The local level model

In a local level model, the underlying level of the time series is assumed to change over time, and the observed values at each time point are influenced by the current level. The model assumes that the changes in the level are random and follow a normal distribution with a mean of zero. This means that the model captures both the trend and the random fluctuations in the data.

non-stationary model: y_t and α_t depend on t

– assumption: the underlying data generating process is a random walk with drift

- observed time series

Level component $\Downarrow$ random walk ~~with a drift~~ ??

random error component $\Downarrow$ noise around the level

– mathematically

observed value at t — unobserved level at t — random error

$$y_t = \mu_t + \varepsilon_t \quad , \quad \varepsilon_t \sim NID(0, \overset{\text{known}}{\sigma_\varepsilon^2})$$

$$\underbrace{\mu_{t+1}}_{\text{states}} = \mu_t + \underbrace{\eta_t}_{\text{random error}} \quad , \quad \eta_t \sim NID(0, \overset{\text{known}}{\sigma_\eta^2})$$

with ε_t, η_s independent $\forall\, t, s \in \mathbb{Z}$ and $\mu_1 \sim N(\overset{\text{known}}{a_1, P})$

Special cases:

- if $\sigma_\varepsilon^2 = 0$ we have a random walk
- if $\sigma_\eta^2 = 0$ we have a white noise with constant mean

Reduced form:

$$y_t = \mu_{t+1} - \eta_t + \varepsilon_t$$

$$y_{t+1} = \mu_{t+1} + \varepsilon_{t+1} = \mu_t + \eta_t + \varepsilon_{t+1}$$

$$y_{t+1} - y_t = \mu_t + \eta_t + \varepsilon_{t+1} - \mu_t - \varepsilon_t$$

$$\Delta y_t = \eta_{t-1} + \varepsilon_t - \varepsilon_{t-1}$$

Autocovariance function

$$\gamma_k = \begin{cases} \sigma_\eta^2 + 2\sigma_\varepsilon^2 & \text{if } k=0 \\ -\sigma_\varepsilon^2 & \text{if } k=1 \\ 0 & \text{if } |k|>1 \end{cases}$$

The autocovariance function measures the covariance between observations at different lags within a time series. It provides information about how the values of a time series at different points in time are related to each other

Autocorrelation function

signal to noise ratio $= \frac{\sigma_\eta^2}{\sigma_\varepsilon^2}$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \rho_0 = 1 \\ \rho_1 = \frac{-\sigma_\varepsilon^2}{\sigma_\eta^2 + 2\sigma_\varepsilon^2} = -\frac{1}{q+2} \\ \rho_k = 0 \quad \text{for } |k|>1 \end{cases}$$

The autocorrelation function is a normalized version of the autocovariance function and measures the linear dependence between observations at different lags in a time series

- **Remarks:**

  - reduced form of Δy_t is a $\underset{\text{moving average}}{MA(1)}$ with $-\frac{1}{2} < \rho_1 < 0$
  - reduced form of y_t is an ARIMA $(0,1,1)$ with $-\frac{1}{2} < \rho_1 < 0$ mmmh...
  - $\Delta y_t = \xi_t - \theta \xi_{t-1}$, $\xi_t \sim N(0, \sigma_\xi^2)$
  - $\theta = \frac{1}{2}\left(2 + q - \sqrt{q^2 + 4q}\right)$, $\theta \in (0,1)$
  - $\sigma_\xi^2 = \frac{\sigma_\eta^2}{(1+\theta)^2}$

- Here we can recall what a MA(1) and ARIMA is

ARIMA $(0,0,1)$

$$X_t = \delta + (1 + \theta_1 \cdot L^1) \cdot \varepsilon_t$$
$$= \delta + \varepsilon_t + \theta_1 \cdot L \cdot \varepsilon_t$$
$$= \delta + \varepsilon_t + \theta_1 \cdot \varepsilon_{t-1} \implies MA(1)$$

$$(1-L)X_t = (1+\theta)\cdot\varepsilon_t$$
$$\Delta X_t = \varepsilon_t + \theta \cdot \varepsilon_{t-1}$$

ARIMA $(0,1,1)$

$$(1-L)X_t = \delta + (1+\theta_1)\cdot\varepsilon_{t-1}$$
$$X_t - X_{t-1} = \delta + \varepsilon_t + \theta\varepsilon_{t-1}$$
$$\Delta X_t = \delta + \varepsilon_t + \theta\varepsilon_{t-1}$$
if $\theta = -1$
$$= \delta + \varepsilon_t - \varepsilon_{t-1}$$

$$\Delta y_t = \eta_{t-1} + \varepsilon_t - \varepsilon_{t-1}$$

ARIMA $(0,1,1)$

$$\underline{\left(1 - \sum^{p} \varphi \cdot L^i\right)\cdot(1-L)^d X_t = \left(1 + \sum^{q} \theta_i L^i\right)\cdot\varepsilon_t}$$
$$(1-L)X_t = (1 + \theta \cdot L)\cdot\varepsilon_t$$
$$\Delta X_t = \varepsilon_t + \theta\cdot\varepsilon_{t-1}$$

Exponential smoothing in the local level model

Exponential smoothing is a widely used technique in time series analysis, including in the context of the local level model. It provides a flexible and efficient way to estimate the level component and make forecasts based on the observed data.

The basic idea behind exponential smoothing is to assign exponentially decreasing weights to past observations, with the most recent observations receiving higher weights.

Let's take

$$\Delta y_{t+1} = \xi_{t+1} - \theta \cdot \xi_t$$

$$y_{t+1} = y_t + \xi_{t+1} - \theta \xi_t$$

which is the best linear predictor of y_{t+1} given $\mathcal{Y}_t$?

$$E(y_{t+1} \mid \mathcal{Y}_t) = y_t - \theta \xi_t$$

now we take the difference in this sense:

$$y_{t+1} - \underbrace{y_{t+1|t}}_{\text{this is}} = v_{t+1} \quad \Rightarrow \text{one step ahead prediction error}$$

$$y_{t+1|t} = y_t - \theta \big(\overbrace{y_t - y_{t|t-1}}^{\xi_t} \big)$$

$$= y_t - \theta y_t + \theta y_{t|t-1}$$

$$\left[y_{t|t-1} = E\left[y_t \mid \mathcal{Y}_{t-1} \right] \overset{?}{=} y_{t-1} - \theta \cdot \xi_{t-1} \right]$$

$$\overset{?}{=} y_t - \overbrace{y_{t-1}}^{y_t - y_{t|t-1}} + \theta \cdot \xi_{t-1}$$

$$\overset{?}{=} \Delta y_t + \theta \cdot \xi_{t-1}$$

$$\overset{\checkmark}{=} \underline{\xi_t} - \cancel{\theta \cdot \xi_{t-1}} + \cancel{\theta \cdot \xi_{t-1}}$$

and setting $\lambda = 1 - \theta$

$$y_{t+1|t} = \lambda \cdot y_t + (1 - \lambda) \cdot y_{t|t-1}$$

$$= (1 - \lambda) \cdot \sum_{j=0}^{\infty} \lambda^j \cdot y_{t-j}$$

$$\Downarrow$$

EWMA: exponential weighted moving average

each data point is assigned a weight that decreases exponentially as we move further into the past. The most recent observation receives a higher weight, while the weights for earlier observations decrease exponentially.

captures recent changes or trends in the data more effectively than traditional moving averages

Exercise 2.13.1

Let $x_t = \Delta y_t = y_t - y_{t-1} = \Delta \alpha_t + \Delta \varepsilon_t$

$E[x_t] =$