CHAPTER – 8

Differentials and Partial Derivatives

Exercise 8.1

Linear Approximation $L(x) = f(x_0) + f'(x_0)(x - x_0)$

1. Let $f(x) = \sqrt[3]{x}$. Find the linear approximation at $x = 27$. Use the linear approximation to approximate $\sqrt[3]{27.2}$.

Solution:

Given $f(x) = x^{\frac{1}{3}}, x_0 = 27$

Linear approximation:

$$L(x) = f(27) + f'(27)(x - 27) \quad \text{----------(1)}$$

$$f(27) = 27^{\frac{1}{3}} = 3$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$f'(x) = \frac{1}{3x^{\frac{2}{3}}}$$

$$f'(27) = \frac{1}{3(27)^{\frac{2}{3}}} = \frac{1}{3(9)} = \frac{1}{27}$$

$$L(x) = 3 + \frac{1}{27}(x - 27) \qquad \text{From (1)}$$

At $x = 27.2$

$$L(27.2) = 3 + \frac{1}{27}(0.2)$$

$$= 3 + 0.0074$$

$$L(27.2) \approx 3.0074$$

Hint: $L(x) \approx f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x$

2. Use the linear approximation to find approximate values

(i) $(123)^{\frac{2}{3}}$

Solution:

Let $f(x) = x^{\frac{2}{3}}, x_0 = 125, \Delta x = -2$

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$$

Linear approximation:

$$(123)^{\frac{2}{3}} = f(125) + f'(125)(-2) \quad \text{----------(1)}$$

$$f(125) = 125^{\frac{2}{3}} = 25$$

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$$

$$f'(x) = \frac{2}{3x^{\frac{1}{3}}}$$

$$f'(125) = \frac{2}{3(5)} = \frac{2}{15}$$

$$(123)^{\frac{2}{3}} = 25 + \frac{2}{15}(-2) \qquad \text{From (1)}$$

$$= 25 - 0.266$$

$$(123)^{\frac{2}{3}} \approx 24.734$$

(ii) $\sqrt[4]{15}$

Solution:

Let $f(x) = x^{\frac{1}{4}}, x_0 = 16, \Delta x = -1$

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$$

Linear approximation:

$$(15)^{\frac{1}{4}} = f(16) + f'(16)(-1) \quad \text{----------(1)}$$

$$f(16) = (16)^{\frac{1}{4}} = 2$$

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4x^{\frac{3}{4}}}$$

$$f'(16) = \frac{1}{4(8)} = \frac{1}{32}$$

$$(15)^{\frac{1}{4}} = 2 + \frac{1}{32}(-1) \qquad \text{From (1)}$$

$$= 2 - 0.03125$$

$$(15)^{\frac{1}{4}} \approx 1.969$$

(iii) $\sqrt[3]{26}$

Solution:

Let $f(x) = x^{\frac{1}{3}}, x_0 = 27, \Delta x = -1$

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$$

Linear approximation:

$$(26)^{\frac{1}{3}} = f(27) + f'(27)(-1) \quad \text{----------(1)}$$

$$f(27) = (27)^{\frac{1}{3}} = 3$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$$

$$f'(27) = \frac{1}{3(9)} = \frac{1}{27}$$

$$(26)^{\frac{1}{3}} = 3 + \frac{1}{27}(-1) \qquad \text{From (1)}$$

$$= 3 - 0.037$$

$$(26)^{\frac{1}{3}} \approx 2.963$$

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3. Find a linear approximation for the following functions at the indicated points.

(i) $f(x) = x^3 - 5x + 12, x_0 = 2$

Solution:

Given $f(x) = x^3 - 5x + 12$ at $x_0 = 2$

$$L(x) = f(x_0) + f'(x_0)(x - x_0) \quad \text{----------(1)}$$

$$f(2) = 8 - 10 + 12 = 10$$

$$f'(x) = 3x^2 - 5$$

$$f'(2) = 12 - 5 = 7$$

$$L(x) = 10 + 7(x - 2) \qquad \text{From (1)}$$

$$= 10 + 7x - 14$$

$$L(x) = 7x - 4$$

(ii) $g(x) = \sqrt{x^2 + 9}, x_0 = -4$

Solution:

Given $g(x) = \sqrt{x^2 + 9}$ at $x_0 = -4$

$$L(x) = g(x_0) + g'(x_0)(x - x_0) \quad \text{----------(1)}$$

$$g(-4) = \sqrt{16 + 9} = 5$$

$$g'(x) = \frac{1}{2\sqrt{x^2+9}}(2x)$$

$$g'(-4) = \frac{-8}{2(5)} = -\frac{4}{5}$$

$$L(x) = 5 - \frac{4}{5}(x + 4) \qquad \text{From (1)}$$

$$= \frac{25 - 4x - 16}{5}$$

$$L(x) = \frac{9 - 4x}{5}$$

(iii) $h(x) = \frac{x}{x+1}, x_0 = 1$

Solution:

Given $h(x) = \frac{x}{x+1}$ at $x_0 = 1$

$$L(x) = h(x_0) + h'(x_0)(x - x_0) \quad \text{-----------(1)}$$

$$h(1) = \frac{1}{2}$$

$$h'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$$

$$h'(1) = \frac{1}{4}$$

$$L(x) = \frac{1}{2} + \frac{1}{4}(x - 1) \qquad \text{From (1)}$$

$$= \frac{2 + x - 1}{4}$$

$$L(x) = \frac{x+1}{4}$$

Hint:

1. Absolute error = Actual value – Approximate value

2. Relative error = $\frac{\textit{Actual value} - \textit{Approximate value}}{\textit{Actual value}}$

3. Percentage error = Relative error × 100

4. The radius of a circular plate is measured as 12.65 cm instead of the actual length 12.5 cm. Find the following in calculating the area of the circular plate:

(i) Absolute error (ii) Relative error (iii) Percentage error

Solution:

Radius, $r = 12.65$

$$\Delta r = 12.65 - 12.50 = 0.15$$

Area of circular plate, $A = \pi r^2$

Absolute error:

Approximate value $= A'(r)\Delta r$

$$= 2\pi r\, \Delta r$$

$$= 2\pi(12.5)(0.15)$$

Approximate value $= 3.75\pi$

$$\text{Actual value} = A(12.65) - A(12.5)$$

$$= \pi(12.65)^2 - \pi(12.5)^2$$

$$= \pi[(12.65)^2 - (12.5)^2]$$

$$= \pi(25.15)(0.15)$$

$$\text{Actual value} = 3.7725\pi$$

$$\text{Absolute error} = \text{Actual value} - \text{Approximate value}$$

$$= 3.7725\pi - 3.75\pi$$

$$\text{Absolute error} = 0.0225\pi \quad \text{----------(1)}$$

$$\text{Relative error} = \frac{0.0225\pi}{3.7725\pi} = 0.006 \quad \text{----------(2)}$$

$$\text{Percentage error} = \text{relative error} \times 100$$

$$= 0.006 \times 100$$

$$\text{Percentage error} = 0.6\% \quad \text{----------(3)}$$

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5. A sphere is made of ice having radius 10 cm. Its radius decreases from 10 cm to 9.8 cm. Find approximations for the following:

(i) change in volume (ii) change in surface area

Solution:

Given $r = 10,\ \Delta r = 10 - 9.8 = 0.2$

(i) Volume $V(r) = \frac{4}{3}\pi r^3$

$$V'(r) = \frac{4}{3}\pi(3r^2)\,\Delta r$$

$$V'(r) = 4\pi r^2 \Delta r$$

Approximate change $= V'(10)\,\Delta r$

$$= 4\pi(100)(0.2)$$

$$= 80\pi\ cm^3$$

(ii) Surface Area, $S(r) = 4\pi r^2$

$$S'(r) = 8\pi r\ \Delta r$$

Approximate change $= S'(10)\ \Delta r$

$$= 8\pi(10)(0.2)$$

$$= 16\pi\ cm^2$$

6. The time T, taken for a complete oscillation of a single pendulum with length l, is given by the equation $T = 2\pi\sqrt{\frac{l}{g}}$, where g is a constant. Find the approximate percentage error in the calculated value of T corresponding to an error of 2 percent in the value of l.

