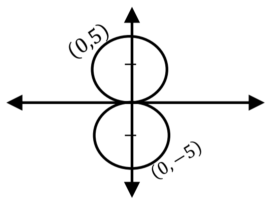
CHAPTER–5

Two Dimensional Analytical Geometry-II

Exercise 5.1

1. Obtain the equation of the circles with radius $5cm$ and touching $x - axis$at the origin in general form.

Solution:



Centre $(h, k) = (0, \pm 5)$

Radius $r = 5cm$

Equation of circle is,

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-0)^2 + (y \pm 5)^2 = 5^2$$

$$x^2 + y^2 \pm 10y + 25 = 25$$

$$x^2 + y^2 \pm 10y = 0$$

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2. Find the equation of the circle with centre $(2, -1)$ and passing through the point $(3, 6)$ in standard form.

Solutions:

Centre $(h, k) = (2, -1)$

Equation of circle is,

$$(x-2)^2 + (y+1)^2 = r^2 \quad \text{----------(1)}$$

Passing through the point at (3,6),

$$(3-2)^2 + (6+1)^2 = r^2$$

$$1^2 + 7^2 = r^2$$

$$r^2 = 50$$

From (1), we get

$$(x-2)^2 + (y+1)^2 = 50$$

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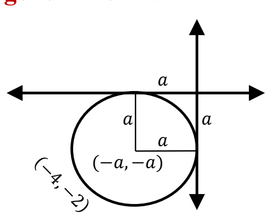
3. Find the equation of circles that touch the both the axes and pass through $(-4, -2)$ in general form.

Solution:



Centre $(h, k) = (-a, -a)$

Radius $r = a$

Equation of circle is,

$$(x+a)^2 + (y+a)^2 = a^2 \quad \text{----------(1)}$$

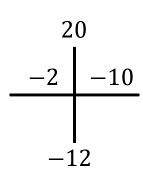
Passing through the point at $(-4, -2)$,

$$(-4+a)^2 + (-2+a)^2 = a^2$$

$$a^2 - 8a + 16 + a^2 - 4a + 4 = a^2$$

$$2a^2 - 12a + 20 = a^2$$

$$a^2 - 12a + 20 = 0$$



$$(a-2)(a-10) = 0$$

$$a - 2 = 0 \text{ or } a - 10 = 0$$

$$a = 2 \text{ or } a = 10$$

When $a = 2$,

$$(x+2)^2 + (y+2)^2 = 2^2$$

$$x^2 + 4x + 4 + y^2 + 4y + 4 = 4$$

$$x^2 + y^2 + 4x + 4y + 8 = 4$$

$$x^2 + y^2 + 4x + 4y + 4 = 0$$

When $a = 10$,

$$(x+10)^2 + (y+10)^2 = 10^2$$

$$x^2 + 20x + 100 + y^2 + 20y + 100 = 100$$

$$x^2 + y^2 + 20x + 20y + 200 = 100$$

$$x^2 + y^2 + 20x + 20y + 100 = 0$$

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4. Find the equation of the circle with centre $(2, 3)$ and passing through the intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$.

Solution:

Given lines are $3x - 2y = 1$ ----------(1)

$4x + y = 27$ ----------(2)

Centre $(h, k) = (2,3)$

Solving (1) and (2), we get

(1) $\quad 3x - 2y = 1$

$(2) \times 2 \quad 8x + 2y = 54$

$$11x = 55$$

$$x = 5$$

Sub $x = 5$ in (1), we get

$$3(5) - 2y = 1$$

$$15 - 2y = 1$$

$$2y = 14$$

$$y = 7$$

∴ The Point is (5,7).

Equation of circle with centre (2,3) is,

$$(x-2)^2 + (y-3)^2 = r^2 \quad \text{----------(3)}$$

Passing through the point at (5,7),

$$(5-2)^2 + (7-3)^2 = r^2$$

$$3^2 + 4^2 = r^2$$

$$9 + 16 = r^2$$

$$r^2 = 25$$

From (3), we get

$$(x-2)^2 + (y-3)^2 = 25$$

$$x^2 - 4x + 4 + y^2 - 6y + 9 = 25$$

$$x^2 + y^2 - 4x - 6y + 13 = 25$$

$$x^2 + y^2 - 4x - 6y - 12 = 0$$

5. Obtain the equation of the circle for which $(3,4)$ and $(2,-7)$ are the ends of a diameter.

Solution:

Given $(x_1, y_1) = (3,4)$ and $(x_2, y_2) = (2,-7)$

Equation of circle is,

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$$

$$(x-3)(x-2) + (y-4)(y+7) = 0$$

$$x^2 - 5x + 6 + y^2 + 3y - 28 = 0$$

$$x^2 + y^2 - 5x + 3y - 22 = 0$$

6. Find the equation of circle through the points $(1,0), (-1,0)$ and $(0,1)$.

Solution:

General equation of circle is,

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{----------(1)}$$

The circle passes through (1,0), (−1,0) and (0,1)

At (1,0),

$$1^2 + 0 + 2g(1) + 2f(0) + c = 0$$

$$2g + c = -1 \quad \text{----------(2)}$$

At (−1,0),

$$(-1)^2 + 0 + 2g(-1) + 2f(0) + c = 0$$

$$-2g + c = -1 \quad \text{----------(3)}$$

At (0,1),

$$0 + 1^2 + 2g(0) + 2f(1) + c = 0$$

$$2f + c = -1 \quad \text{----------(4)}$$

Adding (2) and (3), we get

$$2c = -2$$

$$c = -1$$

Sub $c = -1$ in (2), we get

$$2g - 1 = -1$$

$$2g = 0$$

$$g = 0$$

Sub $c = -1$ in (4), we get

$$2f - 1 = -1$$

$$2f = 0$$

$$f = 0$$

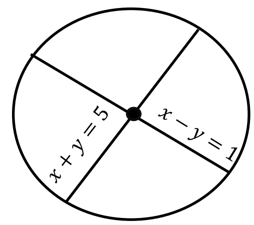
Sub all values in (1), we get

$$x^2 + y^2 + 2(0)x + 2(0)y - 1 = 0$$

$$x^2 + y^2 = 1$$

7. A circle of area 9π square units has two of its diameters along the lines $x + y = 5$ and $x - y = 1$, Find the equation of circle.

Solution:



Area of circle $= 9\pi$

$$\pi r^2 = 9\pi$$

$$r^2 = 9$$

Given $\quad x + y = 5 \quad$ ----------(1)

$\qquad x - y = 1 \quad$ ----------(2)

Adding (1) and (2), we get

$$2x = 6$$

$$x = 3$$

Sub $x = 3$ in (1), we get

$$3 + y = 5$$

$$y = 2$$

∴ centre $(h,k) = (3,2)$

Equation of circle is,

$$(x-3)^2 + (y-2)^2 = 9$$

$$x^2 - 6x + 9 + y^2 - 4y + 4 = 9$$

$$x^2 + y^2 - 6x - 4y + 13 = 9$$

$$x^2 + y^2 - 6x - 4y + 4 = 0$$

8. If $y = 2\sqrt{2} + c$ is a tangent to the circle $x^2 + y^2 = 16$, find the value of c.

Solution:

The condition for the line to be a tangent to the circle is,

$$c^2 = a^2(1 + m^2)$$

Here $a^2 = 16$ and $m = 2\sqrt{2}$

$$c^2 = 16\left(1 + \left(2\sqrt{2}\right)^2\right)$$

$$c^2 = 16(1 + 8)$$

$$c^2 = 16 \times 9$$

$c^2 = 144$

$c = \pm 12$

9. Find the equation of the tangent and normal to the circle $x^2 + y^2 - 6x + 6y - 8 = 0$ at $(2, 2)$.

Solution:

Given eqn is $x^2 + y^2 - 6x + 6y - 8 = 0$

Equation of the tangent is,

$$xx_1 + yy_1 - 6\left(\frac{x+x_1}{2}\right) + 6\left(\frac{y+y_1}{2}\right) - 8 = 0$$

$$xx_1 + yy_1 - 3(x + x_1) + 3(y + y_1) - 8 = 0$$

At $(2,2)$,

$$2x + 2y - 3(x + 2) + 3(y + 2) - 8 = 0$$

$$2x + 2y - 3x - 6 + 3y + 6 - 8 = 0$$

$$-x + 5y - 8 = 0$$

$$x - 5y + 8 = 0$$

Equation of normal is,

$$5x + y + k = 0 \qquad \text{----------(1)}$$

At $(2,2)$,

$$5(2) + 2 + k = 0$$

$$k = -12$$

$\therefore$ Equation of normal is,

$$5x + y - 12 = 0$$

Hint:

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \text{ is } \begin{cases} > 0 & \textit{lies outside} \\ = 0 & \textit{lies on} \\ < 0 & \textit{lies inside} \end{cases}$$

10. Determine whether the points $(-2, 1)$, $(0, 0)$ and $(-4, -3)$ lie outside, on or inside the circle $x^2 + y^2 - 5x + 2y - 5 = 0$.

