CHAPTER – 6

Applications of Vector Algebra

Exercise 6.1

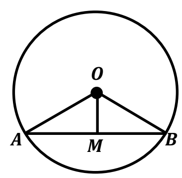
1. Prove by vector method that if a line is drawn from the centre of a circle to the midpoint of a chord, then the line is perpendicular to the chord.

Hint: $\overrightarrow{OA}^2 = OA^2$

Solution:

Let O be the centre of the circle and M be the midpoint of the chord AB.

We have $OA = OB(radii)$



T.P: $\overrightarrow{OM} \perp \overrightarrow{AB}$

$$\overrightarrow{OM}.\overrightarrow{AB} = \left(\frac{\overrightarrow{OB}+\overrightarrow{OA}}{2}\right).\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$

$$= \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OA}\right).\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$

$$= \frac{1}{2}(OB^2 - OA^2)$$

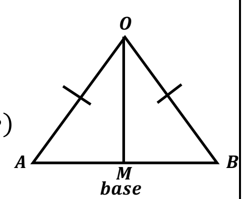
$$= \frac{1}{2}(OA^2 - OA^2) \qquad [\because OA = OB]$$

$$\overrightarrow{OM}.\overrightarrow{AB} = 0$$

Hence $\overrightarrow{OM} \perp \overrightarrow{AB}$.

2. Prove by vector method that the median to the base of an isosceles triangle is perpendicular to the base.

Solution:



Let M be the median of the base AB

We have $OA = OB(isosceles\ triangle)$

T.P: $\overrightarrow{OM} \perp \overrightarrow{AB}$

$$\overrightarrow{OM}.\overrightarrow{AB} = \left(\frac{\overrightarrow{OB}+\overrightarrow{OA}}{2}\right).\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$

$$= \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OA}\right).\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$

$$= \frac{1}{2}(OB^2 - OA^2)$$

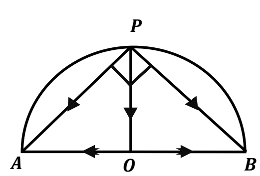
$$= \frac{1}{2}(OA^2 - OA^2) \qquad [\because OA = OB]$$

$$\overrightarrow{OM}.\overrightarrow{AB} = 0$$

Hence $\overrightarrow{OM} \perp \overrightarrow{AB}$.

3. Prove by vector method that an angle is a semi-circle is a right angle.

Solution:



Let O be the origin.

We have $OA = OB = OP(radii)$

T.P: $\angle APB = 90°$

i.e., T.P: $\overrightarrow{PA} \perp \overrightarrow{PB}$

$$\overrightarrow{PA}.\overrightarrow{PB} = \left(\overrightarrow{PO} + \overrightarrow{OA}\right).\left(\overrightarrow{PO} + \overrightarrow{OB}\right)$$

$$= \left(\overrightarrow{PO} + \overrightarrow{OA}\right).\left(\overrightarrow{PO} - \overrightarrow{OA}\right) \ [\because \overrightarrow{OB} = -\overrightarrow{OA}]$$

$$= (PO)^2 - (OA)^2$$

$$= (PO)^2 - (PO)^2$$

$$\overrightarrow{PA}.\overrightarrow{PB} = 0$$

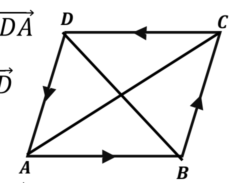
$\Rightarrow$ $\overrightarrow{PA}$ is perpendicular to $\overrightarrow{PB}$

$\Rightarrow \angle APB = 90°$

Hence proved.

4. Prove by vector method that the diagonals of a rhombus bisect each other at right angles.

Solution:



We have $\overrightarrow{AB} = -\overrightarrow{CD}$ and $\overrightarrow{BC} = -\overrightarrow{DA}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$

T.P: $\overrightarrow{AC} \perp \overrightarrow{BD}$

$$\overrightarrow{AC}.\overrightarrow{BD} = \left(\overrightarrow{AB} + \overrightarrow{BC}\right).\left(\overrightarrow{BC} + \overrightarrow{CD}\right)$$

$$= \left(\overrightarrow{BC} + \overrightarrow{AB}\right).\left(\overrightarrow{BC} - \overrightarrow{AB}\right)$$

$$= BC^2 - AB^2$$

$$= AB^2 - AB^2 \quad [\because All\ sides\ are\ equal]$$

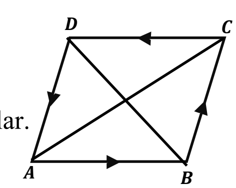
$$\overrightarrow{AC}.\overrightarrow{BD} = 0$$

$\Rightarrow \overrightarrow{AC} \perp \overrightarrow{BD}$

Hence diagonals are perpendicular to each other in rhombus.

5. Using vector method, prove that if the diagonals of a parallelogram are equal, then it is a rectangle.

Solution:



Let $ABCD$ be a parallelogram.

T.P: Adjacent sides are perpendicular.

Given $AC = BD$

Squaring on both sides, we get

$$\Rightarrow \qquad AC^2 = BD^2$$

$$\overrightarrow{AC}^2 = \overrightarrow{BD}^2$$

$$\left(\overrightarrow{AB} + \overrightarrow{BC}\right)^2 = \left(\overrightarrow{BC} + \overrightarrow{CD}\right)^2$$

$$\left(\overrightarrow{BC} + \overrightarrow{AB}\right)^2 = \left(\overrightarrow{BC} - \overrightarrow{AB}\right)^2$$

$\Rightarrow \quad BC^2 + 2\overrightarrow{BC}.\overrightarrow{AB} + AB^2 = BC^2 - 2\overrightarrow{BC}.\overrightarrow{AB} + AB^2$

$\Rightarrow \quad 4\overrightarrow{BC}.\overrightarrow{AB} = 0$

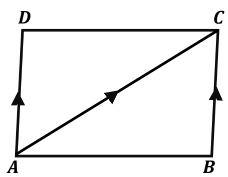
$\Rightarrow \quad \overrightarrow{BC}.\overrightarrow{AB} = 0$

$\Rightarrow \quad \overrightarrow{BC} \perp \overrightarrow{AB}$

$\therefore$ Adjacent sides are perpendicular.

Hence given parallelogram is a rectangle.

6. Prove by vector method that the area of a quadrilateral $ABCD$ having diagonals AC and BD is $\frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{BD}\right|$.



Hint: $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$

Solution:

Let $ABCD$ be a quadrilateral.

Vector area of quadrilateral $ABCD$ = Vector area of ΔABC + Vector area of ΔACD

$= \frac{1}{2}(\overrightarrow{AB} \times \overrightarrow{AC}) + \frac{1}{2}(\overrightarrow{AC} \times \overrightarrow{AD})$

$= -\frac{1}{2}(\overrightarrow{AC} \times \overrightarrow{AB}) + \frac{1}{2}(\overrightarrow{AC} \times \overrightarrow{AD})$

$= \frac{1}{2}\,\overrightarrow{AC} \times [-\overrightarrow{AB} + \overrightarrow{AD}]$

$= \frac{1}{2}\,\overrightarrow{AC} \times [\overrightarrow{BA} + \overrightarrow{AD}]$

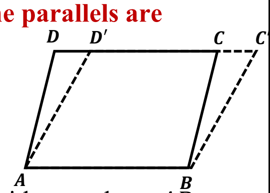
$= \frac{1}{2}\,\overrightarrow{AC} \times \overrightarrow{BD}$

$\therefore$ Vector area of quadrilateral $ABCD = \frac{1}{2}\,\overrightarrow{AC} \times \overrightarrow{BD}$

$\therefore$ Area of the quadrilateral $ABCD = \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{BD}\right|$

Hence proved.

7. Prove by vector method that the parallelograms on the same base and between the same parallels are equal in area.



Solution:

Let $ABCD$ be a parallelogram.

Let $ABC'D'$ be a new parallelogram with same base AB and same parallel AB and CD.

Area of $ABCD = \left|\overrightarrow{AB} \times \overrightarrow{AD}\right|$

$= \left|\overrightarrow{AB} \times \left(\overrightarrow{AD'} + \overrightarrow{DD'}\right)\right|$

$= \left|\overrightarrow{AB} \times \overrightarrow{AD'} + \overrightarrow{AB} \times \overrightarrow{DD'}\right|$

$= \left|\overrightarrow{AB} \times \overrightarrow{AD'} + \vec{0}\right| \quad \left[\because \overrightarrow{AB} \& \overrightarrow{DD'}\, are\ parallel\right]$

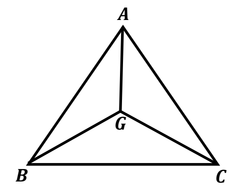
$= \left|\overrightarrow{AB} \times \overrightarrow{AD'}\right|$

$=$ Area of $ABC'D'$

Hence area of $ABCD$ and area of $ABC'D'$ are equal.

8. If G is the centroid of a ΔABC, prove that $(area\ of\ \Delta GAB) = (area\ of\ \Delta GBC) = (area\ of\ \Delta GCA) = \frac{1}{3}(area\ of\ \Delta ABC)$.

Hint: $\vec{A} \times \vec{A} = \vec{0}$



Solution:

Let A be the origin.

$\Rightarrow \quad \overrightarrow{AG} = \frac{\overrightarrow{AA} + \overrightarrow{AB} + \overrightarrow{AC}}{3}$

$\Rightarrow \quad \overrightarrow{AG} = \frac{\overrightarrow{AB} + \overrightarrow{AC}}{3} \qquad [\because \overrightarrow{AA} = \vec{0}]$

Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$ ----------(1)

Area of $\Delta GAB = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AG}\right|$

$= \frac{1}{2}\left|\overrightarrow{AB} \times \left(\frac{\overrightarrow{AB} + \overrightarrow{AC}}{3}\right)\right|$

$= \frac{1}{3}\left(\frac{1}{2}\right)\left|\overrightarrow{AB} \times \overrightarrow{AB} + \overrightarrow{AB} \times \overrightarrow{AC}\right|$

$= \frac{1}{3}\left(\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|\right)$

Area of $\Delta GAB = \frac{1}{3}(area\ of\ \Delta ABC)$ From (1)

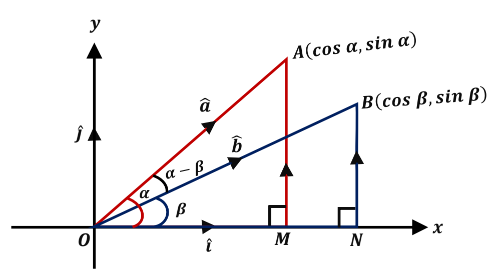
Similarly,

Area of ΔGBC = Area of $\Delta GCA = \frac{1}{3}(area\ of\ \Delta ABC)$

Hence proved.

9. Using vector method, prove that $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

Solution:



Let $\overrightarrow{OA} = \hat{a}$ and $\overrightarrow{OB} = \hat{b}$ be the unit vectors.

$\Rightarrow \quad |\hat{a}| = \left|\hat{b}\right| = 1$

Let $\hat{\imath}$ and $\hat{\jmath}$ be the unit vectors along x, y axis respectively.

The coordinates of A and B be $(\cos\alpha, \sin\alpha)$ and $(\cos\beta, \sin\beta)$ respectively.

Now $\quad \overrightarrow{OA} = \overrightarrow{OM} + \overrightarrow{MA}$

$\hat{a} = \cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}$

Also $\overrightarrow{OB} = \overrightarrow{ON} + \overrightarrow{NB}$

$$\hat{b} = \cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath}$$

$$\hat{b}.\hat{a} = (\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath})\,.(\cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath})$$

$$\hat{b}.\hat{a} = \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad \text{----------(1)}$$

By definition,

$$\hat{b}.\hat{a} = |\hat{b}||\hat{a}|\cos(\alpha-\beta)$$

$$\hat{b}.\hat{a} = (1)(1)\cos(\alpha-\beta)$$

$$\hat{b}.\hat{a} = \cos(\alpha-\beta) \qquad \text{----------(2)}$$

From (1) and (2), we get

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Hence proved.

