Structuring Theories with Implicit Morphisms

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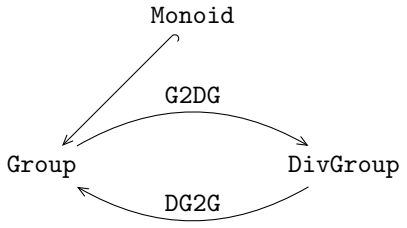
Abstract. We introduce *implicit* morphisms as a concept in formal systems based on theories and theory morphisms. The idea is that there may be at most one implicit morphism from a theory S to a theory T, and if S-expressions are used in T their semantics is obtained by automatically inserting the implicit morphism. The practical appeal of implicit morphisms is that they hit the sweet-spot of being extremely simple to understand and implement while significantly helping with structuring large collections of theories.

1 Introduction

Motivation Theory morphisms have proved an essential tool for managing collections of theories in logics and related formal systems. They can be used to structure theories and build large theories modularly from small components or to relate different theories to each other [**?**,**?**,**?**]. Areas in which tools based on theories and theory morphisms have been developed include specification [**?**,**?**], rewriting [**?**], theorem proving [**?**], and knowledge representation [**?**].

These systems usually use a logic L for the fine-granular formalization of domain knowledge, and a diagram D in the category of L-theories and L-morphisms for the high-level structure of large bodies of knowledge. This diagram is generated by all theories/morphisms defined, induced, or referenced in a user's development.

For example, a user might reference an existing theory `Monoid`, define a new theory `Group` that extends `Monoid`, define a theory `DivGroup` (providing an alternative formulation of groups based on the division operation), and then define two theory morphisms `G2DG` : `Group` ↔ `DivGroup` : `DG2G` that witness an isomorphism between these theories. This would result in the diagram on the right. Note that we use the syntactic direction for the arrows, e.g., an arrow $m : S \to T$ states that any S-expression E (e.g., a sort, term, formula, or proof) can be translated to an T-expression $m(E)$. Crucially, $m(-)$ preserves typing and provability.



The key idea behind implicit morphisms is very simple: We maintain an additional diagram I, which is commutative subdiagram of D and whose morphisms we call *implicit*. The condition of commutativity guarantees that I has at most

one morphism i from theory S to theory T, in which case we write $S \overset{i}{\hookrightarrow} T$. Commutativity makes the following language extension well-defined: if $S \overset{i}{\hookrightarrow} T$, then any identifier c that is visible to S may also be used in T-expressions; and if c is used in a T-expression, the semantics of c is $i(c)$ where i is the uniquely determined implicit morphism $i : S \to T$.

Despite their simplicity, the practical implications of implicit morphism are huge. For example, in the diagram above, we may choose to label `G2DG` implicit. Immediately, every abbreviation or theorem that we have formulated in the theory `Group` becomes available for use in `DivGroup` without any syntactic overhead. We can even label `DG2G` implicit as well if we prove the isomorphism property to ensure that I remains commutative, thus capturing the mathematical intuition that `Group` and `DivGroup` are just different formalizations of the same concept. While these morphisms must be labeled manually, any inclusion morphism like the one from `Monoid` to `Group` is implicit automatically.

Contribution We present a formal system for developing structured theories with implicit morphisms. Our starting point is the MMT language [**?**], which already provides a very general setting for defining and working with theories and morphisms. MMT is foundation-independent in the sense that it allows embedding a large variety of declarative languages (logics, type-theories, etc.). Therefore, all our results are foundation-independent — users of a particular language L can use MMT directly as a structuring formalism for L or can easily transfer our results to a dedicated implementation of L.

More precisely, our contribution is twofold.

Firstly, we give a novel presentation of the syntax and semantics of MMT. Contrary to the original presentation in [**?**], which hard-coded a single structuring mechanism — there called *structures* — our definitions allows for adding arbitrary independent structuring mechanisms. This required a from-scratch redesign of the original presentation in [**?**] (which is why this paper has no preliminaries section).

Based on this extensible definition, we can introduce several additional structuring mechanisms. Many of these have already been implemented in the MMT tool but were not formally defined — a formal definition of their semantics as a part of the MMT language had previously proved too complicated to spell out elegantly. With our reformulation, this is now not only possible but very easy. As an example, we define include declarations, the most important special case of implicit morphisms.

Secondly, we describe implicit morphisms as a novel feature of MMT. Due to our novel presentation of the syntax and semantics, only minor modifications are needed to define the syntax and semantics of implicit morphisms.

A very common situation in developing large libraries (both in formal and informal developments) is to identify two theories S and T via a canonical choice of isomorphisms. In these cases, it is often desirable to identify S and T, e.g., when working with S, we want to be able to use T-syntax directly without

having to apply an isomorphism. One of the major short-comings of formal theories over traditional informal developments is that formal systems usually need to spell out these isomorphisms everywhere. We show that implicit morphisms elegantly allow exactly that kind of intuitive identification. As concrete examples, we consider manual isomorphisms, definitional extensions, and renamings. In particular, we recover the concept of realms [**?**] as a special case of implicit morphisms.