Solution:

Given $\quad dl = \frac{2}{100} l$

$$\frac{dl}{l} = 0.02 \qquad \text{----------(1)}$$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$T = \frac{2\pi}{\sqrt{g}}(l)^{\frac{1}{2}}$$

Taking log on both sides,

$$\log T = \log\frac{2\pi}{\sqrt{g}} + \frac{1}{2}\log l$$

$$\frac{dT}{T} = 0 + \frac{1}{2}\frac{dl}{l}$$

$$\frac{dT}{T} = \frac{1}{2}(0.02) \qquad \text{From (1)}$$

$$\frac{dT}{T} = 0.01$$

$$\frac{dT}{T} \times 100 = 1\%$$

Appropriate percentage error in T is 1%.

7. Show that the percentage error in the n^{th} root of a number is approximately $\frac{1}{n}$ times the percentage error in the number.

Solution:

Let the number be x.

$$y = x^{\frac{1}{n}}$$

Taking log on both sides,

$$\log y = \frac{1}{n}\log x$$

$$\frac{dy}{y} = \frac{1}{n}\frac{dx}{x}$$

$$\frac{dy}{y} \times 100 = \frac{1}{n}\left(\frac{dx}{x} \times 100\right)$$

$$= \frac{1}{n}(percentage\ error\ in\ x)$$

The percentage error in $x^{\frac{1}{n}}$ is $\frac{1}{n}$ times percentage error in x.

Exercise 8.2

1. Find the differentials dy for each of the following functions:

(i) $y = \frac{(1-2x)^3}{3-4x}$

Solution:

$$y = \frac{(1-2x)^3}{3-4x}$$

$$dy = \frac{(3-4x)3(1-2x)^2(-2)-(1-2x)^3(-4)}{(3-4x)^2}\ dx$$

$$= \frac{-6(3-4x)(1-2x)^2+4(1-2x)^3}{(3-4x)^2}\ dx$$

$$= \frac{2(1-2x)^2[-3(3-4x)+2(1-2x)]}{(3-4x)^2}\ dx$$

$$= \frac{2(1-2x)^2[-9+12x+2-4x]}{(3-4x)^2}\ dx$$

$$dy = \frac{2(1-2x)^2(8x-7)}{(3-4x)^2}\ dx$$

(ii) $y = (3 + \sin(2x))^{\frac{2}{3}}$

Solution:

$$y = (3 + \sin(2x))^{\frac{2}{3}}$$

$$dy = \frac{2}{3}(3 + \sin 2x)^{-\frac{1}{3}}(\cos 2x)(2)\ dx$$

$$dy = \frac{4}{3}\frac{(\cos 2x)}{(3+\sin 2x)^{\frac{1}{3}}}\ dx$$

(iii) $y = e^{x^2-5x+7}\cos(x^2-1)$

Solution:

$$y = e^{x^2-5x+7}\cos(x^2-1)$$

$$\frac{dy}{dx} = \cos(x^2-1)\, e^{x^2-5x+7}(2x-5) + e^{x^2-5x+7}(-\sin(x^2-1))(2x)$$

$$dy = e^{x^2-5x+7}\begin{bmatrix}(2x-5)\cos(x^2-1)\\ -2x\sin(x^2-1)\end{bmatrix} dx$$

2. Find df for $f(x) = x^2 + 3x$ and evaluate it for

(i) $x = 2$ and $dx = 0.1$ (ii) $x = 3$ and $dx = 0.02$

Solution:

Given $f(x) = x^2 + 3x$

$$df = (2x+3)\ dx$$

(i) When $x = 2$ and $dx = 0.1$

$df = (2(2)+3)(0.1) = 0.7$

(ii) When $x = 3$ and $dx = 0.02$

$df = (2(3)+3)(0.02) = 0.18$

3. Find Δf and df for the function f for the indicated values of $x, \Delta x$ and compare

(i) $f(x) = x^3 - 2x^2; x = 2, \Delta x = dx = 0.5$

Solution:

Given $f(x) = x^3 - 2x^2$

$df = (3x^2 - 4x)\, dx$

When $x = 2, dx = 0.5$

$df = [3(4) - 4(2)](0.5) = 4(0.5)$

$df = 2$ ----------(1)

$\Delta f = f(x + \Delta x) - f(x)$

$= f(2.5) - f(2)$

$= [(2.5)^3 - 2(2.5)^2] - [(2^3 - 2(2)^2)]$

$= (2.5)^2[2.5 - 2]$

$= (6.25)(2)$

$\Delta f = 3.125$ ----------(2)

Actual change, $\Delta f = 3.125$

Approximate change, $df = 2$

Error $= \Delta f - df = 1.125$

(ii) $f(x) = x^2 + 2x + 3; x = -0.5,\ \Delta x = dx = 0.1$

Solution:

Given $f(x) = x^2 + 2x + 3$

$df = (2x + 2)\, dx$

When $x = -0.5, dx = 0.1$

$df = [2(-0.5) + 2]\,(0.1) = (1)(0.1)$

$df = 0.1$ ----------(1)

$\Delta f = f(x + \Delta x) - f(x)$

$= f(-0.4) - f(-0.5)$

$= [(-0.4)^2 + 2(-0.4) + 3] - [(-0.5)^2 + 2(-0.5) + 3]$

$= [0.16 - 0.8 + 3] - [0.25 - 1 + 3]$

$= 2.36 - 2.25$

$\Delta f = 0.11$ ----------(2)

Actual change, $\Delta f = 0.11$

Approximate change, $df = 0.1$

Error $= \Delta f - df = 0.01$

4. Assuming $\log_{10} e = 0.4343$, find an approximate value of $\log_{10} 1003$.

Solution:

Let $f(x) = \log_{10} x$

$f'(x) = \frac{1}{x}\log_{10} e$

Let $x = 1000$, and $\Delta x = 3$, $x + \Delta x = 1003$

$df \approx f'(x)\, dx$

$= \frac{1}{1000}\log_{10} e\ (3)$

$= 0.003(0.4343)$

$df = 0.0013029$

$f(x + \Delta x) \approx f(x) + df$

$f(1003) = \log_{10} 1000 + df$

$\log_{10} 1003 = \log_{10} 10^3 + df$

$= 3\log_{10} 10 + df$

$= 3 + 0.0013029$

$\log_{10} 1003 = 3.0013029$

5. The trunk of a tree has diameter 30 cm. During the following year, the circumference grew 6 cm.

(i) Approximately, how much did the tree's diameter grow?

(ii) What is the percentage increase in area of the tree's cross section?

Solution:

(i) Circumference $P = 2\pi r$

$dP = 2\pi\, dr$

$6 = 2\pi\, dr$

$dr = \frac{3}{\pi}$

$\therefore$ Approximately change in diameter $= 2 \times \frac{3}{\pi} = \frac{6}{\pi}\ cm.$

(ii) Cross section Area $A = \pi r^2$

$dA = 2\pi r\, dr$

$dA = 2\pi(15)\left(\frac{3}{\pi}\right)$

$dA = 90\ cm^2$

Percentage change $= \frac{dA}{A} \times 100$

$= \frac{2\pi r\, dr}{\pi r^2} \times 100$

$= \frac{2\,dr}{r} \times 100$

$= \frac{2}{15}\left(\frac{3}{\pi}\right) \times 100$

Percentage change $= \frac{40}{\pi}\%$

6. An egg of a particular bird is very nearly spherical. If the radius to the inside of the shell is $5mm$ and radius to the outside of the shell is $5.3mm$, find the volume of the shell approximately.

Solution:

Given $r = 5, dr = 5.3 - 5 = 0.3$

Volume, $V = \frac{4}{3}\pi r^3$

$dV = 4\pi r^2\, dr$

$= 4\pi(5)^2(0.3)$

$= 100\pi(0.3)$

$= 30\pi\ mm^3$

Approximate volume of shell $= 30\pi\ mm^3$

7. Assume that the cross section of the artery of human is circular. A drug is given to a patient to dilate his arteries. If the radius of an artery is increased from $2mm$ to $2.1mm$, how much is cross sectional area increased approximately?

