CHAPTER – 10

Ordinary Differential Equations

Exercise 10.1

Hint: The highest order derivative is Order

The power of highest order derivative is Degree

1. For each of the following differential equations, its order, degree (if exists)

(i) $\frac{dy}{dx} + xy = \cot x$

Solution:

The highest order derivative is $\frac{dy}{dx}$

$\therefore$ The order is 1 and degree is 1.

(ii) $\left(\frac{d^3y}{dx^3}\right)^{\frac{2}{3}} - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4 = 0$

Solution:

$$\left(\frac{d^3y}{dx^3}\right)^{\frac{2}{3}} = 3\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 4$$

$$\left(\frac{d^3y}{dx^2}\right)^2 = \left[3\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 4\right]^3$$

The highest order derivative is $\frac{d^3y}{dx^3}$

$\therefore$ The order is 3 and the degree is 2.

(iii) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{d^2y}{dx^2}\right)$

Solution:

The highest order derivative is $\frac{d^2y}{dx^2}$

$\therefore$ The order is 2.

The given equation is can't be written in a polynomial form.

$\therefore$ The degree of the given equation is not defined.

(iv) $\sqrt{\frac{dy}{dx}} - 4\frac{dy}{dx} - 7x = 0$

Solution:

$$\sqrt{\frac{dy}{dx}} = 4\frac{dy}{dx} + 7x$$

$$\frac{dy}{dx} = \left[4\frac{dy}{dx} + 7x\right]^2$$

The highest order derivative is $\frac{dy}{dx}$

$\therefore$ The order is 1 and the degree is 2.

(v) $y\left(\frac{dy}{dx}\right) = \frac{x}{\left(\frac{dy}{dx}\right)+\left(\frac{dy}{dx}\right)^3}$

Solution:

$$y\left(\frac{dy}{dx}\right)^2 + y\left(\frac{dy}{dx}\right)^4 = x$$

The highest order derivative is $\frac{dy}{dx}$

$\therefore$ The order is 1 and the degree is 4.

(vi) $x^2\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} = 0$

Solution:

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} = -x^2\frac{d^2y}{dx^2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = x^4\left(\frac{d^2y}{dx^2}\right)^2$$

The highest order derivative is $\frac{d^2y}{dx^2}$

$\therefore$ The order is 2 and the degree is 2.

(vii) $\left(\frac{d^2y}{dx^2}\right)^3 = \sqrt{1 + \left(\frac{dy}{dx}\right)}$

Solution:

$$\left(\frac{d^2y}{dx^2}\right)^3 = \sqrt{1 + \left(\frac{dy}{dx}\right)}$$

$$\left(\frac{d^2y}{dx^2}\right)^6 = 1 + \frac{dy}{dx}$$

The highest order derivative is $\frac{d^2y}{dx^2}$

$\therefore$ The order is 2 and the degree is 6.

(viii) $\frac{d^2y}{dx^2} = xy + \cos\left(\frac{dy}{dx}\right)$

Solution:

The highest order derivative is $\frac{d^2y}{dx^2}$

$\therefore$ The order is 2

The given equation can't be written in a polynomial form.

$\therefore$ The degree of the given equation is not defined.

(ix) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + \int y\,dx = x^3$

Solution:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + \int y\,dx = x^3$$

Diff w.r.t. 'x' we get,

$$\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} + y = 3x^2$$

The highest order derivative is $\frac{d^3y}{dx^3}$

$\therefore$ The order is 3 and the degree is 1.

(x) $x = e^{xy\left(\frac{dy}{dx}\right)}$

Solution:

$$x = e^{xy\left(\frac{dy}{dx}\right)}$$

Taking log on both sides we get,

$$\log x = xy\frac{dy}{dx}$$

The highest order derivative is $\frac{dy}{dx}$

$\therefore$ The order is 1 and the degree is 1.

Exercise 10.2

1. Express each of the following physical statements in the form of differential equation.

(i) Radium decays at a rate proportional to the amount Q present.

Solution:

Let the present amount be Q at time t.

$$\frac{dQ}{dt} \propto Q$$

$dQ = kQ$, where k is a constant.

(ii) The population P of a city increases at a rate proportional to the product of population and to the difference between $5,00,000$ and the population.

Solution:

Given P is the population at time t.

$$\frac{dP}{dt} \propto P(500000 - P)$$

$dP = kP(500000 - P)$, where k is a constant.

(iii) For a certain substance, the rate of change of vapor pressure P with respect to temperature T is proportional to the vapor pressure and inversely proportional to the square of the temperature.

Solution:

$$\frac{dP}{dt} \propto \frac{P}{T^2}$$

$dP = k\frac{P}{T^2}$, where k is a constant.

(iv) A saving amount pays 8% interest per year, compounded continuously. In addition, the income from another investment is credited to the amount continuously at the rate of ₹ 400 per year.

Solution:

Let x be the amount invested at time t.

$$\frac{dx}{dt} = \frac{8}{100}x + 400$$

Hint: Evaporates indicates $(-ve)$ sign.

2. Assume that a spherical rain drop evaporates at a rate proportional to its surface area. Form a differential equation involving the rate of change of the radius of the rain drop.

Solution:

Let r be the rain drop at time t.

Let V be the volume of the rain drop and S be the surface area of the rain drop.

$$V = \frac{4}{3}\pi r^3 \text{ and } S = 4\pi r^2$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

Given that, $\frac{dV}{dt} \propto S$

$$\frac{dV}{dt} = -k\,S \qquad \text{(by Hint)}$$

$$4\pi r^2\frac{dr}{dt} = -k\,4\pi r^2$$

$\frac{dr}{dt} = -k$, where k is a constant.

Exercise 10.3

1. Find the differential equation of the family of (i) all non-vertical lines in a plane (ii) all non-horizontal lines in a plane.

Solution:

(i) For non-vertical lines, the equation is

$ax + by + c = 0$ where a, b are constants.$(b \neq 0)$

Diff w.r.t 'x' we get,

$$a + by' = 0$$

$$y' = -\frac{a}{b}$$

Diff w.r.t 'x' we get,

$$y'' = 0$$

(ii) For non-horizontal lines, the equation is

$ax + by + c = 0$ where a, b are constants.$(a \neq 0)$

Diff w.r.t 'y' we get,

$$ax' + b = 0$$

$$x' = -\frac{b}{a}$$

Diff w.r.t 'y' we get,

$$x'' = 0$$

Hint: Equation of tangent to a circle $x^2 + y^2 = a^2$ for a line $y = mx + c$ is $y = mx \pm a\sqrt{1 + m^2}$

2. Form the differential equation of all straight lines touching the circle $x^2 + y^2 = r^2$.

Solution:

Equation of tangent to the circle is

$$y = mx \pm r\sqrt{1 + m^2} \qquad \text{----------(1)}$$

Diff w.r.t 'x' we get,

$$\frac{dy}{dx} = m \qquad \text{----------(2)}$$

Sub (2) in (1) we get,

$$y = x\frac{dy}{dx} \pm r\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$y - x\frac{dy}{dx} = \pm r\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Squaring on both sides we get,

$\left[y - x\frac{dy}{dx}\right]^2 = r^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$, which is the required equation.

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3. Find the differential equation of the family of circles passing through the origin and having their centres on the x-axis.

Solution:

Let the centre of the circle be $(h, 0)$

$\therefore$ Radius $r = h$

Equation of the circle is

$$(x - h)^2 + (y - 0)^2 = h^2 \qquad \text{----------(1)}$$

$$x^2 + h^2 - 2xh + y^2 = h^2$$

$$x^2 - 2xh + y^2 = 0$$

Diff w.r.t 'x' we get,

$$2x - 2h + 2y\frac{dy}{dx} = 0$$

$\div 2$ $$x - h + y\frac{dy}{dx} = 0$$

$$x - h = -y\frac{dy}{dx} \qquad \text{----------(2)}$$

$$h = x + y\frac{dy}{dx} \qquad \text{----------(3)}$$

Sub (2) and (3) in (1) we get,

$$\left(y\frac{dy}{dx}\right)^2 + y^2 = \left(x + y\frac{dy}{dx}\right)^2$$

$$\left(y\frac{dy}{dx}\right)^2 + y^2 = x^2 + \left(y\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}$$

$$y^2 = x^2 + 2xy\frac{dy}{dx}$$

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4. Find the differential equation of the family of all parabolas with latus rectum $4a$ and whose axes are parallel to x-axis.

Solution:

Equation of parabola is

$$(y - k)^2 = 4a(x - h) \qquad \text{----------(1)}$$

Diff w.r.t 'x' we get,

$$2(y - k)y' = 4a$$

$$(y - k)y' = 2a \qquad \text{----------(2)}$$

Diff w.r.t 'x' we get,

$$(y - k)y'' + (y')^2 = 0$$

$$y - k = -\frac{(y')^2}{y''} \qquad \text{----------(3)}$$

Sub (3) in (2) we get,

$$-\frac{(y')^2}{y''}.y' = 2a$$

$$2ay'' = -(y')^3$$

$$2ay'' + (y')^3 = 0$$

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5. Find the differential equation of the family of parabolas with vertex at $(0, -1)$ and having axis along the y-axis.

