CHAPTER–3

Theory of Equations

Exercise 3.1

1. If the sides of a cubic box are increased by 1, 2, 3 units respectively to form a cuboid, then the volume is increased by 52 cubic units. Find the volume of the cuboid.

Solution:

Let the sides of the cubic box are x, x, x

$\therefore$ volume of cube $= x^3$ ----------(1)

Sides are increased by 1, 2, 3 respectively.

$\therefore$ Sides of cuboid are $x+1, x+2, x+3$

$\therefore$ volume of cuboid $= x^3 + 52$ ----------(2)

$$(x+1)(x+2)(x+3) = x^3 + 52$$

$$(x^2+3x+2)(x+3) = x^3 + 52$$

$$x^3 + 3x^2 + 3x^2 + 9x + 2x + 6 = x^3 + 52$$

$$6x^2 + 11x - 46 = 0$$

$$\begin{array}{c|c} \multicolumn{2}{c}{-276} \\ 23 & -12 \\ \hline 6 & 6 \\ \multicolumn{2}{c}{11} \end{array}$$

$$(6x+23)(6x-12) = 0$$

$$6x + 23 = 0 \text{ or } 6x - 12 = 0$$

$$x = 2 \text{ or } x = -\frac{23}{6}\ (not\ possible)$$

$\therefore$ $x = 2$

$\therefore$ volume of cube $= 2^3 = 8$

$\therefore$ volume of cuboid $= 8 + 52 = 60$

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2. Construct a cubic equation with roots

(i) $1, 2, and\ 3$ (ii) $1, 1, and\ -2$ (iii) $2, \frac{1}{2}, and\ 1$

Hint: The cubic Equation is,

$$\boldsymbol{x^3 - x^2(\alpha+\beta+\gamma) + x(\alpha\beta+\beta\gamma+\gamma\alpha) - (\alpha\beta\gamma) = 0}$$

Solutions:

(i) $1, 2, and\ 3$

Given roots are $\alpha = 1, \beta = 2$ and $\gamma = 3$

The cubic equation of the form is,

$$x^3 - x^2(\alpha+\beta+\gamma) + x(\alpha\beta+\beta\gamma+\gamma\alpha) - (\alpha\beta\gamma) = 0$$

$$x^3 - x^2(1+2+3) + x(2+6+3) - (6) = 0$$

$$x^3 - 6x^2 + 11x - 6 = 0$$

(ii) $1, 1, and\ -2$

Given roots are $\alpha = 1, \beta = 1$ and $\gamma = -2$

The cubic equation of the form is,

$$x^3 - x^2(1+1-2) + x(1-2-2) - (-2) = 0$$

$$x^3 - 3x + 2 = 0$$

(iii) $2, \frac{1}{2}, and\ 1$

Given roots are $\alpha = 2, \beta = \frac{1}{2}$ and $\gamma = 1$

The cubic equation of the form is,

$$x^3 - x^2\left(2+\frac{1}{2}+1\right) + x\left(1+\frac{1}{2}+2\right) - (1) = 0$$

$$x^3 - \frac{7}{2}x^2 + \frac{7}{2}x - 1 = 0$$

$$2x^3 - 7x^2 + 7x - 2 = 0$$

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3. If α, β and γ are the roots of the cubic equation $x^3 + 2x^2 + 3x + 4 = 0$, form a cubic equation whose roots are

(i) $2\alpha, 2\beta, 2\gamma$ (ii) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ (iii) $-\alpha, -\beta, -\gamma$

Solutions:

Given $x^3 + 2x^2 + 3x + 4 = 0$

Here $a = 1, b = 2, c = 3$ and $d = 4$

Σ_1: $\alpha+\beta+\gamma = \frac{-b}{a} = -2$ ----------(1)

Σ_2: $\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a} = 3$ ----------(2)

Σ_3: $\alpha\beta\gamma = \frac{-d}{a} = -4$ ----------(3)

(i) $2\alpha, 2\beta, 2\gamma$

Σ_1: $2\alpha + 2\beta + 2\gamma = 2(\alpha+\beta+\gamma) = 2(-2) = -4$

Σ_2: $= (2\alpha)(2\beta) + (2\beta)(2\gamma) + (2\gamma)(2\alpha)$

$= 4\alpha\beta + 4\beta\gamma + 4\gamma\alpha$

$= 4(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= 4(3) = 12$

Σ_3: $(2\alpha)(2\beta)(2\gamma) = 8(\alpha\beta\gamma) = 8(-4) = -32$

The cubic equation is,

$$x^3 - (-4)x^2 + 12x - (-32) = 0$$

$$x^3 + 4x^2 + 12x + 32 = 0$$

(ii) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

Σ_1: $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma} = \frac{3}{-4}$

Σ_2: $\frac{1}{\alpha}\frac{1}{\beta} + \frac{1}{\beta}\frac{1}{\gamma} + \frac{1}{\gamma}\frac{1}{\alpha} = \frac{\gamma+\alpha+\beta}{\alpha\beta\gamma} = \frac{-2}{-4} = \frac{1}{2}$

Σ_3: $\frac{1}{\alpha}\frac{1}{\beta}\frac{1}{\gamma} = \frac{1}{\alpha\beta\gamma} = \frac{1}{-4}$

The cubic equation is,

$$x^3 - \left(\frac{3}{-4}\right)x^2 + \frac{1}{2}x - \left(\frac{1}{-4}\right) = 0$$

$$x^3 + \frac{3}{4}x^2 + \frac{1}{2}x + \frac{1}{4} = 0$$

$$4x^3 + 3x^2 + 2x + 1 = 0$$

(iii) $-\alpha, -\beta, -\gamma$

$\sum_1$: $-\alpha - \beta - \gamma = -(\alpha + \beta + \gamma) = -(-2) = 2$

$\sum_2$: $\alpha\beta + \beta\gamma + \gamma\alpha = 3$

$\sum_3$: $-(\alpha\beta\gamma) = -(-4) = 4$

The cubic equation is,

$$x^3 - 2x^2 + 3x - 4 = 0$$

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4. Solve the equation $3x^3 - 16x^2 + 23x - 6 = 0$ if the product of two roots is 1.

Solution:

Let the roots be α, β *and* γ

Given that Product of two roots is 1.

