CHAPTER – 7

Applications of Differential Calculus

Exercise 7.1

1. A particle moves along a straight line in such a way that after t seconds its distance from the origin is $s = 2t^2 + 3t$ metres.

(i) Find the average velocity between $t = 3$ and $t = 6$ seconds.

(ii) Find the instantaneous velocities at $t = 3$ and $t = 6$ seconds.

Solution:

Given $s(t) = 2t^2 + 3t$

(i) Average velocity $= \frac{s(6)-s(3)}{6-3}$

$$s(6) = 2(36) + 3(6) = 72 + 18 = 90$$

$$s(3) = 2(9) + 3(3) = 18 + 9 = 27$$

Average velocity $= \frac{90-27}{3}$

$$= \frac{63}{3}$$

Average velocity $= 21\ m/s$

(ii) $\frac{ds}{dt} = 4t + 3$

Instantaneous velocity at $t = 3$,

$$\frac{ds}{dt} = 4(3) + 3 = 15\ m/s$$

Instantaneous velocity at $t = 6$,

$$\frac{ds}{dt} = 4(6) + 3 = 27\ m/s$$

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2. A camera is accidently knocked off an edge of a cliff 400 ft high. The camera falls a distance of $s = 16t^2$ in t seconds.

(i) How long does the camera fall before it hits the ground?

(ii) What is the average velocity with which the camera falls during the last 2 seconds?

(iii) What is the instantaneous velocity of the camera when it hits the ground?

Solution:

(i) Given $s(t) = 16t^2$

$$400 = 16t^2$$

$$t^2 = \frac{400}{16}$$

$$t^2 = 25$$

$$t = 5\ sec$$

(ii) Average velocity $= \frac{s(5)-s(3)}{5-3}$

$$= \frac{16(25)-16(9)}{2}$$

$$= \frac{16(25-9)}{2}$$

$$= 8(16)$$

Average velocity $= 128\ ft/sec$

(iii) $\frac{ds}{dt} = 32t$

Instantaneous velocity at $t = 5$,

$$\frac{ds}{dt} = 32(5) = 160\ ft/sec$$

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3. A particle moves along a line according to the law $s(t) = 2t^3 - 9t^2 + 12t - 4$, where $t \geq 0$

(i) At what times the particle changes direction?

(ii) Find the total distance travelled by the particle in the first 4 seconds.

(iii) Find the particles acceleration each time the velocity is zero.

Solution:

(i) $s(t) = 2t^3 - 9t^2 + 12t - 4$

$$v = \frac{ds}{dt} = 6t^2 - 18t + 12$$

$$v = 6(t^2 - 3t + 2) \quad \text{----------(1)}$$

At $v = 0$,

$$6(t^2 - 3t + 2) = 0$$

$$t^2 - 3t + 2 = 0$$

$$(t-1)(t-2) = 0$$

$$t = 1 \text{ or } t = 2$$

Particle changes is direction at $1sec$ and $2sec$

(ii) Distance travelled in first 4sec

$$= |s(1) - s(0)| + |s(2) - s(1)| + |s(4) - s(2)|$$

$s(0) = -4$

$s(1) = 2 - 9 + 12 - 4 = 1$

$s(2) = 16 - 36 + 24 - 4 = 0$

$s(4) = 128 - 144 + 48 - 4 = 28$

Total distance $= |1 + 4| + |0 - 1| + |28 - 0|$

$$= 5 + 1 + 28$$

$\therefore$ Total distance $= 34\ units$

(iii) **Acceleration = rate of change of velocity**

$$a = \frac{dv}{dt}$$

$$a = 6(2t - 3) \qquad \text{From (1)}$$

$$a = 12t - 18$$

At $t = 1$,

$$a = 12 - 18 = -6\ m/s^2$$

At $t = 2$,

$$a = 24 - 18 = 6\ m/s^2$$

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4. If the volume of a cube of side length x is $v = x^3$. Find the rate of change of the volume with respect to x when $x = 5$ units.

Solution:

Given $v = x^3$

$$\frac{dv}{dx} = 3x^2$$

At $x = 5$,

$$\frac{dv}{dx} = 3(25) = 75\ units$$

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5. If the mass $m(x)$ (in kg) of a thin rod length x (in metres) is given by, $m(x) = \sqrt{3x}$ then what is the rate of change of mass with respect to the length when it is $x = 3$ and $x = 27$ metres.

Solution:

Given $m(x) = \sqrt{3x}$

$$m = \sqrt{3}x^{\frac{1}{2}}$$

$$\frac{dm}{dx} = \sqrt{3}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$= \frac{\sqrt{3}}{2x^{\frac{1}{2}}}$$

$$\frac{dm}{dx} = \frac{\sqrt{3}}{2\sqrt{x}}$$

At $x = 3$,

$$\frac{dm}{dx} = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2}\ kg/m$$

At $x = 27$,

$$\frac{dm}{dx} = \frac{\sqrt{3}}{2\sqrt{9 \times 3}} = \frac{1}{2(3)} = \frac{1}{6}\ kg/m$$

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6. A stone is dropped into a pond causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate at 2 cm per second. When the radius is 5cm find the rate of changing of the total area of the disturbed water?

Solution:

Given $\frac{dr}{dt} = 2$ and $r = 5$

Area of circle $A = \pi r^2$

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi(5)\,(2) = 20\pi\ cm^2/sec$$

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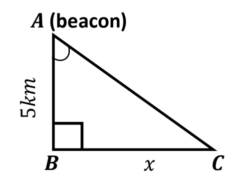
7. A beacon makes one revolution every 10 seconds. It is located on a ship which is anchored 5 km from a straight shore line. How fast is the beam moving along the shore line when it makes an angle of 45° with the shore?

Solution:



Time for 1 rotation = $10sec$

$$\frac{d\theta}{dt} = \frac{2\pi}{10}$$

In ΔABC,

$$\tan\theta = \frac{x}{5}$$

$$x = 5\tan\theta$$

$$\frac{dx}{dt} = 5\sec^2\theta\frac{d\theta}{dt}$$

$$= 5(\sec^2 45°)\left(\frac{2\pi}{10}\right)$$

$$= 5\left(\sqrt{2}\right)^2\left(\frac{2\pi}{10}\right)$$

$$\frac{dx}{dt} = 2\pi\ km/sec$$

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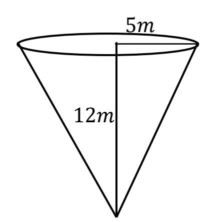
8. A conical water tank with vertex down of 12 metres hight has a radius of 5 metres at the top. If water flows into the tank at a rate 10 cubic m/min, how fast is the depth of the water increases when the water is 8 metres deep?

Solution:



Radius of cone = $5m$

Hight of cone = $12m$

$$\frac{r}{h} = \frac{5}{12}$$

$$r = \frac{5h}{12}$$

Given that $\frac{dV}{dt} = 10\ m^3/min$

Volume of cone $V = \frac{1}{3}\pi r^2 h$

$$= \frac{\pi}{3}\left(\frac{5h}{12}\right)^2 h$$

$$= \frac{\pi}{3}\left(\frac{25h^2}{144}\right)h$$

$$V = \frac{25\pi h^3}{3 \times 144}$$

$$\frac{dV}{dt} = \frac{25\pi}{3\times144}3h^2\frac{dh}{dt}$$

When $h = 8$,

$$10 = \frac{25\pi}{3\times144}(3)(64)\frac{dh}{dt}$$

$$10 = \frac{25\pi}{36}(16)\frac{dh}{dt}$$

$$2 = \frac{5\pi}{9}(4)\frac{dh}{dt}$$

$$1 = \frac{5\pi}{9}(2)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9}{10\pi}\ m/min$$

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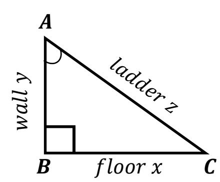
9. A ladder 17 more long is leaning against the wall. The base of the ladder is pulled away from the wall at a rate of 5m/s. when the base of the ladder is 8 metres form the wall,

(i) how fast is the top of the ladder moving down the wall?

(ii) at what rate, the area of the triangle formed by the ladder, wall, and the floor, is changing?

Solution:



Given $\frac{dx}{dt} = 5\ m/sec$

By Pythagoras thm,

$$y^2 + 8^2 = 17^2$$

$$y^2 + 64 = 289$$

$$y^2 = 225$$

$$y = 15$$

Also, $x^2 + y^2 = z^2$

Diff w.r.t 't'we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \qquad (\because z\ is\ constant)$$

$\div 2$, $$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$8(5) + 15\frac{dy}{dt} = 0$$

$$15\frac{dy}{dt} = -40$$

$$\frac{dy}{dt} = -\frac{8}{3}\ m/sec$$

Area of the triangle, $A = \frac{1}{2}bh$

$$A = \frac{1}{2}xy$$

Diff w.r.t 't' we get

$$\frac{dA}{dt} = \frac{1}{2}\left[x\frac{dy}{dt} + y\frac{dx}{dt}\right]$$

$$= \frac{1}{2}\left[8\left(-\frac{8}{3}\right) + 15(5)\right]$$

$$\frac{dA}{dt} = \frac{1}{2}\left[-\frac{64}{3} + 75\right]$$

$$= \frac{1}{2}\left[\frac{-64+225}{3}\right]$$

$$= \frac{1}{2}\left[\frac{161}{3}\right]$$

$$\frac{dA}{dt} = 26.83\ m^2/sec$$

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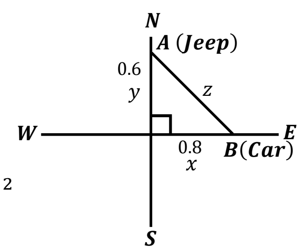
10. A police jeep approaching an orthogonal intersection from the northern direction, is chasing a speeding car that has moving straight east. When the jeep is 0.6 km north of the intersection and the car 0.8 km to the east. The police determine with a radar that the distance between them and the car is increasing at 20 km/hr. if the jeep is moving at 60 km/hr at the instant of measurement, what is the speed of the car?

Solution:



Given $\frac{dz}{dt} = 20\ km/hr$

$$\frac{dy}{dt} = -60\ km/hr$$

By Pythagoras thm,

$$z^2 = (0.6)^2 + (0.8)^2$$

$$z^2 = 0.36 + 0.64$$

$$z^2 = 1$$

$$z = 1$$

Also, $x^2 + y^2 = z^2$

Diff w.r.t 't' we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$\div 2$, $$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$$

$$(0.8)\frac{dx}{dt} + (0.6)(-60) = 1(20)$$

$$0.8\frac{dx}{dt} - 36 = 20$$

$$0.8\frac{dx}{dt} = 56$$

$$\frac{dx}{dt} = \frac{56}{0.8}\times\frac{10}{10}$$

$$= \frac{560}{8}$$

$$\frac{dx}{dt} = 70\ km/hr$$

Exercise 7.2

1. Find the slope of the tangent to the following curves at the respective given points.

(i) $y = x^4 + 2x^2 - x$ at $x = 1$

Solution:

$$y = x^4 + 2x^2 - x$$

$$\frac{dy}{dx} = 4x^3 + 4x - 1$$

At $x = 1$,

$$\frac{dy}{dx} = 4 + 4 - 1 = 7$$

$\therefore$ Slope of the tangent $= 7$

(ii) $x = a\cos^3 t, y = b\sin^3 t$ at $t = \frac{\pi}{2}$.

