CHAPTER - 2

Complex Numbers

Exercise 2.1

Hint: Powers of imaginary unit i:

- $(i)^0 = 1,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1$
- $(i)^n = (i)^{4q+k} = i^k$
- $i + i^2 + i^3 + i^4 = 0$

Simplify the following:

1. $i^{1947} + i^{1950}$

Solution:

$$i^{1947} = i^{4(486)+3} = i^3 = -i$$

$$i^{1950} = i^{4(487)+2} = i^2 = -1$$

$$i^{1947} + i^{1950} = -i - 1$$

2. $i^{1948} - i^{-1869}$

Solution:

$$i^{1948} = i^{4(487)+0} = i^0 = 1$$

$$i^{-1869} = i^{-[4(467)+1]} = i^{-1} = \frac{1}{i}$$

$$= \frac{1}{i} \times \frac{i}{i}$$

$$= \frac{i}{i^2}$$

$$i^{-1869} = -i$$

$$i^{1948} - i^{-1869} = 1 - (-i) = 1 + i$$

3. $\sum_{n=1}^{12} i^n$

Solution:

$$\sum_{n=1}^{12} i^n = (i^1 + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + (i^9 + i^{10} + i^{11} + i^{12})$$

$$= (i - 1 - i + 1) + (i - 1 - i + 1) + (i - 1 - i + 1)$$

$$= 0$$

4. $i^{59} + \frac{1}{i^{59}}$

Solution:

$$i^{59} = i^{4(4)+3} = i^3 = -i$$

$$\frac{1}{i^{59}} = \frac{1}{-i} = \frac{1}{-i} \times \frac{-i}{-i}$$

$$= \frac{-i}{i^2} = \frac{-i}{-1}$$

$$\frac{1}{i^{59}} = i$$

$$\Rightarrow \quad i^{59} + \frac{1}{i^{59}} = -i + i = 0$$

5. $ii^2i^3 \ldots i^{2000}$

Solution:

$$ii^2i^3 \ldots i^{2000} = i^{1+2+3\ldots+2000}$$

$$= i^{\frac{2000(2001)}{2}}$$

$$= (i^{1000})^{2001}$$

$$= (i^{4(250)+0})^{2001}$$

$$= (i^0)^{2001} = 1^{2001}$$

$$ii^2i^3 \ldots i^{2000} = 1$$

6. $\sum_{n=1}^{10} i^{n+50}$

Solution:

$$\sum_{n=1}^{10} i^{n+50} = \sum_{n=1}^{10} i^n\, i^{50}$$

$$= \sum_{n=1}^{10} i^n\, i^{4(12)+2}$$

$$= i^2 \sum_{n=1}^{10} i^n$$

$$= i^2 \left[\begin{array}{c} (i^1 + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) \\ +(i^9 + i^{10}) \end{array} \right]$$

$$= -1[0 + 0 + i - 1]$$

$$= 1 - i$$

$$\sum_{n=1}^{10} i^{n+50} = 1 - i$$

Exercise 2.2

Hint:

- **Rectangular form of complex number is $x + iy$**
- **Two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$**

1. Evaluate the following if $z = 5 - 2i$ and $w = -1 + 3i$

(i) $z + w$	**(ii) $z - iw$**	**(iii) $2z + 3w$**
(iv) zw	**(v) $z^2 + 2wz + w^2$**	**(vi) $(z + w)^2$**

Solutions:

Given $z = 5 - 2i$ and $w = -1 + 3i$

(i) $z + w$

$$z + w = 5 - 2i - 1 + 3i$$

$$z + w = 4 + i$$

(ii) $z - iw$

$$z - iw = (5 - 2i) - i(-1 + 3i)$$

$$= 5 - 2i + i + 3 \qquad [\because i^2 = -1]$$

$$z - iw = 8 - i$$

(iii) $2z + 3w$

$$2z + 3w = 2(5 - 2i) + 3(-1 + 3i)$$
$$= 10 - 4i - 3 + 9i$$
$$2z + 3w = 7 + 5i$$

(iv) zw

$$zw = (5 - 2i)(-1 + 3i)$$
$$= 5(-1) + 5(3i) - 2i(-1) - 2i(3i)$$
$$= -5 + 15i + 2i + 6$$
$$zw = 1 + 17i$$

(v) $z^2 + 2wz + w^2$

$$z^2 + 2wz + w^2 = (z + w)^2$$
$$= (4 + i)^2$$
$$= 16 + 8i - 1$$
$$z^2 + 2wz + w^2 = 15 + 8i$$

(vi) $(z + w)^2$

$$(z + w)^2 = (4 + i)^2 = 15 + 8i$$

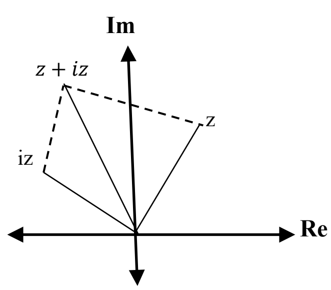
2. Given the complex number $z = 2 + 3i$, represent the complex numbers in Argand diagram.

(i) $z, iz \text{ and } z + iz$ **(ii) $z, -iz \text{ and } z - iz$**

Solutions:

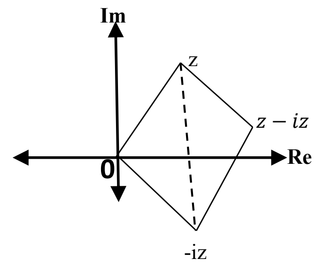
(i) $z, iz \text{ and } z + iz$

$$z = 2 + 3i$$
$$iz = i(2 + 3i) = 2i - 3$$
$$z + iz = 2 + 3i + 2i - 3 = -1 + 5i$$



(ii) $z, -iz \text{ and } z - iz$

$$z = 2 + 3i$$
$$-iz = -i(2 + 3i) = -2i + 3$$
$$z - iz = 2 + 3i - 2i + 3 = 5 + i$$



3. Find the values of the real numbers x and y, if the complex numbers $(3 - i)x - (2 - i)y + 2i + 5$ and $2x + (-1 + 2i)y + 3 + 2i$ are equal.

Solution:

Let $z_1 = (3 - i)x - (2 - i)y + 2i + 5$

$$= 3x - ix - 2y + iy + 2i + 5$$

$$z_1 = (3x - 2y + 5) + i(-x + y + 2) \quad \text{----------(1)}$$

Let $z_2 = 2x + (-1 + 2i)y + 3 + 2i$

$$= 2x - y + i2y + 3 + 2i$$

$$z_2 = (2x - y + 3) + i(2y + 2) \quad \text{----------(2)}$$

Given that the complex numbers are equal.

$\therefore \quad Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$

$$3x - 2y + 5 = 2x - y + 3$$
$$x - y = -2 \quad \text{----------(3)}$$
$$-x + y + 2 = 2y + 2$$
$$x + y = 0 \quad \text{----------(4)}$$

Add (3) and (4) we get,

$$x - y = -2$$
$$x + y = 0$$
$$2x = -2$$
$$x = -1$$

Sub $x = -1$ in (1) we get,

$$-1 - y = -2$$
$$y = 1$$

Hence $x = -1$ and $y = 1$

Exercise 2.3

1. If $z_1 = 1 - 3i$, $z_2 = -4i$, and $z_3 = 5$, show that

(i) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(ii) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

Solutions:

Given $z_1 = 1 - 3i, z_2 = -4i \; and z_3 = 5$

(i) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

LHS = $(z_1 + z_2) + z_3$

$= (1 - 3i - 4i) + 5$

$= 1 - 7i + 5$

$= 6 - 7i$ ----------(1)

RHS $= z_1 + (z_2 + z_3)$

$= 1 - 3i + (-4i + 5)$

$= 6 - 7i$ ----------(2)

From (1) and (2) we get,

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

Hence proved.

(ii) $(z_1 z_2)z_3 = z_1(z_2 z_3)$

LHS = $(z_1 z_2)z_3$

$= [(1 - 3i)(-4i)](5)$

$= (-4i - 12)(5)$

$= -20i - 60$ ----------(1)

RHS $= z_1(z_2 z_3)$

$= (1 - 3i)[(-4i)(5)]$

$= (1 - 3i)(-20i)$

$= -20i - 60$ ----------(2)

From (1) and (2) we get,

$(z_1 z_2)z_3 = z_1(z_2 z_3)$

Hence proved.

---

2. If $z_1 = 3, z_2 = -7i, and \; z_3 = 5 + 4i$, show that

(i) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Solution:

Given $z_1 = 3, z_2 = -7i, and \; z_3 = 5 + 4i$

LHS $= z_1(z_2 + z_3)$

$= (3)[-7i + 5 + 4i]$

$= 3(5 - 3i)$

$= 15 - 9i$ ----------(1)

RHS $= z_1 z_2 + z_1 z_3$

$= 3(-7i) + 3(5 + 4i)$

$= -21i + 15 + 12i$

$= 15 - 9i$ ----------(2)

From (1) and (2) we get,

$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Hence proved.

