CHAPTER – 12

Discrete Mathematics

Exercise 12.1

1. Determine whether $*$ is a binary operation on the sets given below.

(i) $a * b = a.|b|$ on $\mathbb{R}$

Solution:

Let $a, b \in \mathbb{R}$

$|b| \in \mathbb{R}$

$a.|b| \in \mathbb{R}$

$a * b \in \mathbb{R}$

Hence $*$ is binary on $\mathbb{R}$.

(ii) $a * b = \min(a, b)$ on $A = \{1, 2, 3, 4, 5\}$

Solution:

Let $a, b \in A$

$\min(a, b)$ is either a or b, which belongs to A.

$a * b \in A$

Hence $*$ is binary on A.

(iii) $(a * b) = a\sqrt{b}$ is binary on $\mathbb{R}$.

Solution:

Let $a, b \in \mathbb{R}$

$\sqrt{b} \notin \mathbb{R}$ $\quad\left(\because b = -1 \text{ then } \sqrt{-1} \notin \mathbb{R}\right)$

$a\sqrt{b} \notin \mathbb{R}$

$a * b \notin \mathbb{R}$

Hence $*$ not binary on $\mathbb{R}$.

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2. On $\mathbb{Z}$, define $*$ by $(m * n) = m^n + n^m : \forall m, n \in \mathbb{Z}$. Is $*$ binary on $\mathbb{Z}$?

Solution:

Let $m, n \in Z$

$m^n \notin Z$ (if $n < 0$) $\quad\left(m^{-1} = \frac{1}{m} \notin Z\right)$

$m^n + n^m \notin Z$

$m * n \notin Z$

Hence $*$ is not binary on Z.

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3. Let $*$ be defined on $\mathbb{R}$ by $(a * b) = a + b + ab - 7$. Is $*$ binary on $\mathbb{R}$? If so, find $3 * \left(\frac{-7}{15}\right)$.

Solution:

Given $a * b = a + b + ab - 7$

Let $a, b \in \mathbb{R}$

$a + b \in \mathbb{R}$

$ab \in \mathbb{R}$

$-7 \in \mathbb{R}$

$a + b + ab - 7 \in \mathbb{R}$

$a * b \in \mathbb{R}$

Hence $*$ is binary on $\mathbb{R}$.

$$3 * \left(-\frac{7}{15}\right) = 3 - \frac{7}{15} - \frac{21}{15} - 7$$

$$= -4 - \frac{28}{15}$$

$$= \frac{-60-28}{15}$$

$$3 * \left(-\frac{7}{15}\right) = -\frac{88}{15}$$

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4. Let $A = \{a + \sqrt{5}b : a, b \in \mathbb{Z}\}$. Check whether the usual multiplication is a binary operation on A.

Solution:

Let $a + \sqrt{5}b, c + \sqrt{5}d \in A$, where $a, b, c, d \in Z$

$$(a + \sqrt{5}b)(c + \sqrt{5}d) = ac + \sqrt{5}ad + \sqrt{5}bc + 5bd$$

$$= (ac + 5bd) + \sqrt{5}(ad + bc) \in A$$

Hence usual multiplication is binary on A.

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5. (i) Define an operation $*$ on $\mathbb{Q}$ as follows:

$a * b = \left(\frac{a+b}{2}\right)$; $a, b \in \mathbb{Q}$. Examine the closure, commutative, and associative properties satisfied by $*$ on $\mathbb{Q}$.

Solution:

Given $a * b = \frac{a+b}{2}$

Closure Property:

Let $a, b \in \mathbb{Q}$

$a + b \in \mathbb{Q}$

$\frac{a+b}{2} \in \mathbb{Q}$

$a * b \in \mathbb{Q}$

Hence closure property is true.

Commutative Property:

Let $a, b \in \mathbb{Q}$

$a * b = \frac{a+b}{2}$

$$= \frac{b+a}{2}$$

$$a * b = b * a$$

Hence commutative property is true.

