ES VIDEOLEZIONI 2022
LEZ 34

1) - Si lancia una moneta n volte. Determinare la probabilità di ottenere esattamente 2 teste.

<u>Svolgimento</u>

X: n° delle T in n lanci
evento che ci interessa $\{X=2\}$ e cioè $\mathbb{P}(X=2)$

Spettro: $S_X = \{0, 1, 2, \dots, n-1, n\}$ e la $\mathbb{P}(X=2) > 0$ xkè 2 appartiene allo spettro

X ha legge B con parametro n e p (prob testa)

$X \sim B(n, p) \qquad p = \mathbb{P}(T)$ legge binomiale 3 fattori

$$\boxed{\mathbb{P}(n=2) = \underset{1°}{\binom{n}{2}} \cdot \underset{2°}{p^2} \underset{3°}{(1-p)^{n-2}}}$$

1°) n° modi x cui esce teste $\binom{n}{2}$
2°) n successi 2 : $p \cdot p = p^2$
3°) insuccessi $(1-p)^{n-2}$

- Si lancia un dado onesto e indichiamo con N il "punteggio"
N è aleatorio non più come prima
X è il numero delle teste T in N lanci
$\mathbb{P}(X=2)$
Usiamo le $\boxed{\text{alternative}}$ => $\boxed{\text{non so quante volte devo lanciare il dado}}$ e ho legge binomiale
$\{N=1\}, \{N=2\}, \dots, \{N=5\}, \{N=6\}$

$$\mathbb{P}(X=2) = \sum_{n=1}^{6} \mathbb{P}(X=2 \mid N=n) \cdot \mathbb{P}(N=n) = \frac{1}{6} \sum_{n=2}^{6} \mathbb{P}(X=2 \mid N=n) =$$

$$= \frac{p^2}{6} \sum_{n=2}^{6} \binom{n}{2} q^{n-2}$$

3) X al., $F_X(x,\alpha)$ $\alpha > 0$

$$F_X(x,\alpha) = \begin{cases} 0, & x \leq 0 \\ x^\alpha, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

Var. al. di tipo continuo, xhè è l'intervallo

$$f_X(x) = \begin{cases} \alpha x^{\alpha-1} & 0 < x \leq 1 \\ 0, & \text{altrimenti} \end{cases}$$

Calcolare: $K \in \mathbb{N}$, $\mu'_K := \mathbb{E}(X^K)$

Svolg.:

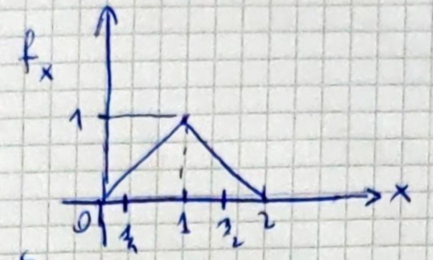
$$\mu'_K = \mathbb{E}(X^K) = \int_0^1 x^K f_X(x,\alpha)\,dx \int_0^1 \alpha x^K \cdot x^{\alpha-1}\,dx =$$

$$= \alpha \int_0^1 x^{K+\alpha-1}\,dx = \alpha \frac{x^{K+\alpha}}{K+\alpha}\Big|_0^1 = \frac{\alpha}{K+\alpha}$$

4) X continua

$$f_X(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \le x < 2 \\ 0, & \text{altr} \end{cases}$$

$S_X = (0,2)$



Determinare il reale ξ per il quale risulta $\mathbb{P}\left(\frac{1}{4} < X \le \xi\right) = \mathbb{P}\left(\xi < X \le \frac{3}{2}\right)$

Svolg: $\boxed{\xi < 1}$

$$\mathbb{P}\left(\frac{1}{4} < X < \xi\right) = \int_{\frac{1}{4}}^{\xi} x\,dx = \frac{x^2}{2}\Big|_{\frac{1}{4}}^{\xi} = \frac{1}{2}\left(\xi^2 - \frac{1}{16}\right)$$ fare i calcoli

$$\mathbb{P}\left(\xi < X \le \frac{3}{2}\right) = \int_{\xi}^{\frac{3}{2}} f_X(x)\,dx = \int_{\xi}^{1} x\,dx + \int_{1}^{3/2} (2-x)\,dx$$