In fact, implicit morphisms work so well that we have refactored the MMT tool in such a way that implicit morphisms are now more primitive than inclusion morphisms. The semantics of inclusion morphisms is obtained by saying that inclusions are implicit morphisms that map all identifiers to themselves. Even the fundamental property that a theory may reference its own identifiers is then just a consequence of the fact that all identity morphisms are implicit. Therefore, surprisingly, adding the new feature of implicit morphisms to the MMT kernel has made its design much simpler.

2 Theories and Theory Morphism

[1] EdN:1

2.1 Overview

Flat Modules MMT provides formal syntax for both structured modules and module expressions and defines their semantics via **flattening**, which defines for every module M a flat module $M^\flat$.

Flat theories are lists of declarations $c : E[= e]$ where E and e are expression, and the latter is optional. We write $\texttt{dom}(T)$ for the set of constant identifiers c in $T^\flat$ and $\texttt{Obj}(T)$ for the set of closed expressions using only the symbols $c \in \texttt{dom}(T)$ (see below for the definition of well-formed expressions). Constant declarations subsume virtually all basic declarations common in formal systems such as type/function/predicate symbols, axioms, theorems, inference rules, etc. In particular, theorems can be represented via the propositions-as-types correspondence as declarations $c : F = P$, which establish theorem F via proof P. Similarly, MMT expressions subsume virtually all objects common in formal systems such as terms, types, formulas, proofs.

Individual formal languages arise as fragments of MMT: they single out the well-formed expressions by defining the two MMT-**judgments** $\vdash_T e : e'$ (typing) and $\vdash_T e \equiv e'$ (equality) for every theory T and $e, e' \in \texttt{Obj}(T)$. The details can be found in [**?**].

[1] EDNOTE: @Dennis: can you formally work through the example from the intro in here? That would be several example environments for (i) after introducing the syntax of theories, the theory Group and DivGroup (one in detail, one sketched), (ii) after introducing morphisms the two isomorphisms (one in detail, one sketched), (iii) after introducing includes, the refactoring of Group using an include, (iv) examples for how the refactored Group is flattened

Flat morphisms from a theory S to a theory T are lists of assignments $c := e$ where $c \in \texttt{dom}(S)$ and $e \in \texttt{Obj}(T)$. Every morphism M induces a **homomorphic extension** $M(-) : \texttt{Obj}(S) \to \texttt{Obj}(T)$, which replaces every $c \in \texttt{dom}(S)$ in an S-expression with the T-expression e such that $c := e$ in M.

The declarations in theories and morphisms are subject to typing conditions that are not essential for our purposes here. We only mention the main theorem that guarantees that well-typed morphisms preserve judgments, i.e., if $\vdash_S e : E$, then $\vdash_T M(e) : M(E)$. This includes the preservation of truth via the propositions-as-types principle where E is a proposition and e its proof.

An Mmt **diagram** consists of a set of named structured module declarations. For a given diagram, we write $\texttt{Thy}$ for the set of theory names. And we write $\texttt{Mor}(S,T)$ for the set of morphisms defined by

- for every declaration $m : S \to T = \{\sigma\}$, we have $m \in \texttt{Mor}(S,T)$,
- for every $T \in \texttt{Thy}$, we have $id_T \in \texttt{Mor}(T,T)$,
- for every $M \in \texttt{Mor}(R,S)$ and $N \in \texttt{Mor}(S,T)$, we have $M;N \in \texttt{Mor}(R,T)$.

$\texttt{Thy}$ and $\texttt{Mor}(S,T)$ form the category of theories and morphisms. Then flattening assigns

- to each $T \in \texttt{Thy}$ the flat theory $T^\flat$,
- to each $M \in \texttt{Mor}(S,T)$ the flat morphism $M^\flat$ from $S^\flat$ to $T^\flat$.

Well-Formed Expressions The inference system to define well-formed expressions is not a primary interest of this paper. Intuitively, the set $\texttt{Obj}(T)$ of expressions over T includes all syntax trees that can be formed using the constants from $T^\flat$. Moreover, well-formed modules may only use expressions that satisfy certain typing and equality judgments. We refer to [**?**] for details.

The only aspect of the inference system for these judgments that is relevant to our purposes here is the rules for constants:

$$\frac{c : E[= e] \text{ in } T^\flat}{\vdash_T c[= e] : E}$$

Here, in order to avoid case distinctions for the case where a definition is present or not, we use a combined typing+equality judgment: $e_1 = e_2 : E$ represents the conjunction of $e_1 : E$, $e_2 : E$, and $e_1 = e_2$. Correspondingly, the definition of the homomorphic extension of a morphism with domain S includes the following case for constants:

$$M(c) = \begin{cases} e & \text{if } (c : E) \in S^\flat, (c := e) \in M^\flat \\ M(e) & \text{if } (c : E = e) \in S^\flat \end{cases}$$

Here if c has a definiens in S, we expand it before applying M.[3]

Note that these base cases introduce a mutual recursion between well-formedness and flattening: the well-formedness of a declaration in a theory depends on the

[3] The Mmt tool accepts $c : e \in M^\flat$ even if $(c : E = e') \in S^\flat$ has a definition. In that case, Mmt checks $\vdash_T M(e') = e$ and puts $M(c) = e$. This is important for efficiency but not essential for our purposes here.

flattening of all preceding declarations; correspondingly, the well-formedness of an assignment in a morphism depends on the homomorphic extension of the morphism obtained by flattening of all preceding assignments. Vice versa, well-formedness is a precondition for defining the flattening — the definition of flattening may become nonsensical if applied to ill-formed modules.