Solution:

Given $x^2 + y^2 - 5x + 2y - 5 = 0$

At $(-2,1)$,

$$\text{LHS} = (-2)^2 + 1^2 - 5(-2) + 2(1) - 5$$

$$= 4 + 1 + 10 + 2 - 5$$

$$= 12 > 0$$

$\therefore$ $(-2,1)$ lies outside the circle.

At $(0,0)$,

$$\text{LHS} = 0 + 0 - 5(0) + 2(0) - 5$$

$$= -5 < 0$$

$\therefore$ $(0,0)$ lies inside the circle.

At $(-4,-3)$,

$$\text{LHS} = (-4)^2 + (-3)^2 - 5(-4) + 2(-3) - 5$$

$$= 16 + 9 + 20 - 6 - 5$$

$$= 45 - 11$$

$$= 34 > 0$$

$\therefore$ $(-4,-3)$ lies outside the circle.

11. find the centre and radius of the following circles.

(i) $x^2 + (y + 2)^2 = 0$

Solution:

Given $(x - 0)^2 + (y + 2)^2 = 0$

Comparing with the circle eqn,

$$(x - h)^2 + (y - k)^2 = r^2$$

Centre $(h, k) = (0, -2)$

Radius $r = 0$

(ii) $x^2 + y^2 + 6x - 4y + 4 = 0$

Solution:

Given $x^2 + y^2 + 6x - 4y + 4 = 0$

Comparing with the circle eqn,

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Centre $(-g, -f) = (-3,2)$

Radius $r = \sqrt{g^2 + f^2 - c}$

$$r = \sqrt{9 + 4 - 4}$$

$$r = \sqrt{9}$$

$$r = 3$$

(iii) $x^2 + y^2 - x + 2y - 3 = 0$

Solution:

Given $x^2 + y^2 - x + 2y - 3 = 0$

Comparing with the circle eqn,

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Centre $(-g, -f) = \left(\frac{1}{2}, -1\right)$

Radius $r = \sqrt{g^2 + f^2 - c}$

$$r = \sqrt{\frac{1}{4} + 1 + 3}$$

$$= \sqrt{\frac{1}{4} + 4}$$

$$r = \sqrt{\frac{17}{4}}$$

$$r = \frac{\sqrt{17}}{2}$$

(iv) $2x^2 + 2y^2 - 6x + 4y + 2 = 0$

Solution:

Given $2x^2 + 2y^2 - 6x + 4y + 2 = 0$

$\div 2$, $x^2 + y^2 - 3x + 2y + 1 = 0$

Comparing with the circle eqn,

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Centre $(-g, -f) = \left(\frac{3}{2}, -1\right)$

Radius $\quad r = \sqrt{g^2 + f^2 - c}$

$$r = \sqrt{\frac{9}{4} + 1 - 1}$$

$$r = \sqrt{\frac{9}{4}}$$

$$r = \frac{3}{2}$$

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12. If the equation $3x^2 + (3 - p)xy + qy^2 - 2px = 8pq$ represent a circle, find p and q. Also determine the centre and radius of the circle.

Solution:

Given $3x^2 + (3 - p)xy + qy^2 - 2px = 8pq$ ----------(1)

$$coefficient\ of\ x^2 = coefficient\ of\ y^2$$

$$3 = q$$

$$coefficient\ of\ xy = 0$$

$$3 - p = 0$$

$$p = 3$$

Sub $p = 3$, and $q = 3$ in (1), we get

$$3x^2 + 3y^2 - 2(3)x = 8(3)(3)$$

$$3x^2 + 3y^2 - 6x = 72$$

$\div 3, \qquad x^2 + y^2 - 2x = 24$

$$x^2 + y^2 - 2x - 24 = 0$$

Comparing with the circle eqn,

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Centre $(-g, -f) = (1,0)$

Radius $\quad r = \sqrt{g^2 + f^2 - c}$

$$r = \sqrt{1 + 0 + 24}$$

$$r = \sqrt{25}$$

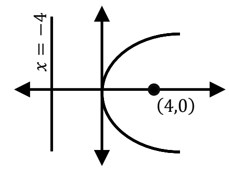
$$r = 5$$

Exercise 5.2

1. Find the equation of parabola in each of the cases given below:

(i) focus $(4, 0)$ and directrix $x = -4$.

Solution:



Given focus $= (4,0)$

$\Rightarrow \qquad a = 4$

Directrix $\quad x = -4$

$\therefore$ Parabola open rightwards.

Equation of parabola is,

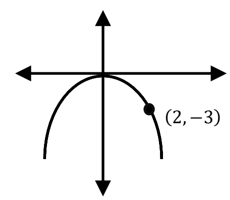
$$y^2 = 4ax$$

$$y^2 = 4(4)x$$

$$y^2 = 16x$$

(ii) passes through $(2, -3)$ and symmetric about y-axis.

Solution:



Given symmetric about y-axis

$\therefore$ Parabola open downwards.

Equation of parabola is,

$$x^2 = -4ay \qquad \text{----------(1)}$$

Passing through $(2, -3)$,

$$2^2 = -4a(-3)$$

$$4 = 12a$$

$$a = \frac{1}{3}$$

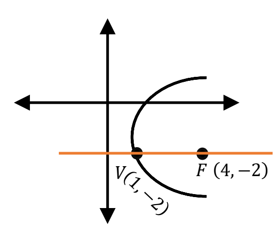
From (1), we get

$$x^2 = -4\left(\frac{1}{3}\right)y$$

$$3x^2 = -4y$$

(iii) vertex $(1, -2)$ and focus $(4, -2)$

Solution:



Vertex $(h, k) = (1, -2)$

Focus $F = (4, -2)$

Here, $\ a = 3$

Parabola open rightwards.

Equation of parabola with vertex (h, k) is,

$$(y - k)^2 = 4a(x - h) \qquad \text{----------(1)}$$

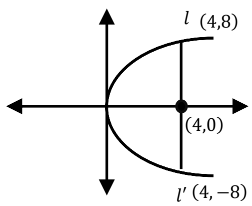
At $(1,-2)$,

$$(y+2)^2 = 4(3)(x-1)^2$$

$$(y+2)^2 = 12(x-1)^2$$

(iv) end points of latus rectum $(4,-8)$ and $(4,8)$

Solution:



Given $l = (4,8)$ and $l' = (4,-8)$

Parabola open rightwards.

$$\text{latus rectum} = 4a$$

$$ll' = 4a$$

$$16 = 4a$$

$$a = 4$$

Equation of parabola is,

$$y^2 = 4ax \qquad \text{----------(1)}$$

$$y^2 = 4(4)x$$

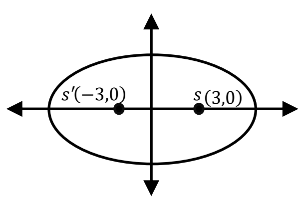
$$y^2 = 16x$$

2. Find the equation of the ellipse in each of the cases given below:

Hint: $focus\ c = ae$

(i) foci $(\pm 3, 0)$, $e = \frac{1}{2}$

Solution:



Given $e = \frac{1}{2}$

Focus $c = 3$

$$ae = 3$$

$$a\left(\frac{1}{2}\right) = 3$$

$$a = 6$$

$$c^2 = a^2 - b^2 \qquad \text{----------(1)}$$

$$9 = 36 - b^2$$

$$b^2 = 36 - 9$$

$$b^2 = 27$$

Since foci $= (\pm 3, 0)$, Major axis is along x-axis.

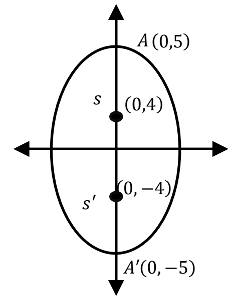
Equation of ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{36} + \frac{y^2}{27} = 1$$

(ii) foci $(0, \pm 4)$ and end points of major axis are $(0, \pm 5)$

Solution:



Given focus $c = 4$

here $a = 5$

$$c^2 = a^2 - b^2$$

$$16 = 25 - b^2$$

$$b^2 = 25 - 16$$

$$b^2 = 9$$

Since foci $= (0, \pm 4)$, Major axis is along y-axis.

Equation of ellipse is,

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$\frac{x^2}{9} + \frac{y^2}{25} = 1$$

(iii) length of latus rectum 8, eccentricity $= \frac{3}{5}$, centre $(0,0)$ and major axis on x-axis.

Solution:

Given $e = \frac{3}{5}$, centre $(0,0)$

Length of latus rectum $= \frac{2b^2}{a}$

$$8 = \frac{2b^2}{a}$$

$$4 = \frac{b^2}{a}$$

$$b^2 = 4a \qquad \text{----------(1)}$$

Wkt, $\quad b^2 = a^2(1 - e^2)$

$$4a = a^2\left(1 - \frac{9}{25}\right)$$

$$4 = a\left(\frac{25-9}{25}\right)$$

$$4(25) = 16a$$

$$4a = 25$$

$$a = \frac{25}{4}$$

Sub $a = \frac{25}{4}$ in (1), we get

$$b^2 = 4\left(\frac{25}{4}\right)$$

$$b^2 = 25$$

Major axis is along x-axis

Equation of ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{\frac{625}{16}} + \frac{y^2}{25} = 1$$

$$\frac{16x^2}{625}+\frac{y^2}{25}=1$$

(iv) length of latus rectum 4, distance between foci $4\sqrt{2}$, centre $(0,0)$ and major axis on y-axis.