Solution:

Given $r = 2, dr = 2.1 - 2 = 0.1$

Cross sectional area, $A = \pi r^2$

$dA = 2\pi r\, dr$

$= 2\pi(2)(0.1)$

$dA = 0.4\pi\ mm^2$

8. In a newly developed city, it is estimated that the voting population (in thousands) will increase according to $V(t) = 30 + 12t^2 - t^3, 0 \leq t \leq 8$ where t is the time in years. Find the increase change in voters for the time change form 4 to $4\frac{1}{6}$ year.

Solution:

Given $t = 4, dt = \frac{1}{6}$

$V(t) = 30 + 12t^2 - t^3$

$dV = (24t - 3t^2)\, dt$

$= [24(4) - 3(4)^2]\left(\frac{1}{6}\right)$

$= (96 - 48)\left(\frac{1}{6}\right)$

$= 48\left(\frac{1}{6}\right)$

$dV = 8$

$\therefore$ Approximate change in voters in thousands $= 8000$

9. The relation between the number of words y a person learns in x hours is given by $y = 52\sqrt{x}, 0 \leq x \leq 9$. What is the approximate number of words learned when x changes from (i) 1 to 1.1 hour? (ii) 4 to 4.1 hour?

Solution:

Given $y = 52\sqrt{x}$

$dy = \frac{52}{2\sqrt{x}}dx$

$dy = \frac{26}{\sqrt{x}}dx$

(i) when $x = 1, dx = 0.1$

$dy = \frac{26}{\sqrt{1}}(0.1) = 2.6$

$dy \approx 3$ words

(ii) when $x = 4, dx = 0.1$

$dy = \frac{26}{\sqrt{4}}(0.1) = \frac{2.6}{2}$

$dy = 1.3$

$dy \approx 1$ word

10. A circular plates expands uniformly under the influence of heat. If it's radius increases from $10.5cm$ to $10.75cm$, then find an approximate change in the area and the approximate percentage changes in the area.

Solution:

Given $r = 10.5, dr = 0.25$

Area of circular plate, $A = \pi r^2$

$dA = 2\pi r\, dr$

$= 2\pi(10.5)(0.25)$

$dA = 5.25\pi$

Percentage change $= \frac{dA}{A} \times 100$

$= \frac{5.25\pi}{\pi r^2} \times 100$

$= \frac{5.25}{(10.5)^2} \times 100$

$= \frac{5.25}{110.25} \times 100$

$= 0.0476 \times 100$

$= 4.76\%$

11. A coat of paint of thickness $0.2\ cm$ is applied to the focus of a cube whose edge is $10\ cm$. Use the

differentials to find approximately how many cubic centimeters of paint is used to paint this cube. Also calculate the exact amount of paint used to paint this cube.

Solution:

Let the focus of the cube be x

Given $x = 10, dx = 0.2$

Volume, $V = x^3$

$dV = 3x^2 dx$

$= 3(10)^2(0.2)$

$dV = 60\ cm^3$

Approximate change in volume $= 60\ cm^3$

Exact amount of paint used $= V(10.2) - V(10)$

$= (10.2)^3 - 10^3$

$= 1061.208 - 1000$

Exact amount of paint used $= 61.208\ cm^3$

Exercise 8.3

1. Evaluate $\lim\limits_{(x,y)\to(1,2)} g(x,y)$, if the limit exists, where $g(x,y) = \frac{3x^2-xy}{x^2+y^2+3}$.

Solution:

$$g(x,y) = \frac{3x^2-xy}{x^2+y^2+3}$$

$$\lim_{(x,y)\to(1,2)} g(x,y) = \frac{\lim\limits_{(x,y)\to(1,2)} 3x^2 - xy}{\lim\limits_{(x,y)\to(1,2)} x^2+y^2+3}$$

$$= \frac{3(1)^2-1(2)}{1^2+2^2+3}$$

$$\lim_{(x,y)\to(1,2)} g(x,y) = \frac{1}{8}$$

2. Evaluate $\lim\limits_{(x,y)\to(0,0)} \cos\left(\frac{x^3+y^2}{x+y+2}\right)$, if the limit exists.

Solution:

$$\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3+y^2}{x+y+2}\right) = \cos\left(\frac{0}{2}\right)$$

$$= \cos 0$$

$$= 1$$

3. Let $f(x,y) = \frac{y^2-xy}{\sqrt{x}-\sqrt{y}}$ for $(x,y) \neq (0,0)$. Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

Solution:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{y(y-x)}{\sqrt{x}-\sqrt{y}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{y\left(\sqrt{y}^2-\sqrt{x}^2\right)}{\sqrt{x}-\sqrt{y}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{y(\sqrt{y}+\sqrt{x})(\sqrt{y}-\sqrt{x})}{\sqrt{x}-\sqrt{y}}$$

$$= \lim_{(x,y)\to(0,0)} -y\left(\sqrt{y}+\sqrt{x}\right)$$

$$= 0$$

Hint: $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

4. Evaluate $\lim\limits_{(x,y)\to(0,0)} \cos\left(\frac{e^x \sin y}{y}\right)$, if the limit exists.

Solution:

$$\lim_{(x,y)\to(0,0)} \cos\left(\frac{e^x \sin y}{y}\right) = \cos\left(\lim_{(x,y)\to(0,0)} e^x . \frac{\sin y}{y}\right)$$

$$= \cos(1.1)$$

$$= \cos 1$$

5. Let $g(x,y) = \frac{x^2y}{x^4+y^2}$ for $(x,y) \neq (0,0)$ and $g(0,0) = 0$.

(i) show that $\lim\limits_{(x,y)\to(0,0)} g(x,y) = 0$ along every line $y = mx, m \in \mathbb{R}$.

(ii) show that $\lim\limits_{(x,y)\to(0,0)} g(x,y) = \frac{k}{1+k^2}$ along every parabola $y = kx^2, k \in \mathbb{R}\backslash\{0\}$.

Solution:

Given $g(x,y) = \frac{x^2y}{x^4+y^2}$

(i) Along $y = mx$

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2.mx}{x^4+m^2x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2(mx)}{x^2(x^2+m^2)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{mx}{x^2+m^2}$$

$$= \frac{0}{m^2}$$

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0$$

(ii) Along $y = kx^2$

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2.kx^2}{x^4+k^2x^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^4k}{x^4(1+k^2)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{k}{1+k^2}$$

$$\lim_{(x,y)\to(0,0)} g(x,y) = \frac{k}{1+k^2}$$

6. Show that $f(x,y)=\frac{x^2-y^2}{y^2+1}$ is continuous at every $(x,y)\in\mathbb{R}^2$.

Solution:

$f(x,y)=\frac{x^2-y^2}{y^2+1}$

Let $(a,b)\in\mathbb{R}^2$

$\Rightarrow \quad f(a,b)=\frac{a^2-b^2}{b^2+1}$ is defined

$$\lim_{(x,y)\to(a,b)} f(x,y)=\lim_{(x,y)\to(a,b)}\frac{x^2-y^2}{y^2+1}$$

$$=\frac{a^2-b^2}{b^2+1}$$

$$=L \text{ exists}$$

$$\lim_{(x,y)\to(a,b)} f(x,y)=L=f(a,b)$$

Hence f is continuous at every $(x,y)\in\mathbb{R}^2$

7. Let $g(x,y)=\frac{e^y\sin x}{x}$, for $(x,y)\neq(0,0)$ and $g(0,0)=1$. Show that g is continuous at $(0,0)$.

Solution:

$g(x,y)=\frac{e^y\sin x}{x}$

$\Rightarrow \quad g(0,0)=1$ (given)

$$\Rightarrow \quad \lim_{(x,y)\to(0,0)} g(x,y)=\lim_{(x,y)\to(0,0)}\frac{e^y\sin x}{x}$$

$$=\lim_{(x,y)\to(0,0)} e^y\cdot\lim_{(x,y)\to(0,0)}\frac{\sin x}{x}$$

$$=1$$

$$=L \text{ exists}$$

$$\Rightarrow \quad \lim_{(x,y)\to(0,0)} g(x,y)=L=g(0,0)$$

Hence $g(x,y)$ is continuous at $(0,0)$

Exercise 8.4

Hint: $f_x=\frac{\partial f}{\partial x}$ and $f_y=\frac{\partial f}{\partial y}$