It is desirable to define well-formedness independently of flattening. But the mutual recursion makes sense from an implementation perspective: typical tools for formal systems first parse, check, and flatten a declaration entirely before moving on to the next declaration. Moreover, often the checking of well-formedness is as difficult or expensive as flattening anyway. In fact, inspecting practical systems with modular features (which often do not have a full rigorous formal specification) shows that our definition elegantly captures the essential commonalities between them.

Therefore, we will make flattening a partial function, i.e., $X^\flat$ is undefined if the module X is not well-formed.

Logics in MMT [2] MMT itself is foundation independent and parmetric in a set of typing rules. We omit the details on implementing type systems and logical frameworks in MMT, and will instead assume theories for the logical framework LF and an implementation of first-order logic for the examples in the remainder of this paper. EdN:2

Specifically, we assume a theory `LF` that implements the dependently-typed lambda calculus LF. This theory provides symbols for dependent function types $\prod_{c:A} B(x)$ (in surface syntax: {x:A}B x), simple function types $A \to B$ (as abbreviation for $\prod_{\_:A} B$, lambda abstractions $\lambda x : A.\ t(x)$ (in surface syntax: [x:A] t x) and function application fa.

The theory `LF` is used as a meta-theory for a theory `FOL` implementing first-order logic with equality. This theory provides symbols for the quantifiers $\forall$ and $\exists$ (implemented via higher-order abstract syntax, i.e. the proposition $\forall x.\ P(x)$ is written as $\forall$[x:U]P x for a fixed type U), equality $\doteq$ and the usual logical connectives $\neg, \wedge, \vee$, etc.

2.2 Syntax

We start with the syntax for theories (which arises as a special case of the one given in [**?**]):

Definition 1 (Theory). *The grammar for theories and expressions is of the form*

$TDec$	::=	$T = \{Dec, \ldots, Dec\}$	*theory declaration*
Dec	::=	$n : E[= E]$	*constant declaration*
c	::=	$t?n$	*qualified constant identifiers*
E	::=	$c \mid \ldots$	*expressions built from constants*

[2] EDNOTE: explain that we use a fixed instance of MMT which adds LF and FOL; only relevant for examples

In a theory declaration, each symbol ***name*** *n may be declared only once, and its* ***type*** *and* ***definiens*** *(if present) must be closed expressions over the previously introduced constants only. We omit the remaining productions for expressions here, which allow forming complex expressions using application, binding, variables, literals, etc.*

Example 1. The (flat) theories `Group` and `DivGroup` from the introduction are:

```
Group =
  U        : type
  op       : U → U → U    # 1 ∘ 2
  unit     : U
  inverse  : U → U  # 1 ⁻¹
  // axioms omitted

DivisionGroup =
  U    : type
  div  : U → U → U # 1 / 2
  unit : U
  // axioms omitted
```

Here we use # <not> to indicate notations; these are part of the language supported by MMT but omitted from the formal grammar above because we only need them here to present legible examples.

Moreover, we use the constants from the theories `LF` and `FOL` described above.

At this point, this grammar only allows forming flat theories, i.e., $\texttt{Thy} = \texttt{thy}$. It is intentionally independent of the concrete choice of modularity principles, i.e., the kinds of module-structuring declarations (e.g., includes) and the module-expression-forming operators (e.g., union). But it already introduces all non-terminals for modularity: new declaration kinds for structured theories arise by adding productions for the non-terminal Dec and new theory expressions are formed by adding productions for the non-terminal T. This makes it easy to incrementally add new declaration kinds and operators to the language.

We proceed accordingly for flat morphisms:

Definition 2 (Morphism). *A* ***morphism*** *is of the form*

$MDec$	$::=$	$m : T \to T = \{Ass, \ldots, Ass\}$	*flat morphism declaration*
Ass	$::=$	$c := o$	*assignment to symbol*
M	$::=$	$m \mid id_T \mid M; M$	*morphism expressions*

such that a morphism declaration contains exactly one assignment $c := o$ for each $c \in \texttt{dom}(S)$, all of which satisfying $o \in \texttt{Obj}(T)$.

Example 2 (Morphisms). We give the morphism `DG2G` between the theories from Ex. 3:

```
DG2G : DivisionGroup -> Group =
  U := U
  div  := [a, b]  a ∘ (b^-1)
  unit  := unit
  // assignments to axioms omitted
```

It maps the universe and unit of a division group to the corresponding notions of a group. And we have $\texttt{DG2G}(a/b) = a \circ b^{-1}$. Additionally, the morphism must maps every axiom of `DivGroup` to a proof in `Group` of the translated statement, but we omit those assignments.

Finally, we define diagrams as collections of modules:

Definition 3 (Diagram). *A **diagram** is a list of theory and morphism declarations:*

$$Dia \quad ::= \quad (TDec \mid MDec)^* \quad \textit{diagrams}$$

A diagram is well-formed if each atomic theory/morphism declaration has a unique name and is well-formed relative to the diagram preceding it.

