Answers (mock exam)

1.

$$Y_n \sim Poisson\left(\frac{1}{\sqrt{n}}\right)$$
$$Y_n \xrightarrow{L_2} \quad ?$$

First we need to decide where it converges. Let's try $E[Y_n]$

$$E[Y_n] = \frac{1}{\sqrt{n}}$$
$$\lim_{n\to\infty} E[Y_n] = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Does it converge to a δ_0? Let's apply the definition

$$\lim_{n\to\infty} E[(Y_n - 0)^2] = \lim_{n\to\infty} E[Y_n^2]$$

The second moment of a Poisson is

$$\lambda = E[Y^2] - E[Y]^2$$
$$\lambda = E[Y^2] - \lambda^2 \Rightarrow E[Y^2] = \lambda + \lambda^2 \Rightarrow E[Y_n^2] = \frac{1}{\sqrt{n}} + \frac{1}{n}$$
$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} + \frac{1}{n} = 0 \Rightarrow Y_n \xrightarrow{L_2} \delta_0$$

2. $X_1 \sim N(0,2)$ and $X_2 \sim N(-2,1)$ and covariance -1. Compute $COV(X_1 + X_2, X_1 - X_2)$.

$$
\begin{aligned}
&COV(X_1 + X_2, X_1 - X_2) = \\
&= COV(X_1, X_1) - \cancel{COV(X_1, X_2)} + \cancel{COV(X_2, X_1)} - COV(X_2, X_2) = \\
&= VAR(X_1) - VAR(X_2) = 2 - 1 = 1
\end{aligned}
$$

3. $X \sim Ber(\frac{1}{2})$

$Y = \frac{1}{2} + \frac{X}{2}$

$$
\begin{aligned}
&M_X(t) = 1 - p + pe^t \\
&M_{aX+b}(t) = e^{bt} M_x(at) \Rightarrow M_Y(t) = e^{\frac{t}{2}} M_X(\frac{t}{2}) = e^{\frac{t}{2}}(1 - p + pe^{\frac{t}{2}})
\end{aligned}
$$

Now if

$$
p = \frac{1}{2} \Rightarrow M_Y(t) = e^{\frac{t}{2}} \left(\frac{1}{2} + \frac{e^{\frac{t}{2}}}{2} \right) = \frac{1}{2} \left(e^{\frac{t}{2}} + e^t \right)
$$

4. $f_X(x) = \frac{x}{2}$, $X \in [0,2]$

$F_Y(y) =?$, $Y = 6X - 3$

$$
\begin{aligned}
F_Y(y) &= \left| \frac{\partial}{\partial y} g^{-1}(y) \right| f_X(g^{-1}(y)) \\
g(X) &= 6X - 3 \qquad g^{-1}(Y) = \frac{Y+3}{6} \\
F_Y(y) &= \frac{1}{6} \left(\frac{Y+3}{6 * 2} \right) = \frac{1}{6} \left(\frac{Y+3}{12} \right)
\end{aligned}
$$

5. $X \sim U(\theta-2, \theta+1)$
$T(X) = \bar{X} + \frac{1}{2}$.
A sufficient condition is that

$$\lim_{n\to\infty} MSE(T_n(X)) = 0$$

We need

$$E[T_n(X)] = E\left[\bar{X} + \frac{1}{2}\right] = E[\bar{X}] + \frac{1}{2} =$$
$$= \frac{\theta - 2 + \theta + 1}{2} + \frac{1}{2} = \frac{2\theta}{2} - \frac{1}{2} + \frac{1}{2} = \theta$$

$$V\left[\bar{X} + \frac{1}{2}\right] = \frac{V(X)}{n} = \frac{1}{12n}(\theta + 1 - \theta + 2)^2 = \frac{9}{12}\frac{1}{n} = \frac{3}{4n}$$

from which $B(T_n(X)) = 0$ and $\lim_{n\to\infty} V(T_n(X)) = 0$

6. $f(X,\theta) = \theta x^{\theta-1}$, for $0 < X < 1$, $(X_1 = 0.2, X_2 = 0.5, X_3 = 0.7, X_4 = 0.8)$.
We need $E(X)$

$$E(X) = \int_0^1 x\theta x^{\theta-1} dx = \int_0^1 \cancel{x}\frac{1}{\cancel{x}}\theta x^{\theta} dx = \theta \int_0^1 x^{\theta} dx =$$
$$= \theta\left[\frac{x^{\theta+1}}{\theta+1}\right]_0^1 = \frac{\theta}{\theta+1}$$

$$\bar{X} = \frac{0.2 + 0.5 + 0.7 + 0.8}{4} = 0.55$$

$$\Rightarrow \frac{\theta}{\theta+1} = 0.55$$

$$\theta = 0.55\theta + 0.55$$
$$0.45\theta = 0.55$$
$$\hat{\theta} = 1.22$$

7. $Y_n \sim Exp(\frac{n}{2})$
   Find n such that $Pr\{Y_n > 0.25\} \leq 0.80.$
   According to the Markov Inequality
   $$P(X \geq \lambda E[X]) \leq \frac{1}{\lambda}$$
   Now we assume $\frac{1}{\lambda} = 0.8 \Rightarrow \lambda = 1.25$
   $E[X] = \frac{2}{n}$ We can write the Markov inequality as
   $$P(X \geq 1.25\frac{2}{n}) \leq 0.8$$
   $$P(X \geq \frac{2.5}{n}) \leq 0.8$$
   $$\frac{2.5}{n} = 0.25$$
   $$n = 10$$

8. $X = (X_1, X_2)^T, \qquad f(x_1, x_2) = kx_2, \qquad 0 < x_1 < x_2 < 1$
   In order to compute k
   $$\int_0^1 \int_0^{x_2} kx_2 dx_1 dx_2 = 1$$

$$\int_0^1 \int_0^{x_2} kx_2 dx_1 dx_2 = k\int_0^1 x_2 \int_0^{x_2} 1 dx_1 dx_2 = k\int_0^1 x_2 [x_1]_0^{x_2} dx_2 =$$
$$= k\int_0^1 x_2^2 dx_2 = k\left[\frac{x_2^3}{3}\right]_0^1 = \frac{k}{3}$$

$\frac{k}{3} = 1, \qquad k = 3$

9.

$$\xrightarrow{L_4} \quad \not\Rightarrow \quad \xrightarrow{a.s}$$