CHAPTER – 9

Applications of Integration

Exercise 9.1

The left-end rule for Reimann sum:

$$S = [f(x_0) + f(x_1) + \cdots + f(x_{n-1})]\,\Delta x$$

The right-end rule for Reimann sum:

$$S = [f(x_1) + f(x_2) + \cdots + f(x_n)]\,\Delta x$$

The mid-point rule for Reimann sum:

$$S = \left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right]\Delta x$$

1. Find an approximate value of $\int_1^{1.5} x\,dx$ by applying the left-end rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$.

Solution:

Here $a = 1, b = 1.5, n = 5$

$f(x) = \int_1^{1.5} x\,dx$

$\Delta x = \frac{b-a}{n} = \frac{1.5-1}{5} = \frac{0.5}{5} = 0.1$

$x_0 = 1, x_1 = 1.1, x_2 = 1.2, x_3 = 1.3, x_4 = 1.4, x_5 = 1.5$

The left-end rule for Reimann sum is,

$S = [f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4)]\,\Delta x$

$= [f(1) + f(1.1) + f(1.2) + f(1.3) + f(1.4)](0.1)$

$= [1 + 1.1 + 1.2 + 1.3 + 1.4](0.1)$

$= (6)(0.1)$

$S = 0.6$

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2. Find an approximate value of $\int_1^{1.5} x^2\,dx$ by applying the right-end rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$.

Solution:

Here $a = 1, b = 1.5, n = 5$

$f(x) = \int_1^{1.5} x^2\,dx$

$\Delta x = \frac{b-a}{n} = \frac{1.5-1}{5} = \frac{0.5}{5} = 0.1$

$x_0 = 1, x_1 = 1.1, x_2 = 1.2, x_3 = 1.3, x_4 = 1.4, x_5 = 1.5$

The right-end rule for Reimann sum is,

$S = [f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]\,\Delta x$

$= \begin{bmatrix} f(1.1) + f(1.2) + f(1.3) + f(1.4) + \\ f(1.5) \end{bmatrix}(0.1)$

$= [(1.1)^2 + (1.2)^2 + (1.3)^2 + (1.4)^2 + (1.5)^2](0.1)$

$= [1.21 + 1.44 + 1.69 + 1.96 + 2.25](0.1)$

$S = (8.55)(0.1)$

$S = 0.855$

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3. Find an approximate value of $\int_1^{1.5} (2-x)\,dx$ by applying the mid-point rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$.

Solution:

Here $a = 1, b = 1.5, n = 5$

$f(x) = \int_1^{1.5} (2-x)\,dx$

$\Delta x = \frac{b-a}{n} = \frac{1.5-1}{5} = \frac{0.5}{5} = 0.1$

$x_0 = 1, x_1 = 1.1, x_2 = 1.2, x_3 = 1.3, x_4 = 1.4, x_5 = 1.5$

The mid-point rule for Reimann sum is,

$S = \left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \cdots + f\left(\frac{x_4+x_5}{2}\right)\right]\Delta x$

$= \left[f\left(\frac{2.1}{2}\right) + f\left(\frac{2.3}{2}\right) + f\left(\frac{2.5}{2}\right) + f\left(\frac{2.7}{2}\right) + f\left(\frac{2.9}{2}\right)\right](0.1)$

$= \begin{bmatrix} f(1.05) + f(1.15) + f(1.25) + f(1.35) \\ +f(1.45) \end{bmatrix}(0.1)$

$= [2 - 1.05 + 2 - 1.15 + 2 - 1.25 + 2 - 1.35 + 2 - 1.45]\,(0.1)$

$= [10 - 6.25](0.1)$

$= (3.75)(0.1)$

$S = 0.375$

Exercise 9.2

Hint: Limit of sum:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left(a + \left(\frac{b-a}{n}\right)r\right)$$

1. Evaluate the following integrals as the limits of sums:

(i) $\int_0^1 (5x+4)\,dx$

Solution:

Here $f(x) = 5x + 4, a = 0$ and $b = 1$

$\frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

$f\left(a + \left(\frac{b-a}{n}\right)r\right) = f\left(0 + \left(\frac{1}{n}\right)r\right)$

$= f\left(\frac{r}{n}\right)$

$f\left(a + \left(\frac{b-a}{n}\right)r\right) = \frac{5r}{n} + 4$ ----------(1)

$$\int_0^1 (5x+4)\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n}\left(\frac{5r}{n}+4\right)$$

$$= \lim_{n\to\infty}\left[\frac{5}{n^2}\sum_{r=1}^{n} r + \frac{4}{n}\sum_{r=1}^{n} 1\right]$$

$$= \lim_{n\to\infty}\left[\frac{5}{n^2}(1+2+\cdots+n) + \frac{4}{n}(n)\right]$$

$$= \lim_{n\to\infty}\left[\frac{5}{n^2}\left(\frac{n(n+1)}{2}\right) + 4\right]$$

$$= \lim_{n\to\infty}\left[\frac{5\left(1+\frac{1}{n}\right)}{2} + 4\right]$$

$$= \frac{5}{2} + 4$$

$$\int_0^1 (5x+4)\,dx = \frac{13}{2}$$

(ii) $\int_1^2 (4x^2-1)\,dx$

Solution:

Here $f(x) = 4x^2 - 1, a = 1$ and $b = 2$

$$\frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$f\left(a + \left(\frac{b-a}{n}\right)r\right) = f\left(1 + \left(\frac{1}{n}\right)r\right)$$

$$= f\left(1 + \frac{r}{n}\right)$$

$$= 4\left(1 + \frac{r}{n}\right)^2 - 1$$

$$= 4\left(1 + \frac{r^2}{n^2} + \frac{2r}{n}\right) - 1$$

$$= 4 + \frac{4r^2}{n^2} + \frac{8r}{n} - 1$$

$$f\left(a + \left(\frac{b-a}{n}\right)r\right) = 3 + \frac{4r^2}{n^2} + \frac{8r}{n} \qquad \text{----------(1)}$$

$$\int_0^1 (5x+4)\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(3 + \frac{4r^2}{n^2} + \frac{8r}{n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{3}{n}\sum_{r=1}^{n} 1 + \frac{4}{n^3}\sum_{r=1}^{n} r^2 + \frac{8}{n^2}\sum_{r=1}^{n} r\right]$$

$$= \lim_{n\to\infty}\left[\begin{array}{c}\frac{3}{n}(n) + \frac{4}{n^3}(1^2+2^2+\cdots+n^2) \\ +\frac{8}{n^2}(1+2+\cdots+n)\end{array}\right]$$

$$= \lim_{n\to\infty}\left[3 + \frac{4}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \lim_{n\to\infty}\left[3 + 4\left(\frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}\right) + 8\left(\frac{1+\frac{1}{n}}{2}\right)\right]$$

$$= 3 + 4\left(\frac{2}{6}\right) + 8\left(\frac{1}{2}\right)$$

$$= 3 + \frac{4}{3} + 4$$

$$= 7 + \frac{4}{3}$$

$$\int_0^1 (5x+4)\,dx = \frac{25}{3}$$

Exercise 9.3

1. Evaluate the definite integrals:

Hint: $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

(i) $\int_3^4 \frac{dx}{x^2-4}$

Solution:

$$\int_3^4 \frac{dx}{x^2-4} = \int_3^4 \frac{dx}{x^2-2^2}$$

$$= \frac{1}{2(2)}\log\left[\left(\frac{x-2}{x+2}\right)\right]_3^4$$

$$= \frac{1}{4}\left[\log\left(\frac{4-2}{4+2}\right) - \log\left(\frac{3-2}{3+2}\right)\right]$$

$$= \frac{1}{4}\left[\log\left(\frac{1}{3}\right) - \log\left(\frac{1}{5}\right)\right]$$

$$= \frac{1}{4}\left[\log\left(\frac{\frac{1}{3}}{\frac{1}{5}}\right)\right] \qquad \left[\because \log a - \log b = \log\left(\frac{a}{b}\right)\right]$$

$$\int_3^4 \frac{dx}{x^2-4} = \frac{1}{4}\log\left(\frac{5}{3}\right)$$

Hint: $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

(ii) $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

Solution:

