M7 Characteristics of Time Series

General Insurance Modelling : Actuarial Modelling III [1]

Professor Benjamin Avanzi

19 May 2023

[1] References: Chapter 2.0-2.3 of Shumway and Stoffer (2017) | → latest slides

1 Introduction (TS 1.0)

- Definition
- Applications
- Process for time series analysis

Definition

- Consider data of the same nature that have been observed at different points in time
- The mere fact that they are of the same nature means that they are likely related in one way or another - let's call those 'correlations' (an acceptable term in this context as we focus on this measure, at least in this course)
- This is in contrast with the usual "i.i.d." assumptions associated with a sample of outcomes of a random variable
- This invalidates some of the techniques we know, and brings additional difficulties, but also opportunities! (such as forecasting)

Definition: "The systematic approach by which one goes about answering the mathematical and statistical questions posed by these time correlations is commonly referred to as **time series analysis**."

1. Introduction (TS 1.0)
   - Definition
   - Applications
   - Process for time series analysis

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Applications

The applications of time series are many, and crucial in many cases:

- Economics: unemployment, GDP, CPI, etc ...
- Finance: share prices, indices, etc ...
- Medicine: COVID-19 cases and fatalities, biometric data for a patient (blood pressure, iron levels, ...), etc ...
- Global warming: ocean temperatures, CO_2 levels, particule levels in the atmosphere, sea levels, all in relation with another, and with many others
- **Actuarial studies**: frequency and severity of claims in a LoB, mortality (at different ages, in different locations, ...), superimposed inflation, IBNR claims, etc ...

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1 Introduction (TS 1.0)

- Definition
- Applications
- Process for time series analysis

Process for time series analysis

Sketch of process:

- Careful examination of data plotted over time (Module 7)
- Compute major statistical indicators (Modules 7 and 8)
- Guess an appropriate method for analysing the data (Modules 8 and 9)
- Fit and assess your model (Module 9)
- Use your model to perform forecasts if relevant (Module 10)

We distinguish two types of approaches:

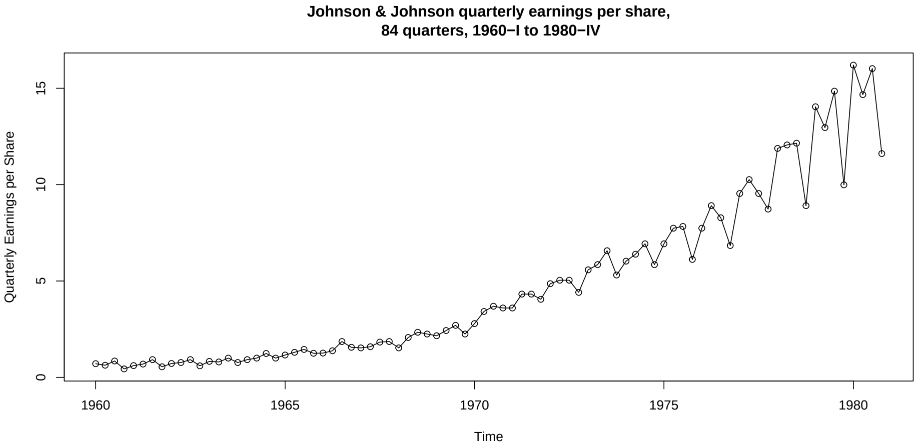
- *Time domain approach*: investigate lagged relationships (impact of today on tomorrow)
- *Frequency domain approach*: investigate cycles (understand regular variations)

In actuarial studies, both are relevant.

2 Examples (TS 1.1)

- Johnson & Johnson quarterly earnings per share
- Global mean land-ocean temperature index
- Dow Jones Industrial Average
- Analysis of two series together: El Niño & fish population
- Signals within noise

Johnson & Johnson quarterly earnings per share



What is the primary pattern?

Can you see any cyclical pattern as well?

How does volatility change over time (if at all)?

2 Examples (TS 1.1)

- Johnson & Johnson quarterly earnings per share
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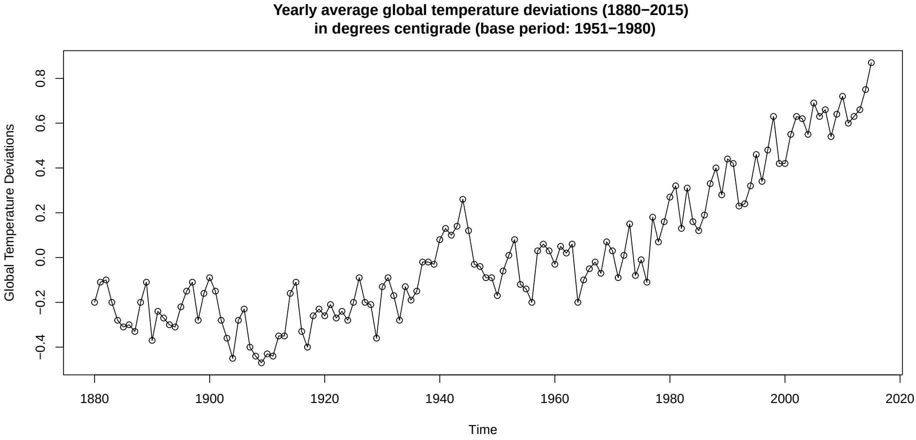
Global mean land-ocean temperature index



Can you see a trend? Are there periods of continuous increase?
What would be the main focus for global warming: trend or cycles?
How does this graph support the global warming thesis?

2 Examples (TS 1.1)

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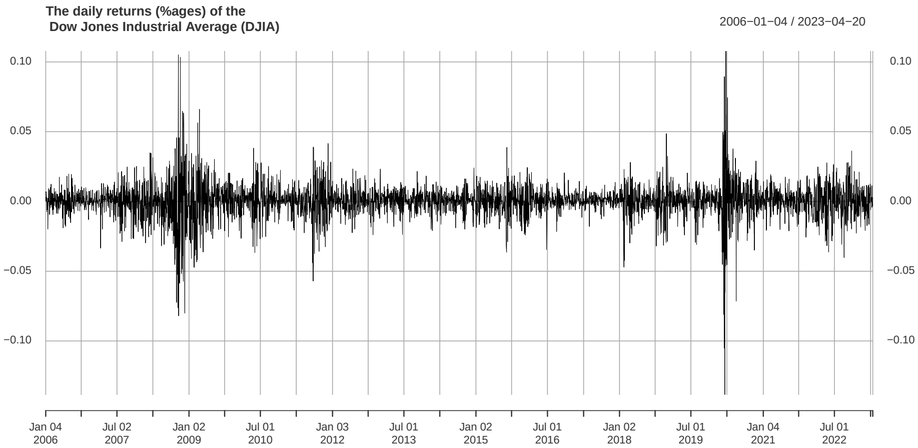
Dow Jones Industrial Average



How is this time series special?