Associative Property:

Let $a, b, c \in \mathbb{Q}$

$$a * (b * c) = a * \left(\frac{b+c}{2}\right)$$

$$= \frac{a + \left(\frac{b+c}{2}\right)}{2}$$

$$a * (b * c) = \frac{2a+b+c}{4} \qquad \text{----------(1)}$$

$$(a * b) * c = \left(\frac{a+b}{2}\right) * c$$

$$= \frac{\left(\frac{a+b}{2}\right) + c}{2}$$

$$(a * b) * c = \frac{a+b+2c}{4} \qquad \text{----------(2)}$$

From (1) and (2) we have

$$a * (b * c) \neq (a * b) * c$$

Hence associative property is not true.

(ii) Define an operation $*$ on $\mathbb{Q}$ as follows: $a * b = \left(\frac{a+b}{2}\right)$; $a, b \in \mathbb{Q}$. Examine the existence of identity and the existence of inverse for the operation $*$ on $\mathbb{Q}$.

Solution:

Given $a * b = \frac{a+b}{2}$

Existence of Identity Property:

Let $a \in \mathbb{Q}$

By definition, $a * e = a$

$$\frac{a+e}{2} = a$$

$$a + e = 2a$$

$e = a$, which is not unique

Hence identity property fails.

Existence of Inverse Property:

$*$ has no identity element.

Hence $*$ has no inverse.

6. Fill in the following table so that the binary operation $*$ on $A = \{a, b, c\}$ is commutative.

$*$	a	b	c
a	b		
b	c	b	a
c	a		c

Solution:

Given $*$ is commutative.

$*$	a	b	c
a	b	c	a
b	c	b	a
c	a	a	c

From table:

$b * a = c = a * b$
$b * c = a = c * b$
$c * a = a = a * c$

7. Consider the binary operation $*$ defined on the set $A = \{a, b, c, d\}$ by the following table:

$*$	a	b	c	d
a	a	c	b	d
b	d	a	b	c
c	c	d	a	a
d	d	b	a	c

Is it commutative and associative?

Solution:

Commutative:

From table, $a * b = c$

$$b * a = d$$

$$a * b \neq b * a$$

Hence $*$ is not commutative on A.

Associative:

$$(a * b) * c = c * c = a \qquad \text{----------(1)}$$

$$a * (b * c) = a * b = c \quad (\because b * c = b) \qquad \text{----------(2)}$$

From (1) and (2) we have

$$(a * b) * c \neq a * (b * c)$$

Hence $*$ is not associative on A.

Hint: $a \vee b = max\{a, b\}$: $a \wedge b = min\{a, b\}$

8. Let $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ be any three Boolean matrices of the same type. Find (i) $A \vee B$ (ii) $A \wedge B$ (iii) $(A \vee B) \wedge C$ (iv) $(A \wedge B) \vee C$.

Solution:

(i) $$A \vee B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$A \vee B = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 & 0 \vee 1 \\ 0 \vee 1 & 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 1 \vee 1 & 0 \vee 0 & 0 \vee 0 & 1 \vee 1 \end{bmatrix}$$

$$A \vee B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(ii) $$A \wedge B = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 & 0 \wedge 1 \\ 0 \wedge 1 & 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 1 \wedge 1 & 0 \wedge 0 & 0 \wedge 0 & 1 \wedge 1 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(iii) $$(A \vee B) \wedge C = \begin{bmatrix} 1 \wedge 1 & 1 \wedge 1 & 1 \wedge 0 & 1 \wedge 1 \\ 1 \wedge 0 & 1 \wedge 1 & 1 \wedge 1 & 1 \wedge 0 \\ 1 \wedge 1 & 0 \wedge 1 & 0 \wedge 1 & 1 \wedge 1 \end{bmatrix}$$

$$(A \vee B) \wedge C = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(iv) $$(A \wedge B) \vee C = \begin{bmatrix} 0 \vee 1 & 0 \vee 1 & 0 \vee 0 & 0 \vee 1 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 & 0 \vee 0 \\ 1 \vee 1 & 0 \vee 1 & 0 \vee 1 & 1 \vee 1 \end{bmatrix}$$

$$(A \wedge B) \vee C = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

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9. (i) Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in \mathbb{R} - \{0\} \right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the commutative and associative properties satisfied by $*$ on M.

