frac{p \cdot x}{p^n \cdot x}x = \frac{\lim_{n\to\infty} y^n}{y}\frac{p \cdot x}{\lim_{n\to\infty} p^n \cdot x}x = x$$

therefore, there exists such a sequence, and thus $(p, y) \rightrightarrows \mathbf{B}(p, y)$ is lower continuous.

Step 3. By Berge's Theorem, and the equivalent relation between upper and lower semi-continuity provided that the correspondence is singleton valued, we can finish the proof.

□

3.3 Solving the CP

1. (Remark) Solving the CP.

$$\mathcal{L} = u(x) + \lambda(y - p \cdot x) + \sum_{\ell=1}^{k} \mu_\ell x_\ell$$

F.O.C.s

$$\begin{cases} \frac{\partial u}{\partial x_\ell} - \lambda p_\ell + \mu_\ell = 0 \\ \lambda(y - p \cdot x) = 0 \\ x_\ell \mu_\ell = 0 \\ \lambda \geqslant 0, \mu_\ell \geqslant 0 \end{cases}$$

If $x_\ell > 0 \Longrightarrow \mu_\ell = 0$, then

$$x_\ell > 0 \Longrightarrow \frac{\partial u}{\partial x_\ell} = \lambda p_\ell x_\ell = 0 \Longrightarrow \frac{\partial u}{\partial x_\ell} \leqslant \lambda p_\ell$$

Therefore, if $x_\ell, x_{\ell'} > 0$, then

$$\frac{u_\ell(x)}{p_\ell} = \lambda = \frac{u_{\ell'}(x)}{p_{\ell'}}$$

if $x_\ell > x_{\ell'} = 0$,

$$\frac{u_\ell(x)}{p_\ell} = \lambda \geqslant \frac{u_{\ell'}(x)}{p_{\ell'}}$$

2. (Example) Consumer chooses zero unit of one of the goods.

$$\begin{aligned} \max_{x_1, x_2} \; & u(x_1, x_2) = x_1 + \sqrt{x_2} \\ \text{s.t. } & x_1 p_1 + x_2 p_2 \leqslant y \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

Form the Lagrangian,

$$\mathcal{L}(x_1, x_2, \lambda) = x_1 + \sqrt{x_2} + \lambda(y - x_1 p_1 - x_2 p_2) + \mu_1 x_1 + \mu_2 x_2$$

F.O.C.s:

$$\begin{aligned} [x_1] &: 1 + \mu_1 = \lambda p_1 \\ [x_2] &: \frac{1}{2\sqrt{x_2}} + \mu_2 = \lambda p_2 \end{aligned}$$

and

$$\mu_1 x_1 = \mu_2 x_2 = 0, \mu_1 \geqslant 0, \mu_2 \geqslant 0$$
$$\lambda (y - x_1 p_1 - x_2 p_2) = 0, \lambda \geqslant 0$$
$$x_1 p_1 + x_2 p_2 \leqslant y$$

then

$$\lambda = \frac{1+\mu_1}{p_1} = \frac{1}{2p_2\sqrt{x_2}} + \frac{\mu_2}{p_2} > 0 \Longrightarrow y = x_1 p_1 + x_2 p_2$$

If $\mu_2 > 0 \Longrightarrow x_2 = 0$, then $x_1 = \frac{y}{p_1} > 0 \Longrightarrow \mu_1 = 0$, therefore the solution is

$$u\left(\frac{y}{p_1}, 0\right) = \frac{y}{p_1}$$

but this requires the budget line to be horizontal, which is precluded, since $p_1, p_2 > 0$. If $\mu_2 = 0 \Longrightarrow x_2 > 0$, if $\mu_1 > 0$, then $x_1 = 0$, therefore, the solution is

$$u\left(0, \frac{y}{p_2}\right) = \sqrt{\frac{y}{p_2}}$$

If $\mu_1 = 0$, then $x_1 > 0$, and thus

$$\frac{1}{p_1} = \frac{1}{2p_2\sqrt{x_2}} \Longrightarrow x_2 = \frac{p_1^2}{4p_2^2} > 0$$

Moreover,

$$x_1 = \frac{1}{p_1}\left(y - \frac{p_1^2}{4p_2}\right) = \frac{4yp_2 - p_1^2}{4p_1 p_2}$$

If $4yp_2 - p_1^2 = 0$, then $x_1 = 0$.

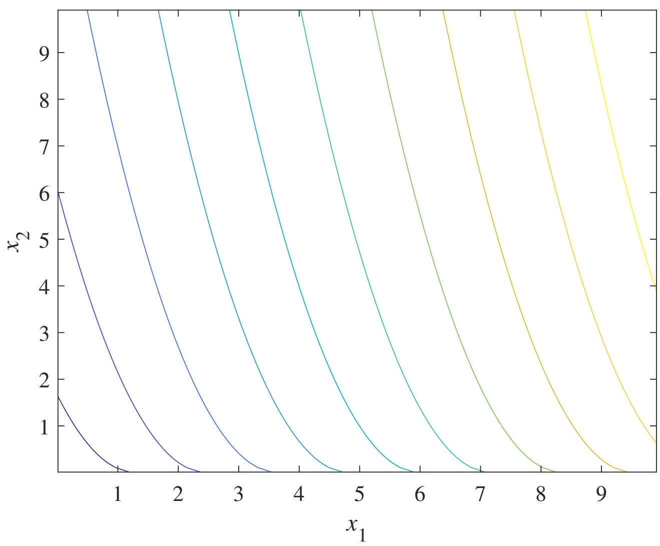

Fig. 3.4. Indifference Curve of $u(x_1, x_2) = x_1 + \sqrt{x_2}$.

3. (Example) $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$

   Form the Lagrangian,

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^{\alpha} x_2^{1-\alpha} + \lambda (y - x_1 p_1 - x_2 p_2) + \mu_1 x_1 + \mu_2 x_2$$

   F.O.C.s:

$$[x_1] : \alpha \left(\frac{x_2}{x_1}\right)^{1-\alpha} + \mu_1 = \lambda p_1$$
$$[x_2] : (1-\alpha) \left(\frac{x_1}{x_2}\right)^{\alpha} + \mu_2 = \lambda p_2$$

   and

$$\mu_1 x_1 = \mu_2 x_2 = 0, \mu_1 \geqslant 0, \mu_2 \geqslant 0$$
$$\lambda (y - x_1 p_1 - x_2 p_2) = 0, \lambda \geqslant 0$$
$$x_1 p_1 + x_2 p_2 \leqslant y$$

   If $x_1 = 0$ or $x_2 = 0$, then $u(x_1, x_2) = 0$, which is not optimal, since we can consumer a little more to get a higher utility which is greater than 0, then

$$x_1 > 0, x_2 > 0 \Longrightarrow \mu_1 = \mu_2 = 0$$

   and

$$\lambda = \frac{\alpha}{p_1} \left(\frac{x_2}{x_1}\right)^{1-\alpha} = \frac{1-\alpha}{p_2} \left(\frac{x_1}{x_2}\right)^{\alpha} > 0 \Longrightarrow x_1 p_1 + x_2 p_2 = y$$

   Therefore,

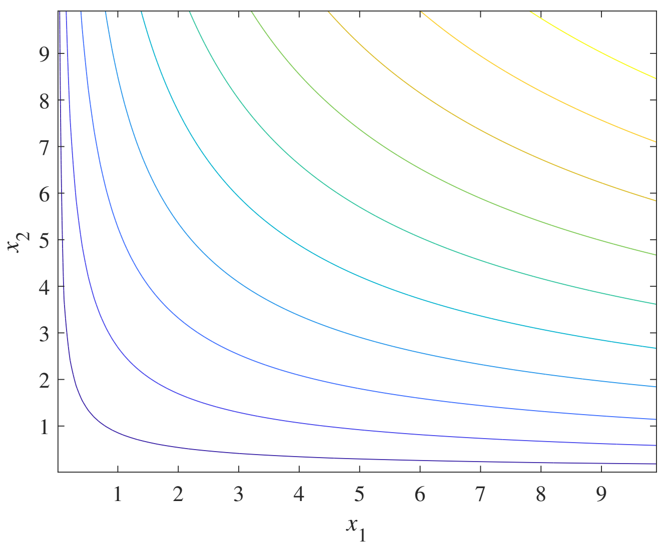

Fig. 3.5. Indifference Curve of $u(x_1, x_2) = x_1^{0.4} x_2^{0.6}$.

$$\frac{p_1}{p_2} = \frac{\alpha}{1-\alpha} \frac{x_2}{x_1} \Longrightarrow \frac{x_1 p_1}{x_2 p_2} = \frac{\alpha}{1-\alpha}$$

   then

$$x_1 = \alpha \frac{y}{p_1}, x_2 = (1-\alpha) \frac{y}{p_2}$$

4. (Example) $u(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$

   Form the Lagrangian,

$$\mathcal{L}(x_1, x_2, \lambda) = \sqrt{x_1} + \sqrt{x_2} + \lambda(y - x_1p_1 - x_2p_2) + \mu_1x_1 + \mu_2x_2$$

   F.O.C.s:

$$[x_1] : \frac{1}{2\sqrt{x_1}} + \mu_1 = \lambda p_1$$
$$[x_2] : \frac{1}{2\sqrt{x_2}} + \mu_2 = \lambda p_2$$

   and

$$\mu_1x_1 = \mu_2x_2 = 0, \mu_1 \geqslant 0, \mu_2 \geqslant 0$$
$$\lambda(y - x_1p_1 - x_2p_2) = 0, \lambda \geqslant 0$$
$$x_1p_1 + x_2p_2 \leqslant y$$

   If $\mu_1 > 0$ or $\mu_2 > 0$, both cases are precluded by $p_1, p_2 > 0$. If $\mu_1 = \mu_2 = 0$, then $x_1 > 0, x_2 > 0$, then

$$\lambda = \frac{1}{2p_2\sqrt{x_2}} = \frac{1}{2p_1\sqrt{x_1}} > 0 \Longrightarrow x_1p_1 + x_2p_2 \leqslant y$$

   and thus

$$\frac{p_2^2}{p_1^2} = \frac{x_1}{x_2} \Longrightarrow \frac{x_1p_1}{x_2p_2} = \frac{p_2}{p_1}$$

   Therefore, the solution is

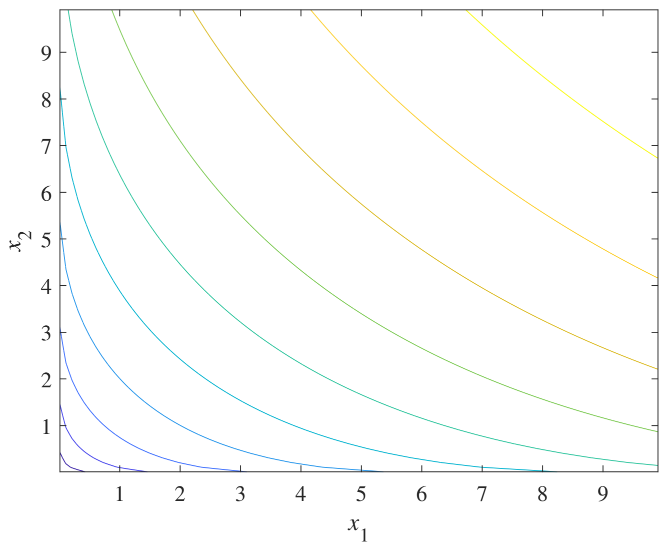

Fig. 3.6. Indifference Curve of $u(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$.

$$x_1 = \frac{p_2}{p_1}\frac{y}{p_1 + p_2}, x_2 = \frac{p_1}{p_2}\frac{y}{p_1 + p_2}$$

   and $u(x_1, x_2) = \sqrt{\frac{y}{p_1\left(1+\frac{p_1}{p_2}\right)}} + \sqrt{\frac{y}{p_2\left(1+\frac{p_2}{p_1}\right)}}$.

5. (Example) $u(x_1, x_2) = (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}}, 0 \neq \rho < 1$

   (a) $\lim_{\rho \to 1} u(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2$.

   (b) $\lim_{\rho \to 0} u(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$.

   (c) $\lim_{\rho \to \infty} u(x_1, x_2) = \min\{\alpha_1 x_1, \alpha_2 x_2\}$.

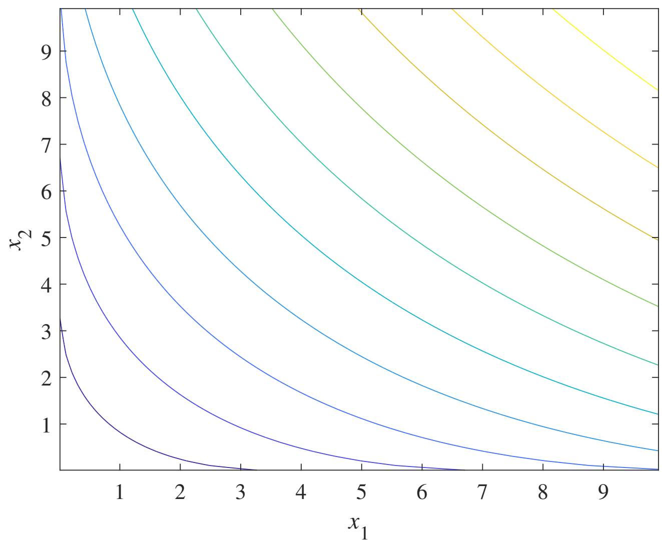

Fig. 3.7. Indifference Curve of $\rho = 0.5$.

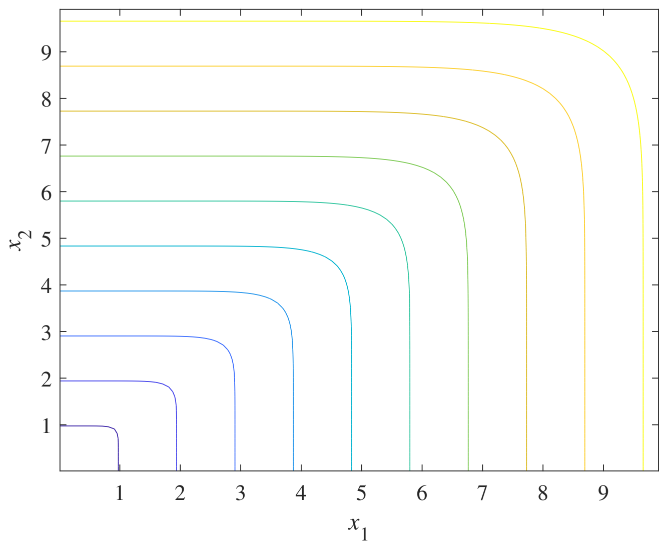

Fig. 3.8. Indifference Curve of $\rho = 10$.

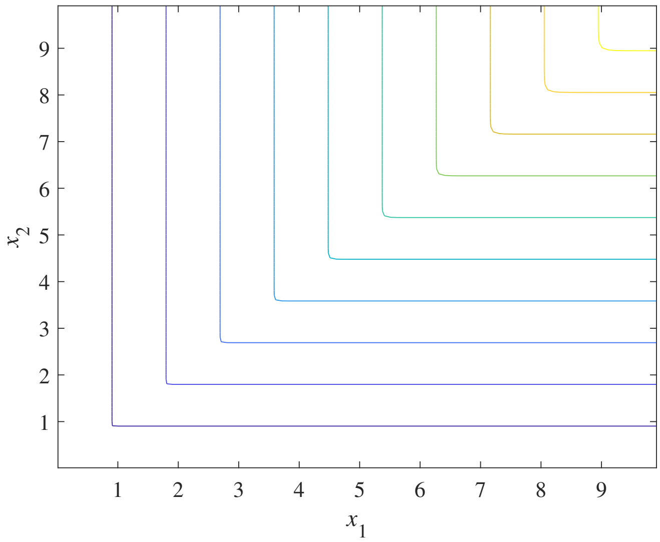

Fig. 3.9. Indifference Curve of $\rho = -100$.

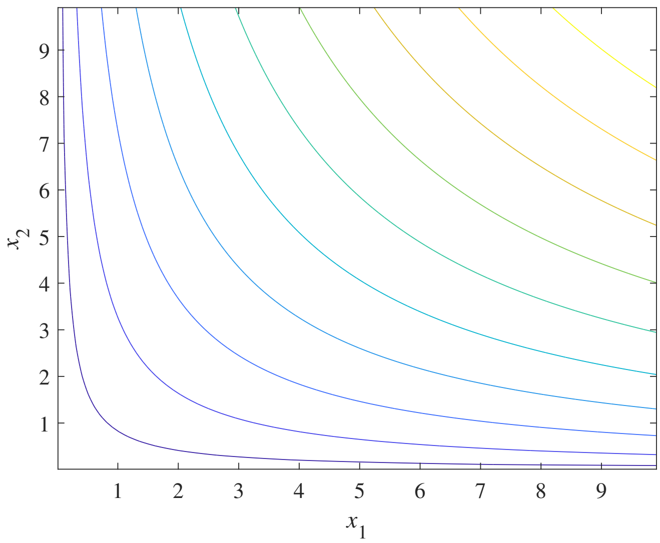

Fig. 3.10. Indifference Curve of $\rho = 0.01$.

Chapter 4

Revealed Preference and Afriat's Theorem

4.1 Afriat's Theorem

1. (Definition)

   (a) Take any **finite** set of **feasible** demand data:

   $$\begin{aligned} x^1 &\geqslant 0 \text{ chosen at } \left(p^1, y^1\right) \\ x^2 &\geqslant 0 \text{ chosen at } \left(p^2, y^2\right) \\ &\vdots \\ x^J &\geqslant 0 \text{ chosen at } \left(p^J, y^J\right) \end{aligned}$$

   where $p^j \cdot x^j \leqslant y^j, j = 1, \cdots, J$. If $p^i \cdot x^j \leqslant y^i$, then we say the data **reveal directly that x^i is weakly preferred to x^j**, written

   $$x^i \succsim^d x^j$$

   If $p^i \cdot x^j < y^i$, then we say the data **reveal directly that x^i is strictly preferred to x^j**, written

   $$x^i \succ^d x^j$$

   (b) For x^i, x^j, if $\exists x^{i_1}, \cdots, x^{i_m}$, with

   $$x^i \succsim^d x^{i_1} \succsim^d \cdots \succsim^d x^{i_m} \succsim^d x^j$$

   then we say the data **indirectly reveal that x^i is weakly preferred to x^j**, written

   $$x^i \succsim^r x^j$$

   Moreover, if $\exists \{x^{i_k}, x^{i_{k+1}}\} \subseteq \{x^{i_1}, \cdots, x^{i_m}\}$, with

   $$x^{i_k} \succ^d x^{i_{k+1}}$$

i.e.,

$$x^i \succsim^d x^{i_1} \succsim^d \cdots x^{i_k} \succ^d x^{i_{k+1}} \cdots \succsim^d x^{i_m} \succsim^d x^j$$

then we say the data **indirectly reveal that x^i is strictly preferred to x^j**, written

$$x^i \succ^r x^j$$

(c) The data **satisfies Generalized Axiom of Revealed Preference (GARP)** if no strict revealed preference cycles exist, i.e., there is no x^i such that

$$x^i \succ x^i$$

2. (Theorem) Suppose a consumer with complete, transitive, and locally insatiable preference $\succsim$ chooses the consumption bundle x^* at (p, y), then we know that

   (a) $\forall x \in \mathbf{B}(p, y)$ with $p \cdot x = y$, $x^* \succsim x$.

   (b) $\forall x \in \mathbf{B}(p, y)$ with $p \cdot x < y$, $x^* \succ x$.

Proof. (a) Since the consumer choose x^*, then $\forall x \in \mathbf{B}(p, y)$, $x^* \succsim x$.

(b) If $p \cdot x < y$, then by locally insatiability, $\forall \varepsilon > 0, \exists x' \in B_\varepsilon(x)$, such that

$$x' \succ x$$

let $\varepsilon = \frac{y - p \cdot x}{\max_{1 \leqslant i \leqslant k} p_k}$, then

$$p \cdot x' = \sum_{i=1}^{k} p_i x_i' < \sum_{i=1}^{k} p_i (x_i + \varepsilon) = p \cdot x + \varepsilon \sum_{i=1}^{k} p_i \leqslant p \cdot x + (y - p \cdot x) = y$$

i.e., $x' \in \mathbf{B}(p, y)$, then

$$x^* \succsim x' \succ x \Longrightarrow x^* \succ x$$

□

3. (Theorem, Afriat's Theorem)

   (a) If a finite set of demand data violates GARP, then these data are inconsistent with choice according to locally insatiable, complete and transitive preference.

   (b) If a finite set of demand data satisfies GARP, then these data are consistent with choice according to complete, transitive, strictly increasing (and thus locally insatiable), continuous and convex preference.

Note: The preference is referred to $\succsim$ rather than $\succsim^r$ or $\succsim^d$.

Proof. Omitted. □

4.2 Slutsky Compensation

1. (Remark) Comparative statics: income

   (a) Inferior: $y \uparrow \Longrightarrow x_i \downarrow$.

   (b) Superior: $y \uparrow \Longrightarrow x_i \uparrow$.

   (c) Luxury: $y \uparrow \Longrightarrow \frac{p_i x_i}{y} \uparrow$.

   (d) Necessities: $y \uparrow \Longrightarrow \frac{p_i x_i}{y} \downarrow$.

2. (Remark) Comparative statics: own-price effects.

   (a) Normal goods: $p_i \uparrow \Longrightarrow x_i \downarrow$.

   (b) Giffen goods: $p_i \uparrow \Longrightarrow x_i \uparrow$.

3. (Theorem, Slutsky Compensation 1) Suppose x is chosen by the consumer at (p, y), and x' is chosen at prices $(p', y') = (p', p' \cdot x)$, where p' is p except for an increase in the price of good i. If these choices are made according to standard model with locally insatiable preference the $x_i' \leqslant x_i$.

   Notes: This is a kind of Law of Demand, but watch out that the income also changes.

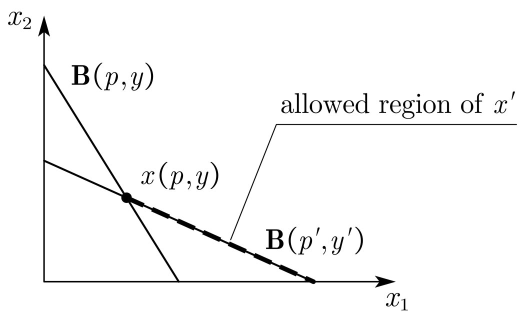

Fig. 4.1. Slutsky Compensation.

Proof. Suppose $x_i' > x_i$, then

$$x \cdot p \leqslant x \cdot p' = y'$$

i.e., $x \in \mathbf{B}(p', y')$, but consumer chooses x', then

$$x' \succsim^d x$$

Since $\succsim$ is locally insatiable, $y' = p' \cdot x' = p' \cdot x$, then

$$p' \cdot x' = \sum_{j \neq i} p_j' x_j' + p_i' x_i' = \sum_{j \neq i} p_j' x_j + p_i' x_i = p' \cdot x$$

and since $\forall j \neq i, p_j' = p_j$,

$$\sum_{j \neq i} p_j' x_j' + p_i' x_i' = \sum_{j \neq i} p_j x_j + p_i' x_i$$

By that $p_i' > p_i$ and assumption $x_i' > x_i$,

$$(p_i' - p_i)\, x_i' > (p_i' - p_i)\, x_i$$

Therefore,

$$(p_i' - p_i)\, x_i' - \sum_{j \neq i} p_j' x_j' - p_i' x_i' > (p_i' - p_i)\, x_i - \sum_{j \neq i} p_j x_j - p_i' x_i$$

which yields

$$p \cdot x' < p \cdot x$$

By local non-satiability,

$$x \succ^d x'$$

a contradiction to $x' \succsim^d x$. $\square$

4. (Theorem) Suppose i is a Giffen good for some preference maximizing consumer with locally insatiable preferences. Then good i must be inferior for this consumer.

   In other words, If $\exists\,(y, p) \in \mathbb{R}_+ \times \mathbb{R}^k_{++}$ and p' is identical to p except that $p_i < p_i'$, let $x \in \underset{x \in \mathbf{B}(p,y)}{\arg\max}\ u(x)$, $x' \in \underset{x \in \mathbf{B}(p',y)}{\arg\max}\ u(x)$, then$x_i < x_i'$, and let $y' = p \cdot x' < y$, if $x'' \in \underset{x \in \mathbf{B}(p,y')}{\arg\max}\ u(x)$, then $x_i'' > x_i$.

   Note: Giffen goods must be inferior.

   Proof. To show $x_i'' > x_i$, we only need to show that

   $$x_i'' \geqslant x_i' > x_i$$

   Since x_i'' is chosen at $(p, y') = (p, p \cdot x')$, x' is chosen at (p', y), p' is identical to p except that $p_i < p_i'$, then by the Slutsky compensation, we have

   $$x_i' \leqslant x_i''$$

   $\square$

5. (Theorem, Slutsky Compensation 2) For a consumer with locally insatiable, complete, and transitive preferences, suppose that x is chosen at (p, y), and x' is chosen at $(p', p' \cdot x)$, for another price p'. Then

   $$(p' - p) \cdot (x' - x) \leqslant 0$$

   Proof. By local non-satiability,

   $$p \cdot x = y, p' \cdot x' = p' \cdot x$$

   Since $x \in \mathbf{B}\,(p', p' \cdot x)$ then

   $$x' \succsim^d x$$

and $p \cdot x' \geqslant y = p \cdot x$, otherwise, suppose $p \cdot x' < y = p \cdot x$, then

$$x \succ^d x'$$

which violates transitivity. Therefore, by

$$\begin{cases} p \cdot x' \geqslant p \cdot x \\ p' \cdot x' = p' \cdot x \end{cases}$$

we have

$$(p' - p) \cdot (x' - x) = p' \cdot x' - p' \cdot x - p \cdot x' + p \cdot x = -p \cdot x' + p \cdot x \leqslant 0$$

□

Chapter 5

Choice under Uncertainty (MWG)

5.1 Lottery Space and Expected Utility

1. (Definition) For finite set X of prices (consequences), we index prices by $n = 1, \cdots, N$.

   (a) A **simple lottery** is a probability distribution $L = (\pi_1, \cdots, \pi_N)$, with $\forall n = 1, \cdots, N$, $\pi_n \in [0, 1]$, and $\sum_{n=1}^{N} \pi_n = 1$.

   (b) Set of all lotteries is called the **lottery space**,

   $$\Pi = \left\{\pi \in \mathbb{R}_+^N : \pi_1 + \cdots + \pi_N = 1\right\}$$

   which is a $N-1$ dimensional simplex.

   (c) Given K simple lotteries $L_K = \left(\pi_1^k, \cdots, \pi_N^k\right), k = 1, \cdots, K$, and a probability distribution $\alpha_k \geqslant 0, \sum_{k=1}^{K} \alpha_k = 1$, the compound lottery is the risky alternative that yields the simple lottery L_k with probability α_k, denoted by $(L_1, \cdots, L_K; \alpha_1, \cdots, \alpha_K)$.

   (d) Naturally, we can write every **compound lottery** $(L_1, \cdots, L_K; \alpha_1, \cdots, \alpha_K)$ to the form of simple lottery $L = (\pi_1, \cdots, \pi_N)$, where

   $$\pi_n = \alpha_1 \pi_n^1 + \cdots + \alpha_K \pi_n^K$$

   we call it **reduced lottery**. We also can write

   $$L = \alpha_1 L_1 + \cdots + \alpha_K L_K$$

2. (Remark)

   (a) Consumer does not choose consequences directly, but instead chooses lotteries.

   (b) The decision maker has a rational preference relation $\succsim$ on Π.

   (c) If we assume that $\succsim$ is rational and continuous, then $\succsim$ can be represented by a continuous utility function $U : \Pi \to \mathbb{R}$, i.e., $\forall \rho, \pi \in \Pi$,

   $$\pi \succsim \rho \Longleftrightarrow U(\pi) \geqslant U(\rho)$$

(d) The lottery space $\Pi = \left\{\pi \in \mathbb{R}_+^N : \pi_1 + \cdots + \pi_N = 1\right\}$ is convex, closed and bounded, and thus, Π is a convex compact set.

3. (Remark)

(a) We have established a utility function $U : \Pi(X) \to \mathbb{R}$.

(b) But we want to know more about the "utility" of outcomes X, so we need to consider "preferences" over outcomes.

4. (Axiom) von Neumann-Morgenstern Axioms.

(a) $\succsim$ is **complete** and **transitive**.

(b) $\succsim$ satisfies **independence**, that is, $\forall L, L', L'' \in \Pi, \forall \alpha \in (0,1)$,

$$L \succsim L' \Longleftrightarrow \alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''$$

(c) $\succsim$ satisfies **continuity**, that is $\forall L, L', L'' \in \Pi$, the following two sets are closed.

$$\{\alpha \in [0,1] : \alpha L + (1-\alpha) L' \succsim L''\} \subseteq [0,1]$$

$$\{\alpha \in [0,1] : L'' \succsim \alpha L + (1-\alpha) L'\} \subseteq [0,1]$$

Note: the continuity ensures that $\exists \alpha^* \in [0,1], \alpha^* L + (1-\alpha^*) L' \sim L''$.

5. (Theorem) $\succsim$ satisfies the von Neumann-Morgenstern Axioms, then it satisfies monotonicity, that is if $L \succ L'$, $p, q \in (0,1)$ then

$$pL + (1-p) L' \succsim qL + (1-q) L' \Longleftrightarrow p \geqslant q$$

Proof. **On the one hand**, if $p \geqslant q$, we have

$$pL + (1-p) L' \sim qL + (p-q) L + (1-p) L' qL + (1-q) L' \sim qL + (p-q) L' + (1-p) L'$$

since they have the same reduced lotteries. Then, by $L \succ L'$ and the independence axiom,

$$(p-q) L + (1-p) L' \succsim (p-q) L' + (1-p) L'$$

then by transitivity,

$$pL + (1-p) L' \succsim qL + (1-q) L'$$

On the other hand, suppose $p < q$, then likewise, by the independence axiom, it holds that

$$qL + (1-q) L' \succ pL + (1-p) L'$$

by contrapositive, we finish the proof. $\square$

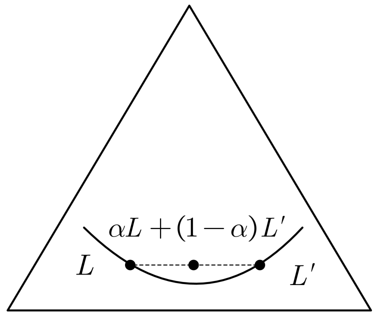

Fig. 5.1. Straight Indifference Curve.

6. (Example) The indifference curves on for $U : \Pi(X) \to \mathbb{R}$ are straight and parallel.

   Take $\Pi = \left\{\pi \in \mathbb{R}_+^3 : \pi_1 + \pi_2 + \pi_3 = 1\right\}$.

   If the indifference curves are not straight, then $\exists L, L' \in \Pi$, with $L \sim L'$, and $\exists \alpha \in (0,1)$ such that $\alpha L+(1-\alpha) L'$ is not indifferent with L or L', W.L.G., assume $\alpha L+(1-\alpha) L' \succ L \sim L'$.

   Since

$$L \sim \alpha L + (1-\alpha) L$$

   then

$$\alpha L + (1-\alpha) L' \succ \alpha L + (1-\alpha) L$$

   by the independence axiom, we have

$$L' \succ L$$

   A contradiction.

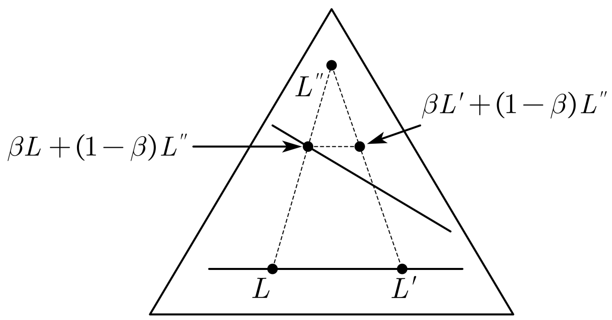

Fig. 5.2. Parallel Indifference Curve.

For $\forall L, L', L'' \in \Pi$, with $L'' \succ L' \sim L$, and $\forall \beta \in (0,1)$, by the independence axiom

$$\beta L + (1-\beta) L'' \sim \beta L' + (1-\beta) L''$$

and thus, the indifference curves are parallel.

7. (Definition) The utility function $U : \Pi \to \mathbb{R}$ has an **expected utility form** if there is an assignment of numbers $u = (u_1, \cdots, u_N)$ to the N outcomes such that $\forall L \in \Pi$, we have

$$U(L) = u \cdot L = u_1\pi_1 + \cdots + u_N\pi_N = \sum_{n=1}^{N} u_n\pi_n$$

A utility function $U : \Pi \to \mathbb{R}$ with the expected utility form is called a **von-Neumann-Morgenstern (v.N-M) expected utility function**.

8. (Theorem) A utility function $U : \Pi \to \mathbb{R}$ has an expected utility form iff it is linear, that is, it has the form, $\forall L_k \in \Pi, k = 1, \cdots, K$ and any probability distribution $(\alpha_1, \cdots, \alpha_K)$,

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k)$$

Proof. (**If**) Let $L^1, \cdots, L^N$ be the canonical basis of $\mathbb{R}^N$, then $\forall L_i \in \Pi$,

$$L = (\pi_1, \cdots, \pi_N) = \sum_{n=1}^{N} \pi_n L^n$$

then, let $u_n = U(L^n)$, and thus,

$$U(L) = U\left(\sum_{n=1}^{N} \pi_n L^n\right) = \sum_{n=1}^{N} \pi_n U(L^n) = \sum_{n=1}^{N} \pi_n u_n$$

(**Only if**) For any compound lottery $(L_1, \cdots, L_K; \alpha_1, \cdots, \alpha_K)$, its reduced lottery is $L' = \sum_{k=1}^{K} L_k\alpha_k$, then

$$\begin{aligned} U(L') &= U\left(\sum_{k=1}^{K} L_k\alpha_k\right) = U\left(\sum_{k=1}^{K}\sum_{n=1}^{N} L^n\pi_n^k\alpha_k\right) \\ &= \sum_{k=1}^{K} \alpha_k \sum_{n=1}^{N} \pi_n^k u_n = \sum_{k=1}^{K} \alpha_k U(L_k) \end{aligned}$$

$\square$

Note: Mathematically, we can regard L as a vector, and U is a function from $\mathbb{R}_+^N$ to $\mathbb{R}$. And for linearity, we can only show that $\forall \alpha, \beta \in [0,1], \alpha + \beta = 1$, $\forall L_1, L_2 \in \Pi$, $U(\alpha L_1 + \beta L_2) = \alpha U(L_1) + \beta U(L_2)$, since by induction, the proof can be simply extended to K.

9. (Theorem) Suppose $U : \Pi \to \mathbb{R}$ is a v.N-M expected utility for the preference relation $\succsim$ on Π. Then $\tilde{U} : \Pi \to \mathbb{R}$ is another v.N-M utility function for $\succsim$ iff $\exists \beta > 0, \gamma \in \mathbb{R}$, such that $\forall L \in \Pi$, $\tilde{U}(L) = \beta U(L) + \gamma$.

Proof. Firstly, on the compact and convex set Π, we can find the maximizer $\bar{L}$ and minimizer L. Then $\forall L \in \Pi$, $\bar{L} \succsim L \succsim L$.

If $\bar{L} \sim L$, then $U(L)$ is a constant, the proof is self-evident.

Now we show the theorem when $\bar{L} \succ L$.

(**If**) Since $U(L)$ is a v.N-M utility function, then take any compound lottery L, it holds that

$$\begin{aligned}\tilde{U}(L) &= \beta U(L) + \gamma = \beta U\left(\sum_{k=1}^{K} \alpha_k L_k\right) + \gamma \\ &= \beta \sum_{k=1}^{K} \alpha_k U(L_k) + \gamma = \sum_{k=1}^{K} \alpha_k \left[\beta U(L_k) + \gamma\right] = \sum_{k=1}^{K} \alpha_k \tilde{U}(L_k)\end{aligned}$$

Note that $\sum_{k=1}^{K} \alpha_k = 1$, so we can take γ into the summation.

(**Only if**) We want to find β, γ to satisfy $\tilde{U}(L) = \beta U(L) + \gamma$.

For any compound lottery $L \in \Pi$, define $\lambda_L \in [0,1]$, such that

$$U(L) = \lambda_L U\left(\bar{L}\right) + (1-\lambda_L) U(L)$$

then

$$\lambda_L = \frac{U(L) - U(L)}{U\left(\bar{L}\right) - U(L)}$$

Since $U(L)$ is a v.N-M utility function (i.e., takes the expected utility form), then

$$\lambda_L U\left(\bar{L}\right) + (1-\lambda_L) U(L) = U\left(\lambda_L \bar{L} + (1-\lambda_L) L\right)$$

and thus,

$$U(L) = U\left(\lambda_L \bar{L} + (1-\lambda_L) L\right) \Longleftrightarrow L \sim \lambda_L \bar{L} + (1-\lambda_L) L$$

Since $\tilde{U}$ also represents $\succsim$ and is also a v.N-M utility function, then

$$\begin{aligned}\tilde{U}(L) &= \tilde{U}\left(\lambda_L \bar{L} + (1-\lambda_L) L\right) \\ &= \lambda_L \tilde{U}\left(\bar{L}\right) + (1-\lambda_L) \tilde{U}(L) \\ &= \lambda_L \left(\tilde{U}\left(\bar{L}\right) - \tilde{U}(L)\right) + \tilde{U}(L)\end{aligned}$$

Plug $\lambda_L = \frac{U(L)-U(L)}{U\left(\bar{L}\right)-U(L)}$ in,

$$\begin{aligned}\tilde{U}(L) &= \frac{U(L) - U(L)}{U\left(\bar{L}\right) - U(L)} \left(\tilde{U}\left(\bar{L}\right) - \tilde{U}(L)\right) + \tilde{U}(L) \\ &= \frac{\tilde{U}\left(\bar{L}\right) - \tilde{U}(L)}{U\left(\bar{L}\right) - U(L)} \left(U(L) - U(L)\right) + \tilde{U}(L)\end{aligned}$$

then

$$\begin{aligned}\beta &= \frac{\tilde{U}\left(\bar{L}\right) - \tilde{U}(L)}{U\left(\bar{L}\right) - U(L)} \\ \gamma &= -\frac{\tilde{U}\left(\bar{L}\right) - \tilde{U}(L)}{U\left(\bar{L}\right) - U(L)} U(L) + \tilde{U}(L)\end{aligned}$$

□

10. (Remark) Let $U(L) = \sum_{n=1}^{N} \pi_n u_n$ represents $\succsim$ on Π.

   (a) $U(L)$ is linear, but u can be nonlinear.

   (b) Any strictly increasing transformation of $U(L)$ still represents $\succsim$, but only linear transformation of $U(L)$ takes the expected utility form.

11. (Remark) Cardinal property and Ordinal property.

   (a) The expected utility property is a cardinal property of utility functions defined on the space of lotteries: Increasing linear transformation does not change the form of the expected utility function.

   (b) Ordinal property of utility function $(e^{U(\cdot)}, \ln U(\cdot))$: Increasing monotone transformation does not change the characteristic of the utility function.

   (c) For example, the statement "the difference in utility between outcomes 1 and 2 is greater than the difference between outcomes 3 and 4" has some meaning. Since

$$u_1 - u_2 > u_3 - u_4 \Longrightarrow \frac{1}{2}u_1 + \frac{1}{2}u_4 > \frac{1}{2}u_3 + \frac{1}{2}u_2$$

   Then let $L = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right), L' = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$.

$$U(L) > U(L')$$

12. (Theorem, Expected Utility Theorem) Given a complete and transitive preference relation $\succsim$ on the lottery space Π. Then $\succsim$ is continuous and satisfies the independence axiom if and only if $\succsim$ admits a utility representation on the expected utility form.