What qualities would a good forecast model need to have?

2 Examples (TS 1.1)

- Johnson & Johnson quarterly earnings per share
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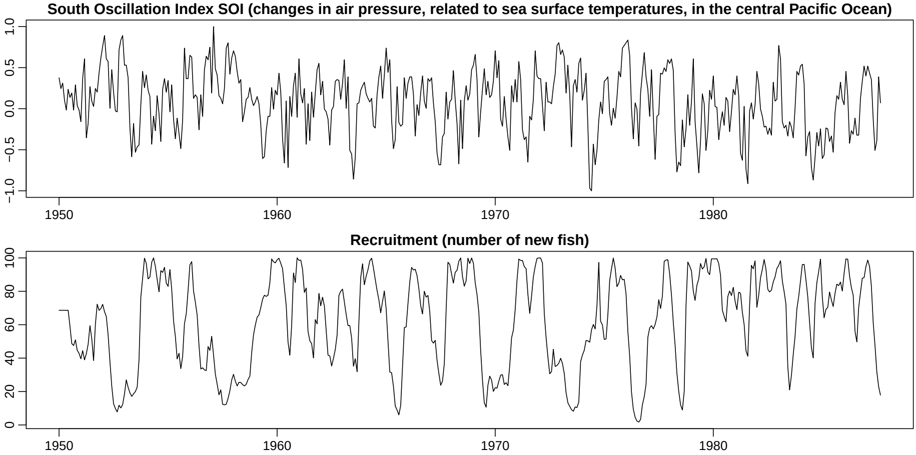
Analysis of two series together: El Niño & fish population



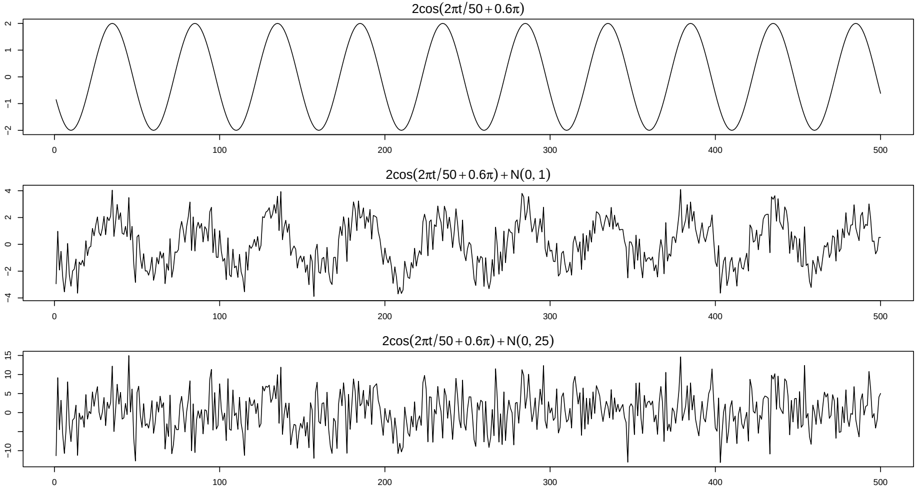
How many cycles can you spot?

Is there a relationship between both series?

2 Examples (TS 1.1)

- Johnson & Johnson quarterly earnings per share
- Global mean land-ocean temperature index
- Dow Jones Industrial Average
- Analysis of two series together: El Niño & fish population
- Signals within noise

Signals within noise



Typically we only see the the signal obscured by noise.

3 Basic models (TS 1.2)

- Preliminaries
- White noise - 3 scales :-)
- Gaussian white noise series and its 3-point moving average
- Filtering (and moving average)
- Autoregressions
- Autoregression examples
- Random walk with drift

Preliminaries

- Our primary objective is to develop mathematical models that provide plausible descriptions for sample data.
- A time series is a sequence of rv's $x_1, x_2, x_3, \ldots,$ **denoted** $\{x_t\}$
- In this course, t will typically be discrete and be $\in \mathbb{N}$ (or subset)
- One set of observed values of $\{x_t\}$ is referred to as a *realisation*
- Time series are usually plotted with time in the x-axis, with observations connected at adjacent periods
- Sampling rate must be sufficient, lest appearance of the data is changed completely (*aliasing*; see also `this` which explains how car wheels can appear to go backwards)
- Smoothness of the time series suggests some level of correlation between adjacent points, or in other words that x_t depends in some way on the past values $x_{t-1}, x_{t-2}, \ldots$. $\rightarrow$ This is a good starting point for imagining appropriate theoretical models!

3 Basic models (TS 1.2)

White noise - 3 scales :-)

Let's define w_t as ***uncorrelated*** random variables w_t with mean 0 and finite variance σ_w^2. This is denoted

$$w_t \sim \mathrm{wn}(0, \sigma_w^2),$$

and is called a **white noise**. Two special cases:

- **White independent noise**: (or iid noise) additional assumption of *iid*, denoted
$$w_t \sim \mathrm{iid}(0, \sigma_w^2).$$
- **Gaussian white noise**: further additional assumption of *normal distribution*, denoted
$$w_t \sim \mathrm{iid}\ N(0, \sigma_w^2).$$

Usually, time series are smoother than that (see bottom graph on the next slide). Ways of introducing *serial correlation* and more *smoothness* into time series include **filtering** and **autoregression**.

3. Basic models (TS 1.2)
    - Preliminaries
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    - Gaussian white noise series and its 3-point moving average
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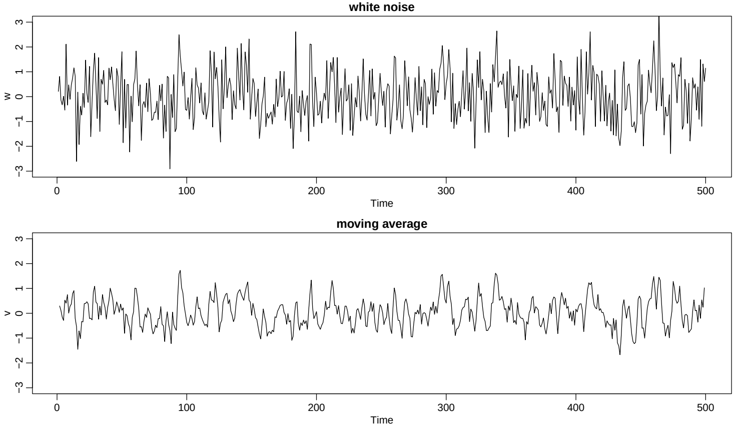
Gaussian white noise series and its 3-point moving average

```r
w <- rnorm(500, 0, 1)  # 500 N(0,1) variates
plot.ts(w, ylim = c(-3, 3), main = "white noise")
v <- stats::filter(w, sides = 2, filter = rep(1/3, 3))  # moving average
plot.ts(v, ylim = c(-3, 3), main = "moving average")
```



3 Basic models (TS 1.2)

- Preliminaries
- White noise - 3 scales :-)
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- **Filtering (and moving average)**
- Autoregressions
- Autoregression examples
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Filtering (and moving average)

A series v_t which is a **linear combination** of values of a more fundamental time series w_t is called a **filtered series**.

