CHAPTER – 4

Inverse Trigonometry Functions

Exercise 4.1

1. Find all values of x such that

(i) $-10\pi \leq x \leq 10\pi$ and $\sin x = 0$

Solution:

Given $\sin x = 0$

$\sin x = \sin 0$

$x = n\pi + (-1)^n(0), n \in R$

$x = n\pi, n = 0, \pm 1, \pm 2, \dots, \pm 10$

(ii) $-3\pi \leq x \leq 3\pi$ and $\sin x = -1$

Solution:

Given $\sin x = -1$

$\sin x = -\sin \frac{\pi}{2}$

$\sin x = \sin\left(\frac{-\pi}{2}\right)$

$x = (4n-1)\frac{\pi}{2}, n \in 0, \pm 1$

Hint: $y = Asin\alpha x$, $Amplitude = |A|$ and $period = \frac{2\pi}{|\alpha|}$

2. Find the period and amplitude of (i) $y = \sin 7x$

Solution:

Comparing with $y = A\sin\alpha x$, we get $A = 1,\ \alpha = 7$

Amplitude $= |A| = |1| = 1$

Period $= \frac{2\pi}{|\alpha|} = \frac{2\pi}{|7|} = \frac{2\pi}{7}$

(ii) $y = -\sin\frac{x}{3}$

Solution:

Comparing with $y = A\sin\alpha x$, we get $A = -1,\ \alpha = \frac{1}{3}$

Amplitude $= |A| = |-1| = 1$

Period $= \frac{2\pi}{|\alpha|} = \frac{2\pi}{\left|\frac{1}{3}\right|} = 6\pi$

(iii) $y = 4\sin(-2x)$

Solution:

Comparing with $y = A\sin\alpha x$, we get $A = 4,\ \alpha = -2$

Amplitude $= |A| = |4| = 4$

Period $= \frac{2\pi}{|\alpha|} = \frac{2\pi}{|-2|} = \frac{2\pi}{2} = \pi$

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3. Sketch the graph of $y = \sin\left(\frac{1}{3}x\right)$ for $0 \leq x \leq 6\pi$.

Solution:

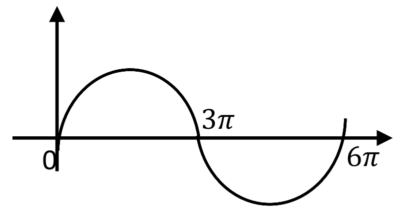
Given $y = \sin\left(\frac{1}{3}x\right)$

Comparing with $y = A\sin\alpha x$, we get $A = 1, \alpha = \frac{1}{3}$

Amplitude $= |A| = 1$

Period $= \frac{2\pi}{\left|\frac{1}{3}\right|} = 6\pi$

The graph is,



Hint: $sin^{-1}(sin\ x) = \begin{cases} x, & if\ x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \\ \pi - x, & if\ x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \end{cases}$

4. Find the value of (i) $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

Solution:

$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \pi - \frac{2\pi}{3}$

$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$

Solution:

$\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right) = \pi - \frac{5\pi}{4}$

$\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right) = -\frac{\pi}{4} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

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5. For what value of x does $\sin x = \sin^{-1} x$?

Solution:

Given $\sin x = \sin^{-1} x$

At $x = 0$,

$\sin x = \sin^{-1} x$

$\sin 0 = \sin^{-1} 0$

$0 = 0$

Only when $x = 0$, then $\sin x = \sin^{-1} x$

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6. Find the domain of the following:

(i) $f(x) = \sin^{-1}\left(\frac{x^2+1}{2x}\right)$

Solution:

$$-1 \le \frac{x^2+1}{2x} \le 1,\ \ x \ne 0$$

$\times\, 2x^2,\quad -2x^2 \le x(x^2+1) \le 2x^2$

Case 1:

$$-2x^2 \le x(x^2+1)$$

$$0 \le x(x^2+1) + 2x^2$$

$$0 \le x(x^2+1+2x)$$

$$0 \le x(x+1)^2 \qquad \text{-----------(1)}$$

$$\begin{array}{c|c} & 1 \\ 1 & 1 \\ \hline & 2 \end{array}$$

Case 2:

$$x(x^2+1) \le 2x^2$$

$$x(x^2+1) - 2x^2 \le 0$$

$$x(x^2+1-2x) \le 0$$

$$x(x-1)^2 \le 0 \qquad \text{----------(2)}$$

$$\begin{array}{c|c} & 1 \\ -1 & -1 \\ \hline & -2 \end{array}$$

The critical numbers are $-1,0,1$

$\therefore$ Domain is $x = \{-1,1\}$

(*ii*) $g(x) = 2\sin^{-1}(2x-1) - \frac{\pi}{4}$

Solution:

$$-1 \le 2x-1 \le 1$$

$+1,\qquad 0 \le 2x \le 2$

$\div\, 2,\qquad 0 \le x \le 1$

$\therefore$ Domain is $x = [0,1]$

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7. Find the value of $\sin^{-1}\left(\sin\frac{5\pi}{9}\cos\frac{\pi}{9} + \cos\frac{5\pi}{9}\sin\frac{\pi}{9}\right)$.

