1. Internal cd-structures

Definition 1. Let $f : A \to B$ and let X be a type. We say that f is *fiberwise orthogonal* to X if every base change $f' : A' \to B'$ of f is orthogonal to X, i.e., $X^{f'} : X^{B'} \to X^{A'}$ is an equivalence.

Proposition 2. *The collection of maps that are fiberwise orthogonal to a fixed X has the following closure properties:*

- (*i*) *It is closed under base change.*
- (*ii*) *It is closed under composition and cancellation on one side: Let $f : A \to B$ and $g : B \to C$. If f is fiberwise orthogonal to X, then g is if and only if gf is.*
- (*iii*) *It is closed under cobase change and sequential transfinite composition.*
- (*iv*) *It is closed under whatever colimits in the arrow category we can perform, which we assume includes at least coproducts, pushouts and sequential colimits.*
- (*v*) *It satisfies descent: Let $f : A \to B$ and suppose $q_i : B_i \to B$, $i : I$, is a jointly surjective family of maps. If each base change $q_i^* f : A_i \to B_i$ is fiberwise orthogonal to X, then so is f.*
- (*vi*) *If $f_1 : A_1 \to B$ is fiberwise orthogonal to X and $f_2 : A_2 \to B$ is any map, then the fiberwise join $f_1 *_B f_2 : A_1 *_B A_2 \to B$ is fiberwise orthogonal to X.*
- (*vii*) *If $f : A \to B$ is fiberwise orthogonal to X, then so is the inclusion of its image, $i : \operatorname{im} f \to B$.*

(Note that they do not satisfy two-out-of-three in general, even though the maps orthogonal to a fixed X do.)

Proof. Most parts follow from the stability of various constructions (e.g., composition, cobase change, and colimits in the arrow category) under base change. Only the last few items require special comment.

For descent, we may first replace the family $(q_i : B_i \to B)_{i:I}$ by the single surjection $q : B' \to B$, where $B' := \sum_{i:I} B_i$. The base change $f' : A' \to B'$ along q is fiberwise orthogonal to X, by (iv). Form the join construction on the map $f' \to f$ (equivalently, form the join construction on $q : B' \to B$, together with its base change along f to A). Each map $f' \times_f f'$, $f' *_f f'$, $f' \times_f (f' *_f f')$, $\ldots$ encountered in the construction is fiberwise orthogonal to X because it is either a base change of f', or a pushout in the arrow category of previous maps. There is a sequential colimit diagram $f' \to f' *_f f' \to f' *_f f' *_f f' \to \cdots \to f$ (because q is surjective), so f is also fiberwise orthogonal to X.

For the fiberwise join, suppose $f_1 : A_1 \to B$ is fiberwise orthogonal to X, and $f_2 : A_2 \to B$ is any map. It is enough to prove that $f_1 *_B f_2$ is *orthogonal* to X, because the construction of the join is stable under base change. The pullback $A_1 \times_B A_2 \to A_2$ of f_1 is orthogonal to X, and therefore so is the inclusion $A_1 \to A_1 *_B A_2$; then apply two-out-of-three to this map and $f_1 : A_1 \to B$.

Finally, let $f : A \to B$ and construct its image factorization through $i : \operatorname{im} f \to B$. By the previous part, the fiberwise joins $f *_B f$, $f *_B f *_B f$, $\ldots$ are fiberwise orthogonal to X, and their colimit is i. □

Definition 3. A *cd-structure* χ is a collection of so-called "distinguished squares": commutative squares of types. A cd-structure is called *complete* if any base change of a distinguished square is again distinguished. For simplicity, **we only consider complete cd-structures** (for now).

Definition 4. To a commutative square

$$\sigma = \begin{pmatrix} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{pmatrix}$$

we associate two maps:

- the *gap map* $g(\sigma) : P \to Y$, where P is the pushout $B \amalg_A X$;
- and its *image* $i(\sigma) : \mathrm{im}(g(\sigma)) \to Y$.

We write $G = G(\chi)$ for $\{ g(\sigma) \mid \sigma \in \chi \}$, and $\mathrm{im}\, G$ for $\{ i(\sigma) \mid \sigma \in \chi \}$,

Proposition 5. *The collections G, $\mathrm{im}\, G$ are closed under base change.*

Proof. Evident, since we require a cd-structure to be complete. □

Definition 6. We say that a type F satisfies

- *excision* for χ, if every morphism of G is (fiberwise) orthogonal to F;
- the *sheaf condition* for χ, if every morphism of $\mathrm{im}\, G$ is (fiberwise) orthogonal to F.

It doesn't matter whether we include the word "fiberwise", by the previous proposition.

Proposition 7. *If F satisfies excision for χ, then it satisfies the sheaf condition for χ.*

Proof. This follows from closure property (vii) of fiberwise orthogonality. □

Our goal is to establish the converse implication under suitable hypotheses on the cd-structure χ. Voevodsky's original paper considers the following condition.