   That is, we can assign a number u_n to each $n = 1, \cdots, N$, such that $\forall L, L' \in \Pi$, note $L = (\pi_1, \cdots, \pi_N), L' = (\pi'_1, \cdots, \pi'_N)$, we have

$$L \succsim L' \Longleftrightarrow \sum_{n=1}^{N} u_n \pi_N \geqslant \sum_{n=1}^{N} u_n \pi'_N$$

   Proof of the only if part. Since Π is compact, then $\exists L, \bar{L} \in \Pi$, such that $\forall L \in \Pi$, $\bar{L} \succ L \succ L$. If $\bar{L} \sim L$, then we can just assign $u_n = 0, \forall n = 1, \cdots, N$. Now suppose $\bar{L} \succ L$.

   Step 1. Show that if $L \succ L', \alpha \in (0,1)$, then $L \succ \alpha L + (1-\alpha) L' \succ L'$.

   By independence axiom,

$$L \sim \alpha L + (1-\alpha) L \succ \alpha L + (1-\alpha) L' \succ \alpha L' + (1-\alpha) L' \sim L'$$

   Step 2. Show that $\alpha, \beta \in (0,1)$, $\beta \bar{L} + (1-\beta) L \succ \alpha \bar{L} + (1-\alpha) L$ iff $\beta > \alpha$. (Just by monotonicity, now we show it again)

(**If**) If $\beta > \alpha$, define γ such that

$$\beta\bar{L} + (1-\beta)\,L = \gamma\bar{L} + (1-\gamma)\left[\alpha\bar{L} + (1-\alpha)\,L\right] \Longrightarrow \gamma = \frac{\beta-\alpha}{1-\alpha} \in (0,1)$$

By step 1,

$$\bar{L} \succ \alpha\bar{L} + (1-\alpha)\,L$$

and by step 1 again,

$$\gamma\bar{L} + (1-\gamma)\left[\alpha\bar{L} + (1-\alpha)\,L\right] \succ \alpha\bar{L} + (1-\alpha)\,L$$

therefore,

$$\beta\bar{L} + (1-\beta)\,L \succ \alpha\bar{L} + (1-\alpha)\,L$$

(**Only if**) Suppose $\beta \leqslant \alpha$. If $\beta = \alpha$, then

$$\beta\bar{L} + (1-\beta)\,L \sim \alpha\bar{L} + (1-\alpha)\,L$$

If $\beta < \alpha$, then

$$\alpha\bar{L} + (1-\alpha)\,L \succ \beta\bar{L} + (1-\beta)\,L$$

Step 3. Show that $\forall L \in \Pi$, $\exists \alpha_L$ which is unique, such that $$\alpha_L\bar{L} + (1-\alpha_L)\,L \sim L$$

Since $\succsim$ is continuous, then

$$\left\{\alpha \in [0,1] : \alpha\bar{L} + (1-\alpha)\,L \succsim L\right\} \subseteq [0,1]$$

is a closed set, then such a α_L exist, just let

$$\alpha_L = \sup\left\{\alpha \in [0,1] : \alpha\bar{L} + (1-\alpha)\,L \succsim L\right\}$$

And uniqueness is because of step 2. If $\alpha \neq \beta$, then $\beta\bar{L} + (1-\beta)\,L$ and $\alpha\bar{L} + (1-\alpha)\,L$ cannot be indifferent.

Step 4. Define the utility function.

Define $U : \Pi \to \mathbb{R}$ as $U : L \mapsto \alpha_L$. Now show that U represents $\succsim$. By step 3, $\forall L, L' \in \Pi$, $\exists \alpha_L, \alpha_{L'}$ such that

$$\alpha_L\bar{L} + (1-\alpha_L)\,L \sim L\alpha_{L'}\bar{L} + (1-\alpha_{L'})\,L \sim L'$$

then

$$L \succsim L' \Longleftrightarrow \alpha_L\bar{L} + (1-\alpha_L)\,L \succsim \alpha_{L'}\bar{L} + (1-\alpha_{L'})\,L$$

By contrapositive statement of step 2, then

$$\alpha_L\bar{L} + (1-\alpha_L)\,L \succsim \alpha_{L'}\bar{L} + (1-\alpha_{L'})\,L \Longleftrightarrow \alpha_L \geqslant \alpha_{L'}$$

Step 5. Show that U is linear and has the expected utility form. (Only show that for any two lotteries, for any K lotteries, the proof is similar.)

$\forall L, L' \in \Pi$, and $\forall \beta \in [0,1]$, we need to show that

$$U\left(\beta L + (1-\beta) L'\right) = \beta U(L) + (1-\beta) U(L')$$

By definition, we have

$$L \sim U(L)\bar{L} + (1 - U(L)) L L' \sim U(L')\bar{L} + (1 - U(L')) L$$

Therefore, by independence axiom,

$$\begin{aligned}\beta L + (1-\beta) L' &\sim \beta \left(U(L)\bar{L} + (1 - U(L)) L\right) + (1-\beta)\left(U(L')\bar{L} + (1-U(L')) L\right) \\ &= \left[\beta U(L) + (1-\beta) U(L')\right]\bar{L} + \left(\beta - \beta U(L) + (1-\beta)(1 - U(L'))\right) L \\ &= \left[\beta U(L) + (1-\beta) U(L')\right]\bar{L} + \left(1 - \beta U(L) - (1-\beta) U(L')\right) L\end{aligned}$$

then by definition,

$$U\left(\beta L + (1-\beta) L'\right) = \beta U(L) + (1-\beta) U(L')$$

Therefore, we have established a utility function U such that U represents $\succsim$ and takes v.N-M utility form.

□

13. (Example, Allais Paradox) There are 3 possible monetary prizes,

First Prize	Second Prize	Third Prize
\$25M	\$5M	\$0

Consider 2 pairs of lotteries:

$$\begin{cases} L_1 = (0,1,0) \\ L_1' = (0.10, 0.89, 0.01) \end{cases}, \begin{cases} L_2 = (0, 0.11, 0.89) \\ L_2' = (0.10, 0, 0.90) \end{cases}$$

Or put it more directly,

	0.89	0.01	0.10
L_1	5	5	5
L_1'	5	0	25
L_2	0	5	5
L_2'	0	0	25

Most people will choose

$$L_1 \succ L_1', L_2' \succ L_2$$

But the independent axiom implies

$$L_1 \succ L_1' \Longleftrightarrow L_2 \succ L_2'$$

Discussion. If the independent axiom holds, then we can construct a v.N-M utility function U, and

$$L_1 \succ L_1' \Longleftrightarrow u_5 > 0.1u_{25} + 0.89u_5 + 0.01u_0$$

adding $0.89u_0 - 0.89u_5$ on both hand sides, then

$$0.11u_5 + 0.89u_0 > 0.1u_{25} + 0.90u_0$$

which implies

$$L_2 \succ L_2'$$

□

14. (Axiom, Savage's Sure Thing Principle, from Kreps 5.1) Suppose a, a', b and b' are 4 acts, and $T \subseteq S$ is a subset of the state space, such that,

$$a(s) = a'(s), b(s) = b'(s), \forall s \in T$$

$$a(s) = b(s), a'(s) = b'(s), \forall s \in T^c$$

Then

$$a \succsim b \Longleftrightarrow a' \succsim b'$$

Note: It is another version of the independence axiom.

	T	T^c
a	$a = a'$	$a = b$
b	$b = b'$	$a = b$
a'	$a' = a$	$a' = b'$
b'	$b' = b$	$a' = b'$

When decide the preference relation between a and b (or, a' and b'), the information in T^c is useless.

15. (Example, Ellsberg Paradox) There are 90 balls, 30 are Yellow, 60 are Red or Green.

Here are two pairs of lotteries.

(a) L_1: get 100 if draw a Yellow.
L_1': get 100 if draw a Red.

(b) L_2: get 100 if draw a Yellow or Green.
L_2': get 100 if draw a Red or Green.

most people will choose

$$L_1' \succ L_1, L_2' \succ L_1$$

and we can get a table.

	Yellow	Red	Green
L_1	100	0	0
L_1'	0	100	0
L_2	100	0	100
L_2'	0	100	100

By Sure thing principle, the state Green is useless when make a choice between L_1 and L_1', so does L_2 and L_2', therefore,

$$L_1 \succ L_1' \Longleftrightarrow L_2 \succ L_2'$$

But a contradiction.

5.2 Utility for Money: Risk Aversion

1. (Remark)

   (a) Previously, we only study a lottery with discrete distribution, now, when considering utility for money, we generalize lottery space to any probability distribution, or in particular, every CDF F on $\mathbb{R}$ is a lottery.

   (b) Assume preferences have expected utility representation:

   $$U(F) = \int u(x)\,\mathrm{d}F(x)$$

   where $u(x)$ is increasing.

   (c) We call U the **von-Neumann-Morgenstern (v.N-M) expected utility function**, and u the **Bernoulli utility function**, just to distinguish the two different things.

   (d) Usually, we assume that u is increasing, $\lim_{x\to\infty} u'(x) = 0$, and $u'' < 0$.

2. (Definition)

   (a) A decision maker is **risk-averse** if for any lottery F

   $$\int u(x)\,\mathrm{d}F(x) \leqslant u\left(\int x\mathrm{d}F(x)\right) = u(\mathbb{E}X)$$

   which is equivalent with that u is concave.

   (b) A decision maker is **strictly risk-averse** if for any non-degenerate lottery F

   $$\int u(x)\,\mathrm{d}F(x) < u\left(\int x\mathrm{d}F(x)\right)$$

   (c) A decision maker is **risk-neutral** if for any lottery F

   $$\int u(x)\,\mathrm{d}F(x) = u\left(\int x\mathrm{d}F(x)\right)$$

(d) A decision maker is **risk-loving** if for any lottery F

$$\int u(x)\,\mathrm{d}F(x) \geqslant u\left(\int x\mathrm{d}F(x)\right)$$

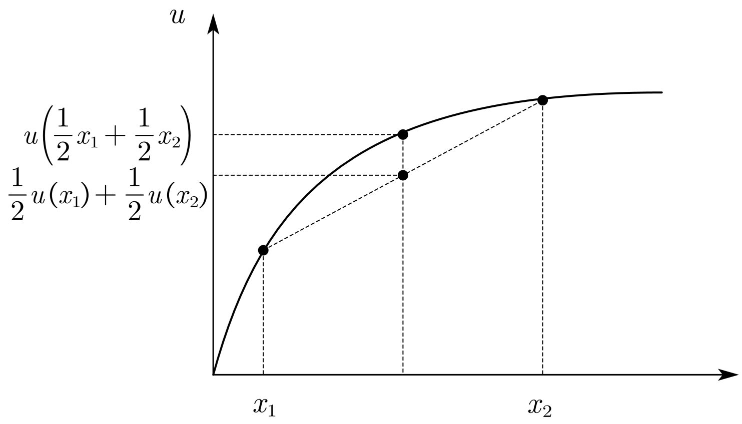

Fig. 5.3. Risk Aversion.

3. (Definition) Given a Bernoulli utility function u.

(a) The **certainty equivalent** of F, denoted $c(F,u)$ is defined as

$$u(c(F,u)) = \int u(x)\,\mathrm{d}F(x)$$

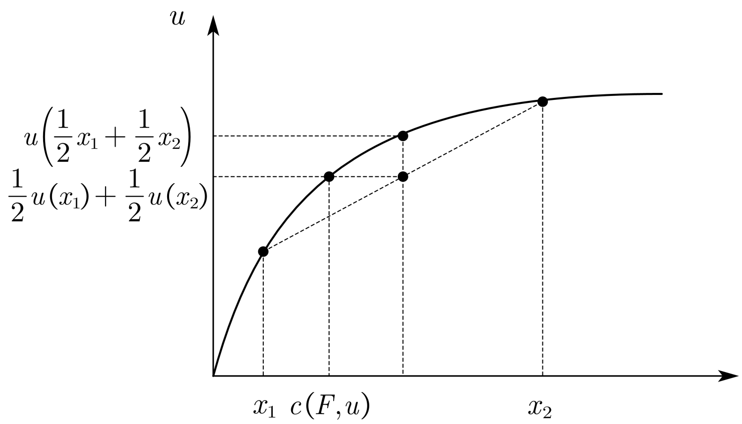

Fig. 5.4. Certainty Equivalent.

(b) For any fixed x and $\varepsilon > 0$, the **probability premium** denoted by $\pi(x,\varepsilon,u)$ is defined as a value such that

$$u(x) = \left(\frac{1}{2}+\pi(x,\varepsilon,u)\right)u(x+\varepsilon)+\left(\frac{1}{2}-\pi(x,\varepsilon,u)\right)u(x-\varepsilon)$$

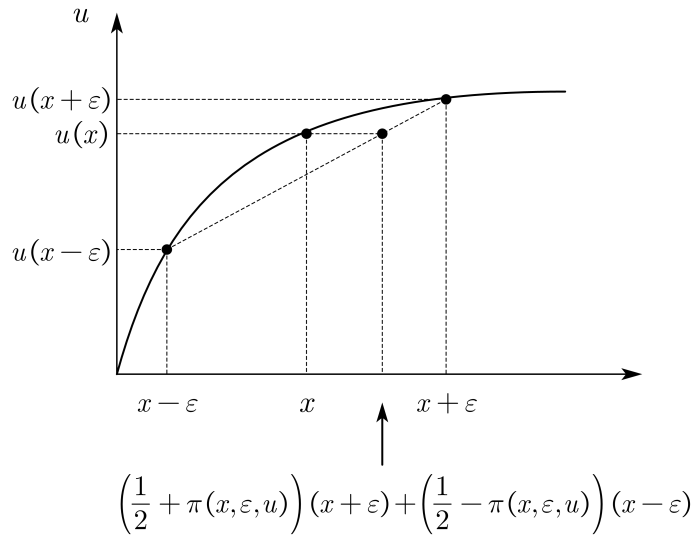

Fig. 5.5. Probability Premium.

4. (Theorem) Let u be the Bernoulli utility function, then the followings are equivalent.

   (a) The decision maker is risk averse.

   (b) u is concave.

   (c) $\forall F, c\,(F,u) \leqslant \int x\mathrm{d}F\,(x)$.

   (d) $\forall x, \varepsilon, \pi\,(x,\varepsilon,u) \geqslant 0$.

Proof. 4 steps to prove.

Step 1. (a)$\Longleftrightarrow$(b)

(a) and (b) are equivalent, because of property of concave functions (Jensen's Inequality).

Step 2. (b)$\Longleftrightarrow$(c)

(c) and (b) are equivalent since by definition and the monotonicity of u, $\forall F$,

$$u\,(c\,(F,u)) = \int u\,(x)\,\mathrm{d}F\,(x) \leqslant u\left(\int x\mathrm{d}F\,(x)\right) \Longleftrightarrow c\,(F,u) \leqslant \int x\mathrm{d}F\,(x)$$

Step 3. (d)$\Longrightarrow$(b)

By definition,

$$u\,(x) = \left(\frac{1}{2} + \pi\,(x,\varepsilon,u)\right) u\,(x+\varepsilon) + \left(\frac{1}{2} - \pi\,(x,\varepsilon,u)\right) u\,(x-\varepsilon)$$

then

$$\pi(x,\varepsilon,u)=\frac{u(x)-\frac{1}{2}u(x+\varepsilon)+\frac{1}{2}u(x-\varepsilon)}{u(x+\varepsilon)-u(x-\varepsilon)}=\frac{u(x)}{u(x+\varepsilon)-u(x-\varepsilon)}-\frac{1}{2}$$

If $\forall x,\varepsilon,\pi(x,\varepsilon,u)\geqslant 0$, then

$$u(x)\geqslant\frac{1}{2}[u(x+\varepsilon)-u(x-\varepsilon)]$$

then we can show that u is concave.

Step 4. (a)$\Longrightarrow$(b)

$\forall\varepsilon>0,\forall x$, define

$$F(z):=\begin{cases}0, & z<x-\varepsilon\\ \frac{1}{2}, & x-\varepsilon\leqslant z<x+\varepsilon\\ 1, & x+\varepsilon\geqslant z\end{cases}$$

$$F_\varepsilon(z):=\begin{cases}0, & z<x-\varepsilon\\ \frac{1}{2}-\pi(x,\varepsilon,u), & x-\varepsilon\leqslant z<x+\varepsilon\\ 1, & x+\varepsilon\geqslant z\end{cases}$$

then by the concavity of u and definition of $\pi(x,\varepsilon,u)$, we have

$$\int z\mathrm{d}F(z)=x\int u(z)\,\mathrm{d}F(z)\leqslant u(x)=\int u(z)\,\mathrm{d}F_\varepsilon(z)$$

and

$$\begin{aligned}\int u(z)\,\mathrm{d}F(z)&=\frac{1}{2}u(x-\varepsilon)+\frac{1}{2}u(x+\varepsilon)\\ \int u(z)\,\mathrm{d}F_\varepsilon(z)&=\left[\frac{1}{2}-\pi(x,\varepsilon,u)\right]u(x-\varepsilon)+\left[\frac{1}{2}+\pi(x,\varepsilon,u)\right]u(x+\varepsilon)\\ &=\frac{1}{2}u(x-\varepsilon)+\frac{1}{2}u(x+\varepsilon)+\pi(x,\varepsilon,u)[u(x+\varepsilon)-u(x-\varepsilon)]\end{aligned}$$

Therefore,

$$\pi(x,\varepsilon,u)[u(x+\varepsilon)-u(x-\varepsilon)]\geqslant 0$$

thus,

$$\pi(x,\varepsilon,u)\geqslant 0$$

$\square$

5. (Definition) Given a twice-differentiable utility function u for money, the **Arrow-Pratt coefficient of absolute risk aversion** at x is defined as

$$r_A(x,u)=-\frac{u''(x)}{u'(x)}$$

6. (Remark) Assume that the risk premium π is differentiable, then by

$$u(x) = \left(\frac{1}{2} + \pi(x, \varepsilon, u)\right) u(x+\varepsilon) + \left(\frac{1}{2} - \pi(x, \varepsilon, u)\right) u(x-\varepsilon)$$

we get (differentiate both sides w.r.t. ε)

$$\begin{aligned} 0 &= u(x+\varepsilon)\frac{\partial}{\partial\varepsilon}\pi(x, \varepsilon, u) + \left(\frac{1}{2} + \pi(x, \varepsilon, u)\right) u'(x+\varepsilon) \\ &\quad - u(x-\varepsilon)\frac{\partial}{\partial\varepsilon}\pi(x, \varepsilon, u) - \left(\frac{1}{2} - \pi(x, \varepsilon, u)\right) u'(x-\varepsilon) \end{aligned}$$

and then

$$\begin{aligned} 0 &= u'(x+\varepsilon)\frac{\partial}{\partial\varepsilon}\pi(x, \varepsilon, u) + u(x+\varepsilon)\frac{\partial^2}{\partial\varepsilon^2}\pi(x, \varepsilon, u) \\ &\quad + u'(x+\varepsilon)\frac{\partial}{\partial\varepsilon}\pi(x, \varepsilon, u) + \left(\frac{1}{2} + \pi(x, \varepsilon, u)\right) u''(x+\varepsilon) \\ &\quad + u'(x-\varepsilon)\frac{\partial}{\partial\varepsilon}\pi(x, \varepsilon, u) - u(x-\varepsilon)\frac{\partial^2}{\partial\varepsilon^2}\pi(x, \varepsilon, u) \\ &\quad + u'(x-\varepsilon)\frac{\partial}{\partial\varepsilon}\pi(x, \varepsilon, u) + \left(\frac{1}{2} - \pi(x, \varepsilon, u)\right) u''(x-\varepsilon) \end{aligned}$$

Let $\varepsilon = 0$,

$$\begin{aligned} 0 &= u'(x)\frac{\partial}{\partial\varepsilon}\pi(x, 0, u) \\ &\quad + u'(x)\frac{\partial}{\partial\varepsilon}\pi(x, 0, u) + \frac{1}{2}u''(x) \\ &\quad + u'(x)\frac{\partial}{\partial\varepsilon}\pi(x, 0, u) \\ &\quad + u'(x)\frac{\partial}{\partial\varepsilon}\pi(x, 0, u) + \frac{1}{2}u''(x) \end{aligned}$$

or

$$0 = 4u'(x)\frac{\partial}{\partial\varepsilon}\pi(x, 0, u) + u''(x)$$

or

$$r_A(x, u) = 4\frac{\partial\pi}{\partial\varepsilon}(x, 0, u)$$

7. (Definition) u_2 is more risk-averse than u_1, if one of the followings holds.

   (a) $\forall x, r_A(x, u_2) \geqslant r_A(x, u_1)$.

   (b) $\exists\psi$, which is increasing and concave (assuming ψ is differentiable), such that $u_2 = \psi \circ u_1$.

   (c) $\forall F, c(F, u_2) \leqslant c(F, u_1)$.

   (d) $\forall x, \varepsilon,\ \pi(x, \varepsilon, u_2) \geqslant \pi(x, \varepsilon, u_1)$ (Assuming π is twice differentiable).

   (e) For a riskless outcome $\bar{x}$,

$$\int u_2(x)\,\mathrm{d}F(x) \geqslant u_2(\bar{x}) \Longrightarrow \int u_1(x)\,\mathrm{d}F(x) \geqslant u_1(\bar{x})$$

In other words, these definitions are equivalent.

Note: u_2 is "more concave" than u_1.

Proof. 8 steps to prove.

Step 1. (b)$\Longrightarrow$(a)

Since $u_2 = \psi(u_1)$, then

$$u_2'(x) = \psi'(u_1(x))\, u_1'(x)$$

and then

$$u_2''(x) = \psi''(u_1(x))\,[u_1'(x)]^2 + \psi'(u_1(x))\, u_1''(x)$$

or

$$\frac{u_2''(x)}{u_1'(x)} = \psi''(u_1(x))\, u_1'(x) + \psi'(u_1(x))\,\frac{u_1''(x)}{u_1'(x)}$$

and since $\frac{u_2'(x)}{\psi'(u_1(x))} = u_1'(x)$, we have

$$-r_A(x,u_2)\,\psi'(u_1(x)) = \psi''(u_1(x))\, u_1'(x) - \psi'(u_1(x))\, r_A(x,u_1)$$

or

$$r_A(x,u_2) = -\frac{\psi''(u_1(x))}{\psi'(u_1(x))}u_1'(x) + r_A(x,u_1)$$

since $\psi' > 0, \psi'' \leqslant 0, u'(x) > 0$, then $r_A(x,u_2) \geqslant r_A(x,u_1)$

Step 2. (a)$\Longrightarrow$(b)

To let $r_A(x,u_2) \geqslant r_A(x,u_1)$, we only need to let $\frac{\psi''(u_1(x))}{\psi'(u_1(x))} \leqslant 0$. Note that $u_2 = \psi(u_1)$, to let u_2 be a utility function for money, ψ must be increasing, then $\psi'' \leqslant 0$ must hold.

Step 3. (b)$\Longrightarrow$(c)

Since $u_2 = \psi(u_1)$, by definition,

$$\psi(u_1(c(F,u_2))) = u_2(c(F,u_2)) = \int u_2(x)\,\mathrm{d}F(x)$$

by the concavity of ψ,

$$\int u_2(x)\,\mathrm{d}F(x) = \int \psi(u_1(x))\,\mathrm{d}F(x) \leqslant \psi\left(\int u_1(x)\,\mathrm{d}F(x)\right) = \psi(u_1(c(F,u_1)))$$

Note that ψ and u_1 are both increasing, therefore,

$$\psi(u_1(c(F,u_2))) \leqslant \psi(u_1(c(F,u_1))) \Longrightarrow c(F,u_2) \leqslant c(F,u_1)$$

Step 4. (c)$\Longrightarrow$(b)

$\forall F, c\left(F, u_2\right) \leqslant c\left(F, u_1\right)$ First note that ψ must be increasing, to make u_2 an increasing function.

$\forall x, y \in \mathbb{R}_+, \forall \lambda \in [0,1]$, we let

$$L := (x, y; \lambda, 1-\lambda)$$

and F be the corresponding CDF. Then

$$\lambda u_1(x) + (1-\lambda) u_1(y) = u_1\left(c\left(F, u_1\right)\right)$$

and then

$$\psi\left(\lambda u_1(x) + (1-\lambda) u_1(y)\right) = \psi\left(u_1\left(c\left(F, u_1\right)\right)\right) = u_2\left(c\left(F, u_1\right)\right)$$

and by definition,

$$u_2\left(c\left(F, u_2\right)\right) = \lambda u_2(x) + (1-\lambda) u_2(y)$$

since u_2 is increasing, and $c\left(F, u_2\right) \leqslant c\left(F, u_1\right)$, we get

$$u_2\left(c\left(F, u_2\right)\right) \leqslant u_2\left(c\left(F, u_1\right)\right)$$

in other words,

$$\psi\left(\lambda u_1(x) + (1-\lambda) u_1(y)\right) \geqslant \lambda u_1(x) + (1-\lambda) u_1(y)$$

Since x, y are arbitrary, then ψ is concave.

Step 5. (c)$\Longrightarrow$(d)

Let $L := \left(x-\varepsilon, x+\varepsilon; \frac{1}{2} - \pi\left(x, \varepsilon, u_2\right), \frac{1}{2} + \pi\left(x, \varepsilon, u_2\right)\right)$, and F be the corresponding CDF, then

$$\begin{aligned}
u\left(c\left(F, u_2\right)\right) &= \int u_2(x)\, \mathrm{d}F \\
&= \left[\frac{1}{2} - \pi\left(x, \varepsilon, u_2\right)\right] u_2(x-\varepsilon) + \left[\frac{1}{2} + \pi\left(x, \varepsilon, u_2\right)\right] u_2(x+\varepsilon) \\
&= u(x)
\end{aligned}$$

then

$$c\left(F, u_2\right) = x$$

and by $c\left(F, u_2\right) \leqslant c\left(F, u_1\right)$, we have

$$c\left(F, u_1\right) \geqslant x$$

Therefore,

$$u_1\left(c\left(F, u_1\right)\right) \geqslant u_1(x)$$

and by definition,

$$u_1\left(c\left(F,u_1\right)\right)=\left[\frac{1}{2}-\pi\left(x,\varepsilon,u_2\right)\right]u_1\left(x-\varepsilon\right)+\left[\frac{1}{2}+\pi\left(x,\varepsilon,u_2\right)\right]u_1\left(x+\varepsilon\right)$$

$$u_1\left(x\right)=\left[\frac{1}{2}-\pi\left(x,\varepsilon,u_1\right)\right]u_1\left(x-\varepsilon\right)+\left[\frac{1}{2}+\pi\left(x,\varepsilon,u_2\right)\right]u_1\left(x+\varepsilon\right)$$

Plug into $u_1\left(c\left(F,u_1\right)\right)\geqslant u_1\left(x\right)$,

$$\pi\left(x,\varepsilon,u_2\right)\left[u_1\left(x+\varepsilon\right)-u_1\left(x-\varepsilon\right)\right]\geqslant\pi\left(x,\varepsilon,u_1\right)\left[u_1\left(x+\varepsilon\right)-u_1\left(x-\varepsilon\right)\right]$$

or

$$\pi\left(x,\varepsilon,u_2\right)\geqslant\pi\left(x,\varepsilon,u_1\right)$$

Step 6. (d)$\Longrightarrow$(a)

By definition, $\pi\left(x,0,u_2\right)=\pi\left(x,0,u_1\right)=0$, therefore, $\pi\left(x,\varepsilon,u_2\right)\geqslant\pi\left(x,\varepsilon,u_1\right)$ implies

$$\frac{\partial\pi}{\partial\varepsilon}\left(x,0,u_2\right)\geqslant\frac{\partial\pi}{\partial\varepsilon}\left(x,0,u_1\right)$$

since

$$r_A\left(x,u\right)=4\frac{\partial\pi}{\partial\varepsilon}\left(x,0,u\right)$$

then

$$r_A\left(x,u_2\right)\geqslant r_A\left(x,u_1\right)$$

Step 7. (c)$\Longrightarrow$(e)

If $\int u_2\left(x\right)\mathrm{d}F\left(x\right)\geqslant u_2\left(\bar{x}\right)$, then $u_2\left(c\left(F,u_2\right)\right)\geqslant u_2\left(\bar{x}\right)\Longrightarrow c\left(F,u_2\right)\geqslant\bar{x}$. Then $c\left(F,u_1\right)\geqslant c\left(F,u_2\right)\Longrightarrow c\left(F,u_1\right)\geqslant\bar{x}$, and thus,

$$u_1\left(c_1\left(F,u_1\right)\right)=\int u_1\left(c\right)\mathrm{d}F\left(x\right)\geqslant u_1\left(\bar{x}\right)$$

Step 8. (e)$\Longrightarrow$(c)

Since

$$\int u_2\left(x\right)\mathrm{d}F\left(x\right)\geqslant u_2\left(\bar{x}\right)\Longleftrightarrow c\left(F,u_2\right)\geqslant\bar{x}$$

$$\int u_1\left(x\right)\mathrm{d}F\left(x\right)\geqslant u_1\left(\bar{x}\right)\Longleftrightarrow c\left(F,u_1\right)\geqslant\bar{x}$$

and then

$$c\left(F,u_2\right)\geqslant\bar{x}\Longrightarrow c\left(F,u_1\right)\geqslant\bar{x}$$

which implies $c\left(F,u_1\right)\geqslant c\left(F,u_2\right)$.

$\square$

8. (Remark) The definition of "riskier" is just a local property, or if we define the relation $\succsim$ by $u_1 \succsim u_2$ iff $\forall x, r_A(x, u_1) \geqslant r_A(x, u_2)$, then $\succsim$ is just a partial order.

   Since there exist u_1, u_2 such that, at a point u_1, u_2, $r_A(x, u_1) > r_A(x, u_2)$, but at another point $x' \neq x$, it holds that $r_A(x', u_1) < r_A(x', u_2)$, therefore, $\succsim$ is not complete.

   Example: $u_1(x) = -e^{-\alpha x}, \alpha > 0$, $u_2(x) = \sqrt{x}$.

   Then
$$r_A(x, u_1) = -\frac{-\alpha^2 e^{-\alpha x}}{\alpha e^{-\alpha x}} = \alpha r_A(x, u_2) = -\frac{-\frac{1}{4}x^{-\frac{3}{2}}}{\frac{1}{2}x^{-\frac{1}{2}}} = \frac{1}{2x}$$

9. (Definition)We say a Bernoulli utility function u **exhibits decreasing (constant, increasing) absolute risk aversion**, if $r_A(x)$ is decreasing (constant, increasing) in x.

   Note: How does risk attitude vary with wealth? It is captured by decreasing absolute risk-aversion.

10. (Theorem) The following properties are equivalent.

    (a) u exhibits decreasing absolute risk aversion.

    (b) $\forall x_1, x_2$ with $x_2 < x_1$, and $\forall z > 0$,$u_2(z) = u(x_2 + z)$ is a concave transformation of $u_1(z) = u(x_1 + z)$.

    (c) $\forall F(z)$, the certainty equivalent of $x+z$, denotes by c_x, is such that $x - c_x$ is decreasing in x. Note that the value of c_x is such that
$$u(c_x) = \int u(x + z)\, \mathrm{d}F(z)$$

    (d) $\pi(x, \varepsilon, u)$ is decreasing in x.

    (e) $\forall F(z)$, if $\int u(x_2 + z)\, \mathrm{d}F(z) \geqslant u(x_2)$ and $x_2 < x_1$, then
$$\int u(x_1 + z)\, \mathrm{d}F(z) \geqslant u(x_1)$$

11. (Definition) Given a utility function u, the **coefficient of relative risk aversion** at x is
$$r_R(x, u) = -\frac{xu''(x)}{u'(x)} = x \cdot r_A(x, u)$$

12. (Remark)Consider a gamble, we are concerned about the relative return $t \cdot x$, but not the absolute return $x \pm w$.

    Let $\tilde{u}(t) := u(tx)$, then
$$r_A(1, \tilde{u}) = \left.\frac{\tilde{u}''(t)}{\tilde{u}'(t)}\right|_{t=1} = \left.\frac{xu''(tx)}{u'(tx)}\right|_{t=1} = \frac{xu''(x)}{u'(x)}$$

13. (Remark) $r_R(x) = xr_A(x)$

    (a) decreasing relative risk-aversion$\Longrightarrow$decreasing absolute risk-aversion.

(b) increasing absolute risk-aversion$\Longrightarrow$increasing relative risk-aversion.

14. (Theorem) The following conditions for a Bernoulli utility function u on amounts of money are equivalent.

    (a) $r_R(x,u)$ is decreasing in x.

    (b) $\forall x_1, x_2$ with $x_2 < x_1$, then $\tilde{u}_2(t) = u(tx_2)$ is a concave transformation of $\tilde{u}_1(t) = u(tx_1)$.

    (c) Given any risk $F(t)$ on $t > 0$, then certainty equivalent c_x defined by $u(c_x) = \int u(tx)\,\mathrm{d}F(t)$ is such that $\frac{x}{c_x}$ is decreasing in x.

Proof. (a)$\Longleftrightarrow$(b) is just by definition.

Now prove that (a)$\Longleftrightarrow$(c).

Define

$$u_x(t) := u(tx)$$

Let c_x be the constant equivalence such that

$$u_x(c_x) = \int u_x(t)\,\mathrm{d}F(t) = \int u(tx)\,\mathrm{d}F(t)$$

For any x, it holds that

$$-\frac{u_x''(t)}{u_x'(t)} = -\frac{xu''(tx)}{u'(tx)}$$

If $r_R(x,u)$ is decreasing in x, then if $x' > x$, then

$$-\frac{u_{x'}''(t)}{u_{x'}'(t)} < -\frac{u_x''(t)}{u_x'(t)}$$

And moreover, it holds iff $c_{x'} > c_x$, and then c_x is increasing in x, and thus, $\frac{x}{c_x}$ is decreasing in x. $\square$

5.3 Stochastic Dominance

1. (Remark)

    (a) Previously, we mainly discuss how to calibrate risk aversion by utility function and compare lotteries, now we compare different payoff distribution.

    (b) We consider the CDFs such that $F(0) = 0, F(\bar{x}) = 1$ for some $\bar{x}$.

textbf

2. (Definition) The distribution F **first-order stochastically dominates** G if, $\forall u : \mathbb{R} \to \mathbb{R}$ and u is nondecreasing, we have

$$\int u(x)\,\mathrm{d}F(x) \geqslant \int u(x)\,\mathrm{d}G(x)$$

Note: If we define $F \succsim G$ iff F first-order stochastically dominates G, then $\succsim$ is reflexive and transitive, but not complete.

3. (Theorem) F first-order stochastically dominates G iff $\forall x, F(x) \leqslant G(x)$.

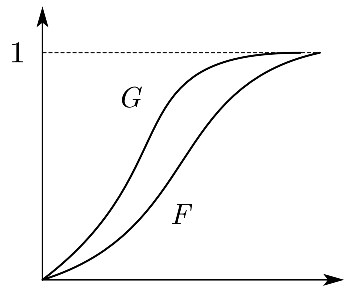

Fig. 5.6. FOSD.

Proof. Let $\bar{x}$ is such that $F(\bar{x}) = G(\bar{x})$. F FOSD G, iff

$$\int u(x)\,\mathrm{d}F(x) \geqslant \int u(x)\,\mathrm{d}G(x)$$

where by integration by parts

$$\begin{aligned}\int u(x)\,\mathrm{d}F(x) &= u(x)\,F(x)\,|_0^{\bar{x}} - \int_0^{\bar{x}} u'(x)\,F(x)\,\mathrm{d}x \\ &= u(\bar{x}) - \int_0^{\bar{x}} u'(x)\,F(x)\,\mathrm{d}x \\ \int u(x)\,\mathrm{d}G(x) &= u(\bar{x}) - \int_0^{\bar{x}} u'(x)\,G(x)\,\mathrm{d}x\end{aligned}$$

then it is equivalent with

$$\begin{aligned}0 &\leqslant \int u(x)\,\mathrm{d}F(x) - \int u(x)\,\mathrm{d}G(x) \\ &= \int_0^{\bar{x}} u'(x)\,G(x)\,\mathrm{d}x - \int_0^{\bar{x}} u'(x)\,F(x)\,\mathrm{d}x \\ &= \int_0^{\bar{x}} u'(x)\,[G(x) - F(x)]\,\mathrm{d}x\end{aligned}$$

(**If**) $\forall x, F(x) \leqslant G(x)$, since $u'(x) \geqslant 0$, then $\int_0^{\bar{x}} u'(x)\,[G(x) - F(x)]\,\mathrm{d}x \geqslant 0$.

(**Only if**) Suppose $\exists x_0, F(x_0) > G(x_0)$, Since F and G are both nondecreasing and right-continuous, then $\exists \delta > 0$ such that

$$\forall x \in (x_0, x_0 + \delta), F(x) > G(x) \Longrightarrow G(x) - F(x) < 0$$

Note that the definition of FOSD is for any u, then we can define

$$u = \begin{cases} x_0 + \delta, & x \geqslant x_0 + \delta \\ x, & x \in (x_0, x_0 + \delta) \\ 0, & x \leqslant x_0 \end{cases}$$

Then

$$\int_0^{\bar{x}} u'(x)[G(x) - F(x)]\,\mathrm{d}x = \int_{x_0}^{x_0+\delta} [G(x) - F(x)]\,\mathrm{d}x < 0$$

Contradicting to the definition of FODS.

□

4. (Remark)

   (a) F FOSD G, if for every amount of money x, F is more likely to yield at least x dollars than G is.

   (b) F FOSD G does not mean that all possible return of F is larger than that of G.

   (c) F FOSD $G \Longrightarrow \int x\mathrm{d}F(x) \geqslant \int x\mathrm{d}G(x)$. The converse is not true.

   Proof.

   $$\int x\mathrm{d}F(x) - \int x\mathrm{d}G(x) = \int [G(x) - F(x)]\,\mathrm{d}x \geqslant 0$$

   □

5. (Example) Upward Probabilistic Shift.

   (a) A compound lottery that pays $x \sim G$.

   (b) Here is another distribution $z \sim H_x$ with $H_x(0) = 0$.

   (c) There is another lottery that pays $y = x + z \sim F$.

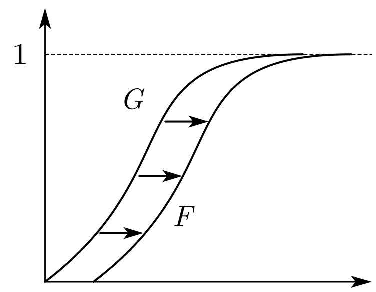

Fig. 5.7. Upward Probabilistic Shift.

Then for non-decreasing u, F FOSD G, since

$$\begin{aligned}\int u(x)\,\mathrm{d}F(x) &= \int\int u(x+z)\,\mathrm{d}H_x(z)\,\mathrm{d}G(x)\\ &\geqslant \int\int u(x)\,\mathrm{d}H_x(z)\,\mathrm{d}G(x)\\ &= \int u(x)\int \mathrm{d}H_x(z)\,\mathrm{d}G(x)\\ &= \int u(x)\,\mathrm{d}G(x)\end{aligned}$$

The converse of the example is also true, that is, if F FOSD G, then it is possible to generate F from G by a "upward probabilistic shift".

6. (Remark) If F FOSD G, then we can think up a scenario, where we draw x from G, but then instead get

$$y(x) = F^{-1}(G(x))$$

since $\forall x, F(x) \leqslant G(x)$, and F, G are both nondecreasing, then

$$y(x) = F^{-1}(G(x)) \geqslant F^{-1}(F(x)) = x$$

Therefore, we actually get more, if our u is nondecreasing, we prefer $y(x)$ to x.

7. (Remark)

   (a) FOSD gives a way to decide which lottery is better for any decision-maker with nondecreasing u.

   (b) For any risk-averse decision-maker, i.e., with concave and nondecreasing u, we need SOSD to make such a decision.