Solution:

$x = a\cos^3 t$	$y = b\sin^3 t$
$\frac{dx}{dt} = a3\cos^2 t\ (-\sin t)$	$\frac{dy}{dt} = b3\sin^2 t\ (\cos t)$
$\frac{dx}{dt} = -3a\cos^2 t\sin t$	$\frac{dy}{dt} = 3b\sin^2 t\cos t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3b\sin^2 t\cos t}{-3a\cos^2 t\sin t}$$

$$\frac{dy}{dx} = -\frac{b\sin t}{a\cos t}$$

$$\frac{dy}{dx} = -\frac{b}{a}\tan t$$

At $t = \frac{\pi}{2}$,

$$\frac{dy}{dx} = \infty$$

$\therefore$ Slope of the tangent $= \infty$

2. Find the point on the curve $y = x^2 - 5x + 4$ at which the tangent is parallel to the line $3x + y = 7$.

Solution:

Given $y = x^2 - 5x + 4$ ----------(1)

$$\frac{dy}{dx} = 2x - 5 = m_1$$

Given $3x + y - 7 = 0$

$$3 + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -3 = m_2$$

Since they are parallel, $m_1 = m_2$

$$2x - 5 = -3$$

$$2x = 2$$

$$x = 1$$

Sub $x = 1$ in (1), we get

$$y = 1 - 5 + 4$$

$$y = 0$$

$\therefore$ The required point is (1,0)

3. Find the points on the curve $y = x^3 - 6x^2 + x + 3$ where the normal is parallel to the line $x + y = 1729$.

Solution:

Given $y = x^3 - 6x^2 + x + 3$ ----------(1)

$$\frac{dy}{dx} = 3x^2 - 12x + 1 \text{ (slope of tangent)}$$

$$slope\ of\ normal = \frac{-1}{slope\ of\ tangent}$$

$$slope\ of\ normal = -\frac{1}{3x^2 - 12x + 1} = m_1$$

Given $x + y - 1729 = 0$

Diff w.r.t 'x' we get

$$1 + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -1 = m_2$$

Since normal is parallel to the line, $m_1 = m_2$

$$\frac{-1}{3x^2 - 12x + 1} = -1$$

$$3x^2 - 12x + 1 = 1$$

$$3x^2 - 12x = 0$$

$$3x(x - 4) = 0$$

$$x = 0 \text{ or } x = 4$$

When $x = 0, y = 3$ From (1)

When $x = 4, y = 64 - 96 + 4 + 3 = -25$

$\therefore$ The required points are $(0, 3)$ and $(4, -25)$

4. Find the points on the curve $y^2 - 4xy = x^2 + 5$ for which the tangent is horizontal.

Solution:

Given $y^2 - 4xy = x^2 + 5$ ----------(1)

Diff w.r.t 'x' we get

$$2y\frac{dy}{dx} - 4\left[x\frac{dy}{dx} + y\frac{dx}{dx}\right] = 2x$$

$$2y\frac{dy}{dx} - 4x\frac{dy}{dx} - 4y = 2x$$

$$2\frac{dy}{dx}(y - 2x) = 2x + 4y$$

$$\frac{dy}{dx} = \frac{2(x + 2y)}{2(y - 2x)}$$

$$\frac{dy}{dx} = \frac{x + 2y}{y - 2x}$$

Since the tangent is horizontal, $\frac{dy}{dx} = 0$

$$\frac{x + 2y}{y - 2x} = 0$$

$$x - 2y = 0$$

$$x = -2y \qquad \text{----------(2)}$$

Sub (2) in (1), we get

$$y^2 - 4(-2y)y = (-2y)^2 + 5$$

$$y^2 + 8y^2 = 4y^2 + 5$$

$$5y^2 = 5$$

$$y^2 = 1$$

$$y = \pm 1$$

When $y = 1, x = -2$ From (2)

When $y = -1, x = 2$

∴ The required points are $(-2, 1)$ and $(2, -1)$

5. Find the tangent and normal to the following curves at the given points on the curve.

(i) $y = x^2 - x^4$ at $(1, 0)$

Solution:

Given $y = x^2 - x^4$

$$\frac{dy}{dx} = 2x - 4x^3$$

At $(1,0)$

$$\frac{dy}{dx} = 2 - 4 = -2$$

Equation of tangent at $(1,0)$ is,

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -2(x - 1)$$

$$y = -2x + 2$$

$$2x + y - 2 = 0$$

Equation of normal is,

$$x - 2y + k = 0$$

At $(1,0)$

$$1 - 2(0) + k = 0$$

$$1 + k = 0$$

$$k = -1$$

Equation of normal is,

$$x - 2y - 1 = 0$$

(ii) $y = x^4 + 2e^x$ at $(0, 2)$

Solution:

Given $y = x^4 + 2e^x$

$$\frac{dy}{dx} = 4x^3 + 2e^x$$

At $(0,2)$

$$\frac{dy}{dx} = 4(0) + 2e^0 = 2$$

Equation of tangent at $(0,2)$ is,

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 2x$$

$$2x - y + 2 = 0$$

Equation of normal is,

$$x + 2y + k = 0$$

At $(0,2)$

$$4 + k = 0$$

$$k = -4$$

Equation of normal is,

$$x + 2y - 4 = 0$$

(iii) $y = x\sin x$ at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

Solution:

Given $y = x \sin x$

$$\frac{dy}{dx} = \sin x + x \cos x$$

At $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\frac{dy}{dx} = \sin\frac{\pi}{2} + \frac{\pi}{2}\cos\frac{\pi}{2} = 1 + \frac{\pi}{2}(0)$$

$$\frac{dy}{dx} = 1$$

Equation of tangent at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ is,

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\pi}{2} = 1\left(x - \frac{\pi}{2}\right)$$

$$y - \frac{\pi}{2} = x - \frac{\pi}{2}$$

$$y = x$$

$$x - y = 0$$

Equation of normal is,

$$x + y + k = 0$$

At $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\frac{\pi}{2} + \frac{\pi}{2} + k = 0$$

$$\pi + k = 0$$

$$k = -\pi$$

Equation of normal is,

$$x + y - \pi = 0$$

(iv) $x = \cos t\,, y = 2sn^2 t$ at $t = \frac{\pi}{3}$

Solution:

Given $x = \cos t\,, y = 2\sin^2 t$

$x = \cos t$	$y = 2\sin^2 t$
$\frac{dx}{dt} = -\sin t$	$\frac{dy}{dt} = 4\sin t\cos t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\sin t\cos t}{-\sin t}$$

$$\frac{dy}{dx} = -4\cos t$$

At $t = \frac{\pi}{3}$

$$\frac{dy}{dx} = -4\cos\frac{\pi}{3} = -4 \times \frac{1}{2}$$

$$\frac{dy}{dx} = -2$$

$\therefore$ The point $(x, y) = (\cos t\,, 2\sin^2 t)$

At $t = \frac{\pi}{3}$

$$(x, y) = \left(\cos\frac{\pi}{3}, 2\sin^2\frac{\pi}{3}\right)$$

$$= \left(\frac{1}{2}, 2\left(\frac{\sqrt{3}}{2}\right)^2\right)$$

$$(x, y) = \left(\frac{1}{2}, \frac{3}{2}\right)$$

Equation of tangents at $\left(\frac{1}{2}, \frac{3}{2}\right)$ is,

$$y - y_1 = m(x - x_1)$$

$$y - \frac{3}{2} = -2\left(x - \frac{1}{2}\right)$$

$$\frac{2y-3}{2} = -2x + 1$$

$$2y - 3 = -4x + 2$$

$$4x + 2y - 5 = 0$$

Equation of normal is,

$$2x - 4y + k = 0$$

At $\left(\frac{1}{2}, \frac{3}{2}\right)$

$$2\left(\frac{1}{2}\right) - 4\left(\frac{3}{2}\right) + k = 0$$

$$1 - 6 + k = 0$$

$$k = 5$$

Equation of normal is,

$$2x - 4y + 5 = 0$$

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6. Find the equations of the tangents to the curve $y = 1 + x^3$ for which the tangent is orthogonal with the line $x + 12y = 12$.

Solution:

Given $y = 1 + x^3$

$$\frac{dy}{dx} = 3x^2 = m_1 \qquad \text{----------(1)}$$

Line is $x + 12y = 12$

$$slope = -\frac{coefficient\ of\ x}{coefficient\ of\ y} = -\frac{1}{12} = m_2$$

Since the tangent is orthogonal,

$$m_1 \times m_2 = -1$$

$$3x^2\left(-\frac{1}{12}\right) = -1$$

$$\frac{x^2}{4} = 1$$

$$x^2 = 4$$

$$x = \pm 2$$

Now, $y = 1 + x^3$

At $x = 2$	At $x = -2$
$y = 1 + 8$	$y = 1 - 8$
$y = 9$	$y = -7$
Point $(2,9)$	Point $(-2, -7)$

From (1) we have,

$$\frac{dy}{dx} = 3x^2$$

At $x = \pm 2$,

$$\frac{dy}{dx} = 3(4) = 12$$

Equation of tangent at $(2,9)$ is

$$y - y_1 = m(x - x_1)$$

$$y - 9 = 12(x - 2)$$

$$y - 9 = 12x - 24$$

$$12x - y - 15 = 0$$

Equation of tangent at $(-2, -7)$ is

$$y + 7 = 12(x + 2)$$

$$y + 7 = 12x + 24$$

$$12x - y + 17 = 0$$

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7. Find the equations of the tangents to the curve $y = \frac{x+1}{x-1}$ which are parallel to the line $x + 2y = 6$.

Solution:

Given $y = \frac{x+1}{x-1}$

$$\frac{dy}{dx} = \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{-2}{(x-1)^2} = m_1 \qquad \text{----------(1)}$$

Line is $x + 2y = 6$

$$slope = -\frac{1}{2} = m_2$$

Since the tangent is parallel, $m_1 = m_2$

$$-\frac{2}{(x-1)^2} = -\frac{1}{2}$$

$$4 = (x-1)^2$$

$$4 = x^2 + 1 - 2x$$

$$x^2 - 2x - 3 = 0$$

$$(x+1)(x-3) = 0$$

$$x = -1, 3$$

$$\begin{array}{c|c} & -3 \\ -3 & 1 \\ \hline & -2 \end{array}$$

Now $y = \frac{x+1}{x-1}$

At $x = -1$	At $x = 3$
$y = 0$	$y = \frac{4}{2} = 2$
Point $(-1,0)$	Point $(3,2)$

Equation of tangent at $(-1,0)$ is

$$y - y_1 = m(x - x_1)$$

$$y = -\frac{1}{2}(x+1)$$

$$2y = -x - 1$$

$$x + 2y + 1 = 0$$

Equation of tangent at $(3,2)$ is

$$y - 2 = -\frac{1}{2}(x-3)$$

$$2y - 4 = -x + 3$$

$$x + 2y - 7 = 0$$

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8. Find the equation of the tangent and normal to the curve given by $x = 7\cos t$ and $y = 2\sin t, t \in R$ at any point on the curve.

Solution:

Given $x = 7\cos t$	$y = 2\sin t$
$\frac{dx}{dt} = -7\sin t$	$\frac{dy}{dt} = 2\cos t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{-7\sin t}$$

$$\frac{dy}{dx} = -\frac{2\cos t}{7\sin t}$$

Equation of tangent is

$$y - y_1 = m(x - x_1)$$

$$y - 2\sin t = -\frac{2\cos t}{7\sin t}(x - 7\cos t)$$

$$7y\sin t - 14\sin^2 t = -2x\cos t + 14\cos^2 t$$

$$2x\cos t + 7y\sin t = 14\cos^2 t + 14\sin^2 t$$

$$2\cos t\, x + 7\sin t\, y = 14$$

Equation of normal is

$$7\sin t\, x - 2\cos t\, y = k$$

At $(7\cos t, 2\sin t)$

$$7\sin t\,(7\cos t) - 2\cos t\,(2\sin t) = k$$

$$49\sin t\cos t - 4\sin t\cos t = k$$

$$45\sin t\cos t = k$$

Equation of normal is

$$7\sin t\, x - 2\cos t\, y = 45\sin t\cos t$$

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9. Find the angle between the rectangular hyperbola $xy = 2$ and the parabola $x^2 + 4y = 0$.

Hint: $\theta = tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

Solution:

Given $xy = 2$

$$y = \frac{2}{x} \qquad \text{----------(1)}$$

$$x^2 + 4y = 0 \qquad \text{----------(2)}$$

Sub (1) in (2) we get,

$$x^2 + 4\left(\frac{2}{x}\right) = 0$$

$$x^3 + 8 = 0$$

$$x^3 = -8 = (-2)^3$$

$$x = -2$$

From (1), $\qquad y = \frac{2}{-2} = -1$

∴ Point of intersection is $(-2, -1)$

Now $xy = 2$

Diff w.r.t 'x' we get,

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

At $(-2, -1)$

$$\frac{dy}{dx} = -\frac{(-1)}{-2} = -\frac{1}{2} = m_1$$

And $x^2 + 4y = 0$

Diff w.r.t 'x' we get,

$$2x + 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{4} = -\frac{x}{2}$$

At $(-2,-1)$, $\frac{dy}{dx} = -\frac{(-2)}{2} = 1 = m_2$

Angle between the hyperbola and parabola is

$$\theta = \tan^{-1}\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$\theta = \tan^{-1}\left|\frac{-\frac{1}{2}-1}{1+\left(-\frac{1}{2}\right)(1)}\right|$$

$$\theta = \tan^{-1}\left|\frac{-\frac{3}{2}}{\frac{1}{2}}\right|$$

$$\theta = \tan^{-1}|-3|$$

$$\theta = \tan^{-1}(3)$$

---

10. Show that the two curves $x^2 - y^2 = r^2$ and $xy = c^2$ where c, r are constants, cut orthogonally.

Hint: If two curves cut orthogonally, then $m_1m_2 = -1$

Solution:

Let the point of intersection be (x_1, y_1)

$$x^2 - y^2 = r^2$$

Diff w.r.t 'x' we get

$$2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{-2y} = \frac{x}{y}$$

At (x_1, y_1)

$$\frac{dy}{dx} = \frac{x_1}{y_1} = m_1$$

Now $xy = c^2$

Diff w.r.t 'x' we get

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

At (x_1, y_1)

$$\frac{dy}{dx} = \frac{-y_1}{x_1} = m_2$$

$$m_1m_2 = \left(\frac{x_1}{y_1}\right)\left(\frac{-y_1}{x_1}\right)$$

$$m_1m_2 = -1$$

Hence two curves cut orthogonally.

Exercise 7.3

1. Explain why Rolle's theorem is not applicable to the following functions in the respective intervals.

(i) $f(x) = \left|\frac{1}{x}\right|, x \in [-1, 1]$

Solution:

Given $f(x) = \left|\frac{1}{x}\right|, x \in [-1,1]$

$f(x)$ is not continuous at $x = 0$.