Gaussiana

5) $Y \sim N(1, \sigma^2)$ e tale che $\mathbb{P}(-1 < Y \le 3) = 0,754$

Determinare $\sigma^2 = \mathbb{D}^2(Y)$ Varianza

Svolg.

$$0,754 = \mathbb{P}(-1 < Y \le 3) = \mathbb{P}\left(-\frac{2}{\sigma} < Z \le \frac{2}{\sigma}\right) = 2\mathbb{P}\left(0 \le Z \le \frac{2}{\sigma}\right)$$

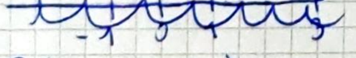


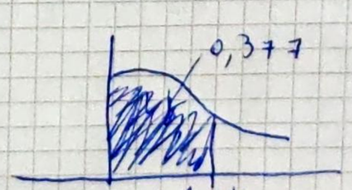
tavola distribuzione gaussiana standard cercare numero $0,754/2$

$$\mathbb{P}\Big(0 < Z < \underbrace{\frac{2}{\sigma}}_{z}\Big) = 0,377$$

$$\frac{2}{\sigma} = z \Leftrightarrow \sigma = \frac{2}{z}$$

$$\int_0^z \varphi(t)\,dt = 0,377$$

$z = 1,16$



$$\frac{2}{\sigma} = 1{,}16 \Leftrightarrow \sigma = \frac{2}{1{,}16} \Leftrightarrow \sigma^2 = \frac{4}{(1{,}16)^2}$$

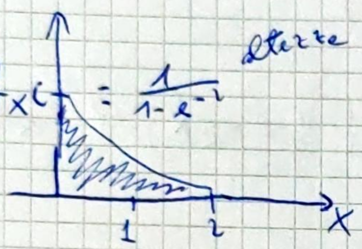
6) Es Standard

L: lato di un quadrato

L v.a. assolutamente continua con funz. den. prob (f.d.p) segnata da

$$f_L(x) = \begin{cases} c \cdot e^{-x} & 0 < x < 2 \\ 0 & \text{altrimenti} \end{cases}$$

Determinare c



$$1 = \int_0^2 c \cdot e^{-x} \, dx = c \cdot \int_0^2 e^{-x} \, dx \iff 1 = c \cdot e^{-x} \Big|_2^0$$

$$\iff 1 = c(1 - e^{-2}) \iff c = \frac{1}{1-e^{-2}}$$

$$f_L(x) = \begin{cases} \dfrac{e^{-x}}{1-e^{-2}}, & 0 < x < 2 \\ 0 & \text{altrimenti} \end{cases}$$

Determinare la Prob che il perimetro del quadrato sia < 2

evento: $\{4L < 2\} \iff \{L < \frac{1}{2}\}$

$$\mathbb{P}(L < \tfrac{1}{2}) = \int_0^{\frac{1}{2}} \frac{e^{-x}}{1-e^{-2}} \, dx = \dots = \frac{1-e^{-1/2}}{1-e^{-2}}$$

Proprietà 2 PdF durante gli ProB.

1) $f_X(x) \geq 0$

2) $\int_R f_X(x)\,dx = 1$

a) $F_X(x)$ è non decrescente

b) $x \in R$, $\lim_{\varepsilon \to 0^+} F_X(x+\varepsilon) = F_X(x)$

c) $\lim_{x \to -\infty} F_X(x) = 0$

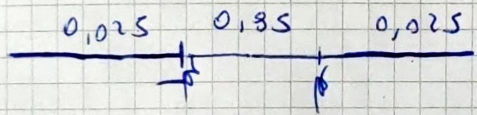
$\lim_{x \to +\infty} F_X(x) = 1$

Altro esercizio su Gaussiane e Tabelle

7) $Z \sim N(0,1)$

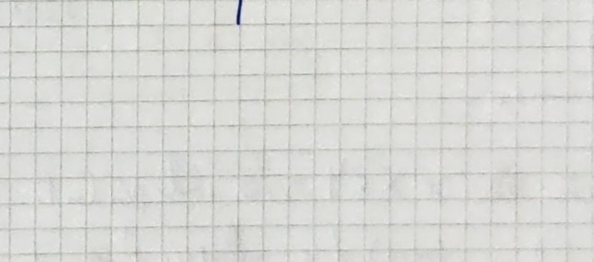
Si determini $\beta \in R^+$. Probabilità che Z assuma un valore esterno dell'intervallo $(-\beta, \beta)$ vale 0.05. × simmetria