Solution:

Given centre (0,0)

Distance between foci $= 4\sqrt{2}$

$$2c = 4\sqrt{2}$$

$$c = 2\sqrt{2}$$

Length of latus rectum $= \frac{2b^2}{a}$

$$4 = \frac{2b^2}{a}$$

$$2 = \frac{b^2}{a}$$

$$b^2 = 2a \qquad \text{----------(1)}$$

Wkt, $\quad c^2 = a^2 - b^2$

$$\left(2\sqrt{2}\right)^2 = a^2 - 2a$$

$$8 = a^2 - 2a$$

$$a^2 - 2a - 8 = 0$$

$$(a-4)(a+2) = 0$$

$$\begin{array}{c|c} \multicolumn{2}{c}{-8} \\ 2 & -4 \\ \hline \multicolumn{2}{c}{-2} \end{array}$$

$a = 4$ or $a = -2$ (not possible)

$\Rightarrow \quad a = 4$

Sub $a = 4$ in (1), we get

$$b^2 = 2a$$

$$b^2 = 8$$

Major axis is along y-axis.

∴ Equation of ellipse is,

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

$$\frac{x^2}{8}+\frac{y^2}{16}=1$$

3. Find the equation of hyperbola in each of the cases given below:

(i) foci $(\pm 2, 0)$, eccentricity $= \frac{3}{2}$.

Solution:

Given $e = \frac{3}{2}$ and foci $(\pm 2,0)$

Focus $c = 2$

$$ae = 2 \qquad [\because c = ae]$$

$$a\left(\frac{3}{2}\right) = 2$$

$$a = \frac{4}{3}$$

$$\Rightarrow \quad a^2 = \frac{16}{9}$$

Now $\quad c^2 = a^2 + b^2$

$$4 = \frac{16}{9} + b^2$$

$$b^2 = 4 - \frac{16}{9} = \frac{36-16}{9}$$

$$\Rightarrow \quad b^2 = \frac{20}{9}$$

Transverse axis is along x-axis.

∴ Equation of hyperbola is,

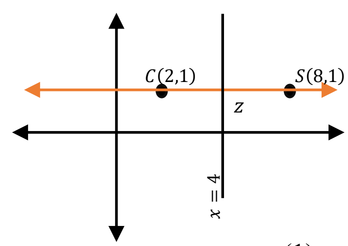
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$\frac{x^2}{\frac{16}{9}}-\frac{y^2}{\frac{20}{9}}=1$$

$$\frac{9x^2}{16}-\frac{9y^2}{20}=1$$

(ii) centre $(2,1)$, one of the foci $(8,1)$ and corresponding directrix $x = 4$.

Solution:



Centre $(h,k) = (2,1)$

Focus $c = CS$

$\Rightarrow \quad c = 6$

$\Rightarrow \quad ae = 6 \qquad$ ----------(1)

Now $\frac{a}{e} = CZ$

$$\frac{a}{e} = 2$$

$\Rightarrow \quad a = 2e$

Sub $a = 2e$ in (1), we get

$$2e(e) = 6$$

$$e^2 = 3$$

$$e = \sqrt{3}$$

Sub $e = \sqrt{3}$ in (1), we get

$$a\left(\sqrt{3}\right) = 6$$

$$a = \frac{2\times\sqrt{3}\sqrt{3}}{\sqrt{3}}$$

$$a = 2\sqrt{3}$$

$$a^2 = \left(2\sqrt{3}\right)^2 = 12$$

Now $c^2 = a^2 + b^2$

$36 = 12 + b^2$

$b^2 = 24$

Transverse axis is along x-axis.

∴ Equation of hyperbola with centre (h, k) is,

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{12} - \frac{(y-1)^2}{24} = 1$$

(iii) passing through $(5, -2)$ and length of the transverse axis along x axis and of length 8 units.

Solution:

Length of transverse axis = 8

$2a = 8$

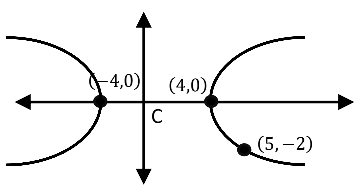
$a = 4$

Transverse axis is along x-axis.

∴ Equation of hyperbola is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{----------(1)}$$

Passes through $(5, -2)$,



$\Rightarrow \quad \frac{5^2}{4^2} - \frac{(-2)^2}{b^2} = 1$

$\frac{25}{16} - \frac{4}{b^2} = 1$

$\frac{25}{16} - 1 = \frac{4}{b^2}$

$\frac{9}{16} = \frac{4}{b^2}$

$b^2 = \frac{4(16)}{9}$

$\Rightarrow \quad b^2 = \frac{64}{9}$

From (1), we get

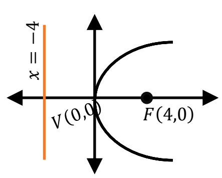
$$\frac{x^2}{16} - \frac{y^2}{\frac{64}{9}} = 1$$

$$\frac{x^2}{16} - \frac{9y^2}{64} = 1$$

4. Find the vertex, focus, equation of directrix and length of the latus rectum of the following:

(i) $y^2 = 16x$

Solution:



Given $y^2 = 16x$

Parabola open rightwards.

Comparing with the parabola eqn,

$(y-k)^2 = 4a(x-h)^2$

Here $\quad 4a = 16$

$\Rightarrow \quad a = 4$

Vertex: $(h, k) = (0,0)$

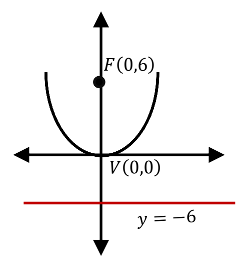
Focus: $(a, 0) = (4,0)$

Directrix: $x = h - a = -4$

Length of latus rectum $= 4a = 16$

(ii) $x^2 = 24y$

Solution:



Given $x^2 = 24y$

Parabola open upwards.

Comparing with the parabola eqn,

$(x-h)^2 = 4a(y-k)$

Here $\quad 4a = 24$

$\Rightarrow \quad a = 6$

Vertex: $(h, k) = (0,0)$

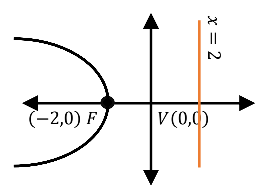
Focus: $(0, a) = (0,6)$

Directrix: $y = k - a = -6$

Length of latus rectum $= 4a = 24$

(iii) $y^2 = -8x$

Solution:



Given $y^2 = -8x$

Parabola open leftwards.

Comparing with the parabola eqn,

$(y-k)^2 = -4a(x-h)$

Here $\quad 4a = 8$

$\Rightarrow \quad a = 2$

Vertex: $(h, k) = (0,0)$

Focus: $(-a, 0) = (-2,0)$

Directrix: $x = h + a = 2$

Length of latus rectum $= 4a = 8$

(iv) $x^2 - 2x + 8y + 17 = 0$

Solution:

Given $x^2 - 2x + 8y + 17 = 0$

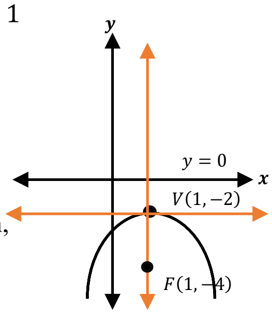
$x^2 - 2x = -8y - 17$

$(x-1)^2 - 1^2 = -8y - 17$

$$(x-1)^2 = -8y - 17 + 1$$

$$(x-1)^2 = -8y - 16$$

$$(x-1)^2 = -8(y+2)$$



Parabola open downwards.

Comparing with the parabola eqn,

$$(x-h)^2 = -4a(y-k)$$

Here $4a = 8$

$\Rightarrow \quad a = 2$

Vertex: $(h,k) = (1,-2)$

Focus: $(0+h, k-a) = (1,-4)$

Directrix: $y = k + a = -2 + 2 = 0$

Length of latus rectum $= 4a = 8$

(v) $y^2 - 4y - 8x + 12 = 0$

Solution:

Given $y^2 - 4y - 8x + 12 = 0$

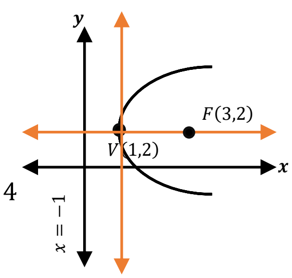
$$y^2 - 4y = 8x - 12$$

$$(y-2)^2 - 2^2 = 8x - 12$$

$$(y-2)^2 = 8x - 12 + 4$$

$$(y-2)^2 = 8x - 8$$

$$(y-2)^2 = 8(x-1)$$



Parabola open rightwards.