Hint: $\sin(A+B) = \sin A\cos B + \cos A\sin B$

Solution:

$$\sin\frac{5\pi}{9}\cos\frac{\pi}{9} + \cos\frac{5\pi}{9}\sin\frac{\pi}{9} = \sin\left(\frac{5\pi}{9} + \frac{\pi}{9}\right)$$

$$= \sin\left(\frac{6\pi}{9}\right)$$

$$= \sin\frac{2\pi}{3}$$

$$= \sin\left(\pi - \frac{2\pi}{3}\right)$$

$$\sin\frac{5\pi}{9}\cos\frac{\pi}{9} + \cos\frac{5\pi}{9}\sin\frac{\pi}{9} = \sin\frac{\pi}{3}$$

$$\sin^{-1}\left(\sin\frac{5\pi}{9}\cos\frac{\pi}{9} + \cos\frac{5\pi}{9}\sin\frac{\pi}{9}\right) = \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$$

Exercise 4.2

1. Find all values of x such that

(i) $-6\pi \le x \le 6\pi$ *and* $\cos x = 0$

Solution:

Given $\cos x = 0$

$$\cos x = \cos\frac{\pi}{2}$$

$$x = (2n+1)\frac{\pi}{2}$$

$$n = 0, \pm1, \pm2, \pm3, \pm4, \pm5, -6$$

(ii) $-5\pi \le x \le 5\pi$ *and* $\cos x = -1$

Solution:

Given $\cos x = -1$

$$\cos x = \cos\pi$$

$$x = (2n+1)\pi$$

$$n = 0, \pm1, \pm2, -3$$

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2. State the reason for $\cos^{-1}\left[\cos\left(\frac{-\pi}{6}\right)\right] \ne -\frac{\pi}{6}$.

Solution:

Wkt, $\cos^{-1}(\cos x) = x$ if $x \in [0,\pi]$

$$\cos^{-1}\left[\cos\left(\frac{-\pi}{6}\right)\right] \ne \frac{-\pi}{6},\ \frac{-\pi}{6} \notin [0,\pi]$$

Reason:

$$\cos^{-1}\left[\cos\left(\frac{-\pi}{6}\right)\right] = \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right] = \frac{\pi}{6},\ \frac{\pi}{6} \in [0,\pi]$$

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3. Is $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$ true? Justify your answer.

Solution:

$$\text{Let } y = \cos^{-1}(-x) \qquad \text{----------(1)}$$

$$\cos y = -x$$

$$-\cos y = x$$

$$\cos(\pi - y) = x \qquad [\because \cos(\pi - x) = -\cos x]$$

$$\pi - y = \cos^{-1}x$$

$$\pi - \cos^{-1}x = y \qquad \text{----------(2)}$$

From (1) and (2), we get

$$\cos^{-1}(-x) = \pi - \cos^{-1}x$$

Hence It is true.

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4. Find the principal value of $\cos^{-1}\left(\frac{1}{2}\right)$.

Solution:

$$\text{Let } y = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\cos y = \frac{1}{2}$$

$$\cos y = \cos\frac{\pi}{3}$$

$y = \frac{\pi}{3} \in [0, \pi]$

$\therefore$ The principal value of $\cos^{-1}\left(\frac{1}{2}\right)$ is $\frac{\pi}{3}$

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5. Find the value of

(i) $2\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$

Solution:

Given $= 2\cos^{-1}\left(\frac{1}{2}\right) + sin^{-1}\left(\frac{1}{2}\right)$

$= 2\left(\frac{\pi}{3}\right) + \frac{\pi}{6}$

$= \frac{4\pi}{6} + \frac{\pi}{6}$

$= \frac{5\pi}{6}$

Hint: $sin^{-1}(-x) = -sin^{-1}x$

(ii) $\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}(-1)$

Solution:

Given $= \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}(-1)$

$= \frac{\pi}{3} - \sin^{-1} 1$

$= \frac{\pi}{3} - \frac{\pi}{2}$

$= \frac{2\pi - 3\pi}{6}$

$= -\frac{\pi}{6}$

Hint: $cos(A+B) = cosAcosB - sinAsinB$

(iii) $\cos^{-1}\left(\cos\frac{\pi}{7}\cos\frac{\pi}{17} - \sin\frac{\pi}{7}\sin\frac{\pi}{17}\right)$

Solution:

Given $= \cos^{-1}\left(\cos\frac{\pi}{7}\cos\frac{\pi}{17} - \sin\frac{\pi}{7}\sin\frac{\pi}{17}\right)$