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5} = \int_{-1}^{1} \frac{dx}{(x+1)^2-1+5}$$

$$= \int_{-1}^{1} \frac{dx}{(x+1)^2+2^2}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{x+1}{2}\right)\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{1+1}{2}\right) - \tan^{-1}\left(\frac{-1+1}{2}\right)\right]$$

$$= \frac{1}{2}\left[\tan^{-1} 1 - \tan^{-1} 0\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - 0\right]$$

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5} = \frac{\pi}{8}$$

Hint: $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$$

(iii) $\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$

Solution:

$$\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx = \int_0^1 \sqrt{\frac{(1-x)}{(1+x)} \times \frac{(1-x)}{(1-x)}}\,dx$$

$$= \int_0^1 \sqrt{\frac{(1-x)^2}{1-x^2}}\,dx$$

$$= \int_0^1 \frac{1-x}{\sqrt{1-x^2}}\,dx$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx + \int_0^1 \frac{-x}{\sqrt{1-x^2}}\,dx$$

$$= [\sin^{-1} x]_0^1 + \frac{1}{2}\int_0^1 \frac{-2x}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1}(1) - \sin^{-1}(0) + \frac{1}{2}\left[2\sqrt{1-x^2}\right]_0^1$$

$$= \frac{\pi}{2} + \frac{1}{2}[0-2]$$

$$\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx = \frac{\pi}{2} - 1$$

Hint: $\mathbf{1 + \cos x = 2\cos^2\frac{x}{2}}$ **and** $\mathbf{\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}}$

$$\int e^x[f(x) + f'(x)]\,dx = e^x f(x)$$

(iv) $\int_0^{\frac{\pi}{2}} e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx$

Solution:

$$\int_0^{\frac{\pi}{2}} e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx = \int_0^{\frac{\pi}{2}} e^x \left[\frac{1}{1+\cos x} + \frac{\sin x}{1+\cos x}\right] dx$$

$$= \int_0^{\frac{\pi}{2}} e^x \left[\frac{1}{2\cos^2\frac{x}{2}} + \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right] dx$$

$$= \int_0^{\frac{\pi}{2}} e^x \left[\frac{\sec^2\frac{x}{2}}{2} + \tan\frac{x}{2}\right] dx$$

$$= \int_0^{\frac{\pi}{2}} e^x \left[\tan\frac{x}{2} + \frac{\sec^2\frac{x}{2}}{2}\right] dx$$

$$= \left[e^x\left(\tan\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \left(e^{\frac{\pi}{2}}\tan\frac{\frac{\pi}{2}}{2}\right) - 0$$

$$\int_0^{\frac{\pi}{2}} e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx = e^{\frac{\pi}{2}}\tan\frac{\pi}{4}$$

Hint: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(v) $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\,\sin^3\theta\,d\theta$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\,\sin^3\theta\,d\theta$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\,\sin^2\theta\,\sin\theta\,d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\,(1-\cos^2\theta)\sin\theta\,d\theta$$

Let $t = \cos\theta$

$dt = -\sin\theta\,d\theta$

θ	0	$\frac{\pi}{2}$
t	1	0

$$I = \int_1^0 \sqrt{t}(1-t^2)(-dt)$$

$$= \int_0^1 \left(t^{\frac{1}{2}} - t^{\frac{5}{2}}\right) dt$$

$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \frac{t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$

$$= \left(\frac{2}{3} - \frac{2}{7}\right) - (0)$$

$$= \frac{14-6}{21}$$

$$I = \frac{8}{21}$$

Hint: $\mathbf{\cos^2\theta - \sin^2\theta = \cos 2\theta}$ **and** $\mathbf{1 + \tan^2\theta = \sec^2\theta}$

(vi) $\int_0^1 \frac{1-x^2}{(1+x^2)^2}\,dx$

Solution:

Let $I = \int_0^1 \frac{1-x^2}{(1+x^2)^2}\,dx$

Let $x = \tan t$

$dx = \sec^2 t\,dt$

x	0	1
t	0	$\frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} \frac{1-tan^2t}{(1+tan^2t)^2}\sec^2 t\,dt$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-tan^2t}{(\sec^2 t)^2}\sec^2 t\,dt$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-tan^2t}{\sec^2 t}\,dt$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\frac{\sin^2 t}{\cos^2 t}}{\frac{1}{\cos^2 t}}\,dt$$

$$= \int_0^{\frac{\pi}{4}} (\cos^2 t - \sin^2 t)\,dt$$

$$= \int_0^{\frac{\pi}{4}} \cos 2t\,dt$$

$$= \left[\frac{\sin 2t}{2}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\sin 2\left(\frac{\pi}{4}\right) - \sin 2(0)\right]$$

$$= \frac{1}{2}\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= \frac{1}{2}[1-0]$$

$$I = \frac{1}{2}$$

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2. Evaluate the following integrals using properties of integration:

Hint: If $f(-x) = \begin{cases} f(x) & even \\ -f(x) & odd \end{cases}$

If $f(x)$ is an odd function, $\int_{-a}^{a} f(x)\,dx = 0$

(i) $\int_{-5}^{5} x\cos\left(\frac{e^x-1}{e^x+1}\right) dx$

Solution:

Let $f(x) = x\cos\left(\frac{e^x-1}{e^x+1}\right)$

$f(-x) = -x\cos\left(\frac{e^{-x}-1}{e^{-x}+1}\right)$

$= -x\cos\left(\frac{\frac{1}{e^x}-1}{\frac{1}{e^x}+1}\right)$

$= -x\cos\left(\frac{1-e^x}{1+e^x}\right)$

$= -x\cos\left[-\left(\frac{e^x-1}{e^x+1}\right)\right]$

$= -x\cos\left(\frac{e^x-1}{e^x+1}\right)$ $\quad [\because \cos(-x) = \cos x]$

$f(-x) = -f(x)$

$\therefore f(x)$ is an odd function.

$\therefore \int_{-5}^{5} x\cos\left(\frac{e^x-1}{e^x+1}\right) dx = 0$

(ii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^5 + x\cos x + \tan^3 x + 1)\, dx$

Solution:

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^5 + x\cos x + \tan^3 x + 1)\, dx$

$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^5 + x\cos x + \tan^3 x)\, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx$ -----(1)

Let $f(x) = x^5 + x\cos x + \tan^3 x$

$f(-x) = (-x)^5 + (-x)\cos(-x) + \tan^3(-x)$

$= -x^5 - x\cos x - \tan^3 x$

$= -[x^5 + x\cos x + \tan^3 x]$

$f(-x) = -f(x)$

$\therefore f(x)$ is an odd function.

$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^5 + x\cos x + \tan^3 x)\, dx = 0$

$I = 0 + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx$ From (1)

$= [x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$

$I = \pi$

Hint: $\cos 2x = 1 - 2\sin^2 x \Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$

If $f(x)$ is even function, $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$

(iii) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\, dx$

Solution:

Let $f(x) = \sin^2 x$

Since $\sin x$ is an odd function, $\sin^2 x$ is even function.

$\therefore \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\, dx = 2\int_0^{\frac{\pi}{4}} \sin^2 x\, dx$

$= 2\int_0^{\frac{\pi}{4}} \left(\frac{1-\cos 2x}{2}\right) dx$

$= \int_0^{\frac{\pi}{4}} (1 - \cos 2x)\, dx$

$= \left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{\pi}{4} - \frac{\sin\frac{\pi}{2}}{2}\right] - [0]$

$= \frac{\pi}{4} - \frac{1}{2}$

$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\, dx = \frac{\pi-2}{4}$

(iv) $\int_0^{2\pi} x\log\left(\frac{3+\cos x}{3-\cos x}\right) dx$

Solution:

Let $I = \int_0^{2\pi} x\log\left(\frac{3+\cos x}{3-\cos x}\right) dx$

Wkt, $\int_0^a x\, f(x)\, dx = \frac{a}{2}\int_0^a f(x)\, dx$ if $f(a-x) = f(x)$

Let $f(x) = \log\left(\frac{3+\cos x}{3-\cos x}\right)$

$f(2\pi - x) = \log\left(\frac{3+\cos(2\pi-x)}{3-\cos(2\pi-x)}\right)$

$= \log\left(\frac{3+\cos x}{3-\cos x}\right)$

$f(2\pi - x) = f(x)$

$\therefore$ $I = \frac{2\pi}{2}\int_0^{2\pi} \log\left(\frac{3+\cos x}{3-\cos x}\right) dx$