Comparing with the parabola eqn,

$$(y-k)^2 = 4a(x-h)$$

Here $4a = 8$

$\Rightarrow \quad a = 2$

Vertex: $(h,k) = (1,2)$

Focus: $(h+a, 0+k) = (3,2)$

Directrix: $x = h - a = 1 - 2 = -1$

Length of latus rectum $= 4a = 8$

5. Identify the type of conic and find centre, foci, vertices, and directrices of each of the following:

(i) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Solution:

Given $\frac{x^2}{25} + \frac{y^2}{9} = 1$

It is an ellipse and major axis is along x-axis.

Comparing with ellipse eqn,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Here $a^2 = 25$ and $b^2 = 9$

$a = 5$ and $b = 3$

$\Rightarrow \quad c^2 = a^2 - b^2$

$c^2 = 25 - 9 = 16$

$\Rightarrow \quad c = 4$

$ae = 4 \qquad [\because c = ae]$

$5e = 4$

$\Rightarrow \quad e = \frac{4}{5}$

Centre $(h,k) = (0,0)$

Vertices $(\pm a, 0) = (\pm 5, 0)$

Foci $(\pm c, 0) = (\pm 4, 0)$

Directrix $x = \pm \frac{a}{e}$

$$x = \pm \frac{5}{\frac{4}{5}}$$

$$x = \pm \frac{25}{4}$$

(ii) $\frac{x^2}{3} + \frac{y^2}{10} = 1$

Solution:

Given $\frac{x^2}{3} + \frac{y^2}{10} = 1$

It is an ellipse and major axis is along y-axis.

Comparing with ellipse eqn,

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

Here $a^2 = 10$ and $b^2 = 3$

$a = \sqrt{10}$ and $b = \sqrt{3}$

$\Rightarrow \quad c^2 = a^2 - b^2$

$c^2 = 10 - 3 = 7$

$\Rightarrow \quad c = \sqrt{7} \qquad$ ----------(1)

$ae = \sqrt{7} \qquad [\because c = ae]$

$\sqrt{10}e = \sqrt{7}$

$\Rightarrow \quad e = \frac{\sqrt{7}}{\sqrt{10}} \qquad$ ----------(2)

Centre $(h,k) = (0,0)$

Vertices $(0, \pm a) = (0, \pm\sqrt{10})$

Foci $(0, \pm c) = (0, \pm\sqrt{7})$

Directrix $y = \pm\frac{a}{e}$

$$y = \pm\frac{\sqrt{10}}{\frac{\sqrt{7}}{\sqrt{10}}}$$

$$y = \pm\frac{10}{\sqrt{7}}$$

(iii) $\frac{x^2}{25} - \frac{y^2}{144} = 1$

Solution:

Given $\frac{x^2}{25} - \frac{y^2}{144} = 1$

It is a hyperbola and transverse axis is along x-axis.

Comparing with ellipse eqn,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Here $a^2 = 25$ and $b^2 = 144$

$a = 5$ and $b = 12$

$\Rightarrow \quad c^2 = a^2 + b^2$

$c^2 = 25 + 144 = 169$

$\Rightarrow \quad c = 13$ ----------(1)

$ae = 13 \qquad [\because c = ae]$

$5e = 13$

$\Rightarrow \quad e = \frac{13}{5}$ ----------(2)

Centre $(h, k) = (0,0)$

Vertices $(\pm a, 0) = (\pm 5,0)$

Foci $(\pm c, 0) = (\pm 13,0)$

Directrix $x = \pm\frac{a}{e}$

$$x = \pm\frac{5}{\frac{13}{5}}$$

$$x = \pm\frac{25}{13}$$

(iv) $\frac{y^2}{16} - \frac{x^2}{9} = 1$

Solution:

Given $\frac{y^2}{16} - \frac{x^2}{9} = 1$

It is a hyperbola and transverse axis along y-axis.

Comparing with ellipse eqn,

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Here $a^2 = 16$ and $b^2 = 9$

$a = 4$ and $b = 3$

$\Rightarrow \quad c^2 = a^2 + b^2$

$c^2 = 16 + 9 = 25$

$c = 5$ -----------(1)

$ae = 5 \qquad [\because c = ae]$

$4e = 5$

$\Rightarrow \quad e = \frac{5}{4}$ ----------(2)

Centre $(h, k) = (0,0)$

Vertices $(0, \pm a) = (0, \pm 4)$

Foci $(0, \pm c) = (0, \pm 5)$

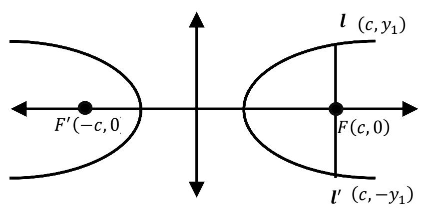
Directrix $y = \pm\frac{a}{e}$

$$y = \pm\frac{4}{\frac{5}{4}}$$

$\Rightarrow \quad y = \pm\frac{16}{5}$

6. Prove that the length of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$.

Solution:



Given $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and transverse axis along x-axis.

Passes through (c, y_1),

$\Rightarrow \quad \frac{c^2}{a^2} - \frac{y_1^2}{b^2} = 1$

$\Rightarrow \quad \frac{c^2}{a^2} - 1 = \frac{y_1^2}{b^2}$

$\frac{c^2 - a^2}{a^2} = \frac{y_1^2}{b^2}$

$\frac{a^2 + b^2 - a^2}{a^2} = \frac{y_1^2}{b^2} \qquad [\because c^2 = a^2 + b^2]$

$\frac{b^2}{a^2} = \frac{y_1^2}{b^2}$

$y_1^2 = \frac{b^4}{a^2}$

$\Rightarrow \quad y_1 = \pm\frac{b^2}{a}$

$\therefore$ The end point of l *and* l' are $\left(c, \frac{b^2}{a}\right)$ and $\left(c, -\frac{b^2}{a}\right)$.

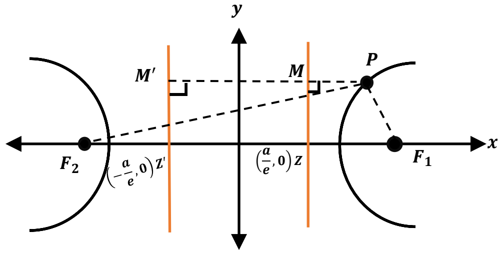
$\therefore$ length of latus rectum $ll' = \frac{2b^2}{a}$

Hence proved.

7. Show that the absolute value of difference of the focal distances of any point P on the hyperbola is the length of its transversal axis.

Solution:

Hyperbola eqn is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



TP: $|F_2P - F_1P| = 2a$ or $|F_1P - F_2P| = 2a$

From figure,

$$e = \frac{F_1P}{PM}$$

$$F_1P = ePM \qquad \text{----------(1)}$$

Also $$e = \frac{F_2P}{PM'}$$

$$F_2P = ePM' \qquad \text{----------(2)}$$

Subtract (2) from (1), we get

$$|F_2P - F_1P| = |ePM' - ePM|$$

$$= |e(PM' - PM)|$$

$$= |e(MM')|$$

$$= |e(ZZ')| \qquad [\because MM' \parallel ZZ']$$

$$= \left|e\left(\frac{2a}{e}\right)\right|$$

$$|F_2P - F_1P| = |2a|$$

$$|F_2P - F_1P| = 2a$$

$|F_2P - F_1P| =$ length of transversal axis

Hence proved.

8. Identify the type of conic and find centre, foci, vertices, and directrices of each of the following:

(i) $\frac{(x-3)^2}{225} + \frac{(y-4)^2}{289} = 1$

Solution:

Given $\frac{(x-3)^2}{225} + \frac{(y-4)^2}{289} = 1$

It is of the form $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, which is an ellipse.

Major axis along y-axis.

Centre $(h, k) = (3,4)$

Here $a^2 = 289$ and $b^2 = 225$

$\Rightarrow \quad a = 17$ and $b = 15$

Now $c^2 = a^2 - b^2$

$c^2 = 289 - 225 = 64$

$\Rightarrow \quad c = 8$

$ae = 8 \qquad [\because c = ae]$

$17e = 8$

$\Rightarrow \quad e = \frac{8}{17}$

$\Rightarrow \quad \frac{a}{e} = \frac{17}{\frac{8}{17}} = \frac{289}{8}$

Centre	(h, k)	$\mathbf{(3,4)}$
Foci	$(h, k \pm c)$ $= (3,4 \pm 8)$	$F_1 = (3,4+8) = \mathbf{(3,12)}$ $F_2 = (3,4-8) = \mathbf{(3,-4)}$
Vertices	$(h, k \pm a)$ $= (3,4 \pm 17)$	$v_1 = (3,4+17) = \mathbf{(3,21)}$ $v_2 = (3,4-17) = \mathbf{(3,-13)}$
Directrix	$y = k \pm \frac{a}{e}$ $y = 4 \pm \frac{289}{8}$	$y = 4 + \frac{289}{8} = \mathbf{\frac{321}{8}}$ (or) $y = 4 - \frac{289}{8} = -\mathbf{\frac{257}{8}}$

(ii) $\frac{(x+1)^2}{100} + \frac{(y-2)^2}{64} = 1$

Solution:

Given $\frac{(x+1)^2}{100} + \frac{(y-2)^2}{64} = 1$

It is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, which is an ellipse

Major axis is along x-axis.