$\therefore\ \alpha\beta = 1$

Given $3x^3 - 16x^2 + 23x - 6 = 0$

$\div 3,\quad x^3 - \frac{16}{3}x^2 + \frac{23}{3}x - 2 = 0$

Here $a = 1, b = -\frac{16}{3}, c = \frac{23}{3}$ and $d = -2$

$\sum_1$: $\alpha + \beta + \gamma = \frac{-b}{a} = \frac{16}{3}$ ---------(1)

$\sum_2$: $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{23}{3}$ ---------(2)

$\sum_3$: $\alpha\beta\gamma = \frac{-d}{a} = 2$ ---------(3)

$\gamma = 2$ $[\because \alpha\beta = 1]$

Substitute $\beta = \frac{1}{\alpha}, \gamma = 2$ in (1), we get

$$\alpha + \frac{1}{\alpha} + 2 = \frac{16}{3}$$

$$\frac{\alpha^2+1}{\alpha} = \frac{16}{3} - 2$$

$$\frac{\alpha^2+1}{\alpha} = \frac{10}{3}$$

$$3\alpha^2 + 3 = 10\alpha$$

$$3\alpha^2 - 10\alpha + 3 = 0$$

$$(3\alpha - 1)(\alpha - 3) = 0$$

$$\alpha = \frac{1}{3}, 3$$

$$\begin{array}{c|c} \multicolumn{2}{c}{9} \\ -1 & -9 \\ \hline 3 & 3 \\ \multicolumn{2}{c}{-10} \end{array}$$

When $\alpha = 3, \beta = \frac{1}{3}$

$\therefore$ The roots are $3, \frac{1}{3}, 2$

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5. Find the sum of squares of roots of the equation $2x^4 - 8x^3 + 6x^2 - 3 = 0$

Hint: $a^2 + b^2 + c^2 + d^2 = (a + b + c + d) - 2(ab + bc + cd + da)$

Solution:

Given $2x^4 - 8x^3 + 6x^2 - 3 = 0$

Let the roots be α, β, γ *and* δ

$\sum_1$: $\alpha + \beta + \gamma + \delta = -\frac{b}{a} = \frac{8}{2} = 4$

$\sum_2$: $\alpha\beta + \beta\gamma + \gamma\alpha + \delta\alpha + \delta\beta + \delta\gamma = \frac{c}{a} = \frac{6}{2} = 3$

To Find: $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

$= (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha + \delta\alpha + \delta\beta + \delta\gamma)$

$= 4^2 - 2(3)$

$= 16 - 6 = 10$

$\Rightarrow\ \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 10$

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6. Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$ if it is given that two of its roots are in the ratio 3 : 2.

Solution:

Given $x^3 - 9x^2 + 14x + 24 = 0$

Sum of odd power coefficient $= 1 + 14 = 15$

Sum of even power coefficient $= -9 + 24 = 15$

Both are equal.

$\therefore -1$ is a root

Since the two roots are in the ratio 3 : 2

$\therefore$ The roots are $-1, 3k, 2k$

$$\alpha + 3k + 2k = \frac{-b}{a}$$

$$-1 + 5k = 9$$

$$5k = 10$$

$$k = 2$$

$\therefore$ The roots are $-1, 3(2), 2(2) = -1, 6, 4$

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7. If $\alpha, \beta,$ *and* γ are the roots of the polynomial equation $ax^3 + bx^2 + cx + d = 0$, find the value of $\sum \frac{\alpha}{\beta\gamma}$ in terms of the coefficients.

Solution:

Given $ax^3 + bx^2 + cx + d = 0$

Let the roots be $\alpha, \beta, and\ \gamma$

$\sum_1$: $\alpha+\beta+\gamma=\frac{-b}{a}$

$\sum_2$: $\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$

$\sum_3$: $\alpha\beta\gamma=\frac{-d}{a}$

$$\sum\frac{\alpha}{\beta\gamma}=\frac{\alpha}{\beta\gamma}+\frac{\beta}{\gamma\alpha}+\frac{\gamma}{\alpha\beta}$$

$$=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}$$

$$=\frac{(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)}{\alpha\beta\gamma}$$

$$\sum\frac{\alpha}{\beta\gamma}=\frac{\left(\frac{-b}{a}\right)^2-2\left(\frac{c}{a}\right)}{\left(\frac{-d}{a}\right)}=\frac{\frac{b^2}{a^2}-\frac{2c}{a}}{-\frac{d}{a}}$$

$$\sum\frac{\alpha}{\beta\gamma}=\frac{2ac-b^2}{ad}$$

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8. If $\alpha, \beta, \gamma, and\ \delta$ are the roots of the polynomial equation $2x^4+5x^3-7x^2+8=0$, find a quadratic equation with integer coefficients whose roots are $\alpha+\beta+\gamma+\delta$ and $\alpha\beta\gamma\delta$.

Solution:

Given $2x^4+5x^3-7x^2+8=0$

Let the roots be $\alpha, \beta, \gamma, and\ \delta$

$\sum_1$: $\alpha+\beta+\gamma+\delta=\frac{-b}{a}=\frac{-5}{2}$

$\sum_4$: $\alpha\beta\gamma\delta=\frac{e}{a}=\frac{8}{2}=4$

The quadratic equation of the form is,

$$x^2-x(s.o.r)+p.o.r=0$$

$$s.o.r=\alpha+\beta+\gamma+\delta+\ \alpha\beta\gamma\delta$$

$$=\frac{-5}{2}+4=\frac{-5+8}{2}$$

$$s.o.r=\frac{3}{2}$$

$$p.o.r=(\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma\delta)$$

$$p.o.r=\left(\frac{-5}{2}\right)(4)=-10$$

The required equation is,

$$x^2-\frac{3}{2}x-10=0$$

$$2x^2-3x-20=0$$

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9. If p and q are the roots of the equation $lx^2+nx+n=0$, show that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0$.

Solution:

Given $lx^2+nx+n=0$

$\sum_1$: $p+q=\frac{-b}{a}=\frac{-n}{l}$

$\sum_2$: $pq=\frac{c}{a}=\frac{n}{l}$

$$\text{LHS}=\frac{\sqrt{p}}{\sqrt{q}}+\frac{\sqrt{q}}{\sqrt{p}}+\sqrt{\frac{n}{l}}$$

$$=\frac{p+q}{\sqrt{pq}}+\sqrt{\frac{n}{l}}$$

$$=\frac{\frac{-n}{l}}{\sqrt{\frac{n}{l}}}+\sqrt{\frac{n}{l}}$$

$$=-\sqrt{\frac{n}{l}}+\sqrt{\frac{n}{l}}$$

$$=0$$

$$=\text{RHS}$$

Hence proved.