Rolle's theorem is not applicable.

(ii) $f(x) = \tan x, x \in [0, \pi]$

Solution:

Given $f(x) = \tan x, x \in [0, \pi]$

$f(x)$ is not continuous at $x = \frac{\pi}{2}$

Rolle's theorem is not applicable.

(iii) $f(x) = x - 2\log x, x \in [2, 7]$

Solution:

Given $f(x) = x - 2\log x, x \in [2,7]$

$f(x)$ is continuous on $[2,7]$

$f(x)$ is differentiable on $(2,7)$

$f(2) = 2 - 2\log 2$

$f(7) = 7 - 2\log 7$

$f(2) \neq f(7)$

$\therefore$ Rolle's theorem is not applicable.

Hint: Tangent is parallel to x-axis, $\frac{dy}{dx} = 0$.

2. Using the Rolle's theorem, determine the values of x which the tangent is parallel to the $x-$ axis for the following functions:

(i) $f(x) = x^2 - x, x \in [0, 1]$

Solution:

Given $f(x) = x^2 - 2x, x \in [0,1]$

$$f'(x) = 2x - 1$$

Now $f'(x) = 0$

$$2x - 1 = 0$$

$$x = \frac{1}{2} \in [0,1]$$

(ii) $f(x) = \frac{x^2-2x}{x+2}, x \in [-1, 6]$

Solution:

Given $f(x) = \frac{x^2-2x}{x+2}, x \in [-1,6]$

$$f'(x) = \frac{(x+2)(2x-2)-(x^2-2x)(1)}{(x+2)^2}$$

$$= \frac{2x^2-2x+4x-4-x^2+2x}{(x+2)^2}$$

$$f'(x) = \frac{x^2+4x-4}{(x+2)^2}$$

Now $f'(x) = 0$

$$\frac{x^2+4x-4}{(x+2)^2} = 0$$

$$x^2 + 4x - 4 = 0$$

Here $a = 1, b = 4$ and $c = -4$

$$x = \frac{-4 \pm \sqrt{16-4(1)(-4)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16+16}}{2}$$

$$= \frac{-4 \pm 4\sqrt{2}}{2}$$

$$x = -2 \pm 2\sqrt{2}$$

$x = -2 + 2\sqrt{2}$ and $x = -2 - 2\sqrt{2}$

Hence $x = -2 + 2\sqrt{2} \in [-1,6]$

(iii) $f(x) = \sqrt{x} - \frac{x}{3}, x \in [0,9]$

Solution:

Given $f(x) = \sqrt{x} - \frac{x}{3}, x \in [0,9]$

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$$

Now $f'(x) = 0$

$$\frac{1}{2\sqrt{x}} - \frac{1}{3} = 0$$

$$\frac{1}{2\sqrt{x}} = \frac{1}{3}$$

$$2\sqrt{x} = 3$$

$$\sqrt{x} = \frac{3}{2}$$

Squaring on both sides, we get

$$x = \frac{9}{4} \in [0,9]$$

---

3. Explain why Lagrange's mean value theorem is not applicable to the following functions in the respective intervals:

(i) $f(x) = \frac{x+1}{x}, x \in [-1,2]$

Solution:

Given $f(x) = \frac{x+1}{x}, x \in [-1,2]$

$f(x)$ is not continuous at $x = 0$

Lagrange's mean value theorem is not applicable.

(ii) $f(x) = |3x + 1|, x \in [-1,3]$

Solution:

Given $f(x) = |3x + 1|, x \in [-1,3]$

$f(x)$ is continuous on $[-1,3]$

Now $3x + 1 = 0$

$$x = -\frac{1}{3}$$

$f(x)$ is not differentiable at $x = -\frac{1}{3}$

Lagrange's mean value theorem is not applicable.

Hint: tangent is parallel to secant $\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$

4. Using the Lagrange's mean value theorem determine the values of x at which the tangent is parallel to the secant line at the end points of the give interval:

(i) $f(x) = x^3 - 3x + 2, x \in [-2,2]$

Solution:

Given $f(x) = x^3 - 3x + 2, x \in [-2,2]$

Here $a = -2$ and $b = 2$

$$f(-2) = -8 + 6 + 2 = 0$$

$$f(2) = 8 - 6 + 2 = 4$$

$$f'(x) = 3x^2 - 3x$$

$$f'(c) = 3c^2 - 3c$$

By Lagrange's mean value theorem,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$3c^2 - 3 = \frac{4-0}{2+2} = \frac{4}{4}$$

$$3c^2 - 3 = 1$$

$$3c^2 = 4$$

$$c^2 = \frac{4}{3}$$

$$c = \pm\frac{2}{\sqrt{3}} \in [-2,2]$$

(ii) $f(x) = (x-2)(x-7), x \in [3,11]$

Solution:

Given $f(x) = (x-2)(x-7), x \in [3,11]$

$$f(x) = x^2 - 9x + 14$$

Here $a = 3$ and $b = 11$

$$f(3) = (1)(-4) = -4$$

$$f(11) = (9)(4) = 36$$

$$f'(x) = 2x - 9$$

$$f'(c) = 2c - 9$$

By Lagrange's mean value theorem,

$$2c - 9 = \frac{36+4}{11-3} = \frac{40}{8}$$

$$2c - 9 = 5$$

$$2c = 14$$

$$c = 7 \in [3,11]$$

---

5. Show that the value in the conclusion of the mean value theorem for

(i) $f(x) = \frac{1}{x}$ on a closed interval of positive numbers $[a, b]$ is $\sqrt{ab}$

Solution:

Given $f(x) = \frac{1}{x}, x \in [a, b]$

$$f(a) = \frac{1}{a} \text{ and } f(b) = \frac{1}{b}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f'(c) = -\frac{1}{c^2}$$

By Lagrange's mean value theorem,

$$\frac{-1}{c^2} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = \frac{a-b}{ab(b-a)}$$

$$\frac{-1}{c^2} = \frac{-(b-a)}{ab(b-a)}$$

$$\frac{1}{c^2} = \frac{1}{ab}$$

$$c^2 = ab$$

$$c = \sqrt{ab}$$

(ii) $f(x) = Ax^2 + Bx + C$ on any interval $[a, b]$ is $\frac{a+b}{2}$.

Solution:

Given $f(x) = Ax^2 + Bx + C, x \in [a, b]$

$$f(a) = Aa^2 + Ba + C \text{ and } f(b) = Ab^2 + Bb + C$$

$$f'(x) = 2Ax + B$$

$$f'(k) = 2Ak + B$$

By Lagrange's mean value theorem,

$$2Ak + B = \frac{Ab^2+Bb+C-Aa^2-Ba-C}{b-a}$$

$$2Ak + B = \frac{A(b^2-a^2)+B(b-a)}{b-a}$$

$$= \frac{A(b+a)(b-a)+B(b-a)}{b-a}$$

$$= \frac{(b-a)[A(b+a)+B]}{b-a}$$

$$2Ak + B = A(b + a) + B$$

$$2Ak = A(b + a)$$

$$k = \frac{b+a}{2}$$

---

6. A race car driver is kilometer stone 20. If his speed never exceeds 150 km/hr, what is the maximum kilometer he can reach in the next two hours.

Solution:

Here the interval is $[0,2]$

$$f(0) = 20 \text{ and } f(2) = ?$$

By Lagrange's mean value theorem,

$$f'(c) \le 150$$

$$\frac{f(2)-f(0)}{2-0} \le 150$$

$$f(2) - 20 \le 300$$

$$f(2) \le 320$$

Hence, $f(2) = 320\ km$

---

7. Suppose that for a function $f(x), f'(x) \le 1$ for all $1 \le x \le 4$. Show that $f(4) - f(1) \le 3$.

Solution:

Here the interval is $[1,4]$

$$f'(x) \le 1$$

By Lagrange's mean value theorem,

$$\frac{f(b)-f(a)}{b-a} \le 1$$

$$\frac{f(4)-f(1)}{4-1} \le 1$$

$$f(4) - f(1) \le 3$$

Hence proved.

---

8. Does there exist a differentiable function $f(x)$ such that $f(0) = -1, f(2) = 4$ and $f'(x) \le 2$ for all x. Justify your answer.

Solution:

Given $f(0) = -1$ and $f(2) = 4$

Here $a = 0$ and $b = 2$

By Lagrange's mean value theorem,

$$f'(x) = \frac{f(b)-f(a)}{b-a}$$

$$= \frac{f(2)-f(0)}{2-0}$$

$$= \frac{4+1}{2} = \frac{5}{2}$$

$$f'(x) = 2.5 \notin [0,2]$$

Hence this is not possible.

9. Show that there lies a point on the curve $f(x) = x(x+3)e^{\frac{-\pi}{2}}, -3 \leq x \leq 0$ where tangent drawn is parallel to the x- axis.

Solution:

Given $f(x) = x(x+3)e^{\frac{\pi}{2}}$

$$f(x) = (x^2+3)e^{\frac{\pi}{2}}$$

Tangent is parallel to x-axis

$(i.e)\quad f'(x) = 0$

$$f'(x) = (2x+3)e^{\frac{\pi}{2}}$$

$$(2x+3)e^{\frac{\pi}{2}} = 0$$

$$x = -\frac{3}{2} \in [-3,0]$$

10. Using mean value theorem prove that for, $a > 0, b > 0, |e^{-a} - e^{-b}| < |a - b|$.

Solution:

Let $f(x) = e^{-x}$

$$f'(x) = -e^{-x}$$

$f(x)$ is continuous on $[a, b]$

$f(x)$ is differentiable on (a, b)

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$-e^{-c} = \frac{e^{-b}-e^{-a}}{b-a} = \frac{-(e^{-a}-e^{-b})}{-(a-b)}$$

$$-e^{-c} = \frac{e^{-a}-e^{-b}}{a-b}$$

Taking modulus on both sides, we get

$$|-e^{-c}| = \left|\frac{e^{-a}-e^{-b}}{a-b}\right|$$

$$|e^{-c}| = \frac{|e^{-a}-e^{-b}|}{|a-b|} \qquad \text{----------(1)}$$

Wkt, $e^{-c} = \frac{1}{e^c} < 1 \qquad [e \approx 2.72]$

$$|e^{-c}| < 1$$

From (1) we get

$$\frac{|e^{-a}-e^{-b}|}{|a-b|} < 1$$

$$|e^{-a} - e^{-b}| < |a - b|$$

Hence proved.

Exercise 7.4

1. Write the Maclaurin series expansion of the following functions:

(i) e^x

Solution:

Let $f(x) = e^x$	$f(0) = 1$
$f'(x) = e^x$	$f'(0) = 1$
$f''(x) = e^x$	$f''(0) = 1$
$f'''(x) = e^x$	$f'''(0) = 1$

The Maclaurin expansion of $f(x)$ is,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(ii) $\sin x$

Solution:

Let $f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$
$f^4(x) = \sin x$	$f^4(0) = 0$
$f^5(x) = \cos x$	$f^5(0) = 1$

The Maclaurin expansion of $f(x)$ is,

$$\sin x = 0 + \frac{x}{1!} + 0 + \frac{-x^3}{3!} + 0 + \frac{x^5}{5!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

(iii) $\cos x$

Solution:

Let $f(x) = \cos x$	$f(0) = 1$
$f'(x) = -\sin x$	$f'(0) = 0$
$f''(x) = -\cos x$	$f''(0) = -1$
$f'''(x) = \sin x$	$f'''(0) = 0$
$f^4(x) = \cos x$	$f^4(0) = 1$

The Maclaurin expansion of $f(x)$ is,

$$\cos x = 1 + 0 + \frac{x^2(-1)}{2!} + 0 + \frac{x^4}{4!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

(iv) $\log(1-x); -1 \leq x < 1$

Solution:

Let $f(x) = \log(1-x)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{1-x}(-1) = \frac{-1}{1-x}$ $\qquad f'(0) = -1$

$f''(x) = -\frac{-1}{(1-x)^2}(-1) = -\frac{1}{(1-x)^2}$ $\qquad f''(0) = -1$

$f'''(x) = -\frac{-2}{(1-x)^3}(-1) = -\frac{2}{(1-x)^3}$ $\qquad f'''(0) = -2$

The Maclaurin expansion of $f(x)$ is,

$$\log(1-x) = 0 + \frac{x(-1)}{1!} + \frac{x^2(-1)}{2!} + \frac{x^3(-2)}{3!} + \cdots$$

$$\log(1-x) = -x - \frac{x^2}{2!} - \frac{2x^3}{3!} - \cdots$$

Hint: $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

(v) $\tan^{-1} x\,; -1 \le x \le 1$

Solution:

Let $f(x) = \tan^{-1} x$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$

$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots$ $\qquad f'(0) = 1$

$f''(x) = -2x + 4x^3 - 6x^5 + \cdots$ $\qquad f''(0) = 0$

$f'''(x) = -2 + 12x^2 - 30x^4 + \cdots$ $\qquad f'''(0) = -2$

$f^4(x) = 24x - 120x^3 + \cdots$ $\qquad f^4(0) = 0$

$f^5(x) = 24 - 360x^2$ $\qquad f^5(0) = 24$

The Maclaurin expansion of $f(x)$ is,

$$\tan^{-1} x = 0 + \frac{x}{1!} + 0 + \frac{x^3(-2)}{3!} + 0 + \frac{x^5(24)}{5!} + \cdots$$

$$\tan^{-1} x = x - \frac{2x^3}{6} + \frac{24x^5}{120} + \cdots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