$= \cos^{-1}\left(\cos\left(\frac{\pi}{7} + \frac{\pi}{17}\right)\right)$

$= \cos^{-1}\left(\cos\left(\frac{17\pi + 7\pi}{119}\right)\right)$

$= \cos^{-1}\left(\cos\left(\frac{24\pi}{119}\right)\right)$

$= \frac{24\pi}{119} \in [0, \pi]$

Hint: $If\ |x| \le a\ then\ -a \le x \le a$

6. Find the domain of

(i) $f(x) = \sin^{-1}\left(\frac{|x|-2}{3}\right) + \cos^{-1}\left(\frac{1-|x|}{4}\right)$

Solution:

Now $\sin^{-1}\left(\frac{|x|-2}{3}\right)$,

$-1 \le \frac{|x|-2}{3} \le 1$

$\times 3,\quad -3 \le |x| - 2 \le 3$

$+2,\quad -1 \le |x| \le 5$ ----------(1)

Now $\cos^{-1}\left(\frac{1-|x|}{4}\right)$,

$-1 \le \frac{1-|x|}{4} \le 1$

$\times 4,\quad -4 \le 1 - |x| \le 4$

$-1,\quad -5 \le -|x| \le 3$

$\times(-1),\quad 5 \ge |x| \ge -3$

$-3 \le |x| \le 5$ ----------(2)

From (1) and (2), we get

$|x| \le 5$

$-5 \le x \le 5$ [by hint]

$x \in [-5, 5]$

$(ii)\ g(x) = \sin^{-1}x + \cos^{-1}x$

Solution:

For $\sin^{-1}x$,

$-1 \le x \le 1$ ----------(1)

For $\cos^{-1}x$,

$-1 \le x \le 1$ ----------(2)

From (1) and (2), we get

$x \in [-1, 1]$

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7. For what values of x, the inequality $\frac{\pi}{2} < \cos^{-1}(3x - 1) < \pi$ holds?

Solution:

$\frac{\pi}{2} < \cos^{-1}(3x - 1) < \pi$

$\times cos,\quad \cos\frac{\pi}{2} > 3x - 1 > \cos\pi$

$0 > 3x - 1 > -1$

$+1,\quad 1 > 3x > 0$

$\div 3,\quad \frac{1}{3} > x > 0$

$0 < x < \frac{1}{3}$

$\therefore$ The inequality holds when $x > 0$ and $x < \frac{1}{3}$

Hint: $cos^{-1}x + sin^{-1}x = \frac{\pi}{2}, x \in [-1, 1]$

8. Find the value of

(i) $\cos\left(\cos^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{4}{5}\right)\right)$

Solution:

$$\text{Given} = \cos\left(\cos^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{4}{5}\right)\right)$$
$$= \cos\left(\frac{\pi}{2}\right)$$
$$= 0$$

Hint: $cos^{-1}(cosx) = \begin{cases} x, & if\ x \in [0,\pi] \\ 2\pi - x, & if\ x \in [\pi, 2\pi] \end{cases}$

(ii) $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)+\cos^{-1}\left(\cos\frac{5\pi}{4}\right)$

Solution:

$$\text{Given} = \cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)+\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$$
$$= 2\pi - \frac{4\pi}{3} + 2\pi - \frac{5\pi}{4}$$
$$= 4\pi - \frac{4\pi}{3} - \frac{5\pi}{4}$$
$$= \frac{48\pi - 16\pi - 15\pi}{12}$$
$$= \frac{17\pi}{12}$$

Exercise 4.3

Hint: For $tan^{-1}x$,

$Domain = R\ and\ range = \left(\frac{-\pi}{2},\frac{\pi}{2}\right), R = (-\infty,\infty)$

1. Find the domain of the following functions:

(i) $tan^{-1}(\sqrt{9-x^2})$

Solution:

$\sqrt{9-x^2} \in R$ $\qquad [\because R = (-\infty,\infty)]$

Wkt, $9 - x^2 \geq 0$

$9 \geq x^2$

$3 \geq |x|$

$|x| \leq 3$

$-3 \leq x \leq 3$

$\therefore$ The domain is $[-3,3]$

(ii) $\frac{1}{2}\tan^{-1}(1-x^2) - \frac{\pi}{4}$

Solution:

$\Rightarrow \quad 1 - x^2 \in R$

$\Rightarrow \quad x \in R$

$\Rightarrow \quad 1 - x^2$ is exist for all $x \in R$

$\therefore$ The domain is R

Hint: $tan^{-1}(tanx) = \begin{cases} x, & if\ x \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right) \\ x - \pi, & otherwise \end{cases}$

2. Find the value of

(i) $\tan^{-1}\left(\tan\frac{5\pi}{4}\right)$

Solution:

$$\tan^{-1}\left(\tan\frac{5\pi}{4}\right) = \frac{5\pi}{4} - \pi$$
$$= \frac{5\pi - 4\pi}{4}$$
$$\tan^{-1}\left(\tan\frac{5\pi}{4}\right) = \frac{\pi}{4} \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right)$$

(ii) $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)$

Solution:

$$\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right) = -\frac{\pi}{6} \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right)$$