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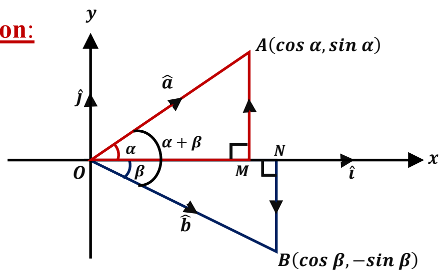
10. Prove by vector method that

$\boldsymbol{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$ **.**

Solution:



Let $\overrightarrow{OA} = \hat{a}$ and $\overrightarrow{OB} = \hat{b}$ be the unit vectors.

$$\Rightarrow \qquad |\hat{a}| = |\hat{b}| = 1$$

Let $\hat{\imath}$ and $\hat{\jmath}$ be the unit vectors along x, y axis respectively.

The coordinates of A and B be $(\cos\alpha, \sin\alpha)$ and $(\cos\beta, -\sin\beta)$ respectively.

Now $\overrightarrow{OA} = \overrightarrow{OM} + \overrightarrow{MA}$

$$\hat{a} = \cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}$$

Also $\overrightarrow{OB} = \overrightarrow{ON} + \overrightarrow{NB}$

$$\hat{b} = \cos\beta\,\hat{\imath} - \sin\beta\,\hat{\jmath}$$

$$\hat{b}\times\hat{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\beta & -\sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$$\hat{b}\times\hat{a} = \hat{k}\,(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \qquad \text{----------(1)}$$

By definition,

$$\hat{b}\times\hat{a} = |\hat{b}||\hat{a}|\sin(\alpha+\beta)\,\hat{k}$$

$$= (1)(1)\sin(\alpha+\beta)\,\hat{k}$$

$$\hat{b}\times\hat{a} = \hat{k}\sin(\alpha+\beta) \qquad \text{----------(2)}$$

From (1) and (2), we get

$$\hat{k}\sin(\alpha+\beta) = \hat{k}\,(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Hence proved.

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11. A particle acted on by constant forces $\boldsymbol{8\hat{\imath} + 2\hat{\jmath} - 6\hat{k}}$ and $\boldsymbol{6\hat{\imath} + 2\hat{\jmath} - 2\hat{k}}$ is displaced from the point $\boldsymbol{(1,2,3)}$ to the point $\boldsymbol{(5,4,1)}$. Find the total work done by the forces.

Hint: work done = $\boldsymbol{\vec{F}.\vec{d}}$

Solution:

Given $\vec{F_1} = 6\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$ and $\vec{F_2} = 8\hat{\imath} + 2\hat{\jmath} - 6\hat{k}$

$$\vec{F} = \vec{F_1} + \vec{F_2}$$

$$\vec{F} = 14\hat{\imath} + 4\hat{\jmath} - 8\hat{k} \qquad \text{----------(1)}$$

Let $\overrightarrow{OA} = \hat{1} + 2\hat{\jmath} + 3\hat{k}$ and $\overrightarrow{OB} = 5\hat{\imath} + 4\hat{\jmath} + \hat{k}$

$$\vec{d} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\vec{d} = 4\hat{\imath} + 2\hat{\jmath} - 2\hat{k} \qquad \text{----------(2)}$$

Work done $= \vec{F}.\vec{d}$

$$= (14\hat{\imath} + 4\hat{\jmath} - 8\hat{k}).(4\hat{\imath} + 2\hat{\jmath} - 2\hat{k})$$

$$= 14(4) + 4(2) - 8(-2)$$

$$= 56 + 8 + 16$$

Work done $= 80\ units$

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12. Forces of magnitudes $\boldsymbol{5\sqrt{2}}$ and $\boldsymbol{10\sqrt{2}}$ units acting in the directions $\boldsymbol{3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}}$ and $\boldsymbol{10\hat{\imath} + 6\hat{\jmath} - 8\hat{k}}$, respectively, act on a particle which is displaced from the point with the position vector $\boldsymbol{4\hat{\imath} - 3\hat{\jmath} - 2\hat{k}}$ to the point with the position vector $\boldsymbol{6\hat{\imath} + \hat{\jmath} - 3\hat{k}}$. Find the work done by the forces.

Hint: direction $\boldsymbol{\hat{a} = \frac{\vec{a}}{|\vec{a}|}}$

Solution:

Given $\vec{a} = 3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$ and $\vec{b} = 10\hat{\imath} + 6\hat{\jmath} - 8\hat{k}$

$$\vec{F} = \vec{F_1} + \vec{F_2}$$

$$\vec{F} = 5\sqrt{2}\hat{a} + 10\sqrt{2}\hat{b} \qquad \text{----------(1)}$$

$$\therefore \qquad |\vec{a}| = \sqrt{9+16+25} = \sqrt{50} = \sqrt{25\times 2}$$

$$|\vec{a}| = 5\sqrt{2}$$

Now $|\vec{b}| = \sqrt{100+36+64} = \sqrt{200} = \sqrt{100\times 2}$

$$|\vec{b}| = 10\sqrt{2}$$

$$\vec{F} = 5\sqrt{2}\left(\frac{3\hat{\imath}+4\hat{\jmath}+5\hat{k}}{5\sqrt{2}}\right) + 10\sqrt{2}\left(\frac{10\hat{\imath}+6\hat{\jmath}-8\hat{k}}{10\sqrt{2}}\right)$$

$$\vec{F} = 3\hat{\imath} + 4\hat{\jmath} + 5\hat{k} + \left(10\hat{\imath} + 6\hat{\jmath} - 8\hat{k}\right)$$

$$\vec{F} = 13\hat{\imath} + 10\hat{\jmath} - 3\hat{k} \qquad \text{----------(2)}$$

Let $\overrightarrow{OA} = 4\hat{\imath} - 3\hat{\jmath} - 2\hat{k}$ and $\overrightarrow{OB} = 6\hat{\imath} + \hat{\jmath} - 3\hat{k}$

$$\vec{d} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= 6\hat{\imath} + \hat{\jmath} - 3\hat{k} - \left(4\hat{\imath} - 3\hat{\jmath} - 2\hat{k}\right)$$

$$\vec{d} = 2\hat{\imath} + 4\hat{\jmath} - \hat{k} \qquad \text{----------(3)}$$

$\therefore$ Work done $= \vec{F}.\vec{d}$

$$= \left(13\hat{\imath} + 10\hat{\jmath} - 3\hat{k}\right).\left(2\hat{\imath} + 4\hat{\jmath} - \hat{k}\right)$$

$$= 13(2) + 10(4) - 3(-1)$$

$$= 26 + 40 + 3$$

$\therefore$ Work done $= 69\ units$

13. Find the magnitude and direction cosines of the torque of a force represented by $3\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$ about the point with position vector $2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ acting through a point whose position vector is $4\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$.

Hint: torque $= \vec{r} \times \vec{F}$

$$\vec{r} = \textbf{\textit{through (or) at a point}} - \textbf{\textit{about the point}}$$

Solution:

Given $\vec{F} = 3\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$ ----------(1)

Let $\overrightarrow{OA} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ and $\overrightarrow{OB} = 4\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$

$$\vec{r} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\vec{r} = 2\hat{\imath} + 5\hat{\jmath} - 7\hat{k} \qquad \text{----------(2)}$$

$\therefore$ torque $= \vec{r} \times \vec{F}$

$$\vec{t} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 5 & -7 \\ 3 & 4 & -5 \end{vmatrix}$$

$$= \hat{\imath}(-25 + 28) - \hat{\jmath}(-10 + 21) + \hat{k}(8 - 15)$$

$$\vec{t} = 3\hat{\imath} - 11\hat{\jmath} - 7\hat{k}$$

Magnitude $|\vec{t}| = \sqrt{9 + 121 + 49} = \sqrt{179}$

Direction cosines $= \left(\frac{3}{\sqrt{179}}, \frac{-11}{\sqrt{179}}, \frac{-7}{\sqrt{179}}\right)$

14. Find the torque of the resultant of the three forces represented by $-3\hat{\imath} + 6\hat{\jmath} - 3\hat{k}, 4\hat{\imath} - 10\hat{\jmath} + 12\hat{k}$ and $4\hat{\imath} + 7\hat{\jmath}$ acting at the point with position vector $8\hat{\imath} - 6\hat{\jmath} - 4\hat{k}$, about the point with position vector $18\hat{\imath} + 3\hat{\jmath} - 9\hat{k}$.

Solution:

Given $\overrightarrow{F_1} = -3\hat{\imath} + 6\hat{\jmath} - 3\hat{k}$

$$\overrightarrow{F_2} = 4\hat{\imath} - 10\hat{\jmath} + 12\hat{k} \text{ and } \overrightarrow{F_3} = 4\hat{\imath} + 7\hat{\jmath}$$

$$\vec{F} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3}$$

$$\vec{F} = 5\hat{\imath} + 3\hat{\jmath} + 9\hat{k} \qquad \text{----------(1)}$$

Let $\overrightarrow{OA} = 8\hat{\imath} - 6\hat{\jmath} - 4\hat{k}$ and $\overrightarrow{OB} = 18\hat{\imath} + 3\hat{\jmath} - 9\hat{k}$

$$\vec{r} = \overrightarrow{OA} - \overrightarrow{OB}$$

$$\vec{r} = -10\hat{\imath} - 9\hat{\jmath} + 5\hat{k} \qquad \text{----------(2)}$$

$\therefore$ torque $= \vec{r} \times \vec{F}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -10 & -9 & 5 \\ 5 & 3 & 9 \end{vmatrix}$$

$$= \hat{\imath}(-81 - 15) - \hat{\jmath}(-90 - 25) + \hat{k}(-30 + 45)$$

$$\Rightarrow \quad \vec{t} = -96\hat{\imath} + 115\hat{\jmath} + 15\hat{k}$$

Exercise 6.2

1. If $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}, \vec{b} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}, \vec{c} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$, find $\vec{a}.\left(\vec{b} \times \vec{c}\right)$.

Solution:

Wkt, $\vec{a}.\left(\vec{b} \times \vec{c}\right) = \left[\vec{a}, \vec{b}, \vec{c}\right]$

$$= \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= 1(1 + 4) + 2(2 + 6) + 3(4 - 3)$$

$$= 5 + 16 + 3$$

$$\vec{a}.\left(\vec{b} \times \vec{c}\right) = 24$$

2. Find the volume of the parallelepiped whose coterminous edges are represented by the vectors $-6\hat{\imath} + 14\hat{\jmath} + 10\hat{k}, 14\hat{\imath} - 10\hat{\jmath} - 6\hat{k}$ and $2\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$.

Solution:

Let $\vec{a} = -6\hat{\imath} + 14\hat{\jmath} + 10\hat{k}$

$$\vec{b} = 14\hat{\imath} - 10\hat{\jmath} - 6\hat{k}$$

$$\vec{c} = 2\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$$

Volume of parallelepiped $= \left[\vec{a}, \vec{b}, \vec{c}\right]$

$$= \begin{vmatrix} -6 & 14 & 10 \\ 14 & -10 & -6 \\ 2 & 4 & -2 \end{vmatrix}$$

$$= -6(20 + 24) - 14(-28 + 12) + 10(56 + 20)$$

$$= -6(44) - 14(-16) + 10(76)$$

$$= -264 + 224 + 760$$

$= 720$

$\therefore$ The volume of parallelepiped is 720 cubic units.

3. The volume of the parallelepiped whose coterminous edges are $7\hat{\imath} + \lambda\hat{\jmath} - 3\hat{k}, \hat{\imath} + 2\hat{\jmath} - \hat{k}, -3\hat{\imath} + 7\hat{\jmath} + 5\hat{k}$ is 90 cubic units. Find the value of λ.

Solution:

Let $\vec{a} = 7\hat{\imath} + \lambda\hat{\jmath} - 3\hat{k}$

$\vec{b} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$

$\vec{c} = -3\hat{\imath} + 7\hat{\jmath} + 5\hat{k}$

Volume of parallelepiped $= [\vec{a}, \vec{b}, \vec{c}]$

$$90 = \begin{vmatrix} 7 & \lambda & -3 \\ 1 & 2 & -1 \\ -3 & 7 & 5 \end{vmatrix}$$

$$90 = 7(10+7) - \lambda(5-3) - 3(7+6)$$

$$90 = 7(17) - \lambda(2) - 3(13)$$

$$90 = 119 - 2\lambda - 39$$

$$90 = 80 - 2\lambda$$

$$10 = -2\lambda$$

$$\lambda = -5$$

4. If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors represented by concurrent edges of a parallelepiped of volume 4 cubic units, find the value of $(\vec{a} + \vec{b}).(\vec{b} \times \vec{c}) + (\vec{b} + \vec{c}).(\vec{c} \times \vec{a}) + (\vec{c} + \vec{a}).(\vec{a} \times \vec{b})$.