Solution:

Given $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in \mathbb{R} - \{0\} \right\}$

Closure:

Let $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}, B = \begin{bmatrix} y & y \\ y & y \end{bmatrix} \in M$ where $x, y \in \mathbb{R} - \{0\}$

$$A * B = \begin{bmatrix} x & x \\ x & x \end{bmatrix} * \begin{bmatrix} y & y \\ y & y \end{bmatrix}$$

$$= \begin{bmatrix} xy + xy & xy + xy \\ xy + xy & xy + xy \end{bmatrix}$$

$$A * B = \begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix} \in M \text{ where } 2xy \in \mathbb{R} - \{0\}$$

Hence $*$ is binary on M.

Commutative property:

$$B * A = \begin{bmatrix} y & y \\ y & y \end{bmatrix} * \begin{bmatrix} x & x \\ x & x \end{bmatrix}$$

$$= \begin{bmatrix} 2yx & 2yx \\ 2yx & 2yx \end{bmatrix}$$

$$= \begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix}$$

$$B * A = A * B$$

Hence commutative property is true.

Associative property:

Let $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}, B = \begin{bmatrix} y & y \\ y & y \end{bmatrix}, C = \begin{bmatrix} z & z \\ z & z \end{bmatrix} \in M$

$$A * (B * C) = \begin{bmatrix} x & x \\ x & x \end{bmatrix} * \left(\begin{bmatrix} y & y \\ y & y \end{bmatrix} * \begin{bmatrix} z & z \\ z & z \end{bmatrix} \right)$$

$$= \begin{bmatrix} x & x \\ x & x \end{bmatrix} * \begin{bmatrix} 2yz & 2yz \\ 2yz & 2yz \end{bmatrix}$$

$$= \begin{bmatrix} 2xyz + 2xyz & 2xyz + 2xyz \\ 2xyz + 2xyz & 2xyz + 2xyz \end{bmatrix}$$

$$= \begin{bmatrix} 4xyz & 4xyz \\ 4xyz & 4xyz \end{bmatrix} \quad \text{----------(1)}$$

$$(A * B) * C = \left(\begin{bmatrix} x & x \\ x & x \end{bmatrix} * \begin{bmatrix} y & y \\ y & y \end{bmatrix} \right) * \begin{bmatrix} z & z \\ z & z \end{bmatrix}$$

$$= \begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix} * \begin{bmatrix} z & z \\ z & z \end{bmatrix}$$

$$= \begin{bmatrix} 4xyz & 4xyz \\ 4xyz & 4xyz \end{bmatrix} \quad \text{----------(2)}$$

From (1) and (2) we have

$$A * (B * C) = (A * B) * C$$

Hence Associative property is true.

(ii) Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in \mathbb{R} - \{0\} \right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the existence of identity, existence of inverse properties for the operation $*$ on M.

Solution:

Given $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in \mathbb{R} - \{0\} \right\}$

Existence of Identity property:

Let $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix} \in M$ where $x \in \mathbb{R} - \{0\}$

Let $E = \begin{bmatrix} e & e \\ e & e \end{bmatrix}$ be the identity element.

By definition, $A * E = A$

$$\begin{bmatrix} x & x \\ x & x \end{bmatrix} * \begin{bmatrix} e & e \\ e & e \end{bmatrix} = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$$

$$\begin{bmatrix} 2xe & 2xe \\ 2xe & 2xe \end{bmatrix} = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$$

$$2xe = x$$

$$2e = 1$$

$$e = \frac{1}{2} \in \mathbb{R} - \{0\}$$

$$E = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \in M$$

Similarly, $E * A = A$

Hence $E = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is the identity element.