1. Find the partial derivatives of the following functions at the indicated points:

(i) $f(x,y)=3x^2-2xy+y^2+5x+2,\ (2,-5)$

Solution:

$f(x,y)=3x^2-2xy+y^2+5x+2$

$f_x=6x-2y+5$

At $(2,-5)$

$f_x=6(2)-2(-5)+5$

$f_x=27$ ----------(1)

$f_y=-2x+2y$

At $(2,-5)$

$f_y=-2(2)+2(-5)$

$f_y=-14$ ----------(2)

(ii) $g(x,y)=3x^2+y^2+5x+2,\ (1,-2)$

Solution:

$g(x,y)=3x^2+y^2+5x+2$

$g_x=6x+5$

At $(1,-2)$

$g_x=6(1)+5=11$ ----------(1)

$g_y=2y$

At $(1,-2)$

$g_y=2(-2)=-4$ ----------(2)

(iii) $h(x,y,z)=x\sin xy+z^2x,\ \left(2,\frac{\pi}{4},1\right)$

Solution:

$h(x,y,z)=x\sin xy+z^2x$

$h_x=\sin xy+xy\cos xy+z^2$

At $\left(2,\frac{\pi}{4},1\right)$

$h_x=\sin\frac{\pi}{2}+\frac{\pi}{2}\cos\frac{\pi}{2}+1$

$h_x=1+0+1=2$ ----------(1)

$h_y=x^2\cos xy$

At $\left(2,\frac{\pi}{4},1\right)$

$h_y=4\cos\frac{\pi}{2}=0$ ----------(2)

$h_z=2zx$

At $\left(2,\frac{\pi}{4},1\right)$

$h_z=2(1)(2)=4$ ----------(3)

(iv) $G(x,y)=e^{x+3y}\log(x^2+y^2),\ (-1,1)$

Solution:

$G(x,y)=e^{x+3y}\log(x^2+y^2)$

$G_x=\log(x^2+y^2)\,e^{x+3y}(1)+e^{x+3y}\frac{2x}{x^2+y^2}$

At $(-1,1)$

$G_x=\log 2\,e^2+e^2\left(-\frac{2}{2}\right)$

$G_x=e^2(\log 2-1)$ ----------(1)

$$G_y = \log(x^2+y^2)\, e^{x+3y}(3) + e^{x+3y}\frac{2y}{x^2+y^2}$$

At $(-1,1)$

$$G_y = 3\log 2\, e^2 + e^2\left(\frac{2}{2}\right)$$

$$= e^2(3\log 2 + 1)$$

$$G_y = e^2(\log 8 + 1) \qquad \text{----------(2)}$$

---

2. For each of the following functions find the f_x, f_y and show that $f_{xy} = f_{yx}$

(i) $f(x,y) = \frac{3x}{y+\sin x}$

Solution:

$$f(x,y) = \frac{3x}{y+\sin x}$$

$$f_x = \frac{(y+\sin x)(3) - 3x\cos x}{(y+\sin x)^2}$$

$$f_y = \frac{-3x}{(y+\sin x)^2}$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{-3x}{(y+\sin x)^2}\right)$$

$$= \frac{(y+\sin x)^2(-3) - (-3x)2(y+\sin x)\cos x}{(y+\sin x)^4}$$

$$= \frac{-3(y+\sin x)^2 + 6x\cos x(y+\sin x)}{(y+\sin x)^4}$$

$$f_{xy} = \frac{-3(y+\sin x) + 6x\cos x}{(y+\sin x)^3} \qquad \text{----------(1)}$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{(y+\sin x)(3) - 3x\cos x}{(y+\sin x)^2}\right)$$

$$= \frac{(y+\sin x)^2(3) - [3(y+\sin x) - 3x\cos x]2(y+\sin x)}{(y+\sin x)^4}$$

$$= \frac{3(y+\sin x)^2 - 6(y+\sin x)^2 + 6x\cos x(y+\sin x)}{(y+\sin x)^4}$$

$$= \frac{3(y+\sin x) - 6(y+\sin x) + 6x\cos x}{(y+\sin x)^3}$$

$$f_{yx} = \frac{-3(y+\sin x) + 6x\cos x}{(y+\sin x)^3} \qquad \text{----------(2)}$$

Hence $f_{xy} = f_{yx}$

(ii) $f(x,y) = \tan^{-1}\left(\frac{x}{y}\right)$

Solution:

$$f(x,y) = \tan^{-1}\left(\frac{x}{y}\right)$$

$$f_x = \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)$$

$$= \frac{y^2}{y^2+x^2}\left(\frac{1}{y}\right)$$

$$f_x = \frac{y}{y^2+x^2} \qquad \text{----------(1)}$$

$$f_y = \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right)$$

$$= \frac{y^2}{y^2+x^2}\left(-\frac{x}{y^2}\right)$$

$$f_y = -\frac{x}{y^2+x^2} \qquad \text{----------(2)}$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(-\frac{x}{y^2+x^2}\right)$$

$$= \frac{-(y^2+x^2) - (-x)(2x)}{(y^2+x^2)^2}$$

$$= \frac{-y^2 - x^2 + 2x^2}{(y^2+x^2)^2}$$

$$f_{xy} = \frac{x^2-y^2}{(y^2+x^2)^2} \qquad \text{----------(3)}$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{y}{y^2+x^2}\right)$$

$$= \frac{(y^2+x^2) - y(2y)}{(y^2+x^2)}$$

$$f_{yx} = \frac{x^2-y^2}{(y^2+x^2)^2} \qquad \text{----------(4)}$$

Hence $f_{xy} = f_{yx}$

(iii) $f(x,y) = \cos(x^2 - 3xy)$

Solution:

$$f(x,y) = \cos(x^2 - 3xy)$$

$$f_x = -\sin(x^2-3xy)\,(2x-3y)$$

$$f_x = \sin(x^2-3xy)\,(3y-2x)$$

$$f_y = -\sin(x^2-3xy)\,(-3x)$$

$$f_y = 3x\sin(x^2-3xy)$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(3x\sin(x^2-3xy))$$

$$f_{xy} = \sin(x^2-3xy)\,3 + 3x\cos(x^2-3xy)\,(2x-3y)$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\sin(x^2-3xy)\,(3y-2x)\right)$$

$$= (3y-2x)\cos(x^2-3xy)\,(-3x) + \sin(x^2-3xy)\,(3)$$

$f_{yx} = 3x(2x - 3y)\cos(x^2 - 3y) + 3\sin(x^2 - 3xy)$

Hence $f_{xy} = f_{yx}$

---

3. If $U(x,y,z) = \frac{x^2+y^2}{xy} + 3z^2y$, find $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}$, and $\frac{\partial U}{\partial z}$.

Solution:

$$U = \frac{x^2}{xy} + \frac{y^2}{xy} + 3z^2y$$

$$U = \frac{x}{y} + \frac{y}{x} + 3z^2y$$

$$\frac{\partial U}{\partial x} = \frac{1}{y} - \frac{y}{x^2}$$

$$\frac{\partial U}{\partial x} = \frac{x^2-y^2}{x^2y}$$

$$\frac{\partial U}{\partial y} = -\frac{x}{y^2} + \frac{1}{x} + 3z^2$$

$$\frac{\partial U}{\partial y} = \frac{y^2-x^2}{y^2x} + 3z^2$$

$$\frac{\partial U}{\partial z} = 6zy$$

---

4. If $U(x,y,z) = \log(x^3 + y^3 + z^3)$, find $\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z}$.

Solution:

Given $U = \log(x^3 + y^3 + z^3)$

$$\frac{\partial U}{\partial x} = \frac{3x^2}{x^3+y^3+z^3}$$

$$\frac{\partial U}{\partial y} = \frac{3y^2}{x^3+y^3+z^3}$$

$$\frac{\partial U}{\partial z} = \frac{3z^2}{x^3+y^3+z^3}$$

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial x} + \frac{\partial U}{\partial x} = \frac{3(x^2+y^2+z^2)}{x^3+y^3+z^3}$$