Hint: $[\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}]$

Solution:

Volume of parallelepiped $= [\vec{a}, \vec{b}, \vec{c}] = 4$

$$(\vec{a} + \vec{b}).(\vec{b} \times \vec{c}) + (\vec{b} + \vec{c}).(\vec{c} \times \vec{a}) + (\vec{c} + \vec{a}).(\vec{a} \times \vec{b})$$

$$= \vec{a}.(\vec{b} \times \vec{c}) + \vec{b}.(\vec{b} \times \vec{c}) + \vec{b}.(\vec{c} \times \vec{a}) + \vec{c}.(\vec{c} \times \vec{a}) + \vec{c}.(\vec{a} \times \vec{b}) + \vec{a}.(\vec{a} \times \vec{b})$$

$$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{b}, \vec{b}, \vec{c}] + [\vec{b}, \vec{c}, \vec{a}] + [\vec{c}, \vec{c}, \vec{a}] + [\vec{c}, \vec{a}, \vec{b}] + [\vec{a}, \vec{a}, \vec{b}]$$

$$= [\vec{a}, \vec{b}, \vec{c}] + 0 + [\vec{a}, \vec{b}, \vec{c}] + 0 + [\vec{a}, \vec{b}, \vec{c}] + 0$$

$$= 4 + 4 + 4$$

$$= 12$$

5.Find the altitude of a parallelepiped determined by the vectors $\vec{a} = -2\hat{\imath} + 5\hat{\jmath} + 3\hat{k}, \vec{b} = \hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\vec{c} = -3\hat{\imath} + \hat{\jmath} + 4\hat{k}$ if the base if taken as the parallelogram determined by $\vec{b}$ and $\vec{c}$.

Hint: $\boldsymbol{Altitude = height}$

Solution:

Given $\vec{a} = -2\hat{\imath} + 5\hat{\jmath} + 3\hat{k}$

$\vec{b} = \hat{\imath} + 3\hat{\jmath} - 2\hat{k}$

$\vec{c} = -3\hat{\imath} + \hat{\jmath} + 4\hat{k}$

Area of parallelepiped $= |\vec{b} \times \vec{c}|$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & -2 \\ -3 & 1 & 4 \end{vmatrix}$$

$$= \hat{\imath}(12+2) - \hat{\jmath}(4-6) + \hat{k}(1+9)$$

$$\vec{b} \times \vec{c} = 14\hat{\imath} + 2\hat{\jmath} + 10\hat{k}$$

$$|\vec{b} \times \vec{c}| = \sqrt{196 + 4 + 100}$$

$$= \sqrt{300}$$

$$= \sqrt{100 \times 3}$$

$$|\vec{b} \times \vec{c}| = 10\sqrt{3} \qquad \text{----------(1)}$$

Volume of parallelepiped $= [\vec{a}, \vec{b}, \vec{c}]$

$$= \begin{vmatrix} -2 & 5 & 3 \\ 1 & 3 & -2 \\ -3 & 1 & 4 \end{vmatrix}$$

$$= -2(12+2) - 5(4-6) + 3(1+9)$$

$$= -2(14) - 5(-2) + 3(10)$$

$$= -28 + 10 + 30$$

$$= 12 \qquad \text{----------(2)}$$

$$volume = Base\ area \times height$$

$$12 = 10\sqrt{3} \times h$$

$$h = \frac{12}{10\sqrt{3}}$$

$$= \frac{6}{5\sqrt{3}} = \frac{2\times3}{5\sqrt{3}}$$

$$h = \frac{2\sqrt{3}}{5}\ units$$

6. Determine whether the three vectors $2\hat{\imath} + 3\hat{k} + \hat{k}, \hat{\imath} - 2\hat{\jmath} + 2\hat{k}$ and $3\hat{\imath} + \hat{\jmath} + 3\hat{k}$ are coplanar.

Solution:

Let $\vec{a} = 2\hat{\imath} + 3\hat{k} + \hat{k}$

$\vec{b} = \hat{\imath} - 2\hat{\jmath} + 2\hat{k}$

$\vec{c} = 3\hat{\imath} + \hat{\jmath} + 3\hat{k}$

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & 1 & 3 \end{vmatrix}$$

$$= 2(-6-2) - 3(3-6) + 1(1+6)$$

$$= 2(-8) - 3(-3) + 1(7)$$

$$= -16 + 9 + 7$$

$$[\vec{a}, \vec{b}, \vec{c}] = 0$$

∴ The three vectors are coplanar.

7. Let $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}, \vec{b} = \hat{\imath}$ and $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$. If $c_1 = 1$ and $c_2 = 2$, find c_3 such that $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

Solution:

Given $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}, \vec{b} = \hat{\imath}$ and $\vec{c} = \hat{\imath} + 2\hat{\jmath} + c_3\hat{k}$

Since $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

$$[\vec{a}, \vec{b}, \vec{c}] = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & c_3 \end{vmatrix} = 0$$

$$1(0) - 1(c_3) + 1(2) = 0$$

$$-c_3 + 2 = 0$$

$$c_3 = 2$$

8. If $\vec{a} = \hat{\imath} - \hat{k}, \vec{b} = x\hat{\imath} + \hat{\jmath} + (1-x)\hat{k}, \vec{c} = y\hat{\imath} + x\hat{\jmath} + (1+x-y)\hat{k}$, show that $[\vec{a}, \vec{b}, \vec{c}]$ depends on neither x nor y.

Solution:

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$$

$$= 1\left(1 + x - y - (x - x^2)\right) - 1(x^2 - y)$$

$$= 1 + x - y - x + x^2 - x^2 + y$$

$$[\vec{a}, \vec{b}, \vec{c}] = 1$$

Hence $[\vec{a}, \vec{b}, \vec{c}]$ depends on neither x nor y.

9. If the vectors $a\hat{\imath} + a\hat{\jmath} + c\hat{k}, \hat{\imath} + \hat{k}$ and $c\hat{\imath} + c\hat{\jmath} + b\hat{k}$ are coplanar, prove that c is the geometric mean of a and b.

Solution:

Let $\vec{a} = a\hat{\imath} + a\hat{\jmath} + c\hat{k}$

$\vec{b} = \hat{\imath} + \hat{k}$

$\vec{c} = c\hat{\imath} + c\hat{\jmath} + b\hat{k}$

Since $\vec{a}, \vec{b}, \vec{c}$ are coplanar, $[\vec{a}, \vec{b}, \vec{c}] = 0$

$$\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$$

$$a(-c) - a(b-c) + c(c) = 0$$

$$-ac - ab + ac + c^2 = 0$$

$$-ab + c^2 = 0$$

$$c^2 = ab$$

$$c = \sqrt{ab}$$

∴ c is the geometric mean of a and b.

Hence proved.

10. Let $\vec{a}, \vec{b}, \vec{c}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, show that $[\vec{a}, \vec{b}, \vec{c}]^2 = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2$.

Solution:

Given $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

∴ $\vec{c}$ is parallel to $\vec{a} \times \vec{b}$.

$$[\vec{a}, \vec{b}, \vec{c}] = (\vec{a} \times \vec{b}) . \vec{c}$$

$$= |\vec{a} \times \vec{b}||\vec{c}| \cos 0° \quad \text{[by defn of } \vec{a}.\vec{b}]$$

$$= |\vec{a} \times \vec{b}|(1)(1) \quad [\because |\vec{c}| = 1]$$

$$= |\vec{a} \times \vec{b}|$$

$$= |\vec{a}||\vec{b}| \sin\frac{\pi}{6}(1) \quad \text{[by defn of } \vec{a} \times \vec{b}]$$

$$[\vec{a}, \vec{b}, \vec{c}] = \frac{1}{2}|\vec{a}||\vec{b}|$$

Squaring on both sides, we get

$$[\vec{a}, \vec{b}, \vec{c}]^2 = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2$$

Hence proved.

Exercise 6.3

1. If $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}, \vec{b} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}, \vec{c} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$, find

(i) $(\vec{a} \times \vec{b}) \times \vec{c}$

Solution:

Given $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}, \vec{b} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}$ and

$\vec{c} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= \hat{\imath}(4-3) - \hat{\jmath}(-2-6) + \hat{k}(1+4)$$

$$\vec{a}\times\vec{b} = \hat{\imath} + 8\hat{\jmath} + 5\hat{k}$$

$$(\vec{a}\times\vec{b})\times\vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 8 & 5 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= \hat{\imath}(8-10) - \hat{\jmath}(1-15) + \hat{k}(2-24)$$

$$(\vec{a}\times\vec{b})\times\vec{c} = -2\hat{\imath} + 14\hat{\jmath} - 22\hat{k}$$

(ii) $\vec{a}\times(\vec{b}\times\vec{c})$

Solution:

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= \hat{\imath}(1+4) - \hat{\jmath}(2+6) + \hat{k}(4-3)$$

$$\vec{b}\times\vec{c} = 5\hat{\imath} - 8\hat{\jmath} + \hat{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 3 \\ 5 & -8 & 1 \end{vmatrix}$$

$$= \hat{\imath}(-2+24) - \hat{\jmath}(1-15) + \hat{k}(-8+10)$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = 22\hat{\imath} + 14\hat{\jmath} + 2\hat{k}$$

2. For any vector $\vec{a}$, prove that $\hat{\imath}\times(\vec{a}\times\hat{\imath}) + \hat{\jmath}\times(\vec{a}\times\hat{\jmath}) + \hat{k}\times(\vec{a}\times\hat{k}) = 2\vec{a}$.

Solution:

Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$$\hat{\imath}\times(\vec{a}\times\hat{\imath}) = (\hat{\imath}.\hat{\imath})\vec{a} - (\hat{\imath}.\vec{a})\hat{\imath}$$

$$= \vec{a} - a_1\hat{\imath} \quad \text{----------(1)}$$

$$\hat{\jmath}\times(\vec{a}\times\hat{\jmath}) = (\hat{\jmath}.\hat{\jmath})\vec{a} - (\hat{\jmath}.\vec{a})\hat{\jmath}$$

$$= \vec{a} - a_2\hat{\jmath} \quad \text{----------(2)}$$

$$\hat{k}\times(\vec{a}\times\hat{k}) = (\hat{k}.\hat{k})\vec{a} - (\hat{k}.\vec{a})\hat{k}$$

$$= \vec{a} - a_3\hat{k} \quad \text{----------(3)}$$

$$\text{LHS} = \vec{a} - a_1\hat{\imath} + \vec{a} - a_2\hat{\jmath} + \vec{a} + a_3\hat{k}$$

$$= 3\vec{a} - (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k})$$

$$= 3\vec{a} - \vec{a}$$

$$= 2\vec{a}$$

$$= \text{RHS}$$

Hence proved.

3. Prove that $[\vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}] = 0$.

Solution:

$$\text{LHS} = (\vec{a}-\vec{b}).[(\vec{b}-\vec{c})\times(\vec{c}-\vec{a})]$$

$$= (\vec{a}-\vec{b}).\left[(\vec{b}\times\vec{c}) - (\vec{b}\times\vec{a}) - (\vec{c}\times\vec{c}) + (\vec{c}\times\vec{a})\right]$$

$$= (\vec{a}-\vec{b}).[(\vec{b}\times\vec{c}) - (\vec{b}\times\vec{a}) + (\vec{c}\times\vec{a})]$$

$$= \vec{a}.(\vec{b}\times\vec{c}) - \vec{a}.(\vec{b}\times\vec{a}) + \vec{a}.(\vec{c}\times\vec{a}) - \vec{b}.(\vec{b}\times\vec{c}) + \vec{b}.(\vec{b}\times\vec{a}) - \vec{b}.(\vec{c}\times\vec{a})$$

$$= [\vec{a},\vec{b},\vec{c}] - [\vec{a},\vec{b},\vec{a}] + [\vec{a},\vec{c},\vec{a}] - [\vec{b},\vec{b},\vec{c}] + [\vec{b},\vec{b},\vec{a}] - [\vec{b},\vec{c},\vec{a}]$$

$$= [\vec{a},\vec{b},\vec{c}] - [\vec{b},\vec{c},\vec{a}]$$

$$= [\vec{a},\vec{b},\vec{c}] - [\vec{a},\vec{b},\vec{c}]$$

$$= 0$$

$$= \text{RHS}$$

Hence proved.