(ii) $(z_1 + z_2)z_3 = z_1 z_3 + z_2 z_3$

Solution:

Given $z_1 = 3, z_2 = -7i, and \; z_3 = 5 + 4i$

LHS = $(z_1 + z_2)z_3$

$= (3 - 7i)(5 + 4i)$

$= 15 + 12i - 35i + 28$

$= 43 - 23i$ ----------(1)

RHS $= z_1 z_3 + z_2 z_3$

$= 3(5 + 4i) + (-7i)(5 + 4i)$

$= 15 + 12i - 35i + 28$

$= 43 - 23i$ ----------(2)

From (1) and (2) we get,

$(z_1 + z_2)z_3 = z_1 z_3 + z_2 z_3$

Hence proved.

---

3. If $z_1 = 2 + 5i, z_2 = -3 - 4i, and \; z_3 = 1 + i$, find the additive and multiplicative inverse of $z_1, z_2, and \; z_3$.

Hint:

- **Additive inverse: $z + (-z) = (-z) + z = 0$**
  $(-z$ *is called the additive inverse of* $z)$
- **Multiplicative inverse: $zw = wz = 1$**
  (*w is called the multiplicative inverse of z. w is denoted by z^{-1} or $\frac{1}{z}$.*)

Solution:

Given $z_1 = 2 + 5i, z_2 = -3 - 4i, and \; z_3 = 1 + i$

(i) **$z_1 = 2 + 5i$**

The additive inverse of z_1:

$z_1 = -z_1 = -(2 + 5i) = -2 - 5i$

The multiplicative inverse of z_1:

$$z_1 = \frac{1}{z_1} = \frac{1}{2+5i}$$

$$= \frac{1}{2+5i} \times \frac{2-5i}{2-5i}$$

$$\frac{1}{z_1} = \frac{2-5i}{4+25}$$

$$\frac{1}{z_1} = \frac{2-5i}{29}$$

$(ii)\ z_2 = -3-4i$

The additive inverse of z_2:

$$z_2 = -z_2 = -(-3-4i) = 3+4i$$

The multiplicative inverse of z_2:

$$z_2 = \frac{1}{z_2}$$

$$= \frac{1}{-(3+4i)}$$

$$= \frac{-1}{3+4i} \times \frac{3-4i}{3-4i}$$

$$\frac{1}{z_2} = \frac{-3+4i}{9+16} = \frac{-3+4i}{25}$$

$(iii)\ z_3 = 1+i$

The additive inverse of z_3:

$$z_3 = -z_3 = -1-i$$

The multiplicative inverse of z_3:

$$z_3 = \frac{1}{z_3} = \frac{1}{1+i}$$

$$\frac{1}{z_3} = \frac{1}{1+i} \times \frac{1-i}{1-i}$$

$$\frac{1}{z_3} = \frac{1-i}{1+1} = \frac{1-i}{2}$$

Exercise 2.4

Properties:

1. $\boldsymbol{if\ z = x+iy\ then\ \bar{z} = x-iy}$
2. $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$
3. $\overline{z_1-z_2} = \overline{z_1} - \overline{z_2}$
4. $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
5. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$
6. $Re(z) = \frac{z+\bar{z}}{2}$
7. $Im(z) = \frac{z-\bar{z}}{2i}$
8. $\bar{\bar{z}} = z$
9. $\overline{z^n} = \bar{z}^n$
10. $z\ is\ real\ if\ and\ only\ if\ z = \bar{z}$
11. $z\ is\ purely\ imaginary\ if\ and\ only\ if\ z = -\bar{z}$

1. Write the following in the rectangular form:

(i) $\overline{(5+9\imath)+(2-4\imath)}$

Solution:

Let $z = \overline{(5+9\imath)+(2-4\imath)}$

$z = \overline{5+9\imath} + \overline{2-4\imath}$ [by Prop 2]

$z = 5-9i+2+4i$ [by Prop 1]

$z = 7-5i$

(ii) $\frac{10-5i}{6+2i}$

Solution:

Let $z = \frac{10-5i}{6+2i}$

$$z = \frac{10-5i}{6+2i} \times \frac{6-2i}{6-2i}$$

$$= \frac{[10(6)+10(-2i)+(-5i)6+(-5i)(-2i)]}{36+4}$$

$$= \frac{(60-20i-30i-10)}{40}$$

$$= \frac{50-50i}{40}$$

$$z = \frac{10(5-5i)}{40} = \frac{5-5i}{4}$$

$$z = \frac{5}{4} - \frac{5}{4}i$$

(iii) $\overline{3\imath} + \frac{1}{2-i}$

Solution:

Let $z = \overline{3\imath} + \frac{1}{2-i}$

$$z = -3i + \frac{1}{2-i}$$

$$z = -3i + \frac{1}{2-i} \times \frac{2+i}{2+i}$$

$$z = -3i + \frac{(2+i)}{4+1}$$

$$z = \frac{-15i+2+i}{5}$$

$$z = \frac{2-14i}{5}$$

$$z = \frac{2}{5} - \frac{14}{5}i$$

---

2. If $z = x+iy$, find the following in rectangular form.

(i) $Re\left(\frac{1}{z}\right)$

Solution:

$$\frac{1}{z} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$$

$$\frac{1}{z} = \frac{x-iy}{x^2+y^2}$$

$$\frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$Re\left(\frac{1}{z}\right) = \frac{x}{x^2+y^2}$$

(ii) $Re(i\bar{z})$

Solution:

$$i\bar{z} = i(x-iy)$$

$$i\bar{z} = ix+y$$

$$Re(i\bar{z}) = y$$

(iii) $Im(3z + 4\bar{z} - 4i)$

Solution:

$$3z + 4\bar{z} - 4i = 3(x + iy) + 4(x - iy) - 4i$$
$$= 3x + i3y + 4x - i4y - 4i$$
$$= 7x + i(3y - 4y - 4)$$
$$3z + 4\bar{z} - 4i = 7x + i(-y - 4)$$
$$Im(3z + 4\bar{z} - 4i) = -y - 4$$

3. If $z_1 = 2 - i$ and $z_2 = -4 + 3i$, find the inverse of z_1z_2 and $\frac{z_1}{z_2}$.

Solution:

Given $z_1 = 2 - i$ and $z_2 = -4 + 3i$

$$z_1z_2 = (2 - i)(-4 + 3i)$$
$$= 2(-4) + 2(3i) + (-i)(-4) + (-i)(3i)$$
$$= -8 + 6i + 4i + 3$$
$$z_1z_2 = -5 + 10i$$

Inverse of $z_1z_2 = \frac{1}{z_1z_2}$

$$\frac{1}{z_1z_2} = \frac{1}{-5+10i} \times \frac{-5-10i}{-5-10i}$$
$$\frac{1}{z_1z_2} = \frac{-5-10i}{25+100}$$
$$\frac{1}{z_1z_2} = \frac{5(-1-2i)}{125} = \frac{-1-2i}{25}$$

Inverse of $\frac{z_1}{z_2} = \frac{1}{\left(\frac{z_1}{z_2}\right)}$

$$= \frac{z_2}{z_1}$$
$$= \frac{-4+3i}{2-i} \times \frac{2+i}{2+i}$$
$$= \frac{(-4)2+(-4)i+(3i)2+(3i)i}{4+1}$$
$$= \frac{-8-4i+6i-3}{5}$$
$$\frac{1}{\left(\frac{z_1}{z_2}\right)} = \frac{-11+2i}{5}$$

4. The complex numbers $u, v,$ and w are related by $\frac{1}{u} = \frac{1}{v} + \frac{1}{w}$. If $v = 3 - 4i$ and $w = 4 + 3i$, find u in rectangular form.

Solution:

Given $v = 3 - 4i$ and $w = 4 + 3i$

$$\frac{1}{u} = \frac{1}{v} + \frac{1}{w}$$
$$= \frac{1}{3-4i} + \frac{1}{4+3i}$$
$$= \frac{4+3i+3-4i}{(3-4i)(4+3i)}$$
$$= \frac{7-i}{12+9i-16i+12}$$
$$\frac{1}{u} = \frac{7-i}{24-7i}$$
$$u = \frac{24-7i}{7-i} \times \frac{7+i}{7+i}$$
$$= \frac{168+24i-49i+7}{49+1}$$
$$= \frac{175-25i}{50}$$
$$= \frac{25(7-i)}{50}$$
$$u = \frac{7-i}{2}$$
$$u = \frac{7}{2} - \frac{1}{2}i$$

5. Prove the following properties:

(i) z *is real if and only if* $z = \bar{z}$

Solution:

Case (i):

Let $z = x + iy$ and $\bar{z} = x - iy$

Given that $z = \bar{z}$

T.P: z is real

$\Rightarrow \quad x + iy = x - iy$

Equating real and imaginary parts we get,

$\Rightarrow \quad x = x$ ----------(1)

$y = -y$ ----------(2)

$\Rightarrow \quad 2y = 0$

$\Rightarrow \quad y = 0$

Sub $y = 0$ in z,we get

$\Rightarrow \quad z = x$

$\therefore$ z is real

Case (ii):

Given that z is real

T.P: $z = \bar{z}$

Let $z = x + iy$ and $\bar{z} = x - iy$

$\Rightarrow \quad z = x$ and $\bar{z} = x \quad [\because z \text{ is real} \Rightarrow y = 0]$

$\Rightarrow \quad z = \bar{z}$

Hence proved.