Hint: $tan(tan^{-1}x) = x,\ \forall\ x \in R$

3. Find the value of

(i) $\tan\left(\tan^{-1}\left(\frac{7\pi}{4}\right)\right)$

Solution:

$$\tan\left(\tan^{-1}\left(\frac{7\pi}{4}\right)\right) = \frac{7\pi}{4} \in R$$

(ii) $\tan(\tan^{-1}(1947))$

Solution:

$$\tan(\tan^{-1}(1947)) = 1947 \in R$$

(iii) $\tan(\tan^{-1}(-0.2021))$

Solution:

$$\tan(\tan^{-1}(-0.2021)) = -0.2021 \in R$$

Hint: $cos^{-1}x + sin^{-1}x = \frac{\pi}{2}, x \in [-1,1]$

4. Find the value of

(i) $\tan\left(\cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(\frac{-1}{2}\right)\right)$

Solution:

$$\text{Given} = \tan\left(\cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(-\frac{1}{2}\right)\right)$$
$$= \tan\left(\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(\frac{1}{2}\right)\right)$$
$$= \tan\left(\frac{\pi}{2}\right)$$
$$= \infty$$

Hint: $sin(sin^{-1}x) = x, if\ x \in [-1,1]$

$$\boldsymbol{sin^{-1}x - sin^{-1}y = sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)}$$

(ii) $\sin\left(\tan^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{4}{5}\right)\right)$

Solution:

For $\tan^{-1}\left(\frac{1}{2}\right)$:	For $\cos^{-1}\left(\frac{4}{5}\right)$:
$h^2 = o^2 + a^2$	$h^2 = o^2 + a^2$
$h^2 = 1^1 + 2^2$	$5^2 = o^2 + 4^2$
$h^2 = 1 + 4 = 5$	$o^2 = 25 - 16 = 9$
$h = \sqrt{5}$	$o = 3$

Given $= \sin\left(\tan^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{4}{5}\right)\right)$

$= \sin\left(\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - sin^{-1}\left(\frac{3}{5}\right)\right)$

$= \sin\left(\sin^{-1}\left(\frac{1}{\sqrt{5}}\sqrt{1-\frac{9}{25}} - \frac{3}{5}\sqrt{1-\frac{1}{5}}\right)\right)$

$= \sin\left(\sin^{-1}\left(\frac{1}{\sqrt{5}}\sqrt{\frac{25-9}{25}} - \frac{3}{5}\sqrt{\frac{5-1}{5}}\right)\right)$

$= \sin\left(\sin^{-1}\left(\frac{1}{\sqrt{5}}\sqrt{\frac{16}{25}} - \frac{3}{5}\sqrt{\frac{4}{5}}\right)\right)$

$= \sin\left(\sin^{-1}\left(\frac{4}{5\sqrt{5}} - \frac{6}{5\sqrt{5}}\right)\right)$

$= \sin\left(\sin^{-1}\left(\frac{-2}{5\sqrt{5}}\right)\right)$

$= \frac{-2}{5\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{-2\sqrt{5}}{25}$

Hint: $\boldsymbol{cos(cos^{-1}x) = x, if\ x \in [-1,1]}$

$$\boldsymbol{cos^{-1}x - cos^{-1}y = cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)}$$

(iii) $\cos\left(\sin^{-1}\left(\frac{4}{5}\right) - \tan^{-1}\left(\frac{3}{4}\right)\right)$

Solution:

For $\sin^{-1}\left(\frac{4}{5}\right)$:	For $\tan^{-1}\left(\frac{3}{4}\right)$:
$h^2 = o^2 + a^2$	$h^2 = o^2 + a^2$
$5^2 = 4^2 + a^2$	$h^2 = 3^2 + 4^2$
$a^2 = 25 - 16 = 9$	$h^2 = 9 + 16 = 25$
$a = 3$	$h = 5$

Given $= \cos\left(\sin^{-1}\left(\frac{4}{5}\right) - \tan^{-1}\left(\frac{3}{4}\right)\right)$

$= \cos\left(\cos^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{4}{5}\right)\right)$

$= \cos\left(\cos^{-1}\left(\frac{3}{5}\times\frac{4}{5} + \sqrt{1-\frac{9}{25}}\sqrt{1-\frac{16}{25}}\right)\right)$

$= \cos\left(\cos^{-1}\left(\frac{12}{25} + \sqrt{\frac{25-9}{25}}\sqrt{\frac{25-16}{25}}\right)\right)$

$= \cos\left(cos^{-1}\left(\frac{12}{25} + \sqrt{\frac{16}{25}}\sqrt{\frac{9}{25}}\right)\right)$

$= \cos\left(\cos^{-1}\left(\frac{12}{25} + \frac{4}{5}\times\frac{3}{5}\right)\right)$

$= \cos\left(\cos^{-1}\left(\frac{12+12}{25}\right)\right)$

$= \cos\left(\cos^{-1}\left(\frac{24}{25}\right)\right)$

$= \frac{24}{25}$

Exercise 4.4

1. Find the principal value of

(i) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Solution:

Let $y = \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

$y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ $\quad\left[\because \sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)\right]$

$y = \frac{\pi}{6} \in [0, \pi]$

(ii) $\cot^{-1}(\sqrt{3})$

Solution:

Let $y = cot^{-1}(\sqrt{3})$

$y = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ $\quad\left[\because \cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)\right]$

$y = \frac{\pi}{6} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(iii) $cosec^{-1}(-\sqrt{2})$

Solution:

Let $y = cosec^{-1}(-\sqrt{2})$

$y = -cosec^{-1}(\sqrt{2})$

$y = -\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ $\quad\left[\because \operatorname{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)\right]$

$y = -\frac{\pi}{4} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

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2. Find the value of

(i) $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$

Solution:

Given $= \tan^{-1}\sqrt{3} - \sec^{-1}(-2)$

$= \frac{\pi}{3} - \cos^{-1}\left(-\frac{1}{2}\right)$

$= \frac{\pi}{3} - \left(\pi - \cos^{-1}\left(\frac{1}{2}\right)\right)$

$= \frac{\pi}{3} - \left(\pi - \frac{\pi}{3}\right)$

$= \frac{\pi}{3} - \frac{(3\pi-\pi)}{3}$

$= \frac{\pi}{3} - \frac{2\pi}{3}$

$= -\frac{\pi}{3}$

(ii) $\sin^{-1}(-1) + \cos^{-1}\left(\frac{1}{2}\right) + \cot^{-1}(2)$

Solution:

Given $= \sin^{-1}(-1) + \cos^{-1}\left(\frac{1}{2}\right) + \cot^{-1}(2)$

$= -\sin^{-1}(1) + \cos^{-1}\left(\frac{1}{2}\right) + \cot^{-1}(2)$

$= -\frac{\pi}{2} + \frac{\pi}{3} + \cot^{-1}(2)$

$= \frac{-3\pi+2\pi}{6} + \cot^{-1}(2)$

$= -\frac{\pi}{6} + \cot^{-1}(2)$

(iii) $\cot^{-1}(1) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sec^{-1}(-\sqrt{2})$

Solution:

Given $= \cot^{-1}(1) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sec^{-1}(-\sqrt{2})$

$= \tan^{-1}(1) - \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

$= \frac{\pi}{4} - \frac{\pi}{3} - \left(\pi - \frac{\pi}{4}\right)$ $[\because \cos^{-1}(-x) = \pi - \cos^{-1}x]$

$= \frac{\pi}{4} - \frac{\pi}{3} - \pi + \frac{\pi}{4}$

$= \frac{3\pi-4\pi-12\pi+3\pi}{12}$

$= \frac{6\pi-16\pi}{12}$

$= -\frac{10\pi}{12}$

$= -\frac{5\pi}{6}$

Exercise 4.5

1. Find the value, if it exists. If not, give the reason for non-existence.

(i) $\sin^{-1}(\cos\pi)$

Solution:

Given $= \sin^{-1}(\cos\pi)$

$= \sin^{-1}(-1)$

$= -\sin^{-1}(1)$

$= -\frac{\pi}{2}$

(ii) $\tan^{-1}\left(\sin\left(-\frac{5\pi}{2}\right)\right)$

Solution:

Given $= \tan^{-1}\left(\sin\left(-\frac{5\pi}{2}\right)\right)$

$= \tan^{-1}\left(-\sin\left(\frac{5\pi}{2}\right)\right)$ $\left[\because \sin\left(2\pi + \frac{\pi}{2}\right) = \sin\frac{\pi}{2}\right]$

$= \tan^{-1}\left(-\sin\frac{\pi}{2}\right)$

$= \tan^{-1}(-1)$

$= -\tan^{-1}(1)$

$= -\frac{\pi}{4}$

Hint: $\sin(2\pi - \theta) = -\sin\theta$ and $-\sin\theta = \sin(-\theta)$

$\sin^{-1}(\sin x) = x, \text{if } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $[-1.57, 1.57]$

(iii) $\sin^{-1}(\sin 5)$

Solution:

Given $= \sin^{-1}(\sin 5) \neq 5 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin^{-1}(\sin 5) = \sin^{-1}(\sin(2\pi - (2\pi - 5)))$

$= \sin^{-1}(-\sin(2\pi - 5))$

$= \sin^{-1}(\sin(5 - 2\pi))$

$\text{Sin}^{-1}(\sin 5) = 5 - 2\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $[\because 5 - 6.28 = -1.28]$

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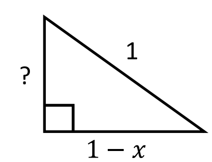
2. Find the value of the expression in terms of x, with the help of a reference triangle.