Centre $(h, k) = (-1,2)$

Here $a^2 = 100$ and $b^2 = 64$

$\Rightarrow \quad a = 10$ and $b = 8$

Now $c^2 = a^2 - b^2$

$\Rightarrow \quad c^2 = 100 - 64 = 36$

$\Rightarrow \quad c = 6$

$ae = 6 \qquad [\because c = ae]$

$10e = 6$

$e = \frac{6}{10} = \frac{3}{5}$

$\frac{a}{e} = \frac{10}{\frac{3}{5}}$

$\frac{a}{e} = \frac{50}{3}$

Centre	(h,k)	$(\mathbf{-1,2})$
Foci	$(h\pm c,k)$ $=(-1\pm 6,2)$	$F_1=(-1+6,2)$ $=(\mathbf{5,2})$ $F_2=(-1-6,2)$ $=(\mathbf{-7,2})$
Vertices	$(h\pm a,k)$ $=(-1\pm 10,2)$	$v_1=(-1+10,2)$ $=(\mathbf{9,2})$ $v_2=(-1-10,2)$ $=(\mathbf{-11,2})$
Directrix	$x=h\pm\frac{a}{e}$ $x=-1\pm\frac{50}{3}$	$x=-1+\frac{50}{3}=\frac{\mathbf{47}}{\mathbf{3}}$ (or) $x=-1-\frac{50}{3}=-\frac{\mathbf{53}}{\mathbf{3}}$

(iii) $\frac{(x+3)^2}{225}-\frac{(y-4)^2}{64}=1$

Solution:

Given $\frac{(x+3)^2}{225}-\frac{(y-4)^2}{64}=1$

It is of the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$,which is a hyperbola.

Transverse axis is along x-axis.

Centre $(h,k)=(-3,4)$

Here $a^2=225$ and $b^2=64$

$\Rightarrow\quad a=15$ and $b=8$

Now $c^2=a^2+b^2$

$c^2=225+64=289$

$c=17$

$ae=17\qquad [\because c=ae]$

$15e=17$

$e=\frac{17}{15}$

$\frac{a}{e}=\frac{15}{\frac{17}{15}}=\frac{225}{17}$

Centre	(h,k)	$(\mathbf{-3,4})$
Foci	$(h\pm c,k)$ $=(-3\pm 17,4)$	$F_1=(-3+17,4)$ $=(\mathbf{14,4})$ $F_2=(-3-17,4)$ $=(\mathbf{-20,4})$
Vertices	$(h\pm a,k)$ $=(-3\pm 15,4)$	$v_1=(-3+15,4)$ $=(\mathbf{12,4})$ $v_2=(-3-15,4)$ $=(\mathbf{-18,4})$
Directrix	$x=h\pm\frac{a}{e}$ $x=-3\pm\frac{225}{17}$	$x=-3+\frac{225}{17}=\frac{\mathbf{174}}{\mathbf{17}}$ (or) $x=-3-\frac{225}{17}=-\frac{\mathbf{276}}{\mathbf{17}}$

(iv) $\frac{(y-2)^2}{25}-\frac{(x+1)^2}{16}=1$

Solution:

Given $\frac{(y-2)^2}{25}-\frac{(x+1)^2}{16}=1$

It is of the form $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$,which is a hyperbola.

Transverse axis is along y-axis.

Centre $(h,k)=(-1,2)$

Here $a^2=25$ and $b^2=16$

$\Rightarrow\quad a=5$ and $b=4$

Now $\quad c^2=a^2+b^2$

$\Rightarrow\quad c^2=25+16=41$

$\Rightarrow\quad c=\sqrt{41}$

$ae=\sqrt{41}\qquad [\because c=ae]$

$5e=\sqrt{41}$

$\Rightarrow\quad e=\frac{\sqrt{41}}{5}$

$\Rightarrow\quad \frac{a}{e}=\frac{5}{\frac{\sqrt{41}}{5}}=\frac{25}{\sqrt{41}}$

Centre	(h,k)	$(\mathbf{-1,2})$
Foci	$(h,k\pm c)$ $=(-1,2\pm\sqrt{41})$	$F_1=(\mathbf{-1,2+\sqrt{41}})$ $F_2=(\mathbf{-1,2-\sqrt{41}})$
Vertices	$(h,k\pm a)$ $=(-1,2\pm 5)$	$v_1=(-1,2+5)=(\mathbf{-1,7})$ $v_2=(-1,2-5)=(\mathbf{-1,-3})$
Directrix	$y=k\pm\frac{a}{e}$ $y=2\pm\frac{25}{\sqrt{41}}$	$y=\mathbf{2}+\frac{\mathbf{25}}{\sqrt{\mathbf{41}}}$ (or) $y=\mathbf{2}-\frac{\mathbf{25}}{\sqrt{\mathbf{41}}}$

(v) $\mathbf{18x^2+12y^2-144x+48y+120=0}$

Solution:

Given $18x^2+12y^2-144x+48y+120=0$

$18x^2-144x+12y^2+48y=-120$

$18(x^2-8x)+12(y^2+4y)=-120$

$18[(x-4)^2-4^2]+12[(y+2)^2-2^2]=-120$

$18(x-4)^2-288+12(y+2)^2-48=-120$

$18(x-4)^2+12(y+2)^2=-120+336=216$

$\div 216,\qquad \frac{(x-4)^2}{12}+\frac{(y+2)^2}{18}=1$

It is of the form $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$, which is an ellipse.

Major axis is along y-axis.

Centre $(h,k) = (4,-2)$

Here $a^2 = 18$ and $b^2 = 12$

$\Rightarrow \quad a = 3\sqrt{2}$ and $b = \sqrt{12}$

Now $c^2 = a^2 - b^2$

$c^2 = 18 - 12 = 6$

$\Rightarrow \quad c = \sqrt{6}$

$ae = \sqrt{6} \qquad [\because c = ae]$

$3\sqrt{2}e = \sqrt{6}$

$e = \frac{\sqrt{6}}{3\sqrt{2}} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad e = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \frac{a}{e} = \frac{3\sqrt{2}}{\frac{1}{\sqrt{3}}} = 3\sqrt{6}$

Centre	(h,k)	$\mathbf{(4,-2)}$
Foci	$(h,k\pm c)$ $=\left(4,-2\pm\sqrt{6}\right)$	$F_1 = \left(\mathbf{4,-2+\sqrt{6}}\right)$ $F_2 = \left(\mathbf{4,-2-\sqrt{6}}\right)$
Vertices	$(h,k\pm a)$ $=\left(4,-2\pm 3\sqrt{2}\right)$	$v_1 = \left(\mathbf{4,-2+3\sqrt{2}}\right)$ $v_2 = \left(\mathbf{4,-2-3\sqrt{2}}\right)$
Directrix	$y = k \pm \frac{a}{e}$ $y = -2 \pm 3\sqrt{6}$	$y = \mathbf{-2+3\sqrt{6}}$ (or) $y = \mathbf{-2-3\sqrt{6}}$

(vi) $9x^2 - y^2 - 36x - 6y + 18 = 0$

Solution:

Given $9x^2 - y^2 - 36x - 6y + 18 = 0$

$$9x^2 - 36x - y^2 - 6y = -18$$

$$9(x^2 - 4x) - (y^2 + 6y) = -18$$

$$9[(x-2)^2 - 2^2] - [(y+3)^2 - 3^2] = -18$$

$$9(x-2)^2 - 36 - (y+3)^2 + 9 = -18$$

$$9(x-2)^2 - (y+3)^2 = 9$$

$\div 9, \qquad \frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$

It is of the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$,which is a hyperbola.

Transverse axis is along x-axis.

Centre $(h,k) = (2,-3)$

Here $a^2 = 1$ and $b^2 = 9$

$\Rightarrow \quad a = 1$ and $b = 3$

Now $c^2 = a^2 + b^2$

$c^2 = 1 + 9 = 10$

$\Rightarrow \quad c = \sqrt{10}$

$ae = \sqrt{10} \qquad [\because c = ae]$

$e = \sqrt{10}$

$\Rightarrow \quad \frac{a}{e} = \frac{1}{\sqrt{10}}$

Centre	(h,k)	$\mathbf{(2,-3)}$
Foci	$(h\pm c,k)$ $=\left(2\pm\sqrt{10},-3\right)$	$F_1 = \left(\mathbf{2+\sqrt{10},-3}\right)$ $F_2 = \left(\mathbf{2-\sqrt{10},-3}\right)$
Vertices	$(h\pm a,k)$ $=(2\pm 1,-3)$	$v_1 = (\mathbf{3,-3})$ $v_2 = (\mathbf{1,-3})$
Directrix	$x = h \pm \frac{a}{e}$ $x = 2 \pm \frac{1}{\sqrt{10}}$	$x = \mathbf{2+\frac{1}{\sqrt{10}}}$ (or) $x = \mathbf{2-\frac{1}{\sqrt{10}}}$

Exercise 5.3

Identify the type of conic section for each of the equations.