(ii) $Re(z) = \frac{z+\bar{z}}{2}$ and $Im(z) = \frac{z-\bar{z}}{2i}$

Solution:

Now $Re(z) = \frac{z+\bar{z}}{2}$

Let $z = x + iy$ and $\bar{z} = x - iy$

$$\frac{z+\bar{z}}{2} = \frac{(x+iy+x-iy)}{2}$$
$$= \frac{2x}{2}$$
$$= x$$
$$= Re(z)$$
$$Re(z) = \frac{z+\bar{z}}{2}$$

Now $Im(z) = \frac{z-\bar{z}}{2i}$

$$\frac{z-\bar{z}}{2i} = \frac{x+iy-x+iy}{2i}$$
$$= \frac{2iy}{2i}$$
$$= y$$
$$= Im(z)$$
$$Im(z) = \frac{z-\bar{z}}{2i}$$

Hence proved.

6. Find the least value of the positive integer n for which $\left(\sqrt{3}+i\right)^n$

(i) real (ii) purely imaginary.

Solution:

When $n = 1$,

$$\left(\sqrt{3}+i\right)^n = \sqrt{3}+i$$

When $n = 2$,

$$\left(\sqrt{3}+i\right)^n = \left(\sqrt{3}+i\right)^2$$
$$= 3 + 2\sqrt{3}i - 1$$
$$= 2 + 2\sqrt{3}i$$

When $n = 3$,

$$\left(\sqrt{3}+i\right)^n = \left(\sqrt{3}+i\right)^3$$
$$= \left(\sqrt{3}+i\right)\left(\sqrt{3}+i\right)^2$$
$$= \left(\sqrt{3}+i\right)\left(2+2\sqrt{3}i\right)$$
$$= 2\sqrt{3} + 6i + 2i - 2\sqrt{3}$$

$\left(\sqrt{3}+i\right)^3 = 8i$, purely imaginary.

Squaring on both sides, we get

$$\left(\left(\sqrt{3}+i\right)^3\right)^2 = (8i)^2$$

$\left(\sqrt{3}+i\right)^6 = -64$, real

Purely imaginary when $n = 3$ and real when $n = 6$.

7. Show that

(i) $\left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}$ is purely imaginary

Solution:

Let $z = \left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}$

T.P: z is purely imaginary

$(i.e)$ $\bar{z} = -z$

$$\bar{z} = \overline{\left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}}$$
$$= \overline{\left(2+i\sqrt{3}\right)^{10}} - \overline{\left(2-i\sqrt{3}\right)^{10}} \quad \text{[by Prop 3]}$$
$$= \left(2-i\sqrt{3}\right)^{10} - \left(2+i\sqrt{3}\right)^{10}$$
$$= -\left[\left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}\right]$$
$$\bar{z} = -z$$

$\therefore z$ is purely imaginary

(ii) $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real

Solution:

Let $z = \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$

T.P: z is real

$(i.e)$ $\bar{z} = z$

$$\frac{19-7i}{9+i} = \frac{19-7i}{9+i} \times \frac{9-i}{9-i}$$
$$= \frac{171-19i-63i-7}{81+1}$$
$$= \frac{164-82i}{82}$$
$$= \frac{82(2-i)}{82}$$
$$\frac{19-7i}{9+i} = 2 - i$$
$$\left(\frac{19-7i}{9+i}\right)^{12} = (2-i)^{12} \qquad \text{-----------(1)}$$
$$\frac{20-5i}{7-6i} = \frac{20-5i}{7-6i} \times \frac{7+6i}{7+6i}$$
$$= \frac{140+120i-35i+30}{49+36}$$
$$= \frac{170+85i}{85}$$

$$= \frac{85(2+i)}{85}$$

$$\frac{20-5i}{7-6i} = 2+i$$

$$\left(\frac{20-5i}{7-6i}\right)^{12} = (2+i)^{12} \qquad \text{----------(2)}$$

From (1) and (2),

$$z = (2-i)^{12} + (2+i)^{12}$$

$$\bar{z} = \overline{(2-i)^{12} + (2+i)^{12}}$$

$$= \overline{(2-i)^{12}} + \overline{(2+i)^{12}} \qquad \text{[by Prop 3]}$$

$$= (2+i)^{12} + (2-i)^{12}$$

$$= (2-i)^{12} + (2+i)^{12}$$

$$\bar{z} = z$$

Hence z is real

Exercise 2.5

Hint: If $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$

Properties:

12. $|z| = |\bar{z}|$
13. $|z_1 + z_2| \le |z_1| + |z_2|$ (*Triangle inequality*)
14. $|z_1 z_2| = |z_1|\,|z_2|$
15. $|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$
16. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \ne 0$
17. $|z^n| = |z|^n$, where n is an integer
18. $Re(z) \le |z|$
19. $Im(z) \le |z|$
20. $|z|^2 = z\bar{z}$
21. $\big||z_1| - |z_2|\big| \le |z_1 + z_2| \le |z_1| + |z_2|$

1. Find the modulus of the following complex numbers

(i) $\frac{2i}{3+4i}$

Solution:

Let $z = \frac{2i}{3+4i}$

$$|z| = \left|\frac{2i}{3+4i}\right|$$

$$= \frac{|2i|}{|3+4i|}$$

$$= \frac{\sqrt{4}}{\sqrt{9+16}}$$

$$|z| = \frac{2}{5}$$

(ii) $\frac{2-i}{1+i} + \frac{1-2i}{1-i}$

Solution:

Let $z = \frac{2-i}{1+i} + \frac{1-2i}{1-i}$

$$= \frac{(2-i)(1-i)+(1-2i)(1+i)}{(1+i)(1-i)}$$

$$= \frac{2-2i-i-1+1+i-2i+2}{1+1}$$

$$= \frac{4-4i}{2} = \frac{2(2-2i)}{2}$$

$$z = 2-2i$$

$$|z| = |2-2i|$$

$$|z| = \sqrt{4+4} = \sqrt{8}$$

$$|z| = 2\sqrt{2}$$

(iii) $(1-i)^{10}$

Solution:

Let $z = (1-i)^{10}$

$$|z| = |(1-i)^{10}|$$

$$|z| = |1-i|^{10} \qquad \text{[by Prop 17]}$$

$$|z| = \left(\sqrt{2}\right)^{10}$$

$$|z| = 2^{\frac{10}{2}} = 2^5$$

$$|z| = 32$$

(iv) $2i(3-4i)(4-3i)$

Solution:

Let $z = 2i(3-4i)(4-3i)$

$$|z| = |(2i)\,(3-4i)\,(4-3i)|$$

$$|z| = |2i|\,|3-4i|\,|4-3i| \qquad \text{[by Prop 14]}$$

$$|z| = \sqrt{4}\,\sqrt{9+16}\,\sqrt{16+9}$$

$$|z| = 2 \times 5 \times 5$$

$$|z| = 50$$

2. For any two complex numbers z_1 and z_2, such that $|z_1| = |z_2| = 1$ and $z_1 z_2 \ne -1$, then show that $\frac{z_1+z_2}{1+z_1 z_2}$ is a real number.