$I = \pi\int_0^{2\pi} \log\left(\frac{3+\cos x}{3-\cos x}\right) dx$ ----------(1)

Wkt, $\int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx$ if $f(2a-x) = f(x)$

$I = 2\pi\int_0^{\pi} \log\left(\frac{3+\cos x}{3-\cos x}\right) dx$ ----------(2)

Wkt, $\int_0^{2a} f(x)\, dx = 0$ if $f(2a-x) = -f(x)$

$f(\pi - x) = \log\left(\frac{3+\cos(\pi-x)}{3-\cos(\pi-x)}\right)$

$= \log\left(\frac{3-\cos x}{3+\cos x}\right)$

$= \log(3-\cos x) - \log(3+\cos x)$

$= -[\log(3+\cos x) - \log(3-\cos x)]$

$= -\log\left(\frac{3+\cos x}{3-\cos x}\right)$

$f(\pi - x) = -f(x)$

$I = 2\pi \times 0$ From (2)

$I = 0$

(v) $\int_0^{2\pi} \sin^4 x \cos^3 x\, dx$

Solution:

Let $I = \int_0^{2\pi} \sin^4 x \cos^3 x\, dx$

Wkt, $\int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx$ if $f(2a-x) = f(x)$

Let $f(x) = \sin^4 x \cos^3 x$

$f(2\pi - x) = \sin^4(2\pi - x)\cos^3(2\pi - x)$

$= \sin^4 x \cos^3 x$

$f(2\pi - x) = f(x)$

$\therefore$ $I = 2\int_0^{\pi} \sin^4 x \cos^3 x\, dx$ ----------(1)

Wkt, $\int_0^{2a} f(x)\, dx = 0$ if $f(2a-x) = -f(x)$

$f(\pi - x) = \sin^4(\pi - x)\cos^3(\pi - x)$

$= \sin^4 x\,(-\cos^3 x)$

$f(\pi - x) = -f(x)$

$\therefore$ $I = 2 \times 0$ From (1)

$I = 0$

(vi) $\int_0^1 |5x - 3|\, dx$

Solution:

Let $I = \int_0^1 |5x-3|\, dx$

$$|5x-3| = \begin{cases} -(5x-3), & when\ 5x-3<0 \\ 5x-3, & when\ 5x-3 \geq 0 \end{cases}$$

$$|5x-3| = \begin{cases} 3-5x, & x < \frac{3}{5} \\ 5x-3, & x \geq \frac{3}{5} \end{cases}$$

$$I = \int_0^{\frac{3}{5}} (3-5x)\, dx + \int_{\frac{3}{5}}^{1} (5x-3)\, dx$$

$$= \left[3x - \frac{5x^2}{2}\right]_0^{\frac{3}{5}} + \left[\frac{5x^2}{2} - 3x\right]_{\frac{3}{5}}^{1}$$

$$= \left[\left(3\left(\frac{3}{5}\right) - \frac{5\left(\frac{9}{25}\right)}{2}\right) - (0)\right] + \left[\left(\frac{5}{2} - 3\right) - \left(\frac{5\left(\frac{9}{25}\right)}{2} - 3\left(\frac{3}{5}\right)\right)\right]$$

$$= \left[\frac{9}{5} - \frac{9}{10}\right] + \left[-\frac{1}{2} - \left(\frac{9}{10} - \frac{9}{5}\right)\right]$$

$$= \frac{9}{5} - \frac{9}{10} - \frac{1}{2} - \frac{9}{10} + \frac{9}{5}$$

$$= \frac{18-9-5-9+18}{10}$$

$$I = \frac{13}{10}$$

Hint: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

(vii) $\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt, x \in \left[0, \frac{\pi}{2}\right]$

Solution:

Let $I = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt$

$I = I_1 + I_2$

For I_1:

Let $t = \sin^2\theta$

$dt = 2\sin\theta\cos\theta\, d\theta$

$dt = \sin 2\theta\, d\theta$

t	0	$\sin^2 x$
θ	0	x

For I_2:

let $t = \cos^2\theta$

$dt = -2\cos\theta\sin\theta\, d\theta$

$dt = -\sin 2\theta\, d\theta$

t	0	$\cos^2 x$
θ	$\frac{\pi}{2}$	x

$$I = \int_0^x \theta\sin 2\theta\, d\theta - \int_{\frac{\pi}{2}}^x \theta \sin 2\theta\, d\theta$$

$$I = \int_0^x \theta\cos\theta\, d\theta + \int_x^{\frac{\pi}{2}} \theta\sin\theta\, d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \theta\sin 2\theta\, d\theta \quad \text{----------(1)}$$

Wkt, $\int u dv\, dx = uv - u'v_1 + u''v_2 - \cdots$

Let $u = \theta$ $\quad dv = \sin 2\theta\, d\theta$

$u' = 1$ $\quad v = -\frac{\cos 2\theta}{2}$

$u'' = 0$ $\quad v_1 = -\frac{\sin 2\theta}{4}$

$$I = \left[-\frac{\theta\cos 2\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$I = \left[-\frac{\frac{\pi}{2}\cos\pi}{2} + \frac{\sin\pi}{4}\right] - [0+0]$$

$$I = \frac{\pi}{4}$$

(viii) $\int_0^1 \frac{\log(1+x)}{1+x^2} dx$

Solution:

Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$

Let $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

x	0	1
θ	0	$\frac{\pi}{4}$

$$\therefore \quad I = \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\, d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\,d\theta \qquad \text{----------(1)}$$

Wkt, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I = \int_0^{\frac{\pi}{4}} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left[1+\left(\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right)\right]d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left[1+\left(\frac{1-\tan\theta}{1+\tan\theta}\right)\right]d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left[\frac{1+\tan\theta+1-tan\theta}{1+\tan\theta}\right]d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\frac{\pi}{4}} [\log 2-\log(1+\tan\theta)]d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log 2\,d\theta - \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\,d\theta$$

$$I = \log 2\int_0^{\frac{\pi}{4}} d\theta - I \qquad \text{From (1)}$$

$$2I = log\,2\,[\theta]_0^{\frac{\pi}{4}}$$

$$2I = \frac{\pi}{4}\log 2$$

$$I = \frac{\pi}{8}\log 2$$

(ix) $\int_0^{\pi} \frac{x\sin x}{1+\sin x}dx$

Solution:

$$\text{Let } I = \int_0^{\pi} \frac{x\sin x}{1+\sin x}dx \qquad \text{----------(1)}$$

Wkt, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}dx$$

$$I = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\sin x}dx \qquad \text{----------(2)}$$

Add (1) and (2) we get,

$$2I = \int_0^{\pi} \frac{x\,sin\,x+(\pi-x)\sin x}{1+\sin x}dx$$

$$= \int_0^{\pi} \frac{\pi\sin x}{1+\sin x}dx$$

$$= \pi\int_0^{\pi} \frac{\sin x+1-1}{1+\sin x}dx$$

$$= \pi\int_0^{\pi} \frac{1+\sin x}{1+\sin x}dx - \pi\int_0^{\pi} \frac{1}{1+\sin x}dx$$

$$= \pi\int_0^{\pi} dx - \pi\int_0^{\pi} \frac{1}{1+\sin x}\times\frac{1-\sin x}{1-\sin x}dx$$

$$= \pi[x]_0^{\pi} - \pi\int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x}dx$$

$$= \pi^2 - \pi\int_0^{\pi} \frac{1-\sin x}{\cos^2 x}dx$$

$$2I = \pi^2 - \pi\int_0^{\pi} (\sec^2 x - \tan x\sec x)\,dx$$

$$2I = \pi^2 - \pi[\tan x - \sec x]_0^{\pi}$$

$$= \pi^2 - \pi[(0+1)-(0-1)]$$

$$2I = \pi^2 - 2\pi$$

$$I = \frac{1}{2}(\pi^2-\pi)$$

(x) $\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\tan x}}dx$

Solution:

$$\text{Let } I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\tan x}}dx \qquad \text{----------(1)}$$

Wkt, $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$$I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{tan\left(\frac{3\pi}{8}+\frac{\pi}{8}-x\right)}}dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{tan\left(\frac{\pi}{2}-x\right)}}dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\cot x}}dx$$

$$I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\frac{1}{\sqrt{\tan x}}}dx$$

$$I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\sqrt{\tan x}}{1+\sqrt{tan\,x}}dx \qquad \text{----------(2)}$$

