CHAPTER – 1

Application of Matrices & Determinants

Exercise 1.1

1.Find the adjoint of the following:

Hint: $adj\ A = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$ $(for\ 2\times 2\ matrix)$

(i) $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$

$A_{ij} = \begin{bmatrix} 2 & -6 \\ -4 & -3 \end{bmatrix}$

$adj\ A = (A_{ij})^T$

$adj\ A = \begin{bmatrix} 2 & -4 \\ -6 & -3 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

$$adj\ A = \begin{bmatrix} +(8-7) & -(6-3) & +(21-12) \\ -(6-7) & +(4-3) & -(14-9) \\ +(3-4) & -(2-3) & +(8-9) \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

Hint: $adj(kA) = k^{n-1}adj(A), k\ is\ a\ scalar.$

(iii) $\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

Solution:

Let $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

$$adj\ A = \left(\frac{1}{3}\right)^2 \begin{bmatrix} +(2+4) & -(-4-2) & +(4-1) \\ -(4+2) & +(4-1) & -(-4-2) \\ +(4-1) & -(4+2) & +(2+4) \end{bmatrix}^T$$

$$adj\ A = \left(\frac{1}{9}\right) \begin{bmatrix} 6 & 6 & 3 \\ -6 & 3 & 6 \\ 3 & -6 & 6 \end{bmatrix}^T$$

$$adj\ A = \left(\frac{1}{3}\right) \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}^T$$

$$adj\ A = \left(\frac{1}{3}\right) \begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

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2. Find the inverse (if it exists) of the following:

Hint: $A^{-1} = \frac{1}{|A|}(adj\ A)$. **where** $|A| \neq 0$.

(i) $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

$|A| = \begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} = 6 - 4 = 2 \neq 0$

∴ A is a Non-singular matrix and A^{-1} exists.

$A^{-1} = \frac{1}{|A|}(adj\ A)$

$adj\ A = \begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$

$A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$

(ii) $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$|A| = \begin{vmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{vmatrix}$

$= 5(25-1) - 1(5-1) + 1(1-5)$

$= 120 - 4 - 4$

$= 112 \neq 0$

∴ A is a Non-singular matrix and A^{-1} exists.

$$adj\ A = \begin{bmatrix} +(25-1) & -(5-1) & +(1-5) \\ -(5-1) & +(25-1) & -(5-1) \\ +(1-5) & -(5-1) & +(25-1) \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$$

$$A^{-1} = \frac{1}{112}\begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$$

$$A^{-1} = \frac{1}{28}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

(iii) $\begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{3} & \mathbf{4} & \mathbf{1} \\ \mathbf{3} & \mathbf{7} & \mathbf{2} \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{vmatrix}$$

$$= 2(8-7) - 3(6-3) + (21-12)$$

$$= 2 - 9 + 9$$

$$= 2 \neq 0$$

$\therefore$ A is a Non-singular matrix and A^{-1} exists.

$$A^{-1} = \frac{1}{|A|}(adj\ A)$$

$$adj\ A = \begin{bmatrix} +(8-7) & -(6-3) & +(21-12) \\ -(6-7) & +(4-3) & -(14-9) \\ +(3-4) & -(2-3) & +(8-9) \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

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3. If $\boldsymbol{F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}}$ **show that**

$\boldsymbol{[F(\alpha)]^{-1} = F(-\alpha).}$

Hint: $\boldsymbol{\sin(-\alpha) = -\sin\alpha}$ **and** $\boldsymbol{\cos(-\alpha) = \cos\alpha}$

Solution:

Given $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$

To Prove: $[F(\alpha)]^{-1} = F(-\alpha)$.

$$F(\alpha) = \begin{vmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{vmatrix}$$

$$= \cos\alpha\,(\cos\alpha - 0) - 0 + \sin\alpha\,(0 + \sin\alpha)$$

$$= \cos^2\alpha + \sin^2\alpha$$

$$= 1 \neq 0$$

$\therefore F(\alpha)$ is Non-singular matrix $[F(\alpha)]^{-1}$ exists.

$$adj\ F(\alpha) = \begin{bmatrix} +\cos\alpha & -0 & +(\sin\alpha) \\ -0 & (\cos^2\alpha + \sin^2\alpha) & -0 \\ +(-\sin\alpha) & -0 & +(\cos\alpha) \end{bmatrix}^T$$

$$adj\ F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}^T$$

$$adj\ F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$[F(\alpha)]^{-1} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \quad \text{----------(1)}$$

$$F(-\alpha) = \begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix}$$

$$F(-\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \quad \text{----------(2)}$$

From (1) and (2) we get, $[F(\alpha)]^{-1} = F(-\alpha)$

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4. If $\boldsymbol{A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}}$, ***show that*** $\boldsymbol{A^2 - 3A - 7I_2 = 0_2}$. **Hence find** $\boldsymbol{A^{-1}}$.

Solution:

Given $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} \times \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 25-3 & 15-6 \\ -5+2 & -3+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$$\text{LHS} = A^2 - 3A - 7I_2$$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 15 & 9 \\ -3 & -6 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 22-15-7 & 9-9-0 \\ -3+3-0 & 1+6-7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 0_2 = \text{RHS}$$

Now $A^2 - 3A - 7I_2 = 0_2$

Post multiply by A^{-1} we get,

$$A^2A^{-1} - 3AA^{-1} - 7I_2A^{-1} = 0_2A^{-1}$$

$$A - 3I - 7A^{-1} = 0_2$$

$$7A^{-1} = A - 3I$$

$$7A^{-1} = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$7A^{-1} = \begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

5. If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$ *prove that* $A^{-1} = A^T$.

Solution:

Given $A^{-1} = A^T$

Pre multiply by A, we get

$$AA^{-1} = AA^T$$

$$I = AA^T \qquad \text{----------(1)}$$

$$\Rightarrow \qquad A^T = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$AA^T = \left(\frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}\right)\left(\frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}\right)$$

$$= \frac{1}{81}\begin{bmatrix} 64+1+16 & -32+4+28 & -8-8+16 \\ -32+4+28 & 16+16+49 & 4-32+28 \\ -8-8+16 & 4-32+28 & 1+64+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AA^T = I$$

Hence proved.

6. If $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$, verify that $A(adj\ A) = (adj\ A)A = |A|I_2$.

Solution:

Given $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 8 & -4 \\ -5 & 3 \end{vmatrix} = 24 - 20 = 4$$

$$|A|I_2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \qquad \text{----------(1)}$$

$$adj\ A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

$$A(adj\ A) = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 24-20 & 32-32 \\ -15+15 & -20+24 \end{bmatrix}$$

$$A(adj\ A) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \qquad \text{----------(2)}$$

$$(adj\ A)A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 24-20 & -12+12 \\ 40-40 & -20+24 \end{bmatrix}$$

$$(adj\ A)A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \qquad \text{---------(3)}$$

From (1), (2) and (3) we get,

$$A(adj\ A) = (adj\ A)A = |A|I_2$$

7. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$, *and* $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$ verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution:

Given $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3+10 & -9+4 \\ -7+25 & -21+10 \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & -5 \\ 18 & -11 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 7 & -5 \\ 18 & -11 \end{vmatrix} = -77 + 90 = 13$$

$$(AB)^{-1} = \frac{1}{|AB|}(adj\ AB)$$

$$(AB)^{-1} = \frac{1}{13}\begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix} \qquad \text{----------(1)}$$

$$|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} = 15 - 14 = 1$$

$$A^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$|B| = \begin{vmatrix} -1 & -3 \\ 5 & 2 \end{vmatrix} = -2 + 15 = 13$$

$$B^{-1} = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} 10-21 & -4+9 \\ -25+7 & 10-3 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{13}\begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix} \quad \text{----------(2)}$$

From (1) and (2) we get, $(AB)^{-1} = B^{-1}A^{-1}$

8. If $adj\ A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$ find A.

Hint: $A = \pm\frac{1}{\sqrt{|adj\ A|}} adj(adj\ A)$

Solution:

Given $adj\ A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$

$$|adj\ A| = \begin{vmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{vmatrix}$$

$$= 2(24-0) + 4(-6-14) + 2(0+24)$$

$$= 48 - 80 + 48$$

$$|adj\ A| = 16$$

$$adj(adj\ A) = \begin{bmatrix} +(24-0) & -(-6-14) & +(0+24) \\ -(-8) & +(4+4) & -(-8) \\ +(28-24) & -(-14+6) & +(24-12) \end{bmatrix}^T$$

$$= \begin{bmatrix} 24 & 20 & 24 \\ 8 & 8 & 8 \\ 4 & 8 & 12 \end{bmatrix}^T$$

$$adj(adj\ A) = \begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$

$$A = \pm\frac{1}{\sqrt{|adj\ A|}} adj(adj\ A)$$

$$A = \pm\frac{1}{\sqrt{16}}\begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$

$$A = \pm\frac{1}{4}\begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$

$$A = \pm\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

9. If $adj\ A = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$ $find\ A^{-1}$.

Hint: $A^{-1} = \pm\frac{1}{\sqrt{|adj\ A|}}(adj\ A)$

Solution:

Given $adj\ A = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$

$$|adj\ A| = \begin{vmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{vmatrix}$$

$$= 0 + 2(36-18) + 0$$

$$|adj\ A| = 36$$

$$A^{-1} = \pm\frac{1}{\sqrt{|adj\ A|}}(adj\ A)$$

$$A^{-1} = \pm\frac{1}{\sqrt{36}}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

$$A^{-1} = \pm\frac{1}{6}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

10. Find $adj((adj\ A))$ $if\ adj\ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

Solution:

Given $adj\ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$adj(adj\ A) = \begin{bmatrix} +(2-0) & -(0-0) & +(0+2) \\ -(0-0) & +(1+1) & -(0-0) \\ +(0-2) & -(0-0) & +(2-0) \end{bmatrix}^T$$

$$adj(adj\ A) = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}^T$$

$$adj(adj\ A) = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

11. $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$**, show that**

$A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

Hint: $\cos 2x = \frac{1-tan^2x}{1+tan^2x}$ **and** $\sin 2x = \frac{2\tan x}{1+tan^2x}$

Solution:

Given $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \qquad \text{----------(1)}$$

$$A^{-1} = \frac{1}{|A|}\,(adj\ A)$$

$$|A| = \begin{vmatrix} 1 & \tan x \\ -\tan x & 1 \end{vmatrix} = 1 + tan^2x$$

$$A^{-1} = \frac{1}{1+tan^2x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \qquad \text{----------(2)}$$

$$A^T A^{-1} = \frac{1}{1+tan^2x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$A^T A^{-1} = \frac{1}{1+tan^2x}\begin{bmatrix} 1 - tan^2x & -\tan x - \tan x \\ \tan x + \tan x & -tan^2x + 1 \end{bmatrix}$$

$$A^T A^{-1} = \begin{bmatrix} \frac{1-tan^2x}{1+tan^2x} & \frac{-2\tan x}{1+tan^2x} \\ \frac{2\tan x}{1+tan^2x} & \frac{-tan^2x+1}{1+tan^2x} \end{bmatrix}$$

$$A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$$

Hence proved.

12. Find the matrix A for which $A\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$.

Solution:

Let $B = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

Given $AB = C$

Post multiply by B^{-1} on both sides we get,

$$ABB^{-1} = CB^{-1}$$

$$A(I) = CB^{-1}$$

$$A = CB^{-1} \qquad \text{----------(1)}$$

Now To Find B^{-1}:

$$|B| = \begin{vmatrix} 5 & 3 \\ -1 & -2 \end{vmatrix} = -10 + 3 = -7$$

$$B^{-1} = -\frac{1}{7}\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$B^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

$$A = \frac{1}{7}\begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix} \qquad \text{From (1)}$$

$$A = \frac{1}{7}\begin{bmatrix} 28-7 & 42-35 \\ 14-7 & 21-35 \end{bmatrix}$$

$$A = \frac{1}{7}\begin{bmatrix} 21 & 7 \\ 7 & -14 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$$

13. Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ **and** $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$**, Find a matrix** X **such that** $AXB = C$.

Solution:

Given $AXB = C$

Pre multiply by A^{-1} and Post multiply by B^{-1} we get,

$$A^{-1}AXBB^{-1} = A^{-1}CB^{-1}$$

$$(I)X(I) = A^{-1}CB^{-1}$$

$$X = A^{-1}CB^{-1}$$

Now To Find B^{-1} :

$$|B| = \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} = 3 + 2 = 5$$

$$B^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

Also Find A^{-1} :

$$|A| = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = 0 + 2 = 2$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$$

$$X = A^{-1}CB^{-1}$$

$$X = \left(\frac{1}{2}\right)\left(\frac{1}{5}\right)\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\right)$$

$$X = \frac{1}{10}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1-1 & 2+3 \\ 2-2 & 4+6 \end{bmatrix}$$

$$X = \frac{1}{10}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 5 \\ 0 & 10 \end{bmatrix}$$

$$X = \frac{1}{10}\begin{bmatrix} 0-0 & 0+10 \\ 0-0 & -10+10 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

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14. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, show that $A^{-1} = \frac{1}{2}(A^2 - 3I)$.

Solution:

Given $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

To Find A^{-1} :

$$A^{-1} = \frac{1}{|A|}(adj\ A)$$

$$|A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 0 - 1(0-1) + 1(1-0)$$

$$= 2$$

$$adj\ A = \begin{bmatrix} +(0-1) & -(0-1) & +(1-0) \\ -(0-1) & +(0-1) & -(0-1) \\ +(1-0) & -(0-1) & +(0-1) \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad \text{----------(1)}$$

Now To Find $\frac{1}{2}(A^2 - 3I)$:

$$A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0+1+1 & 0+0+1 & 0+1+0 \\ 0+0+1 & 1+0+1 & 1+0+0 \\ 0+1+0 & 1+0+0 & 1+1+0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^2 - 3I = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 - 3I = \begin{bmatrix} 2-3 & 1-0 & 1-0 \\ 1-0 & 2-3 & 1-0 \\ 1-0 & 1-0 & 2-3 \end{bmatrix}$$

$$A^2 - 3I = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\frac{1}{2}(A^2 - 3I) = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad \text{----------(2)}$$

From (1) and (2) we get,

$$A^{-1} = \frac{1}{2}(A^2 - 3I)$$

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15. Decrypt the received encoded message $[2 \quad -3]$ $[20 \quad 4]$ with the encryption matrix $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ and the decryption matrix as its inverse, where the system of codes is described by the numbers 1-26 and the letters A-Z respectively, and the number 0 to a blank space.

Solution:

Let $A = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= -1 + 2 = 1$$

$$adj\ A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

Encoded row matrix	**Decoding matrix (A^{-1})**	**Decoded row matrix**
$[2 \quad -3]$	$\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$	$[2 \quad -3]\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$ $= [8 \quad 5]$
$[20 \quad 4]$	$\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$	$[20 \quad 4]\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$ $= [12 \quad 16]$

$\therefore$ The decoded matrices are $[8 \quad 5]\ [12 \quad 16]$

$\therefore$ The receivers read this message as HELP.