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10. If the equations $x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common root, show that it must be equal to $\frac{pq'-p'q}{q-q'}$ or $\frac{q-q'}{p'-p}$.

Solution:

Let the common root be k

$$k^2+pk+q=0 \quad \text{----------(1)}$$

$$k^2+p'k+q'=0 \quad \text{----------(2)}$$

Subtracting (1) from (2), we get

$$(p-p')k+(q-q')=0$$

$$(p-p')k=(q'-q)$$

$$k=\frac{(q'-q)}{-(p'-p)}$$

$$k=\frac{q-q'}{p'-p} \quad \text{----------(3)}$$

$p'\times(1)\Rightarrow\ p'k^2+pp'k+qp'=0$

$p\times(2)\Rightarrow\ pk^2+pp'k+pq'=0$

$$(p'-p)k^2+(qp'-pq')=0$$

$$(p'-p)k^2=-(qp'-pq')$$

$$k^2=\frac{pq'-p'q}{p'-p} \quad \text{----------(4)}$$

Dividing (4) by (3), we get

$$\frac{k^2}{k}=\frac{pq'-p'q}{p'-p}\times\frac{p'-p}{q-q'}$$

$$k=\frac{pq'-p'q}{q-q'}$$

$\therefore$ The common root is $\frac{pq'-p'q}{q-q'}$ or $\frac{q-q'}{p'-p}$

11. A 12 metre tall tree was broken into two parts. It was found that the height of the part which was left standing was the cube root of the length of the part that was cut away. Formulate this into a mathematical problem to find the height of the part which was left standing.

Solution:

Let the two parts of tree be x and $12 - x$

Given that $x = \sqrt[3]{12 - x}$

Cubing on both sides, we get

$$x^3 = 12 - x$$

$$x^3 + x - 12 = 0$$

Exercise 3.2

Nature of roots Δ: ($\Delta = b^2 - 4ac$)

- **$\Delta = 0$, the roots are real and equal.**
- **$\Delta < 0$, the roots are imaginary.**
- **$\Delta > 0$, the roots are real and unequal.**
- **$\Delta = perfect\ square$, the roots are rational.**

1. If k is real, discuss the nature of the roots of the polynomial equation $2x^2 + kx + k = 0$, in terms of k.

Solution:

Given $2x^2 + kx + k = 0$

Here $a = 2, b = k$ and $c = k$

$$\Delta = b^2 - 4ac$$

$$= k^2 - 8k$$

$$\Delta = k(k - 8)$$

The critical numbers are 0,8 $\quad -\infty \quad 0 \quad 8 \quad \infty$

Intervals	$\Delta = b^2 - 4ac$	Nature of roots
$(-\infty, 0)$	> 0	Real and unequal
$(0,8)$	< 0	Imaginary
$(8, \infty)$	> 0	Real and unequal

Case 1: If $k < 0$ *and* $k > 8$ then the roots are real and unequal.

Case 2: If $0 < k < 8$ then the roots are imaginary.

Case 3: If $k = 0$ *or* 8 then the roots are real and equal.

2. Find a polynomial equation of minimum degree with rational coefficients, having $2 + \sqrt{3}i$ as a root.

Solution:

Given root $2 + \sqrt{3}i$

∴ The other root is $2 - \sqrt{3}i$

$$s.o.r = 2 + \sqrt{3}i + 2 - \sqrt{3}i = 4$$

$$p.o.r = (2 + \sqrt{3}i)(2 - \sqrt{3}i) = 4 + 3 = 7$$

The required equation is,

$$x^2 - x(s.o.r) + p.o.r = 0$$

$$x^2 - 4x + 7 = 0$$

3. Find a polynomial equation of minimum degree with rational coefficients, having $2i + 3$ as a root.

Solution:

Given root $3 + 2i$

∴ The other root is $3 - 2i$

$$s.o.r = 3 + 2i + 3 - 2i = 6$$

$$p.o.r = (3 + 2i)(3 - 2i) = 9 + 4 = 13$$

The required equation is,

$$x^2 - 6x + 13 = 0$$

4. Find a polynomial equation of minimum degree with rational coefficients, having $\sqrt{5} - \sqrt{3}$ as a root.

Solution:

Given root is $\sqrt{5} - \sqrt{3}$

∴ The other roots are $(\sqrt{5} + \sqrt{3}), (-\sqrt{5} - \sqrt{3})$ and $(-\sqrt{5} + \sqrt{3})$

Consider the roots $(\sqrt{5} - \sqrt{3}), (\sqrt{5} + \sqrt{3})$

$$s.o.r = \sqrt{5} - \sqrt{3} + \sqrt{5} + \sqrt{3} = 2\sqrt{5}$$

$$p.o.r = (\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3}) = 5 - 3 = 2$$

The quadratic equation is $= x^2 - 2\sqrt{5}x + 2$ ----------(1)

Consider the roots $(-\sqrt{5} - \sqrt{3}), (-\sqrt{5} + \sqrt{3})$

$$s.o.r = -\sqrt{5} - \sqrt{3} - \sqrt{5} + \sqrt{3} = -2\sqrt{5}$$

$$p.o.r = (-\sqrt{5} - \sqrt{3})(-\sqrt{5} + \sqrt{3}) = 5 - 3 = 2$$

The quadratic equation is $= x^2 + 2\sqrt{5}x + 2$ ----------(2)

The required equation with rational coefficients is,

$$(x^2 - 2\sqrt{5}x + 2)(x^2 + 2\sqrt{5}x + 2) = 0$$

$$[(x^2 + 2) - 2\sqrt{5}x]\,[(x^2 + 2) + 2\sqrt{5}x] = 0$$

$$(x^2 + 2)^2 - (2\sqrt{5}x)^2 = 0$$

$$x^4 + 4 + 4x^2 - 20x^2 = 0$$

$$x^4 - 16x^2 + 4 = 0$$

5. Prove that a straight line and parabola cannot intersect at more than two points.

Solution:

Let the equation of straight line be $y = mx + c$ -------(1)

Let the equation of parabola be $y^2 = 4ax$ --------(2)

Sub (1) in (2), we get

$$(mx + c)^2 = 4ax$$

$$m^2x^2 + c^2 + 2mcx = 4ax$$

$$m^2x^2 + (2mc - 4a)x + c^2 = 0$$

Which is a quadratic equation.