Solution:

Equation of parabola with $(0, -1)$ along y-axis is

$$x^2 = 4a(y + 1) \qquad \text{----------(1)}$$

Diff w.r.t 'x' we get,

$$2x = 4ay'$$

$$4a = \frac{2x}{y'} \qquad \text{----------(2)}$$

Sub (2) in (1) we get,

$$x^2 = \frac{2x}{y'}(y + 1)$$

$$xy' = 2y + 2$$

$$xy' - 2y - 2 = 0$$

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6. Find the differential equations of the family of all the ellipses having foci on the y-axis and centre at the origin.

Solution:

Equation of ellipse having foci with y-axis is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

Diff w.r.t 'x' we get,

$$\frac{2x}{b^2}+\frac{2yy'}{a^2}=0$$

$$\frac{x}{b^2}+\frac{yy'}{a^2}=0 \qquad \text{----------(1)}$$

Diff w.r.t 'x' we get,

$$\frac{1}{b^2}+\frac{\left(yy''+(y')^2\right)}{a^2}=0 \qquad \text{----------(2)}$$

Eliminating $\frac{1}{b^2}$ and $\frac{1}{a^2}$ from (1) and (2) we get,

$$\begin{vmatrix} x & yy' \\ 1 & yy''+(y')^2 \end{vmatrix}=0$$

$$xyy''+x(y')^2-yy'=0$$

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7. Find the differential equation corresponding to the family of curves represented by the equation $y=Ae^{8x}+Be^{-8x}$, where A and B are arbitrary constants.

Solution:

Given $y=Ae^{8x}+Be^{-8x}$

Diff w.r.t 'x' we get,

$$y'=8Ae^{8x}-8Be^{-8x}$$

Diff w.r.t 'x' we get,

$$y''=64Ae^{8x}+64Be^{-8x}=64[Ae^{8x}+Be^{-8x}]$$

$$y''=64y$$

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8. Find the differential equation of the curve represented by $xy=ae^x+be^{-x}+x^2$.

Solution:

$$xy=ae^x+be^{-x}+x^2 \qquad \text{----------(1)}$$

Diff w.r.t 'x' we get,

$$xy'+y=ae^x-be^{-x}+2x$$

Diff w.r.t 'x' we get,

$$xy''+y'+y'=ae^x+be^{-x}+2$$

$$xy''+2y'=xy-x^2+2$$

From (1)

$$xy''+2y'+x^2-xy-2=0$$

Exercise 10.4

1. Show that each of the following expressions is a solution of the corresponding given differential equation.

(i) $y=2x^2$; $xy'=2y$

Solution:

$$y=2x^2 \qquad \text{----------(1)}$$

Diff w.r.t 'x' we get,

$$y'=2(2x)$$

Multiply by 'x' we get,

$$xy'=2(2x^2)$$

$$xy'=2y \qquad \text{From (1)}$$

(ii) $y=ae^x+be^{-x}$; $y''-y=0$

Solution:

$$y=ae^x+be^{-x} \qquad \text{----------(1)}$$

Diff w.r.t 'x' we get,

$$y'=ae^x-be^{-x}$$

Diff w.r.t 'x' we get,

$$y''=ae^x+be^{-x}$$

$$y''=y \qquad \text{From (1)}$$

$$y''-y=0$$

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2. Find the value of m so that the function $y=e^{mx}$ is a solution of the given differential equation.

(i) $y'+2y=0$ (ii) $y''-5y'+6y=0$

Solution:

Given $y=e^{mx}$

$$y'=m\,e^{mx}=my \qquad \text{----------(1)}$$

$$y''=m^2\,e^{mx}=m^2y \qquad \text{----------(2)}$$

(i) Given that $y'+2y=0$

$$my+2y=0 \qquad \text{From (1)}$$

$$y(m+2)=0$$

$$m+2=0$$

$$m=-2$$

(ii) Given that $y''-5y'+6y=0$

$$m^2y-5my+6y=0 \qquad \text{From (1) \& (2)}$$

$$y(m^2-5m+6)=0$$

$$m^2-5m+6=0$$

$$(m-3)(m-2)=0$$

$$m=2,3$$

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3. The slope of the tangent to the curve at any point is the reciprocal of four times the ordinate at that point. The curve passes through $(2,5)$. Find the equation of the curve.

Solution:

Given, $\frac{dy}{dx}=\frac{1}{4y}$

$$4y\,dy = dx$$

On integrating both sides, we get

$$4\int y\,dy = \int dx$$

$$4\left(\frac{y^2}{2}\right) = x + c$$

$$2y^2 = x + c \qquad \text{----------(1)}$$

Passing through $(2,5)$

$$2(25) = 2 + c$$

$$c = 48$$

Required equation of the curve is

$$2y^2 = x + 48$$

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4. Show that $y = e^{-x} + mx + n$ is a solution of the differential equation $e^x\left(\frac{d^2y}{dx^2}\right) - 1 = 0$.

Solution:

Given that $y = e^{-x} + mx + n$

$$\frac{dy}{dx} = -e^{-x} + m$$

$$\frac{d^2y}{dx^2} = e^{-x}$$

$$\frac{d^2y}{dx^2} = \frac{1}{e^x}$$

$$e^x\left(\frac{d^2y}{dx^2}\right) = 1$$

$$e^x\left(\frac{d^2y}{dx^2}\right) - 1 = 0$$

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5. Show that $y = ax + \frac{b}{x}, x \neq 0$ is a solution of the differential equation $x^2y'' + xy' - y = 0$.

Solution:

Given that $y = ax + \frac{b}{x}$

$$y' = a - \frac{b}{x^2}$$

$$y'' = \frac{2b}{x^3}$$

$$\text{LHS} = x^2y'' + xy' - y$$

$$= x^2\left(\frac{2b}{x^3}\right) + x\left(a - \frac{b}{x^2}\right) - \left(ax + \frac{b}{x}\right)$$

$$= \frac{2b}{x} + xa - \frac{b}{x} - ax - \frac{b}{x}$$

$$= 0$$

$$= \text{RHS}$$

Hence $y = ax + \frac{b}{x}$ is a solution of the differential equation $x^2y'' + xy' - y = 0$.

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6. Show that $y = ae^{-3x} + b$, where a and b are arbitrary constants, is a solution of the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0.$$

Solution:

Given that $y = ae^{-3x} + b$

$$\frac{dy}{dx} = -3ae^{-3x} \qquad \text{----------(1)}$$

$$\frac{d^2y}{dx^2} = 9ae^{-3x}$$

$$= 3(3ae^{-3x})$$

$$\frac{d^2y}{dx^2} = -3\frac{dy}{dx} \qquad \text{From (1)}$$

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0$$

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7. Show that the differential equation representing the family of curves $y^2 = 2a\left(x + a^{\frac{2}{3}}\right)$, where a is a positive parameter, is $\left(y^2 - 2xy\frac{dy}{dx}\right)^3 = 8\left(y\frac{dy}{dx}\right)^5$.

Solution:

Given $y^2 = 2a\left(x + a^{\frac{2}{3}}\right) \qquad$ ----------(1)

Diff w.r.t 'x' we get,

$$2yy' = 2a \qquad \text{----------(2)}$$

$$yy' = a \qquad \text{----------(3)}$$

Sub (2) and (3) in (1) we get

$$y^2 = 2yy'\left(x + (yy')^{\frac{2}{3}}\right)$$

$$y^2 = 2xyy' + 2(yy')^{\frac{5}{3}}$$

$$y^2 - 2xyy' = 2(yy')^{\frac{5}{3}}$$

$$(y^2 - 2xyy')^3 = 2^3(yy')^5$$

$$\left(y^2 - 2xy\frac{dy}{dx}\right)^3 = 8\left(y\frac{dy}{dx}\right)^5$$

This is the required differential equation.

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8. Show that $y = a\cos bx$ is a solution of the differential equation $\frac{d^2y}{dx^2} + b^2y = 0$.

Solution:

Given $y = a\cos bx$

$$\frac{dy}{dx} = -ab\sin bx$$

$$\frac{d^2y}{dx^2} = -ab^2\cos bx$$

$$= -b^2(a\cos bx)$$

$$\frac{d^2y}{dx^2} = -b^2y$$

$$\frac{d^2y}{dx^2} + b^2y = 0$$

Exercise 10.5

1. If F is the constant force generated by the motor of an automobile of mass M, its velocity V is given by $M\frac{dV}{dt} = F - kV$, where k is a constant. Express V in terms of t given that $V = 0$ when $t = 0$.

Solution:

Given $M\frac{dV}{dt} = F - kV$

$$\frac{M}{F-kV}dv = dt$$

On integrating both sides, we get

$$\int \frac{M}{F-kV}dv = \int dt$$

Multiply and divided by '$-k$'

$$-\frac{M}{k}\int \frac{-k}{F-kV}dv = \int dt$$

$$-\frac{M}{k}\log(F-kV) = t + c \qquad \text{----------(1)}$$

At $t = 0$ and $V = 0$

$$-\frac{M}{k}\log F = c \qquad \text{----------(2)}$$

Sub (2) in (1) we get,

$$-\frac{M}{k}\log(F-kV) = t - \frac{M}{k}\log F$$

$$\frac{M}{k}\log F - \frac{M}{k}\log(F-kV) = t$$

$$\frac{M}{k}[\log F - \log(F-kV)] = t$$

$$\frac{M}{k}\left[\log\left(\frac{F}{F-kV}\right)\right] = t$$

$$\log\left(\frac{F}{F-kV}\right) = \frac{kt}{M}$$

$$\frac{F}{F-kV} = e^{\frac{kt}{M}}$$

$$F = (F-kV)e^{\frac{kt}{M}}$$

$$Fe^{\frac{-kt}{M}} = F - kV$$

$$kV = F - Fe^{\frac{-kt}{M}}$$

$$V = \frac{F}{k}\left(1 - e^{\frac{-kt}{M}}\right)$$

2. The velocity of a parachute falling vertically satisfies the equation $v\frac{dv}{dx} = g\left(1 - \frac{v^2}{k^2}\right)$, where g and k are constants. If v and x are both initially zero, find v in terms of x.