8. (Definition) For two CDF F and G with the same mean, we say F **second-order stochastically dominates (or less risky than)** G if $\forall u : \mathbb{R} \to \mathbb{R}$, and u is nondecreasing and concave, we have

$$\int u(x)\,\mathrm{d}F(x) \geqslant \int u(x)\,\mathrm{d}G(x)$$

9. (Remark) Why this definition?

   (a) Fixing the mean of the distributions F, G, then we let every risk averter compare F and G, if F is preferred to G, then F SOSD G.

   (b) FOSD$\Longrightarrow$SOSD. (Just by definition)

   Let F and G have the same mean, and FOSD implies for all nondecreasing u, $\int u(x)\,\mathrm{d}F(x) \geqslant \int u(x)\,\mathrm{d}G(x)$ holds, then for all nondecreasing and concave u, the inequality must also holds.

10. (Example) Mean-preserving spread.

    (a) Consider a compound lottery that pays $x \sim F$.

    (b) Let $z \sim H_x$, wit $H_x(0) = 0$ and $\int z\mathrm{d}H_x(z) = 0$.

    (c) Let $y = x + z \sim G$.

    Then F SOSD G, since firstly,

$$\int x\mathrm{d}F(x) = \int\int (x+z)\,\mathrm{d}H_x(z)\,\mathrm{d}G(x) = \int x\mathrm{d}G(x)$$

secondly, by concavity of u,

$$\begin{aligned}\int u(x)\,\mathrm{d}G(x) &= \int\int u(x+z)\,\mathrm{d}H_x(z)\,\mathrm{d}F(x)\\ &\leqslant \int u\left(\int (x+z)\,\mathrm{d}H_x(z)\right)\mathrm{d}F(x)\\ &= \int u(x)\,\mathrm{d}F(x)\end{aligned}$$

The converse is also true, that is, if F SOSD G, then G is a mean-preserving spread of F, we can just let $x \sim F, y \sim G$ and $z = y - x$. We actually do not need u is nondecreasing to define SOSD.

11. (Definition) For two CDFs F and G,with $x \sim F$, $y \sim G$, if $\exists z \sim H_x$ with $H_x(0) = 0$ and $\int z\mathrm{d}H_x(z) = 0$ such that $y = x + z$, then we say G is a **mean-preserving spread** of F.

12. (Remark) If F is a mean-preserving spread of G, then G has a larger variance than F. To prove it, let $\mu = \int x\mathrm{d}F(x) = \int x\mathrm{d}G(x)$, and consider

$$\phi(x) = -(x-\mu)^2$$

which is concave.

$$\int -(x-\mu)^2\,\mathrm{d}F(x) \geqslant \int -(x-\mu)^2\,\mathrm{d}G(x)$$

then G has a larger variance than F.

But the converse is not true.

13. (Theorem) Consider two CDFs F and G with the same mean. Then the following statements are equivalent.

    (a) F SOSD G.

    (b) G is a mean-preserving spread of F.

    (c) $\forall x$, $\int_0^x G(t)\,\mathrm{d}t \geqslant \int_0^x F(t)\,\mathrm{d}t$.

Proof. We have already known (a)$\Longleftrightarrow$(b).

Now prove (a)$\Longleftrightarrow$(c).

We first rewrite the definition of SOSD by integration by parts.

To simplify the proof, assume that u is twice differentiable.

Let $\bar{x}$ be such that $F(\bar{x}) = G(\bar{x}) = 1$.

Since

$$\int_0^{\bar{x}} x\mathrm{d}F(x) = \int_0^{\bar{x}} x\mathrm{d}G(x)$$

then

$$xF(x)\,|_0^{\bar{x}} - \int_0^{\bar{x}} F(x)\,\mathrm{d}x = xG(x)\,|_0^{\bar{x}} - \int_0^{\bar{x}} G(x)\,\mathrm{d}x$$

and thus,

$$\int_0^{\bar{x}} F(x)\,\mathrm{d}x = \int_0^{\bar{x}} G(x)\,\mathrm{d}x$$

Define

$$S(x) := \int_0^{x} F(t)\,\mathrm{d}t, T(x) := \int_0^{x} G(t)\,\mathrm{d}t$$

then $S'(x) = F(x), T'(x) = G(x)$, and for F, we have

$$\begin{aligned}
\int_0^{\bar{x}} u(x)\,\mathrm{d}F(x) &= u(x)F(x)\,|_0^{\bar{x}} - \int_0^{\bar{x}} u'(x)F(x)\,\mathrm{d}x \\
&= u(\bar{x}) - \int_0^{\bar{x}} u'(x)\,\mathrm{d}S(x) \\
&= u(\bar{x}) - u'(x)S(x)\,|_0^{\bar{x}} + \int_0^{\bar{x}} S(x)\,\mathrm{d}u'(x) \\
&= u(\bar{x}) - u'(\bar{x})\int_0^{\bar{x}} F(t)\,\mathrm{d}t + \int_0^{\bar{x}} S(x)u''(x)\,\mathrm{d}x
\end{aligned}$$

Likewise,

$$\int_0^{\bar{x}} u(x)\,\mathrm{d}G(x) = u(\bar{x}) - u'(\bar{x})\int_0^{\bar{x}} G(t)\,\mathrm{d}t + \int_0^{\bar{x}} T(x)u''(x)\,\mathrm{d}x$$

By the definition of SODS,

$$\int u(x)\,\mathrm{d}F(x) \geqslant \int u(x)\,\mathrm{d}G(x) \Longleftrightarrow \int u(x)\,\mathrm{d}F(x) - \int u(x)\,\mathrm{d}G(x) \geqslant 0$$

it is equivalent with

$$\int_0^{\bar{x}} [S(x) - T(x)]\,u''(x)\,\mathrm{d}x \geqslant 0$$

Step 1.(c)$\Longrightarrow$(a)

Since $u'' \leqslant 0$, then if $\forall x$, $S(x) - T(x) \leqslant 0$, it holds that

$$\int_0^{\bar{x}} [S(x) - T(x)]\,u''(x)\,\mathrm{d}x \geqslant 0$$

Step 2.(a)$\Longrightarrow$(c)

Suppose $\exists x_0$, such that $S(x_0) - T(x_0) > 0$.

By the fundamental theorem of calculus, $S(x)$ and $T(x)$ are continuous, then $\exists \delta > 0, \forall x \in (x_0, x_0 + \delta)$, $S(x) - T(x) > 0$.

Therefore, we can construct a utility function that is nondecreasing, concave and satisfies, $u'' < 0$ on $(x_0, x_0 + \delta)$, and $u'' = 0$ on other points, then

$$\int_0^{\bar{x}} [S(x) - T(x)]\,u''(x)\,\mathrm{d}x < 0$$

a contradiction to SOSD.

□

14. (Example) Consider the set of lotteries over outcomes in a bounded domain $[a, b] \subseteq \mathbb{R}_{++}$. Define a preference relation $\succsim$ over the set of lotteries as followings. $\forall L, L'$,

    (a) If L dominates L' by FOSD and $L \neq L'$, then $L \succ L'$.

    (b) If L' dominates L by FOSD and $L \neq L'$, then $L' \succ L$.

    (c) Otherwise, $L \sim L'$.

    In the expected utility theory, clearly, completeness is satisfied (since we have "otherwise").

    Transitivity is violated.

    Counterexample 1. On the figure, we can see that $G \succ H, G \sim F$, by transitivity, it must

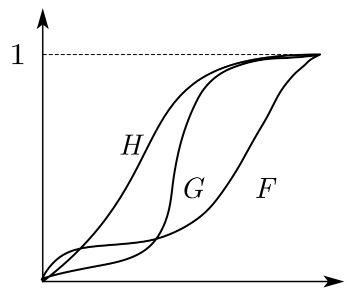

Fig. 5.8. Counterexample.

    hold that $F \succ H$, but $F \sim H$.

    Counterexample 2.

$$\left(3, 0; \frac{1}{2}, \frac{1}{2}\right) \sim (2; 1)$$
$$\left(3, 0; \frac{1}{2}, \frac{1}{2}\right) \sim (1; 1)$$

    but, $(2; 1) \succ (1; 1)$. The axiom of independence is satisfied, since if $F \succ G$, then

$$\forall x \in [a, b]\,, F\,(x) \geqslant G\,(x)$$

    and $\forall \alpha \in [0, 1]$, $\alpha F\,(x) + (1 - \alpha)\,H\,(x) \geqslant \alpha G\,(x) + (1 - \alpha)\,H\,(x)$, which implies $\alpha F\,(x) + (1 - \alpha)\,H\,(x) \succ \alpha G\,(x) + (1 - \alpha)\,H\,(x)$.

    If $F \sim G$, then $\exists x_1, x_2 \in [a, b]$, such that $F\,(x_1) < G\,(x_1)\,, F\,(x_2) > G\,(x_2)$, and this also leads to $\alpha F\,(x) + (1 - \alpha)\,H\,(x) \sim \alpha G\,(x) + (1 - \alpha)\,H\,(x)$.

    Based on MWG's definition, continuity is violated.

    Recall the definition.

    We say $\succsim$ is continuous, if $\forall L, L', L'' \in \Pi$, the following two sets are closed.

$$\{\alpha \in [0, 1] : \alpha L + (1 - \alpha)\,L' \succsim L''\} \subseteq [0, 1]$$

$$\{\alpha \in [0,1] : L'' \succsim \alpha L + (1-\alpha) L'\} \subseteq [0,1]$$

Consider 3 outcomes $x, y, z \in [a,b]$, with $x \succ y \succ z$, and L_x, L_y, L_z give x, y, z with probability 1 respectively.

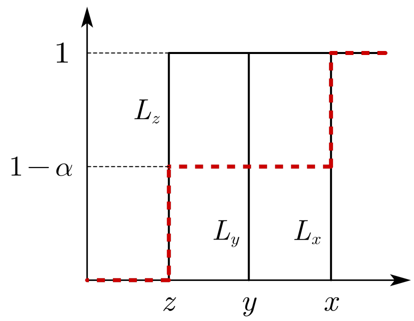

Fig. 5.9. 3 Riskless Lotteries.

And thus,

$$\{\alpha \in [0,1] : \alpha L_x + (1-\alpha) L_z \succsim L_y\} = (0,1]$$

$$\{\alpha \in [0,1] : L_y \succsim \alpha L_x + (1-\alpha) L_z\} = [0,1)$$

is open.

Based on Rubinstein's definition, continuity is satisfied.

The following is the definition.

We say $\succsim$ is continuous, if $p \succ q$, then $\exists \varepsilon_p, \varepsilon_q > 0$ such that $\forall p' \in N_{\varepsilon_p}(p)$ and $\forall q' \in N_{\varepsilon_q}(q)$, $p' \succ q'$.

And the definition implies that if $p \succ q \succ r$, then $\exists \alpha \in (0,1)$, such that

$$q \sim \alpha p + (1-\alpha) r$$

this is also an alternative definition of continuity.

Chapter 6

Social Choice Theory (Rubinstein)

1. (Remark) Set-ups.

   (a) X is the set of alternatives with $|X| \geqslant 3$, it can be finite or infinite.

   (b) There are $n > 1$ individuals in the economy, each individual i has a complete and transitive preference $\succsim_i$ over X.

   (c) Ω is the set of all complete and transitive preference on X.

   (d) We want to find a social preference satisfying some properties, i.e., find a function f such that

   $$f : \Omega^n \to \Omega \ \ (\succsim_1, \succsim_2, \cdots, \succsim_n) \mapsto \succsim$$

   with some properties.

   Note: Since $\forall \succsim \in \Omega$, $\succsim$ is transitive. Then we may want $\succsim$ is transitive, but sometimes together with other good properties, $\succsim$ cannot be transitive, in this case, f is not well-defined by $f : \Omega^n \to \Omega$.

2. (Definition) Properties of f or $\succsim$.

   (a) **Universal Domain**: f is defined on all profiles of preferences Ω^n (i.e., not just for some specific profile $S \subseteq \Omega^n$).

   (b) **Unanimity**: If $\forall i = 1, \cdots, n$, $a \succ_i b$, then $a \succ b$.

   (c) **No-Dictatorship**: There exists no person i such that $\forall a, b \in X$, and $\forall \succsim_1, \cdots, \succsim_{i-1}, \succsim_{i+1}, \cdots, \succsim_n \in \Omega$,

   $$a \succ_i b \Longrightarrow a \succ b$$

   Note:

   i. No-Dictatorship does not mean that the social preferences $\succsim$ cannot coincide with the personal preference $\succsim_i$. The key is that $\succsim$ should not always equal the $\succsim_i$.

   ii. Dictatorship means that $\exists i$, such that $\forall a, b \in X$, and $\forall \succsim_1, \cdots, \succsim_{i-1}, \succsim_{i+1}, \cdots, \succsim_n \in \Omega$,

   $$a \succ_i b \Longrightarrow a \succ b$$

But not require that $a \sim_i b \Longrightarrow a \sim b$.

$$\begin{array}{cc|c} \succsim_i & \succsim_j (j \neq i) & \succsim \\ a & b & a \\ b & a & b \end{array} \xrightarrow{\text{Change}} \begin{array}{cc|c} \succsim_i & \succsim_j (j \neq i) & \succsim \\ b & a & b \\ a & b & a \end{array}$$

(d) **Transitivity**: If $a \succsim b$ and $b \succsim c$, then $a \succsim c$.

(e) **IIA (Independence of Irrelevant Alternatives)**: $\forall a, b \in X$, consider the profile of preferences $\succsim_1, \cdots, \succsim_n$ and $\succsim'_1, \cdots, \succsim'_n$, if $\forall i$,

$$a \succsim_i b \Longleftrightarrow a \succsim'_i b$$

then

$$a \succsim b \Longleftrightarrow a \succsim' b$$

Note:

i. The social ranking of a and b depends only on the way individuals rank a and b.

ii. By contrapositive, IIA can be also defined as $\forall a, b \in X$, if $\forall i$,

$$b \succ_i a \Longleftrightarrow b \succ'_i a$$

then

$$b \succ a \Longleftrightarrow b \succ' a$$

(f) **Majority Voting Rule**: $a \succsim b$ iff $|\{i : a \succ_i b\}| \geqslant |\{i : b \succ_i a\}|$.

Note: Majority rule implies a violation of transitive.

Example (Voting paradox). $X = \{a, b, c\}$, and

$$a \succ_1 b \succ_1 cb \succ_2 c \succ_2 ac \succ_3 a \succ_3 b$$

then majority rule implies $a \succ b, b \succ c, c \succ a$.

3. (Theorem, Arrow's Impossible Theorem) Suppose that there are at least 3 social alternatives and the preferences of all individuals are complete and transitive. Then there is no social ranking rule satisfying

(a) Universal domain.

(b) Completeness and transitivity of $\succsim$.

(c) Unanimity.

(d) No dictatorship.

(e) IIA.

Proof. (From Geanakoplos, 2015, when X is finite)

To prove the theorem, we just need to prove if $\succsim$ satisfies universal domain, unanimity, transitivity, and IIA, then it is dictatorial.

Suppose $\succsim$ is non-dictatorial.

Step 1. Prove a lemma.

(Definition) Outcome b is **extreme** in $\succsim$ if either$\forall a \in X, b \succ a$ or$\forall a \in X, a \succ b$

(Lemma) If $\forall i$, b is extreme in $\succsim_i$, then b is extreme in $\succsim$. Suppose b is not extreme in $\succsim$, then $\exists a, c \in X$ such that $a \succsim b \succsim c$

For individuals, there are 4 types of individuals, as long as b is extreme in $\succsim_i$.

- Type 1. $b \succ_i a \succsim_i c$
- Type 2. $b \succ_i c \succ_i a$
- Type 3. $a \succsim_i c \succ_i b$
- Type 4. $c \succ_i a \succ_i b$

Denote the rest alternatives as $d_1, \cdots, d_k$. Then to make a contradiction, we construct another preference $\succsim_i'$ such that

(a) $\succsim_i'$ preserves the relation of (a, b) and (b, c).

(b) $\forall i, c \succ_i' a$.

(c) b is still extreme in $\succsim_i$.

- Type 1. $b \succ_i' c \succ_i' a \sim_i' d_1 \sim_i' \cdots \sim_i' d_k$
- Type 2. $b \succ_i' c \succ_i' a \sim_i' d_1 \sim_i' \cdots \sim_i' d_k$
- Type 3. $d_1 \sim_i' \cdots \sim_i' d_k \sim_i' c \succ_i' a \succ_i' b$
- Type 4. $d_1 \sim_i' \cdots \sim_i' d_k \sim_i' c \succ_i' a \succ_i' b$

Then, by IIA, social preference between (a, b) and (b, c) is preserved, i.e.,

$$a \succsim' b, b \succsim' c$$

by transitivity,

$$a \succsim' c$$

But by unanimity,

$$c \succ' a$$

A contradiction.

Step 2. Find an agent who controls social preferences for one pair of alternatives.

Start with a profile of preferences $(\succsim_1, \cdots, \succsim_n)$ where $\forall i$, $\forall a \in X$, $a \succ_i b$.

Then we now define a new profile of preferences $(\succsim_1', \cdots, \succsim_n')$ by n steps, such that $\forall i$, $\forall a \in X$, $b \succ_i a$, and relations between other alternatives are preserved.

$$\begin{array}{ccc} \succsim_i & & \succsim_i' \\ d_1 & & b \\ d_2 & \mapsto & d_1 \\ \vdots & & d_2 \\ b & & \vdots \end{array}$$

- Start with $(\succsim_1, \cdots, \succsim_n)$, and by unanimity, b is the worst in social ranking.
- Define $\succsim_1'$, and get $(\succsim_1', \succsim_2 \cdots, \succsim_n)$.
- Define $\succsim_2'$, and get $(\succsim_1', \succsim_2' \cdots, \succsim_n)$
- $\cdots$
- Define $\succsim_i'$, and get $\left(\succsim_1', \succsim_2', \cdots, \succsim_{i-1}', \succsim_i', \succsim_{i+1}, \cdots, \succsim_n\right)$
- $\cdots$
- Define $\succsim_n'$, and get $(\succsim_1', \succsim_2' \cdots, \succsim_n')$, and by unanimity, b is the best in social ranking.

Therefore, $\exists i^*$ such that b is the worst in social ranking $\succsim$ corresponding to the profile , defined as

$$f\left(\succsim_1', \succsim_2', \cdots, \succsim_{i^*-1}', \succsim_{i^*}, \succsim_{i^*+1}, \cdots, \succsim_n\right) = \succsim$$

and b is the best in social ranking $\succsim$ corresponding to the profile defined as

$$f\left(\succsim_1', \succsim_2', \cdots, \succsim_{i^*-1}', \succsim_{i^*}', \succsim_{i^*+1}, \cdots, \succsim_n\right) = \succsim$$

In other words,

$$\forall x \in X, x \succ b, b \succ x$$

Then, we find a i^* that changes the social ranking of b.

Step 3. Show that this agent i^* is a dictator.

We will show that $\forall \succsim_1, \cdots, \succsim_{i^*-1}, \succsim_{i^*+1}, \cdots, \succsim_n \in \Omega$, and $\forall x, x' \in X$, $x \succ_{i^*} x' \Longrightarrow x \succ x'$

Step 3.1.

Consider an arbitrary profile of preferences$(\cdot, \cdot, \cdots, \cdot, \succsim_{i^*}, \cdot, \cdots, \cdot)$, and $\forall a, c \neq b$, W.L.G., assume that $a \succ_{i^*}^* c$, we will show that $a \succ c$. Define profile

$$\left(\succsim_1'', \succsim_2'', \cdots, \succsim_{i^*-1}'', \succsim_{i^*}, \succsim_{i^*+1}'', \cdots, \succsim_n''\right)$$

as

(a) $\forall i$, $a \succ_i'' c$.

(b) $\forall i < i^*$, b is the best in $\succsim_i''$, in particular, $b \succ_i'' a, b \succ_i'' c$.

(c) $\forall i > i^*$, b is the worst in $\succsim_i''$, in particular, $a \succ_i'' b, c \succ_i'' b$

By IIA, the social preferences between (a, c) in profile are the same as in the arbitrary profile.

$\forall i$, the preferences between (a, b) in profile are the same as in profile , then by IIA, $a \succ b \Longrightarrow a \succ b$

$\forall i$, the preferences between (b, c) in profile are the same as in profile , then by IIA, $b \succ c \Longrightarrow b \succ c$

And thus, by transitivity,

$$a \succ b \succ c \Longrightarrow a \succ c$$

Therefore, $\forall a, c \neq b$, the social preference between (a, c) in the arbitrary profile are the same as those of i^*.

> **Step 3.2**. Now, show that $\forall a \in X$, the social preference between (a, b) are the same as the preference of i^*.

Let $c \neq a, b$, then $\exists j^*$, such that j^* is a dictator for all pairs excluding c.

In particular, for (a, b), i^* can affect the social ranking of (a, b). But j^* is a dictator over the pair of (a, b), then it must hold $i^* = j^*$.

Therefore, i^* is a dictator for all pairs of alternatives.

$\square$

4. (Remark)

(a) Let $|X| \geqslant 3$, if there exists a dictator i^*, and $\forall a, b \in X, a \nsim_{i^*} b$, then dictatorship satisfies all of Arrow's axioms except for no dictatorship.

Note that, by assumption, $\succsim_{i^*} \in \Omega$ is complete and transitive.

Discussion. If we don't assume $\forall a, b \in X, a \nsim_{i^*} b$, we cannot get transitivity. Note that $a \sim_{i^*} b \not\Rightarrow a \sim b$. But if $a \sim b$, then $a \sim_{i^*} b$. Since $a \succ_{i^*} b \Longrightarrow a \succ b$ and $b \succ_{i^*} a \Longrightarrow b \succ a$, both cases contradict to $a \sim b$. Consider $\forall x, y, z \in X$, with $x \succsim y, y \succsim z$, then There are 4 cases.

$$\left.\begin{cases} x \succ y \begin{cases} y \succ z \Longrightarrow x \succ_{i^*} y \succ_{i^*} z \Longrightarrow x \succ_{i^*} z \Longrightarrow x \succ z \\ y \sim z \Longrightarrow x \succ_{i^*} y \sim_{i^*} z \Longrightarrow x \succ_{i^*} z \Longrightarrow x \succ z \end{cases} \\ x \sim y \begin{cases} y \succ z \Longrightarrow x \sim_{i^*} y \succ_{i^*} z \Longrightarrow x \succ_{i^*} z \Longrightarrow x \succ z \\ y \sim z \Longrightarrow x \sim_{i^*} y \sim_{i^*} z \Longrightarrow ? \end{cases} \end{cases} \right. \qquad \not\Rightarrow x \succsim z$$

Actually, if $x \sim_{i^*} y \sim_{i^*} z$, then socially, it can be that $x \succ y, y \succ z, x \sim z$. $\square$

(b) Therefore, given the assumption, we cannot find a social preference that violates all of Arrow's axioms.

5. (Example) For each property of the social preferences, there are examples satisfying all other properties but only violating the property.

(a) $|X| \geqslant 3$. Majority voting will satisfy all other properties when $|X| = 2$.

If $|X| = 2$, we don't need to verify transitivity.

(b) Universal Domain.

This case is complicated, the example comes from Sprument (1990). In his example, f is only well defined on the profile in which all preferences are single peaked, and by majority voting, it generates a social preference.

We will discuss it later.

(c) Transitivity. Majority voting.

(d) Unanimity.

Let the social choice be a constant $\succsim_c$ preference that satisfies transitivity.

Example.

$$\begin{array}{ccc|c} \succsim_1 & \succsim_2 & \succsim_3 & \succsim \\ c & c & c & a \\ b & b & b & b \\ a & a & a & c \end{array}$$

(e) IIA.

The example is called Borda counting.

Consider a finite X, for person i, we define $c_i(x) : X \to \mathbb{N}$ as the rank with positive preference order, e.g., if person i, rank $x \in X$ as the m-th, then $c_i(x) = m$.

And social rank function is defined as

$$c(x) := \sum_{i=1}^{n} c_i(x)$$

Here is the example.

$$\begin{array}{cc} \succsim_1 & \succsim_2 \\ x & y \\ z & x \\ y & z \end{array} \Longrightarrow \begin{cases} c(x) = 1 + 2 = 3 \\ c(y) = 3 + 1 = 4 \\ c(z) = 2 + 3 = 5 \end{cases} \Longrightarrow x \succ y \succ z \Longrightarrow \begin{array}{c} \succsim \\ x \\ y \\ z \end{array} \Longrightarrow x \succ y$$

But

$$\begin{array}{cc} \succsim_1 & \succsim_2 \\ x & y \\ y & z \\ z & x \end{array} \Longrightarrow \begin{cases} c(x) = 3 + 1 = 4 \\ c(y) = 2 + 1 = 3 \\ c(z) = 2 + 3 = 5 \end{cases} \Longrightarrow y \succ x \succ z \Longrightarrow \begin{array}{c} \succsim \\ y \\ x \\ z \end{array} \Longrightarrow y \succ z$$

Note that $c(x) \in \mathbb{N}$, then there is no violation of transitivity.

(f) No Dictatorship.

If there is a dictator has no indifference preference, then dictatorship implies all other axioms are satisfied.

6. (Definition) Preference over $X = \mathbb{R}$ are called **single peaked** if $\exists a^* \in \mathbb{R}$ such that $\forall b \in \mathbb{R}$,

   (a) $a^* > b \Longrightarrow a^* \succsim b$.

   (b) $a^* < b \Longrightarrow a^* \succsim b$.

7. (Example) Suppose $a > b > c$.

   $\succsim_1, \succsim_2$ are single peaked, but $\succsim_3$ is not single peaked.

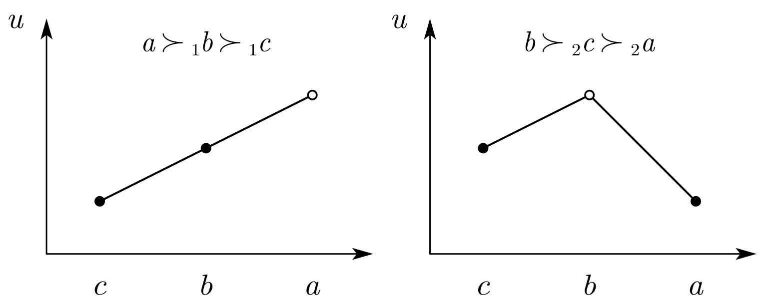

Fig. 6.1. Single Peaked Preferences.

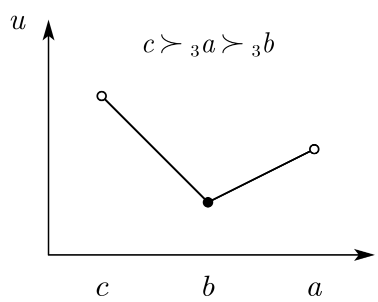

Fig. 6.2. Not a Single Peaked Preference.

8. (Theorem) Suppose $n \geqslant 3$ is odd, and $X = \mathbb{R}$. The individual preference satisfies $\nexists i, \nexists a, b \in \mathbb{R}$, such that $a \sim_i b$. The if we restrict f on $\{\succsim \in \Omega : \succsim \text{ is single peaked}\}^n$, then majority rule works without leading us to nontransitive cycles. Moreover, all other Arrow's axiom is satisfied.

Proof. **Unanimity**. It holds because of majority rule.

IIA. It holds as the social rank of a and b depends on how people rank these two alternatives.

No dictatorship. It holds because of majority rule.

Transitivity. $\forall a, b, c \in \mathbb{R}$ with $a > b > c$. Let

$$\begin{cases} \alpha := |\{i : a \succ_i b\}| \\ \beta := |\{i : b \succ_i c\}| \\ \gamma := |\{i : a \succ_i c\}| \end{cases} \Longrightarrow \begin{cases} n - \alpha = |\{i : b \succ_i a\}| \\ n - \beta = |\{i : c \succ_i b\}| \\ n - \gamma = |\{i : c \succ_i a\}| \end{cases}$$

Since $\forall i$, $\succsim_i$ is single peaked, then

$$\forall i, a \succ_i b \Longrightarrow b \succ_i c, a \succ_i c \Longrightarrow \beta \geqslant \alpha, \gamma \geqslant \alpha$$

$$\forall i, c \succ_i b \Longrightarrow c \succ_i a, b \succ_i a \Longrightarrow n - \gamma \geqslant n - \beta, n - \alpha \geqslant n - \beta \Longrightarrow \gamma \leqslant \beta, \alpha \leqslant \beta$$

(a) $a \succ b \Longrightarrow \alpha > n - \alpha$, since

$$\begin{cases} \beta \geqslant \alpha \\ \gamma \geqslant \alpha \end{cases} \Longrightarrow \begin{cases} \beta > n - \alpha \geqslant n - \beta \\ \gamma > n - \alpha \geqslant n - \gamma \end{cases} \Longrightarrow \begin{cases} \beta > n - \beta \\ \gamma > n - \gamma \end{cases} \Longrightarrow \begin{cases} b \succ c \\ a \succ c \end{cases}$$

(b) $c \succ b \Longrightarrow n - \beta > \beta$, then

$$\begin{cases} \beta \geqslant \alpha \\ \gamma \geqslant \alpha \end{cases} \Longrightarrow \begin{cases} \alpha < n - \beta \leqslant n - \alpha \\ \gamma < n - \beta \leqslant n - \gamma \end{cases} \Longrightarrow \begin{cases} \alpha < n - \alpha \\ \gamma < n - \gamma \end{cases} \Longrightarrow \begin{cases} b \succ a \\ c \succ a \end{cases}$$

(c) $b \succ a$ and $b \succ c$. Whatever relation between (a, c), it will not violate transitivity.

$\square$

9. (Definition) We call the social preference is **liberalism** for each individual i, there $\exists\,(a, b) \in X^2$, such that

$$a \succ_i b \Longrightarrow a \succ b$$

Note:

(a) It's an alternative way to replace IIA with Liberalism.

(b) The intuition of liberalism is that everyone is a dictator over only a small part of the alternatives set.

10. (Example) Sen's Paretian liberal. There are 2 individuals and one book.

(a) a: No one reads it.

(b) b: Only person 1 reads it.

(c) c: Only person 2 reads it.

Rank of person 1 is $a \succ_1 b \succ_1 c$.

Rank of person 2 is $b \succ_2 c \succ_2 a$.

It's evident that person 1 is a dictator over (a, b), and person 2 is a dictator over (a, c), then socially, $a \succ b, c \succ a$.

By unanimity, $b \succ_1 c, b \succ_2 c \Longrightarrow b \succ c$, then this example violates transitivity.

Chapter 7

Competitive and Profit-Maximizing Firms

7.1 Neoclassic Producer Theory

1. (Definition) An economy with k commodities.

   (a) A **production vector** or **production plan** is a vector $z \in \mathbb{R}^k$ that describes the net outputs of the k commodities from a production process.

   Note: For example, $k = 3$ and $z = (-5, 2, 0)$ describes we can produce 2 units of good 2 by using 5 units of good 1.

   (b) A firm is equivalent to its **production possibility set** $Z \subseteq \mathbb{R}^k$.

2. (Assumption) Some requirements on production possibility set Z.

   (a) (Nonempty) $Z \neq \varnothing$.

   (b) (Closedness) Z is closed.

   (c) (No free lunch) $Z \cap \mathbb{R}^k \subseteq \{0\}$, which means no pay, no return.

   (d) (Possibility of inaction, or the ability to shut down) $0 \in Z$.

   (e) (Free disposal) If $z \in Z, z' \leqslant z$ then $z' \in Z$.

   (f) (Irreversibility) If $z \in Z$ and $z \neq 0$, then $-z \notin Z$. This assumption holds trivially if we add the time dimension to commodities.

   (g) (Additivity, or free entry) If $z, z' \in Z$, then $z + z' \in Z$.

3. (Definition) We say a production possibility set Z is

   (a) **constant return to scale** (both IRS and DRS), if $z \in Z$ then $\forall \alpha \geqslant 0, \alpha z \in Z$.

   (b) **decreasing return to scale**, if $z \in Z$ then $\forall \alpha \in [0, 1], \alpha z \in Z$.

   (c) **increasing return to scale**, if $z \in Z$ then $\forall \alpha \geqslant 1, \alpha z \in Z$.

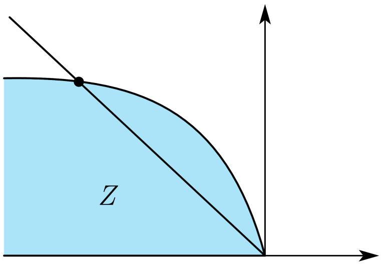

Fig. 7.1. DRS.

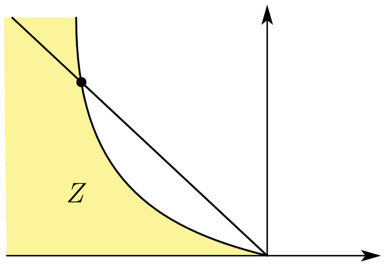

Fig. 7.2. IRS.

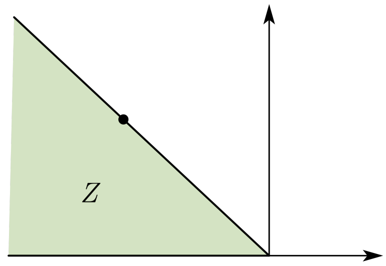

Fig. 7.3. CRS.

(d) **a convex cone**, if Z is convex and CRS, i.e., if $z, z' \in Z$, then $\forall \alpha, \beta \geqslant 0$, then $\alpha z + \beta z' \in Z$.

4. (Definition) The **recession-cone property**. If $\{z^n\}$ is a sequence from Z such that $\|z^n\| \to \infty$, then every accumulation point (not necessarily unique) of the set

$$\left\{ \frac{z^n}{\|z^n\|} : n \in \mathbb{N} \right\}$$

lies in negative orthant of $\mathbb{R}^k$.

Note:

(a) All the points in $\left\{ \frac{z^n}{\|z^n\|} : n \in \mathbb{N} \right\}$ are on the unit circle, and the recession-cone property demands that the normalized vectors will only accumulate in the negative orthant (contain the axes).

(b) The property is inconsistent with IRS and thus, CRS.

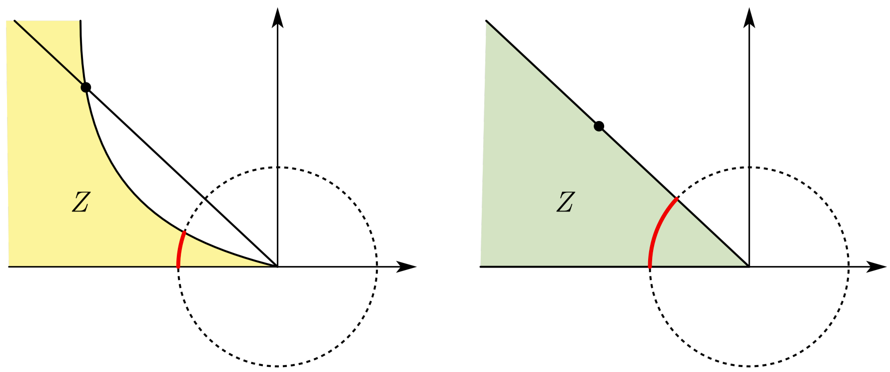

Fig. 7.4. IRS, CRS, and recession-cone property

(c) Why IRS and CRS should be eliminated?

For example, if $p_1 = 1, p_2 = 2$, $(-1, 2) \in Z$, and Z is CRS, then $\forall \alpha \geqslant 0, \alpha(-1, 2) \in Z$. Therefore, we can duplicate the production combination to make infinite profit.

(d) Does the recession-cone property equivalent with DRS?

The answer is no. Recession-cone property doesn't imply DRS. Because we can have a production possibility set like Fig.7.4. The graph shows a Z that satisfies recession-cone property but is not DRS.

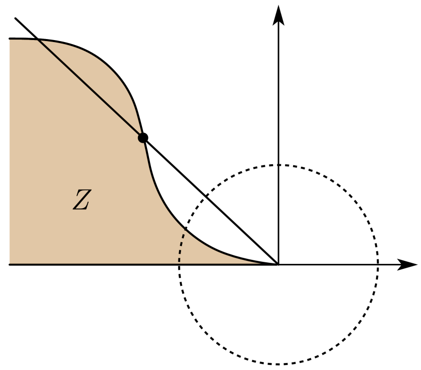

Fig. 7.5. DRS and recession-cone property

5. (Remark) Set-ups.

   (a) Neoclassical firm is a price taker (has no market power).

   (b) Linear prices.

   (c) Firm's objective is the **profit maximization**: For a given $Z \subseteq \mathbb{R}^k$, and $p \in \mathbb{R}^k_{++}$, the firm solves

   $$\max_{z \in Z} z \cdot p$$

   (d) (Definition) Fixing Z, the **optimal production plan** is

   $$Z^* (p) = \arg\max_{z \in Z} p \cdot z$$

   and the **profit function** is

   $$\pi (p) = \sup_{z \in Z} p \cdot z$$

   (e) Since $Z \neq \varnothing, \pi (p) > -\infty$ (Recall that $\sup \varnothing = -\infty$).

   (f) $\pi (p)$ can be ∞, in which case we say $Z^* (p) = \varnothing$

   Again, if $p_1 = 1, p_2 = 2$, $(-1, 2) \in Z$, and Z is CRS, then $\forall \alpha \geqslant 0, \alpha (-1, 2) \in Z$. Therefore, $\pi (p) = \infty$, and $Z^* (p)$ is not well defined. That's why the recession-cone property is important.

6. (Theorem) Simple facts about the PMP.

   (a) Suppose Z exhibits increasing (or constant) return to scale. Then $\forall p \in \mathbb{R}^k_{++}$, if $\pi (p) \geqslant 0$, then either $\pi (p) = 0$ or $\pi (p) = \infty$.

   Proof. If $\pi (p) > 0$, then $\exists z^0 \in Z$, such that $p \cdot z^0 > 0$, then $\forall \alpha \geqslant 1$, $\alpha z^0 \in Z$, then $\alpha z^0 \cdot p = \alpha (z^0 \cdot p) > \alpha \frac{z^0 \cdot p}{2}$, let $\alpha \to \infty$, then $\pi (p) = \infty$.

   $\square$

   (b) The function $\pi (p)$ is homogenous with degree 1 and convex.

   Note: Image all the prices double, then the profit also doubles, it is just a monetary phenomenon.

Proof. For homogeneity, for $\forall \alpha > 0$,

$$\pi(\alpha p) = \sup_{z\in Z} (\alpha p) \cdot z = \alpha \sup_{z\in Z} p \cdot z = \alpha\pi(p)$$

As for $\alpha = 0$, $\forall z \in \mathbb{R}^k$, $(\alpha p) \cdot z = 0$, then $\sup\limits_{z\in Z} (\alpha p) \cdot z = \sup\limits_{z\in Z} 0 = 0$.
But $\pi(\cdot)$ is only well defined for $p \gg 0$. Therefore, we need to extend the definition of $\pi(p)$ to $\mathbb{R}^k$, and then $0 \cdot \pi(p) = 0$ holds for whether $\pi(p)$ is infinite or finite (Define $0 \cdot \infty = 0$ here).

For convexity, $\forall p, p' \in \mathbb{R}^k_{++}$, $\forall \alpha \in [0,1]$, then

$$\begin{aligned}
\alpha\pi(p) + (1-\alpha)\pi(p') &= \alpha \sup_{z\in Z} p \cdot z + (1-\alpha) \sup_{z\in Z} p' \cdot z \\
&= \sup_{z\in Z} \alpha p \cdot z + \sup_{z\in Z} (1-\alpha) p' \cdot z \\
&\leqslant \sup_{z\in Z} [\alpha p \cdot z + (1-\alpha) p' \cdot z] \\
&= \sup_{z\in Z} [\alpha p + (1-\alpha) p'] \cdot z
\end{aligned}$$

The inequality is just by the property of supremum. □

(c) If Z is convex, then $\forall p \in \mathbb{R}^k_{++}$, $Z^*(p)$ is a convex set.