- Example: 3-point moving average (see bottom of previous slide for graph):

$$v_t = \frac{1}{3}(w_{t-1} + w_t + w_{t+1}).$$

- In R, moving averages are implemented through the function

  ```
  filter(x, filter, method = c("convolution"),sides = 2)
  ```

  where `x` is the original series, `filter` is a vector of weights (in reverse time order), `method = c("convolution")` is the default (alternative is `recursive`), and where `sides` is 1 for past values only, and 2 if weights are centered around lag 0 (requires uneven number of weights).

- Moving average smoothers will be further discussed in Module 8.

3 Basic models (TS 1.2)

Autoregressions

A series x_t that depends on some of its past values, as well as a noise w_t is called an **autoregression**, because the formula looks like a regression—not of independent variables, but of its own past values—hence **auto**regression.

- Example: An autoregression of the white noise:

$$x_t = x_{t-1} - 0.9x_{t-2} + w_t.$$

- If the autoregression goes back k periods, one needs k initial conditions (`filter` will use 0's otherwise).
- In R, autoregressions are implemented through the function

```
filter(x, filter, method = c("recursive"),init)
```

  where `x` is the original series, `filter` is a vector of weights (reverse time order) and `init` a vector of initial values (reverse time order).
- Autoregressions will be denoted $AR(p)$ (details in Module 9).

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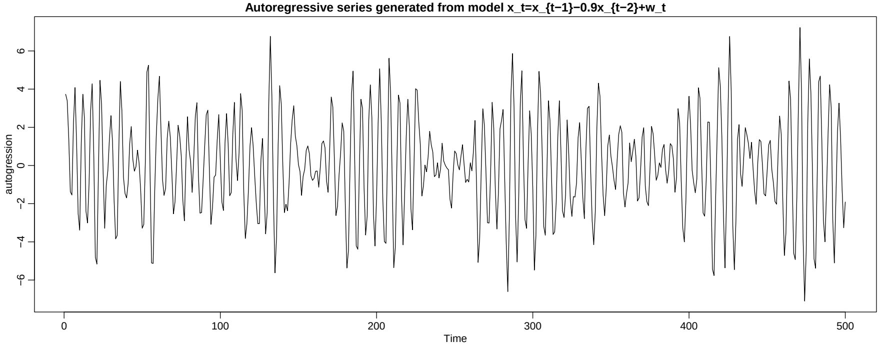
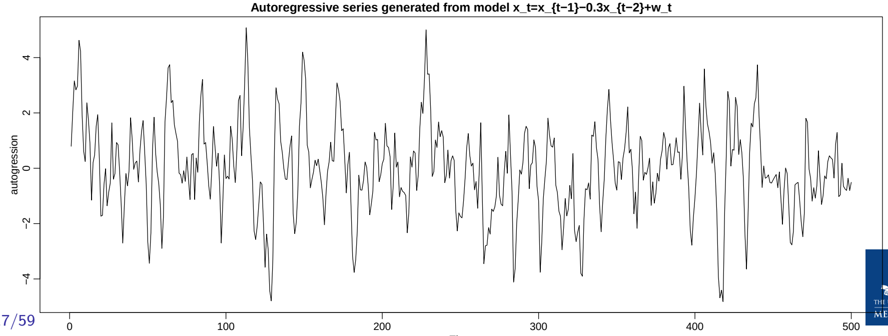
Autoregression examples

```
w = rnorm(550,0,1) # 50 extra to avoid startup problems
x = stats::filter(w, filter=c(1,-.9), method="recursive")[-(1:50)]
# remove first 50
plot.ts(x, ylab="autogression", #
        main="Autoregressive series generated from model x_t=x_{t-1}-0.9x_{
y = stats::filter(w, filter=c(1,-.3), method="recursive")[-(1:50)]
# remove first 50
plot.ts(y, ylab="autogression", #
        main="Autoregressive series generated from model x_t=x_{t-1}-0.3x_{
```


Autoregressive series generated from model x_t=x_{t−1}−0.9x_{t−2}+w_t
autogression
Time
6
4
2
0
−2
−4
−6
0
100
200
300
400
500


Autoregressive series generated from model x_t=x_{t−1}−0.3x_{t−2}+w_t
autogression
4
2
0
−2
−4
0
100
200
300
400
500

3 Basic models (TS 1.2)

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- Autoregressions
- Autoregression examples
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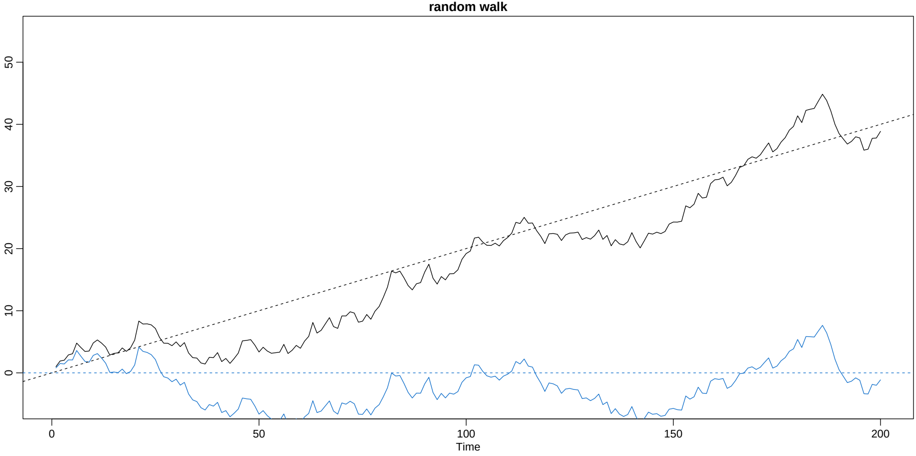
Random walk with drift

- The autoregressions introduced above are all centered around 0 for all t (in the expected sense).
- Assume now that the series increases linearly by δ (called **drift**) every time unit.
- The **random walk with drift** looks back only one time unit:

$$x_t = \delta + x_{t-1} + w_t = \delta t + \sum_{j=1}^{t} w_j \text{ for } t = 1, 2, \ldots$$

  with initial condition $x_0 = 0$ and with w_t a white noise.
- If $\delta = 0$ this is simply called a **random walk**.
- The term can be explained by visualising each increment from t to $t+1$ as a purely random step from wherever the process is at x_t, ignoring what happened before.