Solution:

Given $v\frac{dv}{dx} = g\left(1 - \frac{v^2}{k^2}\right)$

$$v\frac{dv}{dx} = \frac{g}{k^2}(k^2 - v^2)$$

$$\frac{v\,dv}{k^2-v^2} = \frac{g}{k^2}dx$$

On integrating both sides, we get

$$\int \frac{v\,dv}{k^2-v^2} = \frac{g}{k^2}\int dx$$

$$\frac{-1}{2}\log(k^2-v^2) = \frac{gx}{k^2} + c \qquad \text{----------(1)}$$

When $x = 0, v = 0$

$$-\frac{1}{2}\log k^2 = c \qquad \text{----------(2)}$$

Sub (2) in (1) we get,

$$\frac{-1}{2}\log(k^2-v^2) = \frac{gx}{k^2} - \frac{1}{2}\log k^2$$

$$\frac{1}{2}\log k^2 - \frac{1}{2}\log(k^2-v^2) = \frac{gx}{k^2}$$

$$\frac{1}{2}\log\left(\frac{k^2}{k^2-v^2}\right) = \frac{gx}{k^2}$$

$$\log\left(\frac{k^2}{k^2-v^2}\right) = \frac{2gx}{k^2}$$

$$\frac{k^2}{k^2-v^2} = e^{\frac{2gx}{k^2}}$$

$$k^2 = (k^2-v^2)e^{\frac{2gx}{k^2}}$$

$$k^2e^{-\frac{2gx}{k^2}} = k^2 - v^2$$

$$v^2 = k^2\left(1 - e^{-\frac{2gx}{k^2}}\right)$$

Hint: Slope of curve $= \frac{dy}{dx}$

3. Find the equation of the curve whose slope is $\frac{y-1}{x^2+x}$ and which passes through the point $(1, 0)$.

Solution:

Given $\frac{dy}{dx} = \frac{y-1}{x^2+x}$

$$\frac{dy}{y-1} = \frac{dx}{x^2+x}$$

$$\frac{dy}{y-1} = \frac{dx}{x(x+1)}$$

$$\frac{dy}{y-1} = \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

On integrating both sides, we get

$$\int \frac{dy}{y-1} = \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$\log|y-1| = \log|x| - \log|x+1| + \log|c|$$

$$\log|y-1| = \log\left|\frac{cx}{x+1}\right|$$

$$y - 1 = \frac{cx}{x+1} \qquad \text{----------(1)}$$

Passing through (1,0)

$$-1 = \frac{c}{2}$$

$$c = -2 \qquad \text{----------(2)}$$

Sub (2) in (1) we get,

$$y - 1 = -\frac{2x}{x+1}$$

$$y = -\frac{2x}{x+1} + 1 = \frac{-2x+x+1}{x+1}$$

$y = \frac{1-x}{x+1}$, which is a required solution.

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4. Solve the following differential equations:

(i) $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

Solution:

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1-x^2}}$$

On integrating both sides, we get

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{\sqrt{1-x^2}}$$

$$\sin^{-1} y = \sin^{-1} x + C$$

(ii) $ydx + (1+x^2)\tan^{-1} x\, dy = 0$

Solution:

$$(1+x^2)\tan^{-1} x\, dy = -y\, dx$$

$$\frac{dy}{y} = -\frac{dx}{(1+x^2)\tan^{-1} x}$$

On integrating both sides, we get

$$\int \frac{dy}{y} = -\int \frac{\frac{1}{1+x^2}}{\tan^{-1} x} dx$$

$$\log|y| = -\log|\tan^{-1} x| + \log|c|$$

$$\log|y| = \log\left|\frac{c}{\tan^{-1} x}\right|$$

$$y = \frac{c}{\tan^{-1} x}$$

$$y\tan^{-1} x = c$$

(iii) $\sin\frac{dy}{dx} = a, y(0) = 1$

Solution:

$$\frac{dy}{dx} = \sin^{-1} a$$

$$dy = \sin^{-1} a\ dx$$

On integrating both sides, we get

$$\int dy = \int \sin^{-1} a\ dx$$

$$y = x\sin^{-1} a + c \qquad \text{----------(1)}$$

When $x = 0, y = 1$

$$1 = c \qquad \text{----------(2)}$$

Sub (2) in (1) we get,

$$y = x\sin^{-1} a + 1$$

$$y - 1 = x\sin^{-1} a$$

$$\frac{y-1}{x} = \sin^{-1} a$$

$$\sin\left(\frac{y-1}{x}\right) = a$$

(iv) $\frac{dy}{dx} = e^{x+y} + x^3 e^y$

Solution:

$$\frac{dy}{dx} = e^x e^y + x^3 e^y$$

$$\frac{dy}{dx} = e^y(e^x + x^3)$$

$$e^{-y} dy = (e^x + x^3)\, dx$$

On integrating both sides, we get

$$\int e^{-y}\, dy = \int (e^x + x^3)\, dx$$

$$-e^{-y} = e^x + \frac{x^4}{4} + c$$

$$e^x + e^{-y} + \frac{x^4}{4} = c$$

(v) $(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$

Solution:

$$(e^y + 1)\cos x\, dx = -e^y \sin x\, dy$$

$$\frac{\cos x}{\sin x} dx = -\frac{e^y}{e^y+1} dy$$

$$\cot x\, dx = -\frac{e^y}{e^y+1} dy$$

On integrating both sides, we get

$$\int \cot x\, dx = -\int \frac{e^y}{e^y+1} dy$$

$$\log|\sin x| = -\log|e^y + 1| + \log|c|$$

$$\log|\sin x| = \log\left|\frac{c}{e^y+1}\right|$$

$$\sin x = \frac{c}{e^y+1}$$

$$(e^y + 1)\sin x = c$$

(vi) $(ydx - xdy)\cot\left(\frac{x}{y}\right) = ny^2dx$

Solution:

$$\left(\frac{ydx-xdy}{y^2}\right)\cot\left(\frac{x}{y}\right) = ndx$$

$$d\left(\frac{x}{y}\right)\cot\left(\frac{x}{y}\right) = ndx$$

On integrating both sides, we get

$$\int \cot\left(\frac{x}{y}\right) d\left(\frac{x}{y}\right) = \int ndx$$

$$\log\left|\sin\left(\frac{x}{y}\right)\right| = nx + c$$

$$\sin\left(\frac{x}{y}\right) = e^{nx+c}$$

$$\sin\left(\frac{x}{y}\right) = ce^{nx}$$

(vii) $\frac{dy}{dx} - x\sqrt{25-x^2} = 0$

Solution:

$$\frac{dy}{dx} = x\sqrt{25-x^2}$$

$$dy = x\sqrt{25-x^2}\,dx$$

On integrating both sides, we get

$$\int dy = \int x\sqrt{25-x^2}\,dx \qquad \text{----------(1)}$$

Let $t^2 = 25 - x^2$

$2tdt = -2x\,dx$

$tdt = -xdx$

From (1), $\int dy = \int -t^2dt$

$$y = -\frac{t^3}{3} + c$$

$$y = -\frac{(25-x^2)\sqrt{25-x^2}}{3} + c$$

$$3y = -(25-x^2)^{\frac{3}{2}} + c$$

$$3y + (25-x^2)^{\frac{3}{2}} = c$$

(viii) $x\cos y\; dy = e^x(x\log x + 1)dx$

Solution:

$$\cos y\;\; dy = \frac{e^x}{x}(x\log x + 1)dx$$

$$\cos y\, dy = e^x\left(\log x + \frac{1}{x}\right)dx$$

$$\cos y\, dy = e^x \log x\, dx + e^x\frac{1}{x}dx$$

On integrating both sides, we get

$$\int \cos y\, dy = \int e^x \log x\, dx + \int e^x\frac{1}{x}dx \quad \text{----------(1)}$$