Solution:

Given $|z_1| = |z_2| = 1$

$$|z_1| = 1$$

$$z_1\bar{z}_1 = 1$$

$$z_1 = \frac{1}{\bar{z}_1} \qquad \text{-----------(1)}$$

Similarly, $z_2 = \frac{1}{\bar{z}_2}$ -----------(2)

T.P: $\frac{z_1+z_2}{1+z_1 z_2}$ is real number

Now $z_1 + z_2 = \frac{1}{z_1} + \frac{1}{z_2}$ From (1) & (2)

$$z_1 + z_2 = \frac{\overline{z_1} + \overline{z_2}}{\overline{z_1}\,\overline{z_2}}$$

$$\Rightarrow \quad 1 + z_1 z_2 = 1 + \frac{1}{\overline{z_1}\,\overline{z_2}}$$

$$\Rightarrow \quad 1 + z_1 z_2 = \frac{\overline{z_1}\,\overline{z_2} + 1}{\overline{z_1}\,\overline{z_2}}$$

$$\Rightarrow \quad \frac{z_1 + z_2}{1 + z_1 z_2} = \frac{\frac{\overline{z_1} + \overline{z_2}}{\overline{z_1}\,\overline{z_2}}}{\frac{\overline{z_1}\,\overline{z_2} + 1}{\overline{z_1}\,\overline{z_2}}}$$

$$= \frac{\overline{z_1} + \overline{z_2}}{1 + \overline{z_1}\,\overline{z_2}}$$

$$= \frac{\overline{z_1 + z_2}}{\overline{1 + z_1 z_2}} \quad \text{[by Prop 2\&4]}$$

$$\Rightarrow \quad \frac{z_1 + z_2}{1 + z_1 z_2} = \overline{\left(\frac{z_1 + z_2}{1 + z_1 z_2}\right)} \quad \text{[by Prop 5]}$$

$\Rightarrow \quad \frac{z_1 + z_2}{1 + z_1 z_2}$ is a real number. [by Prop 10]

3. Which one of the points $10 - 8i, 11 + 6i$ is closest to $1 + i$.

Solution:

Let $z_1 = 1 + i, z_2 = 10 - 8i$, and $z_3 = 11 + 6i$

$$|z_1 - z_2| = |1 + i - 10 + 8i|$$

$$= |-9 + 9i|$$

$$= \sqrt{81 + 81} = \sqrt{2 \times 81}$$

$$|z_1 - z_2| = 9\sqrt{2} \quad \text{----------(1)}$$

$$|z_1 - z_3| = |1 + i - 11 - 6i|$$

$$= |-10 - 5i|$$

$$= \sqrt{100 + 25} = \sqrt{5 \times 25}$$

$$|z_1 - z_3| = 5\sqrt{5} \quad \text{----------(2)}$$

$$|z_1 - z_2| > |z_1 - z_3|$$

Hence $11 + 6i$ is closest to $1 + i$.

4. If $|z| = 3$, show that $7 \leq |z + 6 - 8i| \leq 13$.

Solution:

Given $|z| = 3$

By Property 21, $\big||z_1| - |z_2|\big| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

$$\big||z| - |6 - 8i|\big| \leq |z + 6 - 8i| \leq |z| + |6 - 8i|$$

$$\left||z| - \sqrt{36 + 64}\right| \leq |z + 6 - 8i| \leq |z| + \sqrt{36 + 64}$$

$$|3 - \sqrt{100}| \leq |z + 6 - 8i| \leq 3 + \sqrt{100}$$

$$|3 - 10| \leq |z + 6 - 8i| \leq 3 + 10$$

$$|-7| \leq |z + 6 - 8i| \leq 13$$

$$7 \leq |z + 6 - 8i| \leq 13$$

Hence proved.

5. If $|z| = 1$, show that $2 \leq |z^2 - 3| \leq 4$.

Solution:

Given $|z| = 1$

By Property 21, $\big||z_1| - |z_2|\big| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

$$\big||z|^2 - |3|\big| \leq |z^2 - 3| \leq |z|^2 + |3|$$

$$|1 - 3| \leq |z^2 - 3| \leq 1 + 3$$

$$|-2| \leq |z^2 - 3| \leq 4$$

$$2 \leq |z^2 - 3| \leq 4$$

Hence proved.

6. If $|z| = 2$, show that $8 \leq |z + 6 + 8i| \leq 12$.

Solution:

Given $|z| = 1$

By Property 21, $\big||z_1| - |z_2|\big| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

$$\big||z| - |6 + 8i|\big| \leq |z + 6 + 8i| \leq |z| + |6 + 8i|$$

$$\left||z| - \sqrt{36 + 64}\right| \leq |z + 6 + 8i| \leq |z| + \sqrt{36 + 64}$$

$$|2 - \sqrt{100}| \leq |z + 6 + 8i| \leq 2 + \sqrt{100}$$

$$|2 - 10| \leq |z + 6 + 8i| \leq 2 + 10$$

$$|-8| \leq |z + 6 + 8i| \leq 12$$

$$8 \leq |z + 6 + 8i| \leq 12$$

Hence proved.

7. If z_1, z_2, and z_3 are three complex numbers such that $|z_1| = 1, |z_2| = 2, |z_3| = 3$ and $|z_1 + z_2 + z_3| = 1$, show that $|9z_1 z_2 + 4z_1 z_3 + z_2 z_3| = 6$.

Solution:

Given $|z_1| = 1, |z_2| = 2, |z_3| = 3$

$\lvert z_1\rvert = 1$	$\lvert z_2\rvert = 2$	$\lvert z_3\rvert = 3$
$\lvert z_1\rvert^2 = 1^2$	$\lvert z_2\rvert^2 = 2^2$	$\lvert z_3\rvert^2 = 3^2$
$z_1\overline{z_1} = 1$	$z_2\overline{z_2} = 4$	$z_3\overline{z_3} = 9$
$z_1 = \frac{1}{\overline{z_1}}$	$z_2 = \frac{4}{\overline{z_2}}$	$z_3 = \frac{9}{\overline{z_3}}$

$$|z_1 + z_2 + z_3| = 1$$

$$\left|\frac{1}{\overline{z_1}} + \frac{4}{\overline{z_2}} + \frac{9}{\overline{z_3}}\right| = 1$$

$$\left|\frac{(\overline{z_2}\,\overline{z_3} + 4\overline{z_1}\,\overline{z_3} + 9\overline{z_1}\,\overline{z_2})}{\overline{z_1}\,\overline{z_2}\,\overline{z_3}}\right| = 1$$

$$\frac{|\overline{z_2z_3+4z_1z_3+9z_1z_2}|}{|z_1z_2z_3|} = 1 \qquad \text{[by Prop 2\&4]}$$

$$|\overline{z_2z_3 + 4z_1z_3 + 9z_1z_2}| = |\overline{z_1z_2z_3}|$$

$$|z_2z_3 + 4z_1z_3 + 9z_1z_2| = |z_1z_2z_3| \text{ [by Prop 12]}$$

$$|z_2z_3 + 4z_1z_3 + 9z_1z_2| = |z_1||z_2||z_3|$$

$$|z_2z_3 + 4z_1z_3 + 9z_1z_2| = 1 \times 2 \times 3$$

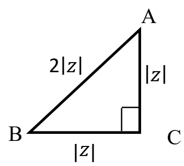
$$|z_2z_3 + 4z_1z_3 + 9z_1z_2| = 6$$

Hence proved.

8. If the area of the triangle formed by the vertices z, iz, and $z + iz$ is 50 square units, find the value of $|z|$.

Solution:

Let $A = z,\ B = iz$ and $C = z + iz$



$$|A - B| = |z - iz|$$
$$= |z(1 - i)|$$
$$= |z||1 - i|$$
$$= |z|\sqrt{1 + 1}$$
$$|A - B| = \sqrt{2}|z| \qquad \text{----------(1)}$$
$$|B - C| = |iz - z - iz|$$
$$= |-z|$$
$$|B - C| = |z| \qquad \text{----------(2)}$$
$$|C - A| = |z + iz - z|$$
$$= |iz| = |i||z|$$
$$|C - A| = |z| \qquad \text{----------(3)}$$

$\therefore AB = \sqrt{2}|z|, BC = |z|$ and $CA = |z|$

$$(BC)^2 + (CA)^2 = |z|^2 + |z|^2$$
$$= 2|z|^2$$
$$(BC)^2 + (CA)^2 = (AB)^2$$

$\therefore$ The ΔABC is isosceles triangle.

$$\text{Area of } \Delta ABC = 50$$
$$\frac{1}{2} \times base \times hight = 50$$
$$\frac{1}{2} \times |z| \times |z| = 50$$
$$|z|^2 = 100$$
$$|z| = 10$$

9. Show that the equation $z^3 + 2\bar{z} = 0$ has five solutions.

Solution:

Given that $z^3 + 2\bar{z} = 0$

$$z^3 = -2\bar{z}$$
$$-\frac{z^3}{2} = \bar{z} \qquad \text{----------(1)}$$

Taking modulus on both sides,

$$|z|^3 = |-2\bar{z}|$$
$$|z|^3 = 2|\bar{z}|$$
$$|z|^3 = 2|z| \qquad \text{[by Prop 12]}$$
$$|z|^3 - 2|z| = 0$$
$$|z|(|z|^2 - 2) = 0$$
$$|z| = 0 \text{ or } |z|^2 - 2 = 0$$

Case (i):

$$|z| = 0$$

$z = 0$ is a solution.

Case (ii):

$$|z|^2 - 2 = 0$$
$$|z|^2 = 2$$
$$z\bar{z} = 2$$
$$z\left(-\frac{z^3}{2}\right) = 2 \qquad \text{From (1)}$$
$$-z^4 = 4$$
$$z^4 = -4$$

$\therefore z$ has four non-zero solutions.

Hence, z has five solutions.