Exercise 1.2

1. Find the rank of the following matrices by minor method:

(i) $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

A is of order 2×2

$\rho(A) \le minimum\{2,2\}$

$\rho(A) \le 2$

$|A| = \begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix}$

$|A| = 4 - 4 = 0$

$\rho(A) \le 1$

Now, consider a minor of $a_{11} = 2 \neq 0$

Hence, $\rho(A) = 1$

(ii) $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

A is of order 3×2

$\rho(A) \le minimum\{3,2\}$

$\rho(A) \le 2$

$|A| = \begin{vmatrix} 4 & -7 \\ 3 & -4 \end{vmatrix}$

$|A| = -16 + 21 = 5 \neq 0$

Hence, $\rho(A) = 2$

(iii) $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

A is of order 2×4

$\rho(A) \le minimum\{2,4\}$

$\rho(A) \le 2$

$|A| = \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}$

$|A| = 1 - 0 = 1 \neq 0$

Hence, $\rho(A) = 2$

(iv) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$

A is of order 3×3

$\rho(A) \le minimum\{3,3\}$

$\rho(A) \le 3$

$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{vmatrix}$

$|A| = 1(-4 + 6) + 2(-2 + 30) + 3(2 - 20)$

$|A| = 2 + 56 - 54 = 4 \neq 0$

Hence, $\rho(A) = 3$

(v) $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

A is of order 3×4

$\rho(A) \le minimum\{3,4\}$

$\rho(A) \le 3$

$|A| = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 2 & 3 \\ 8 & 4 & 2 \end{vmatrix}$

$|A| = 8(3 - 2) = 8 \neq 0$

Hence, $\rho(A) = 3$

---

2. Find the rank of the following matrices by row reduction method:

Hint: Number of Nonzero rows = Rank of matrix

(i) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & -6 & 2 & -4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 2R_1}$

$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2}$

It has two nonzero rows.

$\rho(A) = 2$

(ii) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -4 & 4 \\ 0 & -3 & 2 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}]{R_2 \to R_2 - 3R_1}$

$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow[R_4 \to 7R_4 - 3R_2]{R_3 \to 7R_3 - 4R_2}$

$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_4 \to R_4 + 8R_3}$

It has three nonzero rows.

$\rho(A) = 3$

(iii) $\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

$\sim \begin{bmatrix} 3 & -8 & 5 & 2 \\ 0 & 1 & -7 & 8 \\ 0 & -2 & 14 & -4 \end{bmatrix} \xrightarrow[R_3 \to 3R_3 + R_1]{R_2 \to 3R_2 - 2R_1}$

$\sim \begin{bmatrix} 3 & -8 & 5 & 2 \\ 0 & 1 & -7 & 8 \\ 0 & 0 & 0 & 12 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_2}$

It has three nonzero rows.

$\rho(A) = 3$

3. Find the inverse of the following by Gauss-Jordan method:

(i) $\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$

Applying Gauss-Jordan method we get,

$[A|I_2] = \left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 5 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to \frac{R_1}{2}}$

$= \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 5 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 5R_1}$

$= \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2}$

$= \left[\begin{array}{cc|cc} 1 & 0 & \frac{-4}{2} & 1 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{array}\right] \xrightarrow{R_2 \to 2R_2}$

$[A|I_2] = \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right]$

$A^{-1} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$

Applying Gauss-Jordan method we get,

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 6R_1]{R_2 \to R_2 - R_1}$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 4 & -3 & -6 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - 4R_2}$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_3}$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2}$$

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$

(iii) $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{5} & \mathbf{3} \\ \mathbf{1} & \mathbf{0} & \mathbf{8} \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Applying Gauss-Jordan method we get,

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + 2R_2}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \xrightarrow{R_3 \to (-1)R_3}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \xrightarrow{R_2 \to R_2 + 3R_3}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \xrightarrow{R_1 \to R_1 - 3R_3}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2}$$

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Exercise 1.3

1. Solve the following system of linear equation by matrix inversion method:

(i) $2x + 5y = -2, x + 2y = -3$

Solution:

The matrix of the form is,

$$\begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B \qquad \text{----------(1)}$$

To Find A^{-1}:

$$|A| = \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = 4 - 5 = -1 \neq 0$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \end{bmatrix} \qquad \text{From (1)}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 - 15 \\ -2 + 6 \end{bmatrix} = \begin{bmatrix} -11 \\ 4 \end{bmatrix}$$

Hence $x = -11$ and $y = 4$

(ii) $2x - y = 8, 3x + 2y = -2$

Solution:

The matrix of the form is,

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B \qquad \text{----------(1)}$$

To Find A^{-1}:

$$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4 + 3 = 7 \neq 0$$

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$X = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -2 \end{bmatrix} \qquad \text{From (1)}$$

$$X = \frac{1}{7}\begin{bmatrix} 16-2 \\ -24-4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 14 \\ -28 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

Hence, $x = 2$ and $y = -4$

(iii) $2x + 3y - z = 9, x + y + z = 9, 3x - y - z = -1$

Solution:

The matrix of the form is,

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B \qquad \text{----------(1)}$$

To Find A^{-1}:

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{vmatrix}$$

$$= 2(-1+1) - 3(-1-3) - 1(-1-3)$$

$$= 12 + 4$$

$$= 16 \neq 0$$

$$adj\ A = \begin{bmatrix} +(-1+1) & -(-1-3) & +(-1-3) \\ -(-3-1) & +(-2+3) & -(-2-9) \\ +(3+1) & -(2+1) & +(2-3) \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 0 & 4 & -4 \\ 4 & 1 & 11 \\ 4 & -3 & -1 \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$$

$$X = \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix} \qquad \text{From (1)}$$

$$X = \frac{1}{16}\begin{bmatrix} 0+36-4 \\ 36+9+3 \\ -36+99+1 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 32 \\ 48 \\ 64 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Hence, $x = 2, y = 3\ and\ z = 4.$

$(iv)\ x + y + z - 2 = 0, 6x - 4y + 5z - 31 = 0, 5x + 2y + 2z = 13$

Solution:

The matrix of the form is,

$$\begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B \qquad \text{----------(1)}$$

To Find A^{-1}:

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{vmatrix}$$

$$= 1(-8-10) - 1(12-25) + 1(12+20)$$

$$= -18 + 13 + 32 = 27$$

$$adj\ A = \begin{bmatrix} +(-8-10) & -(12-25) & +(12+20) \\ -(2-2) & +(2-5) & -(2-5) \\ +(5+4) & -(5-6) & +(-4-6) \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} -18 & 13 & 32 \\ 0 & -3 & 3 \\ 9 & 1 & -10 \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$X = \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}\begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix} \qquad \text{From (1)}$$

$$X = \frac{1}{27}\begin{bmatrix} -36+0+117 \\ 26-93+13 \\ 64+93-130 \end{bmatrix}$$

$$X = \frac{1}{27}\begin{bmatrix} 81 \\ -54 \\ 27 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Hence, $x = 3, y = -2\ and\ z = 1.$

---

2. If $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, find the product AB and BA and hence solve the system of equations $x + y + 2z = 1, 3x + 2y + z = 7, 2x + y + 3z = 2$.

Solution:

Given $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} -5+3+6 & -5+2+3 & -10+1+9 \\ 7+3-10 & 7+2-5 & 14+1-15 \\ 1-3+2 & 1-2+1 & 2-1+3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I_3 \qquad \text{----------(1)}$$

$$BA = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} -5+7+2 & 1+1-2 & 3-5+2 \\ -15+14+1 & 3+2-1 & 9-10+1 \\ -10+7+3 & 2+1-3 & 6-5+3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I_3 \qquad \text{----------(2)}$$

From (1) and (2) we get,

$$AB = BA = 4I_3$$

$$AB = 4I_3$$

$$\left(\tfrac{1}{4}A\right)B = I_3$$

Post multiply by B^{-1} we get,

$$\left(\tfrac{1}{4}A\right)BB^{-1} = I_3B^{-1}$$

$$B^{-1} = \tfrac{1}{4}A \qquad \text{-----------(3)}$$

The given system of equation in matrix form is,

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1}\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} \qquad \text{From (3)}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} -5+7+6 \\ 7+7-10 \\ 1-7+2 \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Hence, $x = 2, y = 1$ *and* $z = -1$.