---

5. For each of the following functions find the g_{xy}, g_{xx}, g_{yy}, and g_{yx}.

(i) $g(x,y) = xe^y + 3x^2y$

Solution:

Given $g(x,y) = xe^y + 3x^2y$

$g_x = e^y + 6xy \quad ; \quad g_y = xe^y + 3x^2$

$g_{xx} = 6y \quad ; \quad g_{yy} = xe^y$

$g_{xy} = e^y + 6x \quad ; \quad g_{yx} = e^y + 6x$

(ii) $g(x,y) = \log(5x + 3y)$

Solution:

Given $g(x,y) = \log(5x + 3y)$

$g_x = \frac{5}{5x+3y} \quad ; \quad g_y = \frac{3}{5x+3y}$

$g_{xx} = \frac{-25}{(5x+3y)^2} \quad ; \quad g_{yy} = \frac{-9}{(5x+3y)^2}$

$g_{xy} = \frac{-15}{(5x+3y)^2} \quad ; \quad g_{yx} = \frac{-15}{(5x+3y)^2}$

(iii) $g(x,y) = x^2 + 3xy - 7y + \cos 5x$

Solution:

Given $g(x,y) = x^2 + 3xy - 7y + \cos 5x$

$g_x = 2x + 3y - 5\sin 5x \quad ; \quad g_y = 3x - 7$

$g_{xx} = 2 - 25\cos 5x \quad ; \quad g_{yy} = 0$

$g_{xy} = 3 \quad ; \quad g_{yx} = 3$

Hint: $\frac{\partial^2 W}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial x}\right) = W_{xx}$

6. Let $W(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}, (x,y,z) \neq (0,0,0)$.
Show that $\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 0$.

Solution:

Given $W = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2 + y^2 + z^2)^{\frac{-1}{2}}$

$$W_x = -\frac{1}{2}(x^2 + y^2 + z^2)^{\frac{-3}{2}}(2x)$$

$$W_x = -x(x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$W_{xx} = (x^2 + y^2 + z^2)^{\frac{-3}{2}}(-1) + (-x)\left(\frac{-3}{2}\right)(x^2 + y^2 + z^2)^{\frac{-5}{2}}(2x)$$

$$W_{xx} = (x^2 + y^2 + z^2)^{\frac{-3}{2}}[-1 + 3x^2(x^2 + y^2 + z^2)^{-1}]$$

$$W_{xx} = (x^2 + y^2 + z^2)^{\frac{-3}{2}}\left[-1 + \frac{3x^2}{x^2+y^2+z^2}\right] \quad \text{----------(1)}$$

Similarly,

$$W_{yy} = (x^2 + y^2 + z^2)^{\frac{-3}{2}}\left[-1 + \frac{3y^2}{x^2+y^2+z^2}\right] \quad \text{----------(2)}$$

$$W_{yy} = (x^2 + y^2 + z^2)^{\frac{-3}{2}}\left[-1 + \frac{3z^2}{x^2+y^2+z^2}\right] \quad \text{----------(3)}$$

Now, $W_{xx} + W_{yy} + W_{zz}$

$$= (x^2 + y^2 + z^2)^{\frac{-3}{2}}\left[-3 + \frac{3(x^2+y^2+z^2)}{x^2+y^2+z^2}\right]$$

$$= (x^2 + y^2 + z^2)^{\frac{-3}{2}}[-3 + 3]$$

$$= 0$$

---

7. If $V(x,y) = e^x(x\cos y - y\sin y)$, then prove that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$.

Solution:

Given $V = e^x(x\cos y - y\sin y)$

$$\frac{\partial V}{\partial x} = (x\cos y - y\sin y)e^x + e^x(\cos y)$$

$$\frac{\partial^2 V}{\partial x^2} = e^x(\cos y) + (x\cos y - y\sin y)e^x + e^x\cos y$$

$$= 2e^x\cos y + e^x(x\cos y - y\sin y)$$

$$\frac{\partial^2 V}{\partial x^2} = e^x(2\cos y + x\cos y - y\sin y) \quad \text{----------(1)}$$

$$\frac{\partial V}{\partial y} = e^x(-x\sin y - \sin y - y\cos y)$$

$$\frac{\partial^2 V}{\partial y^2} = e^x(-x\cos y - \cos y - \cos y + y\sin y)$$

$$\frac{\partial^2 V}{\partial y^2} = e^x(-x\cos y - 2\cos y + y\sin y) \quad \text{----------(2)}$$

Now, $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2}$

$$= e^x(2\cos y + x\cos y - y\sin y) + e^x(-x\cos y - 2\cos y + y\sin y)$$

$$= 0$$

Hence proved.

8. If $\mathbf{W(x,y) = xy + \sin xy}$, then prove that $\frac{\partial^2 \mathbf{w}}{\partial y \partial x} = \frac{\partial^2 \mathbf{w}}{\partial x \partial y}$.

Solution:

Given $W = xy + sin\, xy$

$$\frac{\partial W}{\partial x} = y + y\, cos\, xy$$

$$\frac{\partial W}{\partial y} = x + x\, cos\, xy$$

$$\frac{\partial^2 W}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}(y + y\, cos\, xy)$$

$$= 1 + cos\, xy - xy\, sin\, xy \quad \text{----------(1)}$$

$$\frac{\partial^2 W}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(x + x\, cos\, xy)$$

$$= 1 + cos\, xy - xy\, sin\, xy \quad \text{----------(2)}$$

From (1) and (2), $\frac{\partial^2 W}{\partial y \partial x} = \frac{\partial^2 W}{\partial x \partial y}$

9. If $V(x,y,z) = x^3 + y^3 + z^3 + 3xyz$, show that $\frac{\partial^2 V}{\partial y \partial z} = \frac{\partial^2 V}{\partial z \partial y}$.

Solution:

Given $V = x^3 + y^3 + z^3 + 3xyz$

$$\frac{\partial V}{\partial z} = 3z^2 + 3xy$$

$$\frac{\partial^2 V}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z}\right)$$

$$= \frac{\partial}{\partial y}(3z^2 + 3xy)$$

$$\frac{\partial^2 V}{\partial y \partial z} = 3x$$

$$\frac{\partial V}{\partial y} = 3y^2 + 3xz$$

$$\frac{\partial^2 V}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial y}\right)$$

$$= \frac{\partial}{\partial z}(3y^2 + 3xz)$$

$$\frac{\partial^2 V}{\partial z \partial y} = 3x$$

Hence $\frac{\partial^2 V}{\partial y \partial z} = \frac{\partial^2 V}{\partial z \partial y}$

Hint: Profit = Selling price − Cost price

10. A firm produces two types of calculators each week, x number of type A and y number of type B. The weekly revenue and cost functions (in rupees) are $R(x,y) = 80x + 90y + 0.04xy - 0.05x^2 - 0.05y^2$ and $C(x,y) = 8x + 6y + 2000$ respectively.

(i) Find the profit function $P(x,y)$

(ii) Find $\frac{\partial P}{\partial x}(1200, 1800)$ and $\frac{\partial P}{\partial y}(1200, 1800)$ and interpret these results.

Solution:

(i) $P(x,y) = R(x,y) - C(x,y)$

$$P(x,y) = 80x + 90y + 0.04xy - 0.05x^2 - 0.05y^2 - 8x - 6y - 2000$$

$$P(x,y) = 72x + 84y + 0.04xy - 0.05x^2 - 0.05y^2 - 2000$$

(ii) $\frac{\partial P}{\partial x} = 72 + 0.04y - 0.1x$

At (1200,1800)

$$= 72 + 0.04(1800) - 0.1(1200)$$

$$= 72 + 72 - 120$$

$$\frac{\partial P}{\partial x} = 24 \quad \text{----------(1)}$$

$$\frac{\partial P}{\partial y} = 84 + 0.04x - 0.1y$$

At (1200,1800)

$$= 84 + 0.04(1200) - 0.1(1800)$$

$$= 84 + 48 - 180$$

$$\frac{\partial P}{\partial y} = -48 \quad \text{----------(2)}$$

Keeping y constant and increasing x increases profit.