(i) $\sin(\cos^{-1}(1-x))$



Solution:

For $\cos^{-1}(1-x) = \frac{adj}{hyp}$

By pythagoras theorem,

$1^2 = o^2 + (1-x)^2$

$o^2 = 1 - (1-x)^2$

$o^2 = 1 - 1 - x^2 + 2x = -x^2 + 2x$

$o = \sqrt{2x - x^2}$

Given $= \sin(\cos^{-1}(1-x))$

$= \sin\left(\sin^{-1}\left(\frac{\sqrt{2x-x^2}}{1}\right)\right)$

$= \sqrt{2x - x^2}$

(ii) $\cos(\tan^{-1}(3x-1))$

Solution:

For $\tan^{-1}(3x-1) = \frac{opp}{adj}$

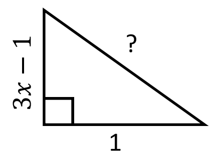
By Pythagoras theorem,

$$h^2 = 1^2 + (3x-1)^2$$

$$h^2 = 1 + 9x^2 - 6x + 1$$

$$h^2 = 9x^2 - 6x + 2$$

$$h = \sqrt{9x^2 - 6x + 2}$$



Given $= \cos(\tan^{-1}(3x-1))$

$$= \cos\left(\cos^{-1}\frac{1}{\sqrt{9x^2-6x+2}}\right)$$

$$= \frac{1}{\sqrt{9x^2-6x+2}}$$

(iii) $\tan\left(\sin^{-1}\left(x+\frac{1}{2}\right)\right)$

Solution:

For $\sin^{-1}\left(x+\frac{1}{2}\right) = \sin^{-1}\left(\frac{2x+1}{2}\right) = \frac{opp}{hyp}$

By Pythagoras theorem,

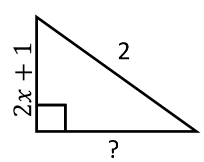
$$2^2 = (2x+1)^2 + a^2$$

$$a^2 = 4 - (2x+1)^2$$

$$a^2 = 4 - 4x^2 - 4x - 1$$

$$a^2 = -4x^2 - 4x + 3$$

$$a = \sqrt{3 - 4x - 4x^2}$$



Given $= \tan\left(\sin^{-1}\left(x+\frac{1}{2}\right)\right)$

$$= \tan\left(\tan^{-1}\left(\frac{2x+1}{\sqrt{3-4x-4x^2}}\right)\right)$$

$$= \frac{2x+1}{\sqrt{3-4x-4x^2}}$$

---

3. Find the value of

(i) $\sin^{-1}\left(\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right)$

Solution:

Given $= \sin^{-1}\left(\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right)$

$$= \sin^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{6}$$

(ii) $\cot\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5}\right)$

Solution:

Given $= \cot\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5}\right)$

$$= \cot\left(\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{3}{5}\right)$$

$$= \cot\left(\frac{\pi}{2}\right)$$

$$= \frac{1}{\tan\left(\frac{\pi}{2}\right)}$$

$$= \frac{1}{\infty}$$

$$= 0$$

For $\sin^{-1}\frac{4}{5} = \frac{opp}{hyp}$

$$5^2 = 4^2 + a^2$$

$$a^2 = 25 - 16 = 9$$

$$a = 3$$

Hint: $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(iii) $\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

Solution:

Given $= \tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

$$= \tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right)\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{\frac{9+8}{12}}{\frac{12-6}{12}}\right)\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{9+8}{12-6}\right)\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{17}{6}\right)\right)$$

$$= \frac{17}{6}$$

For $\sin^{-1}\frac{3}{5} = \frac{opp}{hyp}$

$$5^2 = 3^2 + a^2$$

$$a^2 = 25 - 9 = 16$$

$$a = 4$$

---

4. Prove that

(i) $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

Solution:

LHS $= \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24}$

$$= \tan^{-1}\left(\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\times\frac{7}{24}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{48+77}{264}}{\frac{264-14}{264}}\right)$$

$$= \tan^{-1}\left(\frac{48+77}{264-14}\right)$$

$$= \tan^{-1}\left(\frac{125}{250}\right)$$

$$= \tan^{-1}\left(\frac{1}{2}\right)$$

$$= \text{RHS}$$

Hence Proved.

$$\boldsymbol{sin^{-1}x - sin^{-1}y = sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)}$$

(ii) $\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{16}{65}$

Solution:

For $\cos^{-1}\frac{12}{13} = \frac{adj}{hyp}$

$13^2 = o^2 + 12^2$

$o^2 = 169 - 144 = 25$

$o = 5$

$$\text{LHS} = \sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13}$$

$$= \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{5}{13}$$

$$= \sin^{-1}\left(\frac{3}{5}\sqrt{1-\frac{25}{169}} - \frac{5}{13}\sqrt{1-\frac{9}{25}}\right)$$

$$= \sin^{-1}\left(\frac{3}{5}\sqrt{\frac{169-25}{169}} - \frac{5}{13}\sqrt{\frac{25-9}{25}}\right)$$

$$= \sin^{-1}\left(\frac{3}{5}\sqrt{\frac{144}{169}} - \frac{5}{13}\sqrt{\frac{16}{25}}\right)$$

$$= \sin^{-1}\left(\frac{3}{5}\times\frac{12}{13} - \frac{5}{13}\times\frac{4}{5}\right)$$

$$= \sin^{-1}\left(\frac{36-20}{65}\right)$$

$$= \sin^{-1}\left(\frac{16}{65}\right)$$

$$= \text{RHS}$$

Hence proved.