It cannot have more than two solutions.

Hence line and parabola cannot intersect at more than two points.

Exercise 3.3

1. Solve the cubic equation $2x^3 - x^2 - 18x + 9 = 0$ if sum of two of its roots vanishes.

Solution:

Given $2x^3 - x^2 - 18x + 9 = 0$

Here $a = 2, b = -1, c = -18$ and $d = 0$

Let the roots be $\alpha, -\alpha,$ and β

$\sum_1$: $\alpha - \alpha + \beta = \frac{-b}{a}$

$$\beta = \frac{1}{2}$$

$\sum_3$: $\alpha(-\alpha)\beta = \frac{-d}{a}$

$$-\alpha^2\left(\frac{1}{2}\right) = \frac{-9}{2}$$

$$\alpha^2 = \frac{9}{2} \times 2$$

$$\alpha^2 = 9$$

$$\alpha = \pm 3$$

$\therefore$ The roots are $3, -3, \frac{1}{2}$

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2. Solve the equation $9x^3 - 36x^2 + 44x - 16 = 0$ if the roots form an arithmetic progression.

Solution:

Given $9x^3 - 36x^2 + 44x - 16 = 0$

Here $a = 9, b = -36, c = 44$ and $d = -16$

Let the roots be $\alpha - \beta, \alpha, \alpha + \beta$

$\sum_1$: $\alpha - \beta + \alpha + \alpha + \beta = \frac{-b}{a}$

$$3\alpha = \frac{36}{9}$$

$$\alpha = \frac{4}{3} \qquad \text{----------(1)}$$

$\sum_3$: $(\alpha - \beta)(\alpha)(\alpha + \beta) = \frac{-d}{a}$

$$\left(\frac{4}{3} - \beta\right)\left(\frac{4}{3}\right)\left(\frac{4}{3} + \beta\right) = \frac{16}{9}$$

$$\frac{16}{9} - \beta^2 = \frac{4}{3}$$

$$\beta^2 = \frac{16}{9} - \frac{4}{3} = \frac{4}{9}$$

$$\beta = \pm\frac{2}{3} \qquad \text{----------(2)}$$

At $\alpha = \frac{4}{3}$ and $\beta = \frac{2}{3}$

The roots are $\frac{4}{3} - \frac{2}{3}, \frac{4}{3}, \frac{4}{3} + \frac{2}{3}$

The roots are $\frac{2}{3}, \frac{4}{3}, 2$

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3. Solve the equation $3x^3 - 26x^2 + 52x - 24 = 0$ if its roots form a geometric progression.

Solution:

Given $3x^3 - 26x^2 + 52x - 24 = 0$

Here $a = 3, b = -26, c = 52$ and $d = -24$

Let the roots be $\frac{\alpha}{\beta}, \alpha, \alpha\beta$

$\sum_3$: $\left(\frac{\alpha}{\beta}\right)(\alpha)(\alpha\beta) = \frac{-d}{a}$

$$\alpha^3 = \frac{24}{3} = 8$$

$$\alpha = 2 \qquad \text{----------(1)}$$

$\therefore$ 2 is a root.

$$\begin{array}{c|cccc} 2 & 3 & -26 & 52 & -24 \\ & 0 & 6 & -40 & 24 \\ \hline & 3 & -20 & 12 & \boxed{0} \end{array}$$

$$3x^2 - 20x + 12 = 0$$

$$\begin{array}{c|c} \multicolumn{2}{c}{36} \\ -18 & -2 \\ \hline 3 & 3 \\ \multicolumn{2}{c}{-20} \end{array}$$

$$(x - 6)(3x - 2) = 0$$

$$x = 6, \frac{2}{3}$$

$\therefore$ The roots are $\frac{2}{3}, 2, 6$

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4. Determine k and solve the equation $2x^3 - 6x^2 + 3x + k = 0$ if one of its roots is twice the sum of the other two roots.

Solution:

Given $2x^3 - 6x^2 + 3x + k = 0$

Here $a = 2, b = -6, c = 3$ and $d = k$

Let the roots be $\alpha, \beta, 2(\alpha + \beta)$

$\sum_1$: $\alpha+\beta+2(\alpha+\beta)=\frac{-b}{a}$

$$3\alpha+3\beta=\frac{6}{2}$$

$$3(\alpha+\beta)=3$$

$$\alpha+\beta=1 \qquad \text{----------(1)}$$

Now $2(\alpha+\beta)=2(1)=2$ From (1)

$\therefore$ 2 is a root

$$2(2)^3-6(2)^2+3(2)+k=0$$

$$16-24+6+k=0$$

$$-2+k=0$$

$$\boldsymbol{k=2} \qquad \text{----------(2)}$$

$\therefore$ The equation is $2x^3-6x^2+3x+2=0$

$$\begin{array}{r|rrrr} 2 & 2 & -6 & 3 & 2 \\ & 0 & 4 & -4 & -2 \\ \hline & 2 & -2 & -1 & \boxed{0} \end{array}$$

$$2x^2-2x-1=0$$

Here $a=2, b=-2$ and $c=-1$

$$x=\frac{2\pm\sqrt{4-4(2)(-1)}}{4}$$

$$=\frac{2\pm\sqrt{12}}{4}$$

$$=\frac{2\pm2\sqrt{3}}{4}$$

$$=\frac{2(1\pm\sqrt{3})}{4}$$

$$x=\frac{1\pm\sqrt{3}}{2}$$

$\therefore$ The roots are $2, \frac{1-\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}$ and $k=2$

5. Find all zeros of the polynomial $x^6-3x^5-5x^4+22x^3-39x^2-39x+135$, if it is known that $1+2i$ and $\sqrt{3}$ are two roots.