Exercise 8.5

Hint: Linear Approximation of F at (x_0, y_0, z_0) is

$$L(x,y,z) = F(x_0,y_0,z_0) + \frac{\partial F}{\partial x}(x - x_0) + \frac{\partial F}{\partial y}(y - y_0) + \frac{\partial F}{\partial z}(z - z_0)$$

1. If $w(x,y) = x^3 - 3xy + 2y^2, x, y \in \mathbb{R}$, find the linear approximation for w at $(1,-1)$.

Solution:

Given $w = x^3 - 3xy + 2y^2$

At $(1,-1)$

$$w = 1 + 3 + 2 = 6 \qquad \text{----------(1)}$$

$$\frac{\partial w}{\partial x} = 3x^2 - 3y$$

At $(1,-1)$

$$\frac{\partial w}{\partial x} = 3 + 3 = 6 \qquad \text{----------(2)}$$

$$\frac{\partial w}{\partial y} = -3x + 4y$$

At $(1,-1)$

$$\frac{\partial w}{\partial y} = -3 - 4 = -7 \qquad \text{----------(3)}$$

$$L(x,y) = w(1,-1) + \frac{\partial w}{\partial x}(x-1) + \frac{\partial w}{\partial y}(y+1)$$

$$= 6 + 6(x-1) - 7(y+1)$$

$$= 6 + 6x - 6 - 7y - 7$$

$$L(x,y) = 6x - 7y - 7$$

---

2. Let $z(x,y) = x^2y + 3xy^4, x, y \in \mathbb{R}$. Find linear approximation for z at $(2,-1)$.

Solution:

Given $z = x^2y + 3xy^4$

At $(2,-1)$

$$z = -4 + 6 = 2 \qquad \text{----------(1)}$$

$$\frac{\partial z}{\partial x} = 2xy + 3y^4$$

At $(2,-1)$

$$\frac{\partial z}{\partial x} = -4 + 3 = -1 \qquad \text{----------(2)}$$

$$\frac{\partial z}{\partial y} = x^2 + 12xy^3$$

At $(2,-1)$

$$\frac{\partial z}{\partial y} = 4 - 24 = -20 \qquad \text{----------(3)}$$

$$L(x,y) = z(2,-1) + \frac{\partial z}{\partial x}(x-2) + \frac{\partial z}{\partial y}(y+1)$$

$$= 2 - 1(x-2) - 20(y+1)$$

$$= 2 - x + 2 - 20y - 20$$

$$L(x,y) = -x - 20y - 16$$

---

3. If $v(x,y) = x^2 - xy + \frac{1}{4}y^2 + 7, x, y \in R$, find the differential dv.

Solution:

Given $v = x^2 - xy + \frac{1}{4}y^2 + 7$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= (2x - y)dx + \left(-x + \frac{2y}{4}\right)dy$$

$$dv = (2x - y)dx + \left(-x + \frac{y}{2}\right)dy$$

---

4. Let $V(x,y,z) = xy + yz + zx, x, y, z \in \mathbb{R}$. Find the differential dV.

Solution:

Given $V = xy + yz + zx$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$dV = (y+z)dx + (x+z)dy + (y+x)dz$$

Exercise 8.6

Hint: $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$ **with $u(x,y,z)$.**

1. If $u(x,y) = x^2y + 3xy^4, x = e^t$ and $y = \sin t$, find $\frac{du}{dt}$ and evaluate it at $t = 0$.

Solution:

Given $u = x^2y + 3xy^4, x = e^t$ and $y = \sin t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 3y^4)e^t + (x^2 + 12xy^3)\cos t$$

$$= (2e^t \sin t + 3\sin^4 t)e^t + (e^{2t} + 12e^t \sin^3 t)\cos t$$

$$\frac{du}{dt} = e^t[2e^t \sin t + 3\sin^4 t + e^t \cos t + 12\sin^3 t \cos t]$$

At $t = 0$,

$$\frac{du}{dt} = 1[0 + 0 + 1 + 0]$$

$$\frac{du}{dt} = 1$$

---

2. If $u(x,y,z) = xy^2z^3, x = \sin t, y = \cos t, z = 1 + e^{2t}$, find $\frac{du}{dt}$.

Solution:

Given $u = xy^2z^3, x = \sin t, y = \cos t$ and $z = 1 + e^{2t}$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$\frac{\partial u}{\partial x} = y^2 z^3$ $\quad \frac{\partial u}{\partial y} = 2xyz^3$ $\quad \frac{\partial u}{\partial z} = 3xy^2 z^2$

$\frac{dx}{dt} = \cos t$ $\quad \frac{dy}{dt} = -\sin t$ $\quad \frac{dz}{dt} = 2e^{2t}$

$\frac{du}{dt} = y^2 z^3 \cos t - 2xyz^3 \sin t + 6xy^2 z^2 e^{2t}$

$= \cos^2 t\,(1+e^{2t})^3 \cos t - 2 \sin t \cos t\ (1 + e^{2t})^3 \sin t + 6 \sin t \cos^2 t\,(1+e^{2t})^2 e^{2t}$

$$\frac{du}{dt} = (1+e^{2t})^2 \begin{bmatrix} \cos^3 t\,(1+e^{2t}) - \\ \sin 2t \sin t\,(1+e^{2t}) + \\ 6e^{2t} \sin t \cos^2 t \end{bmatrix}$$

---

3. If $w(x,y,z) = x^2 + y^2 + z^2, x = e^t, y = e^t \sin t$ and $z = e^t \cos t$, find $\frac{dw}{dt}$.

Solution:

Given $w = x^2 + y^2 + z^2$,

` $x = e^t, y = e^t \sin t$ and $z = e^t \cos t$

$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$

$= 2xe^t + 2y(e^t \sin t + e^t \cos t) + 2z(e^t \cos t - e^t \sin t)$

$= 2e^{2t} + 2e^t \sin t\,(e^t \sin t + e^t \cos t) + 2e^t \cos t\,(e^t \cos t - e^t \sin t)$

$= 2e^{2t}\begin{bmatrix} 1 + \sin^2 t + \sin t \cos t + \cos^2 t \\ - \sin t \cos t \end{bmatrix}$

$= 2e^t[1 + \sin^2 t + \cos^2 t]$

$= 2e^{2t}(2)$

$\frac{dw}{dt} = 4e^{2t}$

Hint: $\cos 2A = \cos^2 A - \sin^2 A$; $\sin 2A = 2 \sin A \cos A$

4. Let $U(x,y,z) = xyz, x = e^{-t}, y = e^{-t} \cos t, z = \sin t, t \in \mathbb{R}$. Find $\frac{dU}{dt}$.

Solution:

$U = xyz, x = e^{-t}, y = e^{-t} \cos t$,and $z = \sin t$

$\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt}$

$= -yze^{-t} + xz(-\cos t\, e^{-t} - e^{-t} \sin t) + xy \cos t$

$= -e^{-2t} \cos t \sin t + e^{-t} \sin t \begin{pmatrix} -\cos t\, e^{-t} \\ -e^{-t} \sin t \end{pmatrix} + e^{-2t} \cos^2 t$

$= -e^{-2t} \cos t \sin t - e^{-2t} \cos t \sin t - e^{-2t} \sin^2 t + e^{-2t} \cos^2 t$

$= -e^{-2t}\begin{bmatrix} \cos t \sin t + \cos t \sin t + \sin^2 t \\ - \cos^2 t \end{bmatrix}$

$\frac{dU}{dt} = -e^{-2t}[2 \cos t \sin t - \cos 2t]$

$\frac{dU}{dt} = -e^{-2t}[\sin 2t - \cos 2t]$

---

5. If $w(x,y) = 6x^3 - 3xy + 2y^2, x = e^s, y = \cos s, s \in \mathbb{R}$, find $\frac{dw}{ds}$ and evaluate at $s = 0$.

Solution:

$w(x,y) = 6x^3 - 3xy + 2y^2, x = e^s, y = \cos s$

$\frac{dw}{ds} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{dy}{ds}$

$= (18x^2 - 3y)e^s + (-3x + 4y)(-\sin s)$

$= (18e^{2s} - 3 \cos s)e^s - (-3e^s + 4 \cos s) \sin s$

At $s = 0$,

$\frac{dw}{ds} = (18 - 3) - (-3 + 4)(0)$

$\frac{dw}{dt} = 15$

Hint: $\frac{\partial W}{\partial s} = \frac{\partial W}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial W}{\partial y}\frac{\partial y}{\partial s}$ and $\frac{\partial W}{\partial y} = \frac{\partial W}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial W}{\partial y}\frac{\partial y}{\partial t}$

6. If $z(x,y) = x \tan^{-1}(xy), x = t^2, y = se^t, s,t \in \mathbb{R}$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at $s = t = 1$.