Add (1) and (2) we get,

$$2I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1+\sqrt{\tan x}}{1+\sqrt{tan\,x}}dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} dx$$

$$= [x]_{\frac{\pi}{8}}^{\frac{3\pi}{8}}$$

$$= \frac{3\pi}{8} - \frac{\pi}{8}$$

$$2I = \frac{2\pi}{8}$$

$$I = \frac{\pi}{8}$$

(xi) $\int_0^{\pi} x[\sin^2(\sin x)+\cos^2(\cos x)]\,dx$

Solution:

Let $I = \int_0^{\pi} x[\sin^2(\sin x)+\cos^2(\cos x)]\,dx$

Wkt, $\int_0^a x\,f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$ if $f(a-x) = f(x)$

Let $f(x) = \sin^2(\sin x)+\cos^2(\cos x)$

$$f(\pi-x) = \sin^2(\sin(\pi-x))+\cos^2(\cos(\pi-x))$$

$$= \sin^2(\sin x)+\cos^2(\cos x)$$

$$f(\pi - x) = f(x)$$

$\therefore\ I = \frac{\pi}{2}\int_0^{\pi}[\sin^2(\sin x) + \cos^2(\cos x)]\,dx$

Wkt, $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ if $f(2a-x) = f(x)$

$\therefore\ I = 2 \times \frac{\pi}{2}\int_0^{\frac{\pi}{2}}[sin^2(sin\,x) + cos^2(cos\,x)]\,dx$

$$I = \pi\int_0^{\frac{\pi}{2}}[sin^2(sin\,x) + cos^2(cos\,x)]\,dx \quad \text{----------(1)}$$

Wkt, $\int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx$

$$I = \pi\int_0^{\frac{\pi}{2}}\left[sin^2\left(sin\left(\frac{\pi}{2}-x\right)\right) + cos^2\left(cos\left(\frac{\pi}{2}-x\right)\right)\right]dx$$

$$I = \pi\int_0^{\frac{\pi}{2}}[sin^2(cos\,x) + cos^2(sin\,x)]\,dx \quad \text{----------(2)}$$

Add (1) and (2) we get,

$$2I = \pi\int_0^{\frac{\pi}{2}}\begin{bmatrix} sin^2(sin\,x) + cos^2(cos\,x) \\ +\ sin^2(cos\,x) + cos^2(sin\,x)\end{bmatrix}dx$$

$$2I = \pi\int_0^{\frac{\pi}{2}} 2\,dx$$

$$2I = 2\pi[x]_0^{\frac{\pi}{2}}$$

$$I = \pi\left(\frac{\pi}{2}\right)$$

$$I = \frac{\pi^2}{2}$$

Exercise 9.4

Evaluate the following:

1. $\int_0^1 x^3\, e^{-2x} dx$

Solution:

Let $I = \int_0^1 x^3\, e^{-2x} dx$

Let $u = x^3 \qquad dv = e^{-2x}dx$

$u' = 3x^2 \qquad v = \frac{e^{-2x}}{-2}$

$u'' = 6x \qquad v_1 = \frac{e^{-2x}}{4}$

$u''' = 6 \qquad v_2 = \frac{e^{-2x}}{-8}$

$u^{iv} = 0 \qquad v_3 = \frac{e^{-2x}}{16}$

$$I = [uv - u'v_1 + u''v_2 - u'''v_3]_0^1$$

$$= \left[x^3\left(\frac{e^{-2x}}{-2}\right) - 3x^2\left(\frac{e^{-2x}}{4}\right) + 6x\left(\frac{e^{-2x}}{-8}\right) - 6\left(\frac{e^{-2x}}{16}\right)\right]_0^1$$

$$= \left[e^{-2x}\left(\frac{-x^3}{2} - \frac{3x^2}{4} - \frac{3x}{4} - \frac{3}{8}\right)\right]_0^1$$

$$= \left[e^{-2}\left(-\frac{1}{2} - \frac{3}{4} - \frac{3}{4} - \frac{3}{8}\right)\right] - \left[e^0\left(-\frac{3}{8}\right)\right]$$

$$I = e^{-2}\left(\frac{-4-6-6-3}{8}\right) + \left(\frac{3}{8}\right)$$

$$I = -\frac{19}{8}e^{-2} + \frac{3}{8}$$

2. $\int_0^1 \frac{\sin(3\tan^{-1}x)\tan^{-1}x}{1+x^2}dx$

Solution:

Let $I = \int_0^1 \frac{\sin(3\tan^{-1}x)\tan^{-1}x}{1+x^2}dx$

Let $t = \tan^{-1}x$

$dt = \frac{1}{1+x^2}dx$

x	0	1
t	0	$\frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} t\sin 3t\,dt \quad \text{----------(1)}$$

Let $u = t \qquad dv = \sin 3t\,dt$

$u' = 1 \qquad v = -\frac{\cos 3t}{3}$

$u'' = 0 \qquad v_1 = -\frac{\sin 3t}{9}$

$$I = \left[t\left(-\frac{\cos 3t}{3}\right) - 1\left(-\frac{\sin 3t}{9}\right)\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{4}\left(-\frac{\cos\frac{3\pi}{4}}{3}\right) + \frac{\sin\frac{3\pi}{4}}{9}\right] - [0-0]$$

$$= \frac{\pi}{4}\left(\frac{\frac{1}{\sqrt{2}}}{3}\right) + \frac{\frac{1}{\sqrt{2}}}{9}$$

$$I = \frac{1}{\sqrt{2}}\left(\frac{\pi}{12} + \frac{1}{9}\right)$$

3. $\int_0^{\frac{1}{\sqrt{2}}} \frac{e^{\sin^{-1}x}\sin^{-1}x}{\sqrt{1-x^2}}dx$

Solution:

Let $I = \int_0^{\frac{1}{\sqrt{2}}} \frac{e^{\sin^{-1}x}\sin^{-1}x}{\sqrt{1-x^2}}dx$

Let $t = \sin^{-1}x$

$dt = \frac{1}{\sqrt{1-x^2}}dx$

x	0	$\frac{1}{\sqrt{2}}$
t	0	$\frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} t\,e^t\,dt \quad \text{----------(1)}$$

Let $u = t \qquad dv = e^t dt$

$u' = 1 \qquad v = e^t$

$u'' = 0 \qquad v_1 = e^t$

$$I = [te^t - e^t]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{4}e^{\frac{\pi}{4}} - e^{\frac{\pi}{4}}\right] - [0 - e^0]$$

$$I = e^{\frac{\pi}{4}}\left(\frac{\pi}{4} - 1\right) + 1$$

4. $\int_0^{\frac{\pi}{2}} x^2 \cos 2x\, dx$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} x^2 \cos 2x\, dx$

Let $u = x^2 \qquad dv = \cos 2x\, dx$

$u' = 2x \qquad v = \frac{\sin 2x}{2}$

$u'' = 2 \qquad v_1 = \frac{-\cos 2x}{4}$

$u''' = 0 \qquad v_2 = \frac{-\sin 2x}{8}$

$$I = \left[x^2\left(\frac{\sin 2x}{2}\right) - 2x\left(-\frac{\cos 2x}{4}\right) + 2\left(-\frac{\sin 2x}{8}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{x^2}{2}(\sin 2x) + \frac{x}{2}\cos 2x - \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \left[0 + \frac{\pi}{4}\cos\pi - 0\right] - [0 + 0 - 0]$$

$$I = -\frac{\pi}{4}$$

Exercise 9.5

1. Evaluate the following:

(i) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+5\cos^2 x}$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+5\cos^2 x}$

Dividing both numerator and denominator by $\cos^2 x$,

$$I = \int_0^{\frac{\pi}{2}} \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}+\frac{5\cos^2 x}{\cos^2 x}} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{\sec^2 x+5} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{(1+\tan^2 x)+5} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{6+\tan^2 x} dx$$

Let $t = \tan x$

$dt = \sec^2 x\, dx$

x	0	$\frac{\pi}{2}$
t	0	∞

$$I = \int_0^{\infty} \frac{dt}{6+t^2}$$

$$= \int_0^{\infty} \frac{dt}{\left(\sqrt{6}\right)^2+t^2} \qquad \left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{\sqrt{6}}\left[\tan^{-1}\left(\frac{t}{\sqrt{6}}\right)\right]_0^{\infty}$$

$$= \frac{1}{\sqrt{6}}[\tan^{-1}\infty - \tan^{-1} 0] = \frac{1}{\sqrt{6}}\left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi}{2\sqrt{6}}$$