Definition 8. Given a square

$$\sigma = \begin{pmatrix} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{pmatrix},$$

its *derived square* is

$$\sigma^{\Delta} := \begin{pmatrix} A & \longrightarrow & A \times_X A \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \times_Y B \end{pmatrix}.$$

The cd-structure χ is *regular* if, for each distinguished square σ (with the notation above):

(i) The map $X \to Y$ is a monomorphism.
(ii) The square σ is a pullback.
(iii) The derived square σ^{Δ} is also distinguished.

Remark 9. The squares in a cd-structure come with a particular orientation. More precisely, the transpose of a distinguished square will usually not also be distinguished. The definition of a regular cd-structure is not symmetric under transposition. For no particular reason, we form our derived squares in the "horizontal" direction, even though in the original paper they were formed in the "vertical" direction. This is merely a different typographical convention.

In the original setting, it was also implicit that the objects appearing in (at least some generating family of) distinguished squares are 0-truncated. In this case, Voevodsky proved that for a regular (and complete) cd-structure, the sheaf condition implies the excision condition. We first show why this implication cannot hold in general.

Example 10. Let S be any collection of maps which is stable under base change and passage to the diagonal. The objects with respect to which every map in S is (fiberwise) orthogonal form a left exact localization of the universe, and every left exact localization arises in this way.

Now, consider the cd-structure $\chi(S)$ consisting of all squares of the form

$$\begin{array}{ccc} \varnothing & \longrightarrow & \varnothing \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & Y \end{array}$$

in which the map $f : B \to Y$ belongs to S. The assumptions on S precisely say that this cd-structure is complete and regular. The gap map of the above square is just $B \to Y$ again, so a type F satisfies the excision condition if every map f of S is fiberwise orthogonal to F, while it satisfies the sheaf condition if $\operatorname{im} f \to Y$ is fiberwise orthogonal to F for every map f of S. In general, the latter condition is weaker than the former; the types F satisfying the sheaf condition make up the topological (or monogenic) part of the left exact localization associated to S.

This example also shows that there is no special property enjoyed by the localizations that arise from cd-structures.

Consequently, we cannot prove the implication "sheaf condition $\implies$ excision" for a general regular and complete cd-structure. We need an additional hypothesis.

Definition 11. A cd-structure χ is *truncated* if the gap map $g(\sigma)$ of every distinguished square σ is n-truncated for some n. (Note that n may depend on σ.)

We will prove that for a truncated, regular, complete cd-structure, the sheaf condition implies excision. We need some preparatory lemmas.

Lemma 12 (Wärn)**.** *Let*

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & P \end{array}$$

be a pushout square in which the left map $A \to B$ is a monomorphism. Then:

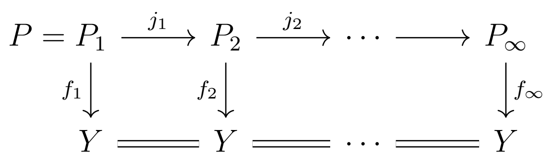
(*i*) *The right map $X \to P$ is also a monomorphism.*
(*ii*) *The square is also a pullback.*
(*iii*) *The derived square*

$$\begin{array}{ccc} A & \longrightarrow & A \times_X A \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \times_P B \end{array}$$

is also a pushout square.

Lemma 13. *Let $f : P \to Y$ be a map and F a type such that both $\Delta_f : P \to P \times_Y P$ and $\operatorname{im} f \to Y$ are fiberwise orthogonal to F. Then, f is also fiberwise orthogonal to F.*

Proof. We need to recall some more details of the join construction on f. We construct a sequence of types over Y by setting $P_1 := P$ and defining $P_{n+1} := P *_Y P_n$. This join comes with a map $j_n : P_n \to P_{n+1}$ of types over Y. We assemble these maps j_n into a diagram and form the colimit P_∞ as a type over Y, as shown.



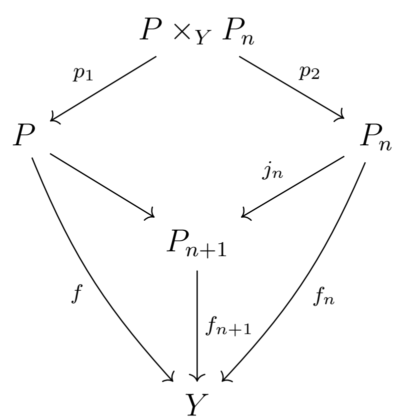
The theorem is that the resulting map $f_\infty : P_\infty \to Y$ can be identified with $\operatorname{im} f \to Y$.

Now we fix F so that the hypotheses are satisfied. We claim that

($*$) if there is any map $h : P \to P_n$ over Y which is fiberwise orthogonal to F, then $j_n : P_n \to P_{n+1}$ is fiberwise orthogonal to F.