Random walk with drift $\delta = 0.2$ and $\sigma_w = 1$



Code used to generate the plot:

```
set.seed(155)  # so you can reproduce the results
w <- rnorm(200)
x <- cumsum(w)
wd <- w + 0.2
xd <- cumsum(wd)
plot.ts(xd, ylim = c(-5, 55), main = "random walk", ylab = "")
lines(x, col = 4)
abline(h = 0, col = 4, lty = 2)
```

4 Describing the behaviour of basic models (TS 1.3)

- Motivation
- Full specification
- Mean function
- Autocovariance function
- The autocorrelation function (ACF)

Motivation

- In this section we would like to develop **theoretical measures** to help describe how times series behave.
- We are particularly interested in describing the **relationships** between observations at different points in time.

4. Describing the behaviour of basic models (TS 1.3)
   - Motivation
   - Full specification
   - Mean function
   - Autocovariance function
   - The autocorrelation function (ACF)

Full specification

- A full specification of a time series of size n at times $t_1, t_2, \ldots, t_n$ for any n would require the full joint distribution function

$$F_{t_1,t_2,\ldots,t_n}(c_1, c_2, \ldots, c_n) = \Pr[x_{t_1} \leq c_1, x_{t_2} \leq c_2, \ldots, x_{t_n} \leq c_n].$$

  This is a quite unwieldy tool for analysis.
- Examination of the margins $F_t(x) = \Pr[x_t \leq x]$ and corresponding pdf $f_t(x)$, *when they exist*, can be informative.
- These are very theoretical. In practice, one often have only **one** realisation for each x_t so that inferring full distributions (let alone their dependence structure) is simply impractical without tricks, manipulations, and assumptions (some of which we will learn).

4 Describing the behaviour of basic models (TS 1.3)

- Motivation
- Full specification
- **Mean function**
- Autocovariance function
- The autocorrelation function (ACF)

Mean function

The **mean function** is defined as

$$\mu_{xt} = E[x_t] = \int_{-\infty}^{\infty} x f_t(x) dx.$$

Examples:

- **Moving Average Series**: we have

$$\mu_{vt} = E[v_t] = \frac{1}{3}\left(E[w_{t-1}] + E[w_t] + E[w_{t+1}]\right) = 0.$$

  Smoothing does not change the mean.
- **Random walk with drift**: we have

$$\mu_{xt} = E[x_t] = \delta t + \sum_{j=1}^{t} E[w_j] = \delta t.$$

4 Describing the behaviour of basic models (TS 1.3)

- Motivation
- Full specification
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- Autocovariance function
- The autocorrelation function (ACF)

Autocovariance function

The **autocovariance function** is defined as the second moment product

$$\gamma_x(s,t) = Cov(x_s, x_t) = E\left[(x_s - \mu_{xs})(x_t - \mu_{xt})\right]$$

for all s and t. Note:

- We will write $\gamma_x(s,t) = \gamma(s,t)$ if no confusion is possible.
- This is a measure of **linear** dependence.
- Smooth series $\rightarrow$ large γ even for t and s far apart
- Choppy series $\rightarrow$ γ is nearly zero for large separations
- [$\gamma_x(s,t) = 0 \Longrightarrow$ independence] $\Longleftarrow$ all variables are normal

For two series x_t and y_t this becomes

$$\gamma_{xy}(s,t) = Cov(x_s, y_t) = E\left[(x_s - \mu_{xs})(y_t - \mu_{yt})\right],$$

called **cross-covariance function**.

Examples of autocovariance functions

White noise: The white noise series w_t has $E[w_t] = 0$ and

$$\gamma_w(s,t) = Cov(w_s, w_t) = \begin{cases} \sigma_w^2 & s = t \\ 0 & s \neq t \end{cases}$$

Remember that if

$$U = \sum_{j=1}^{m} a_j X_j$$

and

$$V = \sum_{k=1}^{r} b_k Y_k$$

then

$$Cov(U, V) = \sum_{j=1}^{m} \sum_{k=1}^{r} a_j b_k Cov(X_j, Y_k).$$

This will be useful for computing γ of filtered series.

Moving average: A 3-point moving average v_t to the white noise series w_t has

$$\gamma_v(s,t) = Cov(v_s, v_t) = \begin{cases} \frac{3}{9}\sigma_w^2 & s = t \\ \frac{2}{9}\sigma_w^2 & |s-t| = 1 \\ \frac{1}{9}\sigma_w^2 & |s-t| = 2 \\ 0 & |s-t| > 2 \end{cases}$$

This only depends on the time separation lag **only**, and not on the absolute location along the series.

This is related to the concept of *weak stationarity* which will introduce later.

Random walk: For the random walk $x_t = \sum_{j=1}^{t} w_j$ we have

$$
\begin{aligned}
\gamma_x(s,t) &= Cov(x_s, x_t) \\
&= Cov\left(\sum_{j=1}^{s} w_j, \sum_{k=1}^{t} w_k\right) \\
&= \min\{s,t\}\sigma_w^2.
\end{aligned}
$$

Contrary to the previous examples, this depends on the absolute location rather than the lag.

Also $Var(x_t) = t\sigma_w^2$ increases without bound as t increases.

4. Describing the behaviour of basic models (TS 1.3)
   - Motivation
   - Full specification
   - Mean function
   - Autocovariance function
   - The autocorrelation function (ACF)

The autocorrelation function (ACF)

The **autocorrelation function (ACF)** is defined as

$$-1 \leq \rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}} \leq 1.$$

- The ACF measures the *linear* predictability of the series at time t, say x_t, using *only* the value x_s.
- If we could do that perfectly, then $\rho(s,t) \pm 1$ and

$$x_t = \beta_0 + \beta_1 x_s$$

  with β_1 of same sign as $\rho(s,t)$.

In the case of two series this becomes

$$-1 \leq \rho_{xy}(s,t) = \frac{\gamma_{xy}(s,t)}{\sqrt{\gamma_x(s,s)\gamma_y(t,t)}} \leq 1,$$

called **cross-correlation function (CCF)**.