Aliter method:

$$\text{LHS} = [\vec{a}-\vec{b}+0\vec{c}, 0\vec{a}+\vec{b}-\vec{c}, -\vec{a}+0\vec{b}+\vec{c}]$$

$$= \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} [\vec{a},\vec{b},\vec{c}]$$

$$= (1(1) + 1(-1) + 0)[\vec{a},\vec{b},\vec{c}]$$

$$= (0)\,[\vec{a},\vec{b},\vec{c}]$$

$$= 0$$

$$= \text{RHS}$$

Hence proved.

4. If $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}, \vec{b} = 3\hat{\imath} + 5\hat{\jmath} + 2\hat{k}, \vec{c} = -\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$, verify that

(i) $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

Solution:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -1 \\ 3 & 5 & 2 \end{vmatrix}$$

$$= \hat{\imath}(6+5) - \hat{\jmath}(4+3) + \hat{k}(10-9)$$

$$\vec{a}\times\vec{b} = 11\hat{\imath} - 7\hat{\jmath} + \hat{k}$$

$$\text{LHS} = (\vec{a}\times\vec{b})\times\vec{c}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 11 & -7 & 1 \\ -1 & -2 & 3 \end{vmatrix}$$

$$= \hat{\imath}(-21+2) - \hat{\jmath}(33+1) + \hat{k}(-22-7)$$

$$= -19\hat{\imath} - 34\hat{\jmath} - 29\hat{k} \quad \text{----------(1)}$$

$\vec{a}.\vec{c} = -2 - 6 - 3 = -11$

$\vec{b}.\vec{c} = -3 - 10 + 6 = -7$

$\text{RHS} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

$= -11(3\hat{\imath} + 5\hat{\jmath} + 2\hat{k}) + 7(2\hat{\imath} + 3\hat{\jmath} - \hat{k})$

$= -33\hat{\imath} - 55\hat{\jmath} - 22\hat{k} + 14\hat{\imath} + 21\hat{\jmath} - 7\hat{k}$

$= -19\hat{\imath} - 34\hat{\jmath} - 29\hat{k}$ ----------(2)

From (1) and (2), we get

$$(\vec{a} \times \vec{b}) \times \vec{c} = (\dot{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$$

Hence verified.

(ii) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Solution:

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 5 & 2 \\ -1 & -2 & 3 \end{vmatrix}$$

$= \hat{\imath}(15 + 4) - \hat{\jmath}(9 + 2) + \hat{k}(-6 + 5)$

$\vec{b} \times \vec{c} = 19\hat{\imath} - 11\hat{\jmath} - \hat{k}$

$\text{LHS} = \vec{a} \times (\vec{b} \times \vec{c})$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -1 \\ 19 & -11 & -1 \end{vmatrix}$$

$= \hat{\imath}(-3 - 11) - \hat{\jmath}(-2 + 19) + \hat{k}(-22 - 57)$

$= -14\hat{\imath} - 17\hat{\jmath} - 79\hat{k}$ ----------(1)

$\vec{a}.\vec{c} = -2 - 6 - 3 = -11$

$\vec{a}.\vec{b} = 6 + 15 - 2 = 19$

$\text{RHS} = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$= -11(3\hat{\imath} + 5\hat{\jmath} + 2\hat{k}) - 19(-\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$

$= -33\hat{\imath} - 55\hat{\jmath} - 22\hat{k} + 19\hat{\imath} + 38\hat{\jmath} - 57\hat{k}$

$= -14\hat{\imath} - 17\hat{\jmath} - 79\hat{k}$ ----------(2)

From (1) and (2), we get

$$(\vec{a} \times \vec{b}) \times \vec{c} = (\dot{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$$

Hence verified.

5. $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}, \vec{b} = -\hat{\imath} + 2\hat{\jmath} - 4\hat{k}, \vec{c} = \hat{\imath} + \hat{\jmath} + \hat{k}$ then fine the value of $(\vec{a} \times \vec{b}) . (\vec{a} \times \vec{c})$

Solution:

$$\Rightarrow \quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -1 \\ -1 & 2 & -4 \end{vmatrix}$$

$= \hat{\imath}(-12 + 2) - \hat{\jmath}(-8 - 1) + \hat{k}(4 + 3)$

$= -10\hat{\imath} + 9\hat{\jmath} + 7\hat{k}$ ----------(1)

$$\Rightarrow \quad \vec{a} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix}$$

$= \hat{\imath}(3 + 1) - \hat{\jmath}(2 + 1) + \hat{k}(2 - 3)$

$= 4\hat{\imath} - 3\hat{\jmath} - \hat{k}$ ----------(2)

$\Rightarrow \quad (\vec{a} \times \vec{b}) . (\vec{a} \times \vec{c}) = -10(4) + 9(-3) + 7(-1)$

$= -40 - 27 - 7$

$\Rightarrow \quad (\vec{a} \times \vec{b}) . (\vec{a} \times \vec{c}) = -74$

6. If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar vectors, show that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$.

Solution:

Given $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar vectors.

$\text{LHS} = (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

$= [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$

$= 0(\vec{c}) - 0(\vec{d})$

$= \vec{0}$

$= \text{RHS}$

Hence proved.

7. If $\vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}, \vec{b} = 2\hat{\imath} - \hat{\jmath} + \hat{k}, \vec{c} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$ and $\vec{a} \times (\vec{b} \times \vec{c}) = l\vec{a} + m\vec{b} + n\vec{c}$, find the values of l, m, n.

Solution:

$$\vec{a} \times (\vec{b} \times \vec{c}) = l\vec{a} + m\vec{b} + n\vec{c}$$

$$(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = l\vec{a} + m\vec{b} + n\vec{c}$$

$$(3 + 4 + 3)\vec{b} - (2 - 2 + 3)\vec{c} = l\vec{a} + m\vec{b} + n\vec{c}$$

$$10\vec{b} - 3\vec{c} = l\vec{a} + m\vec{b} + n\vec{c}$$

Equating $\vec{a}, \vec{b}, \vec{c}$ coefficient on both sides, we get

$\Rightarrow \quad l = 0$

$\Rightarrow \quad m = 10$

$\Rightarrow \quad n = -3$

8. If $\hat{a}, \hat{b}, \hat{c}$ are three unit vectors such that $\hat{b}$ and $\hat{c}$ are non-parallel and $\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$, find the angle between $\hat{a}$ and $\hat{c}$.

Solution:

$$\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$$

$$(\hat{a}.\hat{c})\hat{b} - (\hat{a}.\hat{b})\hat{c} = \frac{1}{2}\hat{b}$$

Equating coefficient of $\hat{b}$, we get

$$\hat{a}.\hat{c} = \frac{1}{2}$$

$$|\hat{a}|.|\hat{c}|\cos\theta = \frac{1}{2}$$

$$(1)(1)\cos\theta = \frac{1}{2} \qquad [\because \hat{a}, \hat{b}, \hat{c} \text{ are unit vectors}]$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{3}$$

$\therefore$ The angle between $\hat{a}$ and $\hat{c}$ is $\frac{\pi}{3}$

Exercise 6.4

1. Find the non-parametric form of vector equation and cartesian equations of the straight line passing through the point with position vector $4\hat{\imath} + 3\hat{\jmath} - 7\hat{k}$ and parallel to the vector $2\hat{\imath} - 6\hat{\jmath} + 7\hat{k}$.

Solution:

Let $\vec{a} = 4\hat{\imath} + 3\hat{\jmath} - 7\hat{k}$ and $\vec{b} = 2\hat{\imath} - 6\hat{\jmath} + 7\hat{k}$

Non-Parametric vector equation of line is,

$$(\vec{r} - \vec{a}) \times \vec{b} = \vec{0}$$

$$\left(\vec{r} - (4\hat{\imath} + 3\hat{\jmath} - 7\hat{k})\right) \times (2\hat{\imath} - 6\hat{\jmath} + 7\hat{k}) = \vec{0}$$

Cartesian equation of line is,

$$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$$

$$(x_1, y_1, z_1) = (4,3,-7) \text{ and } (b_1, b_2, b_3) = (2,-6,7)$$

$$\frac{x-4}{2} = \frac{y-3}{-6} = \frac{z+7}{7}$$

---

2. Find the parametric form of vector equation and cartesian equations of the straight line passing through the point $(-2, 3, 4)$ and parallel to the straight line $\frac{x-1}{-4} = \frac{y+3}{5} = \frac{8-z}{6}$.

Solution:

Let $\vec{a} = -2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ and $\vec{b} = -4\hat{\imath} + 5\hat{\jmath} - 6\hat{k}$

Parametric form of vector equation is,

$$\vec{r} = \vec{a} + t\vec{b}$$

$$\vec{r} = -2\hat{\imath} + 3\hat{\jmath} + 4\hat{k} + t(-4\hat{\imath} + 5\hat{\jmath} - 6\hat{k})$$

Cartesian equation of line is,

$$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$$

$$(x_1, y_1, z_1) = (-2,3,4) \text{ and } (b_1, b_2, b_3) = (-4,5,-6)$$

$$\frac{x+2}{-4} = \frac{y-3}{5} = \frac{z-4}{-6}$$

---

3. Find the points where the straight line passes through $(6, 7, 4)$ and $(8, 4, 9)$ cuts the xz and yz planes.

Solution:

Given $(x_1, y_1, z_1) = (6,7,4)$

$(x_2, y_2, z_2) = (8,4,9)$

Equation of line is,

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-6}{2} = \frac{y-7}{-3} = \frac{z-4}{5} \qquad \text{----------(1)}$$

Equation of xz-plane is $y = 0$,

$$\frac{x-6}{2} = \frac{7}{3} = \frac{z-4}{5} \qquad \text{From (1)}$$

$$\frac{x-6}{2} = \frac{7}{3} \text{ and } \frac{z-4}{5} = \frac{7}{3}$$

$3x - 18 = 14$	$3z - 12 = 35$
$3x = 32$	$3z = 47$
$x = \frac{32}{3}$	$z = \frac{47}{3}$

line cut xz-plane at $\left(\frac{32}{3}, 0, \frac{47}{3}\right)$

Equation of yz-plane is $x = 0$,

$$-3 = \frac{y-7}{-3} = \frac{z-4}{5} \qquad \text{From (1)}$$

$y - 7 = 9$	$z - 4 = -15$
$y = 16$	$z = -11$

Line cut yz-plane at $(0,16,-11)$

4.Find the direction cosines of straight line passing through the points $(5, 6, 7)$ and $(7, 9, 13)$. Also, find the parametric form of the vector equation and cartesian equations of the straight line passing through two given points.

Solution:

Let $\vec{a} = 5\hat{\imath} + 6\hat{\jmath} + 7\hat{k}$ and $\vec{b} = 7\hat{\imath} + 9\hat{\jmath} + 13\hat{k}$

Parametric from of vector equation is,

$$\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$$

$$\vec{r} = 5\hat{\imath} + 6\hat{\jmath} + 7\hat{k} + t(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k})$$

Cartesian equation of line is,

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

$$(x_1,y_1,z_1)=(5,6,7) \text{ and } (x_2,y_2,z_2)=(7,9,13)$$

$$\frac{x-5}{2}=\frac{y-6}{3}=\frac{z-7}{6}$$

$$\vec{b}-\vec{a}=(2,3,6)$$

$$|\vec{b}-\vec{a}|=\sqrt{4+9+36}=\sqrt{49}=7$$

Direction cosines are $\left(\frac{2}{7},\frac{3}{7},\frac{6}{7}\right)$

---

5. Find the acute angle between and following lines.

(i) $\vec{r}=(4\hat{\imath}-\hat{\jmath})+t(\hat{\imath}+2\hat{\jmath}-2\hat{k}), \vec{r}=(\hat{\imath}-2\hat{\jmath}+4\hat{k})+s(-\hat{\imath}-2\hat{\jmath}+2\hat{k})$

Solution:

Given $\vec{b}=\hat{\imath}+2\hat{\jmath}-2\hat{k}$ and $\vec{d}=-\hat{\imath}-2\hat{\jmath}+2\hat{k}$

$$\theta=\cos^{-1}\left(\frac{|\vec{b}.\vec{d}|}{|\vec{b}||\vec{d}|}\right)$$

$$\theta=\cos^{-1}\left(\frac{|-1-4-4|}{\sqrt{1+4+4}\sqrt{1+4+4}}\right)$$

$$\theta=\cos^{-1}\left(\frac{9}{(3)(3)}\right)$$

$$\theta=\cos^{-1}(1)$$

$$\theta=0$$

(ii) $\frac{x+4}{3}=\frac{y-7}{4}=\frac{z+5}{5}, \vec{r}=4\hat{k}+t(2\hat{\imath}+\hat{\jmath}+\hat{k})$.