Proof. $\forall z, z' \in Z^*(p)$, then by the convexity of Z, $\forall \alpha \in [0,1]$,

$$\alpha z + (1-\alpha) z' \in Z$$

Note that $z, z' \in Z^*(p)$ implies

$$\pi(p) = z \cdot p = z' \cdot p$$

Therefore,

$$\alpha z \cdot p + (1-\alpha) z' \cdot p = [\alpha z + (1-\alpha) z'] \cdot p = \pi(p)$$

i.e., $\alpha z + (1-\alpha) z' \in Z^*(p)$. □

(d) $\forall p \in \mathbb{R}^k_{++}$, $\forall \alpha > 0$, $Z^*(p) = Z^*(\alpha p)$.

Proof. By definition, $\forall \alpha > 0$

$$Z^*(p) = \arg\max_{z\in Z} p \cdot z = \arg\max_{z\in Z} \alpha (p \cdot z) = \arg\max_{z\in Z} (\alpha p) \cdot z$$

□

7. (Theorem, price effect) Suppose $z \in Z^*(p)$ and $z' \in Z^*(p')$, then $(p - p') \cdot (z - z') \geqslant 0$.

Proof. Note that $z' \cdot p \leqslant z \cdot p$, and $z \cdot p' \leqslant z' \cdot p'$. Thus,

$$\begin{aligned}(p - p') \cdot (z - z') &= p \cdot z - p \cdot z' - p' \cdot z + p' \cdot z' \\ &= p\,(z - z') + p'\,(z' - z) \\ &\geqslant 0\end{aligned}$$

Note: If we take $z \in Z^*(p)$, and $p - p' = (p_1 - p_1', 0, \cdots, 0)$, and $z' \in Z^*(p')$. Then if $p_1' > p_1$, then by the theorem,

$$(p - p') \cdot (z - z') = (p - p_1') \cdot (z_1 - z_1') \geqslant 0$$

which implies $z_1 - z_1' \leqslant 0$. Simply noting, $p_1 \uparrow \Longrightarrow z_1 \uparrow$. □

8. (Theorem, existence of solutions) Suppose Z is closed and nonempty. For a given$Z$, a solution to $\max_{z \in Z} p \cdot z$ exists for every $p \in \mathbb{R}^k_{++}$ iff Z satisfies the recession-cone property.

Proof. 2 steps.

Step 1. Show the sufficiency.

Given any $p \in \mathbb{R}^k_{++}$, since $Z \neq \varnothing$, then $\exists z^0 \in Z$, we let $L = p \cdot z^0 - 1$, then L is a lower bound of $\{p \cdot z : z \in Z\}$.

Step 1.1. Show that for any convergent sequence $\{z^n\}_{n=1}^{\infty} \in Z^{\mathbb{N}}$, the recession-cone property eliminates the case $\|z^n\| \to \infty$.

Suppose $\|z^n\| \to \infty$, then $\forall K > 0$, $\exists N > 0$ such that $\forall n \geqslant N \quad \|z^n\| > K > 0$.

Let $\hat{z}^m = \frac{z^{N+m}}{\|z^{N+m}\|}$. Since $\forall n \geqslant 1, p \cdot z^n > L$, then $p \cdot \hat{z}^m \geqslant \frac{L}{\|z^{N+m}\|} \to 0$ as $m \to \infty$.

Therefore, for any accumulation point z^* of $\{\hat{z}^m\}$, $p \cdot z^* \geqslant 0$.

But by the recession-cone property, $z^* \leqslant 0$. Note that $p \gg 0$, then $p \cdot z^* \leqslant 0$.

Then, $p \cdot z^* = 0 \Longrightarrow z^* = 0$. But $\forall m \geqslant 1, \|z^m\| > K > 0$, a contradiction.

Step 2.1. Show $\max_{z \in Z} p \cdot z$ exists.

By the closedness of Z, there exists $\{z^n\} \in Z^{\mathbb{N}}$, such that $p \cdot z^n \to \sup_{z \in Z} p \cdot z$ as $n \to \infty$. Note that

$$\lim_{n \to \infty} \|z^n\| < \infty \Longrightarrow \lim_{n \to \infty} \|z_i^n\| < \infty, \forall i = 1, \cdots, k$$

then $\sup_{z \in Z} p \cdot z < \infty$.

Therefore, $\sup_{z \in Z} p \cdot z \in \{p \cdot z : z \in Z\} \Longrightarrow \exists z^* \in Z, p \cdot z^* = \sup_{z \in Z} p \cdot z$.

Step 2. Show the necessity.

Suppose the recession cone property fails. Then $\exists \{z^n\} \in Z^{\mathbb{N}}$, with $\|z^n\| \to \infty$, $\frac{z^n}{\|z^n\|} \to z^*$, but $\exists i$ such that $z_i^* > 0$.

Let p be the price vector that is 1 in all components but the ith, and $p_i = \frac{k+1}{z_i^*}$. Then

$$p \cdot \frac{z^n}{\|z^n\|} = \frac{1}{\|z^n\|}\left[\sum_{j\neq i} z_j^n + \frac{k+1}{z_i^*} z_i^n\right] = \sum_{j\neq i}\frac{z_j^n}{\|z^n\|} + \frac{k+1}{z_i^*}\frac{z_i^n}{\|z^n\|} \to \sum_{j\neq i} z_j^* + k + 1 \geqslant 1$$

then

$$p \cdot z^n \geqslant \|z^n\| \to \infty$$

which implies $Z^*(p) = \varnothing$, a contradiction.

□

9. (Theorem) Suppose Z is closed, nonempty, and has the recession-cone property. Then for any $z^0 \in Z$, the correspondence

$$p \rightrightarrows \left\{z \in Z : p \cdot z \geqslant p \cdot z^0\right\}$$

is locally bounded and upper semi-continuous.

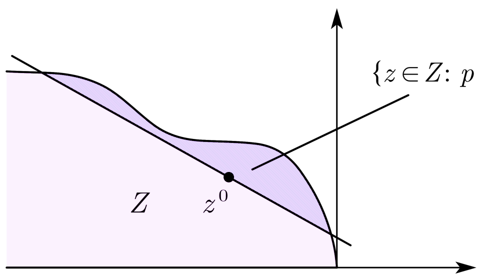

Fig. 7.6

Proof. First prove the upper semi-continuity.

Let $\phi(p) = \{z \in Z : p \cdot z \geqslant p \cdot z^0\}$. $\forall \{p^n\}_{n=1}^{\infty} \subseteq \mathbb{R}_{++}^k$ with $p^n \to p$, then consider $\forall \{z^n\}_{n=1}^{\infty} \subseteq Z$ with $\lim_{n\to\infty} z^n$ exists (let it be z) and $z^n \in \phi(p^n)$, i.e.,

$$p^n \cdot z^n \geqslant p^n \cdot z^0$$

Let $n \to \infty$, we have

$$p \cdot z \geqslant p \cdot z^0 \Longrightarrow z \in \phi(p)$$

Now prove the locally boundedness.

Suppose ϕ is not locally bounded. Then $\exists p \in \mathbb{R}_{++}^k$, $\forall \varepsilon > 0$, $\exists p' \in B_{\varepsilon}(p)$ and there is no bounded set $Z(p) \subseteq Z$ such that $\phi(p') \subseteq Z(p)$, which implies $\phi(p') = \{z \in Z : p' \cdot z \geqslant p' \cdot z^0\}$ is unbounded.

Therefore, $\exists \{z^n\} \subseteq \phi(p')$ with $\|z^n\| \to \infty$, a contradiction to the recession-cone property. $\square$

10. (Theorem) If Z is closed, nonempty, and satisfies the recession-cone property, then the correspondence $p \rightrightarrows Z^*(p)$ is nonempty valued, locally bounded and upper semi-continuous. Moreover, the function $p \to \pi(p)$ is continuous.

    Note: If $Z^*(p)$ is singleton-valued under such Z, then the function $p \to z^*(p)$ is continuous.

    Proof. It's a direct corollary of Berge's Theorem. $\square$

11. (Theorem, Hotelling's Lemma) Suppose the profit function π is continuously differentiable at price p^* and suppose Z is closed. Then $Z^*(p^*)$ is a singleton set, and $z^*(p^*) = \left(\frac{\partial\pi}{\partial p_1}, \frac{\partial\pi}{\partial p_2}, \cdots, \frac{\partial\pi}{\partial p_k}\right)|_{p=p^*} \in Z^*(p^*)$.

    Proof. The proof of $Z^*(p^*)$ is a singleton is omitted. Recall the problem,

$$\pi(p) = \max_{z\in Z} p\cdot z = \max_{z\in Z}\{p_1 z_1 + \cdots + p_k z_k\}\ z^*(p) = \arg\max_{z\in Z} p\cdot z$$

    by the envelope theorem,

$$\frac{\partial\pi}{\partial p_i}(p_i^*) = z_i^*(p_i^*)$$

    $\square$

12. (Theorem) The profit function π^* is differentiable at a price p^0 iff $Z^*(p^0)$ is singleton, in which case $\frac{\mathrm{d}\pi}{\mathrm{d}p}|_{p=p^0} \in Z^*(p^0)$.

    Proof. Omitted. $\square$

7.2 Cost Minimization with a Single Output (MWG)

1. (Remark) Notations.

    (a) q is the amount of output.

    (b) z is a nonnegative vector of inputs.

    (c) $q = f(z)$ is the production function.

    (d) $w \gg 0$ is the vector of input prices.

    (e) The **cost minimization problem** (CMP) is $\min_{z\geqslant 0} w\cdot z$
    s.t. $f(z) \geqslant q$

    (f) $c(w,q)$ is the optimal value function or the **cost function**.

    (g) $z(w,p)$ is the **factor demand correspondence**.

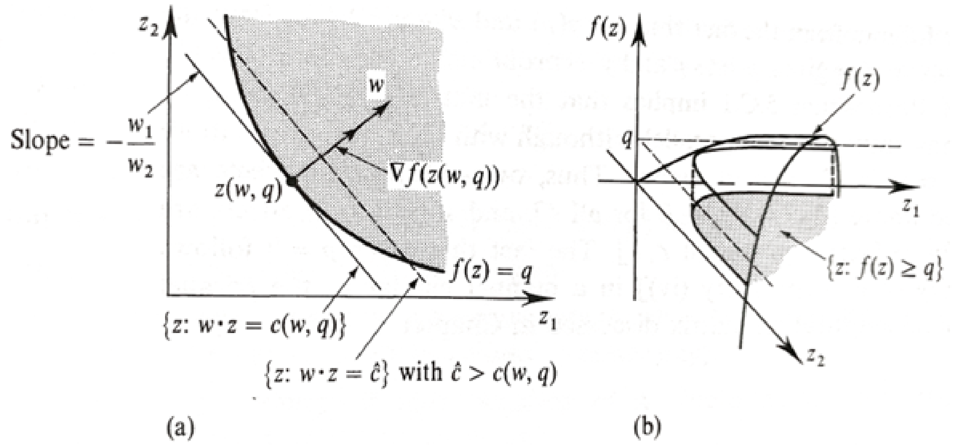

Fig. 7.7

(h) Note: The CMP is still well defined even if the firm is not a price taker in its output market, so long as it is a price taker in its input factors market.

2. (Remark) Let commodity 1 $L-1$ be the input factors, and commodity L be the output.

$$\min_{z \geqslant 0} w \cdot z \text{s.t. } f(z) \geqslant q$$

Then form the Lagrangian,

$$\mathcal{L} = w \cdot z + \lambda (q - f(z))$$

Suppose z^* is optimal, then for $\lambda > 0$, the F.O.C.s are

$$w_\ell \geqslant \lambda \frac{\partial f(z^*)}{\partial z_\ell}, \forall \ell = 1, 2, \cdots L-1 z_\ell^* \cdot \left[w_\ell - \lambda \frac{\partial f(z^*)}{\partial z_\ell} \right] = 0, \forall \ell = 1, 2, \cdots L-1$$

or in matrix form,

$$w \geqslant \lambda \nabla f(z^*) \; [w - \lambda \nabla f(z^*)] \cdot z^* = 0$$

Note that

$$c(w, q) = \min_{z \geqslant 0} \{ w \cdot z + \lambda (q - f(z)) \}$$

then by the Envelop theorem,

$$\frac{\partial c(w, q)}{\partial q} = \lambda$$

3. (Remark)

(a) The marginal rate of technical substitution (MRTS) of input l for input k is defined as

$$\text{MRTS}_{lk} := -\frac{\mathrm{d} z_k}{\mathrm{d} z_l} = \frac{\frac{\partial f}{\partial z_l}(z)}{\frac{\partial f}{\partial z_k}(z)}$$

At the optimum, suppose $z^* \gg 0$,

$$\frac{w_l}{w_k} = \frac{\frac{\partial f}{\partial z_l}(z)}{\frac{\partial f}{\partial z_k}(z)} = \text{MRTS}_{lk}$$

(b) The elasticity of substitution is defined as

$$\sigma_{lk} = -\frac{\mathrm{d}\log\left(\frac{z_l}{z_k}\right)}{\mathrm{d}\log\left(\mathrm{MRTS}_{lk}\right)} = -\frac{\mathrm{d}\log\left(\frac{z_l}{z_k}\right)}{\mathrm{d}\log\left(\frac{w_l}{w_k}\right)} \approx -\frac{\frac{\Delta\left(\frac{z_l}{z_k}\right)}{\frac{z_l}{z_k}}}{\frac{\Delta(\mathrm{MRTS}_{lk})}{\mathrm{MRTS}_{lk}}}$$

(c) (Example) $f(z) = z_1^{\alpha} z_2^{1-\alpha}$, $\sigma_{12} = 1$.

$$\begin{aligned}\sigma_{12} &= -\frac{\mathrm{d}\log\left(\frac{z_l}{z_k}\right)}{\mathrm{d}\log\left(\frac{w_l}{w_k}\right)} = -\frac{\mathrm{d}\log\left(\frac{z_l}{z_k}\right)}{\mathrm{d}\log\left(\frac{\alpha}{1-\alpha}\frac{z_k}{z_l}\right)} \\ &= -\frac{\mathrm{d}\log\left(\frac{z_l}{z_k}\right)}{\mathrm{d}\left[\log\left(\frac{\alpha}{1-\alpha}\right) + \log\left(\frac{z_k}{z_l}\right)\right]} = -\frac{\mathrm{d}\log\left(\frac{z_l}{z_k}\right)}{0 + \mathrm{d}\log\left(\frac{z_k}{z_l}\right)} = 1\end{aligned}$$

4. (Theorem, Properties of the cost function) Suppose that a single-output technology Y is closed and satisfies the free disposal property. Then

(a) $c(w, q)$ is homogenous of degree one in w and non-decreasing in q.

Proof. Recall the Lagrangian function for

$$\min_{z \geqslant 0} w \cdot z \text{s.t. } f(z) \geqslant q$$

is

$$\mathcal{L} = w \cdot z + \lambda (q - f(z))$$

Note that $\lambda (q - f(z)) = 0$ and $\lambda \geqslant 0$, and

$$c(w, q) = \min_{z \geqslant 0} \{w \cdot z + \lambda (q - f(z))\}$$

Then $\forall \alpha \geqslant 0$,

$$c(\alpha w, q) = \min_{z \geqslant 0, f(z) \geqslant q} \alpha w \cdot z = \alpha \min_{z \geqslant 0, f(z) \geqslant q} w \cdot z = \alpha c(w, q)$$

and by the envelop theorem,

$$\frac{\partial c(w, q)}{\partial q} = \lambda \geqslant 0$$

□

(b) $c(w, q)$ is a concave function of w.

Proof. Just by the definition of concavity and minimum (or infimum). □

(c) (Duality) If $\forall q \in \mathbb{R}^{L-1}$, $\{z \geqslant 0 : f(z) \geqslant q\}$ is convex, then

$$Y = \{(z, -q) : w \cdot z \geqslant c(w, q), \forall w \gg 0\}$$

Proof. Omitted. □

(d) If f is concave, then $c(w,q)$ is a convex function of q. (In particular, then the marginal cost is nondecreasing in q).

Proof. $\forall q_1, q_2 \in \mathbb{R}$, and then $\exists z_1 \in z^*(w,p_1), \exists z_2 \in z^*(w,p_2)$, such that

$$f(z_1) = q_1, f(z_2) = q_2$$

then by the convexity of f, $\forall \alpha \in [0,1]$,

$$f(\alpha z_1 + (1-\alpha) z_2) \geqslant \alpha f(z_1) + (1-\alpha) f(z_2) = \alpha q_1 + (1-\alpha) q_2$$

and then

$$\begin{aligned} c(w, \alpha q_1 + (1-\alpha) q_2) &\leqslant c(w, f(\alpha z_1 + (1-\alpha) z_2)) \\ &\leqslant w \cdot [\alpha z_1 + (1-\alpha) z_2] \\ &= \alpha w \cdot z_1 + (1-\alpha) w \cdot z_2 \\ &= \alpha c(w,q_1) + (1-\alpha) c(w,q_2) \end{aligned}$$

□

5. (Theorem, Properties of the factor demand correspondence)

(a) $z(w,q)$ is homogenous of degree zero in w.

Proof. $\forall \alpha > 0$, $\underset{z \geqslant 0, f(z) \geqslant q}{\arg\min}\ w \cdot z = \underset{z \geqslant 0, f(z) \geqslant q}{\arg\min}\ \alpha w \cdot z$. □

(b) If $\{z \geqslant 0 : f(z) \geqslant q\}$ is convex, then $z(w,q)$ is a convex set. Moreover, if $\{z \geqslant 0 : f(z) \geqslant q\}$ is a strictly convex set, then $z(w,q)$ is single valued.

Proof. The first statement is easy, just by the definition of convexity or

$$z(w,q) = \{z \geqslant 0 : f(z) \geqslant q\} \cap \{z \geqslant 0 : w \cdot z = c(w,q)\}$$

Note that $\{z \geqslant 0 : w \cdot z = c(w,q)\}$ is also convex, then $z(w,q)$ is convex.

Now we prove the next statement.

Let $q \geqslant 0, w \gg 0, z \in z(w,q), z' \in z(w,q)$ and $z \neq z'$, by the convexity of $z(w,q)$,

$$\frac{1}{2} z + \frac{1}{2} z' \in z(w,q)$$

since $\{z \geqslant 0 : f(z) \geqslant q\}$is strictly convex, then $\exists z'' \in \{z \geqslant 0 : f(z) \geqslant q\}$, such that

$$\frac{1}{2} z + \frac{1}{2} z' \gg z''$$

thus, $w \cdot \left(\frac{1}{2} z + \frac{1}{2} z'\right) > w \cdot z''$, contradicting to $\frac{1}{2} z + \frac{1}{2} z' \in z(w,q)$. □

(c) (Shepard's Lemma) If $z(\bar{w}, q)$ is a singleton, then $c(w, q)$ is differentiable w.r.t. w at $\bar{w}$, and $\nabla_w c(\bar{w}, q) = z(\bar{w}, q)$.

Proof. The proof for differentiability is omitted. $\nabla_w c(\bar{w}, q) = z(\bar{w}, q)$ is from the Envelope theorem. □

(d) If $z(w, q)$ is differentiable at $\bar{w}$, then $\mathrm{D}_w z(\bar{w}, q) = \mathrm{D}_w^2 c(\bar{w}, q)$ is a symmetric and negative semidefinite matrix with $\mathrm{D}_w z(\bar{w}, q) \cdot \bar{w} = 0$.

Proof. These conclusions are purely from math (constraint optimization part). □

(e) If $f(z)$ is homogeneous of degree one (i.e., CRS) then $c(w, q)$ and $z(w, q)$ are homogenous of degree one in q.

Proof. $\forall \alpha > 0$,

$$
\begin{aligned}
c(w, \alpha q) &= \min_{z \geqslant 0, f(z) \geqslant \alpha q} w \cdot z = \min_{z \geqslant 0, f\left(\frac{z}{\alpha}\right) \geqslant q} w \cdot z = \min_{\alpha x \geqslant 0, f(x) \geqslant q} \alpha w \cdot x \\
&= \alpha \min_{x \geqslant 0, f(x) \geqslant q} w \cdot x = \alpha c(w, q) \\
z(w, \alpha q) &= \arg\min_{z \geqslant 0, f(z) \geqslant \alpha q} w \cdot z = \alpha \arg\min_{x \geqslant 0, f(x) \geqslant q} w \cdot x = \alpha z(w, q)
\end{aligned}
$$

□

6. (Definition) A production vector y is **efficient** if there is no $y' \in Y$ such that $y' \geqslant y$ and $y' \neq y$.

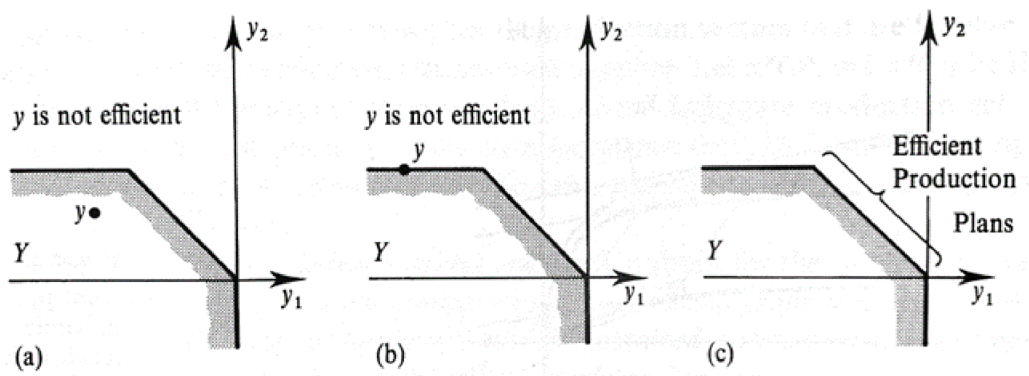

Fig. 7.8

7. (Theorem)

(a) (A version of the first fundamental theorem of welfare economics) If $y \in Y$ is profit maximizing for some $p \gg 0$, then y is efficient.

Proof. Suppose y is not efficient, then $\exists y' \in Y$, such that $y \leqslant y'$ and $y \neq y'$. Therefore,

$$p \cdot y' - p \cdot y = p \cdot (y' - y) > 0$$

A contradiction to that y is a profit maximizer. □

(b) (A version of the second fundamental theorem of welfare economics) Suppose that Y is convex, then every efficient production $y \in Y$ is a profit-maximizing production for some nonzero $p \geqslant 0$.

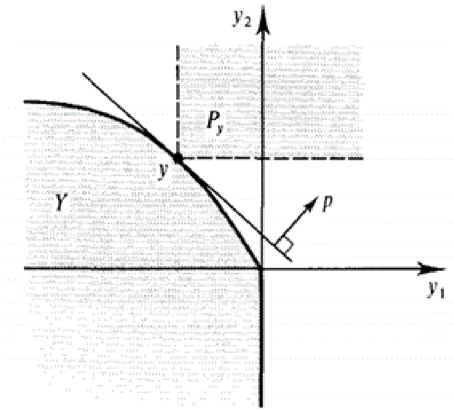

Fig. 7.9

Proof. Suppose y is an efficient production, define

$$P_y = \left\{ y' \in \mathbb{R}^L : y' \gg y \right\}$$

then P_y is convex, and $Y \cap P_y = \varnothing$. Then by the separating hyperplane theorem, $\exists p$, such that

$$p \cdot y' \geqslant p \cdot y'', \forall y' \in P_y, \forall y'' \in Y$$

i.e., $\forall y' \in P_y$,

$$p \cdot y' \geqslant p \cdot y'', \forall y'' \in Y$$

We can prove that $p \geqslant 0$ just as what we did in the pareto optimality part, and then choose a sequence $\{y^n\} \subseteq P_y$, with $y^n \to y$ to conclude that y is a profit-maximizer.

□

7.3 Aggregating Firms

1. (Definition) Let $X_1, X_2, \cdots, X_n$ be finite subsets of $\mathbb{R}^k$. Define the **Minkowski sum** from X_1 to X_n as

$$\sum_{i=1}^{n} X_i := \left\{ x \in \mathbb{R}^k : x = \sum_{i=1}^{n} x_i, (x_1, x_2, \cdots, x_n) \in \prod_{i=1}^{n} X_i \right\}$$

If $\exists i = 1, 2, \cdots, n, X_i = \varnothing$, then $\sum_{i=1}^{n} X_i := \varnothing$.

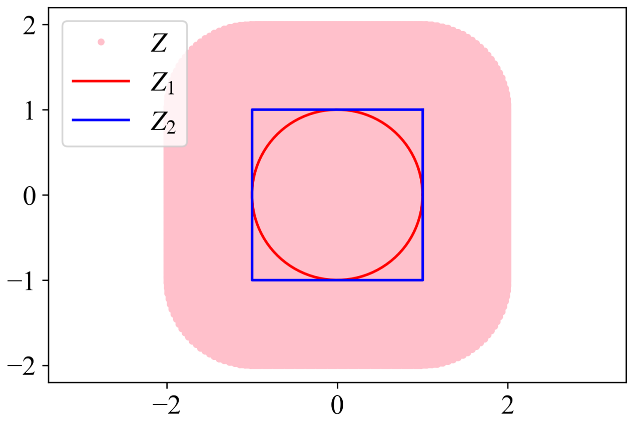

Fig. 7.10. An example of Minkowski sum.

2. (Remark) Set-ups.

   (a) Suppose there are F production units.

   (b) Firm f has a nonempty production-possibility set $\varnothing \neq Z^f \subseteq \mathbb{R}^k$.

   (c) For $p \in \mathbb{R}^k_{++}$, $\pi^f(p) := \sup\limits_{z \in Z^f} p \cdot z$, $Z^{f*}(p) := \arg\max\limits_{z \in Z^f} p \cdot z$.

   (d) The **aggregate production-possibility set** is $Z := \sum_{f=1}^F Z^f$.

   (e) For $p \in \mathbb{R}^k_{++}$,

      i. the **aggregate profit function** is $\pi(p) := \sum_{f=1}^F \pi^f(p)$

      ii. the **aggregate optimal-output set** is $Z^*(p) := \sum_{f=1}^F Z^{f*}(p)$

   (f) Assume that no production externalities.

3. (Theorem)

$$\pi(p) = \sup_{z \in Z} p \cdot z Z^*(p) = \{z \in Z : p \cdot z = \pi(p)\}$$

Note: The sum of the individual-firm profit functions is the profit function of a firm formed by summing the individual-firm production possibility sets. Moreover, the sum of their optimal-netput correspondence is the aggregate optimal-netput correspondence for this megafirm.

Proof of $\pi(p) = \sup\limits_{z \in Z} p \cdot z$. W.L.G., we only need to show the theorem when $F = 2$.

Firstly, note that $\pi^f(p) > -\infty, \forall p$, since $Z^f \neq \varnothing$. And thus, $Z \neq \varnothing$, $\pi(p) > -\infty$.

For each p, by definition,

$$\pi(p) = \pi^1(p) + \pi^2(p)$$

Case 1. $\pi^1(p)$ or $\pi^2(p)$ equals to ∞, then $\pi(p) = \infty$ (is well-defined since $\pi^1(p), \pi^2(p) > -\infty$).

Let $\pi^1(p) = \infty$, then $\exists z^{1,n} \in Z^1$ such that $p \cdot z^{1,n} \uparrow \infty$.

Moreover, $\exists z^2 \in Z^2$, such that $\pi^2(p) \geqslant p \cdot z^2$.

Let $z^n = z^{1,n} + z^2$, and then $p \cdot z^n = p \cdot z^{1,n} + p \cdot z^2 \uparrow \infty$.

Note that $z^n \in Z = Z^1 + Z^2$, and Z is closed, therefore, $\sup\limits_{z \in Z} p \cdot z = \infty$.

Then $\pi(p) = \pi^1(p) + \pi^2(p)$.

Case 2. $\max\{\pi^1(p), \pi^2(p)\} < \infty \Longrightarrow \pi^1(p) < \infty, \pi^2(p) < \infty$.

Let $\pi^0(p) = \sup\limits_{z \in Z} p \cdot z$, we need to show that $\pi^0(p) = \pi(p) := \pi^1(p) + \pi^2(p)$.

Step 2.1. Show that $\pi^0(p) \leqslant \pi^1(p) + \pi^2(p)$

$\exists z^n \in Z$, such that $z^n \cdot p \uparrow \pi^0(p)$, and by definition of Z, $\forall n \geqslant 1$, $\exists (z^{1,n}, z^{2,n}) \in Z^1 \times Z^2$ such that $z^n = z^{1,n} + z^{2,n}$. Then by definition of π^f,

$$z^n \cdot p = z^{1,n} \cdot p + z^{2,n} \cdot p \leqslant \pi^1(p) + \pi^2(p)$$

Let $n \to \infty$, then

$$\pi^0(p) \leqslant \pi^1(p) + \pi^2(p)$$

Step 2.2. Show that $\pi^0(p) \geqslant \pi^1(p) + \pi^2(p)$

$\exists \{(z^{1,n}, z^{2,n})\} \in \left(Z^1 \times Z^2\right)^{\mathbb{N}}$, such that

$$p \cdot z^{1,n} \uparrow \pi^1(p)\, p \cdot z^{2,n} \uparrow \pi^2(p)$$

Then

$$p\left(z^{2,n} + z^{1,n}\right) = p \cdot z^{2,n} + p \cdot z^{1,n} \uparrow \pi^1(p) + \pi^2(p)$$

Since $z^{2,n} + z^{1,n} \in Z$, then by the definition of $\pi^0(p)$,

$$p\left(z^{2,n} + z^{1,n}\right) \leqslant \pi^0(p)$$

Therefore,

$$\pi^0(p) \geqslant \pi^1(p) + \pi^2(p)$$

□

7.4 The Linear Activity Model (MWG)

1. (Example) Suppose that there are three goods, and two Leontief plans

$$a_1 = (1, -1, -2)\, a_2 = (-\beta, 1, -4)\,, \beta > 0$$

Plan a_1 is corresponding to the production function

$$x_1 = \min\left(x_2, \frac{x_3}{2}\right)$$

Plan a_2 is to

$$x_2 = \min\left(\frac{x_1}{\beta}, \frac{x_3}{4}\right)$$

Note that x_3 cannot be produced in this economy.

Suppose that firm 1 plans to produce α_1 units of good 1, and firm 2 plans to produce α_2 units of good 2. Then

$$\begin{cases} \alpha_1 = \min\left(x_2, \frac{x_3}{2}\right) \\ \alpha_2 = \min\left(\frac{x_1}{\beta}, \frac{x_3}{4}\right) \end{cases} \implies \begin{cases} \Delta x_1 = \alpha_1 - \beta\alpha_2 \\ \Delta x_2 = \alpha_2 - \alpha_1 \\ \Delta x_3 = -2\alpha_1 - 4\alpha_2 \end{cases}$$

In matrix form,

$$\Delta x := \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 1 & -\beta \\ -1 & 1 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

Define $b = (2, 4)$, $A = \begin{bmatrix} 0 & \beta \\ 1 & 0 \end{bmatrix}$, then

$$\Delta x = \begin{bmatrix} I - A \\ b \end{bmatrix} \alpha$$

2. (Definition) The **input-output matrix A** is said to be **productive** if $\exists \bar{\alpha} \geqslant 0$,

$$(I - A)\, \bar{\alpha} \gg 0$$

3. (Example) In the previous example, productive A means that the economy can product strictly positive net amount of good 1 and good 2.

$$(I - A)\, \alpha = \begin{bmatrix} 1 & -\beta \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 - \beta\alpha_2 \\ \alpha_2 - \alpha_1 \end{bmatrix} \gg 0$$

Then

$$\begin{cases} \alpha_1 > \beta\alpha_2 \\ \alpha_2 > \alpha_1 \end{cases} \implies \beta < \frac{\alpha_1}{\alpha_2} < 1$$

If $\beta < 1$, then $\frac{1+\beta}{2} \in (\beta, 1)$.

If $\beta > 1$, then $(\beta, 1) = \varnothing$.

Therefore, A is productive iff $\beta < 1$.

4. (Remark) General Leontief's Input-Output Model.
   (a) Each producible good is produced by only one activity. Activity ℓ produces good ℓ, where $\ell = 1, 2, \cdots, L-1$.
   (b) Normalize $a_{\ell\ell} = 1$.
   (c) $(\alpha_1, \cdots, \alpha_{L-1})$ denotes the gross production of good 1 to good $L-1$.
   (d) The input-output matrix is
   $$A = \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1,L-1} \\ -a_{2,1} & 0 & \cdots & -a_{2,L-1} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{L-1,1} & -a_{L-1,2} & \cdots & -a_{L-1,L-1} \end{bmatrix}$$
   (e) The vector $(I-A)\,\alpha$ gives the net production levels of the $L-1$ outputs when the activities are run at levels $\alpha = (\alpha_1, \cdots, \alpha_{L-1})$.
   (f) $b = (-a_{L,1}, -a_{L,2}, \cdots, -a_{L,L-1})$ is the vector of primary factor requirements. The scaler $b \cdot \alpha$ gives the total use of the primary factor.
   (g) Aggregate production set (assuming free disposal and no substitution possibilities) is
   $$Y = \left\{ y \in \mathbb{R}^L : y \leqslant \begin{bmatrix} I-A \\ -b \end{bmatrix} \alpha, \exists \alpha \in \mathbb{R}^L_+ \right\}$$
   (h) If $\exists \bar{\alpha} \geqslant 0$ such that $(I-A)\,\bar{\alpha} \gg 0$, then we say the input-output matrix is **productive**

5. (Theorem) If $A_{L\times L}$ is productive, then $\forall c \in \mathbb{R}^{L-1}_+$, $\exists \alpha \geqslant 0$, such that $(I-A)\,\alpha = c$.

   Note:
   (a) If A is productive, for any nonnegative net amounts of the two producible commodities, there exists a production plan to produce them, provided that there is enough primary factor available.
   (b) For $x \in \mathbb{R}$, we know that if $|x| < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = (1-x)^{-1}$. Now for a matrix A, we have the similar conclusion, if $\rho(A) < 1$, $\sum_{n=0}^{\infty} A^n = (I-A)^{-1}$, where $\rho(A)$ is the spectral radius of matrix A.
   (c) The property of Leontief input-output matrix is called all-or-nothing property.

   Proof. We will show that if A is productive, then the inverse of the matrix $I-A$ exists and is nonnegative.

   To show the claim, we show that if A is productive, then $\lim_{N\to\infty} \sum_{n=0}^{N} A^n$ exists. Because all entries of A are nonnegative, then all entries of $I = A^0, A, A^2, \cdots$ are nonnegative. Thus, every entry of $\sum_{n=0}^{N} A^n$ is nondecreasing with N. Therefore, we only need to show every entry of $\sum_{n=0}^{N} A^n$ is bounded from above.

Since A is productive, then $\exists\bar{\alpha} \geqslant 0$ and $\bar{c} \gg 0$ such that

$$(I - A)\,\bar{\alpha} = \bar{c}$$

Premultiply $\sum_{n=0}^{N} A^n$ both sides of this equality,

$$\left(\sum_{n=0}^{N} A^n - \sum_{n=1}^{N} A^{n+1}\right)\bar{\alpha} = \left(A^0 - A^{N+1}\right)\bar{\alpha} = \left(I - A^{N+1}\right)\bar{\alpha} = \left(\sum_{n=0}^{N} A^n\right)\bar{c}$$

Note that all entries of A^{N+1} are nonnegative, then $\left(I - A^{N+1}\right)\bar{\alpha} \leqslant \bar{\alpha}$, then

$$\left(\sum_{n=0}^{N} A^n\right)\bar{c} \leqslant \bar{\alpha}$$

since $\bar{c} \gg 0$, then any entry of $\sum_{n=0}^{N} A^n$ cannot be larger than $\frac{\max\{\bar{\alpha}_1,\cdots,\bar{\alpha}_{L-1}\}}{\max\{\bar{c}_1,\cdots,\bar{c}_{L-1}\}}$, therefore, $\lim_{N\to\infty}\sum_{n=0}^{N} A^n$ exists.

Moreover, consider

$$\sum_{n=0}^{\infty} A^n (I - A) = \sum_{n=0}^{\infty} A^n - \sum_{n=0}^{\infty} A^{n+1} = A^0 = I$$

Therefore,

$$\sum_{n=0}^{\infty} A^n = (I - A)^{-1}$$

and $\sum_{n=0}^{\infty} A^n$ is nonnegative. $\square$

Chapter 8

General Equilibrium Theory: Some Examples (MWG)

8.1 Edgeworth Box Economy

1. (Definition) Some definitions.

   (a) A **Pure Exchange Economy** is an economy in which there are no production opportunities. Agents possess endowments, economic activity only consists of trading and consumption.

   (b) **Edgeworth Box Economy**:

      i. There are two consumers $i = 1, 2$ and two commodities $\ell = 1, 2$, consumers are price takers.

      ii. Consumer i

         - Consumption vector is $x_i = \begin{pmatrix} x_{1i} \\ x_{2i} \end{pmatrix} \in \mathbb{R}^2_+$.
         - Preference relation $\succsim_i$ is defined on $\mathbb{R}^2_+$.
         - Endowment vector is $\omega_i = \begin{pmatrix} \omega_{1i} \\ \omega_{2i} \end{pmatrix}$.

      iii. Total endowment of good ℓ is $\bar{\omega}_\ell = \omega_{\ell 1} + \omega_{\ell 2}$.

   (c) An **allocation** $x \in \mathbb{R}^4_+$ is an assignment of a nonnegative consumption vector to each consumer.

$$x = (x_1, x_2) = \left(\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}, \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} \right)$$

      i. A feasible allocation is

$$x_{\ell 1} + x_{\ell 2} \leqslant \bar{\omega}_\ell, \ell = 1, 2$$

      ii. A non-wasteful allocation is

$$x_{\ell 1} + x_{\ell 2} = \bar{\omega}_\ell, \ell = 1, 2$$

(d) **Wealth** is determined by prices and endowments as

$$p \cdot \omega_i = p_1 \omega_{1i} + p_2 \omega_{2i}$$

(e) Given endowments, the **budget set** is a function of prices

$$\mathbf{B}_i(p) = \left\{ x_i \in \mathbb{R}_+^2 : p \cdot x_i \leqslant p \cdot \omega_i \right\}$$

	Consumer1	Consumer2	Consumer1	Consumer2
good1	ω_{11}	ω_{12}	x_{11}	x_{12}
good2	ω_{22}	ω_{22}	x_{22}	x_{22}

2. Edgeworth box.

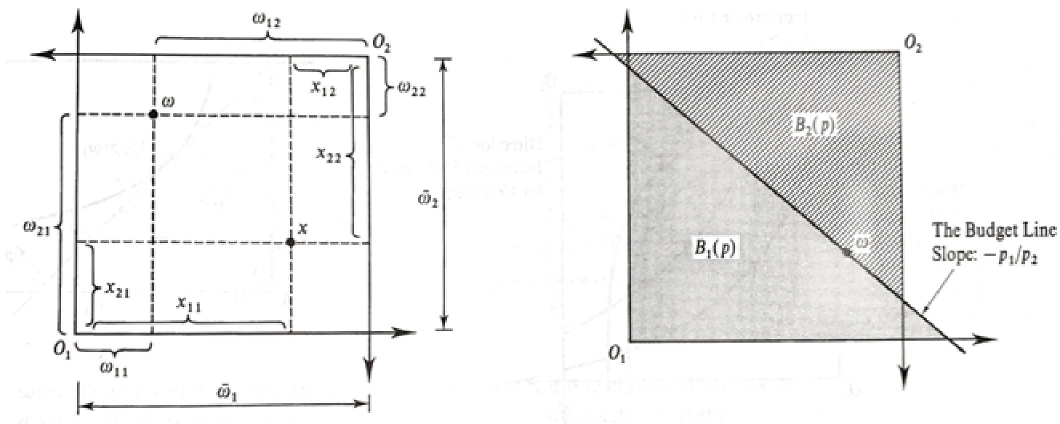

Fig. 8.1

Note that the budget line always cut through the endowment point ω, since ω is always attainable if consumers choose not to trade.