5 Stationary time series (TS 1.4)

- Strict stationarity
- Weak stationarity
- Properties of stationary series
- Examples of (non)-stationarity
- Trend stationarity
- Joint stationarity
- Example of joint stationarity
- Linear process

Strict stationarity

A **strictly stationary** times series is one for which the probabilistic behaviour of every collection of values $\{x_{t_1}, x_{t_2}, \ldots, x_{t_k}\}$ is identical to that of the time shifted set (for any h) $\{x_{t_1+h}, x_{t_2+h}, \ldots, x_{t_k+h}\}$. That is,

$$\Pr[x_{t_1} \leq c_1, x_{t_2} \leq c_2, \ldots, x_{t_k} \leq c_k]$$

$$= \Pr[x_{t_1+h} \leq c_1, x_{t_2+h} \leq c_2, \ldots, x_{t_k+h} \leq c_k]$$

for all $k = 1, 2, \ldots$, all the time points $t_1, t_2, \ldots, t_k$, all numbers $c_1, c_2, \ldots, c_k$ and all time shifts $h = 0, \pm 1, \pm 2, \ldots$. This implies

- identical marginals of dimensions $< k$ for any shift h
- constant mean: $\mu_{xs} = \mu_{xt} \equiv \mu$
- for $k = 2$, an autocovariance function that depends only on $t - s$: $\gamma(s, t) = \gamma(s + h, t + h)$

We need something less constraining, that still allows us to infer properties from a single series.

5 Stationary time series (TS 1.4)

Weak stationarity

A **weakly stationary time series**, x_t, is a finite variance process such that

1. the mean value function, μ_{xt} is **constant** and does not depend on time t, and
2. the autocovariance function, $\gamma(s, t)$ depends on s and t **only through their difference** $|s - t|$.

Note:

- We dropped full distributional requirements. This imposes conditions on the first two moments of the series only.
- Since those completely define a normal distribution, a (weak) stationary Gaussian time series is also strictly stationary.
- We will use the term stationary to mean weakly stationary; if a process is stationary in the strict sense, we will use the term strictly stationary.

Stationarity means we can estimate those two quantities by averaging of a *single* series. This is what we needed.

5. Stationary time series (TS 1.4)
   - Strict stationarity
   - Weak stationarity
   - Properties of stationary series
   - Examples of (non)-stationarity
   - Trend stationarity
   - Joint stationarity
   - Example of joint stationarity
   - Linear process

Properties of stationary series

- Because of condition 1,

$$\mu_t = \mu.$$

- Because of condition 2,

$$\gamma(t+h,t) = Cov(x_{t+h}, x_t) = Cov(x_h, x_0) = \gamma(h,0) \equiv \gamma(h)$$

and the autocovariance of a stationary time series is then

$$\gamma(h) = Cov(x_{t+h}, x_t) = E[(x_{t+h} - \mu)(x_t - \mu)].$$

- $\gamma(h)$ is *non-negative definite*, which means that the variance of linear combinations of variates x_t will never be negative, that is,

$$0 \leq Var(a_1 x_1 + \cdots + a_n x_n) = \sum_{j=1}^{n} \sum_{k=1}^{n} a_j a_k \gamma(j-k).$$

- Furthermore,

$$|\gamma(h)| \leq \gamma(0) \text{ (the variance of the time series)}$$

and

$$\gamma(h) = \gamma(-h).$$

- The autocorrelation function (ACF) of a stationary time series becomes

$$-1 \leq \rho(h) = \frac{\gamma(t+h,t)}{\sqrt{\gamma(t+h,t+h)\gamma(t,t)}} = \frac{\gamma(h)}{\gamma(0)} \leq 1.$$

5 Stationary time series (TS 1.4)

- Strict stationarity
- Weak stationarity
- Properties of stationary series
- **Examples of (non)-stationarity**
- Trend stationarity
- Joint stationarity
- Example of joint stationarity
- Linear process

Examples of (non)-stationarity

White noise: We have

$$\mu_{wt} = 0$$

and

$$\gamma_w(h) = Cov(w_{t+h}, w_t) = \begin{cases} \sigma_w^2 & h = 0, \\ 0 & h \neq 0, \end{cases},$$

which are both independent of time. Hence, the white noise satisfies both conditions and is (weakly) stationary. Furthermore,

$$\rho_w(h) = \begin{cases} 1 & h = 0, \\ 0 & h \neq 0. \end{cases}$$

If in addition $w_t \sim$ iid $\mathrm{N}(0, \sigma_w^2)$, then it is also strictly stationary.

Moving average: For the 3-point MA we have

$$\mu_{vt} = 0 \quad \text{and} \quad \gamma_v(h) = \begin{cases} \frac{3}{9}\sigma_w^2 & h = 0, \\ \frac{2}{9}\sigma_w^2 & h \pm 1, \\ \frac{1}{9}\sigma_w^2 & h \pm 2, \\ 0 & |h| > 2, \end{cases}$$

which are both independent of time. Hence, the 3-point MA satisfies both conditions and is stationary. Furthermore,

$$\rho_v(h) = \begin{cases} 1 & h = 0, \\ \frac{2}{3} & h \pm 1, \\ \frac{1}{3} & h \pm 2, \\ 0 & |h| > 2, \end{cases}$$

which is symmetric around lag 0.

Random walk: For the random walk model $x_t = \sum_{j=1}^{t} w_j$ we have

$$\mu_{xt} = \delta t,$$

which is a function of time t, and

$$\gamma(s,t) = \min\{s,t\}\sigma_w^2,$$

which depends on s and t (not just their difference), so the random walk is **not** stationary.

Furthermore, remember

$$Var(x_t) = \gamma_x(t,t) = t\sigma_w^2$$

which increases without bound as $t \to \infty$.

5 Stationary time series (TS 1.4)

Trend stationarity

- If only the second condition (on the ACF) is satisfied, but not the first condition (on the mean value function), we have **trend stationarity**
- This means that the model has a stationary *behaviour* around its trend.
- Example: if

$$x_t = \alpha + \beta t + y_t \quad \text{where } y_t \text{ is stationary,}$$

then the mean function is

$$\mu_{x,t} = E[x_t] = \alpha + \beta t + \mu_y,$$

which is *not* independent of time. The autocovariance function,

$$\begin{aligned} \gamma_x(h) = Cov(x_{t+h}, x_t) &= E[(x_{t+h} - \mu_{x,t+h})(x_t - \mu_{x,t})] \\ &= E[(y_{t+h} - \mu_y)(y_t - \mu_y)] = \gamma_y(h), \end{aligned}$$

however, is independent of time.