(vi) $\cos^2 x$

Solution:

Let $f(x) = \cos^2 x$ $\qquad f(0) = 1$

$f'(x) = -2\cos x \sin x$

$f'(x) = -\sin 2x$ $\qquad f'(0) = 0$

$f''(x) = -2\cos 2x$ $\qquad f''(0) = -2$

$f'''(x) = 4\sin 2x$ $\qquad f'''(0) = 0$

$f^4(x) = 8\cos 2x$ $\qquad f^4(0) = 8$

The Maclaurin expansion of $f(x)$ is,

$$\cos^2 x = 1 + 0 + \frac{x^2(-2)}{2!} + 0 + \frac{x^4(8)}{4!} + \cdots$$

$$\cos^2 x = 1 - x^2 + \frac{x^4}{3} + \cdots$$

---

2. Write down the Taylor series expansion of the function $\log x$ about $x = 1$ upto three non-zero terms for $x > 0$.

Hint: Taylor series expansion is,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Solution:

Taylor's series of $\log x$ about $x = 1$

$f(x) = \log x$ $\qquad f(1) = 0$

$f'(x) = \frac{1}{x}$ $\qquad f'(1) = 1$

$f''(x) = \frac{-1}{x^2}$ $\qquad f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $\qquad f'''(1) = 2$

Taylor's series expansion of $f(x)$ is,

$$\log x = 0 + \frac{(x-1)}{1!} + \frac{(-1)(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} + \cdots$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots$$

---

3. Expand $\sin x$ in ascending powers $x - \frac{\pi}{4}$ upto three non-zero terms.

Solution:

$\sin x$ in ascending powers at $x - \frac{\pi}{4}$

$\sin x$ expansion at $x = \frac{\pi}{4}$

$f(x) = \sin x$ $\qquad f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x$ $\qquad f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x$ $\qquad f''\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}$

Taylor's series expansion of $f(x)$ is,

$$\sin x = \frac{1}{\sqrt{2}} + \frac{1}{1!\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2!\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 + \cdots$$

$$\sin x = \frac{1}{\sqrt{2}}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 + \cdots\right]$$

---

4. Expand the polynomial $f(x) = x^2 - 3x + 2$ in powers of $x - 1$.

Solution:

Taylor's series expansion of $f(x)$ at $x = 1$

$f(x) = x^2 - 3x + 2$ $\qquad f(1) = 0$

$f'(x) = 2x - 3$ $\qquad f'(1) = -1$

$f''(x) = 2$ $\qquad f''(1) = 2$

$f'''(x) = 0$

Taylor's series expansion of $f(x)$ is,

$$x^2 - 3x + 2 = 0 - \frac{1}{1!}(x-1) + \frac{2}{2!}(x-1)^2 + \cdots$$

$$x^2 - 3x + 2 = -(x-1) + (x-1)^2 + \cdots$$

Exercise 7.5

Evaluate the following limits, if necessary use L'Hopital Rule:

Hint: $\log\infty = \infty$ and $\log 0 = \infty$

Indeterminate forms $\left(\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0^0, 1^\infty, \infty^0, 0 \times \infty\right)$

1. $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2}$

Solution:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} \left(\frac{0}{0}\ form\right)$$

By l'Hopital rule,

$$= \lim_{x\to 0} \frac{\sin x}{2x} \left(\frac{0}{0}\ form\right)$$

$$= \lim_{x\to 0} \frac{\cos x}{2}$$

$$= \frac{1}{2}$$

2. $\lim\limits_{x\to \infty} \frac{2x^2-3}{x^2-5x+3}$

Solution:

$$\lim_{x\to\infty} \frac{2x^2-3}{x^2-5x+3} = \lim_{x\to\infty} \frac{2-\frac{3}{x^2}}{1-\frac{5}{x}+\frac{3}{x^2}} = \frac{2}{1} = 2$$

3. $\lim\limits_{x\to \infty} \frac{x}{\log x}$

Solution:

$$\lim_{x\to\infty} \frac{x}{\log x} \left(\frac{\infty}{\infty}\ form\right)$$

By l'Hopital rule,

$$= \lim_{x\to\infty} \frac{1}{\frac{1}{x}}$$

$$= \infty$$

4. $\lim\limits_{x\to \frac{\pi}{2}} \frac{\sec x}{\tan x}$

Solution:

$$\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{1}{\cos x} \times \frac{\cos x}{\sin x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{1}{\sin x}$$

$$= 1$$

Hint: $e^\infty = \infty$ and $e^{-\infty} = 0$

5. $\lim\limits_{x\to\infty} e^{-x}\sqrt{x}$

Solution:

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^x} \left(\frac{\infty}{\infty}\ form\right)$$

By l'Hopital rule,

$$= \lim_{x\to\infty} \frac{1}{2\sqrt{x}e^x}$$

$$= 0$$

6. $\lim\limits_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

Solution:

$$\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x-\sin x}{x\sin x} \left(\frac{0}{0}\ form\right)$$

By l'Hopiatl rule,

$$= \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} \left(\frac{0}{0}\ form\right)$$

By l'Hopital rule,

$$= \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x + x(-\sin x)}$$

$$= 0$$

7. $\lim\limits_{x\to 1^+} \left(\frac{2}{x^2-1} - \frac{x}{x-1}\right)$

Solution:

$$\operatorname{Lim}_{x\to 1^+} \left(\frac{2}{x^2-1} - \frac{x}{x-1}\right) = \lim_{x\to 1^+} \left(\frac{2}{(x+1)(x-1)} - \frac{x}{x-1}\right)$$

$$= \lim_{x\to 1^+} \frac{2-x(x+1)}{(x+1)(x-1)}$$

$$= \lim_{x\to 1^+} \frac{1-x^2-x}{(x+1)(x-1)} \left(\frac{0}{0}\ form\right)$$

By l'Hopital rule,

$$= \lim_{x\to 1^+} \frac{-2x-1}{2x}$$

$$= -\frac{3}{2}$$

8. $\lim\limits_{x\to 0^+} x^x$

Solution:

$$\lim_{x\to 0^+} x^x \quad (0^0 form)$$

Let $f(x) = x^x$

$$\log f(x) = x\log x = \frac{\log x}{\frac{1}{x}}$$

$$\lim_{x\to 0^+} \log f(x) = \lim_{x\to 0^+} \frac{\log x}{\frac{1}{x}} \left(\frac{\infty}{\infty}\ form\right)$$

By l'Hopital rule,

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}}$$

$$= \lim_{x\to 0^+} (-x)$$

$$\lim_{x\to 0^+} \log f(x) = 0$$

$$\log \lim_{x\to 0^+} x^x = 0$$

$$\lim_{x\to 0^+} x^x = e^0 = 1$$

---

9. $\lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right)^x$

Solution:

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x \ (1^\infty \ form)$$

Let $f(x) = \left(1+\frac{1}{x}\right)^x$

$$\text{Log} f(x) = x\log\left(1+\frac{1}{x}\right)$$

$$\lim_{x\to\infty} \log f(x) = \lim_{x\to\infty} \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \left(\frac{0}{0} \ form\right)$$

By l'Hopital rule,

$$= \lim_{x\to\infty} \frac{\frac{1}{\left(1+\frac{1}{x}\right)}\left(\frac{-1}{x^2}\right)}{\frac{-1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{\left(1+\frac{1}{x}\right)}$$

$$= 1$$

$$\lim_{x\to\infty} \log f(x) = 1$$

$$\log \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = 1$$

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$$

---

10. $\lim\limits_{x\to\frac{\pi}{2}} (\sin x)^{\tan x}$

Solution:

$$\lim_{x\to\frac{\pi}{2}} (\sin x)^{\tan x} \ (1^\infty \ form)$$

Let $f(x) = (\sin x)^{\tan x}$

$$\log f(x) = \tan x \log(\sin x) = \frac{\log(\sin x)}{\frac{1}{\tan x}}$$

$$\lim_{x\to\frac{\pi}{2}} \log f(x) = \lim_{x\to\frac{\pi}{2}} \frac{\log(\sin x)}{\cot x} \ \left(\frac{0}{0} \ form\right)$$

By l'Hopital rule,

$$= \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\sin x}(\cos x)}{-cosec^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}} -\cos x \sin x$$

$$= 0$$

$$\lim_{x\to\frac{\pi}{2}} \log f(x) = 0$$

$$\log \lim_{x\to\frac{\pi}{2}} f(x) = 0$$

$$\lim_{x\to\frac{\pi}{2}} f(x) = e^0 = 1$$

$$\lim_{x\to\frac{\pi}{2}} (\sin x)^{\tan x} = 1$$

---

11. $\lim\limits_{x\to 0^+} (\cos x)^{\frac{1}{x^2}}$

Solution:

$$\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}} \ (1^\infty \ form)$$

Let $f(x) = (\cos x)^{\frac{1}{x^2}}$

$$\log f(x) = \frac{1}{x^2}\log(\cos x)$$

$$\lim_{x\to 0^+} \log f(x) = \lim_{x\to 0^+} \frac{\log(\cos x)}{x^2} \ \left(\frac{0}{0} \ form\right)$$

By l'Hopital rule,

$$= \lim_{x\to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{2x}$$

$$= \lim_{x\to 0^+} \frac{-\tan x}{2x} \ \left(\frac{0}{0} \ form\right)$$

By l'Hopital rule,

$$= \lim_{x\to 0^+} \frac{-\sec^2 x}{2}$$

$$= -\frac{1}{2}$$

$$\lim_{x\to 0^+} \log f(x) = -\frac{1}{2}$$

$$\log \lim_{x\to 0^+} f(x) = -\frac{1}{2}$$

$$\text{Lim}_{x\to 0^+} f(x) = e^{\frac{-1}{2}}$$

$$\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}} = \frac{1}{\sqrt{e}}$$

---

12. If an amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is $A = A_0\left(1+\frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, (that is $n \to \infty$), show that the amount after t years is $A = A_0e^{rt}$.

Solution:

$$\text{Let } y = \left(1+\frac{r}{n}\right)^{nt}$$

$$\log y = nt\log\left(1+\frac{r}{n}\right)$$

$$= \frac{\log\left(1+\frac{r}{n}\right)}{\frac{1}{nt}}$$

$$\lim_{n\to\infty}\log y = \lim_{n\to\infty}\frac{\log\left(1+\frac{r}{n}\right)}{\frac{1}{nt}}\left(\frac{0}{0}\ form\right)$$

By l'Hopital rule,

$$= \lim_{n\to\infty}\frac{\frac{1}{\left(1+\frac{r}{n}\right)}\left(\frac{-r}{n^2}\right)}{\frac{1}{t}\left(\frac{-1}{n^2}\right)}$$

$$= \lim_{n\to\infty}\frac{rt}{\left(1+\frac{r}{n}\right)}$$

$$= rt$$

$$\lim_{n\to\infty}\log y = rt$$

$$\log\lim_{n\to\infty} y = rt$$

$$\lim_{n\to\infty} y = e^{rt} \qquad \text{----------(1)}$$

Now, $A = A_0\left(1+\frac{r}{n}\right)^{rt}$

$$A = A_0 e^{rt} \qquad \text{From (1)}$$

Hence proved.

Exercise 7.6

1. Find the absolute extrema of the following functions on the given closed interval.

Hint: A procedure for finding absolute extrema:

Step 1: Find the critical numbers of $f(x)$ $[f'(x)=0]$

Step 2: Evaluate $f(x)$ at critical numbers and endpoints

Step 3: Largest and smallest value in step 2 is absolute maximum and minimum.

(i) $f(x) = x^2 - 12x + 10;\ [1,7]$

Solution:

Given $f(x) = x^2 - 12x + 10$

$$f'(x) = 2x - 12$$

Put $f'(x) = 0$

$$2x - 12 = 0$$

$$2x = 12$$

$$x = 6 \in [1,7]$$

The critical number is 6

$$f(1) = 1 - 12 + 10 = -1$$

$$f(2) = 4 - 24 + 10 = -10$$

$$f(6) = 36 - 72 + 10 = -26$$

Absolute maximum is -1

Absolute minimum is -26.

(ii) $f(x) = 3x^4 - 4x^3;\ [-1,2]$

Solution:

Given $f(x) = 3x^4 - 4x^3$

$$f'(x) = 12x^3 - 12x^2$$

Put $f'(x) = 0$

$$12x^3 - 12x^2 = 0$$

$$12x^2(x-1) = 0$$

$$x = 0,1 \in [-1,2]$$

The critical numbers are 0,1

$$f(0) = 0$$

$$f(1) = 3 - 4 = -1$$

$$f(-1) = 3 + 4 = 7$$

$$f(2) = 3(16) - 4(8) = 48 - 32 = 16$$

Absolute maximum is 16

Absolute minimum is -1.