Then we can add the preference onto the graph.

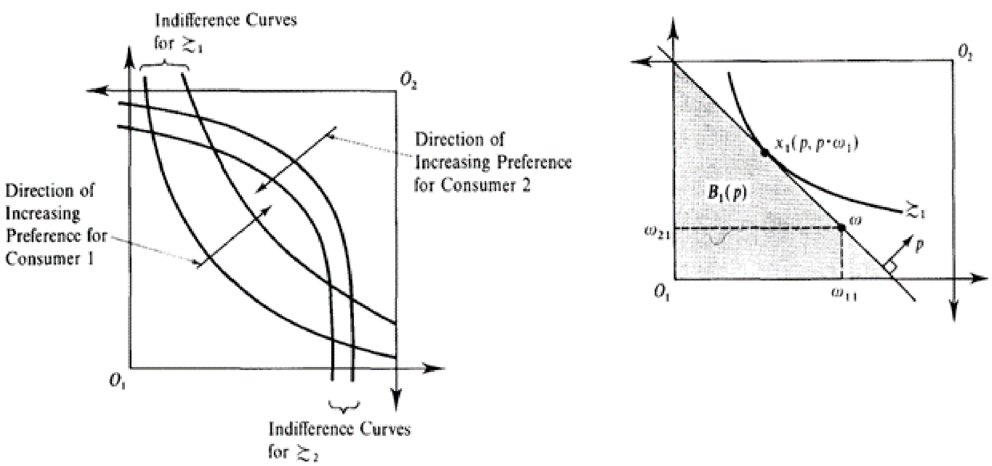

Fig. 8.2

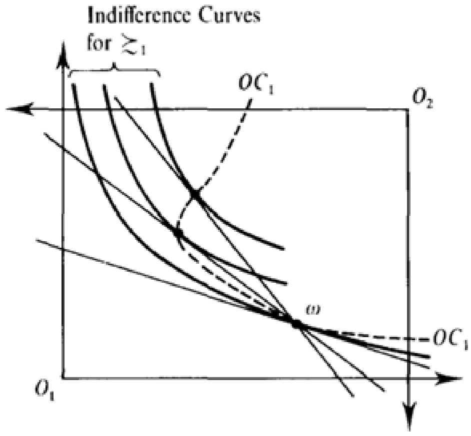

Fig. 8.3

3. (Definition and Remark) **Offer curve**.

    (a) The offer curve of consumer i, is the demand of i given the price p, thus, offer curve if the function of p.

    (b) As p varies, budget line pivots around ω. Given the budget line, consumer demands most preferred point in $\mathbf{B}_i(p)$, the set of demanded consumptions makes up offer curve.

    (c) The offer curve must pass through the endowment point ω. This is because there exists an indifference curve that passes thought ω, and we can draw a tangent line of the indifference curve through.

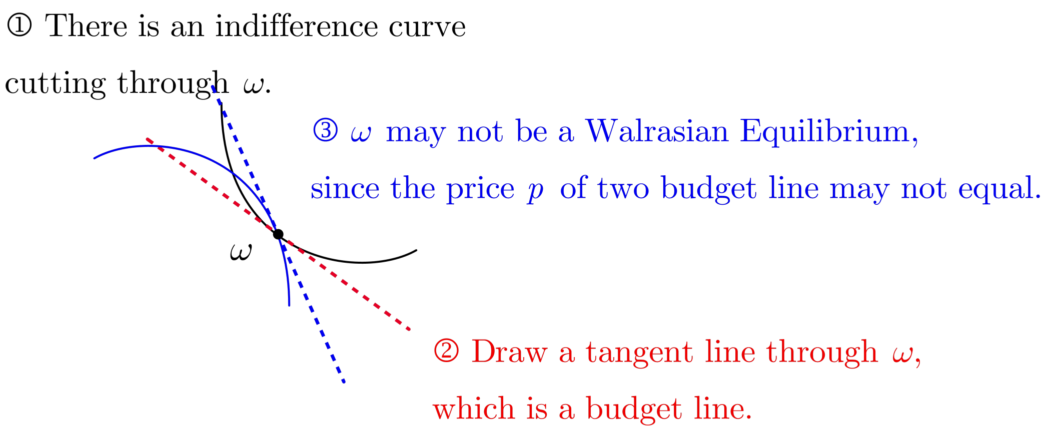

Fig. 8.4

    (d) Since at each point, endowment is affordable, every point on offer curve must be at least as good as ω_i.

    (e) If indifference curves are smooth, the offer curve must be tangent to the consumer's indifference curve at the endowment point.

4. (Definition) A Walrasian (or competitive) equilibrium (WE) for an Edgeworth box economy is a price vector p^* and an allocation $x^* = (x_1^*, x_2^*)$ in the Edgeworth box such that for $i = 1, 2$

$$x_i^* \succsim_i x_i', \forall x_i' \in \mathbf{B}_i(p^*)$$

Note:

(a) At equilibrium, the offer curves of the two consumers intersect.

(b) Any intersection of the offer curves outside of ω corresponds to a WE.

(c) Only relative prices $\frac{p_1^*}{p_2^*}$ are determined in equilibrium, since each consumer's demand is homogenous of degree zero.

$$p^* = (p_1^*, p_2^*) \text{ is a WE} \Longrightarrow \forall \alpha > 0, \alpha p^* = (\alpha p_1^*, \alpha p_2^*) \text{ is a WE.}$$

5. (Example) Consider an Edgeworth box economy. Solve the offer curves, and determine if the intersections are WE.

(a) Preferences: $u_i(x_{1i}, x_{2i}) = x_{1i}^{\alpha} x_{2i}^{1-\alpha}$.

(b) Endowments: $\omega_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \omega_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(c) Suppose the prices are $p = (p_1, p_2)$.

For consumer 1,

$$\max_{x_{11}, x_{21}} x_{11}^{\alpha} x_{21}^{1-\alpha} \text{s.t. } p_1 x_{11} + p_2 x_{21} \leqslant p_1 + 2p_2$$

As we discussed in before, $x_{11}^* > 0, x_{21}^* > 0$, thus,

$$\mathcal{L} = x_{11}^{\alpha} x_{21}^{1-\alpha} + \lambda_1 (p_1 + 2p_2 - p_1 x_{11} - p_2 x_{21})$$

F.O.C.s are

$$\begin{cases} \alpha x_{11}^{\alpha-1} x_{21}^{1-\alpha} - p_1 \lambda_1 = 0 \\ (1-\alpha) x_{11}^{\alpha} x_{21}^{-\alpha} - p_2 \lambda_1 = 0 \end{cases} \Longrightarrow \frac{\alpha}{1-\alpha} \frac{x_{21}}{x_{11}} = \frac{p_1}{p_2} \Longrightarrow \frac{p_2 x_{21}}{p_1 x_{11}} = \frac{1-\alpha}{\alpha}$$

then

$$x_{11}^* = \frac{\alpha(p_1 + 2p_2)}{p_1}, x_{21}^* = \frac{(1-\alpha)(p_1 + 2p_2)}{p_2}$$

Then

$$OC_1(p) = (x_{11}^*, x_{21}^*) = \left[\frac{\alpha(p_1 + 2p_2)}{p_1}, \frac{(1-\alpha)(p_1 + 2p_2)}{p_2}\right]$$

Likewise,

$$OC_2(p) = \left[\frac{\alpha(2p_1 + p_2)}{p_1}, \frac{(1-\alpha)(2p_1 + p_2)}{p_2}\right]$$

Clearing of market 1:

$$\frac{\alpha(p_1^* + 2p_2^*)}{p_1^*} + \frac{\alpha(2p_1^* + p_2^*)}{p_1^*} = 3 \Longrightarrow 1 + \frac{p_2^*}{p_1^*} = \frac{1}{\alpha} \Longrightarrow \frac{p_1^*}{p_2^*} = \frac{\alpha}{1-\alpha}$$

Clearing of market 2:

$$\frac{(1-\alpha)(p_1^*+2p_2^*)}{p_2^*}+\frac{(1-\alpha)(2p_1^*+p_2^*)}{p_2^*}=3\Longrightarrow\frac{p_1^*}{p_2^*}=\frac{\alpha}{1-\alpha}$$

Those 2 conditions meet, therefore, despite of the endowment point ω, the intersection point of the offer curve is a WE.

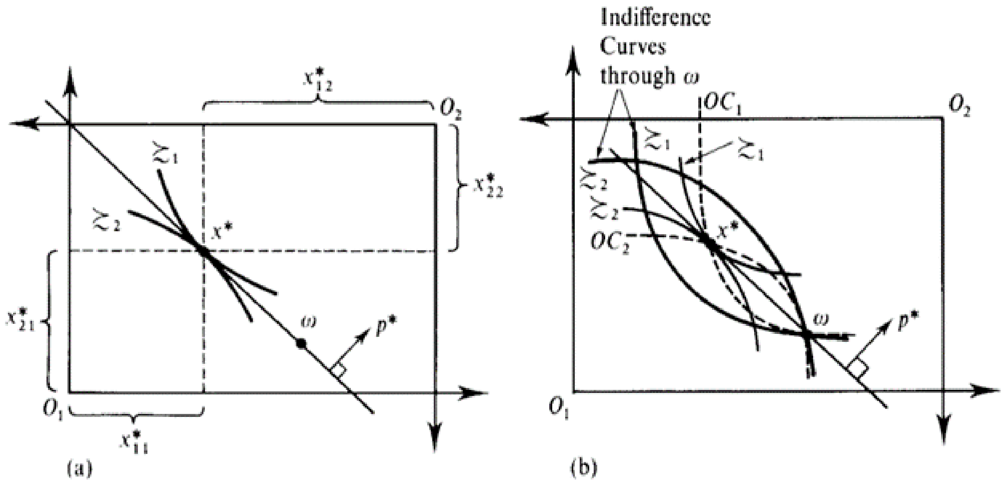

Fig. 8.5

6. (Remark) Solving for WE strategy.

   (a) Optimization: Maximize utility given the budget constraint.

   (b) Demand: Solve for Demand $x_{\ell i}(p)$, i.e., offer curve $OC_i(p)$, as a function of price.

   (c) Market Clearing: Set total demand for a good equal to endowment

   $$x_{\ell 1}+x_{\ell 2}=\omega_\ell, \ell=1,2$$

   (d) Solve for price ratio $\frac{p_1^*}{p_2^*}$.

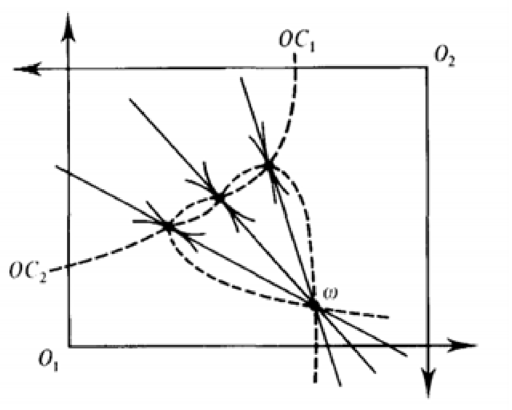

Fig. 8.6

7. (Example) Multiple Walrasian equilibria (Fig 8.6).

8. (Example) Nonexistence of WE.

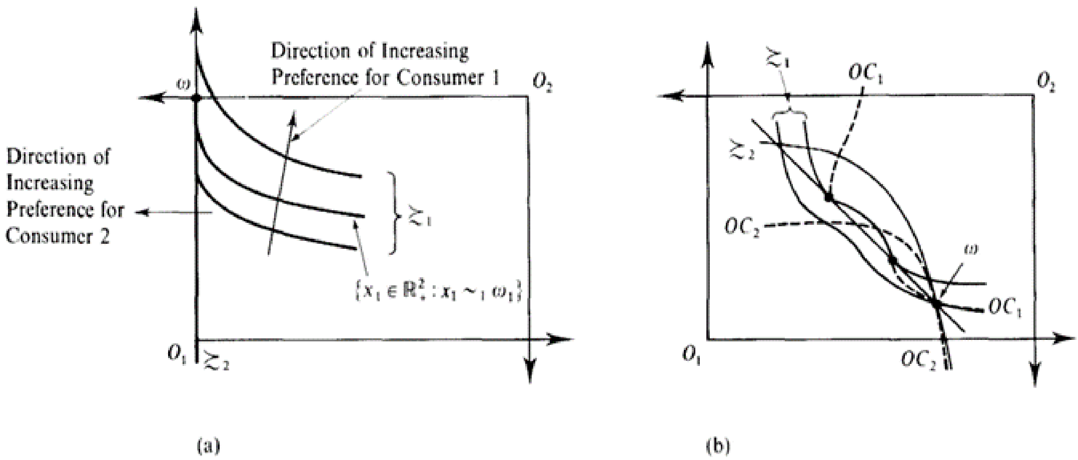

Fig. 8.7

(a) Non strictly monotonic preferences.

i. Endowment on the boundary.

ii. Consumer 2 only desires good 1 and has all good 1.

iii. Consumer 1 has all good 2 but his indifference curve has an infinite slope at ω_1.

iv. Given $p_1 > 0$,

- consumer 2 will keep his initial bundle ω_2 and will not sell good 1,
- But ω_1 is never consumer 1's optimal demand, or consumer 1 will buy good 1 for any p_2. That is, if you draw a budget line through ω, you will find ω is not the optimal choice for consumer 1, even when p_2 is very large.

v. No price at which demands are compatible.

(b) Nonconvex preferences

i. Consumer 1's preference is nonconvex, and thus it has multiple demand points.

ii. Consumer 1's offer curve is not continuous

9. (Definition)

(a)

10. An allocation x in the Edgeworth box is **Pareto optimal** if there doesn't exist other allocation x' in the Edgeworth box such that

$$x_i' \succsim x_i, \forall i = 1, 2, \text{ and } x_i' \succ x_i, \exists i \in \{1, 2\}$$

11. (Remark)

(a) Interior solution: consumers' indifference curves through x must be tangent.

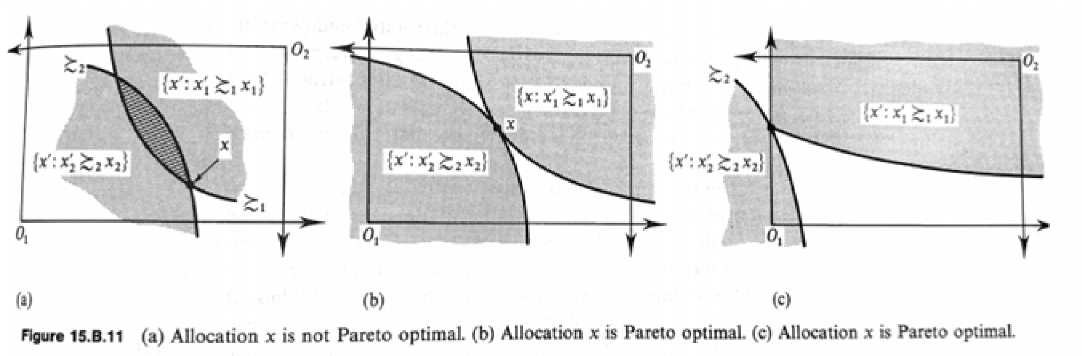

Fig. 8.8

(b) Corner solution: tangency need not hold, i.e., when the Pareto optimum occurs on the boundary, the indifference curve need not to be tangent.

(c) **Pareto set** is the set of all Pareto optimal allocations.

(d) **Contract curve** is the part of Pareto Set where both consumers do at least as well as their initial endowments.

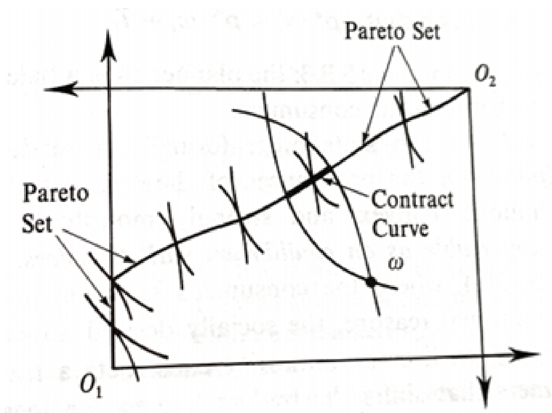

Fig. 8.9

12. (Theorem, the First Welfare Theorem) Any Walrasian equilibrium allocation x^* necessarily belongs to the Pareto set.

Note:

(a) Let $A, B, C \subseteq \mathbb{R}^4_+$, and

$$A = \{\text{All allocations}\}$$
$$B = \{\text{Pareto optimal allocations}\}$$
$$C = \{\text{Equilibrium allocations}\}$$

then

$$C \subseteq B \subseteq A$$

(b) The only possible welfare justification for intervention in the economy is the fulfillment of distributional objectives.

13. (Definition) An allocation x^* is the Edgeworth box is **supportable as an equilibrium with transfers** if there exists a p^* and a wealth transfers T_1 and T_2 satisfying $T_1+T_2=0$, such that $\forall i = 1, 2$, $\forall x_i' \in \mathbb{R}^2_+$ with

$$p^* \cdot x_i' \leqslant p^* \cdot \omega_i + T_i$$

we have

$$x_i^* \succsim_i x_i'$$

14. (Theorem, the Second Welfare Theorem) If the preferences of the two consumers in the Edgeworth box are continuous, convex, and strongly monotone, then any Pareto optimal allocation is supportable as an equilibrium with transfers. Note: There are 2 ways to make the transfer

    (a) Change initial endowments directly, but in reality, there are too many goods to change.

    (b) Change the distribution of money. It is more convenient.

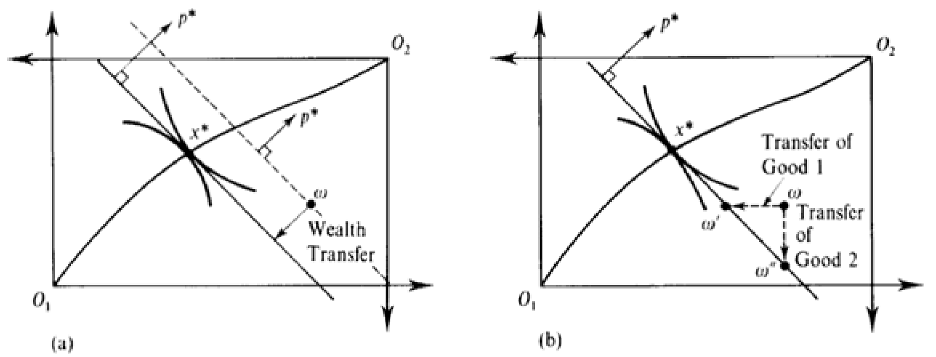

Figure 15.B.13 The second fundamental welfare theorem. (a) Using wealth transfers. (b) Using transfers of endowments.

Fig. 8.10

15. (Example) Non-convex preferences lead to failure of the second welfare theorem. (a) The preference of consumer 1 is non-convex. (b) x^* is a Pareto optimal allocation since any change of x^* should lead to at least one consumer worse off. (c) But x^* is not supportable as an equilibrium with transfers. Since at the budget line, consumer 2 desire x_2^*, but consumer 1 prefers, say x_1' to x_1^*. (d) For example, given an initial endowment ω, and we transfer the budget line to the location we need, then the equilibrium point cannot be x^*, but say x''.

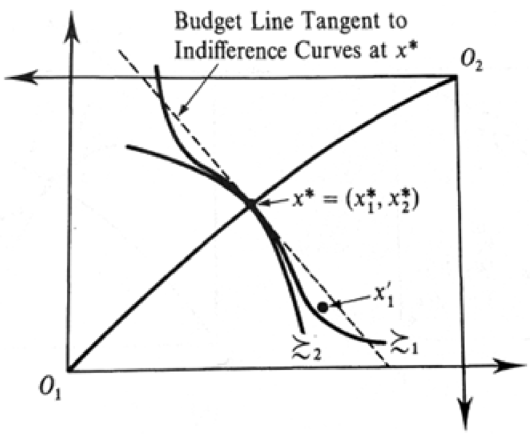

Fig. 8.11

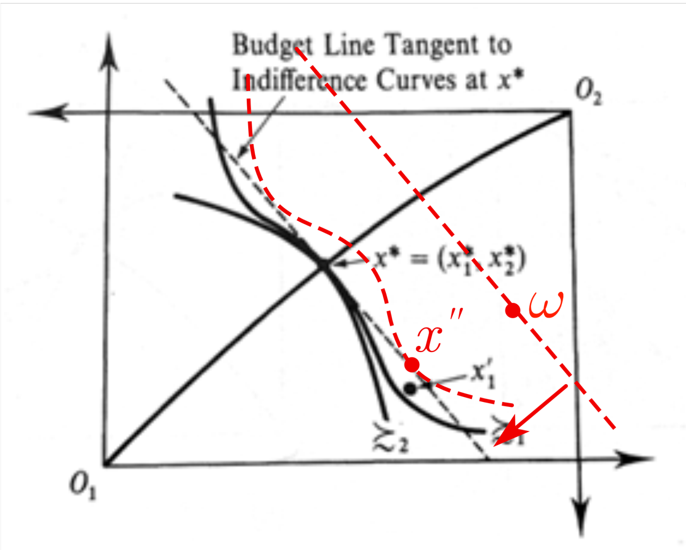

Fig. 8.12

16. (Example) Consider an Edgeworth box economy with 2 goods x, y, and $\bar{\omega}_1 = \bar{\omega}_2 = 2$, $u_1 = u_2 = xy$, $\omega_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \omega_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$. For consumer 1,

$$\max x_1 y_1$$
$$\text{s.t. } p_x x_1 + p_y y_1 \leqslant p_x \cdot 2 + T_1$$

then

$$\begin{cases} y_1 - \lambda p_x = 0 \\ x_1 - \lambda p_y = 0 \end{cases} \implies \frac{y_1}{p_x} = \frac{x_1}{p_y} \implies x_1 p_x = y_1 p_y$$

therefore,

$$x_1 = \frac{2p_x + T_1}{2p_x}, y_1 = \frac{2p_x + T}{2p_y}$$

likewise, we can solve for consumer 2 as

$$x_2 = \frac{2p_y + T_2}{2p_x}, y_2 = \frac{2p_y + T_2}{2p_y}$$

where $T_1 + T_2 = 0$.

Let the market clears,

$$\frac{2p_x + 2p_y}{2p_x} = 2 \Longrightarrow p_x = p_y$$

and thus, $x_1 = y_1, x_2 = y_2$.

In this example, if the government's goal is to change the equilibrium to $\left(\begin{pmatrix} a \\ a \end{pmatrix}, \begin{pmatrix} 2-a \\ 2-a \end{pmatrix}\right)$, then

(a) Firstly, the government should predict the price, as $p^* = p_x = p_y$, and make the lump-sum transfer, which satisfies $\frac{2p^*+T_1}{2p^*} = a \Longrightarrow T_1 = (2a-2)\,p^*$.

(b) Secondly, the market will automatically get to the desired outcome.

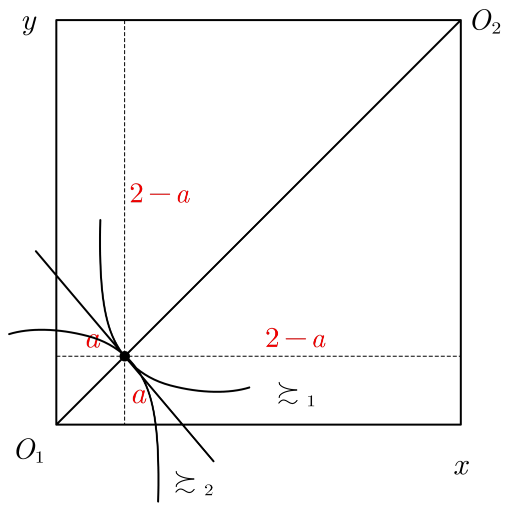

Fig. 8.13

8.2 The One-Consumer, One-Producer Economy

1. Set-ups.

   (a) Single consumer, single firm; both are price taker. x_1 denotes the leisure, and $z = \bar{L} - x_1$ denotes the labor.

   (b) Single consumer, single firm; both are price taker.x_2 denotes the produced goods by the firm.

   (c) Single consumer, single firm; both are price taker. Endowment: $\bar{L}$ of leisure, 0 of x_2.

   (d) Consumer owns the firm and get profits, which are positive.

(e) Consumer's preference $\succsim$ over (x_1, x_2) is assumed to be continuous, convex and strictly increasing.

2. Production.

$$\max_{z \geqslant 0} \; pf(z) - wz$$

where p is the price of x_2, and w is the wage rate. Given (p, w), the firm has

(a) Labor demand $z(p, w)$.

(b) Output $q(p, w)$.

(c) Profit $\pi(p, w)$.

3. Consumption.

$$\max_{(x_1, x_2) \in \mathbb{R}^2_+} \; u(x_1, x_2)$$
$$\text{s.t. } px_2 \leqslant w\left(\bar{L} - x_1\right) + \pi(p, w)$$

Given (p, w), consumer has two demands

(a) $x_1(p, w)$.

(b) $x_2(p, w)$.

4. Walrasian Equilibrium: (p^*, w^*) at which consumption and labor market clears,

$$x_2(p^*, w^*) = q(p^*, w^*) \, \bar{L} - x_1(p^*, w^*) = z(p^*, w^*)$$

Moreover, consumer's and producer's problem are solved.

5. (Remark) Firm has no preferences, then why firm should maximize its profit?

(a) Note that the profit eventually goes to household and will change the budget constraint.

(b) And household has preference and thus should maximize its utility.

6. (Remark) Why the isoprofit line of the producer is the same as the consumer's budget line?

(a) The labor cost and profit of the producer becomes the consumer's income.

(b) Mathematically, they happen to be the same line.

Intuitively,

(a) If the consumer consumes $\bar{L}$ units of leisure (sells no labor), he can purchase $\frac{\pi(p,w)}{p}$ units of x_2. Thus, the budget line must cut the q-axis at $\frac{\pi(p,w)}{p}$.

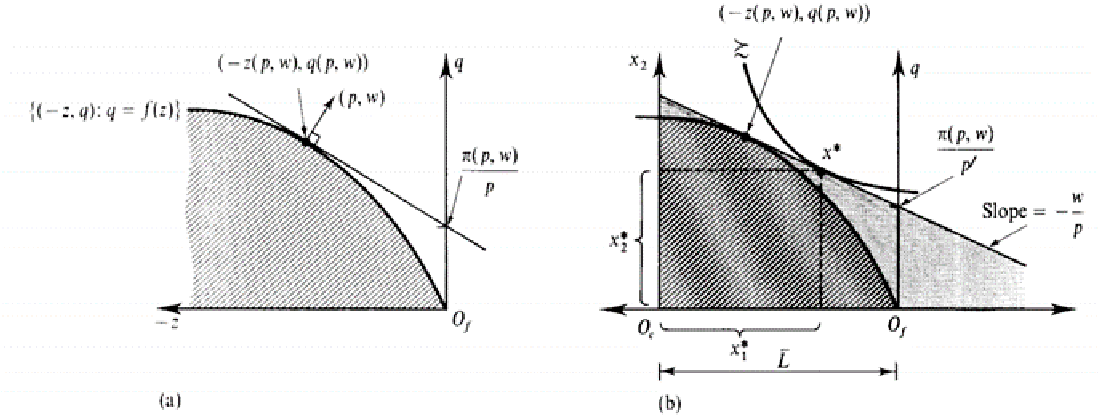

Figure 15.C.1 (a) The firm's problem. (b) The consumer's problem.

Fig. 8.14

(b) Moreover, for each unit of labor he sells, the consumer earns w, and then afford to purchase $\frac{w}{p}$ units of x_2, therefore, the budget line is exactly the isoprofit line associated with the solution to the firm's profit-maximization problem.

(c) Given (w,p), firm has the optimal choice $(z^*, q = f(z^*))$, and then the consumer forms its budget line, and making the optimal choice.

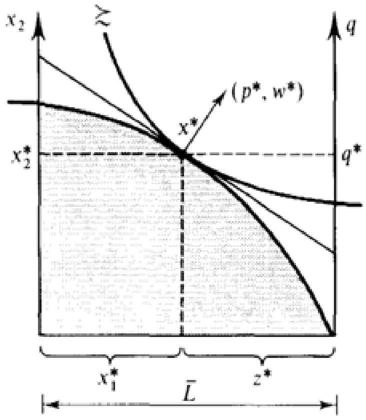

Fig. 8.15

In this figure, (p^*, w^*) is the equilibrium price.

7. (Example) Let $\bar{L} = 1$, $u(x,y) = xy$ and $f(z) = \sqrt{z}$.

For the producer,

$$\max p\sqrt{z} - w \cdot z$$

the F.O.C. is

$$\frac{p}{2\sqrt{z}} - w = 0 \Longrightarrow z^* = \frac{p^2}{4w^2}$$

then

$$\pi^* = p \cdot \frac{p}{2w} - \frac{p^2}{4w} = \frac{p^2}{4w}$$

For the consumer,

$$\max_{x,y}\ xy \text{s.t. } px \leqslant (1-y)\,w+\pi$$

form the Lagrangian,

$$\mathcal{L} = xy + \lambda\,(w+\pi-wy-px)$$

the F.O.C.s are

$$\begin{cases} y-\lambda p=0 \\ x-\lambda w=0 \end{cases} \Longrightarrow \frac{y}{p}=\frac{x}{w} \Longrightarrow xp=yw \Longrightarrow \begin{cases} x^* = \frac{w+\pi}{2p} \\ y^* = \frac{w+\pi}{2w} \end{cases}$$

Let markets clear,

$$\begin{cases} x^* = \sqrt{z^*} \\ y^*+z^*=1 \end{cases} \Longrightarrow \begin{cases} \frac{w+\pi}{2p} = \frac{p}{2w} \\ \frac{w+\pi}{2w} + \frac{p^2}{4w^2} = 1 \end{cases}$$

Plug $\pi^* = \frac{p^2}{4w}$ in, the first equation leads to

$$\frac{w+\frac{p^2}{4w}}{2p} = \frac{p}{2w} \Longrightarrow p^2 = w^2 + \frac{p^2}{4} \Longrightarrow \frac{3}{4}p^2 = w^2$$

the second equation also leads to

$$\frac{w+\frac{p^2}{4w}}{2w} + \frac{p^2}{4w^2} = 1 \Longrightarrow \frac{w^2+\frac{p^2}{4}}{2w^2} + \frac{p^2}{4w^2} = 1 \Longrightarrow w^2 + \frac{3}{4}p^2 = 2w^2$$

Normalize $p=1$, then $w=\frac{\sqrt{3}}{2}$, which is the equilibrium price.

8. (Remark) How to define a Pareto optimality in this scenario?

   There is only 1 consumer, so the Pareto optimal allocation is the state that maximize the consumer's utility subject to the economy's technological endowment constraints.

   And note that the social planner is to maximize the utility rather than the profit of the firm (Since the profit will go to the household eventually).

   (a) x^* is Pareto optimal but cannot be reached by market equilibrium.

   (b) x^* is Pareto optimal and is supportable as a Walrasian equilibrium.

8.3 The 2×2 Production Model

1. (Remark) Set-up of general case.

   (a) J firms, each firm produces a different consumption good q_j.

   (b) L primary inputs.

   (c) Firm j's production technology: $q_j = f_j\,(z_j) \in \mathbb{R}_+$, where $z_j = (z_{1j},\cdots,z_{Lj}) \in \mathbb{R}^L_+$.

      i. Assumption: f_j is strictly increasing, concave and differentiate.

      ii. Total endowment of the L labor inputs: $\bar{z} = (\bar{z}_1,\cdots,\bar{z}_L)$.

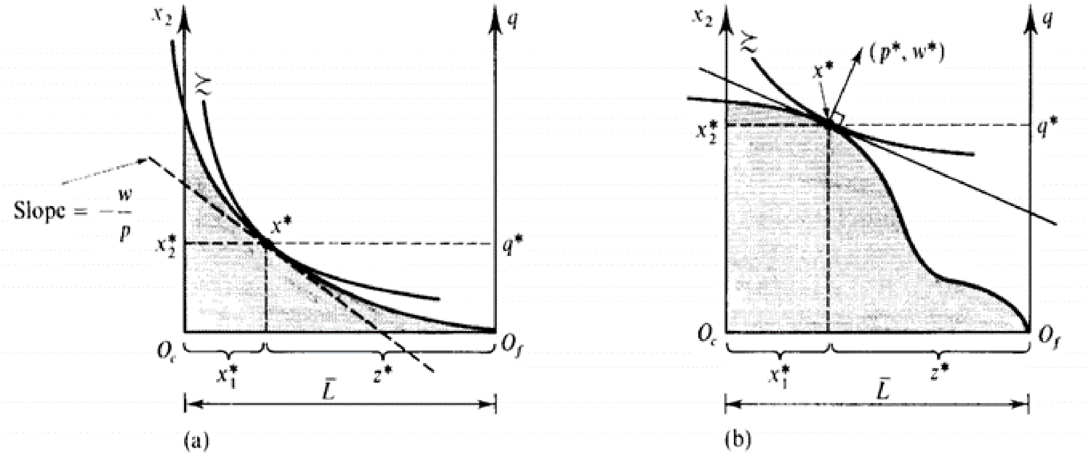

Figure 15.C.3 (a) Failure of the second welfare theorem with a nonconvex technology.
(b) The first welfare theorem applies even with a nonconvex technology.

Fig. 8.16

iii. No intermediate goods.

(d) Denote the input prices as $w = (w_1, \cdots, w_L)$.

(e) Fix output prices $(p_1, \cdots, p_J)$.

2. (Remark) Firm j's problem and market clearing.

$$\max_{z_j \geqslant 0} p_j q_j - w \cdot z_j = p_j f_j(z_j) - w \cdot z_j$$

Assume $z_j^* \gg 0$ is an interior solution, then

$$p_j \frac{\partial f_j(z_j^*)}{\partial z_{\ell j}} = w_\ell^*, \forall j, \ell$$

Market clearing:

$$\sum_{j=1}^{J} z_{\ell j}^* = \bar{z}_\ell, \forall \ell = 1, 2, \cdots, L$$

In the setting, we cannot normalize the input prices w, since the problem or the first order condition depends on the absolute value but not the relative value (firms treat $(p_1, \cdots, p_J)$ as given).

There are $L(J+1)$ variables:

$$z_j = (z_{1j}, \cdots, z_{Lj}), j = 1, 2, \cdots, J w = (w_1, w_2, \cdots, w_L)$$

and has $L(J+1)$ equations:

$$\begin{cases} p_j \frac{\partial f_j(z_j^*)}{\partial z_{\ell j}} = w_\ell^*, \forall j = 1, \cdots, J, \forall \ell = 1, \cdots, L \\ \sum_{j=1}^{J} z_{\ell j}^* = \bar{z}_\ell, \forall \ell = 1, 2, \cdots, L \end{cases}$$

3. (Remark) Social planner's problem.

$$\max_{(z_1,z_2,\cdots,z_J)\geqslant 0}\sum_{j=1}^{J}p_jf_j(z_j)\text{s.t. }\sum_{j=1}^{J}z_j^*=\bar{z}=(\bar{z}_1,\cdots,\bar{z}_L)$$

We claim that social planner's problem is equivalent with market equilibrium.

(a) Firstly note that for the profit maximization problem, we can merge J firms into one firm.

$$\max_{(z_1,z_2,\cdots,z_J)\geqslant 0}\sum_{j=1}^{J}\left[p_jf_j(z_j)-w^*\cdot z_j\right]$$

(b) Those 2 problems equal, since $w^*\cdot\bar{z}$ is a given constant.

$$\sum_{j=1}^{J}\left[p_jf_j(z_j)-w^*\cdot z_j\right]=\sum_{j=1}^{J}p_jf_j(z_j)-w^*\cdot\sum_{j=1}^{J}z_j=\sum_{j=1}^{J}p_jf_j(z_j)-w^*\cdot\bar{z}$$

4. (Remark) A special case: 2×2 production model.

(a) Two productions (output factors), two input factors.

(b) Production with Constant return to scale: $f_1(z_{11},z_{21})$, $f_2(z_{12},z_{22})$.

(c) Unite isoquant of firm j is defined as

$$\left\{(z_{1j}.z_{2j})\in\mathbb{R}_+^2:f_j(z_{1j},z_{2j})=1\right\}$$

(d) $c_j(w)$ is the minimal cost of producing **one unite** of good j when the factor prices are $w=(w_1,w_2)$.

(e) $a_j(w)=(a_{1j}(w),a_{2j}(w))$ is the optimal factor demand.

5. (Example) $f(z_1,z_2)=z_1^{\alpha}z_2^{1-\alpha}$, the unit isoquant is $z_1^{\alpha}z_2^{1-\alpha}=1$. To get the unit cost function, we solve

$$\begin{aligned}\min_{z_1,z_2}\ & w_1z_1+w_2z_2\\ \text{s.t. }\ & z_1^{\alpha}z_2^{1-\alpha}\geqslant 1\end{aligned}$$

then

$$\mathcal{L}=w_1z_1+w_2z_2+\lambda\left(1-z_1^{\alpha}z_2^{1-\alpha}\right)$$

F.O.C.s are (assume $z_1^{\alpha}z_2^{1-\alpha}=1\Longrightarrow z_1=z_2^{\frac{\alpha-1}{\alpha}}$)

$$\begin{cases}w_1=\lambda\alpha z_1^{\alpha-1}z_2^{1-\alpha}\\ w_2=\lambda(1-\alpha)z_1^{\alpha}z_2^{-\alpha}\end{cases}\Longrightarrow\frac{w_1}{w_2}=\frac{\alpha}{1-\alpha}\frac{z_2}{z_1}=\frac{\alpha}{1-\alpha}z_2^{\frac{1}{\alpha}}$$

thus,

$$\begin{aligned}z_2^*&=\left(\frac{w_1}{w_2}\frac{1-\alpha}{\alpha}\right)^{\alpha}\\ z_1^*&=\left(\frac{w_1}{w_2}\frac{1-\alpha}{\alpha}\right)^{\alpha-1}=\left(\frac{w_2}{w_1}\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\end{aligned}$$

Here $a(w) = (a_1(w), a_2(w)) = (z_1^*(w), z_2^*(w))$, then the unit cost function is

$$c(w) = a_1(w)w_1 + a_2(w)w_2 = \left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right] w_1^{\alpha} w_2^{1-\alpha}$$

Moreover, by Shepard's Lemma, or the Envelop theorem,

$$\nabla_w c(w) = (a_1(w), a_2(w))$$

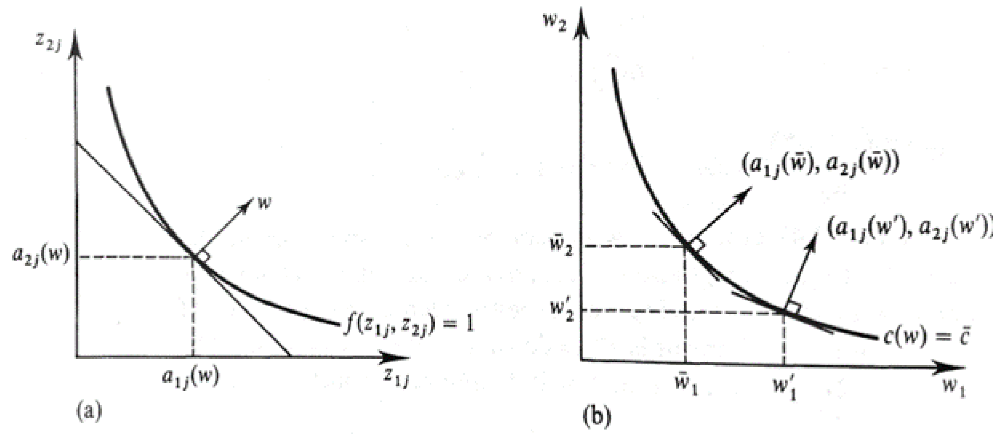

Fig. 8.17

(a) Figure (a) is the unite isoquant.