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3. A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was ₹ 19,800 per month at the end of the first month after 3 years of service and ₹ 23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment. (Use matrix inversion method to solve the problem)

Solution:

Let his monthly salary be ₹ x

And his annual income be ₹ y

Given $\quad x + 3y = 19800 \qquad$ ----------(1)

And $\quad x + 9y = 23400 \qquad$ ----------(2)

The matrix of the form is,

$$\begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19800 \\ 23400 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B \qquad \text{----------(3)}$$

To Find A^{-1}:

$$|A| = \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = 9 - 3 = 6$$

$$A^{-1} = \tfrac{1}{6}\begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$$

$$X = \tfrac{1}{6}\begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 19800 \\ 23400 \end{bmatrix} \qquad \text{From (3)}$$

$$X = \begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3300 \\ 3900 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 29700 - 11700 \\ -3300 + 3900 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18000 \\ 600 \end{bmatrix}$$

Hence $x = 18000\ and\ y = 600$

∴ His starting salary is ₹ 18000 and annual increment is ₹ 600

---

4. Four men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the sae work by using matrix inversion method.

Solution:

Let the work done by man in one day be x

And the work done by woman in one day be y

Therefore, $\frac{4}{x} + \frac{4}{y} = \frac{1}{3}$ ----------(1)

$\frac{2}{x} + \frac{5}{y} = \frac{1}{4}$ ----------(2)

The matrix of the above equation is,

$$\begin{bmatrix} 4 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B \qquad \text{----------(3)}$$

To Find A^{-1}:

$$|A| = \begin{vmatrix} 4 & 4 \\ 2 & 5 \end{vmatrix} = 20 - 8 = 12$$

$$A^{-1} = \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}$$

$$X = \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} \qquad \text{From (3)}$$

$$X = \frac{1}{12}\begin{bmatrix} \frac{5}{3} - 1 \\ -\frac{2}{3} + 1 \end{bmatrix}$$

$$X = \frac{1}{12}\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{18} \\ \frac{1}{36} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \end{bmatrix} = \begin{bmatrix} \frac{1}{18} \\ \frac{1}{36} \end{bmatrix}$$

$$\frac{1}{x} = \frac{1}{18}\ and\ \frac{1}{y} = \frac{1}{36}$$

Hence $x = 18\ days$ and $y = 36\ days$

---

5. The prices of three commodities $A, B\ and\ C$ are ₹ $x, y\ and\ z$ per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C. person Q purchases 2 units of C and sells 3 units of A and one unit of B. person R purchases one unit of A and sells 3 units of B and one unit of C. In the process, $P, Q\ and\ R$ earn ₹ 15000, ₹ 1000 and ₹ 4000 respectively. Find the prices per unit of $A, B\ and\ C$. (Use matrix inversion method to solve the problem)

Solution:

Price of A = ₹ $x\ /unit$

Price of B = ₹ $y\ /unit$

Price of C = ₹ $z\ /unit$

Given $2x - 4y + 5z = 15000$ ----------(1)

$3x + y - 2z = 1000$ ----------(2)

$-x + 3y + z = 4000$ ----------(3)

The matrix form of the above equation is,

$$\begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B \qquad \text{----------(4)}$$

To Find A^{-1}:

$$|A| = \begin{vmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{vmatrix}$$

$$|A| = 2(1 + 6) + 4(3 - 2) + 5(9 + 1)$$

$$|A| = 68$$

$$adj\ A = \begin{bmatrix} +(1+6) & -(3-2) & +(9+1) \\ -(-4-15) & +(2+5) & -(6-4) \\ +(8-5) & -(-4-15) & +(2+12) \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 7 & -1 & 10 \\ 19 & 7 & -2 \\ 3 & 19 & 14 \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$A^{-1} = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$X = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}\begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix} \qquad \text{From (4)}$$

$$X = \frac{1000}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}\begin{bmatrix} 15 \\ 1 \\ 4 \end{bmatrix}$$

$$X = \frac{1000}{68}\begin{bmatrix} 105+19+12 \\ -15+7+76 \\ 150-2+56 \end{bmatrix}$$

$$X = \frac{1000}{68}\begin{bmatrix} 136 \\ 68 \\ 194 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2000 \\ 1000 \\ 3000 \end{bmatrix}$$

Hence, $x = ₹\,2000, y = ₹\,1000\ and\ z = ₹\,3000$

Exercise 1.4

Hint: $\boldsymbol{\Delta} = |\boldsymbol{A}|$

1. Solve the system linear equations in cramer's rule:

(i) $5x - 2y + 16 = 0, x + 3y - 7 = 0$

Solution:

The matrix of the form is,

$$\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -16 \\ 7 \end{bmatrix}$$

$$AX = B$$

$$\Delta = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix} = 15 + 2 = 17 \neq 0$$

$$\Delta_1 = \begin{vmatrix} -16 & -2 \\ 7 & 3 \end{vmatrix} = -48 + 14 = -34$$

$$\Delta_2 = \begin{vmatrix} 5 & -16 \\ 1 & 7 \end{vmatrix} = 35 + 16 = 51$$

$$x = \frac{\Delta_1}{\Delta} = -\frac{34}{17} = -2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{51}{17} = 3$$

Hence, $x = -2\ \ and\ \ y = 3$

(ii) $\frac{3}{x} + 2y = 12, \frac{2}{x} + 3y = 13$

Solution:

The matrix of the form is,

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{x} \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \end{bmatrix}$$

$$AX = B$$

$$\Delta = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4 = 5 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 12 & 2 \\ 13 & 3 \end{vmatrix} = 36 - 26 = 10$$

$$\Delta_2 = \begin{vmatrix} 3 & 12 \\ 2 & 13 \end{vmatrix} = 39 - 24 = 15$$

$$\frac{1}{x} = \frac{\Delta_1}{\Delta} = \frac{10}{5} = 2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{15}{5} = 3$$

Hence, $x = \frac{1}{2}\ \ and\ \ y = 3$

(iii) $3x + 3y - z = 11, 2x - y + 2z = 9, 4x + 3y + 2z = 25$

Solution:

The matrix of the form is,

$$\begin{bmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \\ 25 \end{bmatrix}$$

$$AX = B$$

$$\Delta = \begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{vmatrix}$$

$$= 3(-2-6) - 3(4-8) - 1(6+4)$$

$$= 3(-8) - 3(-4) - 1(10)$$

$$\Delta = -24 + 12 - 10 = -22$$

$$\Delta_1 = \begin{vmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{vmatrix}$$

$$= 11(-2-6) - 3(18-50) - 1(27+25)$$

$$= 11(-8) - 3(-32) - 1(52)$$

$$\Delta_1 = -88 + 96 - 52 = -44$$

$$\Delta_2 = \begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix}$$

$$= 3(18-50) - 11(4-8) - 1(50-36)$$

$= 3(-32) - 11(-4) - 14$

$\Delta_2 = -96 + 44 - 14 = -66$

$$\Delta_3 = \begin{vmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix}$$

$= 3(-25 - 27) - 3(50 - 36) + 11(6 + 4)$

$= 3(-52) - 3(14) + 11(10)$

$\Delta_3 = -156 - 42 + 110 = -88$

$x = \frac{\Delta_1}{\Delta} = \frac{-44}{-22} = 2$

$y = \frac{\Delta_2}{\Delta} = \frac{-66}{-22} = 3$

$z = \frac{\Delta_3}{\Delta} = \frac{-88}{-22} = 4$

Hence, $x = 2,\ y = 3\ and\ z = 4$

(iv) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0, \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0, \frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

Solution:

The matrix of the form is,

$$\begin{bmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{bmatrix} \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$AX = B$$

$$\Delta = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix}$$

$= 3(-8 + 5) + 4(-4 - 2) - 2(-5 - 4)$

$= 3(-3) + 4(-6) - 2(-9)$

$= -9 - 24 + 18$

$\Delta = -15$

$$\Delta_1 = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix}$$

$= 1(-8 + 5) + 4(-8 + 1) - 2(-10 + 2)$

$= -3 + 4(-7) - 2(-8)$

$= -3 - 28 + 16$

$\Delta_1 = -15$

$$\Delta_2 = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix}$$

$= 3(-8 + 1) - 1(-4 - 2) - 2(-1 - 4)$

$= 3(-7) - 1(-6) - 2(-5)$

$= -21 + 6 + 10$

$\Delta_2 = -5$

$$\Delta_3 = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix}$$

$= 3(-2 + 10) + 4(-1 - 4) + 1(-5 - 4)$

$= 3(8) + 4(-5) + 1(-9)$

$= 24 - 20 - 9$

$\Delta_3 = -5$

$\frac{1}{x} = \frac{\Delta_1}{\Delta} = \frac{-15}{-15} = 1$

$\frac{1}{y} = \frac{\Delta_2}{\Delta} = \frac{-5}{-15} = \frac{1}{3}$

$\frac{1}{z} = \frac{\Delta_3}{\Delta} = \frac{-5}{-15} = \frac{1}{3}$

Hence, $x = 1,\ y = 3\ and\ z = 3$

2. In a competitive examination, one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly? (Use cramer's rule to solve the problem)