---

5. Prove that $\boldsymbol{tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]}$.

Solution:

$$\text{LHS} = \tan^{-1}x + \tan^{-1}y + \tan^{-1}z$$

$$= (\tan^{-1}x + \tan^{-1}y) + \tan^{-1}z$$

$$= \tan^{-1}\left(\frac{x+y}{1-xy}\right) + \tan^{-1}z$$

$$= \tan^{-1}\left(\frac{\frac{x+y}{1-xy}+z}{1-\left(\frac{x+y}{1-xy}\right)(z)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{x+y+z(1-xy)}{1-xy}}{\frac{1-xy-(x+y)z}{1-xy}}\right)$$

$$= \tan^{-1}\left(\frac{x+y+(1-xy)z}{1-xy-(x+y)z}\right)$$

$$= \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-zx-yz}\right)$$

LHS = RHS

Hence Proved.

---

6. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, show that $x + y + z = xyz$.

Solution:

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$$

$$\tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right) = \pi$$

$$\frac{x+y+z-xyz}{1-xy-yz-zx} = \tan\pi$$

$$\frac{x+y+z-xyz}{1-xy-yz-zx} = 0$$

$$x+y+z-xyz = 0$$

$$x+y+z = xyz$$

Hence Proved.

7.Prove that $\tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\frac{3x-x^3}{1-3x^2}$, $|x| < \frac{1}{\sqrt{3}}$.

Solution:

$$\text{LHS} = \tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2}$$

$$= \tan^{-1}\left(\frac{x+\frac{2x}{1-x^2}}{1-x\left(\frac{2x}{1-x^2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{(1-x^2)x+2x}{1-x^2}}{\frac{1-x^2-x(2x)}{1-x^2}}\right)$$

$$= \tan^{-1}\left(\frac{(1-x^2)x+2x}{1-x^2-x(2x)}\right)$$

$$= \tan^{-1}\left(\frac{x-x^3+2x}{1-x^2-2x^2}\right)$$

$$= \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$= \text{RHS}$$

Hence Proved.

Hint: $\boldsymbol{tan^{-1}x - tan^{-1}y = tan^{-1}\left(\frac{x-y}{1+xy}\right)}$

8. Simplify $\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{x-y}{x+y}$.

Solution:

$$\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{x-y}{x+y} = \tan^{-1}\left(\frac{\frac{x}{y}-\frac{x-y}{x+y}}{1+\left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right)$$

$$\tan^{-1}\frac{x}{y}-\tan^{-1}\frac{x-y}{x+y}=\tan^{-1}\left(\frac{\frac{x(x+y)-y(x-y)}{y(x+y)}}{\frac{y(x+y)+x(x-y)}{y(x+y)}}\right)$$

$$=\tan^{-1}\left(\frac{x(x+y)-y(x-y)}{y(x+y)+x(x-y)}\right)$$

$$=\tan^{-1}\left(\frac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right)$$

$$=\tan^{-1}(1)$$

$$\tan^{-1}\frac{x}{y}-\tan^{-1}\frac{x-y}{x+y}=\frac{\pi}{4}$$

---

9. Solve:

(i) $\sin^{-1}\frac{5}{x}+\sin^{-1}\frac{12}{x}=\frac{\pi}{2}$

Solution:

$$\sin^{-1}\frac{5}{x}+\sin^{-1}\frac{12}{x}=\frac{\pi}{2}$$

$$\sin^{-1}\frac{5}{x}=\frac{\pi}{2}-\sin^{-1}\frac{12}{x}$$

$$\sin^{-1}\frac{5}{x}=\cos^{-1}\frac{12}{x}\qquad\left[\because \cos^{-1}x+\sin^{-1}x=\frac{\pi}{2}\right]$$

$$\sin^{-1}\frac{5}{x}=\cos^{-1}\frac{12}{x}=\theta\ (say)$$

$\sin^{-1}\frac{5}{x}=\theta$	$\cos^{-1}\frac{12}{x}=\theta$
$\sin\theta=\frac{5}{x}$	$\cos\theta=\frac{12}{x}$

Wkt, $\sin^2\theta+\cos^2\theta=1$

$$\left(\frac{5}{x}\right)^2+\left(\frac{12}{x}\right)^2=1$$

$$\frac{25}{x^2}+\frac{144}{x^2}=1$$

$$\frac{169}{x^2}=1$$

$$x^2=169$$

$$x=\pm 13$$

$$x=13$$

Hint: $2tan^{-1}x=cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), x\geq 0$

