Theory

1. Let y_t be a GARCH(1,1) process

a) Prove that y_t is a martingale difference sequence

Recall that a GARCH (1,1) model is defined as

$$y_t = \sigma_{t|t-1} \cdot z_t \qquad \text{with } z_t \sim \text{iid } N(0,1)$$

$$\sigma^2_{t|t-1} = \omega + \alpha_1 \cdot y^2_{t-1} + \beta \cdot \sigma^2_{t-1|t-2} \qquad \text{with } \begin{array}{l} w > 0 \\ 0 \leq \alpha \cdot \beta \leq 1 \\ \alpha + \beta < 1 \end{array}$$

- $E[y_t \mid \mathcal{Y}_{t-1}] = y_{t-1}$?

$$= E[\sigma_{t|t-1} \cdot z_t \mid \mathcal{Y}_{t-1}]$$

$$= E[\sigma_{t|t-1} \cdot z_{t-1} \mid \mathcal{Y}_{t-1}]$$

$$= z_{t-1} \cdot E[\sigma_{t|t-1} \mid \mathcal{Y}_{t-1}]$$

$$= 0 \cdot E[\sigma_{t|t-1} \mid \mathcal{Y}_{t-1}] = 0$$

- A st. series is a mds if its exp. w.r.t the past is 0

b) Comment on the unconditional properties of y_t

in order to speak about the unconditional properties of y_t let's recall that a GARCH (1,1) can be represented as an ARMA (1,1) model.

- Unconditional moment of y_t

if $\alpha + \beta < 1$ the ARMA (1,1) process is stationary and thus

$$E(y_t^2) = \frac{\omega}{1-(\alpha+\beta)}$$

$$\frac{E(y_t^4)}{E[(y_t^2)]^2} = 3 \cdot \frac{1-(\alpha+\beta)^2}{1-(\alpha+\beta)^2-2\alpha^2} > 3$$

c) Comment on the conditional properties of $y_t \mid \mathcal{F}_{t-1}$

Given $y_t = \sigma_{t|t-1} \cdot z_t$

$E(y_t \mid \mathcal{F}_{t-1})$ is a martingale since

$$= E(\sigma_{t|t-1} \cdot z_t \mid \mathcal{F}_{t-1}) = \sigma_{t|t-1} \cdot E(z_t)$$

$$= 0$$

the notation emphasises the fact that the condit. variance of y_t given the information $(t-1)$ is a time varying function of t

The second moment:

$$E(y_t^2 \mid \mathcal{F}_{t-1}) = \sigma^2_{t|t-1}$$

and since the first moment is 0, $Var(y_t \mid \mathcal{F}_{t-1})$

$$= \sigma^2_{t|t-1}$$

d) Prove that y_t is an ARMA(1, 1) on the squares

Let $v_t = y_t^2 - \sigma^2_{t|t-1}$, so that $\sigma^2_{t-1} = y^2_{t-1} - v_{t-1}$

$$y_t^2 = \sigma^2_{t|t-1} + v_t$$

$$= \omega + \alpha \cdot y^2_{t-1} + \beta \cdot \sigma^2_{t-1} + v_t$$

$$= \omega + \alpha y^2_{t-1} + \beta \left(y^2_{t-1} - v_{t-1} \right) + v_t$$

$$= \omega + \underbrace{(\alpha + \beta) y^2_{t-1}}_{\mathrm{AR}(1)} + \underbrace{v_t - \beta \cdot v_{t-1}}_{\mathrm{MA}(1)}$$

e) Prove that the process $v_t = y_t^2 - \sigma^2_{t|t-1}$ is a mds

$$E\left[v_t \mid \mathcal{I}_{t-1}\right] = E\left[y_t^2 - \sigma^2_{t|t-1} \mid \mathcal{I}_{t-1}\right]$$

$y_t = \sigma_{t|t-1} \cdot z_t$ so, $y_t^2 = \sigma^2_{t|t-1} \cdot z_t^2$

$$= E\left[\sigma^2_{t|t-1} \cdot z_t^2 - \sigma^2_{t|t-1} \mid \mathcal{I}_{t-1}\right]$$

$$= E\left[\sigma^2_{t|t-1} \cdot \left(z_t^2 - 1\right) \mid \mathcal{I}_{t-1}\right]$$

$E(z_t^2) = 1$

$$= (1-1) \cdot E\left[\sigma^2_{t|t-1} \mid \mathcal{I}_{t-1}\right]$$

$$= 0$$

f) Prove that the process $w_t = \frac{y_t^2}{\sigma^2_{t|t-1}} - 1$ is also a mds

$$E[w_t \mid \mathcal{F}_{t-1}] = E\left[\frac{y_t^2}{\sigma^2_{t|t-1}} - 1 \,\middle|\, \mathcal{F}_{t-1}\right]$$

$$= E\left[\frac{\cancel{\sigma^2_{t|t-1}} \cdot z_t^2}{\cancel{\sigma^2_{t|t-1}}} - 1 \,\middle|\, \mathcal{F}_t\right]$$