Solution:

Let the correct answer be x

And let the wrong answer be y

The total number of questions = 100

$\therefore \quad x + y = 100$ ----------(1)

$x - \frac{1}{4}y = 80$

$4x - y = 320$ ----------(2)

The matrix of the form is,

$$\begin{bmatrix}1 & 1\\4 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}100\\320\end{bmatrix}$$

$$AX = B$$

$$\Delta = \begin{vmatrix}1 & 1\\4 & -1\end{vmatrix} = -1-4=-5$$

$$\Delta_1 = \begin{vmatrix}100 & 1\\320 & -1\end{vmatrix} = -100-320=-420$$

$$\Delta_2 = \begin{vmatrix}1 & 100\\4 & 320\end{vmatrix} = 320-400=-80$$

$$x=\frac{\Delta_1}{\Delta}=\frac{-420}{-5}=84$$

$$y=\frac{\Delta_2}{\Delta}=\frac{-80}{-5}=15$$

$\Rightarrow \quad x = 84 \ and\ y = 15$

∴ The number of questions answered correctly = 84

---

3. A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution? (Use cramer's rule to solve the problem)

Solution:

Let the no. of litres in 50% acid used be x

And let the no. of litres in 25% acid used be y

∴ $\qquad x+y=10$ ----------(1)

$$\frac{50}{100}x+\frac{25}{100}y=\frac{40}{100}\times 10$$

$50x+25y=400$ ----------(2)

The matrix of the form is,

$$\begin{bmatrix}1 & 1\\50 & 25\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}10\\400\end{bmatrix}$$

$$AX = B$$

$$\Delta = \begin{vmatrix}1 & 1\\50 & 25\end{vmatrix} = 25-50=-25$$

$$\Delta_1 = \begin{vmatrix}10 & 1\\400 & 25\end{vmatrix} = 250-400=-150$$

$$\Delta_2 = \begin{vmatrix}1 & 10\\50 & 400\end{vmatrix} = 400-500=-100$$

$$x=\frac{\Delta_1}{\Delta}=\frac{-150}{-25}=6$$

$$y=\frac{\Delta_2}{\Delta}=\frac{-100}{-25}=4$$

$\Rightarrow \quad x = 6\ litres\ and\ y = 4\ litres$

∴ We have to mix 6 litres in 50% acid and 4 litres in 25% acid.

---

4. A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself? (Use cramer's rule to solve the problem)

Solution:

Let time taken for pump A to fill the tank be $x\ minutes$

And time taken for pump B to fill the tank be $y\ minutes$

∴ $\qquad \frac{1}{x}+\frac{1}{y}=\frac{1}{10}$ -----------(1)

$\frac{1}{x}-\frac{1}{y}=\frac{1}{30}$ -----------(2)

The matrix of the form is,

$$\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{x}\\\frac{1}{y}\end{bmatrix}=\begin{bmatrix}\frac{1}{10}\\\frac{1}{30}\end{bmatrix}$$

$$AX = B$$

$$\Delta = \begin{vmatrix}1 & 1\\1 & -1\end{vmatrix} = -1-1=-2$$

$$\Delta_1 = \begin{vmatrix}\frac{1}{10} & 1\\\frac{1}{30} & -1\end{vmatrix} = -\frac{1}{10}-\frac{1}{30}=-\frac{4}{30}$$

$$\Delta_2 = \begin{vmatrix}1 & \frac{1}{10}\\1 & \frac{1}{30}\end{vmatrix} = \frac{1}{30}-\frac{1}{10}=-\frac{2}{30}$$

$$\frac{1}{x}=\frac{\Delta_1}{\Delta}=\frac{\frac{-4}{30}}{-2}=\frac{4}{60}=\frac{1}{15}$$

$$\frac{1}{y}=\frac{\Delta_2}{\Delta}=\frac{\frac{-2}{30}}{-2}=\frac{1}{30}$$

$\Rightarrow \quad x = 15\ minutes\ and\ y = 30\ minutes$

∴ Pump A can fill the tank in 15 minutes and Pump B can fill the tank in 30 minutes.

---

5. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is ₹ 150. The cost of two dosai, two idlies and four vadais is ₹ 200. The cost of five dosai, four idlies and two vadais is ₹ 250. The family has ₹ 350 in hand

and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had?

Solution:

Let the cost of 1 dosai be ₹ x

And the cost of 1 idly be ₹ y

And the cost of 1 vadai be ₹ z

$\therefore \quad 2x + 3y + 2z = 150$ ----------(1)

$2x + 2y + 4z = 200$ ----------(2)

$5x + 4Y + 2z = 250$ ----------(3)

The matrix of the form is,

$$\begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & 4 \\ 5 & 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 150 \\ 200 \\ 250 \end{bmatrix}$$

$$AX = B$$

$$\Delta = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & 4 \\ 5 & 4 & 2 \end{vmatrix}$$

$= 2(4-16) - 3(4-20) + 2(8-10)$

$= 2(-12) - 3(-16) + 2(-2)$

$= -24 + 48 - 4$

$\Delta = 20$

$$\Delta_1 = \begin{vmatrix} 150 & 3 & 2 \\ 200 & 2 & 4 \\ 250 & 4 & 2 \end{vmatrix}$$

$= 150(4-16) - 3(400-1000) + 2(800-500)$

$= 150(-12) - 3(-600) + 2(300)$

$= -1800 + 1800 + 600$

$\Delta_1 = 600$

$$\Delta_2 = \begin{vmatrix} 2 & 150 & 2 \\ 2 & 200 & 4 \\ 5 & 250 & 2 \end{vmatrix}$$

$= 2(400-1000) - 150(4-20) + 2(500-1000)$

$= 2(-600) - 150(-16) + 2(-500)$

$= -1200 + 2400 - 1000$

$\Delta_2 = 200$

$$\Delta_3 = \begin{vmatrix} 2 & 3 & 150 \\ 2 & 2 & 200 \\ 5 & 4 & 250 \end{vmatrix}$$

$= 2(500-800) - 3(500-1000) + 150(8-10)$

$= 2(-300) - 3(-500) + 150(-2)$

$= -600 + 1500 - 300$

$\Delta_3 = 600$

$$x = \frac{\Delta_1}{\Delta} = \frac{600}{20} = 30$$

$$y = \frac{\Delta_2}{\Delta} = \frac{200}{20} = 10$$

$$z = \frac{\Delta_3}{\Delta} = \frac{600}{20} = 30$$

Hence, $x = ₹\,30, y = ₹\,10$ *and* $z = ₹\,30$

$\therefore$ The cost of 3 dosai and 6 idlies and 6 vadai is

$= 3 \times 30 + 6 \times 10 + 6 \times 30$

$= 90 + 60 + 180$

$= ₹\,330$

They are having ₹ 350.

$\therefore$ Yes, they will be able to manage the bill.

Exercise 1.5

1. Solve the following systems of equations by Gaussian elimination method:

Hint: reduced to row echelon form

(i) $2x - 2y + 3z = 2, x + 2y - z = 3, 3x - y + 2z = 1$

Solution:

The matrix of the form is,

$$\begin{bmatrix} 2 & -2 & 3 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$AX = B$$

The augmented matrix $[A, B]$ is,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -2 & 3 & 2 \\ 3 & -1 & 2 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -7 & 5 & -8 \end{array}\right] \xrightarrow{R_3 \to 6R_3 - 7R_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & 0 & -5 & -20 \end{array}\right]$$

The equivalent matrix is in echelon form.