(iii) $f(x) = 6x^{\frac{4}{3}} - 3x^{\frac{1}{3}};\ [-1,1]$

Solution:

Given $f(x) = 6x^{\frac{4}{3}} - 3x^{\frac{1}{3}}$

$$f'(x) = 6\left(\frac{4}{3}x^{\frac{1}{3}}\right) - 3\left(\frac{1}{3}x^{\frac{-2}{3}}\right)$$

$$f'(x) = 8x^{\frac{1}{3}} - \frac{1}{x^{\frac{2}{3}}}$$

Put $f'(x) = 0$

$$8x^{\frac{1}{3}} - \frac{1}{x^{\frac{2}{3}}} = 0$$

$$8x^{\frac{1}{3}}.x^{\frac{2}{3}} - 1 = 0$$

$$8x - 1 = 0$$

$$x = \frac{1}{8} \in [-1,1]$$

$f'(x)$ does not exist for $x = 0$

The critical numbers are $0, \frac{1}{8}$

$$f(0) = 0$$

$$f\left(\frac{1}{8}\right) = 6\left(\frac{1}{8}\right)^{\frac{4}{3}} - 3\left(\frac{1}{8}\right)^{\frac{1}{3}}$$

$$f\left(\frac{1}{8}\right) = 6\left(\frac{1}{2}\right)^{3\times\frac{4}{3}} - 3\left(\frac{1}{2}\right)^{3\times\frac{1}{3}}$$

$$= 6\left(\frac{1}{16}\right) - 3\left(\frac{1}{2}\right)$$

$$= \frac{3}{8} - \frac{3}{2}$$

$$f\left(\frac{1}{8}\right) = \frac{3-12}{8} = \frac{-9}{8}$$

$$f(-1) = 6 + 3 = 9$$

$$f(1) = 6 - 3 = 3$$

Absolute maximum is 9

Absolute minimum is $\frac{-9}{8}$.

(iv) $f(x) = 2\cos x + \sin 2x$; $\left[0, \frac{\pi}{2}\right]$

Solution:

Given $f(x) = 2\cos x + \sin 2x$

$$f'(x) = -2\sin x + \cos 2x\,(2)$$

$$f'(x) = -2\sin x + 2\cos 2x$$

Put $f'(x) = 0$

$$-2\sin x + 2\cos 2x = 0$$

$$2\cos 2x = 2\sin x$$

$$1 - 2\sin^2 x = \sin x$$

$$2\sin^2 x + \sin x - 1 = 0$$

$$(\sin x + 1)(2\sin x - 1) = 0$$

$$\sin x = -1 \text{ or } \sin x = \frac{1}{2}$$

$$x = -\frac{\pi}{2} \text{ or } x = \frac{\pi}{6}$$

$$\begin{array}{c|c} & -2 \\ 2 & -1 \\ \hline 2 & 2 \\ & 1 \end{array}$$

$\therefore$ The critical value is $\frac{\pi}{6} \in \left[0, \frac{\pi}{2}\right]$

$$f\left(\frac{\pi}{6}\right) = 2\cos\frac{\pi}{6} + \sin\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

$$f(0) = 2\cos 0 + \sin 0 = 2$$

$$f\left(\frac{\pi}{2}\right) = 2\cos\frac{\pi}{2} + \sin\pi = 0$$

Absolute maximum is $\frac{3\sqrt{3}}{2}$.

Absolute minimum is 0.

---

2. Find the intervals of monotonicities and hence find the local extremum for the following functions:

(i) $f(x) = 2x^3 + 3x^2 - 12x$

Solution:

Given $f(x) = 2x^3 + 3x^2 - 12x$

$$f'(x) = 6x^2 + 6x - 12$$

$$f'(x) = 6(x^2 + x - 2)$$

$$f'(x) = 6(x + 2)(x - 1)$$

$$\begin{array}{c|c} & -2 \\ 2 & -1 \\ \hline & 1 \end{array}$$

Put $f'(x) = 0$

$$6(x + 2)(x - 1) = 0$$

$$x = -2, 1$$

$\therefore$ The critical numbers are $-2, 1$

Intervals	Sign of $f'(x)$	Monotonicity
$(-\infty, -2)$	$+ve$	Strictly increasing
$(-2, 1)$	$-ve$	Strictly decreasing
$(1, \infty)$	$+ve$	Strictly increasing

At $x = -2$,

$f'(x)$ changes sign from $+ve$ to $-ve$

$f(x)$ attains local maximum at $x = -2$

$f(-2) = -16 + 12 + 24 = 20$

Local maximum is 20

At $x = 1$,

$f'(x)$ changes sign from $-ve$ to $+ve$

$f(x)$ attains minimum at $x = 1$

$f(1) = 2 + 3 - 12 = -7$

Local minimum is -7

(ii) $f(x) = \frac{x}{x-5}$

Solution:

Given $f(x) = \frac{x}{x-5}$

$$f'(x) = \frac{(x-5)(1) - x(1)}{(x-5)^2} = \frac{-5}{(x-5)^2}$$

$$f'(x) \neq 0$$

$f'(x)$ does not exist for $x = 5$

$\therefore$ The critical number is 5

Intervals	Sign of $f'(x)$	Monotonicity
$(-\infty, 5)$	$-ve$	Strictly decreasing
$(5, \infty)$	$-ve$	Strictly decreasing

At $x = 5$,

$f'(x)$ does not change its sign.

$\therefore$ There is no local extremum.

(iii) $f(x) = \frac{e^x}{1-e^x}$

Solution:

Given $f(x) = \frac{e^x}{1-e^x}$

$$f'(x) = \frac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2} = \frac{e^x - e^{2x} + e^{2x}}{(1-e^x)^2}$$

$$f'(x) = \frac{e^x}{(1-e^x)^2}$$

$f'(x)$ does not exist for $x = 0$.

∴ The critical number is 0

Intervals	Sign of $f'(x)$	Monotonicity
$(-\infty, 0)$	$+ve$	Strictly increasing
$(0, \infty)$	$+ve$	Strictly increasing

At $x = 0$,

$f'(x)$ does not change the sign.

∴ There is no local extremum.

(iv) $f(x) = \frac{x^3}{3} - \log x$

Solution:

Given $f(x) = \frac{x^3}{3} - \log x; \quad x > 0$

$$f'(x) = \frac{3x^2}{3} - \frac{1}{x}$$

$$f'(x) = x^2 - \frac{1}{x}$$

Put $f'(x) = 0$

$$x^2 - \frac{1}{x} = 0$$

$$x^3 - 1 = 0$$

$$x^3 = 1$$

$$x = 1$$

∴ The critical number is 1

Intervals	Sign of $f'(x)$	Monotonicity
$(0, 1)$	$-ve$	Strictly decreasing
$(1, \infty)$	$+ve$	Strictly increasing

At $x = 1$,

$f'(x)$ changes sign from $-ve$ to $+ve$

$f(x)$ attains local minimum at $x = 1$

$f(1) = \frac{1}{3}$

Local minimum value is $\frac{1}{3}$

(v) $f(x) = \sin x \cos x + 5, x \in [0, 2\pi]$

Hint: $\cos 2x = \cos^2 x - \sin^2 x$

Solution:

Given $f(x) = \sin x \cos x + 5$

$$f'(x) = \cos x\,(\cos x) + \sin x\,(-\sin x)$$

$$f'(x) = \cos^2 x - \sin^2 x$$

$$f'(x) = \cos 2x$$

Put $f'(x) = 0$

$$\cos 2x = 0$$

$$\cos 2x = \cos\frac{\pi}{2}$$

$$2x = (2n+1)\frac{\pi}{2}, n = 0,1,2,3$$

$$x = (2n+1)\frac{\pi}{4}, n = 0,1,2,3$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

∴ The critical values are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Intervals	Sign of $f'(x)$	Monotonicity
$\left(0, \frac{\pi}{4}\right)$	$+ve$	Strictly increasing
$\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$	$-ve$	Strictly decreasing
$\left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$	$+ve$	Strictly increasing
$\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$	$-ve$	Strictly decreasing
$\left(\frac{7\pi}{4}, 2\pi\right)$	$+ve$	Strictly increasing

At $x = \frac{\pi}{4}, \frac{5\pi}{4}$

$f'(x)$ changes sign from $+ve$ to $-ve$

$f(x)$ attains local maximum at $\frac{\pi}{4}, \frac{5\pi}{4}$

$$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 5 = \frac{1}{2} + 5$$

$$f\left(\frac{\pi}{4}\right) = \frac{11}{2}$$

$$f\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{4}\cos\frac{5\pi}{4} + 5$$

$$= \sin\left(\pi + \frac{\pi}{4}\right)\cos\left(\pi + \frac{\pi}{4}\right) + 5$$

$$= \left(-\sin\frac{\pi}{4}\right)\left(-\cos\frac{\pi}{4}\right) + 5$$

$$= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 5 = \frac{1}{2} + 5$$

$$f\left(\frac{5\pi}{4}\right) = \frac{11}{2}$$

Local maximum value is $\frac{11}{2}$

At $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$f'(x)$ changes sign from $-ve$ to $+ve$

$f(x)$ attains local minimum at $\frac{3\pi}{4}, \frac{7\pi}{4}$

$$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4}\cos\frac{3\pi}{4} + 5$$

$$= \sin\left(\pi - \frac{\pi}{4}\right)\cos\left(\pi - \frac{\pi}{4}\right) + 5$$

$$= \sin\frac{\pi}{4}\left(-\cos\frac{\pi}{4}\right) + 5$$

$$= \frac{1}{\sqrt{2}} \times \left(-\frac{1}{\sqrt{2}}\right) + 5 = -\frac{1}{2} + 5$$

$$f\left(\frac{3\pi}{4}\right) = \frac{9}{2}$$

$$f\left(\frac{7\pi}{4}\right) = \sin\left(2\pi - \frac{\pi}{4}\right)\cos\left(2\pi - \frac{\pi}{4}\right) + 5$$

$$= \left(-\sin\frac{\pi}{4}\right)\cos\frac{\pi}{4} + 5$$

$$= \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 5 = -\frac{1}{2} + 5$$

$$f\left(\frac{7\pi}{4}\right) = \frac{9}{2}$$

Local minimum value is $\frac{9}{2}$

Exercise 7.7

1. Find the intervals of concavity and points of inflexion for the following functions:

(i) $f(x) = x(x-4)^3$

Solution:

Given $f(x) = x(x-4)^3$

$$f'(x) = (x-4)^3(1) + x[3(x-4)^2]$$

$$= (x-4)^3 + 3x(x-4)^2$$

$$= (x-4)^2[x - 4 + 3x]$$

$$= (x-4)^2(4x-4)$$

$$f'(x) = 4(x-4)^2(x-1)$$

$$f''(x) = 4[(x-1)\,2(x-4) + (x-4)^2(1)]$$

$$= 4(x-4)[2(x-1) + x - 4]$$

$$= 4(x-4)[3x-6]$$

$$f''(x) = 12(x-4)(x-2)$$

Put $f''(x) = 0$

$$12(x-4)(x-2) = 0$$

$$x = 2, 4$$

$\therefore$ The points of inflection are at $x = 2, 4$

Intervals	Sign of $f''(x)$	Concavity
$(-\infty, 2)$	$+ve$	Concave upward
$(2, 4)$	$-ve$	Concave downward
$(4, \infty)$	$+ve$	Concave upward

At $x = 2, 4$

$f''(x)$ changes its sign

$$f(2) = 2(-8) = -16$$

$$f(4) = 4(0) = 0$$

$\therefore$ The points of inflection are $(2, -16)$ and $(4, 0)$

(ii) $f(x) = \sin x + \cos x, 0 < x < 2\pi$

Hint: $tan\ x$ negative in II and IV Quadrant

Solution:

Given $f(x) = \sin x + \cos x$

$$f'(x) = \cos x - \sin x$$

$$f''(x) = -\sin x - \cos x$$

Put $f''(x) = 0$

$$-\sin x - \cos x = 0$$

$$-\sin x = \cos x$$

$$-\frac{\sin x}{\cos x} = 1$$

$$-\tan x = 1$$

$$\tan x = -1$$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$ [2nd & 4th quadrant]

Intervals	Sign of $f''(x)$	Concavity
$\left(0, \frac{3\pi}{4}\right)$	$-ve$	Concave downward
$\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$	$+ve$	Concave upward
$\left(\frac{7\pi}{4}, 2\pi\right)$	$-ve$	Concave downward

At $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$f''(x)$ changes its sign

$\therefore$ The points of inflection are at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$$f\left(\frac{3\pi}{4}\right) = \sin\left(\pi - \frac{\pi}{4}\right) + \cos\left(\pi - \frac{\pi}{4}\right)$$

$$= \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= 0$$

$$f\left(\frac{7\pi}{4}\right) = \sin\left(2\pi - \frac{\pi}{4}\right) + \cos\left(2\pi - \frac{\pi}{4}\right)$$

$$= -\sin\frac{\pi}{4} + \cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= 0$$

$\therefore$ The points of inflection are $\left(\frac{3\pi}{4}, 0\right)$ and $\left(\frac{7\pi}{4}, 0\right)$

(iii) $f(x) = \frac{1}{2}(e^x - e^{-x})$

Solution:

Given $f(x) = \frac{1}{2}(e^x - e^{-x})$

$$f'(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f''(x) = \frac{1}{2}(e^x - e^{-x})$$

Put $f''(x) = 0$

$$\frac{1}{2}(e^x - e^{-x}) = 0$$

$$e^x - e^{-x} = 0$$

$$e^x - \frac{1}{e^x} = 0$$

$$e^{2x} - 1 = 0$$

$$e^{2x} = 1 = e^0$$

$$2x = 0$$

$$x = 0$$

Intervals	Sign of $f''(x)$	Concavity
$(-\infty, 0)$	$-ve$	Concave downward
$(0, \infty)$	$+ve$	Concave upward

At $x = 0$,

$f''(x)$ changes its sign

∴ The Point of inflection at $x = 0$

$$f(0) = \frac{1}{2}(e^0 - e^0)$$

$$f(0) = 0$$

∴ The Point of inflection is $(0, 0)$

2. Find the local extrema for the following functions using second derivative:

(i) $f(x) = -3x^5 + 5x^3$

Solution:

Given $f(x) = -3x^5 + 5x^3$

$$f'(x) = -15x^4 + 15x^2$$

$$f''(x) = -60x^3 + 30x$$

Put $f'(x) = 0$

$$-15x^4 + 15x^2 = 0$$

$$15x^2(-x^2 + 1) = 0$$

$$x = 0, -1, 1$$

At $x = 0$,

$f''(x) = 0$, No information.