Existence of Inverse property:

Let $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix} \in M$ where $x \in \mathbb{R} - \{0\}$

Let $B = \begin{bmatrix} y & y \\ y & y \end{bmatrix}$ be the inverse of A.

By definition, $A * B = E$

$$\begin{bmatrix} x & x \\ x & x \end{bmatrix} * \begin{bmatrix} y & y \\ y & y \end{bmatrix} = \begin{bmatrix} e & e \\ e & e \end{bmatrix}$$

$$\begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$2xy = \frac{1}{2}$$

$$y = \frac{1}{4x} \in \mathbb{R} - \{0\}$$

Hence the inverse of A is $\begin{bmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{bmatrix}$.

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10. (i) Let A be $\mathbb{Q}\backslash\{1\}$. Define $*$ on A by $x * y = x + y - xy$. Is $*$ binary on A? If so, examine the commutative and associative properties satisfied by $*$ on A.

Solution:

Given $A = \mathbb{Q}\backslash\{1\}$

$$x * y = x + y - xy$$

Closure property:

Let $x, y \in \mathbb{Q}\backslash\{1\}$.

$$x \neq 1 \ ; \qquad y \neq 1$$

$$x - 1 \neq 0 \ ; \ y - 1 \neq 0$$

$$(x-1)(y-1) \neq 0$$

$$xy - x - y + 1 \neq 0$$

$$x + y - xy \neq 1$$

$$x * y \neq 1$$

$$x * y \in A \qquad (\because x, y \in \mathbb{Q})$$

Hence $*$ is binary on A.

Commutative property:

Let $x, y \in A$

$$x * y = x + y - xy$$

$$= y + x - yx$$

$$x * y = y * x$$

Hence commutative property is true.

Associative property:

Let $x, y, z \in A$

$$x * (y * z) = x * (y + z - yz)$$

$$= x + y + z - yz - x(y + z - yz)$$

$$= x + y + z - yz - xy - xz + xyz \quad \text{-------(1)}$$

$$(x * y) * z = (x + y - xy) * z$$

$$= x + y - xy + z - (x + y - xy)z$$

$$= x + y + z - xy - xz - yz + xyz \quad \text{-------(2)}$$

From (1) and (2) we have

$$x * (y * z) = (x * y) * z$$

Hence associative property is true.

(ii) Let A be $\mathbb{Q}\backslash\{1\}$. Define $*$ on A by $x * y = x + y - xy$. Is $*$ binary on A? If so, examine the existence of identity, existence of inverse properties for the operation $*$ on A.

Solution:

Given $A = \mathbb{Q}\backslash\{1\}$

$$x * y = x + y - xy$$

Existence of Identity property:

Let $x \in A$

By definition, $x * e = x$

$$x + e - xe = x$$

$$e - xe = 0$$

$$e(1 - x) = 0$$

$$e = 0 \qquad (\because x \neq 1)$$

$\therefore$ The Identity element is $e = 0 \in A$.

Thus identity property is satisfied.

Existence of Inverse property:

Let $x \in A$

By definition, $x * x^{-1} = e$

$$x + x^{-1} - xx^{-1} = 0$$

$$x + x^{-1}(1 - x) = 0$$

$$x^{-1}(1 - x) = -x$$

$$x^{-1} = \frac{-x}{1-x} \in \mathbb{Q}\backslash\{1\} \qquad (\because x \neq 1)$$

$\therefore$ The Inverse element is $x^{-1} = \frac{-x}{1-x} \in A$

Hence inverse property is satisfied.

Exercise 12.2

1. Let P : Jupiter is a planet and q : India is an island be any two simple statements. Give verbal sentence describing each of the following statements.

(i) $\neg p$ (ii) $p \wedge \neg q$ (iii) $\neg p \vee q$ (iv) $p \to \neg q$ (v) $p \leftrightarrow q$

Solutions:

(i) $\neg p$: Jupiter is not a planet

(ii) $p \wedge \neg q$: Jupiter is a planet and India is not an island

(iii) $\neg p \vee q$: Jupiter is not a planet or India is an island

(iv) $p \to \neg q$: If Jupiter is a planet then India is not an island

(v) $p \leftrightarrow q$: Jupiter is a planet if and only if India is an island

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2. Write each of the following sentences in symbolic form using statement variables p and q.