We give some examples how to add modularity principles to the grammar:

Example 3 (Includes). We extend the grammar with

Dec	::=	**include** S	include a theory into a theory
Ass	::=	**include** M	include a morphism into a morphism

This allows writing the theory `Group` from Ex. 1 by extending a theory of monoids. Again omitting all axioms, this looks as follows:

```
Monoid =
  U: type
  op :  U → U → U        # 1 ∘ 2
  unit   :  U
Group =
  include  Monoid
  inverse  :  U → U  # 1 ^-1
```

For each example, we will define the semantics in the next section.

2.3 Semantics

Flattening Just like the syntax, the definition of flattening is **compositional** in the sense that new modularity principles can be added later independently of each other. In particular, for every module expression X, we define its flattening $X^\flat$ by induction on the syntax. This follows the principle of compositional semantics, e.g., $(f(X,Y))^\flat$ depends only on $X^\flat$ and $Y^\flat$.

For theories T, at this point, this induction only has the base case of a reference to a declared theory t. Further cases arise when we add structuring declarations.

For the base case, of a theory $t = \{\Sigma\}$, we define $t^\flat$ by induction on the declarations in Σ:

$$t^\flat = \Sigma^\flat$$

where

$$\cdot^\flat = \varnothing$$

$$(\Sigma, D)^\flat = \Sigma^\flat \cup D^\flat$$

Here $D^\flat$ is the flattening of declaration d relative to $\Sigma^\flat$.

At this point, we only have one case for declarations D, namely constant declarations. Their flattening is trivial:

$$(n : E[= e])^\flat = \{t?n : E[= e]\}$$

where t is the name of the containing theory.

Correspondingly, for a declared morphism $m : S \to T = \{\sigma\}$, we have $m \in \texttt{Mor}(S, T)$ and $m^\flat$ is defined by induction on the assignments in σ:

$$\cdot^\flat = \varnothing$$

$$(\sigma,\ A)^\flat = \sigma^\flat \cup A^\flat$$

with the trivial base case

$$(c := e)^\flat = \{c := e\}$$

Moreover, for $T \in \texttt{Thy}$ we define

$$id_T^\flat = \{c := c \mid c : E \in T^\flat\}$$

And for $M; N \in \texttt{Mor}(R, T)$, we define

$$(M; N)^\flat = \{c := N^\flat(M^\flat(c)) \mid c : E \in R^\flat\}$$

Note that in both cases, we only have to consider R-constants without definiens.

Example 4 (Includes (continued from Ex. 3)). Includes add new declarations D, so we have to add cases to the definition of $D^\flat$. We do this as follows

$$(\textbf{include}\ S)^\flat = S^\flat$$

This has the effect of copying over all declarations from the included into the including theory.

Note that the definition of $n : E = e^\flat$ qualifies the constant name n with the theory name t to form the identifier $t?n$ in the flattening. Therefore, includes can never lead to name clashes.

Also note, that because $S^\flat$ is a *set* of declarations, the include relation is transitive: if t includes s via two different paths, $t^\flat$ only contains one copy of the declarations of $s^\flat$. The situation is slightly more complicated for morphisms: if

a morphism out of t includes two different morphisms out of s, these have to agree. Therefore, flattening and well-formedness must be distinguished:

$$(\textbf{include}\ M)^\flat = M^\flat$$

under the condition $M^\flat$ agrees with $\sigma^\flat$ on any constant that is in the domain of both.

The flat variant $\texttt{Group}^\flat$ of our refactored theory `Group` from Ex. 3 is hence given by copying all included symbols (from `Monoid`) over to `Group`, resulting in exactly the theory as presented earlier in Example 1 except for global identifiers from `Monoid` – i.e. the prefixes t in $t?n$.

3 Implicit Morphisms

3.1 Overview

The motivation behind implicit morphisms is to generalize the benefits of include declarations even further. Sometimes an atomic morphism is conceptually similar to an inclusion: For example, we might want to declare $m : \texttt{Division} \to \texttt{Group} = \{\texttt{divide} := \lambda x, y.x \circ y^{-1}, \ \ldots\}$ to show how division is "included" into the theory of groups. Other times we have to use structures because we need to duplicate declarations but we would still like to use them like includes. For example, we have to declare the theory of rings using two structures (for the commutative group and the monoid). But we still want a ring to be implicitly converted into, e.g., a commutative group.

Our key idea is to use a commutative[4] subdiagram of the MMT diagram, which we call the *implicit-diagram*. It contains all theories but only some of the morphisms – the ones designated as *implicit*. Because the implicit-diagram commutes, there can be at most one implicit morphism from S to T – if this morphism is M, we write $S \overset{M}{\hookrightarrow} T$.

The implicit-diagram generalizes the inclusion relation $\hookrightarrow$. In particular, all identity and inclusion morphisms are implicit, and we recover $S \hookrightarrow T$ as the special case $S \overset{id_S}{\hookrightarrow} T$. Moreover, the relation "exists M such that $S \overset{M}{\hookrightarrow} T$" is also an order.

Consequently, many of the advantages of inclusions carry over to implicit morphisms.