1. $2x^2 - y^2 = 7$

Solution:

Comparing with the general eqn of conic,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Here $A = 2, C = -1$ *and* $F = -7$

$\Rightarrow A \neq C$, A and C are of opposite signs.

$\therefore$ It is a hyperbola.

2. $3x^2 + 3y^2 - 4x + 3y + 10 = 0$

Solution:

Comparing with the general eqn of conic,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Here $A = 3, C = 3, D = -4, E = 3$ *and* $F = -10$

$\Rightarrow \quad A = C,$ *and* $B = 0$

$\therefore$ It is a circle.

3. $3x^2 + 2y^2 = 14$

Solution:

Comparing with the general eqn of conic,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Here $A = 3, C = 2$ *and* $F = -14$

$\Rightarrow \quad A \neq C, A$ and C are of same signs.

$\therefore$ It is an ellipse.

4. $x^2 + y^2 + x - y = 0$

Solution:

Comparing with the general eqn of conic,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Here $A = 1, C = 1, D = 1$ *and* $F = -1$

$\Rightarrow \quad A = C, and\ B = 0$

$\therefore$ It is a circle.

5. $11x^2 - 25y^2 - 44x + 50y - 256 = 0$

Solution:

Comparing with the general eqn of conic,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Here $A = 11, C = -25, D = -44, E = 50\ and\ F = -256$

$\Rightarrow \quad A \neq C$, A and C are of opposite signs.

$\therefore$ It is a hyperbola.

6. $y^2 + 4x + 3y + 4 = 0$

Solution:

Comparing with the general eqn of conic,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Here $C = 1, D = 4, E = 3\ and\ F = 4$

$\Rightarrow \quad B = 0, and\ A = 0 (either\ A\ or\ C = 0)$

$\therefore$ It is a parabola.

Exercise 5.4

1. Find the equations of the two tangents that can be drawn from $(5, 2)$ to the ellipse $2x^2 + 7y^2 = 14$.

Hint: Eqn of tangent to ellipse is $y = mx \pm \sqrt{a^2m^2 + b^2}$

Solution:

Given $2x^2 + 7y^2 = 14$

$\div 14, \quad \frac{x^2}{7} + \frac{y^2}{2} = 1$

Here $a^2 = 7, b^2 = 2$

Equation of tangent is,

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

$$y = mx \pm \sqrt{7m^2 + 2}$$

At (5,2),

$$2 = 5m \pm \sqrt{7m^2 + 2}$$

$$2 - 5m = \pm\sqrt{7m^2 + 2}$$

Squaring on both sides, we get

$$(2 - 5m)^2 = 7m^2 + 2$$

$$4 - 20m + 25m^2 = 7m^2 + 2$$

$$18m^2 - 20m + 2 = 0$$

$\div 2, \ 9m^2 - 10m + 1 = 0$

$$\begin{array}{c|c} & 9 \\ -1 & -9 \\ \hline 9 & 9 \\ & -10 \end{array}$$

$$(9m - 1)(m - 1) = 0$$

$9m - 1 = 0$ or $m - 1 = 0$

$m = \frac{1}{9}$ or $m = 1$

Equation of tangent is,

$$(y - y_1) = m(x - x_1)$$

At (5,2), $(y - 2) = m(x - 5)$ ----------(1)

When $m = 1$,

$$(y - 2) = 1(x - 5)$$

$$x - 5 - y + 2 = 0$$

$$\boldsymbol{x - y - 3 = 0}$$

When $m = \frac{1}{9}$,

$$(y - 2) = \frac{1}{9}(x - 5)$$

$$9y - 18 = x - 5$$

$$\boldsymbol{x - 9y + 13 = 0}$$

2. Find the equations of tangents to the hyperbola $\frac{x^2}{16} - \frac{y^2}{64} = 1$ which are parallel to $10x - 3y + 9 = 0$.

Hint: Equatoin of tangent to hyperbola is,

$$\boldsymbol{y = mx \pm \sqrt{a^2m^2 - b^2}}$$

Solution:

Given $\frac{x^2}{16} - \frac{y^2}{64} = 1$

Here $a^2 = 16, b^2 = 64$

Also given $10x - 3y + 9 = 0$

$$slope\ m = \frac{-coefficient\ of\ x}{coefficient\ of\ y}$$

$$m = -\left(\frac{10}{-3}\right) = \frac{10}{3}$$

Equation of tangent is parallel to the line, $m = m$

Equation of tangent is,

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

$$y = \frac{10}{3}x \pm \sqrt{16\left(\frac{10}{3}\right)^2 - 64}$$

$$y = \frac{10}{3}x \pm \sqrt{\frac{1600}{9} - 64}$$

$$y = \frac{10}{3}x \pm \sqrt{\frac{1600-576}{9}}$$

$$y = \frac{10}{3}x \pm \sqrt{\frac{1024}{9}}$$

$$y = \frac{10}{3}x \pm \frac{32}{3}$$

$$y = \frac{10x \pm 32}{3}$$

$$3y = 10x \pm 32$$

$$\Rightarrow \quad 10x - 3y \pm 32 = 0$$

$\therefore$ The equation of tangent be $10x - 3y + 32 = 0$ and $10x - 3y - 32 = 0$

3. Show that the line $x - y + 4 = 0$ is a tangent to the ellipse $x^2 + 3y^2 = 12$. Also find the coordinates of the point of contact.

Hint: Point of contact $= \left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$

Solution:

Given $x^2 + 3y^2 = 12$

$\div 12, \quad \frac{x^2}{12} + \frac{y^2}{4} = 1$

Here $a^2 = 12, b^2 = 4$

TP: The line is tangent to the ellipse.

Given $x - y + 4 = 0$

$$y = x + 4$$

Comparing to $y = mx + c$, we get

$$m = 1 \text{ and } c = 4$$

Condition that the line to be tangent to the ellipse is,

$$c^2 = a^2m^2 + b^2$$

$$4^2 = 12(1) + 4$$

$$16 = 16$$

$\therefore$ The line is tangent to the ellipse.

Point of contact $= \left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$

$$= \left(-\frac{12(1)}{4}, \frac{4}{4}\right)$$

Point of contact $= (-3,1)$

4. Find the equation of the tangent to the parabola $y^2 = 16x$ perpendicular to $2x + 2y + 3 = 0$

Hint: Eqn of tangent to parabola is $y = mx + \frac{a}{m}$

Solution:

Given $y^2 = 16x$

Compare with $y^2 = 4ax$, we get

$\Rightarrow \quad a = 4$

Given line is $2x + 2y + 3 = 0$

$$slope\ m = \frac{-coefficient\ of\ x}{coefficient\ of\ y}$$

$$m = -\frac{2}{2}$$

$$m = -1$$

Equation of tangent is perpendicular to the line, $m = -\frac{1}{m}$

$$m = -\frac{1}{(-1)} = 1$$

Equation of tangent is,

$$y = mx + \frac{a}{m}$$

$$y = (1)x + \frac{4}{1}$$

$$y = x + 4$$

$$x - y + 4 = 0$$

5. Find the equation of the tangent $t = 2$ to the parabola $y^2 = 8x$. (Hint: use parametric form)

Solution:

Given $y^2 = 8x$ and $t = 2$

Compare with $y^2 = 4ax$, we get

$\Rightarrow \quad a = 2$

Equation of a tangent to the parabola in parametric form

$$yt = x + at^2$$

$$y(2) = x + 2(2^2)$$

$$2y = x + 8$$

$$x - 2y + 8 = 0$$

6. Find the equation of the tangent and normal to hyperbola $12x^2 - 9y^2 = 108$ at $\theta = \frac{\pi}{3}$. (Hint: use parametric form)

Solution:

Given $12x^2 - 9y^2 = 108$

$\div 108, \quad \frac{x^2}{9} - \frac{y^2}{12} = 1$ ----------(1)

Here $a^2 = 9, b^2 = 12$

$\Rightarrow \quad a = 3, b = 2\sqrt{3}$

Parametric eqn for hyperbola is $x = a\sec\theta, y = b\tan\theta$

$$(a\sec\theta, b\tan\theta) = \left(3\sec\frac{\pi}{3}, 2\sqrt{3}\tan\frac{\pi}{3}\right)$$

$$= \left(3(2), 2\sqrt{3}(\sqrt{3})\right)$$

$$= (6,6)$$

$\therefore$ The point is (6,6)

Equation of tangent is,

$$\frac{xx_1}{9} - \frac{yy_1}{12} = 1 \qquad \text{From (1)}$$

At (6,6)

$$\frac{6x}{9} - \frac{6y}{12} = 1$$

$$\frac{2x}{3} - \frac{y}{2} = 1$$

$$\frac{4x-3y}{6} = 1$$

$$4x - 3y = 6$$

$$4x - 3y - 6 = 0 \qquad \text{----------(2)}$$

Equation of normal is,

$$3x + 4y + k = 0$$

At (6,6)

$$3(6) + 4(6) + k = 0$$

$$18 + 24 + k = 0$$

$$k = -42$$

$\therefore$ Equation of normal is,

$$3x + 4y - 42 = 0 \qquad \text{----------(3)}$$

7. Prove that the point of intersection of the tangent at $'t_1'$ and $'t_2'$ on the parabola $y^2 = 4ax$ is $[at_1t_2, a(t_1 + t_2)]$.