(i) 19 is not a prime number and all the angles of a triangle are equal.

(ii) 19 is a prime number or all the angles of a triangle are not equal.

(iii) 19 is a prime number and all the anlges of a triangle are equal.

(iv) 19 is not a prime number.

Solutions:

p : 19 is a prime number

q : All the angles of a triangle are equal

(i) $\neg p \wedge q$

(ii) $p \vee \neg q$

(iii) $p \wedge q$

(iv) $\neg p$

Hint: Symbolic form

''then'' $(\to)$ **''or''** $(\vee)$ **''and''** $(\wedge)$

3. Determine the truth value of each of the following statements

(i) If $6+2=5$, then the milk is white.

(ii) China is in Europe or $\sqrt{3}$ is an integer.

(iii) It is not true that $5+5=9$ or Earth is a planet.

(iv) 11 is a prime number and all the sides of a rectangle are equal.

Solutions:

S. No	Statement	Symbolic form	Conclusion
(i)	If $6+2=5$, **then** the milk is white	$F \to T$	T
(ii)	China is in Europe **or** $\sqrt{3}$ is an integer	$F \vee F$	F
(iii)	It is not true that $5+5=9$ **or** Earth is a planet	$T \vee T$	T
(iv)	11 is a prime number **and** all the sides of a rectangle are equal	$T \wedge F$	F

Hint: Any declarative sentence is called statement or a 'Proposition' which is either true or false but not both.

4. Which one of the following sentences is a proposition?

(i) $4+7=12$ (ii) What are you doing? (iii) $3^n \leq 81, n \in \mathbb{N}$ (iv) Peacock is our national bird (v) How tall this mountain is!

Solutions:

(i) Proposition

(ii) Not a proposition (It is interrogative)

(iii) Proposition

(iv) Proposition

(v) Not a proposition (It is Exclamatory)

Hint: (i) Converse statement $q \to p$

(ii) Inverse statement $\neg p \to \neg q$

(iii) Contrapositive statement $\neg q \to \neg p$

5. Write the converse, inverse, and contrapositive of each of the following implication.

(i) If x and y are numbers such that $x = y$, then $x^2 = y^2$

Solution:

p : x and y are numbers such that $x = y$

q : $x^2 = y^2$

Converse: $q \to p$

$\Rightarrow$ If $x^2 = y^2$ then $x = y$

Inverse: $\neg p \to \neg q$

$\Rightarrow$ If $x \neq y$ then $x^2 \neq y^2$

Contrapositive: $\neg q \to \neg p$

$\Rightarrow$ If $x^2 \neq y^2$ then $x \neq y$

(ii) If a quadrilateral is a square then it is rectangle

Solution:

p : A quadrilateral is a square

q : A quadrilateral is rectangle

Converse: $q \to p$

If a quadrilateral is a rectangle then it is a square.

Inverse: $\neg p \to \neg q$

If a quadrilateral is not a square then it is not a rectangle.

Contrapositive: $\neg q \to \neg p$

If a quadrilateral is not an rectangle then it is not a square.

6. Construct the truth table for the following statements.

(i) $\neg p \wedge \neg q$

Solution:

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(ii) $\neg(p \wedge \neg q)$

Solution:

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

(iii) $(p \vee q) \vee \neg q$

Solution:

p	q	$\neg q$	$p \vee q$	$(p \vee q) \vee \neg q$
T	T	F	T	T
T	F	T	T	T
F	T	F	T	T
F	F	T	F	T

(iv) $(\neg p \to r) \wedge (p \leftrightarrow q)$

Solution:

p	q	r	$\neg p$	$s: \neg p \to r$	$t: p \leftrightarrow q$	$s \wedge t$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

7. Verify whether the following compound propositions are tautologies or contradictions or contingency

(i) $(p \wedge q) \wedge \neg(p \vee q)$

Solution:

p	q	$s: p \wedge q$	$t: p \vee q$	$\neg t$	$s \wedge \neg t$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

This is a contradiction.