Solution:

Given $\vec{b}=3\hat{\imath}+4\hat{\jmath}+5\hat{k}$ and $\vec{d}=2\hat{\imath}+\hat{\jmath}+\hat{k}$

$$\theta=\cos^{-1}\left(\frac{|\vec{b}.\vec{d}|}{|\vec{b}||\vec{d}|}\right)$$

$$|\vec{b}.\vec{d}|=|6+4+5|=15$$

$$|\vec{b}|=\sqrt{9+16+25}=\sqrt{50}=5\sqrt{2}$$

$$|\vec{d}|=\sqrt{4+1+1}=\sqrt{6}$$

$$\theta=\cos^{-1}\left(\frac{15}{5\sqrt{2}\sqrt{6}}\right)$$

$$\theta=\cos^{1}\left(\frac{3}{\sqrt{12}}\right)$$

$$\theta=\cos^{-1}\left(\frac{3}{2\sqrt{3}}\right)$$

$$\theta=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\theta=\frac{\pi}{6}$$

(iii) $2x=3y=-z$ and $6x=-y=-4z$.

Solution:

Given $2x=3y=-z$

$\div 6,\quad \frac{x}{3}=\frac{y}{2}=\frac{z}{-6}$

$$\vec{b}=3\hat{\imath}+2\hat{\jmath}-6\hat{k} \qquad \text{----------(1)}$$

Also $6x=-y=-4z$

$\div 12,\quad \frac{x}{2}=\frac{y}{-12}=\frac{z}{-3}$

$$\vec{d}=2\hat{\imath}+12\hat{\jmath}-3\hat{k} \qquad \text{----------(2)}$$

$$|\vec{b}.\vec{d}|=|6-24+18|=0$$

$$\theta=\cos^{-1}(0)$$

$$\theta=\frac{\pi}{2}$$

---

6. The vertices of ΔABC are $A(7,2,1), B(6,0,3)$ and $C(4,2,4)$. Find $\angle ABC$.

Solution:

Given $A=(7,2,1), B=(6,0,3)$ and $C=(4,2,4)$

To find: $\angle B$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=-\hat{\imath}-2\hat{\jmath}+2\hat{k}$$

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=-2\hat{\imath}+2\hat{\jmath}+\hat{k}$$

$$\overrightarrow{AB}.\overrightarrow{BC}=2-4+2$$

$$\overrightarrow{AB}.\overrightarrow{BC}=0$$

$\Rightarrow \overrightarrow{AB}$ is perpendicular to $\overrightarrow{BC}$

$\Rightarrow \angle ABC=\frac{\pi}{2}$

---

7. If the straight line joining the points $(2,1,4)$ and $(a-1,4,-1)$ is parallel to the line joining the points $(0,2,b-1)$ and $(5,3,-2)$, find the values of a and b.

Solution:

Let $\overrightarrow{OA}=2\hat{\imath}+\hat{\jmath}+4\hat{k}$, $\overrightarrow{OB}=(a-1)\hat{\imath}+4\hat{\jmath}-\hat{k}$

$\overrightarrow{OC}=2\hat{\jmath}+(b-1)\hat{k}$, $\overrightarrow{OD}=5\hat{\imath}+3\hat{\jmath}-2\hat{k}$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(a-3)\hat{\imath}+3\hat{\jmath}-5\hat{k} \qquad \text{----------(1)}$$

$$\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}=5\hat{\imath}+\hat{\jmath}+(-b-1)\hat{k} \qquad \text{----------(2)}$$

Since $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel,

$$\overrightarrow{AB}=m\,\overrightarrow{CD}$$

$$(a-3)\hat{\imath}+3\hat{\jmath}-5\hat{k}=m\left[5\hat{\imath}+\hat{\jmath}+(-b-1)\hat{k}\right]$$

Equating $\hat{\imath},\hat{\jmath},\hat{k}$ coefficients, we get

$\Rightarrow \quad m=3 \qquad (equating\ \hat{\jmath})$

$\Rightarrow \quad a-3=5m \qquad (equating\ \hat{\imath})$

$\quad a-3=5(3)$

$$a = 15 + 3$$

$\Rightarrow \quad a = 18$

$\Rightarrow \quad -5 = m(-b-1) \qquad (equating\ \hat{k})$

$$-5 = 3(-b-1)$$

$$-\frac{5}{3} = -b - 1$$

$$b = -1 + \frac{5}{3}$$

$\Rightarrow \quad b = \frac{2}{3}$

Hence $a = 18$ and $b = \frac{2}{3}$

8. If the straight line $\frac{x-5}{5m+2} = \frac{2-y}{5} = \frac{1-z}{-1}$ and $x = \frac{2y+1}{4m} = \frac{1-z}{-3}$ are perpendicular to each other, find the values of m.

Solution:

Given $\frac{x-5}{5m+2} = \frac{2-y}{5} = \frac{1-z}{-1}$

$$\frac{x-5}{5m+2} = \frac{y-2}{-5} = \frac{z-1}{1} \qquad \text{----------(1)}$$

Also $x = \frac{2y+1}{4m} = \frac{1-z}{-3}$

$$\frac{x}{1} = \frac{y+\frac{1}{2}}{2m} = \frac{z-1}{3} \qquad \text{----------(2)}$$

Let $\vec{b} = (5m+2)\hat{\imath} - 5\hat{\jmath} + \hat{k}$ and $\vec{d} = \hat{\imath} + 2m\hat{\jmath} + 3\hat{k}$

Since given lines are perpendicular, $\vec{b}.\vec{d} = 0$

$$(5m+2)(1) + (-5)(2m) + (1)(3) = 0$$

$$5m + 2 - 10m + 3 = 0$$

$$-5m = -5$$

$$m = 1$$

9. Show that the points $(2,3,4), (-1,4,5)$ and $(8,1,2)$ are collinear.

Solution:

Let $\overrightarrow{OA} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$

$$\overrightarrow{OB} = -\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$$

$$\overrightarrow{OC} = 8\hat{\imath} + \hat{\jmath} + 2\hat{k}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -3\hat{\imath} + \hat{\jmath} + \hat{k} \qquad \text{----------(1)}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 9\hat{\imath} - 3\hat{\jmath} - 3\hat{k}$$

$$\overrightarrow{BC} = -3(-3\hat{\imath} + \hat{\jmath} + \hat{k})$$

$$\overrightarrow{BC} = -3\,\overrightarrow{AB}$$

$\therefore$ $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel.

Since B is common.

Hence A, B, C are collinear.

Exercise 6.5

1. Find the parametric form of vector equation and cartesian equations of a straight line passing through $(5,2,8)$ and is perpendicular to the straight lines $\vec{r} = (\hat{\imath} + \hat{\jmath} - \hat{k}) + s(2\hat{\imath} - 2\hat{\jmath} + \hat{k})$ and $\vec{r} = (2\hat{\imath} - \hat{\jmath} - 3\hat{k}) + t(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$.

Solution:

Let $\vec{a} = 5\hat{\imath} + 2\hat{\jmath} + 8\hat{k}$

$$\vec{b} = 2\hat{\imath} - 2\hat{\jmath} + \hat{k}$$

$$\vec{d} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= \hat{\imath}(-4-2) - \hat{\jmath}(4-1) + \hat{k}(4+2)$$

$$= -6\hat{\imath} - 3\hat{\jmath} + 6\hat{k}$$

$$\vec{b} \times \vec{d} = -3(2\hat{\imath} + \hat{\jmath} - 2\hat{k})$$

Parametric form of vector equation is,

$$\vec{r} = \vec{a} + t(\vec{b} \times \vec{d})$$

$$\vec{r} = 5\hat{\imath} + 2\hat{\jmath} + 8\hat{k} + t(2\hat{\imath} + \hat{\jmath} - 2\hat{k}), t \in R$$

Cartesian equation of line is,

$$\frac{x-5}{2} = \frac{y-2}{1} = \frac{z-8}{-2}$$

2. Show that the lines $\vec{r} = (6\hat{\imath} + \hat{\jmath} + 2\hat{k}) + s(\hat{\imath} + 2\hat{\jmath} - 3\hat{k})$ and $\vec{r} = (3\hat{\imath} + 2\hat{\jmath} - 2\hat{k}) + t(2\hat{\imath} + 4\hat{\jmath} - 5\hat{k})$ are skew lines and hence find the shortest distance between them.

Solution:

Distance between skew lines,

$$d = \frac{|(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}$$

Let $\vec{a} = 6\hat{\imath} + \hat{\jmath} + 2\hat{k}$ and $\vec{c} = 3\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

$\vec{b} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ and $\vec{d} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$

$$\vec{c} - \vec{a} = -3\hat{\imath} + \hat{\jmath} - 4\hat{k}$$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}$$

$$= \hat{\imath}(-10+12) - \hat{\jmath}(-5+6) + \hat{k}(4-4)$$

$$\vec{b} \times \vec{d} = 2\hat{\imath} - \hat{\jmath}$$

$$|\vec{b}\times\vec{d}| = \sqrt{4+1} = \sqrt{5}$$

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d}) = (-3)(2)+(1)(-1)+(-4)(0)$$

$$= -6-1$$

$$= -7 \neq 0$$

∴ The given lines are skew lines.

$$d = \frac{|-7|}{|\sqrt{5}|}$$

$$d = \frac{7}{\sqrt{5}}\ units$$

3. If the two lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-m}{2} = z$ intersect at a point, find the value of m.

Solution:

Let $\vec{a} = \hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{c} = 3\hat{\imath} + m\hat{\jmath}$

$\vec{b} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ and $\vec{d} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$

$\Rightarrow \quad \vec{c} - \vec{a} = 2\hat{\imath} + (m+1)\hat{\jmath} - \hat{k}$

Since two lines are intersect at a point,

$$[(\vec{c}-\vec{a}), \vec{b}, \vec{d}] = 0$$

$$\begin{vmatrix} 2 & m+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$$2(3-8) - (m+1)(2-4) + (-1)(4-3) = 0$$

$$2(-5) - (m+1)(-2) - 1 = 0$$

$$-10 + 2m + 2 - 1 = 0$$

$$2m - 9 = 0$$

$$m = \frac{9}{2}$$

4. Show that the lines $\frac{x-3}{3} = \frac{y-3}{-1}, z-1=0$ and $\frac{x-6}{2} = \frac{z-1}{3}, y-2=0$ intersect. Also find the point of intersection.

Solution:

Condition for intersecting line is,

$$[(\vec{c}-\vec{a}), \vec{b}, \vec{d}] = 0$$

Let $\vec{a} = 3\hat{\imath} + 3\hat{\jmath} + \hat{k}$ and $\vec{c} = 6\hat{\imath} + 2\hat{\jmath} + \hat{k}$

$\vec{b} = 3\hat{\imath} - \hat{\jmath}$ and $\vec{d} = 2\hat{\imath} + 3\hat{k}$

$\Rightarrow \quad \vec{c} - \vec{a} = 3\hat{\imath} - \hat{\jmath}$

$$[(\vec{c}-\vec{a}), \vec{b}, \vec{d}] = \begin{vmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix}$$

$$= 0 \qquad (\because \textit{two rows are equal})$$

Hence the given lines are intersecting.