Joint stationarity

Two time series, say, x_t and y_t, are said to be **jointly stationary** if they are each stationary, and the cross-covariance function

$$\gamma_{xy}(h) = Cov(x_{t+h}, y_t) = E[(x_{t+h} - \mu_x)(y_t - \mu_y)]$$

is a function only of lag h. The corresponding **cross-correlation function (CCF)** is

$$1 \leq \rho_{xy}(h) = \frac{\gamma_{xy}(h)}{\sqrt{\gamma_x(0)\gamma_y(0)}} \leq 1.$$

Note that because $Cov(x_2, y_1)$ and $Cov(x_1, y_2)$ (for example) need not be the same, it follows that typically

$$\rho_{xy}(h) \neq \rho_{xy}(-h),$$

that is, the CCF is **not generally symmetric about zero**. However, we have

$$\rho_{xy}(h) = \rho_{yx}(-h).$$

5 Stationary time series (TS 1.4)

Example of joint stationarity

Consider

$$x_t = w_t + w_{t-1} \quad \text{and} \quad y_t = w_t - w_{t-1},$$

where w_t are independent with mean 0 and variance σ_w^2. We have then

$$\begin{aligned}
\gamma_x(0) &= \gamma_y(0) = 2\sigma_w^2 \\
\gamma_x(-1) &= \gamma_x(1) = \sigma_w^2 \\
\gamma_y(-1) &= \gamma_y(1) = -\sigma_w^2
\end{aligned}$$

and

$$\gamma_{xy}(-1) = -\sigma_w^2, \quad \gamma_{xy}(0) = 0, \text{ and } \quad \gamma_{xy}(1) = \sigma_w^2,$$

so that

$$\rho_{xy}(h) = \begin{cases} 0 & h = 0, \\ 1/2 & h = 1, \\ -1/2 & h = -1, \\ 0 & |h| \geq 2, \end{cases}$$

which depends only on the lag h, so both series are jointly stationary.

Prediction using cross-correlation

Prediction using cross-correlation: A lagging relation between two series x_t and y_t may be exploited for predictions. For instance, if

$$y_t = Ax_{t-\ell} + w_t,$$

x_t is said to *lead* y_t for $\ell > 0$, and is said to *lag* y_t for $\ell < 0$.

If the relation above holds true, then the lag ℓ can be inferred from the shape of the autocovariance of the input series x_t:

- If w_t is uncorrelated with x_t then

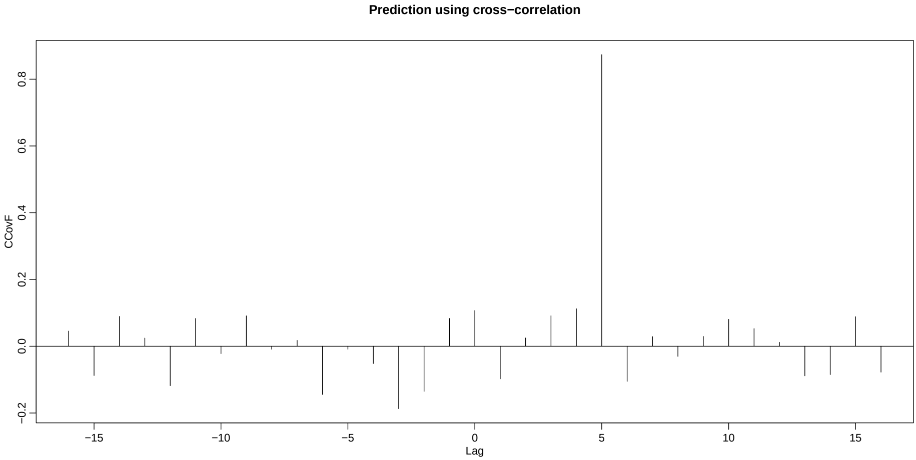
$$\begin{aligned}\gamma_{yx}(h) &= Cov(y_{t+h}, x_t) = Cov(Ax_{t+h-\ell} + w_{t+h}, x_t) \\ &= Cov(Ax_{t+h-\ell}, x_t) = A\gamma_x(h-\ell)\end{aligned}$$

- Since

$$\gamma_x(h-\ell) \leq \gamma_x(0),$$

the peak of $\gamma_{yx}(h)$ should be at $h = \ell$, and
h will be positive if x_t leads y_t, negative if x_t lags y_t.

- Here $\ell = 5$ and x_t leads y_t:



Note this example was simulated and uses the R functions `lag` and `ccf`:

```
x <- rnorm(100)
y <- stats::lag(x, -5) + rnorm(100)
ccf(y, x, ylab = "CCovF", type = "covariance")
```

5 Stationary time series (TS 1.4)

Linear process

A **linear process**, x_t , is defined to be a linear combination of white noise variates w_t, and is given by

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \Psi_j w_{t-j}, \quad \sum_{j=-\infty}^{\infty} |\Psi_j| < \infty$$

This is an important class of models because it encompasses moving averages, autoregressions, and also the combination of both, called *autoregressive moving average (ARMA) processes* which we will introduce later.

Example:

- **Moving average** The 3-point moving average has

$$\Psi_0 = \Psi_{-1} = \Psi_1 = 1/3$$

and is hence a linear process.

Properties of linear processes

- The autocovariance function of a linear process is given by

$$\gamma_x(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \Psi_{j+h}\Psi_j \quad \text{for } h \geq 0.$$

- It has finite variance if $\sum_{j=-\infty}^{\infty} \Psi_j^2 < \infty$.
- In its most general form x_t depends on the future ($j < 0$ components), the present ($j = 0$) and the past ($j > 0$).

For forecasting, a model dependent on the future is useless. We will focus on processes that do not depend on the future. Such processes are called **causal**, that is,

$$x_t \text{ is causal} \iff \Psi_j = 0 \quad \text{for} \quad j < 0,$$

which we will assume unless stated otherwise.

6 Estimation of correlation (TS 1.5)

Background

- One can very rarely hypothetise (specify) time series. In practice, most analyses are performed using sample data.
- Furthermore, **one often has only one realisation of the time series**.
- This means that we don't have n realisations of the time series to estimate its covariance and correlation functions.
- This is why the **assumption of stationarity is essential**: in this case, the assumed 'homogeneity' of the data means we can estimate those functions on one realisation only.
- This also means that one needs to manipulate / de-trend series such that they are arguably stationary before we can fit parameters to them and use them for projections.

6 Estimation of correlation (TS 1.5)

- Background
- Sample mean
- Sample autocovariance function
- Sample autocorrelation function
- Testing for significance of autocorrelation
- Sample cross-covariances and cross-correlations
- Testing for independent cross-whiteness

Sample mean

If a time series is stationary the mean function $\mu_t = \mu$ is constant so that we can estimate it by the sample mean,

$$\overline{x} = \frac{1}{n} \sum_{t=1}^{n} x_t.$$

This estimator is unbiased,

$$E[\overline{x}] = \mu,$$

and has standard error the square root of

$$Var(\overline{x}) = \frac{1}{n^2} Cov\left(\sum_{t=1}^{n} x_t, \sum_{s=1}^{n} x_s\right) = \frac{1}{n} \sum_{h=-n}^{n} \left(1 - \frac{|h|}{n}\right) \gamma_x(h).$$

6 Estimation of correlation (TS 1.5)

- Background
- Sample mean
- **Sample autocovariance function**
- Sample autocorrelation function
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- Sample cross-covariances and cross-correlations
- Testing for independent cross-whiteness

Sample autocovariance function

The **sample autocovariance function** is defined as

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-h}(x_{t+h}-\overline{x})(x_t-\overline{x}) \quad \text{with } \hat{\gamma}(-h) = \hat{\gamma}(h) \text{ for } h = 0, 1, \ldots, n-1.$$

Note:

- The estimator is biased.
- The sum runs over a restricted range $(n-h)$ because x_{t+h} is not available for $t+h > n$.
- One could wonder why the factor of the sum is not $1/(n-h)$ (the number of elements in the sum), but factor $1/n$ is not a mistake. It ensures that the estimate of the variances of linear combinations,

$$\widehat{Var}(a_1 x_1 + \cdots + a_n x_n) = \sum_{j=1}^{n}\sum_{k=1}^{n} a_j a_k \hat{\gamma}(j-k),$$

is non-negative.