$(i.e)$ $\quad x + 2y - z = 3$ ----------(1)

$-6y + 5z = -4$ ----------(2)

$-5z = -20$ ----------(3)

From (3) $\quad z = \frac{20}{5} = 4$

Sub $z = 4$ in (2) we get,

$$-6y + 20 = -4$$

$$-6y = -24$$

$$y = 4$$

Substituting z *and* y values in (1) we get,

$$x + 2(4) - 4 = 3$$

$$x = 3 - 8 + 4$$

$$x = -1$$

$\therefore$ The solution is $x = -1, y = 4, \ z = 4$

(ii) $2x + 4y + 6z = 22, 3x + 8y + 5z = 27, -x + y + 2z = 2$

Solution:

The matrix of the form is,

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 8 & 5 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 27 \\ 2 \end{bmatrix}$$

$$AX = B$$

The augmented matrix $[A, B]$ is,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right] \xrightarrow[R_3 \to R_3 + R1]{R_2 \to R_2 - 3R_1}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{array}\right] \xrightarrow{R_3 \to 2R_3 - 3R_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 0 & 22 & 44 \end{array}\right]$$

The equivalent matrix is in echelon form.

$(i.e)$ $\quad x + 2y + 3z = 11$ ----------(1)

$2y - 4z = -6$ ----------(2)

$22z = 44$ ----------(3)

From (3) $\quad z = \frac{44}{22} = 2$

Sub $z = 2$ in (2) we get,

$$2y - 8 = -6$$

$$2y = 2$$

$$y = 1$$

Substituting z *and* y values in (1) we get,

$$x + 2 + 6 = 11$$

$$x = 3$$

$\therefore$ The solution is $x = 3, y = 1, z = 2$

2. If $ax^2 + bx + c$ is divided by $x + 3, x - 5,$ *and* $x - 1$, the remainders are $21, 61$ *and* 9 respectively. Find a, b *and* c. (Use Gaussian elimination method)

Solution:

Let $P(x) = ax^2 + bx + c$

Now $\quad P(-3) = 21$

$$a(-3)^2 + b(-3) + c = 21$$

$9a - 3b + c = 21$ ----------(1)

Now $\quad P(5) = 61$

$$a(5)^2 + b(5) + c = 61$$

$25a + 5b + c = 61$ ----------(2)

Now $\quad P(1) = 9$

$$a(1)^2 + b(1) + c = 9$$

$a + b + c = 9$ ----------(3)

The matrix of the form is,

$$\begin{bmatrix} 9 & -3 & 1 \\ 25 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 21 \\ 61 \\ 9 \end{bmatrix}$$

$$AX = B$$

The augmented matrix $[A, B]$ is,

$$[A|B] = \left[\begin{array}{ccc|c} 9 & -3 & 1 & 21 \\ 25 & 5 & 1 & 61 \\ 1 & 1 & 1 & 9 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 25 & 5 & 1 & 61 \\ 9 & -3 & 1 & 21 \end{array}\right] \xrightarrow[R_3 \to R_3 - 9R_1]{R_2 \to R_2 - 25R_1}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -20 & -24 & -164 \\ 0 & -12 & -8 & -60 \end{array}\right] \xrightarrow[R_3 \to \frac{R_3}{4}]{R_2 \to \frac{R_2}{4}}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & -3 & -2 & -15 \end{array}\right] \xrightarrow{R_3 \to 5R_3 - 3R_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & 0 & 8 & 48 \end{array}\right]$$

The equivalent matrix is in echelon form.

$(i.e)$ $\quad a + b + c = 9$ ----------(4)

$-5b - 6c = -41$ ----------(5)

$8c = 48$ ----------(6)

From (3) $\quad c = \frac{48}{8} = 6$

Sub $c = 6$ in (5) we get,

$$-5b - 36 = -41$$

$$5b = 5$$

$$b = 1$$

Substituting c *and* b values in (4) we get,

$$a + 1 + 6 = 9$$

$$a = 2$$

$\therefore$ The solution is $a = 2, b = 1, c = 6$

3. An amount of ₹ 65,000 is invested in three bonds at the rates of 6%, 8% and 9% per annum respectively. The total income ₹ 4,800. The income from the third bond is ₹ 600 more than that form the second bond. Determine the price of each bond. (Use Gaussian elimination method)

Solution:

Let the amount invested in 6% bond be ₹ x

And the amount invested in 8% bond be ₹ y

And the amount invested in 9% bond be ₹ z

Now $\quad x + y + z = 65000$ ----------(1)

$$\frac{6}{100}x + \frac{8}{100}y + \frac{9}{100}z = 4800$$

$\Rightarrow \quad 6x + 8y + 9z = 480000$ ----------(2)

Also given that,

$$\frac{9}{100}z = \frac{8}{100}y + 600$$

$$9z = 8y + 60000$$

$-8y + 9z = 60000$ ----------(3)

The matrix of the form is,

$$\begin{bmatrix} 1 & 1 & 1 \\ 6 & 8 & 9 \\ 0 & -8 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 65000 \\ 480000 \\ 60000 \end{bmatrix}$$

$$AX = B$$

The augmented matrix $[A, B]$ is,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 6 & 8 & 9 & 480000 \\ 0 & -8 & 9 & 60000 \end{array}\right] \xrightarrow{R_2 \to R_2 - 6R_1}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 0 & 2 & 3 & 90000 \\ 0 & -8 & 9 & 60000 \end{array}\right] \xrightarrow{R_3 \to R_3 + 4R_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 0 & 2 & 3 & 90000 \\ 0 & 0 & 21 & 420000 \end{array}\right]$$

The equivalent matrix is in echelon form.

$(i.e)$ $\quad x + y + z = 65000$ ----------(4)

$2y + 3z = 90000$ ----------(5)

$21z = 420000$ ----------(6)

$$z = \frac{420000}{21} = 20000$$

Sub $z = 20000$ in (5) we get,

$$2y + 60000 = 90000$$

$$2y = 30000$$

$$y = 15000$$

Substituting z *and* y values in (4) we get,

$$x + 15000 + 20000 = 65000$$

$$x = 30000$$

$\therefore$ The amount invested in,

6% bond $= ₹30000$

8% bond $= ₹15000$

9% bond $= ₹20000$

---

4. A boy is walking along the path $y = ax^2 + bx + c$ through the points $(-6, 8), (-2, -12),$ *and* $(3, 8)$. He wants to meet his friend at $P(7, 60)$. Will he meet his friend? (Use Gaussian elimination method.)

Solution:

Given $\quad y = ax^2 + bx + c \qquad$ ----------(1)

Sub $(-6,8)$ in (1) we get,

$$8 = a(-6)^2 + b(-6) + c$$

$$36a - 6b + c = 8 \qquad \text{----------(2)}$$

Sub $(-2, -12)$ in (1) we get,

$$-12 = a(-2)^2 + b(-2) + c$$

$$4a - 2b + c = -12 \qquad \text{----------(3)}$$

Sub $(3,8)$ in (1) we get,

$$8 = a(3)^2 + b(3) + c$$

$$9a + 3b + c = 8 \qquad \text{----------(4)}$$

The matrix of the form is,

$$\begin{bmatrix} 36 & -6 & 1 \\ 4 & -2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 8 \\ -12 \\ 8 \end{bmatrix}$$

$$AX = B$$

The augmented matrix $[A, B]$ is,

$$[A|B] = \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 9 & 3 & 1 & 8 \end{array}\right] \xrightarrow[R_3 \to 4R_3 - R_1]{R_2 \to 9R_2 - R_1}$$

$$\sim \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -12 & 8 & -116 \\ 0 & 18 & 3 & 24 \end{array}\right] \xrightarrow[R_3 \to \frac{R_3}{3}]{R_2 \to \frac{R_2}{4}}$$

$$\sim \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -3 & 2 & -29 \\ 0 & 6 & 1 & 8 \end{array}\right] \xrightarrow{R_3 \to R_3 + 2R_2}$$

$$\sim \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -3 & 2 & -29 \\ 0 & 0 & 5 & -50 \end{array}\right]$$

The equivalent matrix is in echelon form.