Let $I_1 = \int e^x \log x\, dx$

Put $u = \log x \qquad dv = e^x dx$

$du = \frac{1}{x}dx \qquad v = e^x$

$I_1 = uv - \int vdu$

$$I_1 = \log x\, e^x - \int e^x \frac{1}{x}dx \qquad \text{----------(2)}$$

From (1),

$$\int \cos y\, dy = e^x\log x - \int e^x\frac{1}{x}dx + \int e^x\frac{1}{x}dx$$

$$\int \cos y\, dy = e^x \log x$$

$$\sin y = e^x \log x + c$$

Hint: $\cos(x+y) = \cos x\cos y - \sin x \sin y$

$\cos(x-y) = \cos x\cos y + \sin x\sin y$

(ix) $\tan y\frac{dy}{dx} = \cos(x+y) + \cos(x-y)$

Solution:

$$\tan y\frac{dy}{dx} = 2\cos x\cos y \qquad \text{(by Hint)}$$

$$\frac{\tan y}{\cos y}\,dy = 2\cos x\, dx$$

$$\sec y \tan y\, dy = 2\cos x\, dx$$

On integrating both sides, we get

$$\int \sec y\tan y\, dy = 2\int \cos x\, dx$$

$$\sec y = 2\sin x + c$$

Hint: $\cos 2x = 2\cos^2 x - 1$ and $sec^2x - tan^2x = 1$

(x) $\frac{dy}{dx} = \tan^2(x+y)$

Solution:

$$\frac{dy}{dx} = \tan^2(x+y) \qquad \text{----------(1)}$$

Put $u = x + y$

$$\frac{du}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\frac{du}{dx} - 1 = \tan^2 u \qquad \text{From (1)}$$

$$\frac{du}{dx} = 1 + \tan^2 u$$

$$\frac{du}{dx} = \sec^2 u \qquad \text{(by Hint 2)}$$

$$\frac{du}{\sec^2 u} = dx$$

$$\cos^2 u\, du = dx$$

$$\frac{1+\cos 2u}{2}\, du = dx \qquad \text{(by Hint 1)}$$

On integrating both sides, we get

$$\frac{1}{2}\int (1+\cos 2u)\, du = \int dx$$

$$\frac{1}{2}\left(u + \frac{\sin 2u}{2}\right) = x + c$$

$$\frac{1}{2}\left(u + \frac{2\sin u \cos u}{2}\right) = x + c$$

$$\frac{1}{2}(u + \sin u \cos u) = x + c$$

$$\frac{1}{2}[(x+y) + \sin(x+y)\cos(x+y)] = x + c$$

Exercise 10.6

Solve the following differential equations:

1. $\left[x + y\cos\left(\frac{y}{x}\right)\right] dx = x\cos\left(\frac{y}{x}\right) dy$

Solution:

$$\frac{dy}{dx} = \frac{x + y\cos\left(\frac{y}{x}\right)}{x\cos\left(\frac{y}{x}\right)}$$

$$\frac{dy}{dx} = \frac{x\left[1 + \frac{y}{x}\cos\left(\frac{y}{x}\right)\right]}{x\cos\left(\frac{y}{x}\right)}$$

$$\frac{dy}{dx} = \frac{1 + \frac{y}{x}\cos\left(\frac{y}{x}\right)}{\cos\left(\frac{y}{x}\right)} = g\left(\frac{y}{x}\right) \qquad \text{----------(1)}$$

∴ It is a homogeneous differential equation.

Put $\quad \frac{y}{x} = v \qquad$ ----------(2)

$$y = vx$$

Diff w.r.t 'x' we get,

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$\frac{1+v\cos v}{\cos v} = x\frac{dv}{dx} + v \qquad \text{From (1)}$$

$$x\frac{dv}{dx} = \frac{1+v\cos v}{\cos v} - v$$

$$= \frac{1+v\cos v - v\cos v}{\cos v}$$

$$x\frac{dv}{dx} = \frac{1}{\cos v}$$

$$\cos v\, dv = \frac{dx}{x}$$

On integrating both sides, we get

$$\int \cos v\, dv = \int \frac{dx}{x}$$

$$\sin v = \log|x| + \log|c|$$

$$\sin\left(\frac{y}{x}\right) = \log|cx| \qquad \text{From (2)}$$

2. $(x^3 + y^3)dy - x^2 y dx = 0$

Solution:

$$(x^3 + y^3)dy = x^2 y dx$$

$$\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$$

$$\frac{dx}{dy} = \frac{x^3 + y^3}{x^2 y} \qquad \text{(Taking reciprocal)}$$

$$\frac{dx}{dy} = \frac{y^3\left(\frac{x^3}{y^3} + 1\right)}{x^2 y}$$

$$\frac{dx}{dy} = \frac{\frac{x^3}{y^3} + 1}{\frac{x^2}{y^2}} = g\left(\frac{x}{y}\right) \qquad \text{----------(1)}$$

∴ It is a homogeneous differential equation.

Put $\quad \frac{x}{y} = v \qquad$ ----------(2)

$$x = vy$$

Diff w.r.t 'x' we get,

$$\frac{dx}{dy} = y\frac{dv}{dy} + v$$

$$\frac{v^3+1}{v^2} = y\frac{dv}{dy} + v \qquad \text{From (1)}$$

$$y\frac{dv}{dy} = \frac{v^3+1}{v^2} - v$$

$$= \frac{v^3 + 1 - v^3}{v^2}$$

$$y\frac{dv}{dy} = \frac{1}{v^2}$$

$$v^2 dv = \frac{dy}{y}$$

On integrating both sides, we get

$$\int v^2\, dv = \int \frac{dy}{y}$$

$$\frac{v^3}{3} = \log|y| + \log|c|$$

$$\frac{v^3}{3} = \log|yc|$$

$$yc = e^{\frac{v^3}{3}}$$

$$y = ce^{\frac{x^3}{3y^3}} \qquad \text{From (2)}$$

3. $ye^{\frac{x}{y}} dx = \left(xe^{\frac{x}{y}} + y\right) dy$

Solution:

$$\frac{dx}{dy} = \frac{xe^{\frac{x}{y}} + y}{ye^{\frac{x}{y}}}$$

$$\frac{dx}{dy}=\frac{y\left(\frac{x}{y}e^{\frac{x}{y}}+1\right)}{ye^{\frac{x}{y}}}$$

$$\frac{dx}{dy}=\frac{\frac{x}{y}e^{\frac{x}{y}}+1}{e^{\frac{x}{y}}}=g\left(\frac{x}{y}\right) \qquad \text{----------(1)}$$

∴ It is a homogeneous differential equation.

Put $\quad \frac{x}{y}=v$ ---------(2)

$$x=vy$$

Diff w.r.t 'y' we get,

$$\frac{dx}{dy}=y\frac{dv}{dy}+v$$

$$\frac{ve^v+1}{e^v}=y\frac{dv}{dy}+v \qquad \text{From (1)}$$

$$y\frac{dv}{dy}=\frac{ve^v+1}{e^v}-v$$

$$y\frac{dv}{dy}=\frac{ve^v+1-ve^v}{e^v}$$

$$y\frac{dv}{dy}=\frac{1}{e^v}$$

$$e^v dv=\frac{dy}{y}$$

On integrating both sides, we get

$$\int e^v dv=\int\frac{dy}{y}$$

$$e^v=\log|y|+\log|c|$$

$$e^{\frac{x}{y}}=\log|cy| \qquad \text{From (2)}$$

---

4. $2xydx+(x^2+2y^2)dy=0$

Solution:

$$(x^2+2y^2)dy=-2xydx$$

$$\frac{dy}{dx}=-\frac{2xy}{x^2+2y^2}$$

$$\frac{dx}{dy}=-\frac{x^2+2y^2}{2xy} \qquad \text{(Taking reciprocal)}$$

$$=-\frac{y^2\left(\frac{x^2}{y^2}+2\right)}{2xy}$$

$$\frac{dx}{dy}=-\frac{\left(\frac{x^2}{y^2}+2\right)}{\frac{2x}{y}}=g\left(\frac{x}{y}\right) \qquad \text{----------(1)}$$

∴ It is a homogeneous differential equation.

Put $\quad \frac{x}{y}=v$ ----------(2)

$$x=vy$$

Diff w.r.t 'y' we get,

$$\frac{dx}{dy}=y\frac{dv}{dy}+v$$

$$-\frac{(v^2+2)}{2v}=y\frac{dv}{dy}+v \qquad \text{From (1)}$$

$$y\frac{dv}{dy}=\frac{-v^2-2}{2v}-v$$

$$=\frac{-v^2-2-2v^2}{2v}$$

$$y\frac{dv}{dy}=\frac{-3v^2-2}{2v}$$

$$\frac{2v}{3v^2+2}dv=-\frac{dy}{y}$$

On integrating on both sides, we get

$$\int\frac{2v}{3v^2+2}dv=-\int\frac{dy}{y}$$

$$\frac{2}{6}\log|3v^2+2|=-\log|y|+\log|c|$$

$$\frac{1}{3}\log|3v^2+2|=\log\left|\frac{c}{y}\right|$$

$$\log|3v^2+2|^{\frac{1}{3}}=\log\left|\frac{c}{y}\right|$$

$$(3v^2+2)^{\frac{1}{3}}=\frac{c}{y}$$

$$3v^2+2=\left(\frac{c}{y}\right)^3$$

$$\frac{3x^2}{y^2}+2=\frac{c}{y^3} \qquad \text{From (2)}$$

$$3x^2y+2y^3=c$$

---

5. $(y^2-2xy)dx=(x^2-2xy)dy$

Solution:

$$\frac{dy}{dx}=\frac{y^2-2xy}{x^2-2xy}$$

$$=\frac{x^2\left(\frac{y^2}{x^2}-\frac{2y}{x}\right)}{x^2\left(1-\frac{2y}{x}\right)}$$

$$\frac{dy}{dx}=\frac{\frac{y^2}{x^2}-\frac{2y}{x}}{1-\frac{2y}{x}}=g\left(\frac{y}{x}\right) \qquad \text{----------(1)}$$

∴ It is a homogeneous differential equation.