Solution:

$z = x \tan^{-1}(xy), x = t^2, y = se^t$

$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$

$\frac{\partial z}{\partial x} = \tan^{-1} xy + \frac{xy}{1+x^2y^2}$; $\quad \frac{\partial z}{\partial y} = \frac{x^2}{1+x^2y^2}$

$\frac{\partial x}{\partial s} = 0$; $\quad \frac{\partial y}{\partial s} = e^t$

$\frac{\partial x}{\partial t} = 2t$; $\quad \frac{\partial y}{\partial t} = se^t$

$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$

$= 0 + \left(\frac{x^2}{1+x^2y^2}\right) e^t$

$= \frac{t^4 e^t}{1+t^4 s^2 e^{2t}}$

At $s = t = 1$,

$\frac{\partial z}{\partial s} = \frac{e}{1+e^2}$ ----------(1)

$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

$= \left(tan^{-1} xy + \frac{xy}{1+x^2y^2}\right)(2t) + \left(\frac{x^2}{1+x^2y^2}\right) se^t$

$\frac{\partial z}{\partial t} = 2t\left[\tan^{-1}(t^2 se^t) + \frac{t^2 se^t}{1+t^2 s^2 e^{2t}}\right] + se^t \left[\frac{t^4}{1+t^2 se^{2t}}\right]$

At $s = t = 1$,

$$\frac{\partial z}{\partial t} = 2\left[\tan^{-1}(e) + \frac{e}{1+e^2}\right] + e\left[\frac{1}{1+e^2}\right]$$

$$= 2\tan^{-1} e + \frac{2e}{1+e^2} + \frac{e}{1+e^2}$$

$$\frac{\partial z}{\partial t} = 2\tan^{-1} e + \frac{3e}{1+e^2} \qquad \text{----------(2)}$$

7. Let $U(x, y) = e^x \sin y$, where $x = st^2, y = s^2t$, $s, t \in \mathbb{R}$. Find $\frac{\partial U}{\partial s}, \frac{\partial U}{\partial t}$ and evaluate them at $s = t = 1$.

Solution:

$U = e^x \sin y$ and $x = st^2, y = s^2t$

$$\frac{\partial U}{\partial s} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial U}{\partial x} = e^x \sin y \quad ; \quad \frac{\partial U}{\partial y} = e^x \cos y$$

$$\frac{\partial x}{\partial s} = t^2 \quad ; \quad \frac{\partial y}{\partial s} = 2st$$

$$\frac{\partial x}{\partial t} = 2st \quad ; \quad \frac{\partial y}{\partial t} = s^2$$

$$\frac{\partial U}{\partial s} = e^x \sin y\,(t^2) + e^x \cos y\,(2st)$$

$$\frac{\partial U}{\partial s} = t^2 e^{st^2} \sin(s^2t) + 2st\, e^{st^2} \cos(s^2t)$$

At $s = t = 1$,

$$\frac{\partial U}{\partial s} = e\sin(1) + 2e\cos(1)$$

$$\frac{\partial U}{\partial s} = e[\sin(1) + 2\cos(1)] \qquad \text{-----------(1)}$$

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial t}$$

$$= e^x \sin y\,(2st) + e^x \cos y\,(s^2)$$

$$= 2ste^{st^2} \sin(s^2t) + s^2 e^{st^2} \cos(s^2t)$$

At $s = t = 1$,

$$\frac{\partial U}{\partial t} = 2e\sin(1) + e\cos(1)$$

$$\frac{\partial U}{\partial t} = e[2\sin(1) + \cos(1)] \qquad \text{----------(2)}$$

8. Let $z(x, y) = x^3 - 3x^2y^3$, where $x = se^t, y = se^{-t}$, $s, t \in \mathbb{R}$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution:

$z = x^3 - 3x^2y^3$ and $x = se^t, y = se^{-t}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial x} = 3x^2 - 6xy^3 \quad ; \quad \frac{\partial z}{\partial y} = -9x^2y^2$$

$$\frac{\partial x}{\partial s} = e^t \quad ; \quad \frac{\partial y}{\partial s} = e^{-t}$$

$$\frac{\partial x}{\partial t} = se^t \quad ; \quad \frac{\partial y}{\partial t} = -se^{-t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= [3x^2 - 6xy^3]e^t + [-9x^2y^2]e^{-t}$$

$$= [3s^2e^{2t} - 6se^ts^3e^{-3t}]e^t + [-9s^2e^{2t}s^2e^{-2t}]e^{-t}$$

$$= 3s^2e^{3t} - 6s^4e^{-t} - 9s^4e^{-t}$$

$$\frac{\partial z}{\partial s} = 3s^2e^{3t} - 15s^4e^{-t} \qquad \text{----------(1)}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= [3x^2 - 6xy^3]se^t + [-9x^2y^2](-se^{-t})$$

$$= [3s^2e^{2t} - 6se^ts^3e^{-3t}]se^t + [-9s^2e^{2t}s^2e^{-2t}](-se^{-t})$$

$$= 3s^3e^{3t} - 6s^5e^{-t} + 9s^5e^{-t}$$

$$\frac{\partial z}{\partial t} = 3s^3e^{3t} + 3s^5e^{-t}$$

$$\frac{\partial z}{\partial t} = 3s^3[e^{3t} + s^2e^{-t}] \qquad \text{----------(2)}$$

9. $W(x, y, z) = xy + yz + zx$, $x = u - v, y = uv$ and $z = u + v, u, v \in \mathbb{R}$. Find $\frac{\partial W}{\partial u}, \frac{\partial W}{\partial v}$ and evaluate them at $\left(\frac{1}{2}, 1\right)$.

Solution:

$W = xy + yz + zx$ and $x = u - v, y = uv, z = u + v$

$$\frac{\partial W}{\partial u} = \frac{\partial W}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial W}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial W}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial W}{\partial x} = y + z \qquad \frac{\partial W}{\partial y} = x + z \qquad \frac{\partial W}{\partial z} = y + x$$

$$\frac{\partial x}{\partial u} = 1 \qquad \frac{\partial y}{\partial u} = v \qquad \frac{\partial z}{\partial u} = 1$$

$$\frac{\partial x}{\partial v} = -1 \qquad \frac{\partial y}{\partial v} = u \qquad \frac{\partial z}{\partial v} = 1$$

$$\frac{\partial W}{\partial u} = \frac{\partial W}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial W}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial W}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial W}{\partial u} = (y + z)1 + (x + z)v + (y + x)1$$

$$= uv + u + v + (u - v + u + v)v + uv + u - v$$

$$= uv + u + v + 2uv + uv + u - v$$

$$\frac{\partial W}{\partial u} = 4uv + 2u$$

At $\left(\frac{1}{2}, 1\right)$

$$\frac{\partial W}{\partial u} = 4\left(\frac{1}{2}\right)(1) + 2\left(\frac{1}{2}\right)$$

$$= 2 + 1$$

$$\frac{\partial w}{\partial u} = 3 \qquad \text{----------(1)}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

$$= (y+z)(-1) + (x+z)u + (y+x)1$$

$$= -y - z + (x+z)u + y + x$$

$$= -z + (x+z)u + x$$

$$= -u - v + (u - v + u + v)u + u - v$$

$$= -u - v + 2u^2 + u - v$$

$$\frac{\partial w}{\partial v} = -2v + 2u^2$$

At $\left(\frac{1}{2}, 1\right)$

$$\frac{\partial w}{\partial v} = -2(1) + 2\left(\frac{1}{4}\right)$$

$$= -2 + \frac{1}{2}$$

$$\frac{\partial w}{\partial v} = -\frac{3}{2} \qquad \text{----------(2)}$$

Exercise 8.7

1. In each of the following cases, determine whether the following function is homogeneous of not. If it is so, find the degree.