- It is very easy to maintain the implicit-diagram, e.g., as a partial map that assigns to a pair of theories the implicit morphism between them (if any).
- We can generalize the visibility of identifiers: If $S \overset{M}{\hookrightarrow} T$, we can use all S-identifiers in T as if S were included into T. Any $c \in \texttt{dom}(S)$ is treated as a valid T-identifiers with definiens $M(c)$.
- We can still use canonical identifiers $s?n$. Because there can be at most one implicit morphism $s \overset{M}{\hookrightarrow} T$, using $s?n$ as an identifier in T is unambiguous. Crucially, we can use $s?n$ without bothering what M is.

[4] A diagram commutes if for any two morphisms $M, M' \in \texttt{Mor}(S, T)$, we have $M \equiv M'$.

3.2 Syntax

We introduce the family of sets $\texttt{Mor}^{\texttt{i}}(S,T)$ as a subset of $\texttt{Mor}(S,T)$, holding the *implicit* morphisms. The intuition is that `Thy` and $\texttt{Mor}^{\texttt{i}}(S,T)$ (up to equality of morphisms) form a thin broad subcategory of `Thy` and $\texttt{Mor}(S,T)$.

It remains to define which morphisms are implicit. For that purpose, we allow MMT declarations to carry *attributes*. Precisely, we add the following productions

$MDec$	::=	$Att\ MDec$	attributed morphism
Dec	::=	$Att\ Dec$	attributed declaration
Att	::=	**implicit** $\mid \ldots$	attributes

The set of attributes is itself extensible, and the above grammar only list the one that we use to get started. Additional attributes can be added when adding modularity principles.

Example 5. We can now change the declaration of the morphism `DG2G` from Ex. 2 by adding the attribute **implicit**.

3.3 Semantics

We only have to make two minor changes to the semantics to accommodate implicit morphisms. The first change govern how we obtain implicit morphisms, the second one how we use them.

Obtaining Implicit Morphisms Based on the above attributes, we define the set $\texttt{Mor}^{\texttt{i}}(S,T) \subseteq \texttt{Mor}(S,T)$ of **implicit** morphisms to contain the following elements:

- all declared morphisms $m : S \to T = \{\sigma\}$ whose declaration carries the attribute **implicit**,
- all identity morphisms id_T,
- all compositions $M;N$ of implicit morphisms M and N,
- all morphisms that additional language features designate as implicit based on the use of additional attributes.

To define well-formedness for our extended syntax, we say that adding an attribute to a declaration is only well-formed if there is (up to equality of morphisms) at most one implicit morphism for any pair of theories, i.e., $\texttt{Mor}^{\texttt{i}}(S,T)$ has cardinality 0 or 1. Thus, the implicit morphisms form a thin broad subcategory of theories and morphisms – in other words, a commutative diagram.

Example 6 (Includes (continued from Ex. 4)). For include morphism, we add the following definition: if theory t contains the declaration **include** S, then the induced morphism from S to t is implicit.

Combined with composition morphisms, we see that all transitive includes between theories are implicit. That corresponds to the intuition that anything that is include should be available directly.

Example 7. For our morphism `DG2G` from Ex. 5, this means that all symbols declared in the theory `DivisionGroup` (see Ex. 1) are now visible in the theory `Group` with the definitions provided by `DG2G`.

Using Implicit Morphisms The intuition behind implicit morphisms is that all S-constants $c : E$ that can be mapped into the current theory via an implicit morphism $M : S \to T$ are directly available in T. We can practically realize this by adding new defined constants $c : M(E) = M(c)$ to T. However, physically adding definitions can be inefficient. It is more elegant to modify the typing rules such that $\vdash_T c : M(c) = M(E)$.

To do that, we only have to make a small modification to the original rules of MMT as presented in Sect. 2. To illustrate how simple the modification is, we repeat the original rule first for comparison:

$$\frac{c : E[= e] \text{ in } T^\flat}{\vdash_T c[= e] : E}$$

Now our modified rule is

$$\frac{c : E[= e] \text{ in } S^\flat \qquad S \overset{M}{\hookrightarrow} T}{\vdash_T c = M(c) : M(E)}$$

Note that the modified rule gives every constant a definiens. This is a technical trick to subsume the original rule: if c is already declared in T, we use $M = id_T$ and obtain $\vdash_T c = c : E$. We write $\overline{E}$ for the expression that arises from E by recursively replacing every c with its definiens.

The following theorem is our central theoretical result. It shows that the above modification has the intended properties:

Theorem 1 (Conservativity of Implicit Morphisms). *Consider any combination of modular features from the examples above that includes at least implicit identity morphisms.*

Then for well-formed diagrams and $S, T \in \texttt{Thy}$ and $M \in \texttt{Mor}(S, T)$

1. *Whenever the original system proves $\vdash_T e = e' : E$, so does the modified one.*
2. *Whenever the modified system proves $\vdash_T e = e' : E$, then the original inference system proves $\vdash_T \overline{e} = \overline{e'} : \overline{E}$.*

Proof. For the first claim, we proceed by induction on derivations. We only need to consider case where the original rule was applied. So assume it yields $\vdash_T c[= e] : E$, i.e., $(c : E[= e])$ in $T^\flat$. We apply the modified rule for the special case $T \overset{id_T}{\hookrightarrow} T$. The conclusion reduces to

- if e is absent: $\vdash_T c = c : E$, which is equivalent to $\vdash_T c : E$,
- if e is present: $\vdash_T c = e : E$ because $M(c) = M(e)$ according to the definition of $M(-)$.