At $x = 1$,

$$f''(x) = -30 < 0$$

$f(x)$ attains local maximum at $x = 1$

$$f(1) = -3 + 5 = 2$$

Local maximum value is 2

At $x = -1$,

$$f''(x) = 30 > 0$$

$f(x)$ attains local minimum at $x = -1$

$$f(-1) = 3 - 5 = -2$$

Local minimum value is -2

(ii) $f(x) = x \log x$

Solution:

Given $f(x) = x \log x$

$$f'(x) = \log x + x\left(\frac{1}{x}\right) = \log x + 1$$

$$f''(x) = \frac{1}{x}$$

Put $f'(x) = 0$

$$\log x + 1 = 0$$

$$\log x = -1$$

$$x = e^{-1} = \frac{1}{e}$$

At $x = \frac{1}{e}$, $\quad f''(x) = e > 0$

$f(x)$ attains local minimum at $x = \frac{1}{e}$

$$f\left(\frac{1}{e}\right) = \frac{1}{e}\log\frac{1}{e}$$

$$= \frac{1}{e}\log e^{-1}$$

$$f\left(\frac{1}{e}\right) = -\frac{1}{e}$$

Local minimum value is $\frac{-1}{e}$

(iii) $f(x) = x^2 e^{-2x}$

Solution:

Given $f(x) = x^2 e^{-2x}$

$$f'(x) = e^{-2x}(2x) + x^2 e^{-2x}(-2)$$

$$f'(x) = 2xe^{-2x} - 2x^2 e^{-2x} = e^{-2x}(2x - 2x^2)$$

$$f''(x) = (2x - 2x^2)e^{-2x}(-2) + e^{-2x}(2 - 4x)$$

$$= -4xe^{-2x} + 4x^2 e^{-2x} + 2e^{-2x} - 4xe^{-2x}$$

$$= 4x^2 e^{-2x} - 8xe^{-2x} + 2e^{-2x}$$

$$f''(x) = e^{-2x}(4x^2 - 8x + 2)$$

Put $f'(x) = 0$

$$e^{-2x}(2x - 2x^2) = 0$$

$$2xe^{-2x}(1 - x) = 0$$

$$x(1-x)=0$$
$$x=0,1$$

At $x=0$,

$$f''(x)=2>0$$

$f(x)$ attains local minimum at $x=0$

$$f(0)=0$$

Local minimum value is 0

At $x=1$,

$$f''(x)=e^{-2}(-2)<0$$

$f(x)$ attains local maximum at $x=1$

$$f(1)=e^{-2}=\frac{1}{e^2}$$

Local maximum value is $\frac{1}{e^2}$

---

3. For the function $f(x)=4x^3+3x^2-6x+1$ find the intervals of monotonicity, local extrema, intervals of concavity and points of inflection.

Solution:

Given $f(x)=4x^3+3x^2-6x+1$

$$f'(x)=12x^2+6x-6$$
$$f''(x)=24x+6$$

Put $f'(x)=0$

$$12x^2+6x-6=0$$
$$6(2x^2+x-1)=0$$
$$2x^2+x-1=0$$
$$(2x-1)(x+1)=0$$
$$x=\frac{1}{2},-1$$

∴ The critical number is $-1,\frac{1}{2}$

Intervals	Sign of $f'(x)$	Monotonicity
$(-\infty,-1)$	$+ve$	Strictly increasing
$\left(-1,\frac{1}{2}\right)$	$-ve$	Strictly decreasing
$\left(\frac{1}{2},\infty\right)$	$+ve$	Strictly increasing

At $x=-1$,

$f'(x)$ changes sign from $+ve$ to $-ve$

$f(x)$ attains local maximum at $x=-1$

$$f(-1)=-4+3+6+1=6$$

Local maximum value is 6

At $x=\frac{1}{2}$,

$f'(x)$ changes sign from $-ve$ to $+ve$

$f(x)$ attains local minimum at $x=\frac{1}{2}$

$$f\left(\frac{1}{2}\right)=\frac{4}{8}+\frac{3}{4}-\frac{6}{2}+1=\frac{4+6-24+8}{8}$$
$$f\left(\frac{1}{2}\right)=-\frac{6}{8}=-\frac{3}{4}$$

Local minimum value is $-\frac{3}{4}$

Put $f''(x)=0$

$$24x+6=0$$
$$24x=-6$$
$$x=-\frac{1}{4}$$

Intervals	Sign of $f''(x)$	Concavity
$\left(-\infty,-\frac{1}{4}\right)$	$-ve$	Concave downward
$\left(-\frac{1}{4},\infty\right)$	$+ve$	Concave upward

At $x=-\frac{1}{4}$,

$f''(x)$ changes its sign

∴ The point of inflection at $x=-\frac{1}{4}$

$$f\left(-\frac{1}{4}\right)=4\left(-\frac{1}{64}\right)+3\left(\frac{1}{16}\right)-6\left(-\frac{1}{4}\right)+1$$
$$=-\frac{1}{16}+\frac{3}{16}+\frac{6}{4}+1$$
$$=\frac{-1+3+24+16}{16}=\frac{42}{16}$$
$$f\left(-\frac{1}{4}\right)=\frac{21}{8}$$

∴ The point of inflection is $\left(-\frac{1}{4},\frac{21}{8}\right)$

Exercise 7.8

1. Find two positive numbers whose sum is 12 and their product is maximum.

Solution:

Let the two numbers be x,y

$$x+y=12$$
$$y=12-x$$

Product: $P(x)=x(12-x)=12x-x^2$

$$P'(x)=12-2x$$
$$P''(x)=-2$$

Put $P'(x)=0$

$$12-2x=0$$

$$2x = 12$$

$$x = 6$$

At $x = 6$,

$$P''(x) = -2 < 0$$

$P(x)$ attains maximum at $x = 6$

$$y = 12 - 6 = 6$$

$\therefore$ Two numbers are $6, 6$

2. Find two positive numbers whose product is 20 and their sum is minimum.

Solution:

Let the two positive numbers be x, y

$$xy = 20$$

$$y = \frac{20}{x}$$

Sum: $S(x) = x + \frac{20}{x}$

$$S'(x) = 1 - \frac{20}{x^2}$$

$$S''(x) = \frac{40}{x^3}$$

Put $S'(x) = 0$

$$1 - \frac{20}{x^2} = 0$$

$$1 = \frac{20}{x^2}$$

$$x^2 = 20$$

$$x = 2\sqrt{5}$$

At $x = 2\sqrt{5}$,

$$S''(x) = \frac{40}{(2\sqrt{5})^3} > 0$$

$S(x)$ attains minimum at $x = 2\sqrt{5}$

$$y = \frac{20}{2\sqrt{5}} = \frac{10}{\sqrt{5}}$$

$$= \frac{2\sqrt{5}\sqrt{5}}{\sqrt{5}}$$

$$y = 2\sqrt{5}$$

$\therefore$ Two numbers are $2\sqrt{5}, 2\sqrt{5}$

3. Find the smallest possible value of $x^2 + y^2$ given that $x + y = 10$

Solution:

Given $x + y = 10$

$$y = 10 - x$$

Let $f(x) = x^2 + y^2$

$$f(x) = x^2 + (10 - x)^2$$

$$f'(x) = 2x + 2(10 - x)(-1) = 2x - 20 + 2x$$

$$f'(x) = 4x - 20$$

$$f''(x) = 4$$

Put $f'(x) = 0$

$$4x - 20 = 0$$

$$4x = 20$$

$$x = 5$$

At $x = 5$,

$$f''(x) = 4 > 0$$

$f(x)$ attains minimum at $x = 5$

$$y = 10 - 5 = 5$$

$\therefore$ The minimum value is

$$x^2 + y^2 = 5^2 + 5^2 = 50$$

4. A garden is to be laid out in a rectangular area and protected by wire fence. What is the largest possible area of the fenced garden with 40 metres of wire.

Hint: Perimeter of garden = length of wire

Solution:

Let x and y be the length and breadth of the garden.

$$2x + 2y = 40$$

$$x + y = 20$$

$$y = 20 - x$$

Area: $A = xy$

$$A(x) = x(20 - x) = 20x - x^2$$

$$A'(x) = 20 - 2x$$

$$A''(x) = -2$$

Put $A'(x) = 0$

$$20 - 2x = 0$$

$$2x = 20$$

$$x = 10$$

At $x = 10$,

$$A''(x) = -2 < 0$$

$A(x)$ attains maximum at $x = 10$

$$y = 20 - 10 = 10$$

$\therefore$ Area $= 10 \times 10 = 100m^2$

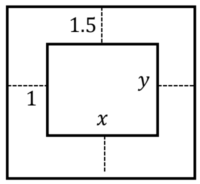
5. A rectangular page is to contain $24cm^2$ of print. The margins at the top and bottom of the page are $1.5cm$ and the margins at other sides of the page is $1cm$. What should be the dimensions of the page so that the area of the paper used is minimum?

Solution:

Let x and y be the dimensions of the printed portion.

$$xy = 24$$

$$y = \frac{24}{x}$$



The poster dimensions are $x+2, y+3$.

Area $A = (x+2)(y+3)$

$$A(x) = (x+2)\left(\frac{24}{x}+3\right)$$

$$A(x) = 24 + 3x + \frac{48}{x} + 6$$

$$A'(x) = 3 - \frac{48}{x^2}$$

$$A''(x) = \frac{96}{x^3}$$

Put $A'(x) = 0$

$$3 - \frac{48}{x^2} = 0$$

$$3 = \frac{48}{x^2}$$

$$x^2 = 16$$

$$x = 4$$

At $x = 4$,

$$A''(x) = \frac{96}{(4)^3} > 0$$

$A(x)$ attains minimum at $x = 4$

$$y = \frac{24}{4} = 6$$

$\therefore$ The dimensions of printed portion are $6cm, 9cm$

---

6. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain $1,80,000\ sq.mtrs$ in order to provide enough grass for herds. No fencing is needed along the river. What is the length of the minimum needed fencing material?

Solution:

Let x and y be the length and breadth of the garden.

$$xy = 1{,}80{,}000$$

$$y = \frac{1{,}80{,}000}{x}$$

Perimeter: $P = 2x + y$

$$P(x) = 2x + \frac{180000}{x}$$

$$P'(x) = 2 - \frac{180000}{x^2}$$

$$P''(x) = \frac{360000}{x^3}$$

Put $P'(x) = 0$

$$2 - \frac{180000}{x^2} = 0$$

$$2 = \frac{180000}{x^2}$$

$$x^2 = 90000$$

$$x = 300$$

At $x = 300$,

$$P''(x) = \frac{360000}{(300)^3} > 0$$

$P(x)$ attains minimum at $x = 300$

$$y = \frac{180000}{300} = 600$$

$\therefore$ The length of the fencing is

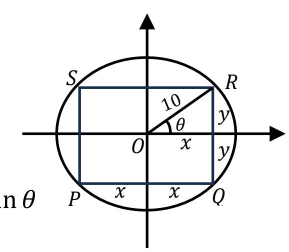
$$2x + y = 2(300) + 600$$

$$= 1200m$$

---

7. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius $10cm$.