Quindi $P(|Z| \leq \beta) = 0{,}95 \iff 2P(Z \leq \beta) = 0{,}95 \iff$



$\iff P(Z \leq \beta) = 0{,}475$

$\beta = 1{,}96$



8) $X \sim N(1,4)$

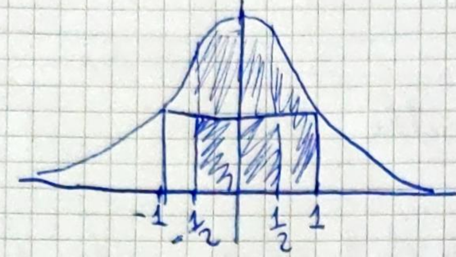
1) $P(0 \le X \le 3) = P\left(\frac{0-1}{2} \le Z \le \frac{3-1}{2}\right) = P\left(-\frac{1}{2} \le Z \le 1\right) =$

$= P\left(-\frac{1}{2} \le Z \le 0\right) + P(0 \le Z \le 1) = P(0 \le Z \le 1/2) + P(0 \le Z \le 1) \approx$

Negativi non ci sono mettiamo 0

$\approx 0,19 + 0,34 \approx 0,53$

Importante



2) $P(X=1 \mid 0 \le X \le 3)$:

$\{X=1\} \cap \{0 \le X \le 3\} = \{X=1\}$

$P(X=1)=0$ stai in condiz. continua

$= \frac{P(X=1)}{P(0 \le X \le 3)} = 0$

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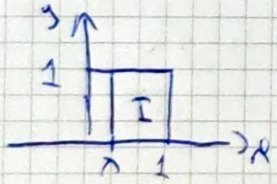
9) (X, Y) Congiuntamente distr.

$$f_{(X,Y)}(x,y) = \begin{cases} x+y, & 0 \le x,y \le 1 \\ 0, & \text{altrimenti} \end{cases}$$



I è il quadrato

a) Si stabilisca se X e Y sono indipendenti: (Non lo sono ci deve fattorizzare essere)

$f_X(x) = \int_0^1 f_{(X,Y)}(x,y)\,dy = \int_0^1 (x+y)\,dy = \int_0^1 x\,dx + \int_0^1 y\,dy$

$= x + \frac{y^2}{2}\Big|_0^1 = x + \frac{1}{2}$

$y \in (0,1)$

$f_Y(y) = y + 1/2$

$x + y \neq (x + \frac{1}{2})(y + \frac{1}{2})$ Non sono indipendenti

b) Vedere se sono correlate

x vederlo bisogna trovare la covarianza, e se è ~~uguale~~ diversa a 0 lo sono.

$$\boxed{Cov(X,Y) = \underline{E}[(X - \mu_X)(Y - \mu_Y)] = E(XY) - \mu_X \mu_Y}$$

$$\boxed{\mu_X = E(X), \quad \mu_Y = E(Y)}$$

$$f_X(x) = \begin{cases} x + \frac{1}{2}, & 0 < x < 1 \\ 0, & \text{altr.} \end{cases} \qquad f_Y(y) = \begin{cases} y + \frac{1}{2}, & 0 < y < 1 \\ 0, & \text{altr} \end{cases}$$

Risulta che X e Y sono somiglianti $\Rightarrow \mu_X = \mu_Y$

Calcoliamo μ_X

$$\mu_X = E(X) = \int_0^1 x\left(x + \frac{1}{2}\right) dx = \int_0^1 x^2 dx + \frac{1}{2}\int_0^1 x\, dx = \frac{x^3}{3}\Big|_0^1 + \frac{1}{2}\frac{x^2}{2}\Big|_0^1 =$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Ora $E(X \cdot Y)$

$$\underline{E}[g(\underline{X})] = \int_{\mathbb{R}} g(\underline{x})\, f_{\underline{X}}(\underline{x})\, d\underline{x}$$

eliminazione

$$E(X \cdot Y) = \int_I \int_I xy(x+y)\, dx\, dy = \underbrace{\int_I \int x^2 y\, dx\, dy}_{*} + \int\int_I xy^2\, dx\, dy = * \text{ Continuo}$$

$$* \int_0^1 dx \int_0^1 dy\, x \cdot y^2 = \int_0^1 x\, dx \int_0^1 y^2 dy = \int_0^1 x\, dx\, \frac{y^3}{3}\Big|_0^1 = \frac{1}{3}\frac{x^2}{2}\Big|_0^1 = \frac{1}{6}$$

$$* \text{Continuo} = \frac{1}{3} \qquad \text{xké } \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