For the second claim, we proceed by induction on derivations. We only need to consider the cases where the modified rule was applied. So assume it yields $\vdash_T c = M(c) : M(E)$ for $(c : E[= e])$ in $S^\flat$ and $S \overset{M}{\hookrightarrow} T$. We distinguish two cases:

- $M(c) = c$, i.e., c does not have a definiens: According to the definition of $M(-)$ this is only possible if e is absent. According to the definition of $T^\flat$, this is only possible if $c = S?n$ for $S \hookrightarrow T$ (including the special case $S = T$). In that case, $S^\flat \subseteq T^\flat$ and M is the include/identity morphism that maps all S-constants to themselves. Now applying the induction hypothesis to the well-formedness derivation of E yields $\vdash_T c : \overline{E}$ as needed.
- $M(c) \neq c$, i.e., c has a non-trivial definiens: Definition expansion eliminates c in favor of e. Clearly $\overline{c} = \overline{M(c)}$, and we only have to show that $\vdash_T \overline{M(c)} : \overline{M(E)}$ That follows from the judgment preservation of morphisms.

4 Applications

4.1 Identifying Theories via Implicit Isomorphisms

In this section, we introduce several language extensions that introduce implicit isomorphisms.

Note that because identity morphisms are implicit, our uniqueness requirement for implicit morphisms implies that two theories S and T must be isomorphic if there are implicit morphisms in both directions. Moreover, making a pair of isomorphisms implicit is well-formed if there are no other implicit morphisms between S and T yet.

Renamings We say that a named morphism $r : S \to T = \{\ldots\}$ is a **renaming** if

- all assignments in its body are of the form $c := c'$ for T-constants c' without definiens
- every such T-constants c' occurs in exactly one assignment.

Clearly, every renaming is an isomorphism. The inverse morphisms contains the flipped assignments $c' := c$.

We make the following extension to syntax and semantics:

- A morphism declaration $r : s \to t = \{\ldots\}$ may carry the attribute **renaming**.
- This is well-formed if there are no implicit morphism between s and t yet.
- In that case, we define $r \in \texttt{Mor}^{\texttt{i}}(s,t)$ and $r^{-1} \in \texttt{Mor}(t,s)$.

Example 8. Consider a variation of the theory `Monoid` from Example 3 in a different library:

```
Monoid2 =
  M: type
  connective : M → M → M      # 1 ∘ 2
  neutral    : M
```

This theory is isomorphic to the previously introduced theory `Monoid` under the trivial renaming

```
Monoids : Monoid2 -> Monoid =
  M          := U
  connective := op
  neutral    := unit
```

Definitional Extensions We say that the named theory t is a **definitional extension** of S if $t = S$ or the body of t contains
- only constant declarations with definiens,
- only include declarations of theories that are definitional extensions of S.

Example 9. Imagine extending the theory `Group` from Example 3 by provable theorems:

```
Group_Theorems =
  include  Group
  inverse_idem  :  ⊢∀[x]  (x^-1)^-1≐x        = (proof)
```

If t is a definitional extension of S, it is easy to prove that t and S are isomorphic: both isomorphisms map all constants without definiens to themselves. In particular, the isomorphism $S \to t$ maps S-constants to themselves and expands the definiens of all other constants.

We make the following extension to syntax and semantics:
- An include declaration **include** s of a named theory s inside theory t may carry the attribute **definitional**.
- In that case, we define $id_s \in \texttt{Mor}^{\texttt{i}}(t, s)$ (in addition to the implicit morphism $id_t \in \texttt{Mor}^{\texttt{i}}(s, t)$ which is induced by the inclusion).

Example 10. We adapt our theory `Group_Theorems` with the newly introduced keyword, which allows us to extend `Group` directly while having the new theorem `inverse_idem` available:

```
Group_Theorems =
  definitional   include  Group
  inverse_idem  :  ⊢∀[x]  (x^-1)^-1≐x        = (proof)

More_Theorems =
  include  ?Group
  some_other_theorem : ⊢ ??       = (proof using  inverse_idem )
```

Remark 1 (Conservative Extensions). A definitional extension is a special case of a conservative extension. More generally, all retractable extensions are conservative, i.e., all extensions $S \hookrightarrow T$ such that there is a morphism $r : T \to S$ such that r is the identity on S.

But we cannot make the retractions implicit morphisms in general because they are not necessarily isomorphisms.

Canonical Isomorphisms If we have isomorphisms $m : s \to t$ and $n : t \to s$, we simply spell them out in morphism declarations and add the keyword **implicit** to both. This requires no language extensions.

Example 11. Having declared the morphism `DG2G` (as in Example 2) implicit, we do the same with the reverse morphism `G2DG`:

```
implicit  G2DG : Group -> DivisionGroup =
  U := U
  op := [a, b]  a / ((unit / unit) / b)
  unit  := unit / unit
  inverse  := [a]  (unit / unit) / a
```

While the first one of these isomorphism declarations is straightforward, the second one requires checking that m and n are actually isomorphisms. Otherwise, the uniqueness condition would be violated. Thus, we have to check $m; n = id_s$ and $n; m = id_t$. In general, the equality of two morphisms $f, g : A \to B$ is equivalent to $\vdash_B f(c) = g(c)$ for all $c : E \in A^\flat$. Thus, if equality of expressions is decidable in the logic that MMT is instantiated with, then MMT can check this directly.