Solution:

Let $P(x)=x^6-3x^5-5x^4+22x^3-39x^2-39x+135$

Given zeros are $1+2i$ and $\sqrt{3}$

$\therefore$ All the zeros are $1+2i, 1-2i, \sqrt{3}, -\sqrt{3}, \alpha, \beta$

$\sum_1$: $1+2i+1-2i+\sqrt{3}-\sqrt{3}+\alpha+\beta=\frac{-b}{a}$

$$2+\alpha+\beta=3$$

$$\alpha+\beta=1$$

$$\beta=1-\alpha \text{ -----(1)}$$

$\sum_6$: $(1+2i)(1-2i)(\sqrt{3})(-\sqrt{3})\alpha\beta=\frac{g}{a}$

$$(1+4)(-3)\alpha\beta=135$$

$$\alpha\beta=\frac{-135}{15}$$

$$\alpha(1-\alpha)=-9 \qquad \text{From (1)}$$

$$\alpha-\alpha^2+9=0$$

$$\alpha^2-\alpha-9=0$$

Here $a=1, b=-1$ and $c=-9$

$$\alpha=\frac{1\pm\sqrt{1-4(1)(-9)}}{2}$$

$$\alpha=\frac{1\pm\sqrt{37}}{2}$$

When $\alpha=\frac{1+\sqrt{37}}{2}$ then $\beta=\frac{1-\sqrt{37}}{2}$

When $\alpha=\frac{1-\sqrt{37}}{2}$ then $\beta=\frac{1+\sqrt{37}}{2}$

$\therefore$ The roots are $1+2i, 1-2i, \sqrt{3}, -\sqrt{3}, \frac{1+\sqrt{37}}{2}, \frac{1-\sqrt{37}}{2}$

6. Solve the cubic equations:

(i) $2x^3-9x^2+10x=3$

Solution:

Given $2x^3-9x^2+10x-3=0$

Sum of all coefficients $=2-9+10-3=0$

$\therefore$ 1 is a root

$$\begin{array}{r|rrrr} 1 & 2 & -9 & 10 & -3 \\ & 0 & 2 & -7 & 3 \\ \hline & 2 & -7 & 3 & \boxed{0} \end{array}$$

$$2x^2-7x+3=0$$

$$(2x-1)(x-3)=0$$

$$x=\frac{1}{2} \text{ or } x=3$$

$$\begin{array}{c|c} \multicolumn{2}{c}{6} \\ -1 & -6 \\ \hline 2 & 2 \\ \multicolumn{2}{c}{-7} \end{array}$$

$\therefore$ The roots are $1, \frac{1}{2}, 3$

(ii) $8x^3-2x^2-7x+3=0$

Solution:

Given $8x^3-2x^2-7x+3=0$

Sum of odd power coefficient $=8-7=1$

Sum of even power coefficient $=-2+3=1$

Both are equal.

$\therefore -1$ is a root

$$\begin{array}{r|rrrr} -1 & 8 & -2 & -7 & 3 \\ & 0 & -8 & 10 & -3 \\ \hline & 8 & -10 & 3 & \boxed{0} \end{array}$$

$$\begin{array}{c|c} \multicolumn{2}{c}{24} \\ -6 & -4 \\ \hline 8 & 8 \\ \multicolumn{2}{c}{-10} \end{array}$$

$8x^2 - 10x + 3 = 0$

$(4x - 3)(2x - 1) = 0$

$x = \frac{3}{4}$ or $x = \frac{1}{2}$

$\therefore$ The roots are $-1, \frac{1}{2}, \frac{3}{4}$

7. Solve the equation $x^4 - 14x^2 + 45 = 0$

Solution:

Given $x^4 - 14x^2 + 45 = 0$

Let $x^2 = y$

$$\begin{array}{c|c} \multicolumn{2}{c}{45} \\ -9 & -5 \\ \hline \multicolumn{2}{c}{-14} \end{array}$$

$y^2 - 14y + 45 = 0$

$(y - 9)(y - 5) = 0$

$y = 9 \; or \; y = 5$

When $y = 9$,

$x^2 = 9 = \pm 3$

When $y = 5$,

$x^2 = 5 = \pm\sqrt{5}$

$\therefore$ The roots are $3, -3, \sqrt{5}, -\sqrt{5}$

Exercise 3.4

1. Solve (i) $(x - 5)(x - 7)(x + 6)(x + 4) = 504$

Solution:

Given $(x - 5)(x - 7)(x + 6)(x + 4) = 504$

$(x - 5)(x + 4)\,(x - 7)(x + 6) = 504$

$(x^2 - x - 20)\,(x^2 - x - 42) = 504$

Put $y = x^2 - x$, we get

$(y - 20)(y - 42) = 504$

$y^2 - 62y + 840 - 504 = 0$

$$\begin{array}{c|c} \multicolumn{2}{c}{336} \\ -56 & -6 \\ \hline \multicolumn{2}{c}{-62} \end{array}$$

$y^2 - 62y + 336 = 0$

$(y - 56)(y - 6) = 0$

$y = 56 \; or \; y = 6$

When $y = 56$,

$$\begin{array}{c|c} \multicolumn{2}{c}{-56} \\ -8 & 7 \\ \hline \multicolumn{2}{c}{-1} \end{array}$$

$x^2 - x = 56$

$x^2 - x - 56 = 0$

$(x - 8)(x + 7) = 0$

$x = 8 \; or \; x = -7$

When $y = 6$,

$$\begin{array}{c|c} \multicolumn{2}{c}{-6} \\ -3 & 2 \\ \hline \multicolumn{2}{c}{-1} \end{array}$$

$x^2 - x = 6$

$x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

$x = 3 \; or \; x = -2$

$\therefore$ The roots are $-7, -2, 3, 8$

(ii) $(x - 4)(x - 7)(x - 2)(x + 1) = 16$

Solution:

Given $(x - 4)(x - 7)(x - 2)(x + 1) = 16$

$(x - 4)(x - 2)(x - 7)(x + 1) = 16$

$(x^2 - 6x + 8)\,(x^2 - 6x - 7) = 16$

Put $y = x^2 - 6x$, we get

$(y + 8)(y - 7) = 16$

$y^2 + y - 56 - 16 = 0$

$$\begin{array}{c|c} \multicolumn{2}{c}{-72} \\ -8 & 9 \\ \hline \multicolumn{2}{c}{1} \end{array}$$

$y^2 + y - 72 = 0$

$(y - 8)(y + 9) = 0$

$y = 8 \; or \; y = -9$

When $y = -9$,

$$\begin{array}{c|c} \multicolumn{2}{c}{9} \\ -3 & -3 \\ \hline \multicolumn{2}{c}{-6} \end{array}$$