6 Estimation of correlation (TS 1.5)

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Sample autocorrelation function

The **sample autocorrelation function (SACF)** is

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}.$$

Under some conditions (see book for details), if x_t is a white noise, then for large n, the SACF $\hat{\rho}(h)$ is approximately normally distributed with zero mean and standard deviation given by

$$\sigma_{\hat{\rho}(h)} = \frac{1}{\sqrt{n}}.$$

6 Estimation of correlation (TS 1.5)

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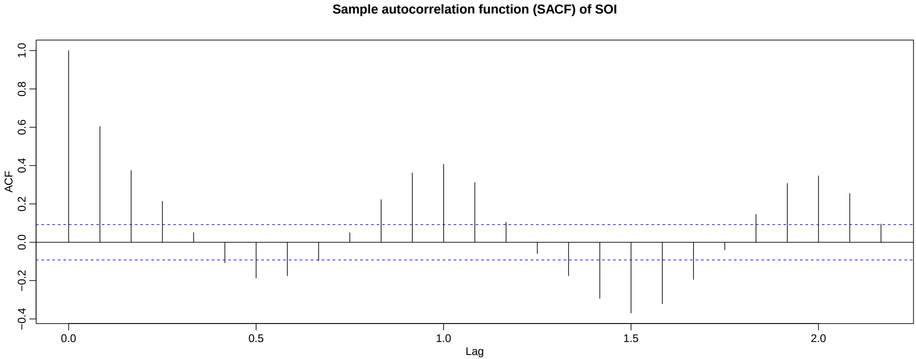
Testing for significance of autocorrelation

The asymptotic result for the variance of the SACF means we can test whether lagged observations are uncorrelated (which is a requirement for white noise):

- test for significance of the $\hat{\rho}$'s at different lags: check how many $\hat{\rho}$'s lie outside the interval $\pm 2/\sqrt{n}$ (a 95% confidence interval)
- One should expect approximately 1 out of 20 to lie outside the interval if the sequence is a white noise. Many more than that would invalidate the whiteness assumption.
- This allows for a recursive approach for manipulating / de-trending series until they are white noise, called **whitening**.
- The R function `acf` automatically displays those bounds with dashed blue lines.

SOI autocorrelation

```r
acf(soi, main = "Sample autocorrelation function (SACF) of SOI")
```

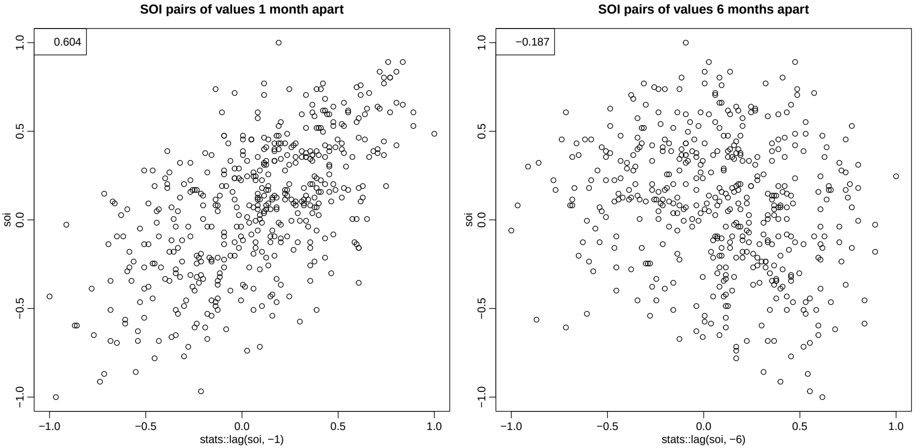



```r
r <- round(acf(soi, 6, plot = FALSE)$acf[-1], 3)  # first 6 sample acf valu
```

```
## [1]  0.604  0.374  0.214  0.050 -0.107 -0.187
```

The SOI series is clearly not a white noise.

```r
plot(stats::lag(soi, -1), soi, main = "SOI pairs of values 1 month apart")
legend("topleft", legend = r[1])
plot(stats::lag(soi, -6), soi, main = "SOI pairs of values 6 months apart")
legend("topleft", legend = r[6])
```



Scatterplots allow to have a visual representation of the dependence (which may not necessarily be linear).

6 Estimation of correlation (TS 1.5)

Sample cross-covariances and cross-correlations

The **sample cross-covariance function** is

$$\hat{\gamma}_{xy}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (x_{t+h} - \overline{x})(y_t - \overline{y}),$$

where $\hat{\gamma}_{xy}(-h) = \hat{\gamma}_{yx}(h)$ determines the function for *negative* lags.

The **sample cross-correlation function** is

$$-1 \leq \hat{\rho}_{xy}(h) = \frac{\hat{\gamma}_{xy}(h)}{\sqrt{\hat{\gamma}_x(0)\hat{\gamma}_y(0)}} \leq 1.$$

Note:

- Graphical examinations of $\hat{\rho}_{xy}(h)$ provide information about the leading or lagging relations in the data.

Testing for independent cross-whiteness

If x_t and y_t are *independent* linear processes then the large sample distribution of $\hat{\rho}_{xy}(h)$ has mean 0 and

$$\sigma_{\hat{\rho}_{xy}} = \frac{1}{\sqrt{n}}$$

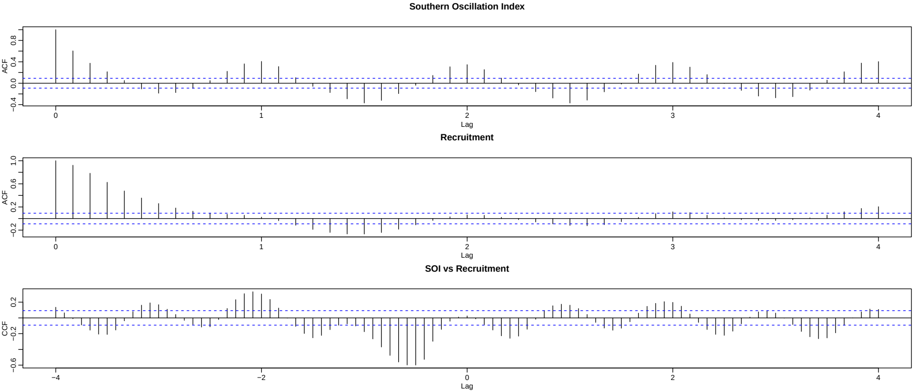
if at least one of the processes is independent white noise.