(i) $f(x,y) = x^2y + 6x^3 + 7$

Solution:

$$f(x,y) = x^2y + 6x^3 + 7$$

$$f(\lambda x, \lambda y) = \lambda^3x^2y + 6\lambda^3x^3 + 7$$

$$\neq \lambda^p f(x,y)$$

$\therefore f$ is not a homogeneous function.

(ii) $h(x,y) = \frac{6x^2y^3 - \pi y^5 + 9x^4y}{2020x^2 + 2019y^2}$

Solution:

$$h(x,y) = \frac{6x^2y^3 - \pi y^5 + 9x^4y}{2020x^2 + 2019y^2}$$

$$h(\lambda x, \lambda y) = \frac{6\lambda^5x^2y^3 - \pi\lambda^5y^5 + 9\lambda^5x^4y}{2020\lambda^2x^2 + 2019\lambda^2y^2}$$

$$= \frac{\lambda^5[6x^2y^3 - \pi y^5 + 9x^4y]}{\lambda^2[2020x^2 + 2019y^2]}$$

$$= \lambda^3 h(x,y)$$

$\therefore h$ is a homogeneous function with degree 3.

(iii) $g(x,y,z) = \frac{\sqrt{3x^2 + 5y^2 + z^2}}{4x + 7y}$

Solution:

$$g(x,y,z) = \frac{\sqrt{3x^2 + 5y^2 + z^2}}{4x + 7y}$$

$$g(\lambda x, \lambda y, \lambda z) = \frac{\sqrt{3\lambda^2x^2 + 5\lambda^2y^2 + \lambda^2z^2}}{4\lambda x + 7\lambda y}$$

$$= \frac{\lambda\sqrt{3x^2 + 5y^2 + z^2}}{\lambda(4x + 7y)}$$

$$= \lambda^0 g(x,y,z)$$

$\therefore g$ is a homogeneous function with degree 0.

(iv) $U(x,y,z) = xy + \sin\left(\frac{y^2 - 2z^2}{xy}\right)$

Solution:

$$U(x,y,z) = xy + \sin\left(\frac{y^2 - 2z^2}{xy}\right)$$

$$U(\lambda x, \lambda y, \lambda z) = \lambda^2xy + \sin\left(\frac{\lambda^2y^2 - 2\lambda^2z^2}{\lambda^2xy}\right)$$

$$= \lambda^2xy + \sin\left(\frac{\lambda^2(y^2 - 2z^2)}{\lambda^2xy}\right)$$

$$= \lambda^2xy + \sin\left(\frac{y^2 - 2z^2}{xy}\right)$$

$$\neq \lambda^p U(x,y,z)$$

$\therefore U$ is not a homogeneous function.

Hint: Euler's Theorem, $x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = pF$

2. Prove that $f(x,y) = x^3 - 2x^2y + 3xy^2 + y^3$ is homogeneous; what is the degree? Verify Euler's Theorem for f.

Solution:

$$f(x,y) = x^3 - 2x^2y + 3xy^2 + y^3$$

$$f(\lambda x, \lambda y) = \lambda^3[x^3 - 2x^2y + 3xy^2 + y^3]$$

$$= \lambda^3 f(x,y)$$

$\therefore f$ is a homogeneous function with degree 3.

To prove: $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f(x,y)$

$$\text{LHS} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$$

$$= x(3x^2 - 4xy + 3y^2) + y(-2x^2 + 6xy + 3y^2)$$

$$= 3x^3 - 4x^2y + 3xy^2 - 2x^2y + 6xy^2 + 3y^3$$

$$= 3x^3 - 6x^2y + 9xy^2 + 3y^3$$

$$= 3(x^3 - 2x^2y + 3xy^2 + y^3)$$

$$= 3\, f(x,y)$$

$$= \text{RHS}$$

Hence Euler's theorem is verified.

3. Prove that $g(x,y) = x\log\left(\frac{y}{x}\right)$ is homogeneous; what is the degree? Verify Euler's Theorem for g.

Solution:

$$g(x,y) = x\log\left(\frac{y}{x}\right)$$

$$g(\lambda x,\lambda y) = \lambda x\log\left(\frac{\lambda y}{\lambda x}\right)$$

$$= \lambda\, g(x,y)$$

$\therefore$ g is a homogeneous function with degree 1.

To prove: $x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = g(x,y)$

$$\frac{\partial g}{\partial x} = \log\left(\frac{y}{x}\right) + x.\frac{1}{\frac{y}{x}}\left(\frac{-y}{x^2}\right)$$

$$= \log\left(\frac{y}{x}\right) + \frac{x^2}{y}\left(\frac{-y}{x^2}\right)$$

$$\frac{\partial g}{\partial x} = \log\left(\frac{y}{x}\right) - 1$$

$$\frac{\partial g}{\partial y} = x.\frac{1}{\frac{y}{x}}\left(\frac{1}{x}\right) = \frac{x^2}{y}\left(\frac{1}{x}\right)$$

$$\frac{\partial g}{\partial y} = \frac{x}{y}$$

$$\text{LHS} = x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y}$$

$$= x\left(\log\left(\frac{y}{x}\right) - 1\right) + y\,\left(\frac{x}{y}\right)$$

$$= x\log\left(\frac{y}{x}\right) - x + x$$

$$= x\log\left(\frac{y}{x}\right)$$

$$= g(x,y)$$

$$= \text{RHS}$$

Hence Euler's theorem is verified.

4. If $u(x,y) = \frac{x^2+y^2}{\sqrt{x+y}}$, prove that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{3}{2}u$.

Solution:

$$u(x,y) = \frac{x^2+y^2}{\sqrt{x+y}}$$

$$u(\lambda x,\lambda y) = \frac{\lambda^2x^2+\lambda^2y^2}{\sqrt{\lambda x+\lambda y}}$$

$$= \frac{\lambda^2(x^2+y^2)}{\sqrt{\lambda}\sqrt{x+y}}$$

$$= \lambda^{\frac{3}{2}}\,u(x,y)$$

$\therefore$ u is a homogeneous function with degree $\frac{3}{2}$.

By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$$

Hence proved.

5. If $v(x,y) = \log\left(\frac{x^2+y^2}{x+y}\right)$, prove that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1$.

Solution:

$$v(x,y) = \log\left(\frac{x^2+y^2}{x+y}\right)$$

$$e^v = \frac{x^2+y^2}{x+y} = f(x,y)\ \text{(say)} \qquad \text{----------(1)}$$

$$f(x,y) = \frac{x^2+y^2}{x+y}$$

$$f(\lambda x,\lambda y) = \frac{\lambda^2(x^2+y^2)}{\lambda(x+y)} = \lambda\, f(x,y)$$

$\therefore$ f is a homogeneous function with degree 1.

By Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f$$

$$x\frac{\partial (e^v)}{\partial x} + y\frac{\partial (e^v)}{\partial y} = e^v \qquad \text{From (1)}$$

$$xe^v\frac{\partial v}{\partial x} + ye^v\frac{\partial v}{\partial y} = e^v$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1$$

Hence proved.

6. If $w(x,y,z) = \log\left(\frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right)$, find $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z}$.

Solution:

$$w(x,y,z) = \log\left(\frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right)$$

$$e^w = \frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2} = f(x,y,z)\ \text{(say)}$$

$$f(\lambda x,\lambda y,\lambda z) = \frac{\lambda^7(5x^3y^4+7y^2xz^4-75y^3z^4)}{\lambda^2(x^2+y^2)}$$

$$= \lambda^5 f(x,y,z)$$

$\therefore$ f is a homogeneous function with degree 5.

By Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 5f$$

$$x\frac{\partial (e^w)}{\partial x} + y\frac{\partial (e^w)}{\partial y} + z\frac{\partial (e^w)}{\partial z} = 5e^w$$

$$xe^w\frac{\partial w}{\partial x} + ye^w\frac{\partial w}{\partial y} + ze^w\frac{\partial w}{\partial z} = 5e^w$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 5$$