Assuming ($*$), we complete the proof as follows. By induction, j_n is fiberwise orthogonal to F, since we may take h to be the composition of all the previous maps $j_{n-1} \cdots j_1$. By closure under sequential compositions, the map $P_1 \to P_\infty$ is fiberwise orthogonal to F. By assumption, the map $\operatorname{im} f \to Y$, which can be identified with $P_\infty \to Y$, is also fiberwise orthogonal to F. Hence, their composition, which is the original map $f : P \to Y$, is fiberwise orthogonal to F.

To prove ($*$), suppose $h : P \to P_n$ is a map over Y which is fiberwise orthogonal to F. We review the step of the join construction that builds P_{n+1} and $j_n : P_n \to P_{n+1}$.



The top square is a pushout, and the outer square is a pullback.

Form the following diagram:

$$\begin{array}{ccc}
P & & \\
{\scriptstyle \Delta_f}\downarrow & & \\
P \times_Y P & \xrightarrow{p_2} & P \\
{\scriptstyle P\times_Y h}\downarrow & & \downarrow{\scriptstyle h} \\
P \times_Y P_n & \xrightarrow{p_2} & P_n \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f_n} \\
P & \xrightarrow{f} & Y
\end{array}$$

Note that, as $f_n h = f$, the outer pullback $P \times_Y P$ is indeed formed using $f : P \to Y$ twice, and therefore $\Delta_f : P \to P \times_Y P$ fits in the diagram. The composition of the three maps on the left is the identity. The top two maps are fiberwise orthogonal to F: the map Δ_f by hypothesis, and the map $P \times_Y h$ because it is a base change of h. Therefore, by cancellation, the projection $p_1 : P \times_Y P_n \to P$ is also fiberwise orthogonal to F. Now, in the join construction, the map j_n is a cobase change of this map $p_1 : P \times_Y P_n \to P$, so it is also fiberwise orthogonal to F, as claimed. □

Theorem 14. *Fix a truncated, regular, complete cd-structure χ. Then if F satisfies the sheaf condition, it also satisfies excision.*

Proof. Assume that F satisfies the sheaf condition. We need to show that the gap map $g(\sigma)$ of any distinguished square is fiberwise orthogonal to F. We can prove this by induction on the truncation level of $g(\sigma)$, since by assumption each $g(\sigma)$ is n-truncated for some n. If $g(\sigma)$ is (-2)-truncated, then it is an isomorphism. So, assume that $g(\sigma)$ is n-truncated and that the claim holds for any σ such that $g(\sigma)$ is $(n-1)$-truncated.

Write

$$\sigma = \left(\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array} \right),$$

$P = B \amalg_A X$, and $g = g(\sigma) : P \to Y$. As F satisfies the sheaf condition, the image $\operatorname{im} g \to Y$ is fiberwise orthogonal to F. By the lemma, then, it is sufficient to prove that $\Delta_g : P \to P \times_Y P$ is also fiberwise orthogonal to F. We will verify this by descent. The type P is covered by its maps from B and X, so $P \times_Y P$ is covered by four maps from $X \times_Y X$, $B \times_Y X$, $X \times_Y B$, $B \times_Y B$ respectively. We check that the base change of Δ_g along each map of this cover is fiberwise orthogonal to F. In each case, the new object produced by base change can also be constructed by forming a pullback over P instead of Y.

- For $X \times_Y X$, we form the pullback square

$$\begin{array}{ccc} X \times_P X & \longrightarrow & X \times_Y X \\ \downarrow & & \downarrow \\ P & \xrightarrow{\Delta_g} & P \times_Y P \end{array}$$

But $X \to Y$ is a monomorphism by the hypothesis that χ is regular, and $X \to P$ is a monomorphism by part (i) of Wärn's lemma. Hence, the top map is an isomorphism (both objects are X).

- For $B \times_Y X$, we form the pullback square

$$\begin{array}{ccc} B \times_P X & \longrightarrow & B \times_Y X \\ \downarrow & & \downarrow \\ P & \xrightarrow{\Delta_g} & P \times_Y P \end{array}$$

But $B \times_Y X = A$ by the hypothesis that χ is regular, and $B \times_P X = A$ by part (ii) of Wärn's lemma. Hence, the top map is an isomorphism (both objects are A). (Likewise for $X \times_Y B$.)

- For $B \times_Y B$, we form the pullback square

$$\begin{array}{ccc} B \times_P B & \longrightarrow & B \times_Y B \\ \downarrow & & \downarrow \\ P & \xrightarrow{\Delta_g} & P \times_Y P \end{array}$$

Part (iii) of Wärn's lemma says that the top map is the gap map $g(\sigma^\Delta)$ of the derived square σ^Δ, which is again distinguished by the hypothesis that χ is regular. The map also has truncation level at most $n - 1$, being a pullback of the diagonal of g. So, by the induction hypothesis, $g(\sigma^\Delta)$ is fiberwise orthogonal to F. □