(b) Figure (b) is the unit cost function, then gradient $(a_1(w), a_2(w))$ is vertical to the tangent line of $c(w)$. It is just a math result. By the implicit function theorem,

$$\frac{\mathrm{d}w_1}{\mathrm{d}w_2} = -\frac{\frac{\partial c}{\partial w_1}}{\frac{\partial c}{\partial w_2}} = -\frac{a_1(w)}{a_2(w)}$$

if $\mathrm{d}w_2 = 1$, then

$$\mathrm{d}w = (\mathrm{d}w_1, \mathrm{d}w_2) = \left(-\frac{a_1(w)}{a_2(w)}, 1\right) \Longrightarrow \mathrm{d}w \cdot a(w) = 0$$

6. (Remark)

(a) Pareto set: the set of factor allocations at which it is not possible, with the given total factor endowments, **to produce more of one good without producing less of the other**.

(b) Any point in the shadow area is not Pareto optimal, since we can shift the isoquant curve to let at least one of them produce more and don't affect the other.

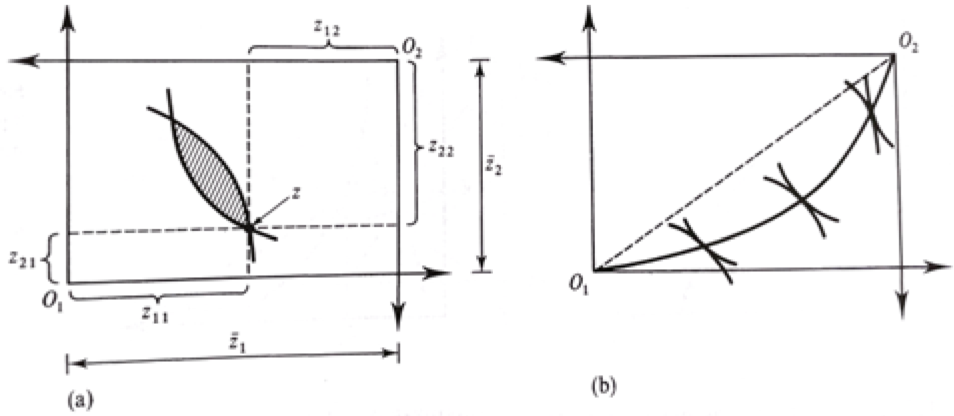

Figure 15.D.2 (a) An inefficient factor allocation. (b) The Pareto set of factor allocations.

Fig. 8.18

7. (Example) The production possibility set. Again, the production functions are

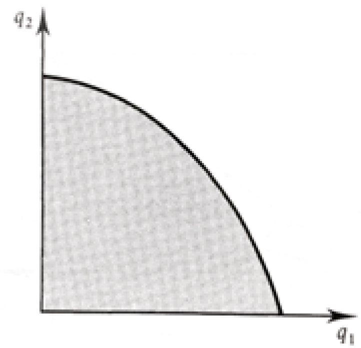

Fig. 8.19

$$f_1(z_{11}, z_{21}) = z_{11}^{\alpha} z_{21}^{1-\alpha}$$
$$f_2(z_{12}, z_{22}) = z_{12}^{\beta} z_{22}^{1-\beta}$$

then by previous computation,

$$a_{11}(w) = \left(\frac{w_2}{w_1}\frac{\alpha}{1-\alpha}\right)^{1-\alpha}, a_{21}(w) = \left(\frac{w_1}{w_2}\frac{1-\alpha}{\alpha}\right)^{\alpha}$$
$$a_{21}(w) = \left(\frac{w_2}{w_1}\frac{\beta}{1-\beta}\right)^{1-\beta}, a_{22}(w) = \left(\frac{w_1}{w_2}\frac{1-\beta}{\beta}\right)^{\beta}$$

The market clearing conditions are

$$\begin{cases} q_1 a_{11}(w) + q_2 a_{12}(w) = z_1 \\ q_1 a_{21}(w) + q_2 a_{22}(w) = z_2 \end{cases}$$

where q_j is the units of firm j's production. Therefore,

$$\begin{cases} q_1\left(\frac{w_2}{w_1}\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + q_2\left(\frac{w_2}{w_1}\frac{\beta}{1-\beta}\right)^{1-\beta} = z_1 \\ q_1\left(\frac{w_1}{w_2}\frac{1-\alpha}{\alpha}\right)^{\alpha} + q_2\left(\frac{w_1}{w_2}\frac{1-\beta}{\beta}\right)^{\beta} = z_2 \end{cases}$$

Given $\frac{w_1}{w_2}$ we can solve for q_1, q_2, then plot the production possibility set.

8. (Definition) The production of good 1 is **relatively more intensive in** factor 1 than is the production of good 2 if, $\forall w = (w_1, w_2) \in \mathbb{R}^2_{++}$,

$$\frac{a_{11}(w)}{a_{21}(w)} > \frac{a_{12}(w)}{a_{22}(w)}$$

Note:

(a) Given CRS, both firms have zero profit, then the unit cost equals to the price (unit revenue).

$$\begin{cases} c_1(w_1^*, w_2^*) = p_1 \\ c_2(w_1^*, w_2^*) = p_2 \end{cases}$$

If those 2 curves have a single cross point, then there is only one equilibrium.

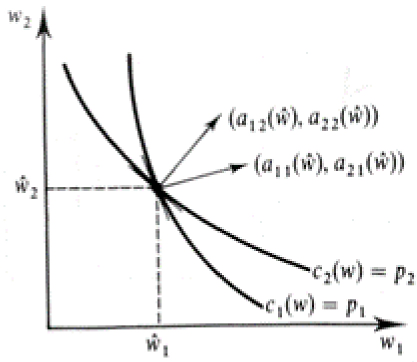

Fig. 8.20

(b) What's the condition of the single crossing?

Intuitively, if the curve $w_2 = f(w_1)$ and $w_2 = g(w_1)$ are continuous, and if $\forall w_1$, the slope of one is greater than another, then they have only one cross point.

For example, consider $c_1(w) = p_2$ and $w = \hat{w}$, the slope at $\hat{w}$ is $-\frac{a_{11}(\hat{w})}{a_{21}(\hat{w})}$, which appears in the definition.

9. (Theorem, Stolper-Samuelson Theorem) In the 2×2 production model with the factor intensity assumption, if p_j increases, then the equilibrium price of the factor more intensively used in the production of good j increases, while the price of the other factor decreases (assuming interior equilibria both before and after the price change).

Note: Factor 1 is more intensive in production good 1, if $p_1 \uparrow$, then the curve $c_1(w) = p_1$ shifts outwards, making the single cross point downward, $w_2 \downarrow$, $w_1 \uparrow$, moreover, $\frac{w_2}{w_1} \downarrow$.

Proof. Differentiate

$$\begin{cases} c_1(w_1^*, w_2^*) = p_1 \\ c_2(w_1^*, w_2^*) = p_2 \end{cases}$$

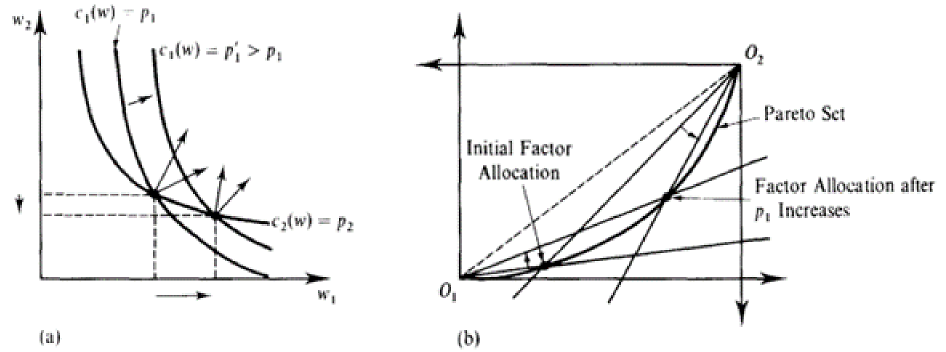

Figure 15.D.6 The Stolper–Samuelson theorem. (a) The change in equilibrium factor prices. (b) The change in the equilibrium factor allocation.

Fig. 8.21

then

$$\begin{cases} \mathrm{d}p_1 = a_{11}\left(w^*\right)\mathrm{d}w_1 + a_{12}\left(w^*\right)\mathrm{d}w_2 \\ \mathrm{d}p_2 = a_{21}\left(w^*\right)\mathrm{d}w_1 + a_{22}\left(w^*\right)\mathrm{d}w_2 \end{cases}$$

or

$$\begin{bmatrix} \mathrm{d}p_1 \\ \mathrm{d}p_2 \end{bmatrix} = \begin{bmatrix} a_{11}\left(w^*\right) & a_{12}\left(w^*\right) \\ a_{21}\left(w^*\right) & a_{22}\left(w^*\right) \end{bmatrix} \begin{bmatrix} \mathrm{d}w_1 \\ \mathrm{d}w_2 \end{bmatrix} := A\mathrm{d}w$$

The intensity assumption is

$$\frac{a_{11}(w)}{a_{21}(w)} > \frac{a_{12}(w)}{a_{22}(w)} \Longleftrightarrow \frac{a_{11}(w)\,a_{22}(w) - a_{12}(w)\,a_{21}(w)}{a_{21}(w)\,a_{22}(w)} > 0 \Longleftrightarrow \det A > 0$$

then A is invertible, thus, we can compute

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} a_{22}\left(w^*\right) & -a_{12}\left(w^*\right) \\ -a_{21}\left(w^*\right) & a_{11}\left(w^*\right) \end{bmatrix}$$

the entries of A^{-1} are positive at the diagonal and negative off the diagonal.

Then if we let $\mathrm{d}p = (1, 0)$, we get $\mathrm{d}w_1 > 0$, $\mathrm{d}w_2 < 0$. □

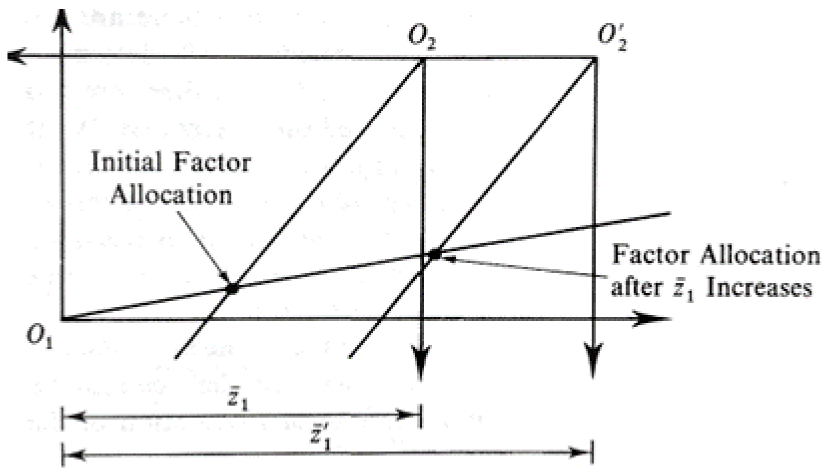

Fig. 8.22

10. (Theorem, Rybcszynski Theorem) In the 2×2 production model with the factor intensity assumption, if the endowment of a factor increases, then the equilibrium price of the factor more intensively used in the production of good j, increases, while the price of the other factor decreases (assuming interior equilibria both before and after the price change).

Proof. Recall the market clearing condition that

$$\begin{cases} q_1 a_{11}(w) + q_2 a_{12}(w) = z_1 \\ q_1 a_{21}(w) + q_2 a_{22}(w) = z_2 \end{cases} \Longrightarrow \begin{bmatrix} a_{11}(w^*) & a_{12}(w^*) \\ a_{21}(w^*) & a_{22}(w^*) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

If $\mathrm{d}z_1 = 1, \mathrm{d}z_2 = 0$, we can similarly prove the Rybcszynski Theorem. $\square$

Chapter 9

Social Efficiency

9.1 Pareto Efficiency

1. (Remark) Set-ups.

   (a) A nonempty finite set of individuals or households $h \in \mathcal{H} = \{1, 2, \cdots, H\}$.

   (b) X is the set of all conceivable social states (don't need to be finite).

   (c) $A \subseteq X$ is the set of feasible social states.

   (d) Individual h's preference $\succsim_h$ (complete and transitive) is defined on X.

   (e) Question: The choice of a social state x affects all the individuals, the choice should take account the preferences of all the individuals. But how?

2. (Definition)

   (a) The social state or outcome x is **Pareto superior** to y (or **Pareto dominates** y), if

   $$x \succsim_h y, \forall h \in \mathcal{H}$$
   $$x \succ_{h_0} y, \exists h_0 \in \mathcal{H}$$

   In this case, we say that y is **Pareto dominated** by or is **Pareto inferior** to x and write $x > y$.

   (b) $x, y \in X$, we say x is **strictly Pareto superior** to y (or **strictly Pareto dominates** y) is

   $$x \succ_h y, \forall h \in \mathcal{H}$$

   and write $x \gg y$.

   (c) $x \in A \subseteq X$ is **Pareto efficient** within A if the is no $y \in A$ such that y Pareto dominates x.

   (d) The set of Pareto efficient point is called the **Pareto frontier** of A.

3. (Theorem)

(a) $>, \gg$ on X are transitive and asymmetric, but not complete and reflexive.

(b) If $x \gg y$ then $x > y$.

Proof. It can easily be proved by the completeness and transitivity of $\succsim_h$. □

4. (Theorem) A given set $A \subseteq X$ has a nonempty Pareto set if either

(a) A is finite.

(b) A is compact and $\succsim_h$ is continuous $\forall h \in \mathcal{H}$, and $X \subseteq \mathbb{R}^H$.

Proof. Let $A_0 = A$.

Step 1. Define a nonempty set.

Define

$$A_1 := c_{\succsim_1}(A_0) = \{y \in A_0 : y \succsim_1 x, \forall x \in A_0\}$$

then $A_1 \neq \varnothing$ (by the finiteness or compactness of A).

For case (a), if $|A_0| < \infty$, then $|A_1| < \infty$.

For case (b) if A_0 is compact, and $\succsim_1$ is continuous, then we show that $A_1 \subseteq A_0 \subseteq \mathbb{R}^H$ is compact.

The boundedness is trivial, since A_0 is bounded.

As for the closedness, let $\{x_n\}_{n=1}^{\infty} \subseteq A_1$ be a convergent sequence in A_1, such that $x_n \to x$. Then by definition of A_1, $\forall y \in A_0, \forall n \geqslant 1, x_n \succsim y$. And since $\succsim_1$ is continuous, then

$$\lim_{n\to\infty} x_n \succsim y, \forall y \in A_0 \Longrightarrow x \succsim y, \forall y \in A_0 \Longrightarrow x_n \to x \in A_1$$

which implies A_1 is closed.

Likewise, define

$$A_2 := c_{\succsim_2}(A_1) = \{y \in A_1 : y \succsim_2 x, \forall x \in A_1\} \subseteq A_1$$

which is nonempty under either (a) or (b).

Continue in this fashion, in the end, we get

$$A_H := c_{\succsim_H}(A_{H-1}) = \{y \in A_{H-1} : y \succsim_H x, \forall x \in A_{H-1}\} \subseteq A_{H-1}$$

which is nonempty under either (a) or (b).

Step 2. Show the defined nonempty set is a part of the Pareto set.

We now show that $\forall x \in A_H$, x is Pareto efficient in A.

Suppose $\exists x \in A_H$, and $\exists y \in A$, such that y is Pareto superior to x. Then

$$y \succsim_h x, \forall h \in \mathcal{H} y \succ_{h_0} x, \exists h_0 \in \mathcal{H}$$

which imply that $x \sim y, y \in A_H \subseteq A_{h_0}$. Together with $y \succ_{h_0} x$, we have $x \notin A_{h_0}$. But $x \in A_H \subseteq A_{h_0}$, a contradiction.

Therefore, $A_H \neq \varnothing$ is a subset of the Pareto set. □

5. (Definition) Let $u_h : X \to \mathbb{R}$ represent $\succsim_h$.

   (a) Define $u(x) := (u_1(x), u_2(x), \cdots, u_H(x))$, which is called the **utility imputation** for the social state x.

   (b) Let $A \subseteq X$ be feasible social states, the set $u(A) = \{u(x) : x \in A\}$ is called the set of **feasible utility imputations**

   Note:

   (a) x is Pareto superior to y iff $u(x) \geqslant u(y)$ and $u(x) \neq u(y)$.

   (b) x is Pareto efficient in $A \subseteq X$ iff there is no $v \in \mathbb{R}^H$, such that

      - $\exists y \in A$, $v = u(y)$,
      - $v \geqslant u(x), v \neq u(x)$.

6. (Definition) A **(Bergson-Samuelson) social welfare (or utility) function** is a function $W : \mathbb{R}^H \to \mathbb{R}$ that assigns a utility value to each possible vector $(u_1, u_2, \cdots, u_H) \in \mathbb{R}^H$ of utility levels for the H individuals in the economy.

7. (Theorem) Suppose that $W : \mathbb{R}^H \to \mathbb{R}$ is strictly increasing. If x is a solution to the problem

$$\max_{x \in X} Wu(x)$$

   then x is Pareto optimal.

   Proof. Suppose x is not Pareto optimal, then $\exists y \in X$ such that

$$u(y) \geqslant u(x), u(x) \neq u(y)$$

   since W is strictly increasing, then $Wu(y) > Wu(x)$. A contradiction. □

8. (Example) Specifically, $Wu(x)$ can be

$$Wu(x) = \sum_{i=1}^{H} \beta_i u_i(x)$$

   where $\beta_i > 0, i = 1, 2, \cdots, n$. Since if $\exists \beta_i = 0$, then W is not strictly increasing.

9. (Theorem, the Separating-Hyperplane Theorem) Suppose X and Y are two disjoint and nonempty convex subset of $\mathbb{R}^k$, then $\exists a \in \mathbb{R}^k$ with $a \neq 0$, and $\exists b \in \mathbb{R}$ such that

$$a \cdot x \geqslant b, \forall x \in X$$
$$a \cdot y \leqslant b, \forall y \in Y$$

10. (Theorem)

 (a) Suppose that the set $\left\{v \in \mathbb{R}^H : \exists x \in X, v \leqslant u(x)\right\}$ is convex. Then every Pareto-efficient point $x^0 \in X$ is the solution to the problem

$$\max_{x \in X} \sum_{h=1}^{H} \alpha_h u_h(x)$$

 for some set of nonnegative weights $(\alpha_h) = (\alpha_1, \alpha_2, \cdots, \alpha_H) \geqslant 0$, and $(\alpha_h) \neq 0$.

 Note: $\sum_{h \in \mathcal{H}} \alpha_h$ doesn't need to equal to 1, since we can do a simple normalization to maximize

$$\max_{x \in X} \sum_{h=1}^{H} \frac{\alpha_h}{\sum_{h \in \mathcal{H}} \alpha_h} u_h(x)$$

 (b) If X is convex and if each $u_h : X \to \mathbb{R}$ is a concave function, then the set $\left\{v \in \mathbb{R}^H : \exists x \in X, v \leqslant u(x)\right\}$ is convex.

 Note: We can use part (b) to get one of the conditions in (a).

Proof. (a) For any Pareto efficient point x^0, then the set

$$V := \left\{v \in \mathbb{R}^H : \exists x \in X, v \leqslant u(x)\right\}$$

and

$$U := \left\{v \in \mathbb{R}^H : v > u\left(x^0\right)\right\}$$

are disjoint.

By hypothesis, V is convex, and U is trivially convex.

By the Separating-Hyperplane theorem, $\exists \beta \in \mathbb{R}^H$ with $\beta \neq 0$, and $\exists \gamma \in \mathbb{R}$ such that

$$\begin{cases} \beta \cdot v \leqslant \gamma, \forall v \in V \\ \beta \cdot v \geqslant \gamma, \forall v \in U \end{cases} \iff \begin{cases} \beta \cdot v \leqslant \gamma, \forall v \in V \\ \beta \cdot v \geqslant \gamma, \forall v > u\left(x^0\right) \end{cases}$$

$$\boxed{\textbf{Step 1}. \text{ Show that } \beta \geqslant 0.}$$

Suppose $\exists h_0$ such that $\beta_{h_0} < 0$, then let

$$v_h = \begin{cases} u_h\left(x^0\right) + 1, & h \neq h_0 \\ u_{h_0}\left(x^0\right) + M, & h = h_0 \end{cases}, \forall M > 0$$

Then $v := (v_h) > u(x^0) \Longrightarrow v \in U \Longrightarrow \beta \cdot v \geqslant \gamma, \forall M \in \mathbb{R}$.

However,

$$\lim_{M\to\infty} \beta \cdot v = \sum_{h=1, h\neq h_0}^{H} \beta_h v_h + \lim_{M\to\infty} \beta_{h_0} \left(u_{h_0}\left(x^0\right) + M\right) = -\infty < \gamma$$

A contradiction, hence, $\beta \geqslant 0$.

Step 2. Show that $\beta \cdot u(x^0) = \gamma$.

Let

$$v^n := u\left(x^0\right) + \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right) > u\left(x^0\right)$$
$$v^n \cdot \beta \geqslant \gamma$$

then

$$\lim_{n\to\infty} v^n \cdot \beta = u\left(x^0\right) \cdot \beta \geqslant \gamma$$

And

$$x^0 \in V \Longrightarrow u\left(x^0\right) \cdot \beta \leqslant \gamma$$

Therefore,

$$\beta \cdot u\left(x^0\right) = \gamma$$

which means x^0 maximize $\beta \cdot u(x)$, for all Pareto efficient point x^0.

(b) For $\forall v, v' \in V := \left\{v \in \mathbb{R}^H : \exists x \in X, v \leqslant u(x)\right\}$,

$$\exists x, x' \in X, v \leqslant u(x), v' \leqslant u(x')$$

which implies $\forall h \in \mathcal{H}, v_h \leqslant u_h(x), v'_h \leqslant u_h(x')$. By convexity of u, $\forall h \in \mathcal{H}$, $\forall \beta \in [0,1]$,

$$u_h(\beta x + (1-\beta) x') \geqslant \beta u_h(x) + (1-\beta) u_h(x') \geqslant \beta v_h + (1-\beta) v'_h$$

Note that $\beta x + (1-\beta) x' \in X$ by the convexity of X. Therefore,

$$\beta v + (1-\beta) v' \in V$$

i.e., V is convex.

□

11. (Remark) The steps to find the Pareto efficient set in 2-dimentional case.

(a) Solve the problem

$$\max_{x\in A} \ \alpha u_1(x) + (1-\alpha) u_2(x)$$

where $\alpha \in (0,1)$. The solution is Pareto efficient.

(b) Solve the problem

$$\max_{x\in A} \ \alpha u_1(x) + (1-\alpha) u_2(x)$$

i. for $\alpha = 0$, and

ii. for $\alpha = 1$,

respectively, then we might get multiple solutions, but only 1 in each case (total 2) is Pareto efficient.

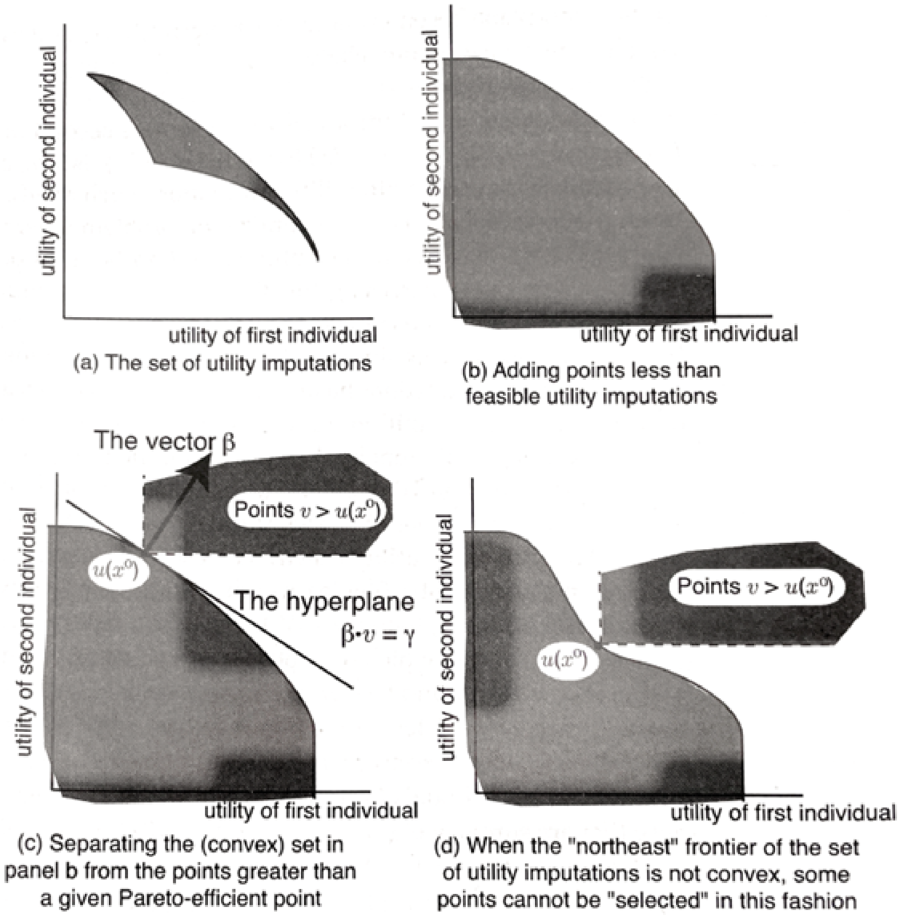

Fig. 9.1

9.2 Syndicate Theory and Efficient Risk Sharing

1. (Remark) Set-ups.

   (a) A finite collection $\mathcal{H}$ of H individuals has formed a syndicate that will share the proceeds from various risky ventures.

   (b) A finite set $\mathcal{S}$ of S states of nature, with generic element s.

   (c) $\mathcal{J}$ is the set of J risky ventures or gambles, where z_{js} is the amount that venture j returns in state s.

   (d) e_{hs} is the endowment of individual h and state s, which is not part of the pooled risky ventures. Note: We can image that there is H individuals, each of who has a little farm, and the economy has a public farm. The production of all the farms depends on the weather, which is a random event.

2. (Definition) A **sharing rule** for the syndicate is an element $(y_{hs}) \in \mathbb{R}^{H\times S}$ such that,

$$\sum_{h\in\mathcal{H}} y_{hs} \leqslant \sum_{j\in\mathcal{J}} z_{js}, \forall s \in \mathcal{S}$$

where y_{hs} is individual h's share of the total in state s.

A **social state** x is one of these sharing rules.

3. (Remark)

(a) We may require $y_{hs} + e_{hs} \geqslant 0, \forall h \in \mathcal{H}, \forall s \in \mathcal{S}$, which means nonnegative wealth or consumption in any state. In this case, one can give all of the endowment to others.

(b) Another alternative constraint is $y_{hs} \geqslant 0, \forall h \in \mathcal{H}, \forall s \in \mathcal{S}$, which implies individuals at least should consume their endowment.

(c) Another alternative constraint is $\sum_{s\in\mathcal{S}} \pi_h(s) U_h(x_{hs}) \leqslant \sum_{s\in\mathcal{S}} \pi_h(s) U_h(e_{hs}), \forall h$.

4. (Remark) More set-ups.

(a) Individual h has (cardinal, or von-Neumann-Morgenstern) utility function $U_h : \mathbb{R}^k \to \mathbb{R}$, where k is the number of goods. And with (subjective) probability assessment $\pi_h(s)$ that state s occurs.

(b) Expected utility is $\sum_{s\in\mathcal{S}} \pi_h(s) U_h(y_{hs} + e_{hs})$.

5. (Theorem) No matter which subset of the three constraints are imposed, the set of feasible sharing rules X is convex, and the individual utility functions are all concave. And therefore, every Pareto efficient point $x^0 \in X$ is the solution to the problem

$$\max_{y\in X} \sum_{h\in\mathcal{H}} \alpha_h u_h(y_{hs} + e_{hs}) = \sum_{h\in\mathcal{H}} \alpha_h \sum_{s\in\mathcal{S}} \pi_h(s) U_h(y_{hs} + e_{hs})$$

for some set of nonnegative weights $(\alpha_h)_{h\in\mathcal{H}}$, not all zero.

6. (Remark) To simplify notations, define

$$x_{hs} := y_{hs} + e_{hs}$$

and

$$W_s := \sum_{h\in\mathcal{H}} e_{hs} + \sum_{j\in\mathcal{J}} z_{js}$$

To find the set of Pareto-efficient sharing rules, solve

$$\max \sum_{h\in\mathcal{H}} \alpha_h \left[\sum_{s\in\mathcal{S}} \pi_h(s) U_h(x_{hs})\right]$$
$$\text{s.t.} \sum_{h\in\mathcal{H}} x_{hs} \leqslant W_s, \forall s \in \mathcal{S}$$
$$x_{hs} \geqslant 0, \forall h \in \mathcal{H}, \forall s \in \mathcal{S}$$

Also note that the budget constraints for different states are independent. For example, $\forall \alpha, \beta \geqslant 0$,

$$\max_{x\in A, y\in B} \{\alpha f(x) + \beta g(x)\} = \alpha \max_{x\in A} f(x) + \beta \max_{x\in B} g(x)$$

Therefore, we have the following proposition.

7. (Theorem) $(x_{hs})_{h\in\mathcal{H}, s\in\mathcal{S}}$ maximizes

$$\begin{aligned} \max \sum_{h\in\mathcal{H}} \alpha_h & \left[\sum_{s\in\mathcal{S}} \pi_h(s) U_h(x_{hs})\right] \\ \text{s.t. } \sum_{h\in\mathcal{H}} x_{hs} & \leqslant W_s, \forall s \in \mathcal{S} \\ x_{hs} & \geqslant 0, \forall h \in \mathcal{H}, \forall s \in \mathcal{S} \end{aligned}$$

iff given any $s \in \mathcal{S}$, $(x_{hs})_{h\in\mathcal{H}}$ maximizes

$$\begin{aligned} \max \sum_{h\in\mathcal{H}} & \alpha_h \pi_h(s) U_h(x_{hs}) \\ \text{s.t. } \sum_{h\in\mathcal{H}} & x_{hs} \leqslant W_s \\ x_{hs} & \geqslant 0, \forall h \in \mathcal{H} \end{aligned}$$

8. (Remark) Suppose individuals have common subjective probability assessment, i.e., $\pi_h(s) = \pi_{h'}(s)\ \forall s \in \mathcal{S}, \forall h, h' \in \mathcal{H}$.

   (a) The common probability assessments are irrelevant to Pareto-efficient sharing rules, since $\max \sum_{h\in\mathcal{H}} \alpha_h \pi_h(s) U_h(x_{hs}) \Longleftrightarrow \max \sum_{h\in\mathcal{H}} \alpha_h U_h(x_{hs})$.
   Namely, The Pareto set stays the same with $\tilde{\pi} \neq \pi$.

   (b) Suppose one individual is risk neutral and the others are strictly risk averse. Suppose as well that the risk-neutral party is not subject to the non-negativity constraint. Then for a risk-averse individual h, x_{hs} is constant in s.

9. (Example) $U_h(x) = -e^{-\mu_h x}$ for each h. Then we need to solve

$$\begin{aligned} \max \sum_{h\in\mathcal{H}} \alpha_h & \left[\sum_{s\in\mathcal{S}} \pi_h(s) U_h(x_{hs})\right] \\ \text{s.t. } \sum_{h\in\mathcal{H}} x_{hs} & \leqslant W_s, \forall s \in \mathcal{S} \\ x_{hs} & \geqslant 0, \forall h \in \mathcal{H}, \forall s \in \mathcal{S} \end{aligned}$$

Suppose the first constraint is binding, but the second constraint is non-binding. The F.O.C.s are

$$\alpha_h \pi_h(s) U_h'(x_{hs}) = \lambda_s \sum_{h\in\mathcal{H}} x_{hs} = W_s$$

Define $\tau_h := \frac{1}{\mu_h}$, then

$$\begin{cases} \frac{\alpha_h}{\tau_h} e^{-\frac{x_{hs}}{\tau_h}} = \frac{\lambda_s}{\pi(s)} \\ \sum_{h\in\mathcal{H}} x_{hs} = W_s \end{cases} \Longrightarrow \begin{cases} x_{hs} = -\tau_h \ln \frac{\tau_h\lambda_s}{\alpha_h\pi(s)} \\ \sum_{h\in\mathcal{H}} x_{hs} = W_s \end{cases} \Longrightarrow \sum_{h\in\mathcal{H}} \tau_h \ln \frac{\alpha_h \pi(s)}{\tau_h \lambda_s} = W_s$$

Then

$$\sum_{h\in\mathcal{H}} \tau_h \ln \frac{\alpha_h \pi(s)}{\tau_h \lambda_s} = \sum_{h\in\mathcal{H}} \tau_h \ln \left(\frac{\alpha_h}{\tau_h}\right) - \ln \left(\frac{\lambda_s}{\pi(s)}\right) \sum_{h\in\mathcal{H}} \tau_h = W_s$$

Define $T := \sum_{h\in\mathcal{H}} \tau_h$, $K := \sum_{h\in\mathcal{H}} \tau_h \ln \left(\frac{\alpha_h}{\tau_h}\right)$, then

$$W_s = K - T \ln \left(\frac{\lambda_s}{\pi(s)}\right) \Longrightarrow \ln \left(\frac{\lambda_s}{\pi(s)}\right) = \frac{K - W_s}{T}$$

Therefore,

$$\begin{aligned} x_{hs} &= \tau_h \ln \frac{\alpha_h \pi(s)}{\tau_h \lambda_s} = \tau_h \ln \frac{\alpha_h}{\tau_h} - \tau_h \ln \frac{\lambda_s}{\pi(s)} \\ &= \tau_h \ln \frac{\alpha_h}{\tau_h} - \tau_h \frac{K - W_s}{T} \\ &= \tau_h \left[\ln \frac{\alpha_h}{\tau_h} - \frac{K}{T}\right] + \frac{\tau_h}{T} W_s \end{aligned}$$

Let $k_h := \tau_h \left[\ln \frac{\alpha_h}{\tau_h} - \frac{K}{T}\right]$, then

$$x_{hs} = \frac{\tau_h}{T} W_s + k_h$$

10. (Example) $U_h(x) = \frac{1-e^{\mu_h x}}{\mu_h}$, if there exists a unique risk neutral household, let it be h_0, i.e., $\frac{\mathrm{d}^2}{\mathrm{d}x^2} U_{h_0}(x) = 0$, we can get that $\mu_{h_0} = 0$, then $\lim_{\mu_h\to 0} U_h(x) = x$. And we can see that $\frac{1}{\tau_{h_0}} \to \infty$, which implies that

$$\frac{\tau_h}{T} = \begin{cases} 0, & h \neq h_0 \\ 1, & h = h_0 \end{cases}$$

then

$$x_{hs} = \begin{cases} k_h, & h \neq h_0 \\ W_s + k_{h_0}, & h = h_0 \end{cases}$$

Chapter 10

General Equilibrium with Quasilinear Preferences (MWG)

10.1 General Equilibrium

1. (Remark) Partial equilibrium.

   In general, a partial equilibrium of one single good price P^* solves the demand-supply equality:

   $$Q_D(P^*) = Q_S(P^*)$$

   However, other factors such as prices of other goods may affect the equilibrium. In general equilibrium, P^* is a price vector, containing prices of all goods.

2. (Remark) General Set-up.

   (a) Economy.

      i. I consumer, $i = 1, 2, \cdots, I$.

      ii. J firms, $j = 1, 2, \cdots, J$.

      iii. L goods, $\ell = 1, 2, \cdots, L$.

   (b) Consumer.

      i. Consumer i's consumption set is X_i, the utility function is $u_i(x_i)$, a consumption bundle is $x_i = (x_{1,i}, \cdots, x_{L,i}) \in X_i$.

      ii. Consumer i's initial endowment is $\omega_i = (\omega_{1i}, \cdots, \omega_{Li})$. Total endowment if good ℓ is $\omega_\ell = \sum_{i=1}^{I} \omega_{\ell i}$

      iii. Consumer i's share of firm j's profit is θ_{ij}, where $\sum_{i=1}^{I} \theta_{ij} = 1$.

   (c) Firm.

      i. Firm j's production set is $Y_j \subseteq \mathbb{R}^L$, $y_j = (y_{1j}, y_{2j}, \cdots .y_{Lj}) \in Y_j$ is a production vector.

ii. Firm j chooses $y_j \in \mathbb{R}^L$, then the net amount of good ℓ available to the economy is $\omega_\ell + \sum_{j=1}^J y_{\ell j}$.

3. (Definition) An **economic allocation** $(x_1, \cdots, x_I, y_1, \cdots, y_J)$ where $x_i \in X_i$, $y_i \in Y_i$. The allocation is **feasible** if

$$\sum_{i=1}^{I} x_{\ell j} \leqslant \omega_\ell + \sum_{j=1}^{J} y_{\ell j}, \forall \ell = 1, 2, \cdots, L$$

4. (Definition) A feasible allocation $(x_1, \cdots, x_I, y_1, \cdots, y_J)$ is **Pareto optimal** or **Pareto efficient** if there is no other feasible allocation $(x_1', \cdots, x_I', y_1', \cdots, y_J')$, such that

$$u_i(x_i') \geqslant u_i(x_i), \forall i = 1, \cdots, I u_i(x_i') > u_i(x_i), \exists i = 1, \cdots, I$$

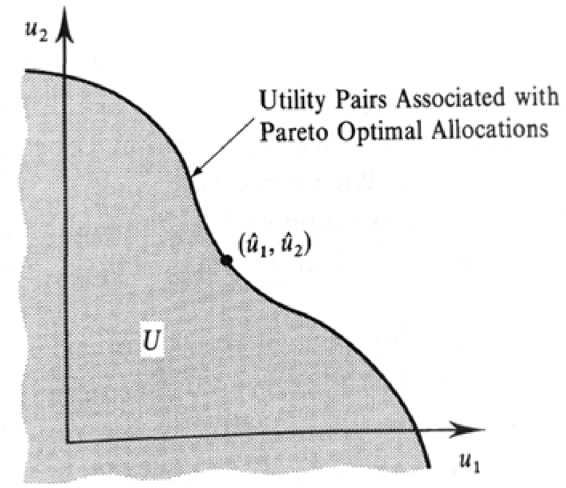

Fig. 10.1

5. (Definition) The allocation $(x_1^*, \cdots, x_I^*, y_1^*, \cdots, y_J^*)$ and price vector $p^* \in \mathbb{R}^L$ constitute a **competitive or Walrasian equilibrium** if the following condition are satisfied:

   (a) Profit maximization: For each firm j, y_j^* solves

$$\max_{y_j \in Y_j} p^* \cdot y_j$$

   (b) Utility maximization: For each consumer i, x_i^* solves

$$\max_{x_i \in X_i} u_i(x_i) \text{ s.t. } p^* \cdot x_i \leqslant p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} \left(p^* \cdot y_j^*\right)$$

   (c) Market clearing: For each good $\ell = 1, 2, \cdots, L$.

$$\sum_{i=1}^{I} x_{\ell i}^* = \omega_\ell + \sum_{j=1}^{J} y_{\ell j}^*$$

6. (Remark) To solve the equilibrium prices p^*, we need to follow the definition of equilibrium.

   (a) Solving the firm's problem gives us $y_j^*(p_1, \cdots, p_L)$.

   (b) Solving the consumer's problem gives us $x_i^*(p_1, \cdots, p_L)$.

   (c) There are L unknows $p_1, \cdots, p_L$, the market clearing condition only gives us $L-1$ equations, since one of them is redundant (Every equation in the market clearing condition can be deduced by the others $L-1$ conditions).

   (d) However, note that the price level doesn't matter, so we can normalize $p_1 = 1$.