Solution:



From diagram,

$$\cos\theta = \frac{x}{10} \qquad \sin\theta = \frac{y}{10}$$

$$x = 10\cos\theta \qquad y = 10\sin\theta$$

Length of rectangle:

$$2x = 2(10\cos\theta) = 20\cos\theta$$

Breadth of rectangle:

$$2y = 2(10\sin\theta) = 20\sin\theta$$

Area: $A = (2x)(2y) = 400\sin\theta\cos\theta$

$$A = 200 \times 2\sin\theta\cos\theta$$

$$A = 200\sin 2\theta$$

$$A' = 400\cos 2\theta$$

$$A'' = -800\sin 2\theta$$

Put $A' = 0$

$$400\cos 2\theta = 0$$

$$\cos 2\theta = 0 = \cos\frac{\pi}{2}$$

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

At $\theta = \frac{\pi}{4}$,

$$A'' = -800 \sin\frac{\pi}{2} = -800 < 0$$

A attains maximum at $\theta = \frac{\pi}{4}$

$\therefore$ The dimensions are $20\cos\frac{\pi}{4}, 20\sin\frac{\pi}{4}$

$$20 \times \frac{1}{\sqrt{2}}, 20 \times \frac{1}{\sqrt{2}}$$

$$10\sqrt{2}, 10\sqrt{2}\ cm$$

8. Prove that among all the rectangles of the given perimeter, the square has the maximum area.

Solution:

Let x and y be the dimension of rectangle.

Perimeter $P = 2x + 2y$

$$2y = P - 2x$$

$$y = \frac{P-2x}{2}$$

Area: $A = xy$

$$A(x) = x\left(\frac{P-2x}{2}\right) = \frac{P}{2}x - x^2$$

$$A'(x) = \frac{P}{2} - 2x$$

$$A''(x) = -2$$

Put $A'(x) = 0$

$$\frac{P}{2} - 2x = 0$$

$$2x = \frac{P}{2}$$

$$x = \frac{P}{4}$$

At $x = \frac{P}{4}$,

$$A''(x) = -2 < 0$$

$A(x)$ attains maximum at $x = \frac{P}{4}$

$$y = \frac{P - \frac{P}{2}}{2} = \frac{\frac{P}{2}}{2}$$

$$y = \frac{P}{4}$$

$\therefore$ The dimensions are $\frac{P}{4}, \frac{P}{4}$

Hence it is a square.

9. Find the dimensions of the largest rectangle that can be inscribed in a semi-circle of radius r cm.

Solution:

From diagram,

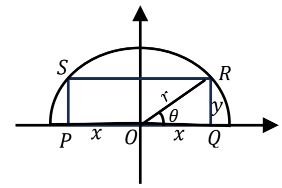
$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

Length of rectangle:

$$2x = 2(r\cos\theta) = 2r\cos\theta$$

Breadth of rectangle:



$$y = r\sin\theta$$

Area: $A = (2x)(y)$

$$A = 2r\cos\theta\,(r\sin\theta)$$

$$A = r^2(2\cos\theta\sin\theta)$$

$$A = r^2\sin 2\theta$$

$$A' = 2r^2\cos 2\theta$$

$$A'' = -4r^2\sin 2\theta$$

Put $A' = 0$

$$2r^2\cos 2\theta = 0$$

$$\cos 2\theta = 0 = \cos\frac{\pi}{2}$$

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

At $x = \frac{\pi}{4}$,

$$A'' = -4r^2\sin\frac{\pi}{2} = -4r^2 < 0$$

A attains maximum at $x = \frac{\pi}{4}$

$\therefore$ The dimensions of rectangle are $2r\cos\frac{\pi}{4}, r\sin\frac{\pi}{4}$

$$2r\left(\frac{1}{\sqrt{2}}\right), r\left(\frac{1}{\sqrt{2}}\right)$$

$$\sqrt{2}r, \frac{r}{\sqrt{2}}\ cm$$

10. A manufacturer wants to design an open box having a square base and a surface area of $108\ sq.cm$. Determine the dimensions of the box for the maximum volume.

Hint: Volume of cuboid $= lbh$

Solution:

Let the length and breadth of the box be x

Let the height of the box be y

$$x^2 + 4xy = 108 \qquad [\because \text{base is square}]$$

$$4xy = 108 - x^2$$

$$y = \frac{108 - x^2}{4x} = \frac{108}{4x} - \frac{x^2}{4x}$$

$$y = \frac{27}{x} - \frac{x}{4}$$

Volume $V = x^2 y$

$$V = x^2\left(\frac{27}{x} - \frac{x}{4}\right) = 27x - \frac{x^3}{4}$$

$$V' = 27 - \frac{3x^2}{4}$$

$$V'' = -\frac{6x}{4} = -\frac{3x}{2}$$

Put $V' = 0$

$$27 - \frac{3x^2}{4} = 0$$

$$\frac{3x^2}{4} = 27$$

$$x^2 = 36$$

$$x = 6$$

At $x = 6$,

$$V'' = -\frac{3}{2}(6) = -9 < 0$$

V attains maximum at $x = 6$

$$y = \frac{27}{6} - \frac{6}{4} = \frac{9}{2} - \frac{3}{2}$$

$$y = 3$$

∴ The length and breadth are $6cm$ and height is $3cm$

---

11. The volume of a cylinder is given by the formula $V = \pi r^2 h$. Find the greatest value of V if $r + h = 6$.

Solution:

Given $V = \pi r^2 h$

$$r + h = 6$$

$$h = 6 - r$$

$$V = \pi r^2(6 - r) = 6\pi r^2 - \pi r^3$$

$$V' = 12\pi r - 3\pi r^2$$

$$V'' = 12\pi - 6\pi r$$

Put $V' = 0$

$$12\pi r - 3\pi r^2 = 0$$

$$3\pi r(4 - r) = 0$$

$$r = 0, 4$$

At $r = 0$, $V = 0$

∴ The minimum volume is 0

At $r = 4$,

$$V'' = 12\pi - 24\pi = -12\pi < 0$$

V attains maximum at $r = 4$

$$h = 6 - 4$$

$$h = 2$$

∴ The maximum volume is,

$$V = \pi(16)(2) = 32\pi$$

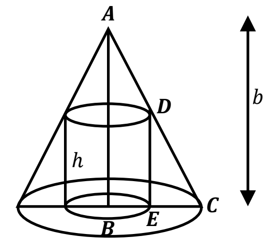
---

12. A hollow cone with base radius a cm and height b cm is placed on a table. Show that the volume of the largest cylinder can be hidden underneath is $\frac{4}{9}$ times volume of cone.

Hint: Volume of cylinder $= \pi r^2 h$

Volume of cone $= \frac{1}{3}\pi r^2 h$

Solution:



Cone:

$$r = BC = a\ cm$$

$$h = AB = b\ cm$$

cylinder:

Let $BE = r$

From diagram,

$$\frac{AB}{BC} = \frac{DE}{EC} \quad [\because \Delta ABC\ \&\ \Delta DEC\ are\ similar]$$

$$\frac{b}{a} = \frac{h}{a-r}$$

$$h = \frac{b(a-r)}{a}$$

Volume $V = \pi r^2 h$

$$V = \pi r^2 \frac{b}{a}(a - r) \qquad \text{----------(1)}$$

$$V = \frac{\pi b}{a}(ar^2 - r^3)$$

$$V' = \frac{\pi b}{a}(2ar - 3r^2)$$

$$V'' = \frac{\pi b}{a}(2a - 6r)$$

Put $V' = 0$

$$\frac{\pi b}{a}(2ar - 3r^2) = 0$$

$$2ar - 3r^2 = 0$$

$$3r^2 = 2ar$$

$$r = \frac{2a}{3}$$

At $r = \frac{2a}{3}$,

$$V'' = \frac{\pi b}{a}\left(2a - 6 \times \frac{2a}{3}\right)$$

$$V'' = \frac{\pi b}{a}(2a - 4a)$$

$$V'' = \frac{\pi b}{a}(-2a) > 0$$

V attains maximum at $r = \frac{2a}{3}$

$$V = \pi\left(\frac{4a^2}{9}\right)\left(\frac{b}{a}\right)\left(a - \frac{2a}{3}\right) \qquad \text{From (1)}$$

$$= \left(\frac{4\pi ab}{9}\right)\left(\frac{a}{3}\right)$$

$$= \frac{4}{9}\left(\frac{1}{3}\pi a^2 b\right)$$

$$V = \frac{4}{9}(volume\ of\ cone)$$

<u>**Exercise 7.9**</u>

1. Vertical Asymptote, which is parallel to y-axis. The line $x = a$ is said to be vertical asymptote for the cure $y = f(x)$ if $\lim\limits_{x\to a^-} f(x) = \pm\infty$ or $\lim\limits_{x\to a^+} f(x) = \pm\infty$.

2. Horizontal Asymptote, which is parallel to x-axis. The line $y = L$ is said to be a horizontal asymptote the curve $y = f(x)$ if either $\lim\limits_{x\to+\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$.

3. Slant Asymptote: degree of numerator > degree of denominator, then dived the numerator by denominator, $y = quotient$.

1. Find the asymptotes of the following curves:

(i) $f(x) = \frac{x^2}{x^2-1}$

<u>**Solution:**</u>

Given $\quad f(x) = \frac{x^2}{x^2-1} = \frac{x^2}{(x+1)(x-1)}$

Vertical Asymptote:

$$\lim_{x\to 1^-} f(x) = -\infty \text{ and } \lim_{x\to 1^+} f(x) = +\infty$$

$$\lim_{x\to -1^-} f(x) = +\infty \text{ and } \lim_{x\to -1^+} f(x) = -\infty$$

$\therefore$ The Vertical asymptotes are $x = -1, 1$

Horizontal Asymptote:

$$f(x) = \frac{x^2}{x^2-1}$$

$$f(x) = \frac{x^2}{x^2\left(1-\frac{1}{x^2}\right)}$$

$$f(x) = \frac{1}{1-\frac{1}{x^2}}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{1-\frac{1}{x^2}} = \frac{1}{1-0} = 1$$

$\therefore$ The horizontal asymptote is $y = 1$

(ii) $f(x) = \frac{x^2}{x+1}$

<u>**Solution:**</u>

Given $f(x) = \frac{x^2}{x+1}$

Vertical Asymptote:

$$\lim_{x\to -1^-} f(x) = -\infty \text{ and } \lim_{x\to -1^+} f(x) = +\infty$$

$\therefore$ The vertical asymptote is $x = -1$

Slant Asymptote:

Divide x^2 by $x - 1$

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2 \\ & x^2 + x \\ \hline & -x \\ & -x-1 \\ \hline & \quad 1 \end{array} \qquad \begin{array}{l} \left(\because \frac{x^2}{x} = x\right) \\ \left(\because \frac{-x}{x} = -1\right) \end{array}$$

$\therefore$ The slant asymptote is $y = x - 1$

(iii) $f(x) = \frac{3x}{\sqrt{x^2+2}}$

<u>**Solution:**</u>

Let $y = \frac{3x}{\sqrt{x^2+2}}$

$$y^2 = \frac{9x^2}{x^2+2}$$

The denominator will not be zero for any real value of x.

Horizontal Asymptote:

$$y^2 = \frac{9x^2}{x^2\left(1+\frac{2}{x^2}\right)}$$

$$y^2 = \frac{9}{1+\frac{2}{x^2}}$$

$$\lim_{x\to\infty} y^2 = \lim_{x\to\infty} \frac{9}{1+\frac{2}{x^2}} = \frac{9}{1-0} = 9$$

$$y^2 = 9$$

$\therefore$ The horizontal asymptote is $y = \pm 3$

(iv) $f(x) = \frac{x^2-6x-1}{x-3}$

<u>**Solution:**</u>

Given $f(x) = \frac{x^2-6x-1}{x+3}$

Vertical Asymptote:

$$\lim_{x\to -3^-} f(x) = -\infty \text{ and } \lim_{x\to -3^+} f(x) = +\infty$$

$\therefore$ The vertical asymptote is $x = -3$

Slant Asymptote:

Dived $x^2 - 6x - 1$ by $x + 3$

$$\begin{array}{r|l} & x-9 \\ \hline x+3 & x^2-6x-1 \\ & x^2+3x \\ \hline & -9x-1 \\ & -9x-27 \\ \hline & 26 \end{array}$$

$\left(\because \frac{x^2}{x} = x\right)$

$\left(\because -\frac{9x}{x} = -9\right)$

$\therefore$ The slant asymptote is $y = x - 9$

(v) $f(x) = \frac{x^2+6x-4}{3x-6}$

Solution:

Given $f(x) = \frac{x^2+6x-4}{3x-6}$

$f(x) = \frac{x^2+6x-4}{3(x-2)}$

Vertical Asymptote:

$$\lim_{x\to 2^-} f(x) = -\infty \text{ and } \lim_{x\to 2^+} f(x) = +\infty$$

$\therefore$ The vertical asymptote is $x = 2$

Slant Asymptote:

Divide $x^2 + 6x - 4$ by $3x - 6$

$$\begin{array}{r|l} & \frac{x}{3}+\frac{8}{3} \\ \hline 3x-6 & x^2+6x-4 \\ & x^2-2x \\ \hline & 8x-4 \\ & 8x-16 \\ \hline & 12 \end{array}$$

$\left(\because \frac{x^2}{3x} = \frac{x}{3}\right)$

$\left(\because \frac{8x}{3x} = \frac{8}{3}\right)$

$\therefore$ The slant asymptote is $y = \frac{x}{3} + \frac{8}{3}$

Hint: Procedure for sketching graph,

1. Domain and Range

2. Intercepts and Symmetry

3. Critical points ($f'(x) = 0$)

4. Local extrema

5. Intervals of Concavity

6. Point of Inflexions

7. Asymptotes

2. Sketch the graphs of the following functions:

(i) $y = -\frac{1}{3}(x^3 - 3x + 2)$

Solution:

Given $y = -\frac{1}{3}(x^3 - 3x + 2)$

(i) Domain and Range $\in R$

(ii) Intercepts:

Put $y = 0$

$-\frac{1}{3}(x^3 - 3x + 2) = 0$

$x^3 - 3x + 2 = 0$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline 1 & 1 & 1 & -2 & \boxed{0} \\ & 0 & 1 & 2 & \\ \hline & 1 & 2 & \boxed{0} & \end{array}$$

$(x - 1)(x - 1)(x + 2) = 0$

$x = 1, 1, -2$

$\therefore$ The x- intercepts are $1, -2$

Put $x = 0$, $\quad y = -\frac{2}{3}$

$\therefore$ The y-intercept is $-\frac{2}{3}$

It has no symmetrical property

(iii) Critical points:

$y = -\frac{1}{3}(x^3 - 3x + 2)$

$y' = -\frac{1}{3}(3x^2 - 3)$

Put $y' = 0$,

$3x^2 - 3 = 0$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

$\therefore$ The critical numbers are $-1, 1$

(iv) Local Extrema:

$y' = -\frac{1}{3}(3x^2 - 3)$

$y'' = -\frac{1}{3}(6x)$

$y'' = -2x$

At $x = 1$,

$y'' = -2 < 0$

y attains local maximum at $x = 1$

$y = -\frac{1}{3}(1 - 3 + 2) = 0$

$\therefore$ The local maximum value is 0.