Solution:

Equation of tangent at t_1 is,

$$yt_1 = x + at_1^2 \qquad \text{----------(1)}$$

Equation of tangent at t_2 is,

$$yt_2 = x + at_2^2 \qquad \text{----------(2)}$$

Solving (1) and (2),

(1) $\Rightarrow \quad yt_1 = x + at_1^2$

(2) $\Rightarrow \quad \underset{(-)}{yt_2} = \underset{(-)}{x} + \underset{(-)}{at_2^2}$

$$y(t_1 - t_2) = a(t_1^2 - t_2^2)$$

$$y(t_1 - t_2) = a(t_1 + t_2)(t_1 - t_2)$$

$$y = a(t_1 + t_2) \qquad \text{----------(3)}$$

Sub (3) in (1), we get

$$a(t_1 + t_2)t_1 = x + at_1^2$$

$$at_1^2 + at_1t_2 = x + at_1^2$$

$$x = at_1t_2$$

$\therefore$ Point of intersection is $[at_1t_2, a(t_1 + t_2)]$

8. If the normal at the point $'t_1'$ on the parabola $y^2 = 4ax$ meets the parabola again at the point $'t_2'$, then prove that $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

Solution:

Equation of normal at t_1 is,

$$y + xt_1 = at_1^3 + 2at_1$$

It meets the point t_2 at $(at_2^2, 2at_2)$,

$$2at_2 + (at_2^2)t_1 = at_1^3 + 2at_1$$

$$2at_2 - 2at_1 = at_1^3 - at_1t_2^2$$

$$2a(t_2 - t_1) = at_1(t_1^2 - t_2^2)$$

$$-2a(t_1 - t_2) = at_1(t_1 + t_2)(t_1 - t_2)$$

$$-2 = t_1(t_1 + t_2)$$

$$t_1 + t_2 = -\frac{2}{t_1}$$

$$t_2 = -\frac{2}{t_1} - t_1$$

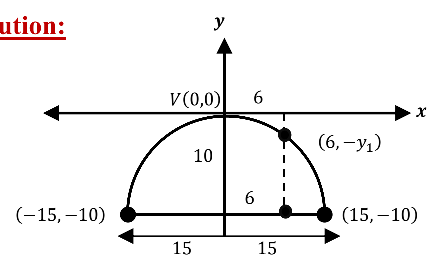
$$t_2 = -\left(\frac{2}{t_1} + t_1\right)$$

Hence proved.

Exercise 5.5

1. A bridge has a parabolic arch that is $10m$ high in the centre and $30m$ wide at the bottom. Find the height of the arch $6m$ from the centre, on either side.

Solution:



Equation of parabolic arch is,

$$x^2 = -4ay \qquad \text{----------(1)}$$

Sub $(15, -10)$ in (1), we get

$$15^2 = -4a(-10)$$

$$\frac{225}{10} = 4a$$

$$\frac{45}{2} = 4a$$

Sub $4a$ value in (1), we get

$$x^2 = -\left(\frac{45}{2}\right)y$$

At $(6, -y_1)$,

$$6^2 = -\left(\frac{45}{2}\right)(-y_1)$$

$$36 = \frac{45}{2}y_1$$

$$12 = \frac{15}{2}y_1$$

$$4 = \frac{5}{2}y_1$$

$$8 = 5y_1$$

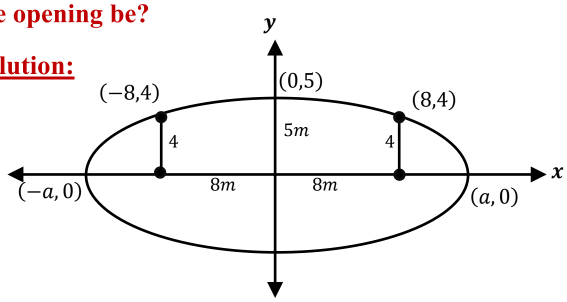
$$y_1 = \frac{8}{5}$$

$$y_1 = 1.6$$

$\therefore$ The required height of arch $= 10 - 1.6 = 8.4m$

2. A tunnel through a mountain for a four lane highway is to have a elliptical opening. The total width of the highway (not be opening) is to be $16m$, and the height at the edge of the road must be sufficient for a truck $4m$ high to clear if the highest point of the opening is to be $5m$ approximately. How wide must the opening be?

Solution:



Equation of ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{----------(1)}$$

Sub $(0,5)$ in (1), we get

$$0 + \frac{y^2}{5^2} = 1$$

$$\frac{5^2}{b^2} = 1$$

$$b^2 = 5^2$$

Sub $(8,4)$ in (1), we get

$$\frac{8^2}{a^2} + \frac{4^2}{5^2} = 1$$

$$\frac{64}{a^2} + \frac{16}{25} = 1$$

$$\frac{64}{a^2} = 1 - \frac{16}{25}$$

$$\frac{64}{a^2} = \frac{9}{25}$$

$$a^2 = \frac{64\times 25}{9}$$

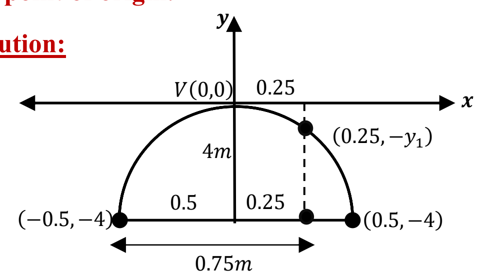
$$a = \frac{8\times 5}{3}$$

$$a = \frac{40}{3}$$

$\therefore$ The required wide for the opening $2a = 2\left(\frac{40}{3}\right) = 26.67m$

3. At a water fountain, water attains a maximum height of $4m$ at horizontal distance of $0.5m$ from its origin. If the path of water is a parabola, find the height of water at a horizontal distance $0.75m$ form the point of origin.

Solution:



Equation of parabola is,

$$x^2 = -4ay \qquad \text{----------(1)}$$

Sub $(-0.5, -4)$ in (1), we get

$$(-0.5)^2 = -4a(-4)$$

$$\frac{0.25}{4} = 4a$$

Sub $4a$ value in (1), we get

$$x^2 = -\left(\frac{0.25}{4}\right)y \qquad \text{----------(2)}$$

At $(0.25, -y_1)$,

$$(0.25)^2 = -\left(\frac{0.25}{4}\right)(-y_1)$$

$$0.25 = \frac{y_1}{4}$$

$$y_1 = 4 \times 0.25$$

$$y_1 = 1$$

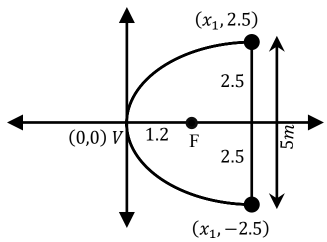
$\therefore$ The required height is $= 4 - 1 = 3m$

4. An engineer designs a satellite dish with a parabolic cross section. The dish is $5m$ wide at the opening, and the focus is placed $1.2m$ from the vertex

(a) Position a coordinate system with the origin at the vertex and the x-axis on the parabola's axis of symmetry and find an equation of the parabola.

(b) Find the depth of the satellite dish at the vertex.

Solution:



Wkt, $VF = a$

$\Rightarrow \quad a = 1.2$

Equation of parabola is,

$$y^2 = 4ax$$

$$y^2 = 4(1.2)x$$

$y^2 = 4.8x$, is the required eqn of parabola.

At $(x_1, 2.5)$,

$$(2.5)^2 = 4.8x_1$$

$$6.25 = 4.8x_1$$

$$x_1 = \frac{6.25}{4.8} \times \frac{100}{100}$$

$$x_1 = \frac{625}{480} = \frac{125}{96}$$

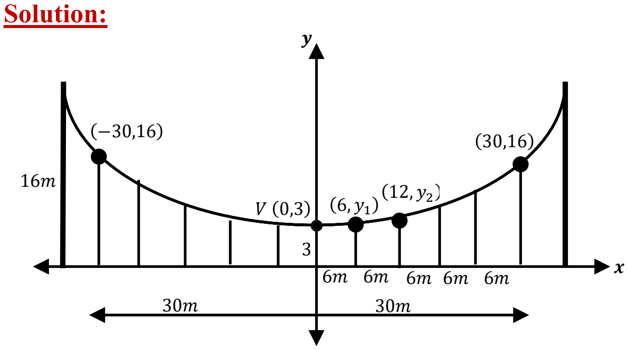
$$x_1 = 1.3m$$

$\therefore$ The depth of the satellite dish is $1.3m$

---

5. Parabolic cable of a $60m$ portion of the roadbed of a suspension bridge are positioned as shown below. Vertical cables are to be spaced every $6m$ along this portion of the roadbed. Calculate the length of first two of these vertical cables from the vertex.