$x^2 - 6x = -9$

$x^2 - 6x + 9 = 0$

$(x - 3)(x - 3) = 0$

$x = 3, 3$

When $y = 8$,

$x^2 - 6x = 8$

$x^2 - 6x - 8 = 0$

Here $a = 1, b = -6 \; and \; c = -8$

$x = \frac{6 \pm \sqrt{36 - 4(1)(-8)}}{2(1)}$

$x = \frac{6 \pm \sqrt{68}}{2}$

$x = \frac{6 \pm 2\sqrt{17}}{2}$

$x = 3 \pm \sqrt{17}$

$\therefore$ The roots are $3, 3, 3 + \sqrt{17}, 3 - \sqrt{17}$

2. Solve $(2x - 1)(x + 3)(x - 2)(2x + 3) + 20 = 0$

Solution:

Given $(2x-1)(x+3)(x-2)(2x+3)+20=0$

$$(2x-1)(2x+3)(x+3)(x-2)+20=0$$

$$(4x^2+6x-2x-3)(x^2+x-6)+20=0$$

$$(4(x^2+x)-3)\,(x^2+x-6)+20=0$$

Put $y=x^2+x$, we get

$$(4y-3)(y-6)+20=0$$

$$4y^2-24y-3y+18+20=0$$

$$4y^2-27y+38=0$$

$$(4y-19)(y-2)=0$$

$$\begin{array}{c|c} \multicolumn{2}{c}{152} \\ -19 & -8 \\ \hline 4 & 4 \\ \multicolumn{2}{c}{-27} \end{array}$$

$$y=\frac{19}{4}\ or\ y=2$$

When $y=2$,

$$x^2+x-2=0$$

$$(x-1)(x+2)=0$$

$$x=1\ or\ x=-2$$

$$\begin{array}{c|c} \multicolumn{2}{c}{-2} \\ 2 & -1 \\ \hline & \\ \multicolumn{2}{c}{1} \end{array}$$

When $y=\frac{19}{4}$,

$$x^2+x-\frac{19}{4}=0$$

$$4x^2+4x-19=0$$

Here $a=4, b=4\ and\ c=-19$

$$x=\frac{-4\pm\sqrt{16-4(4)(-19)}}{2(4)}$$

$$x=\frac{-4\pm\sqrt{16+304}}{8}=\frac{-4\pm\sqrt{320}}{8}$$

$$x=\frac{-4\pm8\sqrt{5}}{8}$$

$$x=\frac{-1\pm2\sqrt{5}}{2}$$

$\therefore$ The roots are $1,-2,\frac{-1+2\sqrt{5}}{2},\frac{-1-2\sqrt{5}}{2}$

Exercise 3.5

1. solve the following equations:

(i) $\sin^2 x-5\sin x+4=0$

Solution:

Given $\sin^2 x-5\sin x+4=0$

Let $sinx=y$

$$y^2-5y+4=0$$

$$(y-1)(y-4)=0$$

$$y=1\ or\ y=4$$

$$\begin{array}{c|c} \multicolumn{2}{c}{4} \\ -1 & -4 \\ \hline & \\ \multicolumn{2}{c}{-5} \end{array}$$

When $y=4$,

$$sinx=4\ (not\ possible)$$

When $y=1$,

$$sinx=1$$

$$sinx=\sin\frac{\pi}{2}$$

$$x=n\pi+(-1)^n\frac{\pi}{2}\ \forall\, n\in Z$$

(ii) $12x^3+8x=29x^2-4$

Solution:

Let $P(x)=12x^3-29x^2+8x+4$

$$p(1)=12-29+8+4=-5\neq0$$

$$p(2)=12(2)^3-29(2)^2+8(2)+4$$

$$=12(8)-29(4)+16+4$$

$$=96-116+20$$

$$P(2)=0$$

$\therefore$ 2 is a root

$$\begin{array}{c|cccc} 2 & 12 & -29 & 8 & 4 \\ & 0 & 24 & -10 & -4 \\ \hline & 12 & -5 & -2 & \boxed{0} \end{array}$$

$$12x^2-5x-2=0$$

$$(3x-2)(4x+1)=0$$

$$x=\frac{2}{3},-\frac{1}{4}$$

$$\begin{array}{c|c} \multicolumn{2}{c}{-24} \\ -8 & 3 \\ \hline 12 & 12 \\ \multicolumn{2}{c}{-5} \end{array}$$

$\therefore$ The roots are $2,\frac{2}{3},\frac{-1}{4}$

---

2. Examine for the rational roots of

(i) $2x^3-x^2-1=0$

Solution:

Given $2x^3-x^2-1=0$

Sum of all coefficients $=2-1-1=0$

$\therefore$ 1 is a root

$$\begin{array}{c|cccc} 1 & 2 & -1 & 0 & -1 \\ & 0 & 2 & 1 & 1 \\ \hline & 2 & 1 & 1 & \boxed{0} \end{array}$$

$$2x^2+x+1=0$$

Here $a=2, b=1\ and\ c=1$

$$\Delta=b^2-4ac$$

$$=1-4(2)(1)$$

$$= 1 - 8$$

$$\Delta = -7 < 0$$

$\therefore$ The roots are imaginary

Hence the rational root is 1

$(ii)\ x^8 - 3x + 1 = 0$

Solution:

Let $P(x) = x^8 - 3x + 1 = 0$

Here $a_n = 1\ and\ a_0 = 1$

$$p \text{ is a factor of } a_0 = \pm 1$$

$$q \text{ is a factor of } a_n = \pm 1$$

$$\frac{p}{q} = \pm 1$$

$\therefore$ The possible values are $1\ and - 1$

$$P(1) = 1 - 3 + 1 = -1 \neq 0$$

$$P(-1) = 1 + 3 + 1 = 5 \neq 0$$

$\therefore$ The equation has no rational roots

---

3. Solve $8x^{\frac{3}{2n}} - 8x^{\frac{-3}{2n}} = 63$

Solution:

Given $8x^{\frac{3}{2n}} - 8x^{\frac{-3}{2n}} = 63$

$$8x^{\frac{3}{2n}} - 8x^{\left(\frac{3}{2n}\right)^{-1}} = 63$$

Let $y = x^{\frac{3}{2n}}$

$$8y - 8y^{-1} = 63$$

$$8y - \frac{8}{y} = 63$$

$$8y^2 - 8 = 63y$$

$$8y^2 - 63y - 8 = 0$$

$$(8y + 1)(y - 8) = 0$$

$$y = \frac{-1}{8}\ or\ y = 8$$

$$\begin{array}{c|c} \multicolumn{2}{c}{-64} \\ 1 & -64 \\ \hline 8 & 8 \\ \multicolumn{2}{c}{-63} \end{array}$$

When $y = 8$,

$$x^{\frac{3}{2n}} = 8 = 2^3$$

$$x = (2^3)^{\frac{2n}{3}}$$

$$x = 2^{2n}$$

$$x = 4^n$$

When $y = -\frac{1}{8}$,

$$x^{\frac{3}{2n}} = -\frac{1}{2^3}$$

$$x = \left(\frac{-1}{2^3}\right)^{\frac{2n}{3}}$$

$$x = \frac{(-1)^{\frac{2n}{3}}}{(2^3)^{\frac{2n}{3}}} = \frac{1}{4^n}\ (not\ possible)$$

$\therefore$ The only possible solution is 4^n.

---

4. Solve $2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}$.

Solution:

Let $y = \sqrt{\frac{x}{a}}$

$$2y + \frac{3}{y} = \frac{b^2 + 6a^2}{ab}$$

$$\frac{2y^2 + 3}{y} = \frac{b^2 + 6a^2}{ab}$$

$$(2y^2 + 3)ab = (b^2 + 6a^2)y$$

$$2aby^2 + 3ab = (b^2 + 6a^2)$$

$$2aby^2 - (b^2 + 6a^2)y + 3ab = 0$$

$$\begin{array}{c|c} \multicolumn{2}{c}{\boldsymbol{6a^2b^2}} \\ \boldsymbol{-b^2} & \boldsymbol{-6a^2} \\ \hline \boldsymbol{2ab} & \boldsymbol{2ab} \\ \multicolumn{2}{c}{\boldsymbol{-(b^2 + 6a^2)}} \end{array}$$

$$(2aby - b^2)(2aby - 6a^2) = 0$$

$$2aby - b^2 = 0$$

$$2aby = b^2$$

$$y = \frac{b^2}{2ab}$$

$$y = \frac{b}{2a}$$

$$2aby - 6a^2 = 0$$

$$2aby = 6a^2$$

$$y = \frac{6a^2}{2ab}$$

$$y = \frac{3a}{b}$$

When $y = \frac{b}{2a}$,

$$\sqrt{\frac{x}{a}} = \frac{b}{2a}$$

Squaring on both sides, we get

$$\frac{x}{a} = \frac{b^2}{4a^2}$$

$$x = \frac{b^2}{4a} \qquad \text{----------(1)}$$

When $y = \frac{3a}{b}$,

$$\sqrt{\frac{x}{a}} = \frac{3a}{b}$$

Squaring on both sides, we get

$$\frac{x}{a} = \frac{9a^2}{b^2}$$

$$x = \frac{9a^3}{b^2} \quad \text{----------(2)}$$

$\therefore$ The solution is $\frac{b^2}{4a}, \frac{9a^3}{b^2}$

---

5. Solve the equations

(i) $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$

Solution:

Given $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$

$\div x^2$, $\quad 6x^2 - 35x + 62 - \frac{35}{x} + \frac{6}{x^2} = 0$

$$6\left(x^2 + \frac{1}{x^2}\right) - 35\left(x + \frac{1}{x}\right) + 62 = 0 \quad \text{----------(1)}$$

Let $y = x + \frac{1}{x}$ ----------(2)

Squaring on both sides, we get

$$y^2 = \left(x + \frac{1}{x}\right)^2$$

$$y^2 = x^2 + \frac{1}{x^2} + 2$$

$$y^2 - 2 = x^2 + \frac{1}{x^2} \quad \text{----------(3)}$$

Sub (2), (3) in (1), we get

$$6(y^2 - 2) - 35y + 62 = 0$$

$$6y^2 - 12 - 35y + 62 = 0$$

$$6y^2 - 35y + 50 = 0$$

$$\begin{array}{c|c} \multicolumn{2}{c}{300} \\ -15 & -20 \\ \hline 6 & 6 \\ \multicolumn{2}{c}{-35} \end{array}$$

$$(2y - 5)(3y - 10) = 0$$

$$y = \frac{5}{2} \text{ or } y = \frac{10}{3}$$

When $y = \frac{5}{2}$,

$$x + \frac{1}{x} = \frac{5}{2}$$

$$2(x^2 + 1) = 5x$$

$$2x^2 - 5x + 2 = 0$$

$$\begin{array}{c|c} \multicolumn{2}{c}{4} \\ -1 & -4 \\ \hline 2 & 2 \\ \multicolumn{2}{c}{-5} \end{array}$$

$$(2x - 1)(x - 2) = 0$$

$$x = \frac{1}{2} \text{ or } x = 2$$

When $y = \frac{10}{3}$,

$$x + \frac{1}{x} = \frac{10}{3}$$

$$3(x^2 + 1) = 10x$$

$$3x^2 - 10x + 3 = 0$$

$$\begin{array}{c|c} \multicolumn{2}{c}{9} \\ -1 & -9 \\ \hline 3 & 3 \\ \multicolumn{2}{c}{-10} \end{array}$$

$$(3x - 1)(x - 3) = 0$$

$$x = \frac{1}{3} \text{ or } x = 3$$

$\therefore$ The roots are $2, \frac{1}{2}, 3, \frac{1}{3}$

(ii) $x^4 + 3x^3 - 3x - 1 = 0$

Solution:

Given $x^4 + 3x^3 - 3x - 1 = 0$

Sum of all coefficients $= 1 + 3 - 3 - 1 = 0$

$\therefore$ 1 is a root

Sum of odd power coefficients $= 1 - 1 = 0$

Sum of even power coefficients $= 3 - 3 = 0$

Both are equal

$\therefore -1$ is a root

$$\begin{array}{r|rrrrr} 1 & 1 & 3 & 0 & -3 & -1 \\ & 0 & 1 & 4 & 4 & 1 \\ \hline -1 & 1 & 4 & 4 & 1 & \boxed{0} \\ & 0 & -1 & -3 & -1 & \\ \hline & 1 & 3 & 1 & \boxed{0} & \end{array}$$

$$x^2 + 3x + 1 = 0$$

Here $a = 1, b = 3$ and $c = 1$

$$x = \frac{-3 \pm \sqrt{9-4}}{2}$$

$$x = \frac{-3 \pm \sqrt{5}}{2}$$

$\therefore$ The roots are $1, -1, \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$