$$= E\left[z_t^2 - 1 \,\middle|\, \mathcal{F}_t\right] \quad \text{and since } z_t \sim N(0,1)$$

$$= 0$$

g) Show that $\sigma^2_{t|t-1} \cdot w_t = v_t$

Recall that $y_t^2 - \sigma^2_{t|t-1} = v_t$

$$\sigma^2_{t|t-1} \cdot w_t = y_t^2 - \sigma^2_{t|t-1}$$

$$\sigma^2_{t|t-1} \cdot \left(\frac{y_t^2}{\sigma^2_{t|t-1}} - 1 \right) = y_t^2 - \sigma^2_{t|t-1}$$

$$y_t^2 - \sigma^2_{t|t-1} = y_t^2 - \sigma^2_{t|t-1}$$

h) Conclude that in a GARCH(1,1) process one can write

$$\sigma^2_{t|t-1} = \underbrace{\omega}^{\text{"omega"}} + (\alpha + \beta) \cdot \sigma^2_{t-1|t-2} + \alpha \cdot v_{t-1}$$

in fact, let $v_t = \sigma^2_{t|t-1} \cdot \underbrace{w_t}^{\text{"w"}}$

and $w_t = \dfrac{y_t^2}{\sigma^2_{t|t-1}} - 1$

So, $\sigma^2_{t|t-1} = \omega + \alpha \cdot \sigma^2_{t-1|t-2} + \beta \sigma^2_{t-1|t-2} + \alpha \sigma^2_{t-1|t-2} \cdot \left(\dfrac{y^2_{t-1}}{\sigma^2_{t-1|t-2}} - 1 \right)$

$$= \omega + \alpha \cdot \cancel{\sigma^2_{t-1|t-2}} + \beta \sigma^2_{t-1|t-2} + \alpha \cdot y^2_{t-1} - \alpha \cdot \cancel{\sigma^2_{t-1|t-2}}$$

$$= \omega + \alpha \cdot y^2_{t-1} + \beta \sigma^2_{t-1|t-2}$$

which is the usual form of the conditional variance of a GARCH(1,1) that relates the past squared observations with the past conditional variance

2)

a) Show that u_t is proportional to the conditional likelihood of the time varying parameter

Recall that

$$u_t = \frac{(\nu+1)\,y_t^2}{(\nu \cdot e^{2\lambda_{t|t-1}} + y_t^2} - 1$$

$$f(y_t;\sigma^2_{t|t-1} \mid \mathcal{Y}_t) \propto \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\cdot\sqrt{\pi}\cdot\sqrt{(\nu-2)\cdot e^{2\lambda_{t|t-1}}}} \cdot \left[1 + \frac{y_t^2}{(\nu-2)\cdot e^{2\lambda_{t|t-1}}}\right]^{-\left(\frac{\nu+1}{2}\right)}$$

$$\log\left(f\left(y_t;\sigma^2_{t|t-1} \mid \mathcal{Y}_{t-1}\right)\right) \propto \log\left(\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\cdot\sqrt{\pi}\cdot\sqrt{(\nu-2)\cdot e^{2\lambda_{t|t-1}}}} \cdot \left[1 + \frac{y_t^2}{(\nu-2)\cdot e^{2\lambda_{t|t-1}}}\right]^{-\left(\frac{\nu+1}{2}\right)}\right)$$

$$l(f(\cdot\,,\cdot)) \propto \log\left(\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\cdot\sqrt{\pi}\cdot\sqrt{\nu-2}}\right) + \log\left(\frac{1}{e^{\lambda_{t|t-1}}}\right) - \frac{\nu+1}{2}\cdot\log\left(1 + \frac{y_t^2}{(\nu-2)e^{2\lambda_{t|t-1}}}\right)$$

Now, taking the derivatives

$$l(f(\cdot)) \propto 0 + \cancel{e^{2\lambda}}_{t|t-1} \cdot \frac{(0 - 2\cancel{e^{2\lambda}}_{t|t-1})}{\cancel{(e^{2\lambda_{t|t-1}})^2}} - \frac{\nu+1}{2}\cdot\frac{1}{1+\frac{y_t^2}{(\nu-2)\cdot e^{2\lambda_{t|t-1}}}}\cdot\left(\frac{-(\nu-2)2e^{2\lambda_{t|t-1}}}{((\nu-2)\cdot e^{2\lambda_{t|t-1}})^2}\right)$$

$$l(f(\cdot)) \propto -2 - \left(\frac{\nu+1}{2}\right)\cdot\frac{\cancel{(\nu-2)}\cdot\cancel{e^{2\lambda_{t|t-1}}}}{(\nu-2)\cdot e^{2\lambda_{t|t-1}} + y_t^2}\cdot\frac{-\cancel{(\nu-2)}\cdot 2\cdot\cancel{e^{2\lambda_{t|t-1}}}}{\cancel{(\nu-2)^2}\cdot\cancel{(e^{2\lambda_{t|t-1}})^2}}$$

$$\propto -2 + \frac{\nu+1}{\cancel{2}}\cdot\frac{\cancel{2}}{(\nu-2)\cdot e^{2\lambda_{t|t-1}} + y_t^2}$$

$$\propto -2 + \frac{\nu+1}{(\nu-2)\cdot e^{2\lambda_{t|t-1}} + y_t^2} \propto -1 + u_t\cdot y_t^2$$