Solution:



Equation of parabola with vertex (h, k) is,

$$(x-h)^2 = 4a(y-k)$$

Centre at (0,3),

$$x^2 = 4a(y-3) \qquad \text{----------(1)}$$

Sub (30,16) in (1), we get

$$30^2 = 4a(16-3)$$

$$900 = 4a(13)$$

$$4a = \frac{900}{13}$$

Sub $4a$ value in (1), we get

$$x^2 = \frac{900}{13}(y-3) \qquad \text{----------(2)}$$

sub $(6, y_1)$ in (2), we get

$$6^2 = \frac{900}{13}(y_1 - 3)$$

$$\frac{36 \times 13}{900} = y_1 - 3$$

$$\frac{13}{25} = y_1 - 3$$

$$y_1 = \frac{13}{25} + 3$$

$$y_1 = \frac{13+75}{25}$$

$$y_1 = \frac{88}{25}$$

$$y_1 = 3.52$$

$\therefore$ The length of first vertical cable is $3.52m$

Sub $(12, y_2)$ in (2), we get

$$12^2 = \frac{900}{13}(y_2 - 3)$$

$$\frac{12 \times 12 \times 13}{900} = y_2 - 3$$

$$\frac{4 \times 13}{25} = y_2 - 3$$

$$y_2 = \frac{52}{25} + 3$$

$$y_2 = \frac{52+75}{25}$$

$$y_2 = \frac{127}{25}$$

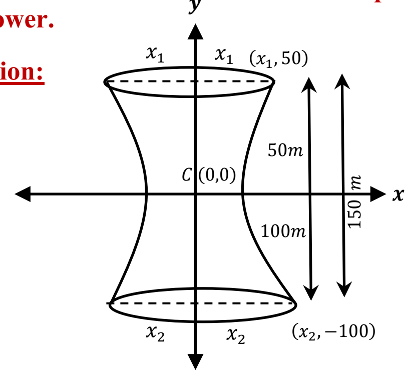
$$y_2 = 5.08$$

$\therefore$ The length of second vertical cable is $5.08m$

---

6. Cross section of a Nuclear cooling tower is in the shape of a hyperbola with equation $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$ .The tower is $150m$ tall and the distance from the top of the tower to the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola. Find the diameter of the top and base of the tower.

Solution:



Equation of hyperbola is,

$$\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1 \qquad \text{----------(1)}$$

Sub $(x_1, 50)$ in (1), we get

$$\frac{x_1^2}{30^2} - \frac{50^2}{44^2} = 1$$

$$\frac{x_1^2}{30^2} = 1 + \frac{2500}{1936}$$

$$\frac{x_1^2}{30^2} = \frac{1936+2500}{1936} = \frac{4436}{1936}$$

$$x_1^2 = \frac{4436}{44^2} \times 30^2$$

$$x_1 = \frac{66.6}{44} \times 30$$

$$x_1 = 45.41$$

∴ The diameter of top is $2x_1 = 2(45.41) = 90.82m$

Sub $(x_1, -100)$ in (1), we get

$$\frac{x_2^2}{30^2} - \frac{(-100)^2}{44^2} = 1$$

$$\frac{x_2^2}{30^2} = 1 + \frac{10000}{1936}$$

$$\frac{x_2^2}{30^2} = \frac{1936+10000}{1936} = \frac{11936}{1936}$$

$$x_2^2 = \frac{11936}{44^2} \times 30^2$$

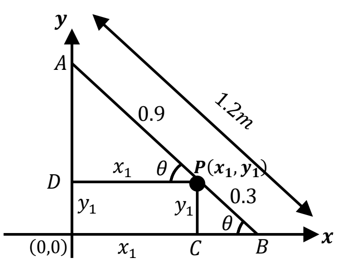
$$x_2 = \frac{109.25}{44} \times 30$$

$$x_2 = 74.49$$

∴ The diameter of base is $2x_2 = 2(74.49) = 148.98m$

7. A rod of length $1.2m$ moves with its ends always touching the coordinate axes. The locus of a point P on the rod, which is $0.3m$ from the end in contact with x-axis is an ellipse. Find the eccentricity.

Solution:



In ΔADP,

$$\cos\theta = \frac{adj}{hyp} = \frac{x_1}{0.9}$$

In ΔPCB,

$$\sin\theta = \frac{opp}{hyp} = \frac{y_1}{0.3}$$

Wkt, $\cos^2\theta + \sin^2\theta = 1$

$$\frac{x_1^2}{(0.9)^2} + \frac{y_1^2}{(0.3)^2} = 1, \text{ which is an ellipse}$$

∴ The locus of $P(x_1, y_1)$ is an ellipse.

Here $a^2 = (0.9)^2$ and $b^2 = (0.3)^2$

$$c^2 = a^2 - b^2$$

$$c^2 = (0.9)^2 - (0.3)^2$$

$$c^2 = 0.81 - 0.09$$

$$a^2e^2 = 0.72 \qquad [\because c = ae]$$

$$(0.81)e^2 = 0.72$$

$$e^2 = \frac{0.72}{0.81}$$

$$= \frac{72}{81}$$

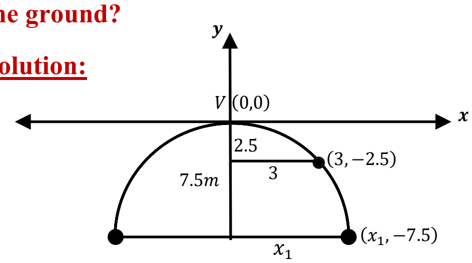
$$e^2 = \frac{8}{9}$$

$$e = \frac{2\sqrt{2}}{3}$$

∴ The eccentricity, $e = \frac{2\sqrt{2}}{3}$.

8. Assume that water issuing from the end of a horizontal pipe, $7.5m$ above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position $2.5m$ below the line of the pipe, the flow of water has curved outward $3m$ beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?

Solution:



Equation of parabola is,

$$x^2 = -4ay \qquad \text{-----------(1)}$$

Sub $(3, -2.5)$ in (1), we get

$$3^2 = -4a(-2.5)$$

$$\frac{9}{2.5} = 4a$$

Sub $4a$ value in (1), we get

$$x^2 = -\left(\frac{9}{2.5}\right)y$$

At $(x_1, -7.5)$,

$$x_1^2 = \left(-\frac{9}{2.5}\right)(-7.5)$$

$$x_1^2 = 9 \times 3$$

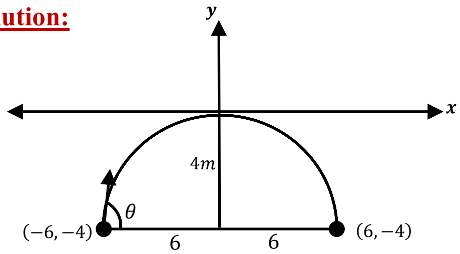
$$x_1 = 3\sqrt{3}$$

∴ Water strikes the ground $3\sqrt{3}m$ beyond the vertical line.

9. On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of $4m$

when it is $6m$ away from the point of projection. Finally it reaches the ground $12m$ away from the starting point. Find the angle of projection.

Solution:



Equation of parabola is,

$$x^2 = -4ay \qquad \text{----------(1)}$$

Sub $(-6,-4)$ in (1), we get

$$(-6)^2 = -4a(-4)$$

$$\frac{36}{4} = 4a$$

$$9 = 4a$$

Sub $4a$ value in (1), we get

$$x^2 = -9y$$

$$y = -\frac{x^2}{9}$$

Diff w.r.t 'x', we get

$$\frac{dy}{dx} = -\frac{2x}{9}$$

At $(-6,-4)$,

$$\frac{dy}{dx} = -\frac{2(-6)}{9}$$

$$\frac{dy}{dx} = \frac{4}{3}$$

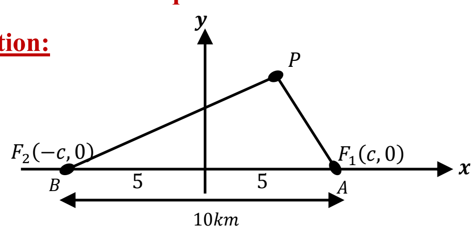
$$\tan\theta = \frac{4}{3} \qquad \left[\because m = \tan\theta = \frac{dy}{dx}\right]$$

$$\theta = \tan^{-1}\frac{4}{3}$$

$\therefore$ The angle of projection is $\tan^{-1}\frac{4}{3}$.

10. Points A and B are $10km$ apart and it is determined from the sound of an explosion heard at those points at different times that the location of the explosion is $6km$ closer to A than B. Show that the location of the explosion is restricted to a particular curve and find an equation of it.

Solution:



From the diagram,

$$F_1P - F_2P = 6$$

$$2a = 6 \qquad [by\ Ex\ 5.2(7)]$$

$$a = 3$$

$\therefore$ The curve is hyperbola.

Now $F_1F_2 = 10$

$$2c = 10$$

$$c = 5$$

$$c^2 = a^2 + b^2$$

$$5^2 = 3^2 + b^2$$

$$b^2 = 25 - 9$$

$$b^2 = 16$$

$\therefore$ The required equation is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$