Given that $z-1=0$ and $y-2=0$

$z = 1$ and $\quad y = 2$

From first line,

$$\frac{x-3}{3} = \frac{y-3}{-1}$$

$$\frac{x-3}{3} = \frac{-1}{-1} = 1$$

$$x - 3 = 3$$

$$x = 6$$

From second line,

$$\frac{x-6}{2} = \frac{z-1}{3}$$

$$\frac{x-6}{2} = \frac{1-1}{3} = 0$$

$$x - 6 = 0$$

$$x = 6$$

∴ The point of intersection is (6,2,1)

5. Show that the straight lines $x+1 = 2y = -12z$ and $x = y+2 = 6z-6$ are skew and hence find the shortest distance between them.

Solution:

Given $x + 1 = 2y = -12z$

$\div 12, \quad \frac{x+1}{12} = \frac{y}{6} = \frac{z}{-1}$ ----------(1)

$$x = y + 2 = 6z - 6$$

$\div 6, \quad \frac{x}{6} = \frac{y+2}{6} = \frac{z-1}{1}$ ----------(2)

Shortest distance between skew lines,

$$d = \frac{|(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}$$

Let $\vec{a} = -\hat{\imath}$ and $\vec{b} = 12\hat{\imath} + 6\hat{\jmath} - \hat{k}$

Let $\vec{c} = -2\hat{\jmath} + \hat{k}$ and $\vec{d} = 6\hat{\imath} + 6\hat{\jmath} + \hat{k}$

$\vec{c} - \vec{a} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$ ----------(3)

$$\vec{b}\times\vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 12 & 6 & -1 \\ 6 & 6 & 1 \end{vmatrix}$$

$$= \hat{\imath}(6+6) - \hat{\jmath}(12+6) + \hat{k}(72-36)$$

$$= 12\hat{\imath} - 18\hat{\jmath} + 36\hat{k}$$

$\vec{b}\times\vec{d} = 6(2\hat{\imath} - 3\hat{\jmath} + 6\hat{k})$ --------- (4)

$$|\vec{b}\times\vec{d}| = 6\sqrt{4+9+36} = 6(7) = 42$$

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d}) = (1)(12) + (-2)(-18) + (1)(36)$$

$$= 12 + 36 + 36$$

$$= 84 \neq 0$$

∴ Given lines are skew lines.

$$d = \frac{|84|}{42} = \frac{84}{42}$$

$$d = 2\ units$$

6. Find the parametric form of vector equation of the straight line passing through $(-1, 2, 1)$ and parallel to the straight line $\vec{r} = (2\hat{\imath} + 3\hat{\jmath} - \hat{k}) + t(\hat{\imath} - 2\hat{\jmath} + \hat{k})$ and hence find the shortest distance between the lines.

Solution:

Let $\vec{a} = -\hat{\imath} + 2\hat{\jmath} + \hat{k}$

$\vec{b} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$

$\vec{c} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$

Parametric form of vector equation is,

$$\vec{r} = \vec{a} + t\vec{b}$$

$$\vec{r} = -\hat{\imath} + 2\hat{\jmath} + \hat{k} + t(\hat{\imath} - 2\hat{\jmath} + \hat{k})$$

Shortest distance between the lines,

$$d = \frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|} \qquad \text{----------(1)}$$

$$\vec{c} - \vec{a} = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$$

$$(\vec{c} - \vec{a}) \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= \hat{\imath}(1-4) - \hat{\jmath}(3+2) + \hat{k}(-6-1)$$

$$(\vec{c} - \vec{a}) \times \vec{b} = -3\hat{\imath} - 5\hat{\jmath} - 7\hat{k}$$

$$|(\vec{c} - \vec{a}) \times \vec{b}| = \sqrt{9 + 25 + 49} = \sqrt{83}$$

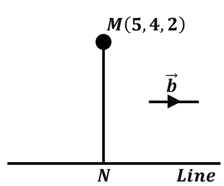
$$|\vec{b}| = \sqrt{1 + 4 + 1} = \sqrt{6}$$

$$d = \frac{\sqrt{83}}{\sqrt{6}}\ units \qquad \text{From (1)}$$

---

7. Find the foot of the perpendicular drawn from the point $(5, 4, 2)$ to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1}$. Also, find the equation of the perpendicular.

Solution:



Let $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$

Any point on the line is,

$$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1} = s\ (say)$$

The point $N = (2s - 1, 3s + 3, -s + 1)$ ----------(1)

$$\overrightarrow{ON} = (2s-1)\hat{\imath} + (3s+3)\hat{\jmath} + (-s+1)\hat{k}$$

$$\overrightarrow{OM} = 5\hat{\imath} + 4\hat{\jmath} + 2\hat{k}$$

$$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$$

$$\overrightarrow{MN} = (2s-6)\hat{\imath} + (3s-1)\hat{\jmath} + (-s-1)\hat{k}$$

Since $\vec{b}$ is perpendicular to $\overrightarrow{MN}$, $\quad \overrightarrow{MN}.\vec{b} = 0$

$$(2s-6)(2) + (3s-1)(3) + (-s-1)(-1) = 0$$

$$4s - 12 + 9s - 3 + s + 1 = 0$$

$$14s = 14$$

$$s = 1$$

$\therefore$ The Coordinates of N is $(1,6,0)$

Cartesian equation of line $M(5,4,2)$ and $N(1,6,0)$ is,

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-5}{-4} = \frac{y-4}{2} = \frac{z-2}{-2}$$

Exercise 6.6

1. Find the vector equation of a plane which is at a distance of 7 units from the origin having $3, -4, 5$ as direction ratios of a normal to it.

Solution:

Given $p = 7$

$$\vec{d} = 3\hat{\imath} - 4\hat{\jmath} + 5\hat{k}$$

$$|\vec{d}| = \sqrt{9 + 16 + 25} = \sqrt{25 \times 2} = 5\sqrt{2}$$

$$\hat{d} = \frac{\vec{d}}{|\vec{d}|} = \frac{1}{5\sqrt{2}}(3\hat{\imath} - 4\hat{\jmath} + 5\hat{k})$$

Vector equation of plane is,

$$\vec{r}.\hat{d} = p$$

$$\vec{r}.\left(\frac{3\hat{\imath} - 4\hat{\jmath} + 5\hat{k}}{5\sqrt{2}}\right) = 7$$

---

2. Find the direction cosines of the normal to the plane $12x + 3y - 4z = 65$. Also, find the non-parametric form of a vector equation of a plane and the length of the perpendicular to the plane from the origin.

Solution:

Given $12x + 3y - 4z = 65$

Vector equation of plane is,

$$\vec{r}.(12\hat{\imath} + 3\hat{\jmath} - 4\hat{k}) = 65$$

$$\vec{r}.\frac{12\hat{\imath} + 3\hat{\jmath} - 4\hat{k}}{\sqrt{144+9+16}} = \frac{65}{\sqrt{144+9+16}}$$

$$\vec{r}.\left(\frac{12\hat{\imath} + 3\hat{\jmath} - 4\hat{k}}{13}\right) = \frac{65}{13}$$

$$\vec{r}.\left(\frac{12\hat{\imath} + 3\hat{\jmath} - 4\hat{k}}{13}\right) = 5$$

Comparing with $\vec{r}.\hat{d} = p$, we get

The length of the $\perp$ to the plane from the origin is 5.

Direction cosines $\left(\frac{12}{13}, \frac{3}{13}, -\frac{4}{13}\right)$

---

3. Find the vector and cartesian equations of the plane passing through the point with position vector $2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}$ and normal to the vector $\hat{\imath} + 3\hat{\jmath} + 5\hat{k}$.

Solution:

Let $\vec{a} = 2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}$

$\vec{n} = \hat{\imath} + 3\hat{\jmath} + 5\hat{k}$

Vector equation of plane is,

$$\vec{r}.\vec{n} = \vec{a}.\vec{n}$$

$$\vec{r}.\left(\hat{\imath} + 3\hat{\jmath} + 5\hat{k}\right) = 2 + 18 + 15$$

$$\vec{r}.\left(\hat{\imath} + 3\hat{\jmath} + 5\hat{k}\right) = 35$$

Cartesian equation of the plane is,

$$\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right).\left(\hat{\imath} + 3\hat{\jmath} + 5\hat{k}\right) = 35$$

$$x + 3y + 5z = 35$$

4. A plane passes through the point $(-1, 1, 2)$ and the normal to the plane of magnitude $3\sqrt{3}$ makes equal acute angle with the coordinate axes. Find the equation of the plane.

Solution:

Let $\vec{a} = -\hat{\imath} + \hat{\jmath} + 2\hat{k}$

Magnitude $= 3\sqrt{3}$

Wkt, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

The normal to the plane makes an equal angle

$$\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1 \qquad (\alpha = \beta = \gamma)$$

$$3\cos^2\alpha = 1$$

$$\cos^2\alpha = \frac{1}{3}$$

$$\cos\alpha = \frac{1}{\sqrt{3}}$$

Direction cosines $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$\hat{n} = \frac{1}{\sqrt{3}}\hat{\imath} + \frac{1}{\sqrt{3}}\hat{\jmath} + \frac{1}{\sqrt{3}}\hat{k}$$

$$\hat{n} = \frac{1}{\sqrt{3}}\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right)$$

$$\vec{n} = 3\sqrt{3}\,\hat{n} \qquad \left(\because \hat{n} = \frac{\vec{n}}{|\vec{n}|}\right)$$

$$\vec{n} = 3\sqrt{3}\left[\frac{1}{\sqrt{3}}\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right)\right]$$

$$\vec{n} = 3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}$$

Vector equation of plane is,

$$\vec{r}.\vec{n} = \vec{a}.\vec{n}$$

$$\vec{r}.\left(3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}\right) = -3 + 3 + 6$$

$$\vec{r}.3\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right) = 6$$

$$\vec{r}.\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right) = 2$$

Cartesian equation of plane is,

$$\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right).\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right) = 6$$

$$x + y + z = 6$$

5. Find the intercepts cut off by the plane $\vec{r}.\left(6\hat{\imath} + 4\hat{\jmath} - 3\hat{k}\right) = 12$ on the coordinate axes.

Solution:

Given $\vec{r}.\left(6\hat{\imath} + 4\hat{\jmath} - 3\hat{k}\right) = 12$

Cartesian equation of plane is,

$$6x + 4y - 3z = 12$$

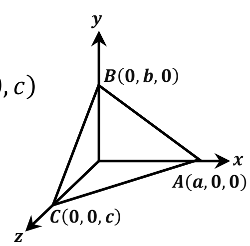
$$\div 12, \qquad \frac{x}{2} + \frac{y}{3} - \frac{z}{4} = 1$$

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{-4} = 1$$

$\therefore$ x-intercept $= 2$, y-intercept $= 3$, z-intercept $= -4$

6. If a plane meets the coordinate axes at A, B, C such that the centroid of the triangle ABC is the point (u, v, w), find the equation of the plane.

Solution:



Let $A(a, 0,0), B(0, b, 0)$ and $C(0,0, c)$

Centroid of $\Delta ABC = \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$

$$(u, v, w) = \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$$

Equating the coordinates, we get

$u = \frac{a}{3}$	$v = \frac{b}{3}$	$w = \frac{c}{3}$
$3u = a$	$3v = b$	$3w = c$

Equation of plane is,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$\frac{x}{3u} + \frac{y}{3v} + \frac{z}{3w} = 1$$

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 3$$

Exercise 6.7

1. Find the non-parametric form of vector equation, and cartesian equations of the plane passing through the point $(2, 3, 6)$ and parallel to the straight lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-3}{1}$ and $\frac{x+3}{2} = \frac{y-3}{-5} = \frac{z+1}{-3}$

Solution:

$\Rightarrow \quad (x_1, y_1, z_1) = (2,3,6)$

$\Rightarrow \quad (l_1, m_1, n_1) = (2,3,1)$ and $(l_2, m_2, n_2) = (2, -5, -3)$

Cartesian equation of plane is,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2 & y-3 & z-6 \\ 2 & 3 & 1 \\ 2 & -5 & -3 \end{vmatrix} = 0$$

$$(x-2)(-4)-(y-3)(-8)+(z-6)(-16)=0$$

$$-4x+8+8y-24-16z+96=0$$

$$-4x+8y-16z+80=0$$

$\div 4,$ $$-x+2y-4z+20=0$$

$$\boldsymbol{x-2y+4z-20=0}$$

$$x-2y+4z=20$$

Non-parametric form of vector equation of plane is,

$$\boldsymbol{\vec{r}.\left(\hat{\imath}-2\hat{\jmath}+4\hat{k}\right)=20}$$

---

2. Find the non-parametric form of vector equation, and cartesian equations of the plane passing through the points $(2,2,1)$, $(9,3,6)$ and perpendicular to the plane $2x+6y+6z=9$.