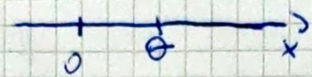
Quindi $E(XY) - \mu_X \mu_Y = \frac{1}{3} - \frac{49}{144} = -\frac{1}{144} \neq 0$

Cov $\neq 0$ sono correlati, e lo sono in negativo, se uno è > 0 l'altro è maggiormente ... e c'è una tendenza negativa

LEZ 35

1) $\underline{X} = (X_1, X_2, \dots, X_n)$, X $f_X(x,\theta) = \begin{cases} e^{-(x-\theta)}, & x>\theta \\ 0, & \text{altr} \end{cases}$ $\theta>0$, $\Theta=(0,+\infty)$

i.i.d.



Preliminare

$$\int_{\theta}^{+\infty} e^{-(x-\theta)}\,dx = \int_{0}^{+\infty} e^{-(x-\theta)}\,d(x-\theta) = e^{-(x-\theta)}\Big|_{+\infty}^{0} = 1-0 = \underline{1}$$

PdP verificata

2) X è v.a. discreta con $S_X = \{-N, -N+1, \dots, 0, 1, \dots, 2N-1, 2N\}$

Determinare PdB: $k \in S_X$ $f_X(k) \; \mathbb{P}(X = k) = c \cdot 2^{-|k|}$ con $c > 0$

$$1 = c\sum_{j=-N}^{2N} 2^{-|j|} = c\left[\sum_{j=-N}^{-1} 2^{j} + \sum_{j=0}^{2N} 2^{-j}\right]$$ mettiamo $j = -i$

$$= c\left[\sum_{i=1}^{N} 2^{-i} + \sum_{j=0}^{2N} 2^{-j}\right] = c\left[\sum_{i=1}^{N} \left(\tfrac{1}{2}\right)^{i} + \sum_{j=0}^{2N} \left(\tfrac{1}{2}\right)^{j}\right] = \cancel{1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}}$$

$c = \dfrac{1}{\dots}$

Determinare la legge di $Y = X^2$ quando $N = 4$

$S_X = \{\cancel{-4}, \cancel{-3}, -2, -1, 0, 1, 2, 3, 4\}$

$S_Y = \{\underbrace{4, \underbrace{1, 0, 1}_{2\text{ valori}}, 4}_{2\text{ valori}}, 9, 16\}$

$S_Y = \{0, 1, 4, 9, 16\}$

$\mathbb{P}(Y = 16) = \mathbb{P}(X = 4) = c \cdot 2^{-4}$

$\mathbb{P}(Y = 9) = \mathbb{P}(X = 3) = c \cdot 2^{-3}$

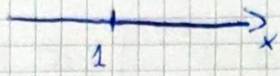
$\mathbb{P}(Y = 4) = \mathbb{P}(X = -2) + \mathbb{P}(X = 2) = c \cdot 2^{-2} + c \cdot 2^{2}$

$\mathbb{P}(Y = 0) = \mathbb{P}(X = 0) = c \cdot 1 = c$

$\mathbb{P}(Y = 1) = \mathbb{P}(X = -1) + \mathbb{P}(X = 1) = c \cdot 2^{-1} + c \cdot 2^{+1}$ IMPORTANTE

3) $\alpha > 0$ è dato (*) IMP.

$$F_X(x) = \begin{cases} 1 - \frac{1}{x^{\alpha}}, & x > 1 \\ 0, & \text{altrimenti} \end{cases}$$