10. Find the square roots of

Hint: Formula for finding square root,

$$\sqrt{a + ib} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right), \text{ where } z = a + ib$$

(i) $4 + 3i$

Solution:

$$|4 + 3i| = \sqrt{16 + 9} = \sqrt{25} = 5$$

Applying the square root formula, we get

$$\sqrt{4 + 3i} = \pm\left(\sqrt{\frac{5+4}{2}} + i\frac{3}{|3|}\sqrt{\frac{5-4}{2}}\right)$$
$$= \pm\left(\sqrt{\frac{9}{2}} + i\sqrt{\frac{1}{2}}\right)$$
$$\sqrt{4 + 3i} = \pm\left(\frac{3}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

(ii) $-6+8i$

Solution:

$$|-6+8i| = \sqrt{36+64} = \sqrt{100} = 10$$

Applying square root formula, we get

$$\sqrt{-6+8i} = \pm\left(\sqrt{\frac{10-6}{2}} + i\frac{8}{|8|}\sqrt{\frac{10+6}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{4}{2}} + i\sqrt{\frac{16}{2}}\right)$$

$$= \pm\left(\sqrt{2} + i\sqrt{8}\right)$$

$$= \pm\left(\sqrt{2} + i2\sqrt{2}\right)$$

$$\sqrt{-6+8i} = \pm\sqrt{2}\,(1+2i)$$

(iii) $-5-12i$

Solution:

$$|-5-12i| = \sqrt{25+144} = \sqrt{169} = 13$$

Applying square root formula, we get

$$\sqrt{-5-12i} = \pm\left(\sqrt{\frac{13-5}{2}} + i\frac{-12}{|-12|}\sqrt{\frac{13+5}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{8}{2}} - i\sqrt{\frac{18}{2}}\right)$$

$$= \pm\left(\sqrt{4} - i\sqrt{9}\right)$$

$$\sqrt{-5-12i} = \pm(2-3i)$$

Exercise 2.6

1. If $z = x+iy$ is a complex number such that $\left|\frac{z-4i}{z+4i}\right| = 1$ show that the locus of z is real axis.

Solution:

Given that $\left|\frac{z-4i}{z+4i}\right| = 1$

$$\frac{|z-4i|}{|z+4i|} = 1 \qquad \text{[by Prop 5]}$$

$$|z-4i| = |z+4i|$$

Let $z = x+iy$ ----------(1)

$$|x+iy-4i| = |x+iy+4i|$$

$$|x+i(y-4)| = |x+i(y+4)|$$

$$\sqrt{x^2+(y-4)^2} = \sqrt{x^2+(y+4)^2}$$

Squaring on both sides, we get

$$x^2+(y-4)^2 = x^2+(y+4)^2$$

$$y^2-8y+16 = y^2+8y+16$$

$$-8y-8y = 0$$

$$-16y = 0$$

$$y = 0$$

Sub $y=0$ in (1), we get

$$z = x+i(0)$$

$$z = x$$

∴ The locus of z is real axis.

2. If $z = x+iy$ is a complex number such that $Im\left(\frac{2z+1}{iz+1}\right) = 0$, show that the locus of z is $2x^2+2y^2+x-2y=0$.

Solution:

Let $z = x+iy$

$$\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1}$$

$$= \frac{2x+2iy+1}{ix-y+1}$$

$$= \frac{(2x+1)+i2y}{(1-y)+ix}$$

$$= \frac{(2x+1)+i2y}{(1-y)+ix} \times \frac{(1-y)-ix}{(1-y)-ix}$$

$$\frac{2z+1}{iz+1} = \frac{(2x+1)(1-y)+2xy+i(2y(1-y)-x(2x+1)}{(1-y)^2+x^2}$$

Given that $Im\left(\frac{2z+1}{iz+1}\right) = 0$

$$\frac{2y(1-y)-x(2x+1)}{(1-y)^2+x^2} = 0$$

$$2y-2y^2-2x^2-x = 0$$

$$2x^2+2y^2+x-2y = 0$$

Hence Proved.

3. Obtain the Cartesian form of the locus of $z = x+iy$ in each of the following cases:

(i) $[Re(iz)]^2 = 3$

Solution:

Given $z = x+iy$

$$iz = i(x+iy) = ix-y$$

$$Re(iz) = -y$$

$$[Re(iz)]^2 = y^2$$

$$y^2 = 3$$

(ii) $Im[(1-i)z+1] = 0$

Solution:

Given $z = x+iy$

$$(1-i)z+1 = (1-i)(x+iy)+1$$
$$= x+iy-ix+y+1$$
$$(1-i)z+1 = x+y+1+i(y-x)$$
$$Im[(1-i)z+1] = y-x$$
$$y-x=0$$
$$x-y=0$$

(iii) $|z+i| = |z-1|$

Solution:

Given $z = x+iy$

$$|z+i| = |z-1|$$
$$|x+iy+i| = |x+iy-1|$$
$$|x+i(y+1)| = |(x-1)+iy|$$
$$\sqrt{x^2+(y+1)^2} = \sqrt{(x-1)^2+y^2}$$

Squaring on both sides, we get

$$x^2+(y+1)^2 = (x-1)^2+y^2$$
$$x^2+y^2+2y+1 = x^2-2x+1+y^2$$
$$2y = -2x$$
$$2x+2y=0$$
$$x+y=0$$

(iv) $\bar{z} = z^{-1}$

Solution:

Given $z = x+iy$

$$\bar{z} = \frac{1}{z}$$
$$z\bar{z} = 1$$
$$|z|^2 = 1 \qquad \text{[by Prop 20]}$$
$$|x+iy|^2 = 1$$
$$x^2+y^2=1$$

---

4. Show that the following equations represent a circle and find its centre and radius.

Hint: $|z-z_0| = r$, where z_0 is centre and r is radius

(i) $|z-2-i| = 3$

Solution:

$\Rightarrow \quad |z-(2+i)| = 3$

This is of the form $|z-z_0| = r$ and it represents a circle.

Centre is $(2+i)$ and radius $= 3$ units

(ii) $|2z+2-4i| = 2$

Solution:

$\Rightarrow \quad |2z+2-4i| = 2$

$\Rightarrow \quad 2|z+1-2i| = 2$

$\Rightarrow \quad |z-(-1+2i)| = 1$

This is of the form $|z-z_0| = r$ and it represents a circle.

Centre is $(-1+2i)$ and radius $= 1$ unit

(iii) $|3z-6+12i| = 8$

Solution:

$\Rightarrow \quad 3|z-2+4i| = 8$

$\Rightarrow \quad |z-(2-4i)| = \frac{8}{3}$

This is of the form $|z-z_0| = r$ and it represents a circle.

Centre is $(2-4i)$ and radius $= \frac{8}{3}$ units

---

5. Obtain the Cartesian equation for the locus of $z = x+iy$ in each of the following cases:

(i) $|z-4| = 16$

Solution:

Given $z = x+iy$

$$|x+iy-4| = 16$$
$$|(x-4)+iy| = 16$$
$$\sqrt{(x-4)^2+y^2} = 16$$

Squaring on both sides, we get

$$(x-4)^2+y^2 = 256$$
$$x^2-8x+16+y^2 = 256$$
$$x^2+y^2-8x-240=0$$

(ii) $|z-4|^2 - |z-1|^2| = 16$

Solution:

Given $z = x+iy$

$$|x+iy-4|^2 - |x+iy-1|^2 = 16$$
$$|(x-4)+iy|^2 - |(x-1)+iy|^2 = 16$$
$$(x-4)^2+y^2-[(x-1)^2+y^2] = 16$$
$$x^2-8x+16+1-[x^2-2x+1+1] = 16$$
$$-6x+15 = 16$$
$$-6x-1=0$$
$$6x+1=0$$

Exercise 2.7

Polar form of complex number is $z = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{x^2 + y^2}$

Euler form of complex number is $e^{i\theta} = \cos\theta + i\sin\theta$

Properties of polar form:

22. **If $z = r(\cos\theta + i\sin\theta)$ then $z^{-1} = \frac{1}{r}(\cos\theta - i\sin\theta)$**
23. **If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_1 + i\sin\theta_2)$ then $z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$**
24. **If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_1 + i\sin\theta_2)$ then $\frac{z_1}{z_2} = \frac{r_1}{r_2}[(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$**

---

1. Write in polar form of the following complex numbers

(i) $2 + i2\sqrt{3}$

Solution:

Let $2 + i2\sqrt{3} = r(\cos\theta + i\sin\theta)$

$$r = \sqrt{x^2+y^2} = \sqrt{4+12} = \sqrt{16} = 4$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right|$$

$$\alpha = \tan^{-1}\left|\frac{2\sqrt{3}}{2}\right|$$

$$\alpha = \tan^{-1}\left|\sqrt{3}\right|$$

$$\alpha = \frac{\pi}{3}$$

Since $2 + i2\sqrt{3}$ lies in I quadrant, $\theta = \alpha$

$$\theta = \frac{\pi}{3}$$

The polar form of $2 + i2\sqrt{3}$ can be written as

$$= 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= 4\left(\cos\left(2k\pi + \frac{\pi}{3}\right) + i\sin\left(2k\pi + \frac{\pi}{3}\right)\right), k \in Z$$