$(i.e) \qquad 36a - 6b + c = 8 \qquad$ ----------(5)

$$-3b + 2c = -29 \qquad \text{----------(6)}$$

$$5c = -50 \qquad \text{----------(7)}$$

From (7), $\qquad c = -10$

Sub $c = -10$ in (6) we get,

$$-3b - 20 = -29$$

$$-3b = -9$$

$$b = 3$$

Substituting c *and* b values in (5) we get,

$$36a - 18 - 10 = 8$$

$$36a = 36$$

$$a = 1$$

Sub a, b *and* c values in (1) we get,

$$y = x^2 + 3x - 10$$

At $x = 7$,

$$y = 7^2 + 3(7) - 10$$

$$= 49 + 21 - 10$$

$$y = 60$$

$\therefore$ (7,60) is a point on the path.

$\therefore$ Yes, he will meet his friend.

Exercise 1.6

Non-homogeneous Linear Equation

Hint: (i) If $\rho(A) = \rho(A,B) = number\ of\ unknowns$ then the system of equation is consistent and has a unique solution.

(ii) If $\rho(A) = \rho(A,B) < number\ of\ unknowns$ then the system of equation is consistent and has infinite number of solutions.

(iii) If $\rho(A) \neq \rho(A,B)$ then the system of equation is inconsistent and has no solution.

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1. Test for consistency and is possible, solve the following systems of equations by rank method:

(i) $x - y + 2z = 2, 2x + y + 4z = 7, 4x - y + z = 4$

Solution:

Here the number of unknowns $= 3$

The matrix form of the system is,

$$AX = B$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 4 \\ 4 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$

The augmented matrix $[A, B]$ is,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 2 & 1 & 4 & 7 \\ 4 & -1 & 1 & 4 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 3 & -7 & -4 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & -7 & -7 \end{array}\right] \quad \text{----------(1)}$$

The above matrix is in echelon form.

$\Rightarrow \quad \rho(A,B) = \rho(A) = 3 = number\ of\ unknowns$

$\Rightarrow$ The system of equation is consistent and has a unique solution.

From (1) we get,

$$x - y + 2z = 2 \quad \text{----------(2)}$$

$$3y = 3 \quad \text{----------(3)}$$

$$-7z = -7 \quad \text{----------(4)}$$

From (4) $\quad z = 1$

From (3) $\quad y = 1$

Sub $z\ and\ y$ values in (2) we get,

$$x - 1 + 2 = 2$$

$$x = 1$$

Hence, $x = 1, y = 1\ and\ z = 1$

(ii) $3x + y + z = 2, x - 3y + 2z = 1, 7x - y + 4z = 5$

Solution:

Here the number of unknowns $= 3$

The matrix form of the system is,

$$AX = B$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & -3 & 2 \\ 7 & -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

The augmented matrix $[A, B]$ is,

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 1 & 1 & 2 \\ 1 & -3 & 2 & 1 \\ 7 & -1 & 4 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 3 & 1 & 1 & 2 \\ 7 & -1 & 4 & 5 \end{array}\right] \xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - 3R_1}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 20 & -10 & -2 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The above matrix is in echelon form.

$\Rightarrow \quad \rho(A,B) = \rho(A) = 2 < number\ of\ unknowns$

$\Rightarrow$ The system of equation is consistent and has infinite number of solutions.

$(i.e) \quad x - 3y + 2z = 1 \quad$ ------------(1)

$$10y - 5z = -1 \quad \text{------------(2)}$$

$$z = t,\ t \in R$$

From (2) we get,

$$10y - 5t = -1$$

$$10y = -1 + 5t$$

$$y = \frac{-1+5t}{10}$$

Sub $y = \frac{-1+5t}{10}$ in (1) we get,

$$x = 1 + 3y - 2z$$

$$x = 1 + 3\left(\frac{-1+5t}{10}\right) - 2t$$

$$x = \frac{10-3+15t-20t}{10}$$

$$x = \frac{7-5t}{10}$$

Hence, $x = \frac{7-5t}{10}, y = \frac{-1+5t}{10}$ $and\ z = t, where\ t \in R$

(iii) $2x + 2y + z = 5, x - y + z = 1, 3x + y + 2z = 4$

Solution:

Here the number of unknowns $= 3$

The matrix form of the system is,

$$AX = B$$

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

The augmented matrix $[A, B]$ is,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 5 \\ 1 & -1 & 1 & 1 \\ 3 & 1 & 2 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 2 & 1 & 5 \\ 3 & 1 & 2 & 4 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 4 & -1 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

The above matrix is in echelon form.

$\Rightarrow \quad \rho(A) = 2\ and\ \rho(A, B) = 3$

$\Rightarrow \quad \rho(A) \neq \rho(A, B)$

∴ The system of equation is inconsistent and has no solutions.

(iv) $2x - y + z = 2, 6x - 3y + 3z = 6, 4x - 2y + 2z = 4$

Solution:

Here the number of unknowns $= 3$

The matrix form of the system is,

$$AX = B$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 6 & -3 & 3 \\ 4 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

The augmented matrix of $[A, B]$ is,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -1 & 1 & 2 \\ 6 & -3 & 3 & 6 \\ 4 & -2 & 2 & 4 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1}$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1}$$

The above matrix is in echelon form.

$\Rightarrow \quad \rho(A) = \rho(A, B) = 1 < number\ of\ unknowns$

∴ The system of equation is consistent and has infinite number of solutions.

$(i.e) \qquad 2x - y + z = 2 \qquad$ ----------(1)

Taking $z = t\ and\ y = s\ where\ t, s \in R$

From (1) we get,

$$2x - s + t = 2$$

$$2x = 2 + s - t$$

$$x = \frac{1}{2}(2 + s - t) \text{ where } t, s \in R$$

Hence, $x = \frac{1}{2}(2 + s - t), y = s\ and\ z = t$

2. Find the value of k for which the equations $kx - 2y + z = 1, x - 2ky + z = -2, x - 2y + kz = 1$ have,

(i) no solution (ii) unique solution (iii) infinitely many solution

Solution:

Here the number of unknowns $= 3$

The matrix form of the system is,

$$AX = B$$

$$\begin{bmatrix} k & -2 & 1 \\ 1 & -2k & 1 \\ 1 & -2 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The augmented matrix of $[A, B]$ is,

$$[A|B] = \left[\begin{array}{ccc|c} k & -2 & 1 & 1 \\ 1 & -2k & 1 & -2 \\ 1 & -2 & k & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 1 & -2k & 1 & -2 \\ k & -2 & 1 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - kR_1]{R_2 \to R_2 - R_1}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & -2+2k & 1-k^2 & 1-k \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2k & 1 & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & 0 & 2-k-k^2 & -2-k \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2k & 1 & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & 0 & -(k+2)(k-1) & -(k+2) \end{array}\right]$$

$[\because 2 - k - k^2 = -(k^2 + k - 2) = -(k+2)(k-1)]$

The above matrix is in echelon form.

(i) <u>No solution:</u> when $\boldsymbol{k = 1}$,

$$\rho(A) = 1 \; and \; \rho(A, B) = 3$$

$$\rho(A) \neq \rho(A, B)$$

∴ The system of equation is inconsistent and has no solution.

(ii) <u>Unique solution:</u> when $\boldsymbol{k \neq -2}$, $\boldsymbol{k \neq 1}$,

$$\rho(A) = 3 \; and \; \rho(A, B) = 3$$

$$\rho(A) = \rho(A, B) = number \; of \; unknowns$$

∴ The system of equation is consistent and has a unique solution.

(iii) <u>Infinitely many solutions:</u> when $\boldsymbol{k = -2}$,

$$\rho(A) = 2 \; and \; \rho(A, B) = 2$$

$$\rho(A) = \rho(A, B) < number \; of \; unknowns$$

∴ The system of equation is consistent and has infinitely many solutions.

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3. Investigate the values of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ the system of linear equations $\boldsymbol{2x + 3y + 5z = 9, 7x + 3y - 5z = 8, 2x + 3y + \lambda z = \mu}$, have

(i) no solution (ii) a unique solution (iii) an infinite number of solutions.

<u>**Solution:**</u>

Here the number of unknowns = 3

The matrix form of the system is,

$$AX = B$$

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -5 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

The augmented matrix of $[A, B]$ is,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to 2R_2 - 7R_1}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 0 & -15 & -45 & -47 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right]$$

The above matrix is in echelon form.