However, this does not work in practice. Already elementary examples require stronger, undecidable equality relations are used:

Example 12. Consider the isomorphism from Ex. 11. The result of mapping $x \circ y$ from `Group` to `DivGroup` and back is $x \circ (unit \circ y^{-1})^{-1}$. Clearly, the group axioms imply that this is equal to $x \circ y$. But formally that requires working with the undecidable equality of first-order logic.

Therefore, in our running example, we only make one of the two isomorphisms implicit.

In the sequel, we design a general solution to this problem. It allows systematically proving the equality of two morphisms and using that to make both isomorphisms implicit. This is novel work, but it requires significant prerequisites and is only peripherally related to implicit morphisms. Therefore, we only sketch the idea and leave the details to future work.

We add a language feature to MMT to prove equalities between morphisms: We add the productions

$$\begin{array}{lll} Dia & ::= & (TDec \mid MDec \mid MEq)^* \\ MEq & ::= & \textbf{equal}\, M = M : T \to Tby\{Ass^*\} \end{array}$$

We define the declaration $M = N : S \to Tby\{\sigma\}$ to be well-formed iff
- $M : S \to T$ and $N : S \to T$ are well-formed morphisms
- σ contains exactly one assignment $c := p$ for every $(c : E) \in S^\flat$
- for each of these assignments $c := p$, the term p is a proof of $\vdash_T M(c) = N(c)$.

To make m and n from above implicit isomorphisms, we have to do three things: define m and n, prove the equalities of $m; n = id_s$ and $n; m = id_t$, and make m and n implicit. Note that we cannot make both m and n implicit right away because that is only well-formed after proving the equalities.) Thus, we define a new attribute **implicit-later**, which states that a morphism should be considered implicit as soon as subsequent equality proves make it well-formed to do so.

Example 13 (Isomorphisms). We can now add declarations

$$\textbf{implicit} : \texttt{DG2G} : \texttt{DivGroup} \to \texttt{Group} = \text{(as above)}$$

$$\textbf{implicit-later} : \texttt{G2DG} : \texttt{Group} \to \texttt{DivGroup} = \text{(as above)}$$

$$\textbf{equal}\, \texttt{G2DG}; \texttt{DG2G} = id_{\texttt{Group}} : \texttt{Group} \to \texttt{Group} = \text{(omitted)}$$

$$\textbf{equal}\, \texttt{DG2D}; \texttt{G2DG} = id_{\texttt{DivGroup}} : \texttt{DivGroup} \to \texttt{DivGroup} = \text{(omitted)}$$

where the isomorphisms are as above and we omit all the equality proofs.

4.2 Fine-Granular and Flexible Theory Hierarchies

A common problem when defining modular theory hierarchies is that the most natural include-hierarchy for the most important theories is not necessarily the same as the most comprehensive hierarchy. For example, Ex. 3 defines `Group` with an include from `Monoid`. Instead, we could include `Monoid` into the intermediate theory `CancellationMonoid` and include that into `Group`. This change is not possible in retrospect — changing the theory hierarchy (which is one of the most fundamental structures of a library) usually presents an insurmountable refactoring problem. So instead we could systematically build a hierarchy that uses every intermediate theory (as done in [**?**]). But this yields a very deep and complex hierarchy that is hard to navigate for casual users. Moreover, it does not protect us from later on discovering yet another intermediate theory that should have been added.

Implicit morphisms provide a simple solution to this problem because they behave effectively like inclusions but can be added later on. In the above example, we would

- define `CancellationMonoid` with an include from `Monoid`,
- keep `Group` as it is, i.e., also with an include `Monoid`,
- add an implicit morphism `CancellationMonoid` → `Group`.

In the sequel, we develop a similar, more complex example: we build a hierarchy of theories between `Band` and `Semilattice`. [3] All of these theories are of the form $t = \{\textbf{include}\ \texttt{Band}, a :\vdash F\}$, where F is a universally quantified equational axiom.[5] In particular, $\texttt{SemiLattice} = \{\textbf{include}\ \texttt{Band}, a :\vdash \forall[x]\forall[y] x \circ y \doteq y \circ x\}$. Fig. 1 shows these theories. EdN:3

There are various morphisms between them that describe the lattice structure of the corresponding varieties of bands. All of these map the constants from `Band` to themselves and the axiom a to a proof.[6] We make all of these morphisms

[3] EdNote: @DM: this is not quite the lattice from the Wikipedia page: it consists of the 3x3 square plus Bands at the top

[5] All varieties of bands can be axiomatized in this way.

[6] Our formalization of the lattice of bands using implict morphisms can be found at https://gl.mathhub.info/MitM/smglom/blob/devel/source/algebra/bands.mmt and https://gl.mathhub.info/MitM/smglom/blob/devel/source/algebra/lattice.mmt

implicit. It is straightforward to prove that the diagram commutes: any two morphisms are identical except for assignment for the axiom. Here, MMT can discharge the proof obligation easily using proof irrelevance.