Put $\quad \frac{y}{x}=v$ ----------(2)

$$y=vx$$

Diff w.r.t 'x' we get,

$$\frac{dy}{dx}=x\frac{dv}{dx}+v$$

$$\frac{v^2-2v}{1-2v}=x\frac{dv}{dx}+v \qquad \text{From (1)}$$

$$x\frac{dv}{dx}=\frac{v^2-2v}{1-2v}-v$$

$$=\frac{v^2-2v-v+2v^2}{1-2v}$$

$$x\frac{dv}{dx} = \frac{3v^2-3v}{1-2v} = -3\left(\frac{v^2-v}{2v+1}\right)$$

$$\frac{2v+1}{v^2-v}dv = -3\frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{2v+1}{v^2-v}dv = -3\int \frac{dx}{x}$$

$$\log|v^2-v| = -3\log|x| + \log|c|$$

$$\log|v^2-v| = \log\left|\frac{c}{x^3}\right|$$

$$v^2 - v = \frac{c}{x^3}$$

$$\frac{y^2}{x^2} - \frac{y}{x} = \frac{c}{x^3} \qquad \text{From (2)}$$

$$y^2x - yx^2 = c$$

6. $x\frac{dy}{dx} = y - x\cos^2\left(\frac{y}{x}\right)$

Solution:

$$\frac{dy}{dx} = \frac{y}{x} - \cos^2\left(\frac{y}{x}\right) = g\left(\frac{y}{x}\right) \qquad \text{----------(1)}$$

∴ It is a homogeneous differential equation.

Put $\frac{y}{x} = v$ ----------(2)

$$y = vx$$

Diff w.r.t 'x' we get,

$$\frac{dy}{dx} = x\frac{dv}{x} + v$$

$$v - \cos^2 v = x\frac{dv}{dx} + v \qquad \text{From (1)}$$

$$x\frac{dv}{dx} = -\cos^2 v$$

$$\frac{dv}{\cos^2 v} = -\frac{dx}{x}$$

On integrating both sides, we get

$$\int \sec^2 v\, dv = -\int \frac{dx}{x}$$

$$\tan v = -\log|x| + \log|c|$$

$$\tan v = \log\left|\frac{c}{x}\right|$$

$$e^{\tan v} = \frac{c}{x}$$

$$x\, e^{\tan\frac{y}{x}} = c \qquad \text{From (2)}$$

7. $\left(1 + 3e^{\frac{y}{x}}\right)dy + 3e^{\frac{y}{x}}\left(1 - \frac{y}{x}\right)dx = 0$, given that $y = 0$ when $x = 1$

Solution:

$$\left(1 + 3e^{\frac{y}{x}}\right)dy = -3e^{\frac{y}{x}}\left(1 - \frac{y}{x}\right)dx$$

$$\frac{dy}{dx} = -\frac{3e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)}{\left(1+3e^{\frac{y}{x}}\right)} = g\left(\frac{y}{x}\right) \qquad \text{----------(1)}$$

∴ It is a homogeneous differential equation.

Put $\frac{y}{x} = v$ ----------(2)

$$y = vx$$

Diff w.r.t 'x' we get,

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$-\frac{3e^v(1-v)}{1+3e^v} = x\frac{dv}{dx} + v \qquad \text{From (1)}$$

$$x\frac{dv}{dx} = \frac{-3e^v+3ve^v}{1+3e^v} - v$$

$$= \frac{-3e^v+3ve^v-v-3ve^v}{1+3e^v}$$

$$x\frac{dv}{dx} = \frac{-3e^v-v}{1+3e^v}$$

$$\frac{1+3e^v}{3e^v+v}dv = -\frac{dx}{x}$$

On integrating on both sides, we get

$$\int \frac{3e^v+1}{3e^v+v}dv = -\int \frac{dx}{x}$$

$$\log|3e^v + v| = -\log|x| + \log|c|$$

$$\log|3e^v + v| = \log\left|\frac{c}{x}\right|$$

$$3e^v + v = \frac{c}{x}$$

$$3e^{\frac{y}{x}} + \frac{y}{x} = \frac{c}{x} \qquad \text{From (2)}$$

$$3xe^{\frac{y}{x}} + y = c \qquad \text{----------(3)}$$

When $y = 0$ and $x = 1$

$$3 = c$$

$$3xe^{\frac{y}{x}} + y = 3 \qquad \text{From (3)}$$

8. $(x^2 + y^2)dy = xy\, dx$. It is given that $y(1) = 1$and $y(x_0) = e$. Find the value of x_0.

Solution:

$$\frac{dx}{dy} = \frac{x^2+y^2}{xy} = \frac{x}{y} + \frac{y}{x}$$

$$\frac{dx}{dy} = \frac{x}{y} + \frac{1}{\frac{x}{y}} = g\left(\frac{x}{y}\right) \qquad \text{----------(1)}$$

∴ It is a homogeneous differential equation.

Put $\frac{x}{y} = v$ ----------(2)

$$x = vy$$

Diff w.r.t 'y' we get,

$$\frac{dx}{dy} = y\frac{dv}{dy} + v \qquad \text{From (1)}$$

$$v + \frac{1}{v} = y\frac{dv}{dy} + v$$

$$y\frac{dv}{dy} = \frac{1}{v}$$

$$v\,dv = \frac{dy}{y}$$

On integrating both sides, we get

$$\int v\,dv = \int \frac{dy}{y}$$

$$\frac{v^2}{2} = \log|y| + \log|c|$$

$$\frac{v^2}{2} = \log|cy|$$

$$e^{\frac{v^2}{2}} = cy$$

$$e^{\frac{x^2}{2y^2}} = cy \qquad \text{----------(3)}$$

When $x = 1, y = 1$

$$c = e^{\frac{1}{2}}$$

$$e^{\frac{x^2}{2y^2}} = e^{\frac{1}{2}}y \qquad \text{From (3)}$$

When $x = x_0, y = e$

$$e^{\frac{x_0^2}{2e^2}} = e^{\frac{3}{2}}$$

$$\frac{x_0^2}{2e^2} = \frac{3}{2}$$

$$x_0^2 = 3e^2$$

$$x_0 = \pm\sqrt{3}e$$

Exercise 10.7

Hint: The first order differential equation of the form is

$$\frac{dy}{dx} + Py = Q \quad \text{(or)} \quad \frac{dx}{dy} + Px = Q$$

The solution of the differential equation is,

$$ye^{\int pdx} = \int Qe^{\int pdx}dx + C \qquad \text{(or)}$$

$$xe^{\int pdy} = \int Qe^{\int pdy}dy + C$$

Where, Integrating Factor (I.F) is $e^{\int pdx}$ or $e^{\int pdy}$

Solve the following linear differential equations:

1. $\cos x\frac{dy}{dx} + y\sin x = 1$

Solution:

$$\cos x\frac{dy}{dx} + y\sin x = 1$$

$$\div \cos x, \qquad \frac{dy}{dx} + \frac{y\sin x}{\cos x} = \frac{1}{\cos x}$$

$$\frac{dy}{dx} + y\tan x = \sec x$$

Where, $P = \tan x$ and $Q = \sec x$

$$I.F = e^{\int pdx}$$

$$= e^{\int \tan x dx}$$

$$= e^{\log|\sec x|}$$

$$I.F = \sec x$$

The solution of the differential equation is,

$$y(I.F) = \int Q(I.F)\,dx + c$$

$$y\sec x = \int \sec^2 x\,dx + c$$

$$y\sec x = \tan x + c$$

$$\frac{y}{\cos x} = \tan x + c$$

$$y = \sin x + C\cos x$$

---

2. $(1 - x^2)\frac{dy}{dx} - xy = 1$

Solution:

$$(1 - x^2)\frac{dy}{dx} - xy = 1$$

$$\div (1 - x^2), \quad \frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{1}{1-x^2}$$

Where $P = -\frac{x}{1-x^2}$ and $Q = \frac{1}{1-x^2}$

$$I.F = e^{\int pdx}$$

$$= e^{\int \frac{-x}{1-x^2}dx}$$

$$= e^{\frac{1}{2}\log|1-x^2|}$$

$$= e^{\log|1-x^2|^{\frac{1}{2}}}$$

$$I.F = \sqrt{1 - x^2}$$

The solution of the differential equation is,

$$y\,(I.F) = \int Q(I.F)\,dx + c$$

$$y\sqrt{1 - x^2} = \int \frac{1}{1-x^2}\sqrt{1 - x^2}dx + c$$

$$y\sqrt{1 - x^2} = \int \frac{1}{\sqrt{1-x^2}}dx + c$$

$$y\sqrt{1 - x^2} = \sin^{-1} x + c$$

$$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}} + c(1 - x^2)^{-\frac{1}{2}}$$