(ii) $2\tan^{-1}x=cos^{-1}\frac{1-a^2}{1+a^2}-\cos^{-1}\frac{1-b^2}{1+b^2}\ , a>0, b>0$

Solution:

$$2\tan^{-1}x=\cos^{-1}\frac{1-a^2}{1+a^2}-\cos^{-1}\frac{1-b^2}{1+b^2}$$

$$2\tan^{-1}x=2\tan^{-1}a-2\tan^{-1}b$$

$\div 2,\quad \tan^{-1}x=\tan^{-1}a-\tan^{-1}b$

$$\tan^{-1}x=\tan^{-1}\left(\frac{a-b}{1+ab}\right)$$

$$x=\frac{a-b}{1+ab}$$

Hint: $2tan^{-1}x=tan^{-1}\left(\frac{2x}{1-x^2}\right), |x|<1$

(iii) $2\tan^{-1}(\cos x)=\tan^{-1}(2cosec\ x)$

Solution:

$$2\tan^{-1}(\cos x)=\tan^{-1}(2cosec\ x)$$

$$\tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right)=\tan^{-1}(2cosec\ x)$$

$$\frac{2\cos x}{1-\cos^2 x}=2cosec\ x$$

$$\frac{\cos x}{\sin^2 x}=\frac{1}{\sin x}\qquad[\because \sin^2 x+\cos^2 x=1]$$

$$\sin x\cos x=\sin^2 x$$

$$\sin x\cos x-\sin^2 x=0$$

$$\sin x(\cos x-\sin x)=0$$

When $\sin x=0$

$$\boldsymbol{x=n\pi, n\in Z}\qquad\text{----------(1)}$$

When $\cos x-\sin x=0$

$$-\sin x=-\cos x$$

$$\frac{\sin x}{\cos x}=1$$

$$\tan x=1$$

$$\boldsymbol{x=n\pi+\frac{\pi}{4}, n\in Z}\qquad\text{----------(2)}$$

Hint: $tanA-tanB=\frac{tanA-tanB}{1+tanAtanB}$

(iv) $\cot^{-1}x-\cot^{-1}(x+2)=\frac{\pi}{12}, x>0$

Solution:

$$\cot^{-1}x-cot^{-1}(x+2)=\frac{\pi}{12}$$

$$\tan^{-1}\frac{1}{x}-\tan^{-1}\frac{1}{x+2}=\frac{\pi}{12}$$

$$\tan^{-1}\left(\frac{\frac{1}{x}-\frac{1}{x+2}}{1+\left(\frac{1}{x}\right)\left(\frac{1}{x+2}\right)}\right)=\frac{\pi}{12}$$

$$\tan^{-1}\left(\frac{\frac{x+2-x}{x(x+2)}}{\frac{x(x+2)+1}{x(x+2)}}\right)=\frac{\pi}{12}$$

$$\tan^{-1}\left(\frac{x+2-x}{x(x+2)+1}\right)=\frac{\pi}{12}$$

$$\frac{2}{x^2+2x+1}=\tan\frac{\pi}{12}=\tan 15°$$

$$\frac{2}{(x+1)^2}=\tan(45°-30°)$$

$$\frac{2}{(x+1)^2}=\frac{\tan 45°-\tan 30°}{1+\tan 45°\tan 30°}\qquad\text{[by Hint]}$$

$$\frac{2}{(x+1)^2}=\frac{1-\frac{1}{\sqrt{3}}}{1+(1)\left(\frac{1}{\sqrt{3}}\right)}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$\frac{2}{(x+1)^2} = \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$\frac{2}{(x+1)^2} = \frac{\sqrt{3}^2-1^2}{\left(\sqrt{3}+1\right)^2}$$

$$\frac{2}{(x+1)^2} = \frac{2}{\left(\sqrt{3}+1\right)^2}$$

$$(x+1)^2 = \left(\sqrt{3}+1\right)^2$$

$$x+1 = \sqrt{3}+1$$

$$x = \sqrt{3}$$

---

10. Find the number of solutions of the equation $\tan^{-1}(x-1) + tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}(3x)$.

Solution:

$$\tan^{-1}(x-1) + \tan^{-1} x + \tan^{-1}(x+1) = \tan^{-1}(3x)$$

$$\tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}(3x) - \tan^{-1} x$$

$$\tan^{-1}\left(\frac{x-1+x+1}{1-(x-1)(x+1)}\right) = \tan^{-1}\left(\frac{3x-x}{1+(3x)(x)}\right)$$

$$\frac{2x}{1-(x^2-1)} = \frac{2x}{1+3x^2}$$

$$x(1+3x^2) = x\left(1-(x^2-1)\right)$$

$$x+3x^3 = x(1-x^2+1)$$

$$x+3x^3 = x(2-x^2)$$

$$x+3x^3 = 2x - x^3$$

$$4x^3 - x = 0$$

which is a cubic equation.

∴ The number of solutions is 3.