At $x = -1$,

$$y'' = 2 > 0$$

y attains local minimum at $x = -1$

$$y = -\frac{1}{3}(-1 + 3 + 2)$$

$$y = -\frac{4}{3}$$

$\therefore$ The local minimum value is $-\frac{4}{3}$.

(v) Intervals of Concavity:

Put $y'' = 0$

$$-2x = 0$$

$$x = 0$$

Intervals	**Sign of y''**	**Concavity**
$(-\infty, 0)$	$+ve$	Concave upward
$(0, \infty)$	$-ve$	Concave downward

(vi) Point of Inflection:

At $x = 0$,

y'' changes its sign

$\therefore$ The point of inflection at $x = 0$

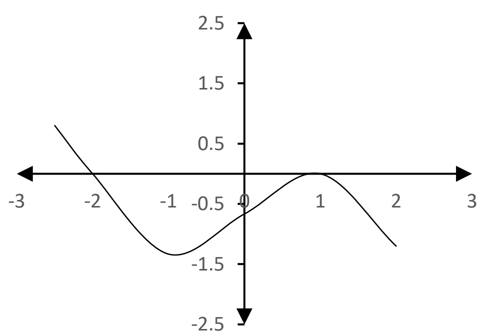
$$y = -\frac{2}{3}$$

$\therefore$ The point of inflection is $\left(0, -\frac{2}{3}\right)$

(vii) Asymptotes:

It has no asymptote

The curve is,



(ii) $y = x\sqrt{4 - x}$

Solution:

Given $y = x\sqrt{4 - x}$

(i) Domain:

$$4 - x \geq 0$$

$$4 \geq x$$

Domain: $(-\infty, 4]$

Range: $\left(-\infty, \frac{16}{3\sqrt{3}}\right]$

(ii) Intercepts:

Put $y = 0$

$$x\sqrt{4 - x} = 0$$

$$x = 0, 4$$

$\therefore$ The x- intercepts are $0, 4$

Put $x = 0$, $\quad y = 0$

$\therefore$ The y-intercept is 0

It has no symmetrical property

(iii) Critical points:

$$y = x\sqrt{4 - x}$$

$$y' = \sqrt{4 - x}\,(1) + x\frac{1}{2\sqrt{4-x}}(-1)$$

$$= \sqrt{4 - x} - \frac{x}{2\sqrt{4-x}}$$

$$= \frac{2(4-x)-x}{2\sqrt{4-x}}$$

$$y' = \frac{8-3x}{2\sqrt{4-x}}$$

Put $y' = 0$,

$$\frac{8-3x}{2\sqrt{4-x}} = 0$$

$$x = \frac{8}{3}, 4$$

$\therefore$ The critical number is $\frac{8}{3}, 4$

(iv) Local Extrema:

$$y' = \frac{8-3x}{2\sqrt{4-x}}$$

$$y'' = \frac{2\sqrt{4-x}(-3)-(8-3x)\frac{2}{2\sqrt{4-x}}(-1)}{\left(2\sqrt{4-x}\right)^2}$$

$$y'' = \frac{-6\sqrt{4-x}+\frac{8-3x}{\sqrt{4-x}}}{4(4-x)}$$

$$y'' = \frac{-6(4-x)+(8-3x)}{4(4-x)\sqrt{4-x}}$$

$$y'' = \frac{3x-16}{4(4-x)^{\frac{3}{2}}}$$

At $x = \frac{8}{3}$,

$$y'' < 0$$

y attains local maximum at $x = \frac{8}{3}$

$$y = \frac{8}{3}\sqrt{4 - \frac{8}{3}} = \frac{8}{3}\sqrt{\frac{4}{3}}$$

$$y = \frac{16}{3\sqrt{3}}$$

$\therefore$ The local maximum value is $\frac{16}{3\sqrt{3}}$.

(v) Intervals of Concavity:

Put $y'' = 0$

$$\frac{3x-16}{4(4-x)^{\frac{3}{2}}} = 0$$

$x = 4$ | $3x - 16 = 0$

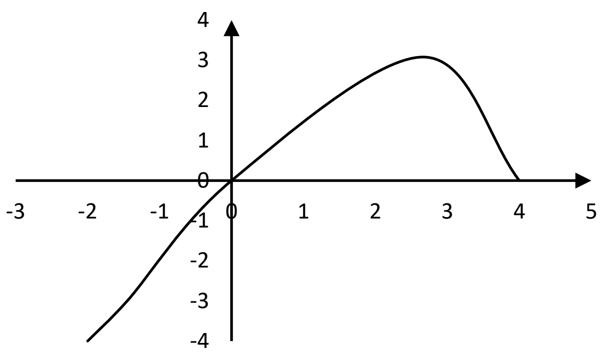
$x = \frac{16}{3} \notin (-\infty, 4]$

Intervals	**Sign of y''**	**Concavity**
$(-\infty, 4)$	$-ve$	Concave downward

(vi) No point of inflection.

(vii) It has no asymptote

The curve is,



(iii) $y = \frac{x^2+1}{x^2-4}$

Solution:

Given $y = \frac{x^2+1}{x^2-4}$

(i) Domain: $\mathbb{R} - \{-2,2\}$

Range: $(-\infty, \infty)$

(ii) Intercepts:

Put $y = 0$

$$\frac{x^2+1}{x^2-4} = 0$$

$$x^2 + 1 = 0$$

$$x^2 = -1$$

The curve does not cut the x-axis.

Put $x = 0$, $y = -\frac{1}{4}$

$\therefore$ The y-intercept is $-\frac{1}{4}$

Symmetry:

$$f(x) = f(-x)$$

$\therefore$ The curve is symmetrical about y-axis.

(iii) Critical Points:

$$y = \frac{x^2+1}{x^2-4}$$

$$y' = \frac{(x^2-4)(2x)-(x^2+1)(2x)}{(x^2-4)^2}$$

$$= \frac{2x^3-8x-2x^3-2x}{(x^2-4)^2}$$

$$y' = \frac{-10x}{(x^2-4)^2}$$

Put $y' = 0$

$$\frac{-10x}{(x^2-4)^2} = 0$$

$-10x = 0$ | $(x^2 - 4) = 0$

$x = 0$ | $x^2 = 4$

$x = \pm 2 \notin$ Domain

$\therefore$ The critical point is $x = 0$

(iv) Local Extrema:

$$y' = \frac{-10x}{(x^2-4)^2}$$

$$y'' = \frac{(x^2-4)^2(-10)-(-10x)2(x^2-4)(2x)}{(x^2-4)^4}$$

$$= \frac{(x^2-4)[-10x^2+40+40x^2]}{(x^2-4)^4}$$

$$y'' = \frac{(x^2-4)(30x^2+40)}{(x^2-4)^4}$$

At $x = 0$,

$$y'' < 0$$

y attains local maximum at $x = 0$

$$y = -\frac{1}{4}$$

$\therefore$ The local maximum value is $-\frac{1}{4}$

(v) Intervals of concavity:

Put $y'' = 0$

$$\frac{(x^2-4)(30x^2+40)}{(x^2-4)^4} = 0$$

$x^2 - 4 = 0$ | $30x^2 + 40 = 0$

$x = \pm 2$ | $x^2 = -\frac{4}{3}$ (not possible)

Intervals	**Sign of y''**	**Concavity**
$(-\infty,-2)$	$-ve$	Concave downward
$(-2,2)$	$+ve$	Concave upward
$(2,\infty)$	$-ve$	Concave downward

(vi) Points of Inflection:

Since $-2, 2$ are not in the domain.

$\therefore$ No point of inflection.

(vii) Asymptotes:

Given $y=\frac{x^2+1}{x^2-4}=\frac{x^2+1}{(x+2)(x-2)}$

$\therefore$ The vertical asymptote is $x=-2,2$

Horizontal Asymptote:

$$y=\frac{x^2+1}{x^2-4}$$

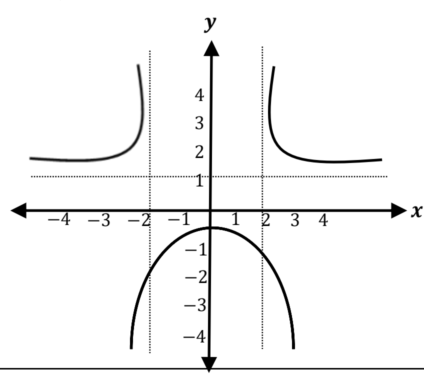
$$y=\frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2\left(1-\frac{4}{x^2}\right)}$$

$$y=\frac{1+\frac{1}{x^2}}{1-\frac{4}{x^2}}$$

$$\lim_{x\to\infty} y=\lim_{x\to\infty}\frac{1+\frac{1}{x^2}}{1-\frac{4}{x^2}}=1$$

$\therefore$ The horizontal asymptote is $y=1$

The curve is,



(iv) $y=\frac{1}{1+e^{-x}}$

Solution:

Given $y=\frac{1}{1+e^{-x}}$

(i) Domain: $(-\infty,\infty)$

As $x\to-\infty, y=0$

As $x\to\infty, y=1$

Range: $(0,1)$

(ii) Intercepts:

Put $y=0$, not possible

Put $x=0$,

$$y=\frac{1}{1+1}=\frac{1}{2}$$

$\therefore$ The y-intercept is $\frac{1}{2}$

It has no symmetrical property.

(iii) Critical points:

$$y=\frac{1}{1+e^{-x}}=(1+e^{-x})^{-1}$$

$$y'=-(1+e^{-x})^{-2}(e^{-x})(-1)$$

$$y'=\frac{e^{-x}}{(1+e^{-x})^2}$$

Put $y'=0$

$$\frac{e^{-x}}{(1+e^{-x})^2}=0$$

$$e^{-x}=0 \text{ (not possible)}$$

$\therefore$ It is an increasing function.

(iv) Local Extrema:

$$y'=\frac{e^{-x}}{(1+e^{-x})^2}$$

$$y''=\frac{(1+e^{-x})^2(-e^{-x})-e^{-x}2(1+e^{-x})(-e^{-x})}{(1+e^{-x})^4}$$

$$=\frac{(1+e^{-x})[-e^{-x}(1+e^{-x})+2e^{-2x}]}{(1+e^{-x})^4}$$

$$=\frac{(1+e^{-x})[-e^{-x}-e^{-2x}+2e^{-2x}+]}{(1+e^{-x})^4}$$

$$=\frac{(-e^{-x}+e^{-2x})}{(1+e^{-x})^3}$$

$$y''=\frac{e^{-x}(-1+e^{-x})}{(1+e^{-x})^3}$$

Put $y''=0$

$$\frac{e^{-x}(-1+e^{-x})}{(1+e^{-x})^3}=0$$

$$-1+e^{-x}=0$$

$$e^{-x}=1=e^0$$

$$x=0$$

(vi) Point of inflections:

At $x=0$,

y'' changes its sign

$\therefore$ The point of inflection at $x=0$

$$y=\frac{1}{2}$$

$\therefore$ The point of inflection is $\left(0, \frac{1}{2}\right)$

(vii) Asymptotes:

Horizontal Asymptote:

$$y = \frac{1}{1+e^{-x}}$$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{1}{1+e^{-\infty}} = \frac{1}{1-0} = 1$$

$$\lim_{x\to-\infty} y = \lim_{x\to\infty} \frac{1}{1+e^{\infty}} = \frac{1}{\infty} = 0$$

$\therefore$ The horizontal asymptote is $y = 0, 1$

The curve is,