(ii) $3 - i\sqrt{3}$

Solution:

Let $3 - i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$$r = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$$

Now $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

$$\alpha = \tan^{-1}\left|\frac{-\sqrt{3}}{3}\right|$$

$$\alpha = \tan^{-1}\left|\frac{-1}{\sqrt{3}}\right|$$

$$\alpha = \frac{\pi}{6}$$

Since $3 - i\sqrt{3}$ lies in IV quadrant, $\theta = -\alpha$

$$\theta = -\frac{\pi}{6}$$

The polar form of $3 - i\sqrt{3}$ can be written as

$$= 2\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$= 2\sqrt{3}\left(\cos\left(2k\pi - \frac{\pi}{6}\right) + i\sin\left(2k\pi - \frac{\pi}{6}\right)\right), k \in Z$$

(iii) $-2 - i2$

Solution:

Let $-2 - i2 = r(\cos\theta + i\sin\theta)$

$$r = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right|$$

$$\alpha = \tan^{-1}\left|\frac{-2}{-2}\right|$$

$$\alpha = \tan^{-1}|1|$$

$$\alpha = \frac{\pi}{4}$$

Since $-2 - i2$ lies in III quadrant, $\theta = \alpha - \pi$

$$\theta = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$$

$$\theta = -\frac{3\pi}{4}$$

The polar form of $-2 - i2$ can be written as

$$= 2\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$$

$$= 2\sqrt{2}\left(\cos\left(2k\pi - \frac{3\pi}{4}\right) + i\sin\left(2k\pi - \frac{3\pi}{4}\right)\right), k \in Z$$

(iv) $\frac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$

Solution:

Let $i - 1 = r(\cos\theta + i\sin\theta)$

$$r = \sqrt{1+1} = \sqrt{2}$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right|$$

$$\alpha = \tan^{-1}\left|\frac{-1}{1}\right|$$

$$\alpha = \tan^{-1}|1|$$

$$\alpha = \frac{\pi}{4}$$

Since $i - 1$ lies in II quadrant, $\theta = \pi - \alpha$

$$\theta = \pi - \frac{\pi}{4}$$

$$\theta = \frac{3\pi}{4}$$

The polar form of $\frac{i-1}{cos\frac{\pi}{3}+i\,sin\frac{\pi}{3}}$ can be written as

$$= \frac{\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)}{cos\frac{\pi}{3}+i\,sin\frac{\pi}{3}}$$

$$= \sqrt{2}\left[\cos\left(\frac{3\pi}{4}-\frac{\pi}{3}\right)+i\sin\left(\frac{3\pi}{4}-\frac{\pi}{3}\right)\right] \quad \text{[by Prop 24]}$$

$$= \sqrt{2}\left[\cos\left(\frac{9\pi-4\pi}{12}\right)+i\sin\left(\frac{9\pi-4\pi}{12}\right)\right]$$

$$= \sqrt{2}\left[\cos\left(\frac{5\pi}{12}\right)+i\sin\left(\frac{5\pi}{12}\right)\right]$$

$$= \sqrt{2}\left[\cos\left(2k\pi+\frac{5\pi}{12}\right)+i\sin\left(2k\pi+\frac{5\pi}{12}\right)\right], k \in Z$$

2. Find the rectangular form of the complex numbers

(i) $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$

Solution:

$$\text{Let } z_1 = \left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

$$z_2 = \left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$$

$$z_1z_2 = \cos\left(\frac{\pi}{6}+\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{6}+\frac{\pi}{12}\right) \text{ [by Prop 23]}$$

$$= \cos\frac{3\pi}{12}+i\sin\frac{3\pi}{12}$$

$$z_1z_2 = \cos\frac{\pi}{4}+i\sin\frac{\pi}{4}$$

$$z_1z_2 = \frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}$$

$$z_1z_2 = \frac{1}{\sqrt{2}}(1+i)$$

(ii) $\frac{\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)}{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}$

Solution:

$$\text{Let } z_1 = \cos\frac{\pi}{6}-i\sin\frac{\pi}{6} = \cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)$$

$$z_2 = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

$$\frac{z_1}{z_2} = \frac{1}{2}\left[\cos\left(-\frac{\pi}{6}-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{6}-\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(-\frac{3\pi}{6}\right)+i\sin\left(-\frac{3\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right)-i\sin\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{2}[0-i(1)]$$

$$\frac{z_1}{z_2} = -\frac{i}{2}$$

3. If $(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\dots(x_n+iy_n) = a+ib$, Show that

$(x_1^2+y_1^2)(x_2^2+y_2^2)(x_3^2+y_3^2)\dots(x_n^2+y_n^2) = a^2+b^2$

Solution:

Given that

$$(x_1+iy_1)(x_2+iy_2)\dots(x_n+iy_n) = a+ib$$

Taking modulus on both sides, we get

$$|(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\dots(x_n+iy_n)| = |a+ib|$$

$$|(x_1+iy_1)||(x_2+iy_2)||(x_3+iy_3)|\dots|(x_n+iy_n)| = |a+ib|$$

$$\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}\sqrt{x_3^2+y_3^2}\dots\sqrt{x_n^2+y_n^2} = \sqrt{a^2+b^2}$$

Squaring on both sides, we get

$$(x_1^2+y_1^2)(x_2^2+y_2^2)\dots(x_n^2+y_n^2) = a^2+b^2$$

(ii) $\sum_{r=1}^{n}\tan^{-1}\left(\frac{y_r}{x_r}\right) = \tan^{-1}\left(\frac{b}{a}\right)+2k\pi,\ k \in Z$

Solution:

Given $(x_1+iy_1)(x_2+iy_2)\dots(x_n+iy_n) = a+ib$

Taking argument on both sides, we get

$$\arg[(x_1+iy_1)(x_2+iy_2)\dots(x_n+iy_n)] = \arg(a+ib)$$

$$\arg(x_1+iy_1)+\arg(x_2+iy_2)+\dots+\arg(x_n+iy_n) = \arg(a+ib)$$

$$\tan^{-1}\left(\frac{y_1}{x_1}\right)+\tan^{-1}\left(\frac{y_2}{x_2}\right)+\dots+\tan^{-1}\left(\frac{y_n}{x_n}\right) = \tan^{-1}\left(\frac{b}{a}\right)$$

Hence $\sum_{r=1}^{n}\tan^{-1}\left(\frac{y_r}{x_r}\right) = 2k\pi+\tan^{-1}\left(\frac{b}{a}\right), k \in Z$

4. If $\frac{1+z}{1-z} = \cos 2\theta + i\sin 2\theta$, show that $z = i\tan\theta$.

Hint: $\sin 2\theta = 2\sin\theta\cos\theta$, $1-\cos 2\theta = 2\sin^2\theta$ and $1+\cos 2\theta = 2\cos^2\theta$

Dividendo and componendo rule, $\frac{Nr-Dr}{Nr+Dr}$

Solution:

$$\frac{1+z}{1-z} = \cos 2\theta + i\sin 2\theta$$

By dividendo and componendo theorem, we get

$$\frac{1+z-1+z}{1+z+1-z} = \frac{(\cos 2\theta+i\sin 2\theta-1)}{(\cos 2\theta+i\sin 2\theta+1)}$$

$$\frac{2z}{2} = \frac{(1-2\sin^2\theta+2i\sin\theta\cos\theta-1)}{(2\cos^2\theta-1+2i\sin\theta\cos\theta+1)}$$

$$z = \frac{2\sin\theta(i\cos\theta-\sin\theta)}{2\cos\theta(\cos\theta+i\sin\theta)}$$

$$z = \tan\theta\frac{(i\cos\theta-\sin\theta)}{(\cos\theta+i\sin\theta)}\times\frac{i}{i}$$

$$z = \tan\theta \frac{(i\cos\theta - \sin\theta)i}{(i\cos\theta - \sin\theta)}$$

$$z = i\tan\theta$$

5. If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$, show that

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

Hint: If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

de moivre's theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Solution:

Given that

$$\cos\alpha + \cos\beta + \cos\gamma = 0 \qquad \text{----------(1)}$$

$$\sin\alpha + \sin\beta + \sin\gamma = 0$$

Multiply by i, we get

$$i\sin\alpha + i\sin\beta + i\sin\gamma = 0 \qquad \text{----------(2)}$$

Adding (1) and (2), we get

$$(\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta) + (\cos\gamma + i\sin\gamma) = 0$$

Let $z_1 = (\cos\alpha + i\sin\alpha)$, $z_2 = (\cos\beta + i\sin\beta)$, and

$z_3 = (\cos\gamma + i\sin\gamma)$

If $z_1 + z_2 + z_3 = 0$, then $z_1^3 + z_2^3 + z_3^3 = 3z_1z_2z_3$.

$$(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3 = 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$$

By de Moivre's theorem,

$$(\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta) + (\cos 3\gamma + i\sin 3\gamma) = 3[\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)]$$

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma) = 3\cos(\alpha + \beta + \gamma) + 3i\sin(\alpha + \beta + \gamma)$$

Equating real and imaginary parts, we get

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$$

$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$$

Hence proved.