(ii) $((p \vee q) \wedge \neg p) \to q$

Solution:

p	q	$s: p \vee q$	$t: \neg p$	$r: s \wedge t$	$r \to q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

It is a tautology.

(iii) $(p \to q) \leftrightarrow (\neg p \to q)$

Solution:

p	q	$s: p \to q$	$t: \neg p$	$r: \neg p \to q$	$s \leftrightarrow r$
T	T	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Hence It is contingency.

(iv) $((p \to q) \wedge (q \to r)) \to (p \to r)$

Solution:

p	q	r	s: $p \to q$	t: $q \to r$	u: $p \to r$	w: $s \wedge t$	$w \to u$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

It is a tautology.

8. Show that (i) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Solution:

p	q	$s: p \wedge q$	$\neg s$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

From the table, $\neg(p \wedge q) \equiv \neg p \vee \neg q$

(ii) $\neg(p \to q) \equiv p \wedge \neg q$

Solution:

p	q	$s: p \to q$	$\neg s$	$\neg q$	$p \wedge \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

From the table, $\neg(p \to q) \equiv p \wedge \neg q$

9. Prove that $q \to p \equiv \neg p \vee \neg q$

Solution:

p	q	$q \to p$	$\neg p$	$\neg q$	$\neg p \to \neg q$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	T	T

From the table, $q \to p \equiv \neg p \vee \neg q$

10. Show that $p \to q$ and $q \to p$ are not equivalent.

Solution:

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

From the table, $p \to q \neq q \to p$.

11. Show that $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Solution:

p	q	$s: p \leftrightarrow q$	$\neg s$	$\neg q$	$p \leftrightarrow \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	F	T
F	F	T	F	T	F

From the table, $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Hint: $p \to q \equiv \neg p \vee q$

12. Check whether the statement $p \to (q \to p)$ is a tautology or a contradiction without using the truth table.

Solution:

$$
\begin{aligned}
p \to (q \to p) &\equiv p \to (\neg q \vee p) \\
&\equiv \neg p \vee (\neg q \vee p) \\
&\equiv \neg p \vee (p \vee \neg q) \text{ (By commutative law)} \\
&\equiv (\neg p \vee q) \vee \neg q \text{ (By associative law)} \\
&\equiv T \vee \neg q \\
p \to (q \to p) &\equiv T
\end{aligned}
$$

$\therefore$ It is a tautology.

13. Using the truth table check whether the statements $\neg(p \vee q) \vee (\neg p \wedge q)$ and $\neg p$ are logically equivalent.

Solution:

p	q	$s: p \vee q$	$\neg s$	$\neg p$	$t: \neg p \wedge q$	$\neg s \vee t$
T	T	T	F	F	F	F
T	F	T	F	F	F	F
F	T	T	F	T	T	T
F	F	F	T	T	F	T

Hence both statements are logically equivalent.

Hint: $p \to q \equiv \neg p \vee q$

14. Prove $p \to (q \to r) \equiv (p \wedge q) \to r$ without using truth table.

Solution:

$$
\begin{aligned}
p \to (q \to r) &\equiv p \to (\neg q \vee r) \\
&\equiv \neg p \vee (\neg q \vee r) \\
&\equiv (\neg p \vee \neg q) \vee r \text{ (By associative law)} \\
&\equiv \neg(p \wedge q) \vee r \text{ (by De Morgan's law)} \\
p \to (q \to r) &\equiv (p \wedge q) \to r
\end{aligned}
$$

Hence proved.

15. Prove that $p \rightarrow (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$ using truth table.

Solution:

p	q	r	$\neg p$	$\neg q$	s: $\neg q \vee r$	t: $p \rightarrow s$	u: $\neg p \vee s$
T	T	T	F	F	T	T	T
T	T	F	F	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	F	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the table, $p \rightarrow (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$