$K \in \mathbb{N}$

Def. f.d.p. Formula da usare $M_k = E(X^k) = \int_{1}^{+\infty} f_X(x)\,dx$

$$f_X(x) = \begin{cases} \alpha\, x^{-\alpha-1}, & x > 1 \\ 0, & \text{altrimenti} \end{cases} = \begin{cases} \alpha \cdot \frac{1}{x^{\alpha+1}}, & x > 1 \\ 0, & \text{altrimenti} \end{cases} \quad \Big\{ \text{Con } \alpha = 2$$

$$M_k = E(x^k) = \int_1^{+\infty} x^k f_X(x)\,dx = \int_1^{+\infty} x^k \frac{\alpha}{x^{\alpha+1}}\,dx = \alpha \int_1^{+\infty} x^{-(\alpha+1)+k}\,dx =$$

$$= \alpha \left. \frac{x^{-(\alpha+1)+k+1}}{-(\alpha+1)+k+1} \right|_1^{+\infty} = \alpha \left. \frac{x^{k-\alpha}}{k-\alpha} \right|_1^{\infty} = \begin{cases} \frac{\alpha}{k-\alpha} & k < \alpha \\ \infty & k \geq \alpha \end{cases}$$

Se $k = \alpha$

$$\alpha \int_1^{+\infty} x^{-\alpha-1+\alpha}$$

Quando $k < \alpha$ esiste M_k Metodo dei momenti

LEZ 36

1) X a.c. $\exists$ f.d.p. — Questa quindi è di LAPLACE ([illegible])

$$f_X(x;\lambda,\beta) = \frac{1}{2\beta} e^{-\frac{|x-\lambda|}{\beta}}$$

$x \in \mathbb{R}$

$\lambda \in \mathbb{R}$ fattore di localizzazione

$\beta > 0$ fattore di scala

Alcuni risultati:

$$\begin{cases} E(X) = \lambda \\ D^2(X) = 2\beta^2 \Rightarrow \mu_2' - (\mu_1')^2 = 2\beta^2 \end{cases} \Rightarrow \begin{cases} \mu_1' = \lambda \\ \mu_2' = (\mu_1')^2 + 2\beta^2 \end{cases}$$

$p = 2$, $s_1 = 1$, $s_2 = 2$

$$\begin{cases} \lambda = \mu_1' \\ 2\beta^2 = \mu_2' - (\mu_1')^2 \Rightarrow \beta^2 = \frac{1}{2}\left[\mu_2' - (\mu_1')^2\right] \Rightarrow \beta = \sqrt{\frac{\mu_2' - (\mu_1')^2}{2}} \end{cases}$$

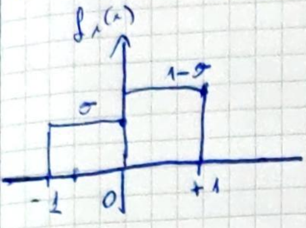
$$\hat{A}_{MM} = \overline{X}, \quad \hat{B}_{MM} = \sqrt{\frac{\overline{X^{(2)}} - (\overline{X})^2}{2}}$$

7) leggi di Probabilità

X v.a. a. continua $\theta \in (0,1)$

$$f(x) = \begin{cases} \theta , & -1 < x \leq 0 \\ 1-\theta , & 0 < x \leq 1 \\ 0 , & \text{altrimenti} \end{cases}$$

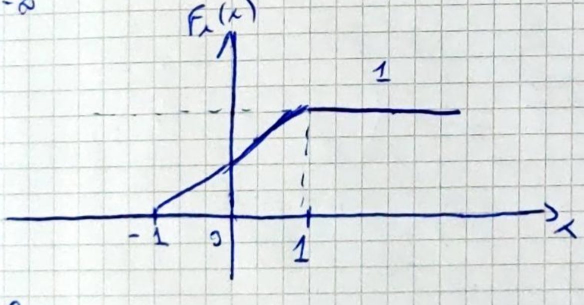
Determinare e graficare della f.d. di X.



$$F_x(x) = \begin{cases} 0 , & x \leq -1 \\ \theta(x+1) , & -1 < x \leq 0 \\ \theta + (1-\theta)\,x & 0 < x \leq 1 \\ 1 , & x > 1 \end{cases}$$

$$\int_{-\infty} f_x(x)dx = \int_{-1}^{x} \theta dx = \theta(x+1)$$

$$\int_{-\infty} f_x(x)dx = 0 + 0 + \int_{0}^{x} (1-\theta)\,dx = \theta + (1-\theta)\, x$$



Calcolare

Determiniamo la mediana

$F_x(Me) = \frac{1}{2}$ se $\theta \leq \frac{1}{2}$

$$\theta + (1-\theta)\,Me = \frac{1}{2} \Rightarrow Me = \frac{\frac{1}{2}-\theta}{1-\theta} = \frac{1-2\theta}{2(1-\theta)}$$