Solution:

$(x_1,y_1,z_1)=(2,2,1)$ and $(x_2,y_2,z_2)=(9,3,6)$

$(l,m,n)=(2,6,6)$

Cartesian equation of plane is,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l & m & n \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix} = 0$$

$$(x-2)(-24)-(y-2)(32)+(z-1)(40)=0$$

$$-24x+48-32y+64+40z-40=0$$

$$-24x-32y+40z+72=0$$

$\div 8,$ $$-3x-4y+5z+9=0$$

$$\boldsymbol{3x+4y-5z-9=0}$$

$$3x+4y-5z=9$$

Non-parametric form of vector equation of plane is,

$$\boldsymbol{\vec{r}.\left(3\hat{\imath}+4\hat{\jmath}-5\hat{k}\right)=9}$$

---

3. Find parametric form of vector equation and cartesian equations of the plane passing through the points $(2,2,1)$, $(1,-2,3)$ and parallel to the straight line passing through the points $(2,1,-3)$ and $(-1,5,-8)$.

Solution:

Let $\vec{a}=2\hat{\imath}+2\hat{\jmath}+\hat{k}$

$\vec{b}=\hat{\imath}-2\hat{\jmath}+3\hat{k}$

Parallel to $\vec{c}=-3\hat{\imath}+4\hat{\jmath}-5\hat{k}$

Parametric form of vector equation is,

$$\vec{r}=\vec{a}+s\left(\vec{b}-\vec{a}\right)+t\vec{c}, \text{ where } s,t\in R$$

$$\vec{r}=2\hat{\imath}+2\hat{\jmath}+\hat{k}+s\left(-\hat{\imath}-4\hat{\jmath}+2\hat{k}\right)+ t\left(-3\hat{\imath}+4\hat{\jmath}-5\hat{k}\right)$$

Cartesian equation plane is,

$(x_1,y_1,z_1)=(2,2,1)$ and $(x_2,y_2,z_2)=(1,-2,3)$

$(l,m,n)=(-3,4,-5)$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l & m & n \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ -1 & -4 & 2 \\ -3 & 4 & -5 \end{vmatrix} = 0$$

$$(x-2)(12)-(y-2)(11)+(z-1)(-16)=0$$

$$12x-24-11y+22-16z+16=0$$

$$12x-11y-16z+14=0$$

---

4. Find the non-parametric form of vector equation and cartesian equation of the plane passing through the point $(1,-2,4)$ and perpendicular to the plane $x+2y-3z=11$ and parallel to the line $\frac{x+7}{3}=\frac{y+3}{-1}=\frac{z}{1}$.

Solution:

$(x_1,y_1,z_1)=(1,-2,4)$

$(l_1,m_1,n_1)=(1,2,-3)$ and $(l_2,m_2,n_2)=(3,-1,1)$

Cartesian equation of plane is,

$$\begin{vmatrix} x\_x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-1 & y+2 & z-4 \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix} = 0$$

$$(x-1)(-1)-(y+2)(10)+(z-4)(-7)=0$$

$$-x+1-10y-20-7z+28=0$$

$$-x-10y-7z+9=0$$

$$\boldsymbol{x+10y+7z-9=0}$$

$$x + 10y + 7z = 9$$

Non-parametric form of vector equation of plane is,

$$\boldsymbol{\vec{r}.\left(\hat{\imath} + 10\hat{\jmath} + 7\hat{k}\right) = 9}$$

5. Find the parametric form of vector equation, and cartesian equations of the plane containing the line $\vec{r} = \left(\hat{\imath} - \hat{\jmath} + 3\hat{k}\right) + t\left(2\hat{\imath} - \hat{\jmath} + 4\hat{k}\right)$ and perpendicular to plane $\vec{r}.\left(\hat{\imath} + 2\hat{\jmath} + \hat{k}\right) = 8$.

Solution:

Let $\vec{a} = \hat{\imath} - \hat{\jmath} + 3\hat{k}$

Parallel to $\vec{b} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$ and $\vec{c} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$

Parametric form of vector equation is,

$\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$, where $s, t \in R$

$$\vec{r} = \hat{\imath} - \hat{\jmath} + 3\hat{k} + s\left(2\hat{\imath} - \hat{\jmath} + 4\hat{k}\right) + t\left(\hat{\imath} + 2\hat{\jmath} + \hat{k}\right)$$

Cartesian equation of plane is,

$(x_1, y_1, z_1) = (1, -1, 3)$

$(l_1, m_1, n_1) = (2, -1, 4)$ and $(l_2, m_2, n_2) = (1, 2, 1)$

Cartesian equation of plane is,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 1 & y + 1 & z - 3 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$$(x - 1)(-9) - (y + 1)(-2) + (z - 3)(5) = 0$$

$$-9x + 9 + 2y + 2 + 5z - 15 = 0$$

$$-9x + 2y + 5z - 4 = 0$$

$$9x - 2y - 5z + 4 = 0$$

6. Find the parametric vector, non-parametric vector and cartesian form of the equations of the plane passing through the three non-collinear points $(3, 6, -2), (-1, -2, 6)$ and $(6, 4, -2)$.

Solution:

Let $\vec{a} = 3\hat{\imath} + 6\hat{\jmath} - 2\hat{k}$

$\vec{b} = -\hat{\imath} - 2\hat{\jmath} + 6\hat{k}$

$\vec{c} = 6\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$

Parametric form of vector equation is,

$\vec{r} = \vec{a} + s\left(\vec{b} - \vec{a}\right) + t(\vec{c} - \vec{a})$, where $s, t \in R$

$$\vec{r} = 3\hat{\imath} + 6\hat{\jmath} - 2\hat{k} + s\left(-4\hat{\imath} - 8\hat{\jmath} + 8\hat{k}\right) + t(3\hat{\imath} + 2\hat{\jmath})$$

Cartesian equation of plane is,

$(x_1, y_1, z_1) = (3, 6, -2)$, $(x_2, y_2, z_2) = (-1, -2, 6)$ and

$(x_3, y_3, z_3) = (6, 4, -2)$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 3 & y - 6 & z + 2 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$$

$$(x - 3)(16) - (y - 6)(-24) + (z + 2)(32) = 0$$

$\div 8,$ $\quad (x - 3)(2) - (y - 6)(-3) + (z + 2)(4) = 0$

$$2x - 6 + 3y - 18 + 4z + 8 = 0$$

$$\boldsymbol{2x + 3y + 4z - 16 = 0}$$

$$2x + 3y + 4z = 16$$

Non-parametric form of vector equation of plane is,

$$\boldsymbol{\vec{r}.\left(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}\right) = 16}$$

7. Find the non-parametric form of vector equation, and cartesian equations of the plane $\vec{r} = \left(6\hat{\imath} - \hat{\jmath} + \hat{k}\right) - s\left(-\hat{\imath} + 2\hat{\jmath} + \hat{k}\right) + t\left(-5\hat{\imath} - 4\hat{\jmath} - 5\hat{k}\right)$.

Solution:

$(x_1, y_1, z_1) = (6, -1, 1)$

$(l_1, m_1, n_1) = (-1, 2, 1)$ and $(l_2, m_2, n_2) = (-5, -4, -5)$

Cartesian equation of plane is,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 6 & y + 1 & z - 1 \\ -1 & 2 & 1 \\ -5 & -4 & -5 \end{vmatrix} = 0$$

$$(x - 6)(-6) - (y + 1)(10) + (z - 1)(14) = 0$$

$$-6x + 36 - 10y - 10 + 14z - 14 = 0$$

$$-6x - 10y + 14z + 12 = 0$$

$\div 2,$ $\quad -3x - 5y + 7z + 6 = 0$

$$\boldsymbol{3x + 5y - 7z - 6 = 0}$$

$$3x + 5y - 7z = 6$$

Non-parametric form of vector equation of plane is,

$$\boldsymbol{\vec{r}.\left(3\hat{\imath} + 5\hat{\jmath} - 7\hat{k}\right) = 6}$$

Exercise 6.8

1. Show that the straight lines $\vec{r} = \left(5\hat{\imath} + 7\hat{\jmath} - 3\hat{k}\right) + s\left(4\hat{\imath} + 4\hat{\jmath} - 5\hat{k}\right)$ and $\vec{r} = \left(8\hat{\imath} + 4\hat{\jmath} + 5\hat{k}\right) + t\left(7\hat{\imath} + \hat{\jmath} +$

$3\hat{k})$ are coplanar. Find the vector equations of the plane in which they lie.

Solution:

Condition for coplanar is, $[\vec{c}-\vec{a},\vec{b},\vec{d}]=0$

Let $\vec{a}=5\hat{\imath}+7\hat{\jmath}-3\hat{k}$, $\vec{b}=4\hat{\imath}+4\hat{\jmath}-5\hat{k}$

And $\vec{c}=8\hat{\imath}+4\hat{\jmath}+5\hat{k}$, $\vec{d}=7\hat{\imath}+\hat{\jmath}+3\hat{k}$

$$\vec{c}-\vec{a}=3\hat{\imath}-3\hat{\jmath}+8\hat{k}$$

$$[\vec{c}-\vec{a},\vec{b},\vec{d}]=\begin{vmatrix}3 & -3 & 8\\ 4 & 4 & -5\\ 7 & 1 & 3\end{vmatrix}$$

$$=3(17)+3(47)+8(-24)$$

$$=51+141-192$$

$$=192-192$$

$$=0$$

Hence given lines are coplanar.

Equation of plane is, $[\vec{r}-\vec{a},\vec{b},\vec{d}]=0$

$$\begin{vmatrix}x-5 & y-7 & z+3\\ 4 & 4 & -5\\ 7 & 1 & 3\end{vmatrix}=0$$

$$(x-5)(17)-(y-7)(47)+(z+3)(-24)=0$$

$$17x-85-47y+329-24z-72=0$$

$$17x-47y-24z+172=0$$

$$17x-47y-24z=-172$$

Non-parametric form of vector equation is,

$$\boldsymbol{\vec{r}.(17\hat{\imath}-47\hat{\jmath}-24\hat{k})=-172}$$

---

2. Show that the lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{3}$ and $\frac{x-1}{-3}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar. Also, find the plane containing these lines.

Solution:

Condition for coplanar is,

$$[\vec{c}-\vec{a},\vec{b},\vec{d}]=0$$

Let $\vec{a}=2\hat{\imath}+3\hat{\jmath}+4\hat{k}$, $\vec{b}=\hat{\imath}+\hat{\jmath}+3\hat{k}$

And $\vec{c}=\hat{\imath}+4\hat{\jmath}+5\hat{k}$, $\vec{d}=-3\hat{\imath}+2\hat{\jmath}+\hat{k}$

$$\vec{c}-\vec{a}=-\hat{\imath}+\hat{\jmath}+\hat{k}$$

$$[\vec{c}-\vec{a},\vec{b},\vec{d}]=\begin{vmatrix}-1 & 1 & 1\\ 1 & 1 & 3\\ -3 & 2 & 1\end{vmatrix}$$

$$=-1(-5)-1(10)+1(5)$$

$$=5-10+5$$

$$=10-10$$

$$=0$$

Hence given lines are coplanar.

Equation of plane is, $[\vec{r}-\vec{a},\vec{b},\vec{d}]=0$

$$\begin{vmatrix}x-2 & y-3 & z-4\\ 1 & 1 & 3\\ -3 & 2 & 1\end{vmatrix}=0$$

$$(x-2)(-5)-(y-3)(10)+(z-4)(5)=0$$

$$-5x+10-10y+30+5z-20=0$$

$$-5x-10y+5z+20=0$$

$\div 5,$ $$-x-2y+z+4=0$$

$$x+2y-z-4=0$$

---

3. If the straight lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{m^2}$ and $\frac{x-3}{1}=\frac{y-2}{m^2}=\frac{z-1}{2}$ are coplanar, find the distinct real values of m.