Hint: $\int uvdx = uv_1 - u'v_2 + u''v_3 - \cdots$

3. $\frac{dy}{dx}+\frac{y}{x}=\sin x$

Solution:

$$\frac{dy}{dx}+\frac{y}{x}=\sin x$$

Where $P=\frac{1}{x}$ and $Q=\sin x$

$u=x\ \ ; dv=\sin x\,dx$
$u'=1\ \ ; v_1=-\cos x$
$u''=0\ \ ; v_2=-\sin x$

$$I.F=e^{\int\frac{1}{x}dx}$$

$$=e^{\log x}$$

$$I.F=x$$

The solution of the differential equation is,

$$yx=\int x\sin x\,dx+c$$

$$yx=-x\cos x+\sin x+c$$

Hint: $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|$

4. $(x^2+1)\frac{dy}{dx}+2xy=\sqrt{x^2+4}$

Solution:

$$(x^2+1)\frac{dy}{dx}+2xy=\sqrt{x^2+4}$$

$\div(x^2+1)$, $\quad \frac{dy}{dx}+\frac{2xy}{x^2+1}=\frac{\sqrt{x^2+4}}{x^2+1}$

Where $P=\frac{2x}{x^2+1}$ and $Q=\frac{\sqrt{x^2+4}}{x^2+1}$

$$I.F=e^{\int\frac{2x}{x^2+1}dx}$$

$$=e^{\log|x^2+1|}$$

$$I.F=x^2+1$$

The solution of the differential equation is,

$$y(x^2+1)=\int\frac{\sqrt{x^2+4}}{x^2+1}(x^2+1)dx+c$$

$$=\int\sqrt{x^2+4}\,dx+c$$

$$=\frac{x}{2}\sqrt{x^2+4}+\frac{4}{2}\log\left|x+\sqrt{x^2+4}\right|+c$$

$$y(x^2+1)=\frac{x}{2}\sqrt{x^2+4}+2\log\left|x^2+\sqrt{x^2+4}\right|+c$$

5. $(2x-10y^3)dy+ydx=0$

Solution:

$$ydx=-(2x-10y^3)dy$$

$$\frac{dx}{dy}=-\frac{2x}{y}+\frac{10y^3}{y}$$

$$\frac{dx}{dy}+\frac{2x}{y}=10y^2$$

Where $P=\frac{2}{y}$ and $Q=10y^2$

$$I.F=e^{\int pdy}$$

$$=e^{\int\frac{2}{y}dy}$$

$$=e^{2\log|y|}=e^{\log|y|^2}$$

$$I.F=y^2$$

The solution of the differential equation is,

$$x(I.F)=\int Q(I.F)\,dy+c$$

$$xy^2=\int 10y^2\,y^2dy+c$$

$$=10\int y^4\,dy+c$$

$$=10\left(\frac{y^5}{5}\right)+c$$

$$xy^2=2y^5+c$$

6. $x\sin x\frac{dy}{dx}+(x\cos x+\sin x)y=\sin x$

Solution:

$$x\sin x\frac{dy}{dx}+(x\cos x+\sin x)y=\sin x$$

$\div(x\sin x)$, $\quad \frac{dy}{dx}+\left(\frac{x\cos x+\sin x}{x\sin x}\right)y=\frac{1}{x}$

Where $P=\frac{x\cos x+\sin x}{x\sin x}$ and $Q=\frac{1}{x}$

$$I.F=e^{\int\frac{x\cos x+\sin x}{x\sin x}dx}$$

$$=e^{\log|x\sin x|}$$

$$I.F=x\sin x$$

The solution of the differential equation is,

$$y(x\sin x)=\int\frac{1}{x}x\sin x\,dx+c$$

$$xy\sin x=\int\sin x\,dx+c$$

$$xy\sin x=-\cos x+c$$

7. $\left(y-e^{\sin^{-1}x}\right)\frac{dx}{dy}+\sqrt{1-x^2}=0$

Solution:

Given $\left(y-e^{\sin^{-1}x}\right)\frac{dx}{dy}+\sqrt{1-x^2}=0$

$\div\left(y-e^{\sin^{-1}x}\right)$,

$$\frac{dx}{dy}+\frac{\sqrt{1-x^2}}{y-e^{\sin^{-1}x}}=0$$

$$\frac{dy}{dx}+\frac{y-e^{\sin^{-1}x}}{\sqrt{1-x^2}}=0 \qquad \text{(Taking reciprocal)}$$

$$\frac{dy}{dx}+\frac{y}{\sqrt{1-x^2}}=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

Where $P = \frac{1}{\sqrt{1-x^2}}$ and $Q = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

$$I.F = e^{\int \frac{1}{\sqrt{1-x^2}}dx} = e^{\sin^{-1}x}$$

The solution of the differential equation is,

$$ye^{\sin^{-1}x} = \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}} e^{\sin^{-1}x}\, dx + c$$

$$= \int \frac{\left(e^{\sin^{-1}x}\right)^2}{\sqrt{1-x^2}} dx + c \qquad \text{----------(1)}$$

Let $t = \sin^{-1}x$

$$dt = \frac{1}{\sqrt{1-x^2}} dx$$

$$ye^{\sin^{-1}x} = \int e^{2t}\, dt + c \qquad \text{From (1)}$$

$$= \frac{e^{2t}}{2} + c$$

$$ye^{\sin^{-1}x} = \frac{e^{2\sin^{-1}x}}{2} + c$$

Hint: $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|$

8. $\frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$

Solution:

Where $P = \frac{1}{(1-x)\sqrt{x}}$ and $Q = 1 - \sqrt{x}$

$$I.F = e^{\int \frac{dx}{(1-x)\sqrt{x}}}$$

Let $u = \sqrt{x}$, then $u^2 = x$

$$du = \frac{1}{2\sqrt{x}} dx$$

$$2du = \frac{dx}{\sqrt{x}}$$

$$I.F = e^{\int \frac{2du}{1-u^2}}$$

$$= e^{2\times\frac{1}{2}\log\left|\frac{1+u}{1-u}\right|}$$

$$I.F = \frac{1+u}{1-u} = \frac{1+\sqrt{x}}{1-\sqrt{x}}$$

The solution of the differential equation is,

$$y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = \int\left(1-\sqrt{x}\right)\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) dx + c$$

$$= \int\left(1+\sqrt{x}\right) dx + c$$

$$= x + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = x + \frac{2}{3}x\sqrt{x} + c$$

9. $(1 + x + xy^2)\frac{dy}{dx} + (y + y^3) = 0$

Solution:

$\div (1 + x + xy^2)$, $\frac{dy}{dx} + \frac{(y+y^3)}{1+x+xy^2} = 0$

$$\frac{dy}{dx} = -\frac{y(1+y^2)}{1+x(1+y^2)}$$

$$\frac{dx}{dy} = -\frac{1+x(1+y^2)}{y(1+y^2)} \qquad \text{(Taking reciprocal)}$$

$$\frac{dx}{dy} = -\frac{1}{y(1+y^2)} - \frac{x}{y}$$

$$\frac{dx}{dy} + \frac{x}{y} = -\frac{1}{y(1+y^2)}$$

Where $P = \frac{1}{y}$ and $Q = -\frac{1}{y(1+y^2)}$

$$I.F = e^{\int \frac{1}{y}dy} = e^{\log|y|}$$

$$I.F = y$$

The solution of the differential equation is,

$$xy = \int \frac{-1}{y(1+y^2)}(y)dy + c$$

$$= -\int \frac{1}{1+y^2} dy + c$$

$$= -\tan^{-1}y + c$$

$$xy + \tan^{-1}y = c$$

10. $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{\sin 2x}{\log x}$

Solution:

Where $P = \frac{1}{x\log x}$ and $Q = \frac{\sin 2x}{\log x}$

$$I.F = e^{\int \frac{\frac{1}{x}}{\log x}dx}$$

$$= e^{\log|\log x|}$$

$$I.F = \log x$$

The solution of the differential equation is,

$$y\log x = \int \frac{\sin 2x}{\log x}\log x\, dx + c$$

$$= \int \sin 2x\, dx + c$$

$$= -\frac{\cos 2x}{2} + c$$

$$y\log x + \frac{\cos 2x}{2} = c$$

11. $(x + a)\frac{dy}{dx} - 2y = (x + a)^4$

Solution:

$\div (x + a)$, $\quad \frac{dy}{dx} - \frac{2y}{x+a} = (x + a)^3$

Where $P = -\frac{2}{x+a}$ and $Q = (x + a)^3$

$$I.F = e^{-2\int\frac{dx}{x+a}}$$

$$= e^{-2\log|x+a|}$$

$$= e^{\log|x+a|^{-2}}$$

$$I.F = (x+a)^{-2} = \frac{1}{(x+a)^2}$$

The solution of the differential equation is,

$$\frac{y}{(x+a)^2} = \int (x+a)^3 \frac{1}{(x+a)^2} dx + c$$

$$= \int (x+a)\, dx + c$$

$$\frac{y}{(x+a)^2} = \frac{(x+a)^2}{2} + c$$

$$y = \frac{(x+a)^4}{2} + c$$

Hint: $\cos 2x = 1 - 2\sin^2 x$

12. $\frac{dy}{dx} = \frac{\sin^2 x}{1+x^3} - \frac{3x^2}{1+x^3} y$

Solution:

$$\frac{dy}{dx} + \frac{3x^2}{1+x^3} y = \frac{\sin^2 x}{1+x^3}$$

Where $P = \frac{3x^2}{1+x^3}$ and $Q = \frac{\sin^2 x}{1+x^3}$

$$I.F = e^{\int\frac{3x^2}{1+x^3}dx}$$

$$= e^{\log|1+x^3|}$$

$$I.F = 1 + x^3$$

The solution of the differential equation is,

$$y(1+x^3) = \int \frac{\sin^2 x}{1+x^3}(1+x^3)dx + c$$

$$y(1+x^3) = \int \sin^2 x\, dx + c$$

$$= \int \left(\frac{1-\cos 2x}{2}\right) dx + c$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$$

$$y(1+x^3) = \frac{x}{2} - \frac{\sin 2x}{4} + c$$

---

13. $x\frac{dy}{dx} + y = x\log x$

Solution:

$\div x,\quad \frac{dy}{dx} + \frac{y}{x} = \log x$

Where $P = \frac{1}{x}$ and $Q = \log x$

$$I.F = e^{\int\frac{1}{x}dx}$$

$$= e^{\log|x|}$$

$$I.F = x$$

$$\boxed{\begin{aligned} &\int u\,dv = uv - \int v\,du \\ &u = \log x\,;\ dv = x\,dx \\ &du = \frac{1}{x}dx\,;\ \ v = \frac{x^2}{2}\end{aligned}}$$

The solution of the differential equation is,

$$xy = \int x\log x\, dx + c$$

$$= \frac{x^2}{2}\log x - \int \frac{x^2}{2}\left(\frac{1}{x}\right) dx + c$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\int x\, dx + c$$

$$xy = \frac{x^2}{2}\log x - \frac{x^2}{4} + c$$

$$4xy = 2x^2\log x - x^2 + c$$

---

14. $x\frac{dy}{dx} + 2y - x^2\log x = 0$

Solution:

$\div x,\quad \frac{dy}{dx} + \frac{2y}{x} = x\log x$

Where $P = \frac{2}{x}$ and $Q = x\log x$

$$I.F = e^{2\int\frac{1}{x}dx}$$

$$= e^{2\log|x|}$$

$$= e^{\log|x|^2}$$

$$I.F = x^2$$

$$\boxed{\begin{aligned} &\int u\,dv = uv - \int v\,du \\ &u = \log x\,;\ dv = x^3\,dx \\ &du = \frac{1}{x}dx\ ;\ \ v = \frac{x^4}{4}\end{aligned}}$$

The solution of the differential equation is,

$$yx^2 = \int x^3\log x\, dx + c$$

$$= \frac{x^4}{4}\log x - \int \frac{x^4}{4}\left(\frac{1}{x}\right) dx + c$$

$$= \frac{x^4}{4}\log x - \frac{1}{4}\int x^3\, dx + c$$

$$yx^2 = \frac{x^4}{4}\log x - \frac{x^4}{16}dx + c$$

---

15. $\frac{dy}{dx} + \frac{3y}{x} = \frac{1}{x^2}$, given that $y = 2$ when $x = 1$

Solution:

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{1}{x^2}$$

Where $P = \frac{3}{x}$ and $Q = \frac{1}{x^2}$

$$I.F = e^{3\int\frac{1}{x}dx}$$

$$= e^{3\log|x|} = e^{\log|x|^3}$$

$$I.F = x^3$$

The solution of the differential equation is,

$$x^3y = \int \frac{1}{x^2}(x^3)dx + c$$

$$= \int x\, dx + c$$

$$x^3y = \frac{x^2}{2} + c \qquad \text{----------(1)}$$

When $x = 1$ and $y = 2$

$$2 = \frac{1}{2} + c$$

$$c = 2 - \frac{1}{2}$$

$$c = \frac{3}{2}$$

$$x^3y = \frac{x^2}{2} + \frac{3}{2} \qquad \text{From (1)}$$

$$2x^3y = x^2 + 3$$

Exercise 10.8

1. The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given that the number triples in 5 hours, find how many bacteria will be present after 10 hours?

Solution:

Let x be the number of bacteria at any time t.

$$\frac{dx}{dt} \propto x$$

$$\frac{dx}{dt} = kx$$

$$\frac{dx}{x} = kdt$$

On integrating both sides, we get

$$\int \frac{dx}{x} = \int kdt$$

$$\log|x| = kt + c$$

$$x = e^{kt+c}$$

$$x = ce^{kt} \qquad \text{----------(1)}$$

When $t = 0, x = x_0$

$$x_0 = c$$

From (1) we have,

$$x = x_0e^{kt} \qquad \text{----------(2)}$$

When $t = 5, x = 3x_0$

$$3x_0 = x_0e^{5k}$$

$$e^{5k} = 3$$

When $t = 10$,

$$x = x_0e^{10k} \qquad \text{From (2)}$$

$$x = x_0(e^{5k})^2 = x_0(3)^2$$

$$x = 9x_0$$

∴ After 10 hours the number of bacteria is 9 times the original number of bacteria.

---

2. Find the population of a city at any time t, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years the population increased from 3,00,000 to 4,00,000.

Solution:

Let P be the population of a city at any time t.

$$\frac{dP}{dt} \propto P$$

$$\frac{dP}{dt} = kP$$

$$\frac{dP}{P} = kdt$$

On integrating both sides, we get

$$\int \frac{dP}{P} = \int kdt$$

$$\log|P| = kt + c$$

$$P = e^{kt+c}$$

$$P = ce^{kt} \qquad \text{----------(1)}$$

When $t = 0, P = 300000$

$$300000 = c$$

From (1) we have,

$$P = 300000e^{kt} \qquad \text{----------(2)}$$

When $t = 40, P = 400000$

$$400000 = 300000e^{40k}$$

$$e^{40k} = \frac{4}{3}$$

$$e^k = \left(\frac{4}{3}\right)^{\frac{1}{40}}$$

From (2) we have,

$$P = 300000\left(\frac{4}{3}\right)^{\frac{t}{40}}$$

---

3. The equation of electromotive force for an electric circuit containing resistance and self-inductance is $E = Ri + L\frac{di}{dt}$, where E is the electromotive force given to the circuit, R the resistance and L, the coefficient of induction. Find the current i at time t when $E = 0$.

Solution:

Given that $E = Ri + L\frac{di}{dt}$

$\div L$, $$\frac{E}{L} = \frac{Ri}{L} + \frac{di}{dt}$$

$$\frac{di}{dt} - \frac{R}{L}i = \frac{E}{L}$$

This is linear differential equation in i.

Where $P = \frac{R}{L}$ and $Q = \frac{E}{L}$

$$I.F = e^{\int pdt}$$

$$= e^{\int \frac{R}{L} dt}$$

$$I.F = e^{\frac{R}{L}t}$$

The solution of the differential equation is,

$$i(I.F) = \int Q\,(I.F)dt + c$$

$$i\,e^{\frac{R}{L}t} = \int \frac{E}{L} e^{\frac{R}{L}t} dt + c$$

$$= \frac{E}{L}\left(\frac{e^{\frac{R}{L}t}}{\frac{R}{L}}\right) + c$$

$$i\,e^{\frac{R}{L}t} = \frac{E}{R} e^{\frac{R}{L}t} + c$$

$$i = \frac{E}{R} + ce^{-\frac{R}{L}t} \qquad \text{----------(1)}$$

When $E = 0$,

$$i = ce^{-\frac{R}{L}t}$$

---

4. The engine of a motor boat moving at 10 m/s is shut off. Given that the retardation at any subsequent time (after shutting off the engine) equal to the velocity at that time. Find the velocity after 2 seconds of switching off the engine.

Solution:

Let v be the velocity of the motor boat at any time t

$$-\frac{dv}{dt} = v$$

$$\frac{dv}{v} = -dt$$

On integrating both sides, we get

$$\int \frac{dv}{v} = \int -dt$$

$$\log|v| = -t + c$$

$$v = e^{-t+c}$$

$$v = e^{-t}e^{c}$$

$$v = ce^{-t} \qquad \text{----------(1)}$$

When $t = 0, v = 10$

$$10 = c$$

From (1) we have,

$$v = 10e^{-t} \qquad \text{----------(2)}$$

When $t = 2$,

$$v = 10e^{-2}$$

$$v = \frac{10}{e^2} m/s$$

---

5. Suppose a person deposits 10000 Indian rupees in a bank account at the rate of 5% per annum compounded continuously. How much money will be in his bank account 18 months later?

Solution:

Let P be the principal amount at any time t (in years)

$$\frac{dP}{dt} = \frac{5}{100}P$$

$$\frac{dP}{P} = 0.05dt$$

On integrating both sides, we get

$$\int \frac{dP}{P} = \int 0.05\,dt$$

$$\log|P| = 0.05t + c$$

$$P = e^{0.05t+c}$$

$$P = ce^{0.05t} \qquad \text{----------(1)}$$

When $t = 0, P = 10000$

$$10000 = c$$

From (1) we have,

$$P = 10000e^{0.05t}$$

When $t = 18\,months = \frac{3}{2}yrs$

$$P = 10000e^{0.0075}$$

---

6. Assume that the rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. What percentage of the original radioactive nuclei will remain after 1000 years?