Se $\theta = \frac{1}{4}$ $\qquad \frac{1-\frac{2}{4}}{2\cdot\frac{3}{4}} = \frac{1}{3}$

Se $\theta > \frac{1}{2}$

$\theta(Me+1) = \frac{1}{2}$

$Me + 1 = \frac{1}{2\theta}$

$Me = \frac{1}{2\theta} - 1 = \frac{1-2\theta}{2\theta}$

- Determinare la Media $E(x)$

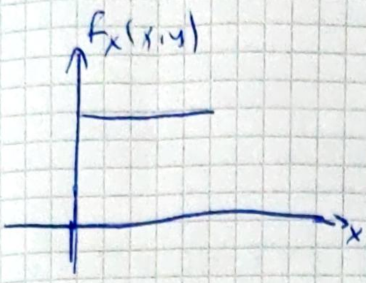
$$Me = \begin{cases} \frac{1-2\theta}{2\theta}, & \theta > \frac{1}{2} \\ \frac{1-2\theta}{2(1-\theta)}, & \theta \leq \frac{1}{2} \end{cases}$$

$$E(x) = \int_{-1}^{1} x\, f_x(x)\, dx = \int_{-1}^{0} x \cdot \theta\, dx + \int_{0}^{1} x(1-\theta)\, dx = \theta \left.\frac{x^2}{2}\right|_{-1}^{0} + (1-\theta)\left.\frac{x^2}{2}\right|_{0}^{1} =$$

$$= \frac{1}{2}\left[-\theta + (1-\theta)\right] = \frac{1-2\theta}{2} = \frac{1}{2} - \theta$$

Solo $\theta = \frac{1}{2}$ valore

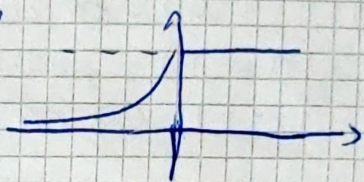
3) $F_X(x,\lambda) = \begin{cases} \frac{1}{1-\lambda x}, & x<0 \\ 1, & x \geq 0 \end{cases}$ Per quali valori di λ F_X è una f.d.



$F'(x,\lambda) = \frac{\cancel{\ldots}\, 0-(\lambda)}{(1-\lambda x)^2} = \frac{\lambda}{(1-\lambda x)^2}$ Se $\lambda>0$ la funz. è crescente quindi deve $\lambda>0$

Se $\lambda=0$, rimane 1 e non è una f.d. $F(x,\lambda) = \frac{\lambda}{(1-\lambda x)^2} \iff \underline{\lambda>0}$

$\lim_{x\to 0^-} \frac{1}{1-\lambda x} = 1$ lim sinistro



Aggiungiamo una v.a. $Y = \frac{1}{X}$, Det. f.d. di Y.

$F_Y(y) = P(Y\leq y) = P(\frac{1}{X} \leq y) = P(X \geq \frac{1}{y}) = P(\frac{1}{y} \leq X < 0) =$ 1°

$= F_X(0) - F_X(\frac{1}{y}) = 1 - \frac{1}{1-\frac{\lambda}{y}} = \frac{\lambda}{\lambda - y}$

$F_Y(y) = \begin{cases} \frac{\lambda}{\lambda-y}, & y \leq 0 & 1° \\ 1, & y \geq 0 & 2° \end{cases}$

2° Con $y \geq 0$

$F_Y(y) = P(Y\leq y) = P(\frac{1}{X} \leq y) =$
$= P(X \geq \frac{1}{y}) + P(X<0) =$
$= 0 + F_X(0) = \underline{1}$

Caso $\lambda = 1$, $F_X(\lambda;1) = F_Y(\overset{y=1/x}{y};1)$ $\frac{1}{1-x}$ e $\frac{1}{1-y}$

$F_X(\lambda;1) = F_Y(x;1)$

$X \overset{\lambda=1}{\sim} Y$

$\frac{1}{1-\lambda M_e} = \frac{1}{2} \iff 1-\lambda M_e = 2 \iff 1-2 = \lambda M_e \iff -\frac{1}{\lambda} = M_e$

$\int_{-\infty}^{0} \frac{x\,\lambda}{(1-\lambda x)^2} dx$