---

6. Find all real numbers satisfying $4^x - 3(2^{x+2}) + 2^5 = 0$

Solution:

Given $4^x - 3(2^{x+2}) + 2^5 = 0$

$$2^{2x} - 3(2^x.2^2) + 2^5 = 0$$

Let $y = 2^x$

$$y^2 - 3(4y) + 2^5 = 0$$

$$y^2 - 12y + 32 = 0$$

$$\begin{array}{c|c} \multicolumn{2}{c}{32} \\ -4 & -8 \\ \hline & \\ \multicolumn{2}{c}{-12} \end{array}$$

$$(y - 4)(y - 8) = 0$$

$$y = 4 \text{ or } y = 8$$

When $y = 4$,

$$2^x = 4 = 2^2$$

$$x = 2$$

When $y = 8$,

$$2^x = 8 = 2^3$$

$$x = 3$$

$\therefore$ The roots are 2,3

7. Solve the equation $6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0$ if it is known that $\frac{1}{3}$ is a solution.

Solution:

Given $\frac{1}{3}$ is a solution.

$\therefore$ The other root is 3 [$\because$ It is a reciprocal equation]

$\frac{1}{3}$	6	−5	−38	−5	6
	0	2	−1	−13	−6
3	6	−3	−39	−18	0
	0	18	45	18	
	6	15	6	0	

$$6x^2 + 15x + 6 = 0$$

$\div 3,$ $$2x^2 + 5x + 2 = 0$$

	4	
	1	4
	2	2
	5	

$$(2x + 1)(x + 2) = 0$$

$$x = \frac{-1}{2} \quad or \quad x = -2$$

$\therefore$ The roots are $3, \frac{1}{3}, -2, \frac{-1}{2}$

Exercise 3.6

1. Discuss the maximum possible number of positive and negative roots of the polynomial equation $9x^9 - 4x^8 + 4x^7 - 3x^6 + 2x^5 + x^3 + 7x^2 + 7x + 2 = 0$.

Solution:

Let $P(x) = 9x^9 - 4x^8 + 4x^7 - 3x^6 + 2x^5 + x^3 + 7x^2 + 7x + 2 = 0$

Number of sign changes $= 4$

Maximum number of positive roots $= 4$

$P(-x) = -9x^9 - 4x^8 - 4x^7 - 3x^6 - 2x^5 - x^3 + 7x^2 - 7x + 2 = 0$

Number of sign changes $= 3$

Maximum number of negative roots $= 3$

Hence, It has atmost four positive roots and atmost three negative roots.

2. Discuss the maximum possible number of positive and negative zeros of the polynomials $x^2 - 5x + 6$ and $x^2 - 5x + 16$. Also draw rough sketch of the graphs.

Solution:

Let $P(x) = x^2 - 5x + 6$

Number of sign changes $= 2$

Maximum number of positive zeros $= 2$

Now $P(-x) = x^2 + 5x + 6$

Number of sign changes $= 0$

It has no negative zeros

Let $q(x) = x^2 - 5x + 16$

Number of sign changes $= 2$

Maximum number of positive zeros $= 2$

Now $q(-x) = x^2 + 5x + 16$

Number of sign changes $= 0$

It has no negative zeros

Now $y = x^2 - 5x + 6$

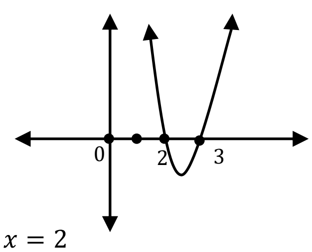
At $x = 0, y = 6$

At $y = 0,$

$$x^2 - 5x + 6 = 0$$

$$(x - 3)(x - 2) = 0$$

$$x = 3 \text{ or } x = 2$$



Hence, It has atmost two positive roots and no negative roots.

3. Show that the equation $x^9 - 5x^5 + 4x^4 + 2x^2 + 1 = 0$ has atleast 6 imaginary solutions.

Solution:

Let $P(x) = x^9 - 5x^5 + 4x^4 + 2x^2 + 1 = 0$

Number of sign changes $= 2$

Maximum number of positive roots $= 2$

$P(-x) = -x^9 + 5x^5 + 4x^4 + 2x^2 + 1 = 0$

Number of sign changes $= 1$

Maximum number of negative roots $= 1$

Now $P(0) = 1 \neq 0$

$\therefore$ 0 is not a root

Total number of roots $= 9$

Number of positive and negative roots $= 2 + 1 = 3$

Imaginary roots $= 9 - 3 = 6$

Hence the given equation has atleast 6 imaginary roots.

4. Determine the number of positive and negative roots of the equation $x^9 - 5x^8 - 14x^7 = 0$.

Solution:

Let $P(x) = x^9 - 5x^8 - 14x^7 = 0$

Number of sign changes $= 1$

Maximum number of positive roots $= 1$

Now $P(-x) = -x^9 - 5x^8 + 14x^7 = 0$

Number of sign changes $= 1$

Maximum number of negative roots $= 1$

Hence, it has one positive real root and one negative real root.

---

5. Find the exact number of real zeros and imaginary of the polynomial $x^9 + 9x^7 + 7x^5 + 5x^3 + 3x$.

Solution:

Let $P(x) = x^9 + 9x^7 + 7x^5 + 5x^3 + 3x$

Number of sign changes $= 0$

Maximum number of positive zeros $= 0$

Now $P(-x) = -x^9 - 9x^7 - 7x^5 - 5x^3 - 3x$

Number of sign changes $= 0$

Maximum number of negative zeros $= 0$

Now $P(0) = 0$

$\therefore$ 0 is a zero

$\Rightarrow$ Maximum number of real zeros $= 1$

$\Rightarrow$ Total number of zeros $= 9$

$\Rightarrow$ Imaginary zeros $= 9 - 1 = 8$

$\therefore$ The given equation has 8 imaginary zeros and it has no positive real roots and no negative real roots.