7. (Theorem, Walras' Rule) If the allocation $(x_1, \cdots, x_I, y_1, \cdots, y_J)$ and price vector $p \gg 0$ satisfies the market clearing condition for all goods $\ell \neq k$, and if every consumer's budget constraint is binding, i.e.,

$$p^* \cdot x_i = p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} \left(p^* \cdot y_j^*\right)$$

then the market for good k also clears.

Proof. Adding up the consumers' budget constraints over $i = 1, \cdots, I$,

$$\sum_{i=1}^{I} p^* \cdot x_i^* = \sum_{i=1}^{I} p^* \cdot \omega_i + \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_{ij} \left(p^* \cdot y_j^*\right)$$

then

$$\sum_{i=1}^{I} p^* \cdot x_i^* = \sum_{i=1}^{I} p^* \cdot \omega_i + \sum_{j=1}^{J} \left(p^* \cdot y_j^*\right) \sum_{i=1}^{I} \theta_{ij} = \sum_{i=1}^{I} p^* \cdot \omega_i + \sum_{j=1}^{J} \left(p^* \cdot y_j^*\right)$$

or

$$\begin{aligned} p^* \cdot \sum_{i=1}^{I} \left(x_{1i}^*, \cdots, x_{Li}^*\right) &= p^* \cdot \sum_{i=1}^{I} \left(\omega_{1i}, \cdots, \omega_{Li}\right) + p^* \cdot \sum_{j=1}^{J} \left(y_{1j}^*, \cdots, y_{Lj}^*\right) \\ &= p^* \cdot \left(\omega_1, \cdots, \omega_L\right) + p^* \cdot \sum_{j=1}^{J} \left(y_{1j}^*, \cdots, y_{Lj}^*\right) \end{aligned}$$

or

$$\sum_{\ell=1}^{L} \left(p_\ell^* \sum_{i=1}^{I} x_{\ell i}^*\right) = \sum_{\ell=1}^{L} p_\ell^* \omega_\ell + \sum_{\ell=1}^{L} \left(p_\ell^* \cdot \sum_{j=1}^{J} y_{\ell j}^*\right)$$

By the market clearing conditions for all goods $\ell \neq k$,

$$\sum_{i=1}^{I} x_{\ell i}^* = \omega_\ell + \sum_{j=1}^{J} y_{\ell j}^*$$

or

$$p_\ell^* \cdot \sum_{i=1}^{I} x_{\ell i}^* = p_\ell^* \cdot \omega_\ell + p_\ell^* \cdot \sum_{j=1}^{J} y_{\ell j}^*, \forall \ell \neq k$$

then

$$\sum_{\ell\neq k}^{L}\left(p_\ell^*\sum_{i=1}^{I}x_{\ell i}^*\right)=\sum_{\ell\neq k}^{L}p_\ell^*\omega_\ell+\sum_{\ell\neq k}^{L}\left(p_\ell^*\cdot\sum_{j=1}^{J}y_{\ell j}^*\right)$$

Therefore,

$$p_k^*\sum_{i=1}^{I}x_{ki}^*=p_k^*\omega_k+p_k^*\cdot\sum_{j=1}^{J}y_{kj}^*\Longrightarrow\sum_{i=1}^{I}x_{ki}^*=\omega_k+\sum_{j=1}^{J}y_{kj}^*$$

□

10.2 Quasi-linear Preference in a Two-Good Economy

1. (Remark) Set-ups.

   (a) Two goods ℓ and m.

   (b) Consumer i's preference is quasi-linear.

   $$u_i(m_i,x_i)=m_i+\phi_i(x_i)$$

   thus, good m can measure the utility difference, and every consumer likes good m.

   (c) Consumption of m can be negative, to avoid boundary problems.

   (d) Endowments.

      i. No initial endowments of good ℓ, it must be produced by firms.

      ii. Consumer i's initial endowment of the numeraire m is $\omega_{m,i}>0$.

      iii. Aggregate endowment of m is $\omega_m=\sum_{i=1}^{I}\omega_{m,i}$.

   (e) Firms are able to produce good ℓ from m. The amount of numeraire required by firm j to produce $q_j>0$ units of good ℓ is given by the cost function $c_j(q_j)$.

   The production possibility set is

   $$Y_j=\{(-z_j,q_j):z_j\geqslant c_j(q_j),q_j\geqslant 0\}$$

2. (Remark) Decentralized Equilibrium.

   (a) Firm j's problem

   $$\max_{q_j\geqslant 0}p^*q_j-c_j(q_j)$$

   $$\begin{aligned}p^*&<c_j'\left(q_j^*\right),\ \text{if } q_j^*=0\\ p^*&=c_j'\left(q_j^*\right),\ \text{if } q_j^*>0\end{aligned}$$

   Recall the conclusion that if f is concave, then $c(w,q)$ is a convex function of q. Thus, if we assume the production function is concave, then $p^*q_j-c_j(q_j)$ is a concave function of q_j, which means the F.O.C. is both sufficient and necessary.

(b) Consumer i's problem

$$\max_{x_i \in \mathbb{R}_+, m_i \in \mathbb{R}} \phi_i(x_i) + m_i$$
$$\text{s.t. } m_i + p^* x_i \leqslant \omega_{m,i} + \sum_{j=1}^{J} \theta_{ij}\left(p^* q_j - c_j\left(q_j^*\right)\right)$$

F.O.C. is

$$p^* > \phi_i'(x_i^*), \text{ if } x_i^* = 0$$
$$p^* = \phi_i'(x_i^*), \text{ if } x_i^* > 0$$

(c) Market clearing

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*$$

(d) Note that we normalize price of good m to 1. Then the aggregate demand and supply depend on p only, i.e.,

$$x(p) = \sum_{i=1}^{I} x_i(p)$$

$$q(p) = \sum_{j=1}^{J} q_j(p)$$

(e) If the equality of F.O.C.s all hold, then

$$\phi_i'(x_i^*) = p^* = c_j'\left(q_j^*\right), \forall i, j$$

3. (Example) Suppose the production function is given by $y_j = \alpha_j \sqrt{m_j}$.

Then the cost function is

$$c_j(q_j) = \left(\frac{q_j}{\alpha_j}\right)^2$$

Firm's problem:

$$\max_{q_j \geqslant 0} p^* q_j - \left(\frac{q}{\alpha_j}\right)^2$$

F.O.C. is

$$p^* - \frac{2q}{\alpha_j^2} = 0 \Longrightarrow q^* = \frac{\alpha_j^2}{2} p^*$$

Suppose $\phi_i(x_i) = \beta_i \sqrt{x_i}$, then

$$\max_{x_i \in \mathbb{R}_+, m_i \in \mathbb{R}} \beta_i \sqrt{x_i} + m_i$$
$$\text{s.t. } m_i + p^* x_i \leqslant \omega_{m,i} + \sum_{j=1}^{J} \theta_{ij}\left(p^* q_j - c_j\left(q_j^*\right)\right)$$

F.O.C. is

$$\frac{\beta_i}{2\sqrt{x_i}} = p^* \Longrightarrow x_i^* = \left(\frac{\beta_i}{2p^*}\right)^2$$

Market clearing:

$$p^* \cdot \sum_{j=1}^{J} \frac{\alpha_j^2}{2} = \left(\frac{1}{p^*}\right)^2 \cdot \sum_{i=1}^{I} \frac{\beta_i^2}{4} \Longrightarrow p^* = \left(\sum_{i=1}^{I} \frac{\beta_i^2}{4} \Big/ \sum_{j=1}^{J} \frac{\alpha_j^2}{2}\right)^{\frac{1}{3}}$$

4. (Remark) Centralized social planner's problem. Suppose that the planner decides on the production first.

   (a) Choose production level $\bar{q}_j$ for firm j.

   (b) Total output available: $\sum_{j=1}^{J} \bar{q}_j$.

   (c) Total numeraire available: $\omega_m - \sum_{j=1}^{J} c_j(\bar{q}_j)$

   Next, the planner decides level $\bar{x}_i$ for consumer i, subject to

$$\sum_{i=1}^{I} \bar{x}_i \leqslant \sum_{j=1}^{J} \bar{q}_j \sum_{i=1}^{I} m_i \leqslant \omega_m - \sum_{j=1}^{J} c_j(\bar{q}_j)$$

   The set of possible utilities is (assume ϕ_i is non-decreasing)

$$\left\{(u_1, u_2, \cdots, u_I) \Big| \sum_{i=1}^{I} u_i \leqslant \sum_{i=1}^{I} \phi_i(\bar{x}_i) + \omega_m - \sum_{j=1}^{J} c_j(\bar{q}_j)\right\}$$

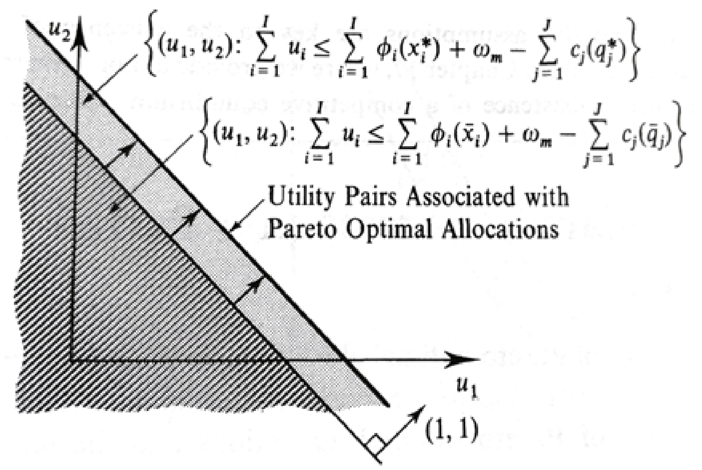

Fig. 10.2

The social optimal (Pareto efficient) allocation solves

$$\max_{\substack{(x_1, x_2, \cdots, x_I) \geqslant 0 \\ (q_1, q_2, \cdots, q_J) \geqslant 0}} \sum_{i=1}^{I} \phi_i(x_i) + \omega_m - \sum_{j=1}^{J} c_j(q_j) \text{ s.t. } 0 = \sum_{i=1}^{I} x_i - \sum_{j=1}^{J} q_j$$

The F.O.C.s are

$$\phi_i'(x_i) = \lambda, \forall i = 1, 2, \cdots, I$$
$$c_j'(q_j) = \lambda, \forall j = 1, 2, \cdots, J$$

Therefore,

$$\phi_i'(x_i) = \lambda = c_j'(q_j), \forall i, j$$

which coincidences with the decentralized equilibrium condition.

5. (Theorem, The First Fundamental Theorem of Welfare Economics) If the price p^* and allocation $(x_1^*, \cdots, x_I^*, q_1^*, \cdots, q_J^*)$ constitutes a competitive equilibrium, then this allocation is Pareto optimal, i.e., $\sum_{i=1}^{I} u_i$ is optimal.

6. (Theorem, The Second Fundamental Theorem of Welfare Economics) For any Pareto optimal levels of utility $(u_1^*, \cdots, u_I^*)$, there are transfers of numeraire commodity $(T_1, \cdots, T_I)$ satisfying $\sum_{i=1}^{I} T_i = 0$ such that a competitive equilibrium reached from the endowments $(\omega_{m,1} + T_1, \cdots, \omega_{m,I} + T_I)$ yields the utilities $(u_1^*, \cdots, u_I^*)$.

   Note: The intuition for this theorem is that everyone in the economy loves the numeraire commodity m.

Chapter 11

General Equilibrium

11.1 Walrasian Equilibrium and Its Properties

1. (Definition) An **economy** $\mathcal{E}$ consists of

   (a) A commodity space $\mathbb{R}^k$, where $k < \infty$.

   (b) F firms, where $F < \infty$. Firm f has a production-possibility set $\varnothing \neq Z^f \subseteq \mathbb{R}^k$.

   (c) H consumers, where $H < \infty$. Consumer h is characterized by

      i. a consumption space $\varnothing \neq X^h \subseteq \mathbb{R}^k_+$.

      ii. a utility function $u^h : X^h \to \mathbb{R}$.

      iii. a commodity endowment $e^h \in X^h$.

      iv. F shareholdings s^{fh}, $f = 1, \cdots, F$, and $\sum_{h=1}^{H} s^{fh} = 1, \forall f$.

   Note: we often assume that consumer's utility function u^h is continuous and firm's production-possibility set Z^f is nonempty and closed.

2. (Definition) A **Walrasian equilibrium** for the economy $\mathcal{E}$ is a consumption allocation $\left\{x^h : h = 1, \cdots, H\right\}$, a production plans $\left\{z^f : f = 1, \cdots, F\right\}$ and a price vector $p \in \mathbb{R}^k$, such that

   (a) For each consumer h, $x^h \in X^h$ and solves the problem

   $$\max_{x \in X^h} u^h(x) \text{ s.t. } p \cdot x \leqslant p \cdot e^h + \sum_{f=1}^{F} s^{fh} \left(p \cdot z^f\right)$$

   (b) For each firm f, $z^f \in Z^f$ and solves the problem

   $$\max_{z \in Z^f} p \cdot z$$

   (c) Markets clear:

   $$\sum_{h=1}^{H} x^h \leqslant \sum_{h=1}^{H} e^h + \sum_{f=1}^{F} z^f$$

3. (Remark)

   (a) It is part of the definition of a Walrasian equilibrium that prices are arranged so that consumers and firms can solve their respective problems (the solution set is not empty).

   (b) For a pure-exchange economy, we can either omit the firms in the definition, or set $Z^f = \{0\}$.

   (c) Both consumers and firms are price-takers.

4. (Remark) Change notations.

   (a) Let $\boldsymbol{X} = \prod_{h=1}^{H} X^h$ be **the space of consumption allocation vector**, for $\boldsymbol{x} \in \boldsymbol{X}$, write $\boldsymbol{x}^h$ for h's part of the allocation.

   (b) Let $\boldsymbol{Z} = \prod_{f=1}^{F} Z^f$ be **the space of production plans for all firms**, for $\boldsymbol{z} \in \boldsymbol{Z}$, write $\boldsymbol{z}^f$ for f's par of the production plan.

   (c) A Walrasian equilibrium consists of a triple $(p, \boldsymbol{x}, \boldsymbol{z})$ or $\left(\{p_i\}_{i=1}^{k}, \{\boldsymbol{x}^h\}_{h=1}^{H}, \{\boldsymbol{z}^f\}_{f=1}^{F}\right)$, where p is the equilibrium price vector, $\boldsymbol{x}$ is the consumption allocation and $\boldsymbol{z}$ is the production plan.

5. (Definition) For a given economy $\mathcal{E}$, **the space of socially feasible consumption allocations**, denoted $\boldsymbol{X}^*$, is defined as

$$\boldsymbol{X}^* = \left\{ \boldsymbol{x} \in \boldsymbol{X} : \sum_{h=1}^{H} \boldsymbol{x}^h \leqslant \sum_{h=1}^{H} e^h + \sum_{f=1}^{F} \boldsymbol{z}^f \cdot p, \text{ for some } \boldsymbol{z} \in \boldsymbol{Z} \right\}$$

   Note that the entries of $\boldsymbol{x}^h$ can be negative, i.e., we allow for the disposal of goods.

6. (Theorem) Properties of Walrasian Equilibria.

   (a) If $(p, \boldsymbol{x}, \boldsymbol{z})$ is a Walrasian equilibrium for some economy, then so is $(\lambda p, \boldsymbol{x}, \boldsymbol{z})$ for all $\lambda > 0$.

   (b) If any consumer is globally insatiable, then $p \neq 0$.

   (c) If any consumer has a nondecreasing and globally insatiable utility function, or if any firm has a free-disposal technology, then $p > 0$.

   (d) If every consumer is locally insatiable,

$$p \cdot \left(\sum_{h=1}^{H} \boldsymbol{x}^h \right) = p \cdot \left(\sum_{h=1}^{H} e^h \right) + p \cdot \left(\sum_{f=1}^{F} \boldsymbol{z}^f \right)$$

   If, in addition, $p \geqslant 0$, then for each commodity i,

$$\sum_{h=1}^{H} \boldsymbol{x}_i^h < \sum_{h=1}^{H} e_i^h + \sum_{f=1}^{F} \boldsymbol{z}_i^f \Longrightarrow p_i = 0$$

Proof. (a) Just check that the budget constraint in the definition is unchanged.

(b) Suppose consumer h is globally insatiable and $p = 0$, then consumer h could offer any bundle and therefore, her utility-maximization problem will have no solution, a contradiction to the definition of Walrasian equilibrium.

(c) Suppose p is an equilibrium price vector and $\exists i$, $p_i < 0$.

Step 1. Suppose consumer h has a nondecreasing and globally insatiable utility function.

Let $\boldsymbol{x}^h$ be h's equilibrium consumption, then $\exists \hat{\boldsymbol{x}}^h \in X^h$, such that

$$u^h\left(\hat{\boldsymbol{x}}^h\right) > u^h\left(\boldsymbol{x}^h\right)$$

If $p \cdot \hat{\boldsymbol{x}}^h \leqslant p \cdot \boldsymbol{x}^h$, then $\hat{\boldsymbol{x}}^h$ satisfies the budget constraint, contradicting to the $\boldsymbol{x}^h$ solves h's utility maximization problem.

If $p \cdot \hat{\boldsymbol{x}}^h \leqslant p \cdot \boldsymbol{x}^h$, define $M := p \cdot \boldsymbol{x}^h - p \cdot \hat{\boldsymbol{x}}^h > 0$. Let b be a zero vector except the ith component is 1. Then

$$\hat{\boldsymbol{x}}^h + \frac{M}{|p_i|} b \geqslant \hat{\boldsymbol{x}}^h$$

and thus,

$$u\left(\hat{\boldsymbol{x}}^h + \frac{M}{|p_i|} b\right) > u\left(\hat{\boldsymbol{x}}^h\right)$$

but

$$p \cdot \left(\hat{\boldsymbol{x}}^h + \frac{M}{|p_i|} b\right) = p \cdot \hat{\boldsymbol{x}}^h + \frac{M}{|p_i|} p_i = p \cdot \hat{\boldsymbol{x}}^h - M = p \cdot \boldsymbol{x}^h$$

Again, $\hat{\boldsymbol{x}}^h + \frac{M}{|p_i|} b$ satisfies the budget constraint, contradicting to the $\boldsymbol{x}^h$ solves h's utility maximization problem.

Step 2. Suppose firm f has a free-disposal technology.

Let b be a zero vector except the ith component is 1.

For any optimal production plan $\boldsymbol{z}^f$, the plan $\boldsymbol{z}^f - b$ us feasible. And thus, the plan $\boldsymbol{z}^f - b$ givens $-p_i$ more profit than $\boldsymbol{z}^f$, contradicting $\boldsymbol{z}^f$ is optimal.

(d) By local insatiability, each consumer will spend all her money, i.e.,

$$p \cdot \boldsymbol{x}^h = p \cdot e^h + \sum_{f=1}^{F} s^{fh}\left(p \cdot \boldsymbol{z}^f\right)$$

Summing across h gives us the first half, or

$$p \cdot \left(\sum_{h=1}^{H} \boldsymbol{x}^h - \sum_{h=1}^{H} e^h - \sum_{f=1}^{F} \boldsymbol{z}^f\right) = 0$$

Note that

$$\sum_{h=1}^{H} \boldsymbol{x}_i^h - \left(\sum_{h=1}^{H} e_i^h + \sum_{f=1}^{F} \boldsymbol{z}_i^f\right) \geqslant 0, \forall h$$

Suppose $p \geqslant 0$, then [1]

$$p_i \cdot \left(\sum_{h=1}^{H} \boldsymbol{x}_i^h - \sum_{h=1}^{H} e_i^h - \sum_{f=1}^{F} \boldsymbol{z}_i^f\right) \geqslant 0 \Longrightarrow p_i \cdot \left(\sum_{h=1}^{H} \boldsymbol{x}_i^h - \sum_{h=1}^{H} e_i^h - \sum_{f=1}^{F} \boldsymbol{z}_i^f\right) = 0$$

Now suppose that for commodity i,

$$\sum_{h=1}^{H} \boldsymbol{x}_i^h < \sum_{h=1}^{H} e_i^h + \sum_{f=1}^{F} \boldsymbol{z}_i^f$$

then

$$p_i = 0$$

□

7. (Theorem) Suppose that $\exists h$, u^h is strictly increasing in the consumption of commodity i, i.e., x, x' satisfy that $x_i > x_i', x_j = x_j', j \neq i$, then $u^h(x) > u^h(x')$. Then $p_i > 0$.

   Proof. If $p_i \leqslant 0$, then the utility maximization problem for consumer h has no solution. □

8. (Theorem) Suppose $p_i > 0$ for every Walrasian equilibrium, and suppose $\exists j \neq i$, $\exists f$ such that $z \in Z^f$, then $\exists \hat{z} \in Z^f$ such that

$$\begin{cases} \hat{z}_i - z_i > 0 \\ \hat{z}_j - z_j \leqslant 0 \\ \hat{z}_\ell = z_\ell, \forall \ell \notin \{i, j\} \end{cases}$$

   Then $p_j > 0$ for every Walrasian equilibrium price p.

   Note:

   - Scenario 1: We use j to produce i. If $p_i > 0$, then $p_j > 0$. Intuitively, if the production has positive value, then so is the factor producing it.
   - Scenario 2: Both i, j are inputs, but i and j can be substituted with each other. If $p_i > 0$, then $p_j > 0$.

   Proof. Suppose $p_j \leqslant 0$, then

$$p_i (z_i - \hat{z}_i) < 0, p_j (z_j - \hat{z}_j) \leqslant 0$$

   thus,

$$p \cdot z - p \cdot \hat{z} = p \cdot (z - \hat{z}) = p_i (z_i - \hat{z}_i) + p_j (z_j - \hat{z}_j) < 0$$

   contradicting to z is an optimal production plan. □

[1] $x, y \in \mathbb{R}, x \geqslant 0, y \geqslant 0, x + y = 0 \Longrightarrow x = y = 0$.

11.2 Existence of Walrasian Equilibria

1. (Remark) A quick review.

   (a) Let $\phi : X \rightrightarrows Y$, we say

   i. ϕ is **locally bounded** if $\forall x \in X$, $\exists \varepsilon > 0$, $\forall x' \in B_\varepsilon(x)$ there is a bounded subset $Y(x) \subseteq Y$ such that $\phi(x') \subseteq Y(x)$.

   ii. ϕ is **upper semi-continuous**, if $\forall \{x_n\}_{n=1}^\infty \in X^\mathbb{N}$ with $x_n \to x \in X$, and $\forall \{y_n\}_{n=1}^\infty \in Y^\mathbb{N}$, with $\forall n \geqslant 1, y_n \in \phi(x_n)$, and $\lim_{n\to\infty} y_n$ exists, then $\lim_{n\to\infty} y_n = y \in \phi(x)$.

   iii. ϕ is **lower semi-continuous**, if $\forall x \in X, \forall \{x_n\}_{n=1}^\infty \in X^\mathbb{N}$ with $x_n \to x \in X$, and $\forall y \in \phi(x)$, $\exists \{y_n\}_{n=1}^\infty \in Y^\mathbb{N}$ with $\forall n \geqslant 1, y_n \in \phi(x_n)$, such that $\lim_{n\to\infty} y_n = y$.

   iv. ϕ is **continuous**, if ϕ is both upper and lower semi-continuous.

   (b) (Berge's Theorem) When the objection function $f(x,t)$ is continuous and the constrain correspondence $t \rightrightarrows A(t)$ is continuous and locally bounded, then the value function $\max_{x\in A(t)} f(x,t)$is continuous and the correspondence of maximizers

$$\arg\max_{x\in A(t)} f(x,t)$$

   is non-empty, upper semi-continuous, and locally bounded.

   Simply put, if $t^n \to t^*, x^n \in \arg\max_{x\in A(t^n)} f(x,t^n)$, then $x^n \to x^* \in A(t^*)$. Therefore, $x^* \in \arg\max_{x\in A(t^*)} f(x,t^*)$ and $\max_{x\in A(t^n)} f(x,t^n) \to \max_{x\in A(t^*)} f(x,t^*)$.

2. (Theorem, Kakutani's Fixed-point Theorem) Suppose that X is a nonempty, compact, convex subset of $\mathbb{R}^n$, and $F : X \rightrightarrows X$ is an upper semicontinuous, convex, and nonempty valued correspondence. Then $\exists x \in X$ such that $x \in F(x)$.

3. (Definition) An **n-player generalized game $G = \{A_\ell, C_\ell, u_\ell\}_{\ell=1}^n$** for a finite integer n consists of, for each $\ell = 1, \cdots, n$,

   (a) a set of strategies or actions A_ℓ,

   (b) a constraint correspondence $C_\ell : \prod_{m\neq\ell}^n A_m \rightrightarrows A_\ell$ (given other people's choice, your strategies are limited), and

   (c) a utility function $u_\ell : \prod_{\ell=1}^n A_\ell \to \mathbb{R}$.

4. (Definition) A **Nash equilibrium** for an n-player generalized game is a strategy profile $(a_\ell^*)_{\ell=1}^n \in \prod_{\ell=1}^n A_\ell$ such that, $\forall \ell = 1, 2, \cdots, n$,

   (a) (Feasible) $a_\ell^* \in C_\ell\left((a_m^*)_{m\neq\ell}\right)$ and

   (b) (Optimal) a_ℓ^* solves $\max_{a_\ell \in C_\ell\left((a_m^*)_{m\neq\ell}\right)} u_\ell\left(a_\ell, (a_m^*)_{m\neq l}\right)$.

5. (Example) There are two players, $n = 2$. $A_\ell = \{H, T\}, \forall \ell = 1, 2$.

$$C_1(H) = \{H\}, C_1(T) = \{T\}\, C_2(H) = \{T\}, C_2(T) = \{H\}$$

There is no Nash equilibrium, if it has, then there are two cases.

 (a) If 2 chooses H, then 1 will choose H, which restrict 2 to choose T, leading to a contradiction to $a_\ell^* \in C_\ell\left((a_m^*)_{m \neq \ell}\right)$.

 (b) If 2 chooses T, then 1 will choose T, which restrict 2 to choose H, again, leading to a contradiction to $a_\ell^* \in C_\ell\left((a_m^*)_{m \neq \ell}\right)$.

6. (Theorem) Suppose that G is a generalized game for which

 (a) each A_ℓ is a nonempty, compact, convex subset of $\mathbb{R}^{k_\ell}$ for some integer k_ℓ,

 (b) each $C_\ell(a_{\neg\ell})$ is a continuous, nonempty-valued, and convex-valued correspondence, and

 (c) each u_ℓ is jointly continuous in $(a_1, \cdots, a_n)$ and quasi-concave in a_ℓ (for each fixed $a_{\neg\ell} := (a_m)_{m \neq \ell}$.

Proof. Two steps.

Step 1. Apply Berge's Thereon.

For each $\ell = 1, 2, \cdots, n$, consider the parametric maximization problem

$$\max_{a_\ell \in C_\ell(a_{\neg\ell})} u_\ell(a_\ell; a_{\neg\ell})$$

Let $u_\ell^*(a_{\neg\ell})$ be the value of the maximized function, and $A_\ell^*(a_{\neg\ell})$ be the set of maximizers.

Note that A_ℓ is compact in $\mathbb{R}^{k_\ell}$ and thus bounded.

Then u is continuous, $C_\ell(a_{\neg\ell}) \subseteq A_\ell$ is bounded, continuous. By Berge's theorem, $a_{\neg\ell} \to u_\ell^*(a_{\neg\ell})$ is continuous, and $a_{\neg\ell} \rightrightarrows A_\ell^*(a_{\neg\ell})$ is upper semi-continuous.

Step 2. Apply Kakutani's Fixed-point Theorem.

By the convexity of $C_\ell(a_{\neg\ell})$ and quasi-concavity of u_ℓ in a_ℓ, we know that $A_\ell^*(a_{\neg\ell})$ is convex.

Define a self-correspondence A^* as

$$A^* : \prod_{\ell=1}^n A_\ell \rightrightarrows \prod_{\ell=1}^n A_\ell \;\; (a_1, \cdots, a_n) \mapsto \{(a_1^*, \cdots, a_n^*)\} \subseteq \prod_{\ell=1}^n A_\ell^*(a_{\neg\ell})$$

Recall that A_ℓ is a nonempty, compact, convex subset of $\mathbb{R}^{k_\ell}$, then A^* a nonempty, compact, convex subset of $\mathbb{R}^{k_1 + \cdots + k_n}$.

Now show that A^* is an upper semi-continuous, convex, and nonempty valued correspondence.

(a) upper semi-continuity: Suppose $\{a^i = (a_1^i, a_2^i, \cdots, a_n^i)\}_{i=1}^{\infty}$ is a sequence in $\prod_{\ell=1}^n A_\ell$, with

$$a^i = \left(a_1^i, a_2^i, \cdots, a_n^i\right) \to a = (a_1, \cdots, a_n)$$

which implies

$$a_\ell^i \to a_\ell, \forall \ell$$

By the upper semi-continuity of each $A_\ell^* (a_{\neg\ell})$, $a_\ell \in A_\ell^* (a_{\neg\ell})$, therefore,

$$(a_1, \cdots, a_n) \in \prod_{\ell=1}^{n} A_\ell^* (a_{\neg\ell})$$

(b) convexity: Fix $a = (a_1, \cdots, a_n)$, $\forall \hat{a}^*, \tilde{a}^* \in A^* (a)$, $\forall \lambda \in (0, 1)$, consider

$$\lambda \hat{a}^* + (1-\lambda) \tilde{a}^* = (\lambda \hat{a}_1^* + (1-\lambda) \tilde{a}_1^*, \cdots, \lambda \hat{a}_n^* + (1-\lambda) \tilde{a}_n^*)$$

By the convexity of $A_\ell^* (a_{\neg\ell})$, $\forall \ell$, we can see that

$$\lambda \hat{a}_\ell^* + (1-\lambda) \tilde{a}_\ell^* \in A_\ell^* (a_{\neg\ell}), \forall \ell$$

then

$$\lambda \hat{a}^* + (1-\lambda) \tilde{a}^* \in A^* (a)$$

(c) nonempty valued: Fix $a = (a_1, \cdots, a_n)$. Since $\forall \ell$, $A_\ell^* (a_{\neg\ell})$ is nonempty, then $\exists a_\ell^* \in A_\ell^* (a_{\neg\ell})$, then $\exists (a_1^*, \cdots, a_n^*) \in A^* (a)$.

Then by Kakutani's Fixed-point Theorem, $\exists a^* \in \prod_{\ell=1}^n A_\ell$, such that $a^* \in A^* (a^*)$, which implies that $a_\ell^* \in A_\ell^* (a_{\neg\ell}^*)$ and meets the definition of Nash equilibrium.

See table 11.1.

□

7. (Theorem, A very bounded economy) Suppose the economy $\mathcal{E}$ satisfies the following conditions:

(a) $\forall h$, X_h is a compact and convex subset of $\mathbb{R}_+^k$, $e^h \in \text{int}\left(X^h\right)$, and $\succsim^h$ is continuous and convex.

(b) $\forall f$, Z^f is compact and convex and $0 \in Z^f$. Then $\mathcal{E}$ has a Walrasian equilibrium.

Idea of Proof. The number of players in the game in $1 + H + F$:

(a) A mythical auctioneer.

i. Action set: $P = \{p : p_1 + \cdots + p_k = 1\}$

ii. No constraint.

iii. The utility function is the social net trade.

$$v\left(p, \left(x^h\right), \left(z^f\right)\right) = \sum_{h=1}^{H} p \cdot x^h - \sum_{h=1}^{H} p \cdot e^h - \sum_{f=1}^{F} p \cdot z^f$$

(b) H consumers.

i. Action set: X^h

ii. Constraint set is $\left\{x \in X^h : p \cdot x \leqslant p \cdot e^h + \sum_{f=1}^{F} s^{fh} \pi^f(p)\right\}$

iii. Consumer h's utility function is u^h.

(c) F firms.

i. Action set: Z^f

ii. No constraint.

iii. Firm f's utility function is $p \cdot z^f$

□

8. (Theorem, A somewhat bounded economy) Suppose the economy $\mathcal{E}$ satisfies the following conditions:

(a) $\forall h$, X^h is a convex subset of $\mathbb{R}_+^k$, $e^h \in \operatorname{int}\left(X^h\right)$, and $\succsim^h$ is continuous and semi-strictly convex.

(b) $\forall f$, Z^f is convex and $0 \in Z^f$.

(c) $\exists \beta$, such that: if $\left(z^f\right)_{f=1}^{F}$ is a selection of production plans, where $z^f \in Z^f$ with $\sum_{h=1}^{H} e^h + \sum_{f=1}^{F} z^f \geqslant 0$, then $\left\|z^f\right\| \leqslant \beta, \forall f$.

Then $\mathcal{E}$ has a Walrasian equilibrium.

Note: β depends on e^h, for example, $k = 2$, $x_2 = \sqrt{x_1}$, $\bar{e} = (10, 2)$, then

$$\beta \geqslant \max_{0 \leqslant x_1 \leqslant 10} \left\|(-x_1, \sqrt{x_1})\right\|$$

The condition (c) is the boundness condition we impose. It says, more or less, that production technologies are not such that an unbounded amount of any good can be produced from the resources that can be provided by the economy. If $\mathcal{E}$ is a pure-exchange economy, the condition (c) holds trivially.

Idea of proof. Use the somewhat bounded condition to create an artificial economy $\hat{\mathcal{E}}$ from $\mathcal{E}$, letting $\hat{\mathcal{E}}$ be a very bounded economy. Then $\hat{\mathcal{E}}$ has an equilibrium $(p, \boldsymbol{x}, \boldsymbol{z})$, then verify that $(p, \boldsymbol{x}, \boldsymbol{z})$ is also an equilibrium of $\mathcal{E}$. □

Table 11.1: Check the conditions.

	Action Set	Constraint Correspondence			Utility Function	
	nonempty, compact, convex	Nonempty	Continuous	Convex	continuous	Quasi-concave in own choice
Auctioneer	By definitions or by assumptions	No constraint, hold trivially			By definitions or by assumptions	Linear in own choice
Firms						Linear in own choice
Consumers		By definitions or by assumptions	**Main efforts for proof**	Easy to verify	By assumptions	

Chapter 12

General Equilibrium, Efficiency, and the Core

12.1 The First Theorem of Welfare Economics

1. (Remark) Assumption and notations:

   (a) $\forall h$, $X^h \subseteq \mathbb{R}^k_+$, u^h is continuous.

   (b) $\forall f$, Z^f is closed and nonempty.

   (c) $\boldsymbol{X} = \prod_{h=1}^H X^h$ and $\boldsymbol{Z} = \prod_{f=1}^F Z^f$.

   (d) The space of socially feasible consumption allocations

   $$\boldsymbol{X}^* = \left\{ \boldsymbol{x} \in \boldsymbol{X} : \sum_{h=1}^H \boldsymbol{x}^h \leqslant \sum_{h=1}^H e^h + \sum_{f=1}^F \boldsymbol{z}^f \cdot p, \text{ for some } \boldsymbol{z} \in \boldsymbol{Z} \right\}$$

   (e) Define

   $$X^* = \left\{ x \in \mathbb{R}^k : x \leqslant \sum_{h=1}^H e^h + \sum_{f=1}^F \boldsymbol{z}^f, \text{ for some } \boldsymbol{z} \in \boldsymbol{Z} \right\}$$

   Note: X^* consists of bundles of goods that the economy is capable of producing. Moreover, if $\boldsymbol{x} \in X^*$ then $\boldsymbol{x}^h \in X^h \subseteq \mathbb{R}^k_+$, but if $x \in X^*$, x can have negative components.

2. (Theorem, The First Theorem of Welfare Economics) Suppose $(p, \boldsymbol{x}, \boldsymbol{z})$ is a Walrasian equilibrium for an economy with locally insatiable consumers, such that $p \geqslant 0$. Then $\boldsymbol{x}$ is Pareto efficient within X^*.

   Note: We do not impose any condition on consumer's preference $\succsim^h$, for example, convexity. However, the big assumption is that the Walrasian equilibrium exists.

   Idea of Proof. Let $\boldsymbol{x}$ be a Walrasian equilibrium. Show that the hyperplane

   $$\left\{ x \in \mathbb{R}^k : p \cdot x = Y \right\} \text{ where } Y = \sum_{h=1}^H p \cdot \boldsymbol{x}^h$$

strictly separates bundles of goods that the economy is capable of producing from bundles of goods that can be distributed to the consumers in a fashion that is Pareto superior to $\boldsymbol{x}$.

Step 1. Show that $\forall x \in X^*, p \cdot x \leqslant Y$. (By $p \geqslant 0$, local insatiability, and $x \leqslant \sum_{h=1}^{H} e^h + \sum_{f=1}^{F} z^f$).

Step 2. Define

$$\mathcal{PS}(\boldsymbol{x}) = \left\{ x \in \mathbb{R}_+^k : x = \sum_{h=1}^{H} \hat{\boldsymbol{x}}^h \text{ for some } \hat{\boldsymbol{x}} \text{ is Pareto superior to } \boldsymbol{x} \right\}$$

Show that

$$\forall x \in \mathcal{PS}(\boldsymbol{x}), p \cdot x > Y$$

Step 3. Show that if $\boldsymbol{x}$ is not Pareto efficient, then $\exists \hat{\boldsymbol{x}} \in X^*$, such that $\hat{\boldsymbol{x}}$ is Pareto superior to $\boldsymbol{x}$. Let $x = \sum_{h=1}^{H} \hat{\boldsymbol{x}}^h$, then $p \cdot x > Y$. However, $\hat{\boldsymbol{x}} \in X^*$ implies $p \cdot x \leqslant Y$. □

12.2 The Second Theorem of Welfare Economics

1. (Definition) A **Walrasian quasi-equilibrium** for an economy $\mathcal{E}$ is a triple $(p, \boldsymbol{x}, \boldsymbol{z})$ such that $p \neq 0$, where

   (a) $\forall f, \boldsymbol{z}^f \in \arg\max_{z \in Z^f} p \cdot z$.

   (b) $p \cdot \boldsymbol{x}^h \leqslant p \cdot e^h + \sum_{f=1}^{F} s^{fh} (p \cdot \boldsymbol{z}^f)$.

   (c) If $x^h \in X^h$, such that $x^h \succ^h \boldsymbol{x}^h$, then

   $$p \cdot x^h \geqslant p \cdot e^h + \sum_{f=1}^{F} s^{fh} (p \cdot \boldsymbol{z}^f)$$

   (d) $p \cdot x^h \geqslant p \cdot e^h + \sum_{f=1}^{F} s^{fh} (p \cdot \boldsymbol{z}^f)$.

   Note: Intuitively, if $\boldsymbol{x}^h$ solves consumer h's maximization problem, and $x^h \succ \boldsymbol{x}^h$, then x^h must disobey the budget constraint, i.e.,

   $$p \cdot x^h > p \cdot e^h + \sum_{f=1}^{F} s^{fh} (p \cdot \boldsymbol{z}^f)$$

   However, the definition of quasi-equilibrium weakens such strict inequality.

   The following theorem states the intuition in a rigorous way.

2. (Theorem)

   (a) If $(p, \boldsymbol{x}, \boldsymbol{z})$ is a Walrasian equilibrium for economy $\mathcal{E}$, then it is a Walrasian quasi-equilibrium.

(b) If $(p, \boldsymbol{x}, \boldsymbol{z})$ is a Walrasian quasi-equilibrium for economy $\mathcal{E}$, and if $\exists h$, such that $p \cdot e^h + \sum_{f=1}^{F} s^{fh} \left(p \cdot \boldsymbol{z}^f\right) > 0$, then for consumer h, if $x^h \succ^h \boldsymbol{x}^h$ then

$$p \cdot x^h > p \cdot e^h + \sum_{f=1}^{F} s^{fh} \left(p \cdot \boldsymbol{z}^f\right)$$

Or, put the other way around, for consumer h, $\boldsymbol{x}^h$ maximizes her preferences over her budget set. So, if each consumer's net wealth at the prices p is strictly positive, the quasi-equilibrium is a Walrasian equilibrium.