(ii) $\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\sin^2 x}$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \frac{dx}{5+4\sin^2 x}$

Dividing both numerator and denominator by $\cos^2 x$,

$$I = \int_0^{\frac{\pi}{2}} \frac{\frac{1}{\cos^2 x}}{\frac{5}{\cos^2 x}+\frac{4\sin^2 x}{\cos^2 x}} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{5\sec^2 x+4\tan^2 x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{5(1+\tan^2 x)+4\tan^2 x} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{5+9\tan^2 x} dx$$

Let $t = \tan x$

$dt = \sec^2 x\, dx$

x	0	$\frac{\pi}{2}$
t	0	∞

$$I = \int_0^{\infty} \frac{dt}{5+9t^2}$$

$$= \int_0^{\infty} \frac{dt}{\left(\sqrt{5}\right)^2+(3t)^2}$$

$$= \frac{1}{\sqrt{5}} \times \frac{1}{3}\left[\tan^{-1}\left(\frac{3t}{\sqrt{5}}\right)\right]_0^{\infty}$$

$$= \frac{1}{3\sqrt{5}}[\tan^{-1}\infty - \tan^{-1} 0]$$

$$= \frac{1}{3\sqrt{5}}\left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi}{6\sqrt{5}}$$

Exercise 9.6

1. Evaluate the following:

(i) $\int_0^{\frac{\pi}{2}} \sin^{10} x\, dx$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \sin^{10} x\, dx$

$$= \frac{9}{10} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$I = \frac{63\pi}{512}$$

(ii) $\int_0^{\frac{\pi}{2}} \cos^7 x\, dx$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \cos^7 x\, dx$

$$= \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3}$$

$$I = \frac{16}{35}$$

(iii) $\int_0^{\frac{\pi}{4}} sin^6 2x\, dx$

Solution:

Let $I = \int_0^{\frac{\pi}{4}} sin^6 2x\, dx$

Let $t = 2x$

$dt = 2\, dx$

x	0	$\frac{\pi}{4}$
t	0	$\frac{\pi}{2}$

$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin^6 t\, dt$

$I = \frac{1}{2}\left[\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}\right] = \frac{5\pi}{64}$

(iv) $\int_0^{\frac{\pi}{6}} sin^5 3x\, dx$

Solution:

Let $I = \int_0^{\frac{\pi}{6}} sin^5 3x\, dx$

Let $t = 3x$

$dt = 3\, dx$

x	0	$\frac{\pi}{6}$
t	0	$\frac{\pi}{2}$

$I = \frac{1}{3}\int_0^{\frac{\pi}{2}} sin^5 t\, dt$

$= \frac{1}{3}\left[\frac{4}{5}\times\frac{2}{3}\right]$

$I = \frac{8}{45}$

Hint: If m and n is odd then,

$$\int_0^{\frac{\pi}{2}} sin^n\theta\, cos^m\theta\, d\theta = \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdots\frac{2}{n+3}\cdot\frac{1}{n+1}$$

(v) $\int_0^{\frac{\pi}{2}} sin^2 x\, cos^4 x\, dx$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} sin^2 x\, cos^4 x\, dx$

$= \frac{3}{6}\times\frac{1}{4}\times\frac{1}{2}\times\frac{\pi}{2}$

$I = \frac{\pi}{32}$

Aliter Method:

Let $I = \int_0^{\frac{\pi}{2}} sin^2 x\, cos^4 x\, dx$

$I = \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\, cos^4 x\, dx$

$I = \int_0^{\frac{\pi}{2}} cos^4 x\, dx - \int_0^{\frac{\pi}{2}} cos^6 x\, dx$

$= \frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} - \frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$

$= \frac{3\pi}{16} - \frac{5\pi}{32}$

$I = \frac{\pi}{32}$

(vi) $\int_0^{2\pi} sin^7\frac{x}{4} dx$

Solution:

Let $I = \int_0^{2\pi} sin^7\frac{x}{4} dx$

Let $t = \frac{x}{4}$

$dt = \frac{dx}{4}$

$dx = 4\, dt$

x	0	2π
t	0	$\frac{\pi}{2}$

$I = 4\int_0^{\frac{\pi}{2}} sin^7 t\, dt$

$= 4\left[\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\right]$

$I = \frac{64}{35}$

Hint: If m and n is odd then,

$$\int_0^{\frac{\pi}{2}} sin^n\theta\, cos^m\theta\, d\theta = \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdots\frac{2}{n+3}\cdot\frac{1}{n+1}$$

(vii) $\int_0^{\frac{\pi}{2}} sin^3\theta\, cos^5\theta\, d\theta$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} sin^3\theta\, cos^5\theta\, d\theta$

$= \frac{4}{8}\times\frac{2}{6}\times\frac{1}{4}$

$I = \frac{1}{24}$

Hint: $\int_0^1 x^m(1-x)^n\, dx = \frac{m!\times n!}{(m+n+1)!}$

(viii) $\int_0^1 x^2(1-x)^3\, dx$

Solution:

$I = \frac{2!\times 3!}{(2+3+1)!}$

$= \frac{2\times 6}{6\times5\times4\times3\times2\times1}$

$I = \frac{1}{60}$

Exercise 9.7

Hint: $\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$

Evaluate the following:

1. (i) $\int_0^{\infty} x^5 e^{-3x} dx$

Solution:

Here $n = 5, a = 3$

$\int_0^{\infty} x^5 e^{-3x} dx = \frac{5!}{3^{5+1}} = \frac{5!}{3^6}$

Hint: $\int_0^\infty x^n e^{-x}\,dx = n!$

(ii) $\int_0^{\frac{\pi}{2}} \frac{e^{-\tan x}}{\cos^6 x}\,dx$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \frac{e^{-\tan x}}{\cos^6 x}\,dx$

$= \int_0^{\frac{\pi}{2}} e^{-\tan x} \sec^6 x\,dx$

$= \int_0^{\frac{\pi}{2}} e^{-\tan x} (\sec^2 x)^2 \sec^2 x\,dx$

$I = \int_0^{\frac{\pi}{2}} e^{-\tan x}(1+\tan^2 x)^2 \sec^2 x\,dx$

Let $t = \tan x$

$dt = \sec^2 x\,dx$

x	0	$\frac{\pi}{2}$
t	0	∞

$I = \int_0^\infty e^{-t}(1+t^2)^2 dt$

$= \int_0^\infty e^{-t}(1+t^4+2t^2)dt$

$= \int_0^\infty t^0 e^{-t}\,dt + \int_0^\infty t^4 e^{-t}\,dt + 2\int_0^\infty t^2 e^{-t}\,dt$

$= 0! + 4! + 2!$

$= 1 + 24 + 4$

$I = 28$

2. If $\int_0^\infty e^{-\alpha x^2} x^3\,dx = 32$, find α.

Solution:

Given $\int_0^\infty e^{-\alpha x^2} x^3\,dx = 32$

$\int_0^\infty e^{-\alpha x^2} x^2 x\,dx = 32$ ----------(1)

Let $t = x^2$

$dt = 2x\,dx$

$\frac{dt}{2} = x\,dx$

x	0	∞
t	0	∞

From (1), we get

$\frac{1}{2}\int_0^\infty t\,e^{-\alpha t}\,dt = 32$

$\frac{1}{2}\left[\frac{1!}{\alpha^{1+1}}\right] = 32$ $\quad \left[\because \int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}\right]$

$\frac{1}{\alpha^2} = 64$

$\alpha^2 = \frac{1}{64}$

$\alpha = \frac{1}{8}$ $\quad [\because \alpha > 0]$

Exercise 9.8

1. Find the area of the region bounded by $3x - 2y + 6 = 0$, $x = -3$, $x = 1$ and x-axis.

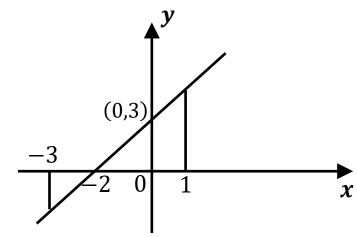
Solution:

Given $3x - 2y + 6 = 0$

$2y = 3x + 6$

$y = \frac{3}{2}(x+2)$ ----------(1)

x	0	-2
y	3	0



$A = \int_{-3}^{-2} -y\,dx + \int_{-2}^{1} y\,dx$

$= -\frac{3}{2}\int_{-3}^{-2}(x+2)dx + \frac{3}{2}\int_{-2}^{1}(x+2)dx$

$= -\frac{3}{2}\left[\frac{x^2}{2}+2x\right]_{-3}^{-2} + \frac{3}{2}\left[\frac{x^2}{2}+2x\right]_{-2}^{1}$

$= -\frac{3}{2}\left[(2-4)-\left(\frac{9}{2}-6\right)\right] + \frac{3}{2}\left[\left(\frac{1}{2}+2\right)-(2-4)\right]$

$= -\frac{3}{2}\left[-2+\frac{3}{2}\right] + \frac{3}{2}\left[\frac{5}{2}+2\right]$

$= -\frac{3}{2}\left[-\frac{1}{2}\right] + \frac{3}{2}\left[\frac{9}{2}\right]$

$= \frac{3}{4} + \frac{27}{4}$

$= \frac{30}{4}$

$A = \frac{15}{2}$

2. Find the area of the region bounded by $2x - y + 1 = 0$, $y = -1$, $y = 3$ and y-axis.

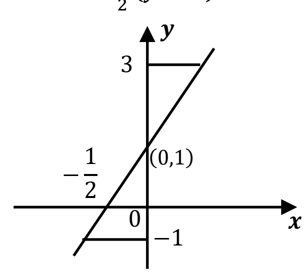
Solution:

Given $2x - y + 1 = 0$

$2x = y - 1$

$x = \frac{1}{2}(y-1)$ ----------(1)

x	0	$-\frac{1}{2}$
y	1	0



$A = \int_{-1}^{1} -x\,dy + \int_{1}^{3} x\,dy$

$= -\frac{1}{2}\int_{-1}^{1}(y-1)dy + \frac{1}{2}\int_{1}^{3}(y-1)dy$

$= -\frac{1}{2}\left[\frac{(y-1)^2}{2}\right]_{-1}^{1} + \frac{1}{2}\left[\frac{(y-1)^2}{2}\right]_{1}^{3}$

$= -\frac{1}{2}\left[0-\left(\frac{4}{2}\right)\right]+\frac{1}{2}\left[\frac{4}{2}-0\right]$

$= -\frac{1}{2}(-2)+\frac{1}{2}(2)$

$= 1+1$

$A = 2$

---

3. Find the area of the region bounded by the curve $2+x-x^2+y=0$, x-axis, $x=-3$ and $x=3$.

Solution:

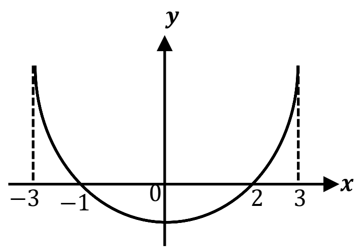
Given $2+x-x^2+y=0$

$y = x^2-x-2$

$y = (x+1)(x-2)$ ----------(1)

Put $y=0$, $x=-1,2$

Put $x=0$, $y=-2$



$A = \int_{-3}^{-1} y\,dx + \int_{-1}^{2} -y\,dx + \int_{2}^{3} y\,dx$

$= \int_{-3}^{-1}(x^2-x-2)dx - \int_{-1}^{2}(x^2-x-2)dx + \int_{2}^{3}(x^2-x-2)dx$

$= \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-3}^{-1} - \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-1}^{2} + \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{2}^{3}$

$= \begin{bmatrix}\left(-\frac{1}{3}-\frac{1}{2}+2\right)\\-\left(-9-\frac{9}{2}+6\right)\end{bmatrix} - \begin{bmatrix}\left(\frac{8}{3}-2-4\right)\\-\left(-\frac{1}{3}-\frac{1}{2}+2\right)\end{bmatrix} + \begin{bmatrix}\left(9-\frac{9}{2}-6\right)\\-\left(\frac{8}{3}-2-4\right)\end{bmatrix}$

$= \left[\left(\frac{7}{6}\right)-\left(-3-\frac{9}{2}\right)\right] - \left[\left(\frac{8}{3}-6\right)-\left(\frac{7}{6}\right)\right] + \left[\left(3-\frac{9}{2}\right)-\left(\frac{8}{3}-6\right)\right]$

$= \left[\frac{7}{6}+\frac{15}{2}\right] - \left[-\frac{10}{3}-\frac{7}{6}\right] + \left[-\frac{3}{2}+\frac{10}{3}\right]$

$= \frac{52}{6}+\frac{27}{6}+\frac{11}{6}$

$= \frac{90}{6}$

$A = 15$

---

4. Find the area of the region bounded by the line $y=2x+5$ and the parabola $y=x^2-2x$.

Solution:

Line: $y = 2x+5$ ----------(1)

x	0	$-\frac{5}{2}$
y	5	0

Parabola: $y = x^2-2x$ ----------(2)

From (1) and (2), we get

$x^2-2x = 2x+5$

$x^2-4x-5 = 0$

$(x+1)(x-5) = 0$

$x = -1, 5$

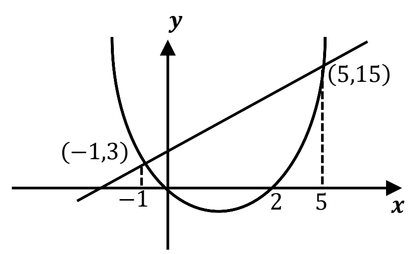
At $x=-1$,

$y = (-1)^2-2(-1) = 3$

At $x=5$,

$y = 5^2-2(5) = 15$

$\therefore$ The point of intersection is $(-1,3)$ and $(5,15)$



$A = \int_{-1}^{5}(y_u-y_l)dx$

$= \int_{-1}^{5}[(2x+5)-(x^2-2x)]dx$

$= \int_{-1}^{5}[5+4x-x^2]dx$

$= \left[5x+\frac{4x^2}{2}-\frac{x^3}{3}\right]_{-1}^{5}$

$= \left[5x+2x^2-\frac{x^3}{3}\right]_{-1}^{5}$

$= \left[\left(25+50-\frac{125}{3}\right)-\left(-5+2+\frac{1}{3}\right)\right]$

$= \left(75-\frac{125}{3}\right)-\left(-3+\frac{1}{3}\right)$

$= \left(\frac{225-125}{3}\right)-\left(\frac{-9+1}{3}\right)$

$= \frac{100}{3}+\frac{8}{3} = \frac{108}{3}$

$A = 36$

---

5. Find the area of the region bounded between the curves $y=\sin x$ and $y=\cos x$ and the lines $x=0$ and $x=\pi$.

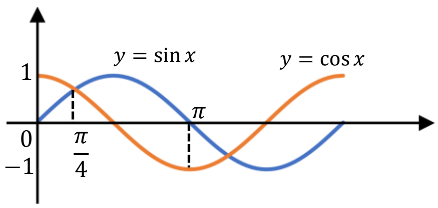
Solution:

Given $y = \sin x$ and $y = \cos x$

i.e., $\sin x = \cos x$

$$x = \frac{\pi}{4} \in [0, \pi]$$

$\therefore$ The point of intersection is $x = \frac{\pi}{4}$



$$A = \int_0^{\pi} (y_u - y_l)\, dx$$

$$A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x)\, dx$$

$$= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\cos x - \sin x]_{\frac{\pi}{4}}^{\pi}$$

$$= \left[\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 + \cos 0)\right]$$

$$+ \left[(-\cos\pi - \sin\pi) - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)\right]$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0 + 1)\right] + \left[(1 - 0) - \left(\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{2}{\sqrt{2}} - 1 + 1 + \frac{2}{\sqrt{2}}$$

$$= \sqrt{2} + \sqrt{2}$$

$$A = 2\sqrt{2}$$

6. Find the area of the region bounded by $y = \tan x, y = \cot x$ and the lines $x = 0, x = \frac{\pi}{2}, y = 0$.

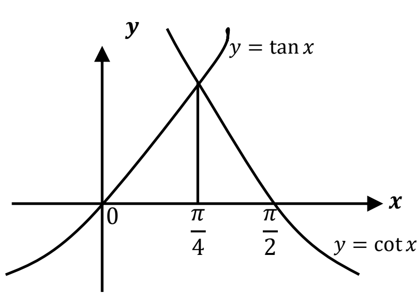
Solution:

Given $y = \tan x$ and $y = \cot x$

i.e, $\tan x = \cot x$

$$x = \frac{\pi}{4} \in [0, \pi]$$

$\therefore$ The point of intersection is $x = \frac{\pi}{4}$



$$A = \int_0^{\frac{\pi}{4}} y\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y\, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan x\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\, dx$$

$$= [\log \sec x]_0^{\frac{\pi}{4}} + [\log \sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \log \sec\frac{\pi}{4} - \log \sec 0 + \log \sin\frac{\pi}{2} - \log \sin\frac{\pi}{4}$$

$$= \log\sqrt{2} - \log 1 + \log 1 - \log\frac{1}{\sqrt{2}}$$

$$= \log\sqrt{2} - \log\frac{1}{\sqrt{2}}$$

$$= \log\left(\frac{\sqrt{2}}{\frac{1}{\sqrt{2}}}\right)$$

$$A = \log 2$$

7. Find the area of the region bounded by the parabola $y^2 = x$ and the line $y = x - 2$.

Solution:

Line: $y = x - 2$

$x = y + 2$ ----------(1)

x	0	2
y	-2	0

Parabola: $x = y^2$ ----------(2)

From (1) and (2), we get

$$y = y^2 - 2$$

$$y^2 - y - 2 = 0$$

$$(y + 1)(y - 2) = 0$$

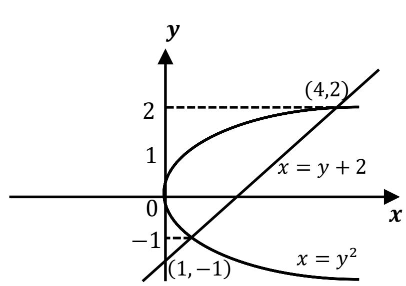
$$y = -1, 2$$

From (2), we get

At $y = -1$, $x = 1$

At $y = 2$, $x = 4$

$\therefore$ The point of intersection is $(1, -1)$ and $(4, 2)$



$$A = \int_{-1}^{2} (x_R - x_l)\, dy$$

$A = \int_{-1}^{2}(x_R - x_l)dy$

$= \int_{-1}^{2}[(y+2) - y^2]dy$

$= \left[\frac{(y+2)^2}{2} - \frac{y^3}{3}\right]_{-1}^{2}$

$= \left[\frac{16}{2} - \frac{8}{3}\right] - \left[\frac{1}{2} + \frac{1}{3}\right]$

$= 8 - \frac{8}{3} - \frac{1}{2} - \frac{1}{3}$

$= \frac{48-16-3-2}{6}$

$= \frac{27}{6}$

$A = \frac{9}{2}$

---

8. Father of a family wishes to divide his square field bounded by $x = 0, x = 4, y = 4$ and $y = 0$ along the curve $y^2 = 4x$ and $x^2 = 4y$ into three equal parts for his wife, daughter and son. Is it possible to divide? If so, find the area to be divided among them.

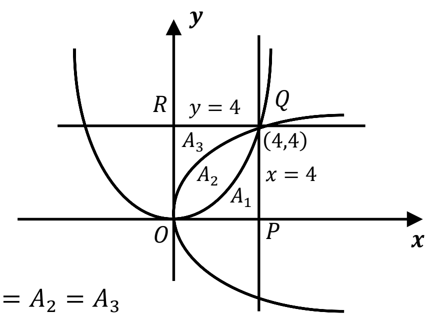
Solution:

Parabola: $x^2 = 4y$

$y = \frac{x^2}{4}$ ----------(1)

Parabola: $y^2 = 4x$

$y = 2\sqrt{x}$ ----------(2)



T.P: $A_1 = A_2 = A_3$

$A_1 = \int_0^4 y\,dx$

$= \int_0^4 \frac{x^2}{4}dx$

$= \left[\frac{x^3}{12}\right]_0^4 = \frac{64}{12} - 0$

$A_1 = \frac{16}{3}$ ----------(3)

$A_2 = \int_0^4 (y_u - y_l)dx$

$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx$

$= \left[\frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4$

$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4$

$= \left[\frac{4}{3}(8) - \frac{64}{12}\right] - [0]$

$= \frac{32}{3} - \frac{16}{3}$

$A_2 = \frac{16}{3}$ ----------(4)

$A_3 = \int_0^4 x\,dy$

$= \int_0^4 \frac{y^2}{4}dy$

$= \left[\frac{y^3}{12}\right]_0^4 = \frac{64}{12} - 0$

$A_3 = \frac{16}{3}$ ----------(5)

Yes, it is possible to divide among them and the area is $\frac{16}{3}$ sq. units

---

9. The curve $y = (x-2)^2 + 1$ has a minimum point at P. A point Q on the curve is such that the slope of PQ is 2. Find the area bounded by the curve and the chord PQ.

Solution:

$y = (x-2)^2 + 1$ ----------(1)

$(x-2)^2 = (y-1)$

It represents a parabola with vertex $(2,1)$

$\therefore P$ is $(2,1)$

Slope of the line PQ is 2.

$y = mx + c$

$y = 2x + c$

Passing through $(2,1)$

$1 = 4 + c$

$c = -3$

$\therefore$ The line equation is,

$y = 2x - 3$ ----------(2)

From (1) and (2), we get

$2x - 3 = (x-2)^2 + 1$

$2x - 4 = x^2 - 4x + 4$

$x^2 - 6x + 8 = 0$

$(x-4)(x-2) = 0$

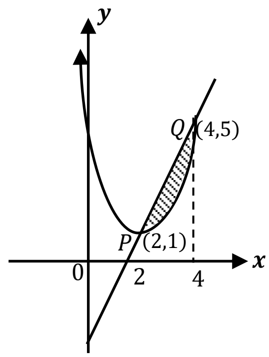
$x = 4, 2$

At $x = 4, \quad y = 2^2 + 1 = 5$ From (1)

At $x = 2, \quad y = 0 + 1 = 1$

$\therefore$ The point of intersection is $(4,5)$ and $(2,1)$

$\therefore Q$ is $(4,5)$



$$A = \int_2^4 (y_u - y_l)dx$$

$$= \int_2^4 [(2x-3) - ((x-2)^2 + 1)]dx$$

$$= \int_2^4 [(2x-3) - (x^2 - 4x + 4 + 1)]dx$$

$$= \int_2^4 [2x - 3 - x^2 + 4x - 5]dx$$

$$= \int_2^4 [-x^2 + 6x - 8]dx$$

$$= \left[-\frac{x^3}{3} + \frac{6x^2}{2} - 8x\right]_2^4$$

$$= \left[-\frac{64}{3} + 48 - 32\right] - \left[-\frac{8}{3} + 12 - 16\right]$$

$$= -\frac{64}{3} + 16 + \frac{8}{3} + 4$$

$$= 20 - \frac{56}{3}$$

$$A = \frac{4}{3}$$

Hint: $\boldsymbol{\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c}$

10. Find the area of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$.

Solution:

Circle: $x^2 + y^2 = 16$ ----------(1)

$$y^2 = 16 - x^2$$

$$y = \sqrt{16 - x^2}$$

Compare with $x^2 + y^2 = r^2$, we get $r = 4$

Parabola: $y^2 = 6x$ ----------(2)

$$y = \sqrt{6x}$$

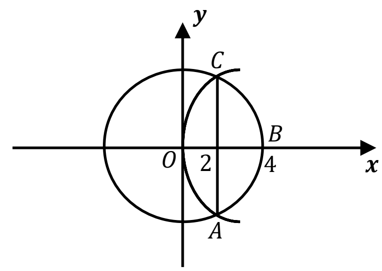
From (1) and (2), we get

$$x^2 + 6x = 16$$

$$x^2 + 6x - 16 = 0$$

$$(x+8)(x-2) = 0$$

$$x = -8, 2$$



$$A = 2\int_0^2 y_1\, dx + 2\int_2^4 y_2\, dx$$

$$= 2\int_0^2 \sqrt{6x}\, dx + 2\int_2^4 \sqrt{16 - x^2}\, dx$$

$$= 2\sqrt{6}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 + 2\left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_2^4$$

$$= \frac{4\sqrt{6}}{3}\left[2\sqrt{2} - 0\right] + 2\begin{bmatrix}(0 + 8\sin^{-1}1) - \\ \left(2\sqrt{3} + 8\sin^{-1}\frac{1}{2}\right)\end{bmatrix}$$

