Curvature and Shear Vorticity in Cartesian Coordinates

Sharan Majumdar, 6/10/20. Using Bleck (JAS, 1991)[1]

Cartesian Coordinates

$$\boldsymbol{x} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k} = (x, y, z)$$

Horizontal Velocity

$$\boldsymbol{v} = u\,\boldsymbol{i} + v\,\boldsymbol{j} = (u, v, 0)$$

Relative Vorticity

$$\zeta = \boldsymbol{k} \cdot (\nabla \times \boldsymbol{v}) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Natural Coordinates

Unit tangent vector

$$\boldsymbol{t} = \frac{1}{\sqrt{u^2 + v^2}}\,\boldsymbol{v} = \frac{\boldsymbol{v}}{V}$$

Unit normal vector

$$\boldsymbol{n} = \boldsymbol{k} \times \boldsymbol{t}$$

Velocity

$$\boldsymbol{v} = V\,\boldsymbol{t}$$

Relative Vorticity

$$\begin{aligned} \zeta &= -\frac{\partial V}{\partial n} + \frac{V}{R_c} \\ &= \zeta_s + \zeta_c \\ &= \text{Shear vorticity} + \text{Curvature Vorticity} \end{aligned}$$

Calculus Identities for below

$$\begin{aligned} \frac{\partial}{\partial n} &\equiv \boldsymbol{n} \cdot \nabla \\ \nabla \times (A\boldsymbol{b}) &\equiv A\,\nabla \times \boldsymbol{b} + (\nabla \mathrm{A}) \times \boldsymbol{b} \\ \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) &\equiv -\boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{c}) \end{aligned}$$

Rewrite relative vorticity as

$$\zeta = \boldsymbol{k} \cdot (\nabla \times V\boldsymbol{t})$$

Shear vorticity

$$\begin{aligned} \zeta_s &= -\frac{\partial V}{\partial n} = -\boldsymbol{n} \cdot \nabla V = -\nabla V \cdot \boldsymbol{n} \\ &= -\nabla V \cdot (\boldsymbol{k} \times \boldsymbol{t}) \\ &= \boldsymbol{k} \cdot (\nabla V \times \boldsymbol{t}) \end{aligned}$$

We use this to reformulate the curvature vorticity (on next page):

[1] https://journals.ametsoc.org/jas/article/48/8/1124/22892/Tendency-Equations-for-Shear-and-Curvature

$$
\begin{aligned}
\zeta_c &= \zeta - \zeta_s \\
&= \boldsymbol{k}\cdot(\nabla\times V\boldsymbol{t}) - \boldsymbol{k}\cdot(\nabla V\times\boldsymbol{t}) \\
&= V\,\boldsymbol{k}\cdot(\nabla\times\boldsymbol{t}) \\
&= V\,\boldsymbol{k}\cdot\left\{\nabla\times\left(\frac{1}{\sqrt{u^2+v^2}}\,\boldsymbol{v}\right)\right\} \\
&= V\,\boldsymbol{k}\cdot\left\{\left(\frac{1}{\sqrt{u^2+v^2}}\,\nabla\times\boldsymbol{v}\right)+\nabla\left(\frac{1}{\sqrt{u^2+v^2}}\right)\times\boldsymbol{v}\right\} \\
&= \zeta + V\boldsymbol{k}\cdot\left\{\nabla\left(\frac{1}{\sqrt{u^2+v^2}}\right)\times\boldsymbol{v}\right\} \\
&= \zeta + V\boldsymbol{k}\cdot\left\{-\frac{1}{V^3}\left(u\frac{\partial u}{\partial x}+v\,\frac{\partial v}{\partial x},u\frac{\partial u}{\partial y}+v\,\frac{\partial v}{\partial y}\right)\times\boldsymbol{v}\right\} \\
&= \zeta - \frac{1}{V^2}\left(vu\frac{\partial u}{\partial x}+vv\frac{\partial v}{\partial x}-uu\frac{\partial u}{\partial y}-uv\frac{\partial v}{\partial y}\right) \\
&= \zeta - \frac{1}{V^2}(u\ v)\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix}v\\ -u\end{pmatrix}
\end{aligned}
$$

Hence, the shear vorticity

$$
\begin{aligned}
\zeta_s &= \frac{1}{V^2}\left(vu\frac{\partial u}{\partial x}+vv\frac{\partial v}{\partial x}-uu\frac{\partial u}{\partial y}-uv\frac{\partial v}{\partial y}\right) \\
&= \frac{1}{V^2}(u\ v)\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix}v\\ -u\end{pmatrix}
\end{aligned}
$$

Continuing with the curvature vorticity:

$$
\begin{aligned}
\zeta_c &= \zeta - \frac{1}{V^2}\left(vu\frac{\partial u}{\partial x}+vv\frac{\partial v}{\partial x}-uu\frac{\partial u}{\partial y}-uv\frac{\partial v}{\partial y}\right) \\
&= \frac{1}{V^2}(uu+vv)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)-\frac{1}{V^2}\left(vu\frac{\partial u}{\partial x}+vv\frac{\partial v}{\partial x}-uu\frac{\partial u}{\partial y}-uv\frac{\partial v}{\partial y}\right)
\end{aligned}
$$

Hence, the curvature vorticity

$$
\begin{aligned}
\zeta_c &= \frac{1}{V^2}\left(uu\frac{\partial v}{\partial x}-vv\frac{\partial u}{\partial y}-vu\frac{\partial u}{\partial x}+uv\frac{\partial v}{\partial y}\right) \\
&= \frac{1}{V^2}(-v\ u)\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}
\end{aligned}
$$

Adding the shear vorticity and curvature vorticity:

$$
\begin{aligned}
\zeta_s + \zeta_c &= \frac{1}{V^2}\left(\cancel{vu\frac{\partial u}{\partial x}}+vv\frac{\partial v}{\partial x}-uu\frac{\partial u}{\partial y}\cancel{-uv\frac{\partial v}{\partial y}}+uu\frac{\partial v}{\partial x}-vv\frac{\partial u}{\partial y}\cancel{-vu\frac{\partial u}{\partial x}}\cancel{+uv\frac{\partial v}{\partial y}}\right) \\
&= \frac{1}{V^2}(u^2+v^2)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) = \zeta
\end{aligned}
$$