(i) <u>No solution:</u> when $\boldsymbol{\lambda = 5, \mu \neq 9}$,

$$\rho(A) = 2 \; and \; \rho(A, B) = 3$$

$$\rho(A) \neq \rho(A, B)$$

∴ The system of equation is inconsistent and has no solution.

(ii) <u>A unique solution:</u> when $\boldsymbol{\lambda \neq 5 \; and \; \mu \in R}$,

$$\rho(A) = 3 \; and \; \rho(A, B) = 3$$

$$\rho(A) = \rho(A, B) = 3 = number \; of \; unknowns$$

∴ The system of equation is consistent and has a unique solution.

(iii) <u>An infinite number of solutions:</u> when $\boldsymbol{\lambda = 5}$, $\boldsymbol{\mu = 9}$,

$$\rho(A) = 2 \; and \; \rho(A, B) = 2$$

$$\rho(A) = \rho(A, B) = 2 < number \; of \; unknowns$$

∴ The system of equation is consistent and has infinitely many solutions.

Exercise 1.7

Homogeneous Equation

(i) If $\rho(A) = \rho(A, O) = number\ of\ unknowns$ then it has a unique solution and it is a trivial solution $|A| \neq 0$.

(ii) If $\rho(A) = \rho(A, O) < number\ of\ unknowns$ the system has a non-trivial solution iff $|A| = 0$.

1. solve the following system of homogeneous equations.

(i) $3x + 2y + 7z = 0$, $4x - 3y - 2z = 0$, $5x + 9y + 23z = 0$

Solution:

Here the number of unknowns $= 3$

The matrix form of the system is,

$$AX = B$$

$$\begin{bmatrix} 3 & 2 & 7 \\ 4 & -3 & -2 \\ 5 & 9 & 23 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of $[A, O]$ is,

$$[A|O] = \left[\begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 4 & -3 & -2 & 0 \\ 5 & 9 & 23 & 0 \end{array}\right] \xrightarrow[R_3 \to 3R_3 - 5R_1]{R_2 \to 3R_2 - 4R_1}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 0 & -17 & -34 & 0 \\ 0 & 17 & 34 & 0 \end{array}\right] \xrightarrow[R_3 \to 3R_3 - 5R_1]{R_2 \to 3R_2 - 4R_1}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 0 & -17 & -34 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2}$$

The above matrix is in echelon form.

$$\rho(A) = 2\ and\ \rho(A, O) = 2$$

$$\rho(A) = \rho(A, O) < number\ of\ unknowns$$

$\therefore$ The system of equation is consistent and has infinite number of solutions.

$(i.e)$ $\quad 3x + 2y + 7z = 0$ ----------(1)

$-17y - 34z = 0$ ----------(2)

$0 = 0$ ----------(3)

Taking $z = t, where\ t \in R$

From (2) we get,

$$-17y = 34t$$

$$y = -\frac{34t}{17}$$

$$y = -2t$$

Sub $z = t\ and\ y = -2t$ in (1) we get,

$$3x + 2(-2t) + 7t = 0$$

$$3x - 4t + 7t = 0$$

$$3x + 3t = 0$$

$$3x = -3t$$

$$x = -t$$

$\therefore$ The solution is $x = -t, y = -2t\ and\ z = t, t \in R$

(ii) $2x + 3y - z = 0$, $x - y - 2z = 0$, $3x + y + 3z = 0$

Solution:

Here the number of unknowns $= 3$

The matrix form of the system is,

$$AX = B$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of $[A, O]$ is,

$$[A|O] = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & -1 & -2 & 0 \\ 3 & 1 & 3 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 2 & 3 & -1 & 0 \\ 3 & 1 & 3 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 4 & 9 & 0 \end{array}\right] \xrightarrow{R_3 \to 5R_3 - 4R_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & 33 & 0 \end{array}\right]$$

The above matrix is in echelon form.

$$\rho(A) = 3\ and\ \rho(A, O) = 3$$

$$\rho(A) = \rho(A, O) = 3 = number\ of\ unknowns$$

$\therefore$ The system of equation is consistent and has a trivial solution.

$\Rightarrow \quad x = 0, y = 0 \; and \; z = 0$

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2. Determine the values of λ for which the following system of equations $x + y + 3z = 0, 4x + 3y + \lambda z = 0, 2x + y + 2z = 0$ has (i) a unique solution (ii) a non-trivial solution

Solution:

Here the number of unknowns = 3

The matrix form of the equation is,

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 4 & 3 & \lambda \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of $[A, O]$ is,

$$[A|O] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 4 & 3 & \lambda & 0 \\ 2 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 3 & \lambda & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -1 & \lambda - 12 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & \lambda - 8 & 0 \end{array}\right]$$

The above matrix is in echelon form.

(i) <u>A unique solution:</u> when $\boldsymbol{\lambda \neq 8}$,

$$\rho(A) = 3 \; and \; \rho(A, O) = 3$$

$$\rho(A) = \rho(A, O) = 3 = number \; of \; unknowns$$

$\therefore$ The system of equation is consistent and has a unique solution.

(ii) <u>A non-trivial solution:</u> when $\boldsymbol{\lambda = 8}$,

$$\rho(A) = 2 \; and \; \rho(A, O) = 2$$

$$\rho(A) = \rho(A, O) = 2 < number \; of \; unknowns$$

$\therefore$ The system of equation is consistent and has a non-trivial solution.

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3. By using Gaussian elimination method, balance the chemical reaction equation: $C_2H_6 + O_2 \to H_2O + CO_2$

Solution:

The balanced equation is,

$$x_1 C_2H_6 + x_2 O_2 \to x_3 H_2O + x_4 CO_2$$

Comparing $C, H \; and \; O$ atoms in both sides we get,

$$2x_1 = x_4$$

$$2x_1 - x_4 = 0 \qquad \text{----------(1)}$$

$$6x_1 = 2x_3$$

$$6x_1 - 2x_3 = 0$$

$$\div 2 \qquad 3x_1 - x_3 = 0 \qquad \text{----------(2)}$$

$$2x_2 = x_3 + 2x_4$$

$$2x_2 - x_3 - 2x_4 = 0 \qquad \text{----------(3)}$$

Here the number of unknowns = 4

The augmented matrix of $[A, O]$ is,

$$[A|O] = \left[\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array}\right] \xrightarrow{R_2 \to 2R_2 - 3R_1}$$

$$\sim \left[\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\sim \left[\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & -2 & 0 \\ 0 & 0 & -2 & 3 & 0 \end{array}\right]$$

The above matrix is in echelon form.

$$\rho(A) = 3 \; and \; \rho(A, O) = 3$$

$$\rho(A) = \rho(A, O) = 3 < number \; of \; unknowns$$

$\therefore$ The system of equation is consistent and has infinite number of solutions.

$$(i.e) \qquad 2x_1 - x_4 = 0 \qquad \text{----------(4)}$$

$$2x_2 - x_3 - 2x_4 = 0 \qquad \text{----------(5)}$$

$$-2x_3 + 3x_4 = 0 \qquad \text{----------(6)}$$

Taking $x_4 = t, t \in R$

From (6) we get,

$$-2x_3 + 3t = 0$$

$$-2x_3 = -3t$$

$$x_3 = \frac{3}{2}t$$

From (4) we get,

$$2x_1 - t = 0$$

$$x_1 = \frac{t}{2}$$

Sub $x_3 = \frac{3}{2}t \ and \ x_4 = t$ in (5) we get,

$$2x_2 - \frac{3}{2}t - 2t = 0$$

$$2x_2 = \frac{3}{2}t + 2t$$

$$2x_2 = \frac{1}{2}(3t + 4t)$$

$$x_2 = \frac{7}{4}t$$

Let $t = 4$ we get, $x_1 = 2, x_2 = 7, x_3 = 6 \ and \ x_4 = 4$

∴ The balanced equation is,

$$2C_2H_6 + 7O_2 \rightarrow 6H_2O + 4CO_2$$