$$= \frac{8\sqrt{12}}{3} + 2\left[\frac{8\pi}{2} - \left(2\sqrt{3} + \frac{8\pi}{6}\right)\right]$$

$$= \frac{16\sqrt{3}}{3} + 2\left[4\pi - 2\sqrt{3} - \frac{4\pi}{3}\right]$$

$$= \frac{16\sqrt{3}}{3} + 8\pi - 4\sqrt{3} - \frac{8\pi}{3}$$

$$= \frac{16\sqrt{3} + 24\pi - 12\sqrt{3} - 8\pi}{3}$$

$$= \frac{4\sqrt{3} + 16\pi}{3}$$

$$A = \frac{4}{3}(\sqrt{3} + 4\pi)$$

Exercise 9.9

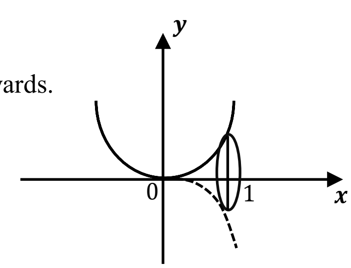
Hint: Volume $\boldsymbol{V = \pi\int_a^b y^2 dx}$ (or) $\boldsymbol{V = \pi\int_a^b x^2 dy}$

1. Find by integration, the volume of the solid generated by revolving about the x-axis, the region enclosed by $y = 2x^2$, $y = 0$ and $x = 1$.

Solution:

Given $y = 2x^2$

Which is a parabola open upwards.



$$V = \pi\int_0^1 y^2 dx$$

$$= \pi\int_0^1 (2x^2)^2 dx$$

$$= \pi\int_0^1 (4x^4)dx$$

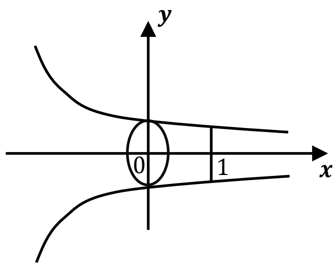
$$= 4\pi\left[\frac{x^5}{5}\right]_0^1$$

$$= 4\pi\left[\frac{1}{5} - 0\right]$$

$$V = \frac{4\pi}{5}$$

2. Find by integration, the volume of the solid generated by revolving about x-axis, the region enclosed by $y = e^{-2x}, y = 0, x = 0$ and $x = 1$.

Solution:



Given $y = e^{-2x}$

$$V = \pi\int_0^1 y^2 dx$$

$$= \pi\int_0^1 (e^{-2x})^2 dx$$

$$= \pi\int_0^1 e^{-4x} dx$$

$$= \pi\left[\frac{e^{-4x}}{-4}\right]_0^1$$

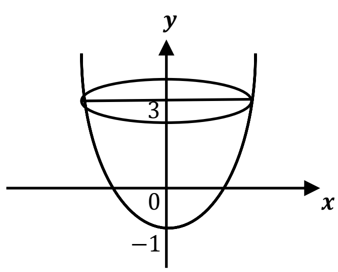
$$= \frac{\pi}{4}\left[-e^{-4x}\right]_0^1$$

$$= \frac{\pi}{4}\left[-e^{-4} + e^{0}\right]$$

$$V = \frac{\pi}{4}\left[1 - e^{-4}\right]$$

3. Find by integration, the volume of the solid generated by revolving about the y-axis, the region enclosed by $x^2 = 1 + y$ and $y = 3$.

Solution:



Given $x^2 = 1 + y$

Put $x = 0, y = -1$

$$V = \pi\int_{-1}^{3} x^2 dy$$

$$= \pi\int_{-1}^{3} (1 + y) dy$$

$$= \pi\left[y + \frac{y^2}{2}\right]_{-1}^{3}$$

$$= \pi\left[\left(3 + \frac{9}{2}\right) - \left(-1 + \frac{1}{2}\right)\right]$$

$$= \pi\left[\frac{15}{2} + \frac{1}{2}\right]$$

$$= \pi\left(\frac{16}{2}\right)$$

$$V = 8\pi$$

4. The region enclosed between the graph of $y = x$ and $y = x^2$ is denoted by R, find the volume generated when R is rotated through $360°$ about x-axis.

Solution:

Line: $y = x$ ----------(1)

Parabola: $y = x^2$ ----------(2)

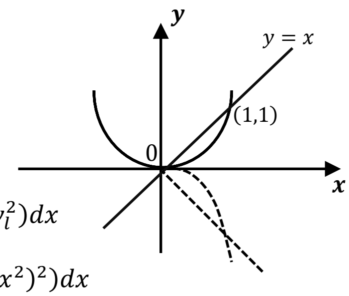
From (1) and (2),

$$x^2 = x$$

$$x^2 - x = 0$$

$$x(x - 1) = 0$$

$$x = 0, 1$$



$$V = \pi\int_0^1 (y_u^2 - y_l^2) dx$$

$$= \pi\int_0^1 (x^2 - (x^2)^2) dx$$

$$= \pi\int_0^1 (x^2 - x^4) dx$$

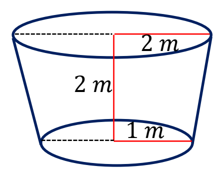
$$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \pi\left[\left(\frac{1}{3} - \frac{1}{5}\right) - 0\right]$$

$$= \pi\left(\frac{5-3}{15}\right)$$

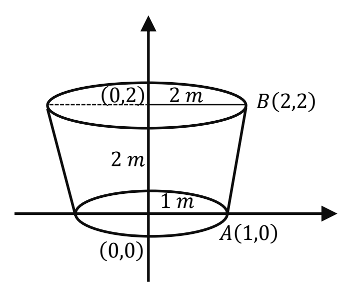
$$V = \frac{2\pi}{15}$$

5. Find by integration, the volume of the container which is in the shape of a right circular conical frustum as shown in figure.



Solution:

The equation of line AB is,



$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-0}{2-0} = \frac{x-1}{2-1}$$

$$\frac{y}{2} = x - 1$$

$$x = \frac{y}{2} + 1$$

$$x = \frac{y+2}{2}$$

$$V = \pi\int_0^2 x^2 dy$$

$$= \pi\int_0^2 \left(\frac{y+2}{2}\right)^2 dy$$

$$= \pi\int_0^2 \frac{(y+2)^2}{4} dy$$

$$= \frac{\pi}{4}\left[\frac{(y+2)^3}{3}\right]_0^2$$

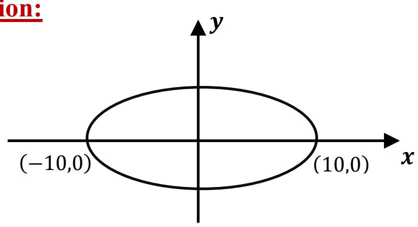
$$= \frac{\pi}{4}\left[\frac{64}{3} - \frac{8}{3}\right]$$

$$= \frac{\pi}{4}\left(\frac{56}{3}\right)$$

$$V = \frac{14\pi}{3}$$

6. A watermelon has an ellipsoid shape which can be obtained by revolving an ellipse with major axis 20 cm and minor axis 10 cm about its major axis. Find its volume using integration:

Solution:



Major axis: $2a = 20$

$a = 10$

Minor axis: $2b = 10$

$b = 5$

∴ The equation of ellipse is,

$$\frac{x^2}{100} + \frac{y^2}{25} = 1$$

$$\frac{y^2}{25} = 1 - \frac{x^2}{100}$$

$$y^2 = 25\left(\frac{100-x^2}{100}\right)$$

$$y^2 = \frac{100-x^2}{4}$$

$$V = \pi\int_{-10}^{10} y^2 dx$$

$$= \pi\int_{-10}^{10}\left(\frac{100-x^2}{4}\right)dx$$

$$= \frac{\pi}{4}\left[100x - \frac{x^3}{3}\right]_{-10}^{10}$$

$$= \frac{\pi}{4}\left[\left(1000 - \frac{1000}{3}\right) - \left(-1000 + \frac{1000}{3}\right)\right]$$

$$= \frac{\pi}{4}\left[\frac{2000}{3} - \left(-\frac{2000}{3}\right)\right]$$

$$= \frac{\pi}{4}\left(\frac{4000}{3}\right)$$

$$V = \frac{1000\pi}{3}$$