Idea of Proof. (a) Just check the definition.

(b) Suppose $(p, \boldsymbol{x}, \boldsymbol{z})$ is a Walrasian quasi-equilibrium and satisfies the conditions. Suppose $\boldsymbol{x}^h$ does not maximize u^h, then $\exists x^h$, such that x^h in the budget set and $x^h \succ^h \boldsymbol{x}^h$.

First show that $p \cdot x^h = p \cdot e^h + \sum_{f=1}^{F} s^{fh} \left(p \cdot \boldsymbol{z}^f\right)$.

By continuity, $\exists \alpha$, which is very close to 1, such that $\alpha x^h \succ^h \boldsymbol{x}^h$, then check that $p \cdot \left(\alpha x^h\right) < p \cdot x^h = p \cdot e^h + \sum_{f=1}^{F} s^{fh} \left(p \cdot \boldsymbol{z}^f\right)$, contradicting to the definition of quasi-equilibrium.

□

3. (Theorem, The Second Theorem of Welfare Economics) Suppose that, for a given economy $\mathcal{E}$, each Z^f convex, and each $\succsim^h$ is convex and locally insatiable. Suppose $\boldsymbol{x} \in X^*$ is Pareto efficient within X^*. Then there exists a way to reallocate endowments and shareholdings among consumers such that, for the economy with the reallocated endowments and shareholdings, there are production plans $\boldsymbol{z}$ and nonzero price vector p such that $(p, \boldsymbol{x}, \boldsymbol{z})$ is a Walrasian quasi-equilibrium.

Proof. Omitted. □

12.3 The Core in a Pure-Exchange Economy

1. (Definition) A subset of consumer is called a **coalition** of consumers.

   Note: Each coalition is able to distribute among its members the endowments of those members.

2. (Definition) For a pure-exchange economy $\mathcal{E}$,

   (a) For each nonempty subset $\mathcal{J} \subseteq \mathcal{H} = \{1, 2, \cdots, H\}$ define

$$\boldsymbol{X}^{\mathcal{J}} := \left\{ \left(x^j\right)_{j \in \mathcal{J}} \in \left(\mathbb{R}_+^k\right)^{\mathcal{J}} : \sum_{j \in \mathcal{J}} x^j \leqslant \sum_{j \in \mathcal{J}} e^j \right\}$$

(b) A feasible allocation $\boldsymbol{x} \in X^*$ is a **core** (allocation) for $\mathcal{E}$ if, $\forall \mathcal{J} \subseteq \mathcal{H}$ and $\forall \left(\hat{x}^j\right)_{j\in\mathcal{J}} \in \boldsymbol{X}^{\mathcal{J}}$, if $\forall j \in \mathcal{J}, \hat{x}^j \succsim^j \boldsymbol{x}^j$ then $\forall j \in \mathcal{J}, \hat{x}^j \sim^j \boldsymbol{x}^j$.

Note: By definition, any core allocation is Pareto efficient, since the coalition of the whole consumers generate the Pareto efficient allocation.

3. (Theorem) If $(p, \boldsymbol{x})$ is a Walrasian equilibrium for a pure-exchange economy $\mathcal{E}$ in which all consumers are locally insatiable and $p \geqslant 0$, then $\boldsymbol{x}$ is a core allocation of $\mathcal{E}$.

Note: W.E.Allocations $\subseteq$ CoreAllocations $\subseteq$ P.E.Allocations.

Idea of Proof. Similar to the proof of the First Theorem of Welfare Economics. Suppose $(p, \boldsymbol{x})$ is a Walrasian equilibrium, let $Y^{\mathcal{J}} = \sum_{j\in\mathcal{J}} p \cdot e^h$.

First, show that

$$p \cdot \left(\sum_{j\in\mathcal{J}} \hat{x}^j\right) \leqslant Y^{\mathcal{J}}, \forall \left(\hat{x}_j\right)_{j\in\mathcal{J}} \in \boldsymbol{X}^{\mathcal{J}}$$

Second, show that if $\left(\hat{x}_j\right)_{j\in\mathcal{J}}$ is such that

$$\hat{x}^j \succsim^j \boldsymbol{x}^j, \forall j \in \mathcal{J} \hat{x}^j \succ^j \boldsymbol{x}^j, \exists j \in \mathcal{J}$$

then $p \cdot \left(\sum_{j\in\mathcal{J}} \hat{x}^j\right) > Y^{\mathcal{J}}$.

Third, suppose $\boldsymbol{x}$ is not a core, then a contradiction. $\square$

4. (Example) Suppose there are two types of consumers in the economy, A and B, they have identical utility function $u\left(x_1, x_2\right) = \sqrt{x_1 x_2}$. Type A enjoys the endowment $e^A = (1, 0)$, type B enjoys $e^B = (0, 1)$ instead. Let u^A $\left(u^B\right)$ denote individual consumer of type A (B)'s utility in equilibrium.

(a) Without deviation:

$$u^A + u^B = 1$$

(b) If 2 consumers of type A and 1 of type B deviate:

$$e^{\{A,A,B\}} = (2, 1) \text{ and } x^A = (2x, x), x^B = (2 - 2x, 1 - x)$$

then

$$\begin{cases} u_A = \sqrt{\frac{x}{2} \cdot x} = \frac{x}{\sqrt{2}} \\ u_B = \sqrt{2}\,(1 - x) \end{cases} \implies \sqrt{2} u_A + \frac{u_B}{\sqrt{2}} = 1$$

(c) If 2 consumers of type B and 1 of type A deviate:

$$\frac{u_A}{\sqrt{2}} + \sqrt{2} u_B = 1$$

The dotted line is the survive part. Continue in this fashion, we can add more people into the coalition, then the dotted part will shrink into a point $\left(\frac{1}{2}, \frac{1}{2}\right)$, which is the Walrasian equilibrium allocation.

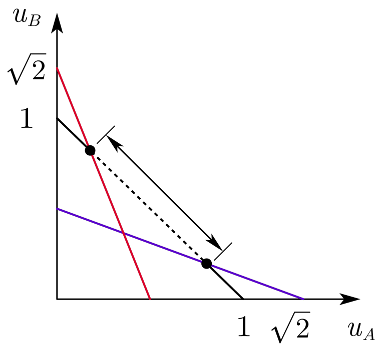

Fig. 12.1

5. (Remark) Set-ups for a replica economy.

   (a) Begin with a finite, pure-exchange economy $\mathcal{E}$. There are H types of consumers. In particular, consumer h has utility function u^h and an endowment e^h.

   (b) The N-replica version of $\mathcal{E}$, denote $\mathcal{E}^N$, is an economy with N-replicas of each the H consumers in the original economy $\mathcal{E}$. Therefore, there are $N \cdot H$ consumers, and thus N consumers of type h, for each h.

   (c) A general allocation for $\mathcal{E}^N$ is a point $\left(x^{hn}\right)_{h\in\mathcal{H};n\in\mathcal{N}} \in \left(\mathbb{R}^k_+\right)^{HN}$ such that

   $$\sum_{n=1}^{N}\sum_{h=1}^{H} x^{hn} \leqslant N \sum_{h=1}^{H} e^h$$

   (d) (Definition) An **equal-treatment allocation** is an allocation $\left(x^{hn}\right)$ such that, for any fixed type h,

   $$x^{hn} = x^{hn'}, \forall n, n'$$

6. (Theorem) Suppose that all consumers have strictly convex preference. Then any allocation $\left(x^{hn}\right)$ that is in the core of N-replica economy $\mathcal{E}^N$ (for all N) is an equal-treatment allocation.

   Proof. Suppose a core allocation is not an equal-treatment allocation, then there exist two consumers of the same type h enjoys different consumption bundle. Then these N consumers of type h can choose to deviate to make a Pareto improvement (ensured by the strict convexity of preferences) among them.

   Since $x^{h1}, \cdots, x^{hN}$ are not all equal, by Jensen's inequality,

   $$\frac{1}{N}\left(u\left(x^{h1}\right) + \cdots + u\left(x^{hN}\right)\right) < u\left(\frac{1}{N}\sum_{n=1}^{N} x^{hn}\right)$$

   Then $\exists i$, such that $u\left(x^{hi}\right) < u\left(\frac{1}{N}\sum_{n=1}^{N} x^{hn}\right)$. □

7. (Remark)

(a) If the equal-treatment allocation x is not in the core of $\mathcal{E}^N$, then it is not in the core of $\mathcal{E}^M$, $\forall M > N$.

(b) Suppose $p \geqslant 0$, Walrasian equilibrium allocations $\boldsymbol{x}$ for $\mathcal{E}$ are, when viewed as equal-treatment allocations, in the core of $\mathcal{E}^N, \forall N$.

8. (Theorem, The Debreu-Scarf Theorem) Suppose $\sum_h e^h \gg 0$ and u^h is nondecreasing and strictly convex for each h. If $\boldsymbol{x} \in \left(\mathbb{R}_+^k\right)^H$, viewed as an equal-treatment allocation, is in the core of $\mathcal{E}^N, \forall N$, then $\exists p \gg 0$, such that $\boldsymbol{x}$ is a Walrasian-equilibrium allocation for the base economy $\mathcal{E}$ (and hence, for $\mathcal{E}^N, \forall N$).

   Note: Under certain conditions, a core allocation is a Walrasian equilibrium allocation.

Chapter 13

General Equilibrium, Time, and Uncertainty

13.1 A Framework for Time and Uncertainty

1. (Remark) Set-up for time and uncertainty.

   (a) Discrete time: $t = 0, 1, 2, \cdots, T$.

   (b) A set Ω contains all states of nature, but only $\omega \in \Omega$ will be the true path.

   (c) Let $\mathcal{F}_t$ be a partition of Ω, representing the information at time t.

   Note that $\mathcal{F}_t$ is not a σ-algebra here. Also note that $\mathcal{F}_0 = \{\Omega\}$, $\mathcal{F}_T = \{\{\omega\}\}$.

   (d) If $f_{t+1} \in \mathcal{F}_{t+1}$, then $\exists f_t \in \mathcal{F}_t$, such that $f_{t+1} \subset f_t$. We call f_t is the **immediate predecessor** of f_{t+1}, and f_{t+1} is **one of immediate successors** of f_t.

   (e) Elements of $\mathcal{F}_t$ are called **time-t contingencies**.

2. (Example)

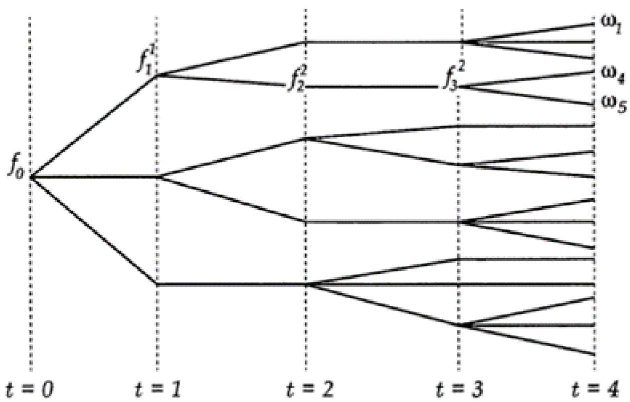

Fig. 13.1

For example, $\mathcal{F}_0 = \{f_0\}, \mathcal{F}_1 = \{f_1^1, f_2^1, f_1^2\}$, where

$$\begin{aligned} f_0 &= \{\omega_1, \cdots, \omega_{16}\} \\ f_1^1 &= \{\omega_1, \cdots, \omega_5\} \\ f_2^1 &= \{\omega_6, \cdots, \omega_{11}\} \\ f_3^1 &= \{\omega_{12}, \cdots, \omega_{16}\} \end{aligned}$$

3. (Remark) Discussion on notations.

   (a) To make time explicit, we should think of f_t being the pair $(t, \{\cdot\})$, in previous example,
   $$f_2^2 = (2, \{\omega_4, \omega_5\}) \neq (3, \{\omega_4, \omega_5\}) = f_3^2$$

   (b) In such notation with time, $\mathcal{F}_t \cap \mathcal{F}_{t'} = \varnothing$.

   (c) Suppose that the number of time-t contingencies is N_t , then the total number of contingencies is $N = N_0 + N_1 + \cdots + N_T$. In previous example,
   $$N = 1 + 3 + 5 + 8 + 16 = 33$$

   (d) However, sometimes we'd like to drop the time notation (after we get familiar with notations), then we write, for example, $\omega \in f_t$ or $f_t \subseteq f'_{t'}$.

   (e) Let $\hat{f}_t$ be the immediate predecessor of f_t.
   The set of all predecessors is denoted by $\boldsymbol{P}(f_t) = \left\{\hat{f}_t, \hat{\hat{f}}_t, \cdots\right\}$.
   The set of all successors of f_t is denoted by $\boldsymbol{S}(f_t) = \{g : f_t \in \boldsymbol{P}(g)\}$.

13.2 All-at-once Market Structure

1. (Remark) Set-up for the economy.

   (a) Time/uncertainty/information (t/u/i) structure specified by the time index $t = 0, 1, \cdots, T$, set of states of nature Ω and information-partition structure $\{\mathcal{F}_t\}_{t=0}^T$.

   (b) There are k basic commodities. Each commodity is expanded into N different "**contingency-labeled commodities**". The consumption space and preferences are defined over $\mathbb{R}_+^{kN}$.

   (c) Production possibility sets $Z^f \subseteq \mathbb{R}^{kN}$.

   (d) Consumption bundle x:

      i. the amount of good i in contingency f_t is x_{if_t};

      ii. $x_{f_t} \in \mathbb{R}_+^k$ denotes the consumption bundle in contingency f_t

   (e) The endowment of consumer h is $e^h \in \mathbb{R}_+^{kN}$, similarly, with components $e_{if_t}^h$.

2. (Remark) All-at-once market structure.

   The all-at-once market structure for economy $\mathcal{E}$ consists of kN markets, all conduced at time $t = 0$, with one market in each of the kN contingency-stamped goods.

   (a) Let p be a price vector for these markets, with components denoted p_{if_t}

   (b) Firm f's objective:
   $$\max_{z^f \in Z^f} p \cdot z^f$$

   (c) Consumer h's objective:
   $$\max_{x \in \mathbb{R}_+^{kN}} \ u^h(x)$$
   $$\text{s.t. } p \cdot x \leqslant p \cdot e^h + \sum_{f=1}^{F} s^{fh} \left(p \cdot z^f\right)$$

   (d) Markets clearing:
   $$\sum_{h=1}^{H} x^h \leqslant \sum_{h=1}^{H} e^h + \sum_{f=1}^{F} z^f$$

3. (Theorem) In any Walrasian equilibrium of the all-at-once market with assumptions

   (a) Each consumer has continuous, nondecreasing, and locally insatiable (and thus, globally insatiable) preferences.

   (b) Each contingency f_t, some consumer is globally insatiable in the consumption of contingency-f_t commodity $i = 1$. In other words, $\forall f_t, \exists h$, such that $\forall x \in \mathbb{R}_+^{kN}$, $\exists x' \in \mathbb{R}_+^{kN}$ such that x and x' are identical except on coordinate $1f_t$, and $u^h(x') > u^h(x)$.

   We hereafter refer to basic commodity 1 as wheat (a numeraire that everyone accepts).

   We have

   (a) Equilibrium prices are nonnegative.

   (b) The price of contingency-f_t wheat is strictly positive for all f_t.

   (c) Every consumer satisfies her budget constraint with equality.

   (d) Every Walrasian-equilibrium allocation is Pareto efficient.

Proof. It's directly from previous theorems on general equilibrium. $\square$

13.3 EPPPE: Two Alternative Market Structures

1. (Remark) From now on, we look only at pure-exchange economies

2. (Remark) alternative I consists of

   (a) Contingency f_t spot markets: In each contingency f_t markets will open in each of the k basic commodities.

   Let the spot prices be denoted by $r_{f_t} \in \mathbb{R}_+^k$.

   (b) Contingency f_t future markets: At time 0, $N-1$ future markets in which contracts to deliver f_t $(f_t \neq f_0)$ wheat are traded.

   Let $q \in \mathbb{R}_+^{N-1}$ be the prices of future contracts, where q_{f_t} is the price at date 0 for the promised delivery of a unite of wheat in contingency f_t.

3. (Definition) An **equilibrium of plans, prices, and price expectations (EPPPE)** for I market is a tuple $(r, q, \boldsymbol{x}, \boldsymbol{y})$ where

   (a) For each contingency f_t (including f_0), a price vector $r_{f_t} \in \mathbb{R}_+^k$, where r_{f_t} gives the equilibrium prices (at time t) for spot commodities in the f_t spot market;

   (b) q_{f_t} for each contingency f_t other than f_0 is the price (at time 0) for the contingency-f_t future contract in wheat;

   (c) $\forall h$, a consumption plan $\boldsymbol{x}^h \in \mathbb{R}_+^{kN}$, where $\boldsymbol{x}_{if_t}^h$ is the amount of contingency-f_t good i that h consumes in equilibrium; and

   (d) $\forall h$, a contingent-futures-market position $\boldsymbol{y}^h \in \mathbb{R}^{N-1}$, where $\boldsymbol{y}_{f_t}^h$ is the position taken in the contingent future market for f_t contingent wheat by h.

   Note: If $\boldsymbol{y}_{f_t}^h > 0$, then h is a buyer of f_t contingent contract, and h will get $\boldsymbol{y}_{f_t}^h$ units of wheat in f_t from the seller of the contract. Likewise, if $\boldsymbol{y}_{f_t}^h < 0$, h will give $\boldsymbol{y}_{f_t}^h$ units of wheat in f_t to the buyer of the f_t contingent contract.

   such that

   (a) $\forall h$, $\boldsymbol{x}^h$ maximizes u^h, for which there is a corresponding future-market position $\boldsymbol{y}^h \in \mathbb{R}^{N-1}$ that, together with $\boldsymbol{x}^h$, satisfy the contingency-by-contingency budget constraint.

   $$r_{f_0} \cdot \boldsymbol{x}_{f_0}^h + q \cdot \boldsymbol{y}^h \leqslant r_{f_0} \cdot e_{f_0}^h, \text{ and } r_{f_t} \cdot \boldsymbol{x}_{f_t}^h \leqslant r_{1f_t} \cdot \boldsymbol{y}_{f_t}^h + r_{f_t} \cdot e_{f_t}^h \text{ for } f_t \neq f_0$$

   where $e_{f_t}^h$ is h's endowment of commodities in contingency f_t.

   (b) All markets clear:

   $$\sum_{h=1}^{H} \boldsymbol{x}_{f_t}^h \leqslant \sum_{h=1}^{H} e_{f_t}^h, \forall f_t$$
   $$\sum_{h=1}^{H} \boldsymbol{y}^h = 0$$

4. (Theorem)

   (a) Suppose $(p, \boldsymbol{x})$ is a Walrasian equilibrium for the all-at-once market structure. Then $(r, q, \boldsymbol{x}, \boldsymbol{y})$ where

   $$r_{if_t} = p_{if_t}, q_{f_t} = p_{1f_t}, \text{ and } \boldsymbol{y}^h_{f_t} = p_{f_t} \frac{\boldsymbol{x}^h_{f_t} - e^h_{f_t}}{p_{1f_t}}$$

   is an EPPPE for the economy with contingent future markets (alternative I).

   (b) If $(r, q, \boldsymbol{x}, \boldsymbol{y})$ is an EPPPE for the economy with contingent future markets (alternative I), then $(p, \boldsymbol{x})$ where

   $$p_{f_0} = r_{f_0} \text{ and } p_{if_t} = q_{f_t} \frac{r_{if_t}}{r_{1f_t}} \text{ for } f_t \neq f_0$$

   is a Walrasian equilibrium for the all-at-once market structure.

   Note: $\frac{r_{if_t}}{r_{1f_t}}$ is the price of good i at f_t normalized by the price of wheat at f_t. Multiplying q_{f_t}, the price of wheat future at time 0, converts the normalized price to time 0.

5. (Remark) alternative II (time-0 trading) consists of

   (a) Contingency f_t spot markets.

   (b) $N - 1$ financial-claims markets: one market exists for each $f_t \neq f_0$ at time 0. The financial claim is to give one unite of "money" or numeraire currency in the f_t spot market.

   If consumer h purchases y_{f_t} of contingent f_t financial claims, then the contingency-by-contingency budget constraint becomes

   $$r_{f_0} x^h_{f_0} + q \cdot y^h \leqslant r_{f_0} e^h_{f_0}, \text{and} r_{f_t} x^h_{f_t} \leqslant r_{f_t} e^h_{f_t} + y^h_{f_t}, \forall f_t \neq f_0$$

   where q is the price of each financial claims (Arrow-Debreu securities).

   Note: In alternative I market, the price level of each contingent market f_t is independent with other market, only the price ratio matter. Intuitively, the agents are allowed to trade in different contingency market by different currency (units of value). However, in alternative II market, the value measurement is the same, i.e., the money or numeraire currency.

   (c) (Assumption) Each consumer has continuous, nondecreasing, and locally insatiable preferences. For each f_t, there is some consumer h is globally insatiable in f_t-contingent consumption: $\exists h$, $\forall x \in \mathbb{R}^{kN}_+$, $\exists x' \in \mathbb{R}^{kN}_+$ with x and x' are identical expect component f_t, such that $u^h(x') > u^h(x)$.

6. (Remark) Complex Financial Securities. (Alternative with sequential trading)

   (a) A **security** s is described by two things:

i. A subset of contingencies $\mathcal{F}(s) \subseteq \mathcal{F}$ at which the security trades

ii. A dividend structure $d_s : \mathcal{F} \to \mathbb{R}_+$ or $d_s \in \mathbb{R}_+^N$.

(b) Securities are financial: dividends are paid in "numeraire."

(c) (Assumption)

i. Securities are traded **ex dividend**: Dividends are paid to who held the securities just prior to any contingency f_t trading.
Example: If $f_t \in \mathcal{F}(s)$ and $d_s(f_t) = 4$, consumer enters contingency f_t holding 6 units of s and sells 2 units. Her dividend payment is 24 rather than 16.

ii. No security trades at date T.

iii. If $d_s(f_t) \neq 0$, then $\exists f'_{t'}$ such that $f'_{t'} \in \mathcal{F}(s)$, i.e., $f'_{t'}$ precedes f_t at which s trades.

7. (Definition) Fix a dynamic economy: a list of basic commodities; a t/u/i structure; consumers with preferences (utility functions) and endowments; a finite set $\mathcal{S}$ of securities; and a market structure consisting of whatever trade is permitted by the securities, plus spot markets in all basic commodities, in every contingency.

An **equilibrium of plans, prices, and price expectations (EPPPE)** the economy is a tuple $(r, q, \boldsymbol{x}, \boldsymbol{y})$ where

(a) prices for each basic commodity in each contingency's spot market, given by r_{if_t};

(b) prices for each security $s \in \mathcal{S}$ in $f_t \in \mathcal{F}(s)$, given by q_{sf_t};

(c) for each consumer, $\boldsymbol{x}^h \in \mathbb{R}_+^{kN}$ is the consumption plan; and

(d) for each consumer, $\boldsymbol{y}^h : \mathcal{F} \times \mathcal{S} \to \mathbb{R}$ is the trading plan, where $\boldsymbol{y}^h(f_t, s)$ represents the number of units of security s held by h in contingency f_t, **after trading takes place**.

such that

(a) Each consumer's trading plan must be **legitimate**, respect the trading constraints

$$\begin{aligned} &\boldsymbol{y}^h(f_0, s) \neq 0 \text{ only if } f_0 \in \mathcal{F}(s) \\ &\boldsymbol{y}^h(f_t, s) \neq \boldsymbol{y}^h\left(\hat{f}_t, s\right) \text{ only if } f_t \in \mathcal{F}(s), \forall f_t \neq f_0 \end{aligned}$$

(b) Each consumer h maximizes her utility, subject to her budget constraints, one for each contingency.

To simplify notations, set

$$\boldsymbol{y}^h\left(\hat{f}_0, s\right) = 0, \forall s \in \mathcal{S}$$

and if $f_t \notin \mathcal{F}(s)$, set

$$\boldsymbol{y}^h\left(\hat{f}_t, s\right) = \boldsymbol{y}^h(f_t, s)$$

The budget constraints are

$$r_{f_t} \cdot \boldsymbol{x}^h_{f_t} + \sum_{s \in \mathcal{S}} q_{sf_t} \left[\boldsymbol{y}^h (f_t, s) - \boldsymbol{y}^h \left(\hat{f}_t, s \right) \right] \leqslant r_{f_t} \cdot e^h_{f_t} + \sum_{s \in \mathcal{S}} d_{sf_t} \boldsymbol{y}^h \left(\hat{f}_t, s \right)$$

(c) All markets clear:

$$\sum_{h=1}^{H} \boldsymbol{x}^h_{f_t} \leqslant \sum_{h=1}^{H} e^h_{f_t}, \text{ and } \sum_{h=1}^{H} \boldsymbol{y}^h (f_t, s) = 0, \forall f_t, s$$

8. (Definition) Arbitrage and the subspace of feasible wealth transfers

(a) **The set of feasible wealth transfers** for a given set of securities $\mathcal{S}$ and prices q for those securities is the set $M(\mathcal{S}, q)$ of $\xi \in \mathbb{R}^N$, such that for some legitimate trading plan y,

$$\xi (f_t) = \sum_{s \in \mathcal{S}} \left(q_{sf_t} \left[y \left(\hat{f}_t, s \right) - y (f_t, s) \right] + y \left(\hat{f}_t, s \right) d_s (f_t) \right)$$

Note $0 \in M(\mathcal{S}, q)$, since trading plan $y = 0$ is legitimate and feasible.

(b) If $\exists \xi \in M(\mathcal{S}, q)$, such that $\xi \geqslant 0$ and $\xi \neq 0$, we say that the securities $\mathcal{S}$ and prices q **admit an arbitrage opportunity**

(c) If $M(\mathcal{S}, q) \cap \mathbb{R}^N_+ = \{0\}$, then we say that the securities $\mathcal{S}$ and prices q **do not admit an arbitrage opportunity**.

(d) If there is some full (pure-exchange) economy with securities $\mathcal{S}$ and an EPPPE in which the equilibrium prices for the securities are given by q, we say that $\mathcal{S}$ and q are a **viable** model of the securities markets, while if there is no such economy, we say that $\mathcal{S}$ and q are **not viable**.

9. (Theorem)

(a) $M(\mathcal{S}, q)$ is a linear subspace of $\mathbb{R}^N$.

(b) $\mathcal{S}$ and q are viable $\Longleftrightarrow M(\mathcal{S}, q) \cap \mathbb{R}^N_+ = \{0\} \Longleftrightarrow \exists \pi \in \mathbb{R}^N_{++}$ such that $\forall \xi \in M(\mathcal{S}, q)$, $\pi \cdot \xi = 0$.

(c) Suppose $\mathcal{S}$ and q are viable, let

$$\Pi(\mathcal{S}, q) := \left\{ \pi \in \mathbb{R}^N_{++} : \pi \cdot \xi = 0, \forall \xi \in M(\mathcal{S}, q) \right\}$$

then $\Pi(\mathcal{S}, q)$ is a cone without the origin, and

$$\dim (M(\mathcal{S}, q)) + \dim (\Pi(\mathcal{S}, q)) = N$$

Moreover,

$$\xi \in M(\mathcal{S}, q) \Longleftrightarrow \pi \cdot \xi = 0, \forall \pi \in \Pi(\mathcal{S}, q)$$

Note: The theorems all comes from linear algebra.

If $Y \subseteq \mathbb{R}^N$ is a linear subspace, let $Y^{\perp} = \left\{z \in \mathbb{R}^N : \langle z, y\rangle = 0, \forall y \in Y\right\}$, then $\dim(Y) + \dim\left(Y^{\perp}\right) = N$. The vector π is sometimes called stochastic discount factor.

10. (Theorem)

    (a) Suppose $\mathcal{S}$ and q are viable, and $\dim(M(\mathcal{S}, q)) = N - 1$. Choose π from $\Pi(\mathcal{S}, q)$. Let r be the spot market prices, consumer h, facing the prices q and r can attain the consumption bundle $x \in \mathbb{R}^{kN}$ iff

    $$\sum_{f_t \in \mathcal{F}} \pi_{f_t} r_{f_t} \cdot \left(x_{f_t} - e_{f_t}^h\right) \leqslant 0$$

    (b) Suppose $(r, q, \boldsymbol{x}, \boldsymbol{y})$ is an EPPPE. If $\dim(M(\mathcal{S}, q)) = N - 1$, choose π from $\Pi(\mathcal{S}, q)$, then x is the equilibrium allocation of a Walrasian equilibrium for the all-markets-at-once economy, where the Walrasian-equilibrium prices p are given by

    $$p_{i f_t} = \pi_{f_t} r_{i f_t}$$

Chapter 14

General Equilibrium with Asymmetric Information

14.1 Adverse Selection: An Example Model

1. (Remark) Set-up.

   (a) Many identical potential firms that can hire workers.

   (b) Each produces the same output using an identical CRS technology in which labor is the only input, e.g., $y = \alpha L$.

   (c) The firms are risk neutral, seek to maximize their expected profits, and act as price takers.

   (d) Workers differ in the number of units of output they produce if hired by a firm, which we denote by $\theta = \left[\underline{\theta}, \bar{\theta}\right]$, the set of possible worker productivity levels.

   (e) The distribution of θ is given by the CDF $F(\theta)$.

   (f) A worker can choose to work either at a firm or at home, and we suppose that a worker of type θ can earn $r(\theta)$ on her own through home production.

   (g) The total number (or measure) of workers is N.

2. (Remark) Complete Information Benchmark.

   (a) Consider first the competitive equilibrium arising in this model when workers productivity levels are publicly observable.

   (b) Because the labor of each different type of worker is a distinct good, there is a distinct equilibrium wage $w^*(\theta)$ for each θ.

   (c) Given the competitive, constant returns nature of the firms (thus, zero profit), in a competitive equilibrium we have $w^*(\theta) = \theta, \forall \theta$.

   (d) The set of workers accepting employment in a firm is

$$\{\theta : r(\theta) \leqslant w^*(\theta)\} = \{\theta : r(\theta) \leqslant \theta\}$$

3. (Example) Competitive Equilibrium with Adverse Selection.

   Suppose $r(\theta) = \alpha\theta, \alpha < 1$, thus, $r(\theta) < \theta$ if $\theta > 0$.

   Let $\theta \sim U[0,2]$, then
   $$\mathbb{E}[\theta|\, r(\theta) \leqslant w] = \mathbb{E}\left[\theta|\, \theta \leqslant \frac{w}{\alpha}\right] = \frac{w}{2\alpha}$$
   If $\alpha > \frac{1}{2}$, then
   $$\mathbb{E}[\theta|\, r(\theta) \leqslant w] = \frac{w}{2\alpha} < w$$
   If the market has an equilibrium, then $w = \theta$, but
   $$\mathbb{E}[\theta|\, r(\theta) \leqslant w] < \theta$$
   which means all workers stay at home for home production and deviate from the market.

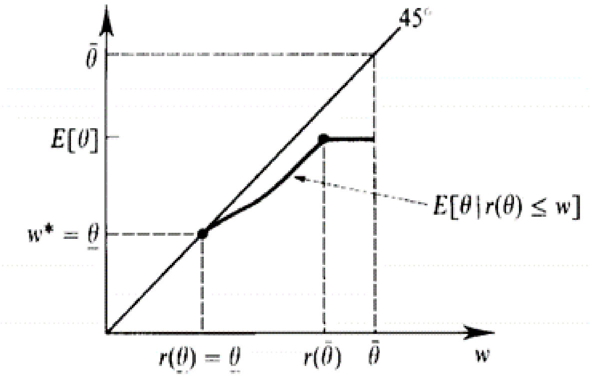

Fig. 14.1

   For example, if $\alpha = \frac{2}{3}$,

   (a) If all workers go to work, then $\mathbb{E}(\theta) = 1, w = 1$.

   (b) But $r(\theta) = \frac{2}{3}\theta$, then the worker with $\frac{2}{3}\theta > 1 \Longrightarrow \theta > \frac{3}{2}$ will choose to deviate, then $w = \mathbb{E}[\theta|\, r(\theta) \leqslant w] = \mathbb{E}\left[\theta|\, \theta \leqslant \frac{3}{2}\right] = \frac{3}{4}$.

   (c) But worker with $\frac{2}{3}\theta > \frac{3}{4} \Longrightarrow \theta > \frac{9}{8}$ will chose to deviate, then $w = \frac{9}{16}$.

   (d) But worker with $\theta > \frac{9}{16} \cdot \frac{3}{2} = \frac{27}{32}$ will deviate, then $w = \frac{27}{64}$.

   (e) Eventually, $w = \lim_{n\to\infty} \left(\frac{3}{4}\right)^n = 0$.

   If we change the parameters in the example, there can be a unique equilibrium.

4. (Example) Competitive Equilibrium with Adverse Selection. Firms earn zero profits in any equilibrium, and workers are better off if the wage rate is higher (those workers who do not accept employment are indifferent; all other workers are strictly better off).

5. (Remark) New set-ups for screening.

   The uninformed parties take steps to try to distinguish, or screen, the various types of individuals on the other side of the market

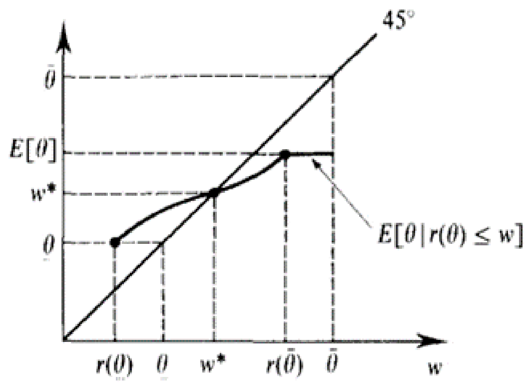

Fig. 14.2

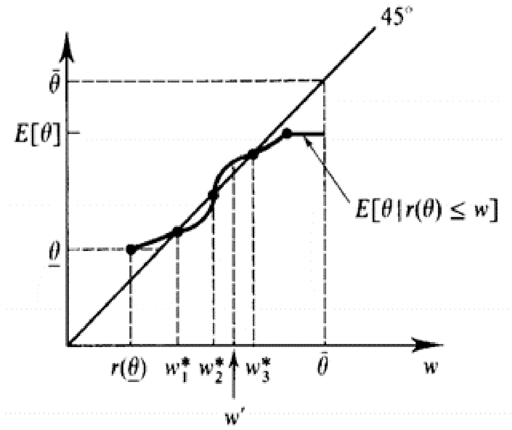

Fig. 14.3

(a) two types of workers, θ_L and θ_H, with $\theta_H > \theta_L > 0$. A worker at birth makes a draw of θ, with $\mathbb{P}(\theta = \theta_H) = \lambda, \mathbb{P}(\theta = \theta_L) = 1 - \lambda$.

(b) Workers earn nothing if they do not accept employment in a firm, i.e., $r(\theta_H) = r(\theta_L) = 0$.

(c) The screening tool: jobs may differ in the "task level" t required of the worker. The firm uses the screening tool to identify the worker type. Assume that higher task levels add nothing to the output of the worker; rather, their only effect is to lower the utility of the worker.

(d) The utility of a type θ worker who receives wage w and faces task level $t > 0$ is

$$u(w, t|\, \theta) = w - c(t, \theta)$$

where $c(0, \theta) = 0$, $c_t(t, \theta) > 0$, $c_{tt}(t, \theta) > 0$, $c_\theta(t, \theta) < 0$ and $c_{t\theta}(t, \theta) < 0$.

(e) Firms:

i. The task level t serves to distinguish among types.

ii. The payoff of a firm is $\theta - w$.

iii. There are a large number of firms (free entry), so that an additional firm could always enter if a profitable contracting opportunity existed.

iv. Each firm offers only a single contract (w, t)

(f) Equilibrium:

i. All workers have jobs.

ii. No firm leaves or enters the job market (If there is a positive profit, then other firms will enter).

6. (Remark) Benchmark: If worker types were observable (full information), then there are two types of contracts in equilibrium

$$(w_i^*, t_i^*) = (\theta_i, 0) \text{ for } i \in \{L, H\}$$

7. (Remark) No pooling equilibria exist.

(a) Pooling equilibria: both types of workers sign the same contract.

(b) The two solid line is the indifference curve of two types of workers. The condition $c_{t\theta}(t, \theta) < 0$ ensures that they have only one crossing point.

(c) The only possible candidate of equilibrium is (w^p, t^p).

(d) Consider the point $\left(\tilde{w}, \tilde{l}\right)$, workers of type θ_L won't choose the contract, but workers of θ_H will.

(e) However, $\left(\tilde{w}, \tilde{l}\right)$ is not a pooling equilibrium. Firstly, type-θ_L workers have no job. Secondly, $\left(\tilde{w}, \tilde{l}\right)$ generates positive profit for the firm, so other firm will enter the market.

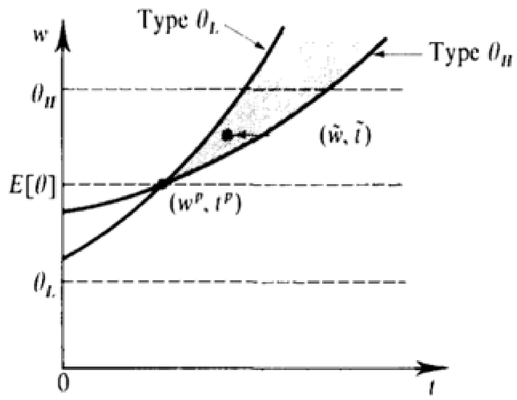

Fig. 14.4

8. Separating equilibria.

(a) In Figure (a), equilibrium exists. It happened when λ is low.

Firms offer contract (w_H, t_H) to the workers with θ_H, and offer contract (w_L, t_L) to the workers with θ_L.

For workers with θ_H, $(w_H, t_H) \succ_H (w_L, t_L)$.

For workers with θ_L, $(w_H, t_H) \sim_L (w_L, t_L)$, so they are not motivated to deviate from the offer (w_L, t_L) or (w_H, t_H).

(b) In figure (b), no equilibrium exists: a profitable deviation to a pooling contract such as $\left(\tilde{w}, \tilde{t}\right)$ exists. It happened when λ is high, $\mathbb{E}\theta$ is high, and thus, $\left(\tilde{w}, \tilde{t}\right)$ givens the firms positive profit.

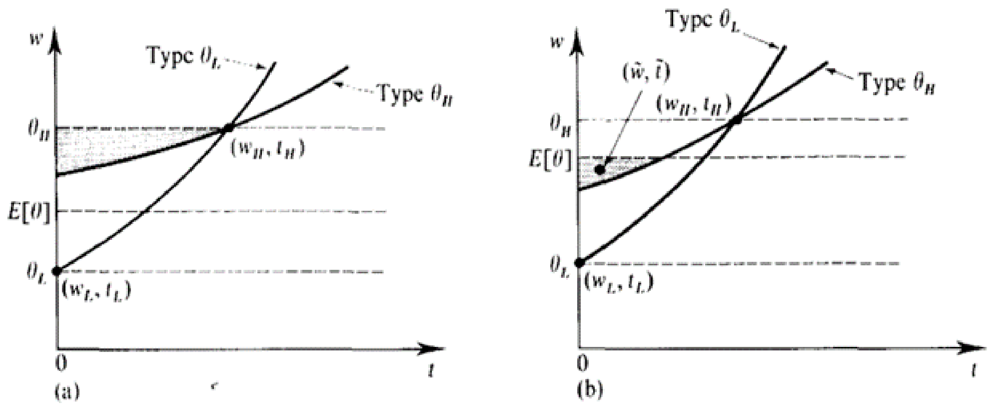

Fig. 14.5

14.2 Insurance Markets: Rothschild and Stiglitz (1976)

Unfinished.