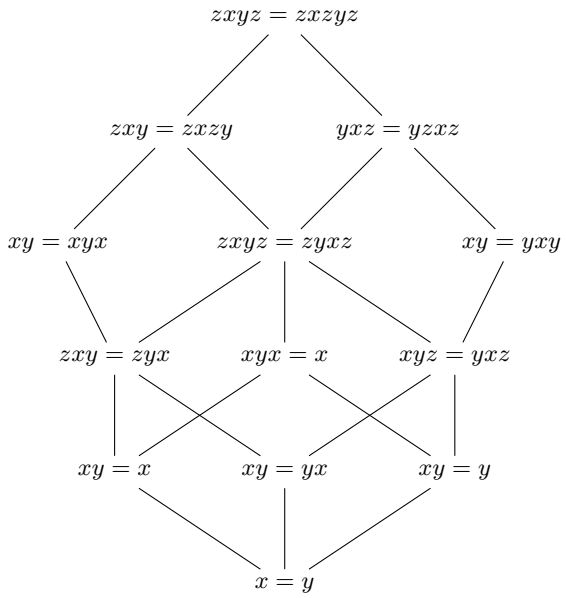


Fig. 1. The Lattice of Varieties of Bands

4.3 Transparent Refactoring

A major drawback of using modular theories is that it can preclude transparent refactoring. For example, consider a theory $t = \{\textbf{include}\ r, \textbf{include}\ s\}$, and assume we want to move a constant declaration D for the name n from r to s. Thus, the change to should be straightforward as it does not change the semantics of t.

However, this is not a local change. It also requires updating every qualified reference from $r?n$ to $s?n$. Such references can occur anywhere where t is used. That may include theories that the person who does the refactoring does not know or does not have access to. Even if the source files always use the unqualified reference n (because the checker is smart enough to dynamically disambiguate them), this still requires a global rebuild to reach a consistent state again.

With implicit morphisms, we can solve this problem by making only the following local changes:

1. We rename s to s'.

2. We delete the declaration $n : E$ from s'.
3. We create new theories $r' = \{\textbf{include } r,\ D\}$ and $s = \{\textbf{include } s', D\}$.
4. We change t to $t = \{\textbf{include } r', \textbf{include } s'\}$.
5. We add an implicit morphism $s \to t$ that maps $s?n$ to $r'?n$.

Now t has the desired new structure. But, all old references to $s?n$ stay well-formed so that no global changes are needed, as in Figure 2.

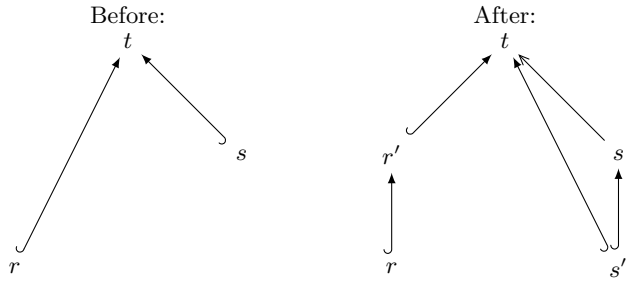


Fig. 2. An Example for Transparent Refactoring

5 Conclusion

5.1 Related Work

[4] EdN:4

[?] defines a realm (essentially) as a certain cluster of theories:

- a set of isomorphic theories $B_1, \ldots,$
- for each B_i, a set of retractable extensions $E^i_1, \ldots,$
- a theory F including all of the above.

The intuition is that the B_i are alternative definitions of a theory; the E^i_j add definitions and theorems; and F aggregates everything into the outwardly visible interface (the face) of the realm.

A good example is the realm of topological spaces with the B_i corresponding to the various possible definitions (via neighborhoods, open sets, closure operator, etc.).

Users of the realm include only the face and thus (i) gain access to all derived knowledge in the theory of topological spaces, and (ii) do not have to commit to one particular definition. Indeed, this corresponds more closely to the way how conventional mathematics sees the theory of topological spaces.

[?] calls for an implementation of realms as a new primitive concept in addition to theories and morphisms. However, in the presence of implicit morphisms, we do not need it: All we have to do is to designate the isomorphisms and the retraction is implicit and include any one of the B_i into F.

[4] EdNote: finish

Canonical Structures Coq [**?**]

Type Classes unnamed implict conversions, uniqueness problem does not arise

Implicit Conversions in Scala The Scala programming language [**?**] has a clean implementation of implicit conversions between any two types. If we think of functions between classes as theory morphisms, this corresponds to implicit theory morphisms.

Contrary to MMT, a conversion is only available if explicitly imported into the current namespace. Moreover, Scala does not chain conversions, i.e, the composition of implicit morphisms is not implicit. Scala's limitations are intentional: Their goal is to prevent programmers from overusing implicit conversions because that can be very confusing to readers.

These limitations are lifted in MMT's conversions, since they can only act between theories anyway. Thus, their potential to confuse is smaller. Moreover, the use of qualified identifiers in MMT means that readers can always tell where an identifier is implicitly-included from, and the MMT IDE can always show the implicit morphism along which it is imported.

5.2 Future Work

EdN:5 [5]

build big library, necessary to investigate scoping problem

Scoping drawback: global data structure

problem in big libraries?

We anticipate that this will result in an evolution of our solution that allows more localized control over which morphisms are implicit. But we defer this until the current implementation has been used to conduct very large case studies.

References

[5] EDNOTE: finish