6. If $z = x + iy$ and $\arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$, show that $x^2 + y^2 + 3x - 3y + 2 = 0$.

Hint: $arg\,(z) = tan^{-1}\left|\frac{y}{x}\right|$

Solution:

Let $z = x + iy$

$$\frac{z-i}{z+2} = \frac{x+iy-i}{x+iy+2}$$

$$= \frac{x+i(y-1)}{(x+2)+iy} \times \frac{(x+2)-iy}{(x+2)-iy}$$

$$= \frac{x(x+2) - ixy + i(y-1)(x+2) + y(y-1)}{(x+2)^2 + y^2}$$

$$= \frac{x^2 + 2x + y^2 - y - ixy + i(xy + 2y - x - 2)}{(x+2)^2 + y^2}$$

$$= \frac{x^2 + y^2 + 2x - y + i(-xy + xy + 2y - x - 2)}{(x+2)^2 + y^2}$$

$$\frac{z-i}{z+2} = \frac{(x^2 + y^2 + 2x - y) + i(2y - x - 2)}{(x+2)^2 + y^2}$$

$$\arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$$

$$tan^{-1}\left|\frac{2y - x - 2}{x^2 + y^2 + 2x - y}\right| = \frac{\pi}{4}$$

$$\frac{2y - x - 2}{x^2 + y^2 + 2x - y} = \tan\frac{\pi}{4}$$

$$\frac{2y - x - 2}{x^2 + y^2 + 2x - y} = 1$$

$$2y - x - 2 = x^2 + y^2 + 2x - y$$

$$x^2 + y^2 + 3x - 3y + 2 = 0$$

Hence proved.

Exercise 2.8

Hint:

- $z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right], k \in Z$
- $1 + \omega + \omega^2 = 0, \omega^3 = 1 \text{ and } \omega^4 = \omega$

1. If $\omega \neq 1$ is a cube root of unity, show that

$\frac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2} + \frac{a + b\omega + c\omega^2}{c + a\omega + b\omega^2} = -1.$

Solution:

Since ω is a cube root of unity, we have

$\Rightarrow \quad \omega^3 = 1$ and $1 + \omega + \omega^2 = 0$

$$\text{LHS} = \frac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2} + \frac{a + b\omega + c\omega^2}{c + a\omega + b\omega^2}$$

$$= \frac{\omega^2(a + b\omega + c\omega^2)}{\omega^2(b + c\omega + a\omega^2)} + \frac{\omega^2(a + b\omega + c\omega^2)}{\omega^2(c + a\omega + b\omega^2)}$$

$$= \frac{a\omega^2 + b + c\omega}{\omega^2(b + c\omega + a\omega^2)} + \frac{\omega^2(a + b\omega + c\omega^2)}{c\omega^2 + a + b\omega}$$

$$= \frac{1}{\omega^2} + \omega^2 = \frac{1 + \omega^4}{\omega^2}$$

$$= \frac{1 + \omega}{\omega^2} = \frac{-\omega^2}{\omega^2} \qquad [\because 1 + \omega = -\omega^2]$$

$= -1$

= RHS

$\Rightarrow \quad \frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} + \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} = -1$

Hence proved.

2. show that $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5 = -\sqrt{3}$.

Solution:

Polar form of $\frac{\sqrt{3}}{2}+\frac{i}{2} = r\,(\cos\theta + i\sin\theta)$

$$r = \sqrt{x^2+y^2} = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right|$$

$$\alpha = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right| = \frac{\pi}{6}$$

Since $\frac{\sqrt{3}}{2}+\frac{i}{2}$ lies in I quadrant, $\theta = \alpha$

$$\theta = \frac{\pi}{6}$$

$\Rightarrow \quad \frac{\sqrt{3}}{2}+\frac{i}{2} = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

Similarly, $\frac{\sqrt{3}}{2}-\frac{i}{2} = \left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

$$\text{LHS} = \left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^5 + \left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^5$$

$$= \left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right) + \left(\cos\frac{5\pi}{6}-i\sin\frac{5\pi}{6}\right)$$

$$= 2\cos\left(\frac{5\pi}{6}\right)$$

$$= 2\cos\left(\pi-\frac{\pi}{6}\right)$$

$$= -2\cos\left(\frac{\pi}{6}\right)$$

$$= -2\times\frac{\sqrt{3}}{2}$$

$$= -\sqrt{3}$$

$$= \text{RHS}$$

$\Rightarrow \quad \left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5 = -\sqrt{3}$

Hence proved.

3. Find the value of $\left(\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10}$.

Solution:

Let $z = \sin\frac{\pi}{10} + i\cos\frac{\pi}{10}$

$\therefore \quad \frac{1}{z} = \sin\frac{\pi}{10} - i\cos\frac{\pi}{10}$

$$\left(\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10} = \left(\frac{1+z}{1+\frac{1}{z}}\right)^{10}$$

$$= \left(\frac{1+z}{\frac{z+1}{z}}\right)^{10}$$

$$= z^{10}$$

$$= \left(\sin\frac{\pi}{10} + i\cos\frac{\pi}{10}\right)^{10}$$

$$= i^{10}\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{10}$$

$$= i^8 i^2\,(\cos\pi + i\sin\pi)$$

$$= (-1)(-1)$$

$$\left(\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10} = 1$$

4. If $2\cos\alpha = x+\frac{1}{x}$ and $2\cos\beta = y+\frac{1}{y}$, show that

(i) $\frac{x}{y}+\frac{y}{x} = 2\cos(\alpha-\beta)$ (ii) $xy-\frac{1}{xy} = 2i\sin(\alpha+\beta)$

(iii) $\frac{x^m}{y^n}-\frac{y^n}{x^m} = 2i\sin(m\alpha-n\beta)$

(iv) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha+n\beta)$

Solutions:

Given $2\cos\alpha = x+\frac{1}{x}$ and $2\cos\beta = y+\frac{1}{y}$

$$2\cos\alpha = \frac{x^2+1}{x}$$

$$2x\cos\alpha = x^2+1$$

$$x^2 - 2x\cos\alpha + 1 = 0$$

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Here $a = 1, b = -2\cos\alpha$ and $c = 1$

$$x = \frac{2\cos\alpha\pm\sqrt{4\cos^2\alpha-4}}{2}$$

$$= \frac{2\cos\alpha\pm\sqrt{4(\cos^2\alpha-1)}}{2}$$

$$= \frac{2\cos\alpha\pm\sqrt{-4\sin^2\alpha}}{2}$$

$$= \frac{2(\cos\alpha\pm i\sin\alpha)}{2}$$

$$x = \cos\alpha \pm i\sin\alpha$$

Similarly, $y = \cos\beta \pm i\sin\beta$

Let $x = \cos\alpha + i\sin\alpha$ & $y = \cos\beta + i\sin\beta$ --------(1)

(i) $\frac{x}{y}+\frac{y}{x} = 2\cos(\alpha-\beta)$

$$\frac{x}{y} = \frac{(\cos\alpha+i\sin\alpha)}{(\cos\beta+i\sin\beta)}$$

$\frac{x}{y} = \cos(\alpha-\beta) + i\sin(\alpha-\beta) \qquad$ [by Prop 24]

$$\frac{y}{x} = \frac{(\cos\beta + i\sin\beta)}{(\cos\alpha + i\sin\alpha)}$$

$$= \cos(\beta - \alpha) + i\sin(\beta - \alpha)$$

$$\frac{y}{x} = \cos(\alpha - \beta) - i\sin(\alpha - \beta) \quad \text{[by Prop 24]}$$

$$\frac{x}{y} + \frac{y}{x} = \cos(\alpha - \beta) + i\sin(\alpha - \beta) + \cos(\alpha - \beta) - i\sin(\alpha - \beta)$$

$$\frac{x}{y} + \frac{y}{x} = 2\cos(\alpha - \beta)$$

Hence proved.

(ii) $xy - \frac{1}{xy} = 2isin\,(\alpha + \beta)$

$$xy = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$xy = \cos(\alpha + \beta) + i\sin(\alpha + \beta)$$

$$\frac{1}{xy} = \frac{1}{\cos(\alpha+\beta) + i\sin(\alpha+\beta)}$$

$$\frac{1}{xy} = \cos(\alpha + \beta) - i\sin(\alpha + \beta) \quad \text{[by Prop 22]}$$

$$xy - \frac{1}{xy} = \cos(\alpha + \beta) + i\sin(\alpha + \beta) - \cos(\alpha + \beta) + i\sin(\alpha + \beta)$$

$$xy - \frac{1}{xy} = 2i\sin(\alpha + \beta)$$

Hence proved.