Solution:

Let $\vec{a}=\hat{\imath}+2\hat{\jmath}+3\hat{k}$, $\vec{b}=\hat{\imath}+2\hat{\jmath}+m^2\hat{k}$

And $\vec{c}=3\hat{\imath}+2\hat{\jmath}+\hat{k}$, $\vec{d}=\hat{\imath}+m^2\hat{\jmath}+2\hat{k}$

$\Rightarrow$ $\vec{c}-\vec{a}=2\hat{\imath}-2\hat{k}$

Since given lines are coplanar,

$$[\vec{c}-\vec{a},\vec{b},\vec{d}]=0$$

$$\begin{vmatrix}2 & 0 & -2\\ 1 & 2 & m^2\\ 1 & m^2 & 2\end{vmatrix}=0$$

$$2(4-m^4)-2(m^2-2)=0$$

$$8-2m^4-2m^2+4=0$$

$$-2m^4-2m^2+12=0$$

$\div -2,$ $$m^4+m^2-6=0$$

$$m^4+m^2=6$$

$$m^2(m^2+1)=2\times 3$$

$$m^2=2$$

$$m=\pm\sqrt{2}$$

$\therefore$ Distinct real values of m are $\sqrt{2},-\sqrt{2}$

---

4. If the straight lines $\frac{x-1}{2}=\frac{y+1}{\lambda}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{\lambda}$ are coplanar, find λ and equations of the planes containing these two lines.

Solution:

Let $\vec{a} = \hat{\imath} - \hat{\jmath}, \vec{b} = 2\hat{\imath} + \lambda\hat{\jmath} + 2\hat{k}$

And $\vec{c} = -\hat{\imath} - \hat{\jmath}, \vec{d} = 5\hat{\imath} + 2\hat{\jmath} + \lambda\hat{k}$

$\Rightarrow \quad \vec{c} - \vec{a} = -2\hat{\imath}$

Since given lines are coplanar,

$$[\vec{c} - \vec{a}, \vec{b}, \vec{d}] = 0$$

$$\begin{vmatrix} -2 & 0 & 0 \\ 2 & \lambda & 2 \\ 5 & 2 & \lambda \end{vmatrix} = 0$$

$$-2(\lambda^2 - 4) = 0$$

$$\lambda^2 - 4 = 0$$

$$\lambda^2 = 4$$

$$\lambda = \pm 2$$

Equation of plane is,

$$[\vec{r} - \vec{a}, \vec{b}, \vec{d}] = 0$$

$$\begin{vmatrix} x-1 & y+1 & z \\ 2 & \lambda & 2 \\ 5 & 2 & \lambda \end{vmatrix} = 0$$

$$(x-1)(\lambda^2 - 4) - (y+1)(2\lambda - 10) + z(4 - 5\lambda) = 0$$

At $\lambda = 2$,

$$-(y+1)(-6) + z(-6) = 0$$

$\div 6$, $\quad y + 1 - z = 0$

$$\boldsymbol{y - z + 1 = 0}$$

At $\lambda = -2$,

$$-(y+1)(-14) + z(14) = 0$$

$\div 14$, $\quad y + 1 + z = 0$

$$\boldsymbol{y + z + 1 = 0}$$

Exercise 6.9

1. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.(2\hat{\imath} - 7\hat{\jmath} + 4\hat{k}) = 3$ and $3x - 5y + 4z + 11 = 0$, and the point $(-2, 1, 3)$.

Solution:

Given $2x - 7y + 4z - 3 = 0$ and

$$3x - 5y + 4z + 11 = 0$$

Equation of plane is,

$$2x - 7y + 4z - 3 + \lambda(3x - 5y + 4z + 11) = 0 \text{ -----(1)}$$

Passing through $(-2,1,3)$

$$(-4 - 7 + 12 - 3) + \lambda(-6 - 5 + 12 + 11) = 0$$

$$-2 + \lambda(12) = 0$$

$$12\lambda = 2$$

$$\lambda = \frac{1}{6}$$

From (1), we get

$$2x - 7y + 4z - 3 + \left(\frac{1}{6}\right)(3x - 5y + 4z + 11) = 0$$

$$12x - 42y + 24z - 18 + 3x - 5y + 4z + 11 = 0$$

$$15x - 47y + 28z - 7 = 0$$

---

2. Find the equation of the plane passing through the line of intersection of the planes $x + 2y + 3z = 2$ and $x - y + z = 3$, and at a distance $\frac{2}{\sqrt{3}}$ from the point $(3, 1, -1)$.

Solution:

Equation of plane is,

$$x + 2y + 3z - 2 + \lambda(x - y + z - 3) = 0 \text{ ------(1)}$$

$$(1+\lambda)x + (2-\lambda)y + (3+\lambda)z - 2 - 3\lambda = 0$$

$$\left|\frac{(1+\lambda)3 + (2-\lambda)1 + (3+\lambda)(-1) - 2 - 3\lambda}{\sqrt{(1+\lambda)^2 + (2-\lambda)^2 + (3+\lambda)^2}}\right| = \frac{2}{\sqrt{3}}$$

$$\left|\frac{3 + 3\lambda + 2 - \lambda - 3 - \lambda - 2 - 3\lambda}{\sqrt{1 + \lambda^2 + 2\lambda + 4 + \lambda^2 - 4\lambda + 9 + \lambda^2 + 6\lambda}}\right| = \frac{2}{\sqrt{3}}$$

$$\left|\frac{-2\lambda}{\sqrt{3\lambda^2 + 4\lambda + 14}}\right| = \frac{2}{\sqrt{3}}$$

Squaring on both sides, we get

$$\frac{4\lambda^2}{3\lambda^2 + 4\lambda + 14} = \frac{4}{3}$$

$$3\lambda^2 = 3\lambda^2 + 4\lambda + 14$$

$$4\lambda + 14 = 0$$

$$4\lambda = -14$$

$$\lambda = -\frac{7}{2}$$

From (1), we get

$$x + 2y + 3z - 2 + \left(-\frac{7}{2}\right)(x - y + z - 3) = 0$$

$$2x + 4y + 6z - 4 - 7x + 7y - 7z + 21 = 0$$

$$-5x + 11y - z + 17 = 0$$

$$5x - 11y + z - 17 = 0$$

---

3. Find the angle between the line $\vec{r} = (2\hat{\imath} - \hat{\jmath} + \hat{k}) + t(\hat{\imath} + 2\hat{\jmath} - 2\hat{k})$ and the plane $\vec{r}.(6\hat{\imath} + 3\hat{\jmath} + 2\hat{k}) = 8$

Solution:

Angle between the line and a plane is,

$$\theta = \sin^{-1}\left(\frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}\right) \quad \text{----------(1)}$$

Let $\vec{b} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$ and $\vec{n} = 6\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$

$$\vec{b}.\vec{n} = 1(6) + 2(3) - 2(2)$$

$$= 6 + 6 - 4$$

$$\vec{b}.\vec{n} = 8$$

$$|\vec{b}| = \sqrt{1+4+4} = 3$$

$$|\vec{n}| = \sqrt{36+9+4} = 7$$

From (1), we get

$$\theta = \sin^{-1}\left(\frac{8}{3(7)}\right)$$

$$\theta = \sin^{-1}\left(\frac{8}{21}\right)$$

4. Find the angle between the plane $\vec{r}.\left(\hat{\imath} + \hat{\jmath} - 2\hat{k}\right) = 3$ and $2x - 2y + z = 2$.

Solution:

The Angle between the two planes is,

$$\theta = \cos^{-1}\left(\frac{|\vec{n_1}.\vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}\right) \quad \text{----------(1)}$$

Let $\vec{n_1} = \hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{n_2} = 2\hat{\imath} - 2\hat{\jmath} + \hat{k}$

$$\vec{n_1}.\vec{n_2} = 1(2) + 1(-2) - 2(1)$$

$$= 2 - 2 - 2$$

$$\vec{n_1}.\vec{n_2} = -2$$

$$|\vec{n_1}| = \sqrt{1+1+4} = \sqrt{6}$$

$$|\vec{n_2}| = \sqrt{4+4+1} = 3$$

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{6}}\right) \quad \text{From (1)}$$

5. Find the equation of the plane which passes through the point $(3, 4, -1)$ and is parallel to the plane $2x - 3y + 5z + 7 = 0$. Also, find the distance between the two planes.

Solution:

Given $2x - 3y + 5z + 7 = 0$ ----------(1)

Parallel to the plane is,

$$2x - 3y + 5z + k = 0 \quad \text{----------(2)}$$

Passing through $(3,4,-1)$

$$6 - 12 - 5 + k = 0$$

$$-11 + k = 0$$

$$k = 11$$

From (2), we get

$$2x - 3y + 5z + 11 = 0 \quad \text{----------(3)}$$

Distance between two parallel planes,

$$d = \left|\frac{d_1 - d_2}{\sqrt{a^2+b^2+c^2}}\right|$$

$a = 2, b = -3, c = 5$ and $d_1 = 7, d_2 = 11$

$$d = \left|\frac{7-11}{\sqrt{4+9+25}}\right|$$

$$d = \left|\frac{-4}{\sqrt{38}}\right|$$

$$d = \frac{4}{\sqrt{38}}\ units$$

6. Find the length of the perpendicular from the point $(1, -2, 3)$ to the plane $x - y + z = 5$.

Solution:

Given $(x_1, y_1, z_1) = (1, -2, 3)$

$\Rightarrow \quad a = 1, b = -1, c = 1, and\ d = -5$

Distance between the point and a plane is,

$$d = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2+b^2+c^2}}\right|$$

$$d = \left|\frac{1+2+3-5}{\sqrt{1+1+1}}\right|$$

$$d = \left|\frac{1}{\sqrt{3}}\right|$$

$$d = \frac{1}{\sqrt{3}}\ units$$

7. Find the point of intersection of the line $x - 1 = \frac{y}{2} = z + 1$ with the plane $2x - y + 2z = 2$. Also, find the angle between the line and the plane.

Solution:

$$\frac{x-1}{1} = \frac{y}{2} = \frac{z+1}{1} = t$$

$$(x, y, z) = (t+1, 2t, t-1)$$

Given plane is $2x - y + 2z = 2$

At $(t+1, 2t, t-1)$

$$2(t+1) - 2t + 2(t-1) = 2$$

$$2t + 2 - 2t + 2t - 2 = 2$$

$$2t = 2$$

$$t = 1$$

$\therefore$ Point of intersection is $(2,2,0)$

Angle between the line and a plane is,

$$\theta = \sin^{-1}\left(\frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}\right) \quad \text{----------(1)}$$

Let $\vec{b} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$ and $\vec{n} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$

$$\vec{b}.\vec{n} = 2 - 2 + 2 = 2$$

$$|\vec{b}| = \sqrt{1+4+1} = \sqrt{6}$$

$$|\vec{n}| = \sqrt{4+1+4} = 3$$

From (1), we get

$$\theta = \sin^{-1}\left(\frac{|2|}{3\sqrt{6}}\right)$$

$$\theta = \sin^{-1}\left(\frac{2}{3\sqrt{6}}\right)$$

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8. Find the coordinates of the foot of the perpendicular and length of the perpendicular from the point $(4,3,2)$ to the plane $x + 2y + 3z = 2$.

Solution:

The direction of normal plane is $(1,2,3)$

Equation of line is,

$$\Rightarrow \quad \frac{x-4}{1} = \frac{y-3}{2} = \frac{z-2}{3} = t$$

$$\Rightarrow \quad (x,y,z) = (t+4, 2t+3, 3t+2)$$

Given plane is $x + 2y + 3z - 2 = 0$

At $(t+4, 2t+3, 3t+2)$

$$t + 4 + 2(2t+3) + 3(3t+2) - 2 = 0$$

$$t + 4 + 4t + 6 + 9t + 6 - 2 = 0$$

$$14t + 14 = 0$$

$$14t = -14$$

$$t = -1$$

∴ Point of intersection is $(3,1,-1)$

Distance between $(4,3,2)$ and $(3,1,-1)$

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$$d = \sqrt{(-1)^2 + (-2)^2 + (-3)^2}$$

$$d = \sqrt{1+4+9}$$

$$d = \sqrt{14} \text{ units}$$