Solution:

Let A be the amount of radioactive nuclei present at any time t

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = -kA \qquad (\because \text{It is decay})$$

$$\frac{da}{A} = -kdt$$

On integrating both sides, we get

$$\int \frac{dA}{A} = \int -kdt$$

$$\log|A| = -kt + c$$

$$A = e^{-kt+c}$$

$$A = ce^{-kt} \qquad \text{----------(1)}$$

When $t = 0, A = A_0$

$$A_0 = c$$

From (1) we have,

$$A = A_0 e^{-kt} \quad \text{----------(2)}$$

When $t = 100, A = 90\%\ of\ A_0$

$$\frac{90}{100} A_0 = A_0 e^{-100k}$$

$$e^{-100k} = \frac{9}{10}$$

When $t = 1000$,

$$A = A_0 e^{-1000k} \quad \text{From (2)}$$

$$A = A_0 (e^{-100k})^{10}$$

$$A = A_0 \left(\frac{9}{10}\right)^{10}$$

$$\%\ of\ A = \frac{A_0\left(\frac{9}{10}\right)^{10}}{A_0} \times 100$$

$$\%\ of\ A = \frac{9^{10}}{10^8}$$

The radioactive nuclei remain after 1000 years is $\frac{9^{10}}{10^8}\%$

7. Water at temperature $100°C$ cools in 10 minutes to $80°C$ in a room temperature of $25°C$. Find

(i) The temperature of water after 20 minutes

(ii) The time when the temperature is $40°C$

$$\left[\log_e \frac{11}{15} = -0.3101; \log_e 5 = 1.6094\right]$$

Solution:

Let T be the temperature of water at any time t minutes

By Newton's law of cooling,

$$\frac{dT}{dt} \propto T - S$$

$$\frac{dT}{dt} = k(T - S)$$

$$\frac{dT}{T-S} = kdt$$

On integrating both sides, we get

$$\int \frac{dT}{T-S} = \int kdt$$

$$\log|T - S| = kt + c$$

$$T - S = e^{kt+c}$$

$$T - 25 = ce^{kt} \quad \text{----------(1)}$$

At $t = 0, T = 100$

$$100 - 25 = c$$

$$75 = c$$

From (1) we have,

$$T - 25 = 75e^{kt} \quad \text{----------(2)}$$

At $t = 10, T = 80$

$$55 = 75e^{10k}$$

$$e^{10k} = \frac{11}{15} \quad \text{----------(3)}$$

(i) At $t = 20$,

$$T - 25 = 75e^{20k}$$

$$T = 25 + 75(e^{10k})^2$$

$$= 25 + 75\left(\frac{11}{15}\right)^2$$

$$= 25 + \frac{121}{3}$$

$$= 25 + 40.33$$

$$T = 65.33$$

∴ Temperature of water after 20 minutes is $65.33°C$ (approximately)

(ii) From (3) we have

$$e^{10k} = \frac{11}{15}$$

Take log on both sides, we get

$$10k = \log\frac{11}{15}$$

$$10k = -0.3101$$

$$k = -0.03101 \quad \text{----------(4)}$$

At $T = 40$

$$15 = 75e^{kt}$$

$$\frac{1}{e^{kt}} = 5$$

$$e^{-kt} = 5$$

Take log on both sides, we get

$$-kt = \log 5$$

$$-kt = 1.6094$$

$$0.03101t = 1.6094$$

$$t = \frac{1.6094}{0.03101}$$

$$t = 51.89$$

Hence the time when the temperature is $40°C$ is approximately 51.9 minutes.

8. At 10.00 A.M. a woman took a cup of hot instant coffee from her microwave oven and placed it on a nearby Kitchen counter to cool. At this instant the temperature of the coffee was $180°F$, and 10 minutes later it was $160°F$. Assume the constant temperature of the kitchen was $70°F$.

(i) What was the temperature of the coffee at 10.15 A.M.? $\left[\log\frac{9}{11}=-0.2006,\ \log\frac{7}{11}=-0.452\right]$

(ii)The woman likes to drink coffee when its temperature is between $130°F$ and $140°F$. Between what times should she have drunk the coffee?$\left[\log\frac{6}{11}=-0.6061\right]$

Solution:

Let T be the temperature of water at any time t minutes

By Newton's law of cooling,

$$\frac{dT}{dt}\propto T-S$$

$$\frac{dT}{dt}=k(T-S)$$

$$\frac{dT}{T-S}=kdt$$

On integrating both sides, we get

$$\int\frac{dT}{T-S}=\int kdt$$

$$\log|T-S|=kt+c$$

$$T-S=e^{kt+c}$$

$$T-70=ce^{kt} \qquad \text{----------(1)}$$

At $t=0, T=180$

$$110=c$$

$$T-70=110e^{kt} \qquad \text{----------(2)}$$

At $t=10, T=160$

$$90=110e^{10k}$$

$$e^{10k}=\frac{9}{11} \qquad \text{----------(3)}$$

(i) At $t=15$,

$$T-70=110e^{15k} \qquad \text{From (2)}$$

$$=110e^{10k\times\frac{15}{10}}$$

$$=110(e^{10k})^{\frac{3}{2}}$$

$$=110\left(\frac{9}{11}\right)^{\frac{3}{2}}$$

$$=110\left(\frac{9}{11}\right)\left(\frac{9}{11}\right)^{\frac{1}{2}}$$

$$=90\left(\frac{9}{11}\right)^{\frac{1}{2}}$$

$$=90(0.9045)$$

$$T-70=81.4$$

$$T=151.4°F$$

(ii) From (3) we have, $e^{10k}=\frac{9}{11}$

$$10k=\log\frac{9}{11}$$

$$10k=-0.2006$$

$$k=-0.02006$$

At $T=130$,

$$60=110e^{kt} \qquad \text{From (2)}$$

$$e^{kt}=\frac{6}{11}$$

$$kt=\log\frac{6}{11}$$

$$kt=-0.6061$$

$$t=\frac{-0.6061}{-0.02006}$$

$$t=30.216 \qquad \text{----------(4)}$$

At $T=140$,

$$70=110e^{kt} \qquad \text{From (2)}$$

$$e^{kt}=\frac{7}{11}$$

$$kt=\log\frac{7}{11}$$

$$kt=-0.452$$

$$t=\frac{-0.452}{-0.02006}$$

$$t=22.53 \qquad \text{----------(5)}$$

∴ She drunk the coffee between 10.22 $A.M.$ and 10.30 $A.M.$ approximately.

---

9. A pot of boiling water at $100°C$ is removed from a stove at time $t=0$ and left to cool in the kitchen. After 5 minutes, the water temperature has decreased to $80°C$, and another 5 minutes later it has dropped to $65°C$. Determine the temperature of the kitchen.

Solution:

Let T be the temperature of boiling water at any time t

By Newton's law of cooling,

$$\frac{dT}{dt}\propto T-S$$

$$\frac{dT}{dt}=k(T-S)$$

$$\frac{dT}{T-S}=kdt$$

On integrating both sides, we get

$$\int\frac{dT}{T-S}=\int kdt$$

$$\log|T-S|=kt+c$$

$$T-S=e^{kt+c}$$

$$T - S = ce^{kt} \quad \text{----------(1)}$$

At $t = 0, T = 100$

$$100 - S = c$$

$$T - S = (100 - S)e^{kt} \quad \text{From (1)}$$

At $t = 5, T = 80$

$$80 - S = (100 - S)e^{5k}$$

$$e^{5k} = \frac{80-S}{100-S} \quad \text{----------(2)}$$

At $t = 10, T = 65$

$$65 - S = (100 - S)e^{10k}$$

$$65 - S = (100 - S)(e^{5k})^2$$

$$65 - S = (100 - S)\left(\frac{80-S}{100-S}\right)^2$$

$$65 - S = \frac{(80-S)^2}{100-S}$$

$$(65 - S)(100 - S) = (80 - S)^2$$

$$6500 - 165S + S^2 = 6400 + S^2 - 160S$$

$$6500 - 6400 = -160S + 165S$$

$$100 = 5S$$

$$S = 20$$

∴ The temperature of the kitchen is $20°C$.

10. A tank initially contains 50 litres of pure water. Starting time at time $t = 0$ a brine containing with 2 grams of dissolved salt per litre flows into the tank at the rate of 3 litres per minute. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate. Find the amount of salt present in the tank at any time $t > 0$.

Solution:

Let x denote the amount of salt in the tank at any time t

$$\frac{dx}{dt} = inflow\ rate - outflow\ rate$$

$$\frac{dx}{dt} = 6 - \frac{3x}{50}$$

$$\frac{dx}{dt} = -3\left(\frac{x}{50} - 2\right)$$

$$\frac{dx}{dt} = -3\left(\frac{x-100}{50}\right)$$

$$\frac{dx}{x-100} = -\frac{3}{50}dt$$

On integrating both sides, we get

$$\int \frac{dx}{x-100} = -\frac{3}{50}\int dt$$

$$\log|x - 100| = -\frac{3}{50}t + c$$

$$x - 100 = e^{\frac{-3}{50}t + c}$$

$$x - 100 = ce^{\frac{-3}{50}t} \quad \text{----------(1)}$$

At $t = 0, x = 0$

$$-100 = c$$

$$x - 100 = -100e^{\frac{-3}{50}t} \quad \text{From (1)}$$

$$x = 100 - 100e^{\frac{-3}{50}t}$$

$$x = 100\left(1 - e^{\frac{-3}{50}t}\right)$$