(iii) $\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2i\sin(m\alpha - n\beta)$

$$\frac{x^m}{y^n} = \frac{(\cos\alpha + i\sin\alpha)^m}{(\cos\beta + i\sin\beta)^n}$$

By de Moivre's theorem, we get

$$\frac{x^m}{y^n} = \frac{\cos m\alpha + i\sin m\alpha}{\cos n\beta + i\sin n\beta}$$

$$\frac{x^m}{y^n} = \cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$$

$$\frac{y^n}{x^m} = \frac{1}{\left(\frac{x^m}{y^n}\right)} \quad \text{[by Prop 24]}$$

$$\frac{y^n}{x^m} = \cos(m\alpha - n\beta) - i\sin(m\alpha - n\beta)$$

$$\frac{x^m}{y^n} - \frac{y^n}{x^m} = \cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta) - \cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$$

$$\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2i\sin(m\alpha - n\beta)$$

Hence proved.

(iv) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$

$$x^m y^n = (\cos\alpha + i\sin\alpha)^m(\cos\beta + i\sin\beta)^n$$

$$x^m y^n = [\cos(m\alpha) + i\sin(m\alpha)]\ [\cos(n\beta) + i\sin(n\beta)]$$

$$x^m y^n = \cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta)$$

$$\frac{1}{x^m y^n} = \cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta)$$

$$x^m y^n + \frac{1}{x^m y^n} = \cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta) + \cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta)$$

$$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$$

Hence proved.

Hint: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

5. Solve the equation $z^3 + 27 = 0$.

Solution:

$$z^3 + 27 = 0$$

$$z^3 + 3^3 = 0$$

$$(z + 3)(z^2 - 3z + 9) = 0$$

Case (i):

$$z + 3 = 0$$

$$z = -3$$

Case (ii):

$$z^2 - 3z + 9 = 0$$

Here $a = 1, b = -3, c = 9$

$$z = \frac{3 \pm \sqrt{9 - 4(9)}}{2}$$

$$= \frac{3 \pm \sqrt{-3(9)}}{2}$$

$$= \frac{3 \pm 3i\sqrt{3}}{2}$$

$$z = \frac{3}{2}\left(1 \pm i\sqrt{3}\right)$$

∴ The values of z is $-3, \frac{3}{2}\left(1 \pm i\sqrt{3}\right)$

Hint:

- **Polar form of $1 = (cos\ 0 + i\ sin\ 0)$**
- **$1 = cos\ 2k\pi + isin\ 2k\pi$**
- **$\omega = e^{\frac{i2\pi}{n}}$**

6. If $\omega \neq 1$ is a cube root of unity, show that the roots of the equation $(z - 1)^3 + 8 = 0$ are $-1, 1 - 2\omega, 1 - 2\omega^2$.

Solution:

Given $(z - 1)^3 + 8 = 0$

$$(z - 1)^3 = -8$$

$$(z - 1)^3 = (-2)^3 \times 1$$

$$z - 1 = -2(1)^{\frac{1}{3}}$$

$$z-1=-2\,[\cos(2k\pi)+i\sin\,(2k\pi)]^{\frac{1}{3}}$$

$$z-1=-2\left[\cos\left(\frac{2k\pi}{3}\right)+i\sin\left(\frac{2k\pi}{3}\right)\right], k=0,1,2.$$

When $k=0$,

$$z-1=-2\,[\cos 0+i\sin 0]$$

$$z=-2(1)+1$$

$$z=-1$$

When $k=1$,

$$z-1=-2\left[\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right]$$

$$z-1=-2\omega$$

$$z=1-2\omega$$

When $k=2$,

$$z-1=-2\left[\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right]$$

$$z-1=-2\omega^2$$

$$z=1-2\omega^2$$

$\therefore$ The values of z are $-1, 1-2\omega, 1-2\omega^2$

---

7. Find the value of $\sum_{k=1}^{8}\left(\cos\frac{2k\pi}{9}+i\sin\frac{2k\pi}{9}\right)$.

Hint: $1+\omega+\omega^2+\cdots+\omega^{n-1}=0$, where $\omega=cis\frac{2\pi}{n}$

$cis\,\theta=cos\theta+i\,sin\theta$

Solution:

Wkt, $1+\omega+\omega^2+\cdots+\omega^{n-1}=0$, where $\omega=cis\frac{2\pi}{n}$

Put $n=9$,

$$1+\omega+\omega^2+\cdots+\omega^{9-1}=0, \text{ where } \omega=cis\frac{2\pi}{9}$$

$$1+\omega+\omega^2+\cdots+\omega^8=0$$

$$\omega+\omega^2+\cdots+\omega^8=-1$$

$$cis\frac{2\pi}{9}+\left(cis\frac{2\pi}{9}\right)^2+\cdots+\left(cis\frac{2\pi}{9}\right)^8=-1$$

$$cis\frac{2\pi}{9}+cis\frac{4\pi}{9}+\cdots+cis\frac{16\pi}{9}=-1$$

$$\sum_{k=1}^{8}\left(cis\frac{2k\pi}{9}\right)=-1$$

---

8. If $\omega\neq 1$ is a cube root of unity, show that

(i) $(1-\omega+\omega^2)^6+(1+\omega-\omega^2)^6=128$

Solution:

Wkt $1+\omega+\omega^2=0$ and $\omega^3=1$

$$\text{LHS}=(1+\omega^2-\omega)^6+(1+\omega-\omega^2)^6$$

$$=(-\omega-\omega)^6+(-\omega^2-\omega^2)^6$$

$$=(-2\omega)^6+(-2\omega^2)^6$$

$$=2^6\omega^6+2^6\omega^{12}$$

$$=2^6(\omega^6+\omega^{12})$$

$$=2^6[(\omega^3)^2+(\omega^3)^4] \qquad [\because \omega^3=1]$$

$$=64(1+1)$$

$$=128$$

$$=\text{RHS}$$

$$(1-\omega+\omega^2)^6+(1+\omega-\omega^2)^6=128$$

Hence proved.

(ii) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots\left(1+\omega^{2^{11}}\right)=1$

Solution:

$$\text{LHS}=(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots(1+\omega^{1024})(1+\omega^{2048})$$

$$=(1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)\ldots(1+\omega)(1+\omega^2)$$

$$=(-\omega^2)(-\omega)(-\omega^2)(-\omega)\ldots(-\omega^2)(-\omega)$$

$$=\omega^3\omega^3\ldots\omega^3$$

$$=1.1\ldots1$$

$$=1$$

$$=\text{RHS}$$

$$(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots\left(1+\omega^{2^{11}}\right)=1$$

Hence proved.

---

9. If $z=2-2i$, find the rotation of z by θ radians in the counter clockwise direction about the origin when

(i) $\theta=\frac{\pi}{3}$ (ii) $\theta=\frac{2\pi}{3}$ (iii) $\theta=\frac{3\pi}{2}$.

Solution:

Let $z=2-2i$

Polar form of $2-2i=r(\cos\theta+i\sin\theta)$

$$r=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}$$

$$\alpha=\tan^{-1}\left|-\frac{2}{2}\right|=\tan^{-1}1$$

$$\alpha=\frac{\pi}{4}$$

Since $2-2i$ lies in IV quadrant, $\theta=-\alpha$

$$\theta=-\frac{\pi}{4}$$

$$z=2\sqrt{2}\,\left(\cos\left(\frac{-\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right)$$

(i) $\boldsymbol{\theta = \frac{\pi}{3}}$

z is rotated by $\theta = \frac{\pi}{3}$ in the counter clockwise direction.

$$z = 2\sqrt{2}\left[\left(\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\right)\right]$$

$$z = 2\sqrt{2}\left[\left(\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right)\right]$$

(ii) $\boldsymbol{\theta = \frac{2\pi}{3}}$

z is rotated by $\theta = \frac{2\pi}{3}$ in the counter clockwise direction.

$$z = 2\sqrt{2}\left[\left(\cos\left(\frac{2\pi}{3} - \frac{\pi}{4}\right) + i\sin\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)\right)\right]$$

$$z = 2\sqrt{2}\left[\left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right)\right]$$

(iii) $\boldsymbol{\theta = \frac{3\pi}{2}}$

z is rotated by $\theta = \frac{3\pi}{2}$ in the counter clockwise direction.

$$z = 2\sqrt{2}\left[\left(\cos\left(\frac{3\pi}{2} - \frac{\pi}{4}\right) + i\sin\left(\frac{3\pi}{2} - \frac{\pi}{4}\right)\right)\right]$$

$$z = 2\sqrt{2}\left[\left(\cos\left(\frac{12\pi - 2\pi}{8}\right) + i\sin\left(\frac{12\pi - 2\pi}{8}\right)\right)\right]$$

$$z = 2\sqrt{2}\left[\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right)\right]$$