State space models

A state space model consists of two key components:

- observation equation: $y_t = Z_t \cdot \alpha_t + G_t \cdot \varepsilon_t \qquad , \varepsilon_t \sim NID(0, 1)$
- state equation: $\alpha_{t+1} = T_t \cdot \alpha_t + H_t \cdot \eta_t \qquad , \eta_t \sim NID(0, 1)$

- y_t observations can be univariate or multivariate
- α_t is unobserved
- Z_t, T_t, G_t, H_t, Q_t system matrices or parameters, often static

Examples

- Unobserved components models:
  - LLM, LLT, AR(1) plus noise
- ARMA models
  - ARMA(2,1)

- The LLM case:

in order to rewrite an LLM model in a state space form let's recall that the LLM is

$$y_t = \mu_t + \varepsilon_t$$

$$\mu_{t+1} = \mu_t + \eta_t$$

Instead a state space model is

$$y_t = Z_t \cdot \alpha_t + G_t \cdot \varepsilon_t$$

$$\alpha_{t+1} = T_t \cdot \alpha_t + H_t \cdot \eta_t$$

So, let's say that $\mu_t = \alpha_t$, if

$$\begin{cases} \cdot\ Z_t = 1 & \cdot\ H_t = \sigma_\eta \\ \cdot\ G_t = \sigma_\varepsilon & \cdot\ Q_t = 1 \\ \cdot\ T_t = 1 & \end{cases}$$

the local level model can be written as a state space model.

- The AR(1) + noise

As the LLM doesn't differ that much from the AR(1) + noise model we can use the results seen before and in addiction:

- $T_t = \phi$

- The LLT case

- $\alpha_t = \begin{bmatrix} \mu_t \\ \beta_t \end{bmatrix}$ - $H_t = \begin{bmatrix} \sigma_\eta & 0 \\ 0 & \sigma_\xi \end{bmatrix}$ - $Q_t = 1$

- $T_t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ - $Z_t = \begin{bmatrix} 1 & 0 \end{bmatrix}$ - $G_t = \sigma_\varepsilon$

- The ARMA (2, 1)

$$y_t = \phi_1 \cdot y_{t-1} + \phi_2 \cdot y_{t-2} + \zeta_t + \theta \cdot \zeta_{t-1} \quad , \quad \zeta_t \sim N(0, \sigma^2_\zeta)$$

and in a state space form

$$\begin{cases} - \alpha_t = \begin{bmatrix} y_t \\ \phi_2 \cdot y_{t-1} + \theta \zeta_t \end{bmatrix} \\ - T_t = \begin{bmatrix} \phi_1 & 1 \\ \phi_2 & 0 \end{bmatrix} \\ - Z_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \end{cases} \quad \begin{cases} - H_t = \begin{bmatrix} 1 \\ \theta \end{bmatrix} \\ - Q_t = \sigma^2_\zeta \\ - \eta_t = \zeta_{t+1} \end{cases} \quad \begin{cases} - G_t = 0 \end{cases}$$