This is very useful, and adds to the toolbox:

- This provides feedback about the quality of our explanation of the relationship between both time series: if we have explained the trends and relationships between both processes, then their residuals should be independent white noise.
- After each improvement of our model, significance of the $\hat{\rho}_{xy}$'s of the residuals can be tested: if we have independent cross-whiteness then we have a good model. If the $\hat{\rho}_{xy}$'s are still significant (outside the boundaries) then we still have things to explain (to add).

SOI and recruitment correlation analysis

```r
acf(soi, 48, main = "Southern Oscillation Index")
acf(rec, 48, main = "Recruitment")
ccf(soi, rec, 48, main = "SOI vs Recruitment", ylab = "CCF")
```



The SCCF (bottom) has a different cycle, and peak at $h = -6$ suggests SOI leads Recruitment by 6 months (negatively).

Idea of prewhitening

- to use the test of cross-whiteness one needs to "prewhiten" at least one of the series
- for the SOI vs recruitment example, there is strong seasonality which, upon removal, may whiten the series
- we look at an example here that looks like the SOI vs recruitment example, and show how this seasonality could be removed with the help of sin and cos functions

Example:

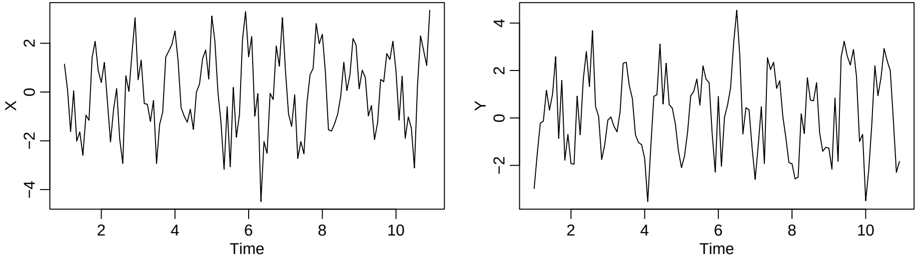
- Let us generate two series x_t and y_t, for $t = 1, \dots, 120$, independently as

$$x_t = 2\cos\left(2\pi t \frac{1}{12}\right) + w_{t1} \quad \text{and} \quad y_t = 2\cos\left(2\pi[t+5])\frac{1}{12}\right) + w_{t2},$$

where $\{w_{t1}, w_{t2}; t = 1, \dots, 120\}$ are all independent standard normals.

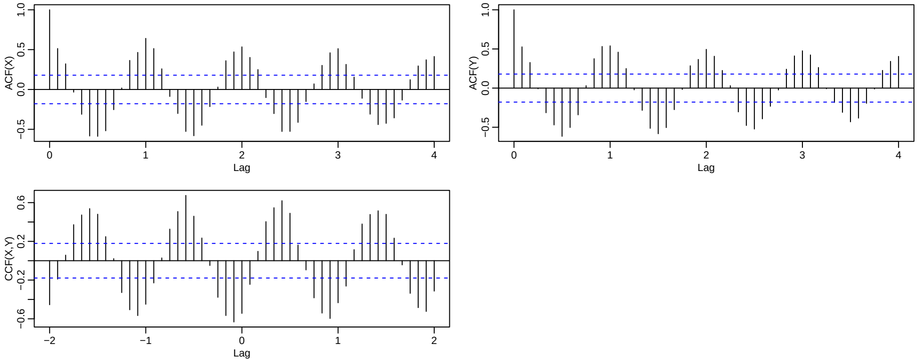
- this generates the data and plots it:

```
set.seed(1492)
num <- 120
t <- 1:num
X <- ts(2 * cos(2 * pi * t/12) + rnorm(num), freq = 12)
Y <- ts(2 * cos(2 * pi * (t + 5)/12) + rnorm(num), freq = 12)
par(mfrow = c(1, 2), mgp = c(1.6, 0.6, 0), mar = c(3, 3, 1, 1))
plot(X)
plot(Y)
```



- looking at the ACFs one can see seasonality

```
par(mfrow = c(3, 2), mgp = c(1.6, 0.6, 0), mar = c(3, 3, 1, 1))
acf(X, 48, ylab = "ACF(X)")
acf(Y, 48, ylab = "ACF(Y)")
ccf(X, Y, 24, ylab = "CCF(X,Y)")
```



- furthermore the CCF suggests cross-correlation even though the series are independent

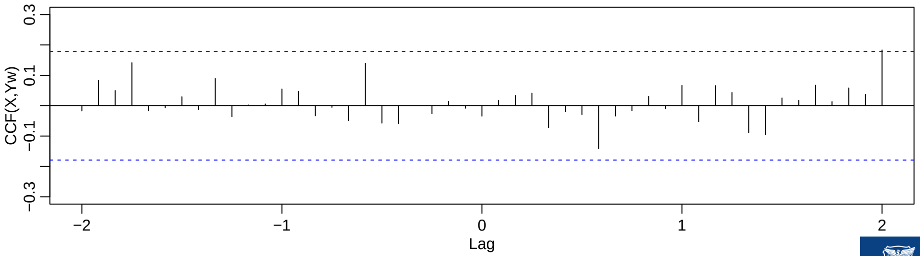
- what we do now is to "prewhiten" y_t by removing the signal from the data by running a regression of y_t on $\cos(2\pi t)$ and $\sin(2\pi t)$ and then putting

$$\tilde{y} = y_t - \hat{y}_t,$$

where $\hat{y}_t$ are the predicted values from the regression.

- in the R code below, Yw is $\tilde{y}$

```
par(mgp = c(1.6, 0.6, 0), mar = c(3, 3, 1, 1))
Yw <- resid(lm(Y ~ cos(2 * pi * t/12) + sin(2 * pi * t/12), na.action = NUL
ccf(X, Yw, 24, ylab = "CCF(X,Yw)", ylim = c(-0.3, 0.3))
```



- the updated CCF now suggests cross-independence, as it should

References I

Shumway, Robert H., and David S. Stoffer. 2017. *Time Series Analysis and Its Applications: With r Examples*. Springer.