Let's take now a Beta-t-GARCH

$$f(y_t \mid \mathcal{F}_{t-1}) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\cdot\sqrt{\pi(\nu-2)}\cdot\sqrt{\sigma_t^2}} \cdot \left[1 + \frac{(y_t-\mu_{t|t-1})^2}{(\nu-2)\cdot\sigma^2_{t|t-1}}\right]^{-\frac{(\nu+1)}{2}}$$

with $\mu_{t|t-1} = 0$

we take the log

$$\log(f(\cdot)) \propto \log\left(\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\cdot\sqrt{\pi(\nu-2)}}\right) + \log\left(\frac{1}{\sqrt{\sigma_t^2}}\right) - \left(\frac{\nu+1}{2}\right)\cdot\log\left(1+\frac{y_t^2}{(\nu-2)\cdot\sigma^2_{t|t-1}}\right)^{-\frac{\nu+1}{2}}$$

taking the derivatives we get

$$\frac{d\log(f(\cdot))}{d\sigma_t^2} \propto 0 + \sigma_t^2 - \left(\frac{\nu+1}{2}\right)\cdot\frac{1}{1+\frac{y_t^2}{(\nu-2)\cdot\sigma^2_{t|t-1}}}\cdot\left(\frac{-y_t^2\cdot\cancel{(\nu-2)}}{(\nu-2)^{\cancel{2}}\sigma^4_{t|t-1}}\right)$$

$$\propto \sigma_t^2 - \left(\frac{\nu+1}{2}\right)\cdot\frac{\cancel{(\nu-2)}\,\cancel{\sigma^2_{t|t-1}}}{(\nu-2)\cdot\sigma^2_{t|t-1}+y_t^2}\cdot\left(\frac{-y_t^2}{\cancel{(\nu-2)}\cdot\sigma^{\cancel{4}\,2}_{t|t-1}}\right)$$

$$\propto \sigma_t^2 + \frac{(\nu+1)\cdot y_t^2}{(\nu-2)\cdot\sigma^2_{t|t-1}+y_t}\cdot\frac{1}{2}\cdot\frac{1}{\sigma^2_{t|t-1}}$$

$$\propto \sigma_t^2 + u_t\cdot\frac{1}{2\sigma^2_{t|t-1}}$$

b) Show that u_t is a beta distributed r.v.

Recall that for a Beta-t-EGARCH

$$u_t = \frac{(\nu+1)\cdot y_t^2}{\nu \cdot \exp(\lambda_{t|t-1}) + y_t^2} - 1$$

and $u_t = (\nu+1)\cdot b_t - 1$

with $$b_t = \frac{y_t^2 / \nu \cdot \exp(\lambda_{t|t-1})}{1 + y_t^2 / \nu\cdot\exp(\lambda_{t|t-1})}$$

We need to show that b_t is distributed as a $\text{Beta}\left(\frac{1}{2} ; \frac{\nu}{2}\right)$

and $$\frac{y_t^2}{\nu \cdot \exp(\lambda_{t|t-1})} \sim F(1, \nu)$$

therefore

$$b_t \sim \text{Beta}\left(\frac{1}{2}, \frac{\nu}{2}\right)$$

Similarly for the Beta-t-GARCH

$$u_t = \frac{(\nu+1)\cdot y_t^2}{(\nu-2)\cdot \sigma^2_{t|t-1} + y_t^2} - 1$$

$$u_t = (\nu+1) b_t - 1$$

$$\frac{(\nu+1)\cdot y_t^2}{(\nu-2)\cdot \sigma^2_{t|t-1} + y_t^2} - \cancel{1} = (\nu+1)\cdot b_t - \cancel{1}$$

$$\frac{\frac{(\nu+1)\cdot y_t^2}{(\nu-2)\cdot\sigma^2_{t|t-1} + y_t^2}}{(\nu+1)} = b_t$$

$$b_t = \frac{y_t^2}{(\nu-2)\cdot\sigma^2_{t|t-1} + y_t^2}$$

$$= \frac{\frac{y_t^2}{(\nu-2)\cdot\sigma^2_{t|t-1}}}{1 + \frac{y_t^2}{(\nu-2)\cdot\sigma^2_{t|t-1}}} \sim \frac{F(1, \nu-2)}{1 + F(1, \nu-2)}$$

$$\Downarrow$$

$$\sim \mathrm{Beta}\left(\frac{1}{2}; \frac{\nu-2}{2}\right)$$

$\downarrow$

of the first kind

c) Discuss the relation with GARCH processes

Both models, GARCH and Beta-t-(E)-GARCH, are models used to predict volatility.
However, when data presents higher variability, leading to extreme observations the Gaussian assumption might not be the more suitable.
Therefore the variation proposed by the Beta-t-GAR of taking the transformed variable as a Beta distr. let the conditional variance more resistant to extreme observations