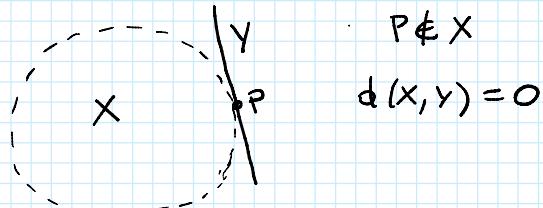


$$(\vec{E}, \mathcal{E}, \pi) \quad \dim \mathcal{E} = m \quad \mathcal{R} = (O, \mathcal{B})$$

$$P, Q \in \mathcal{E}, \quad d(P, Q) \stackrel{\text{def}}{=} \| \overrightarrow{PQ} \|$$

$$X, Y \subseteq \mathcal{E}, \quad d(X, Y) \stackrel{\text{def}}{=} \inf \{d(P, Q) \mid P \in X, Q \in Y\}$$



Siano $P \in \mathcal{E}$ e \mathcal{H} un iper piano

$$\text{Se } P \in \mathcal{H}, \text{ allora } d(P, \mathcal{H}) = d(P, P) = 0$$

Se $P \notin \mathcal{H}$, allora:



Esempio: $m=4$

$$\mathcal{R} = (O, \mathcal{B})$$

$$\mathcal{H}: x_1 - 3x_2 - x_3 + 2x_4 - 1 = 0 \quad w(1, -3, -1, 2) \perp \mathcal{H}, \quad \overrightarrow{\mathcal{H}} = \mathcal{L}(w)$$

$$P(0, 1, 2, 0)$$

$$\mathcal{R}: \begin{cases} x_1 = t \\ x_2 = 1 - 3t \\ x_3 = 2 - t \\ x_4 = 2t \end{cases} \quad T(t, 1-3t, 2-t, 2t)$$

$$\mathcal{R} \cap \mathcal{H}: t - 3 + 9t - 2 + t + 4t - 1 = 0 \quad 15t - 6 = 0 \Rightarrow \bar{t} = \frac{6}{15} = \frac{2}{5}$$

$$\bar{P}\left(\frac{2}{5}, -\frac{1}{5}, \frac{8}{5}, \frac{4}{5}\right)$$

$$d(P, \mathcal{H}) = d(P, \bar{P}) = \|\overrightarrow{P\bar{P}}\| = \sqrt{\frac{4}{25} + \frac{36}{25} + \frac{64}{25} + \frac{16}{25}} = \sqrt{\frac{60}{25}} = \frac{\sqrt{60}}{5}$$

$$\overrightarrow{P\bar{P}}\left(\frac{2}{5}, -\frac{6}{5}, -\frac{2}{5}, \frac{4}{5}\right)$$

In generale:

$$\mathcal{H}: a_1x_1 + a_2x_2 + \dots + a_nx_n - b = 0 \quad P(c_1, c_2, \dots, c_m) \quad (a_1, a_2, \dots, a_m) \neq 0$$

$$w(a_1, a_2, \dots, a_m) \quad \overrightarrow{\mathcal{H}} = \mathcal{L}(w)$$

$$\mathcal{R}: \begin{cases} x_1 = c_1 + a_1t \\ x_2 = c_2 + a_2t \\ \vdots \\ x_m = c_m + a_mt \end{cases} \quad \mathcal{R} \perp \mathcal{H} \quad \text{e} \quad P \in \mathcal{R}$$

$$T(c_1 + \alpha_1 t, c_2 + \alpha_2 t, \dots, c_m + \alpha_m t)$$

$$\alpha_1(c_1 + \alpha_1 t) + \alpha_2(c_2 + \alpha_2 t) + \dots + \alpha_m(c_m + \alpha_m t) - b = 0$$

$$\alpha_1 c_1 + \alpha_1^2 t + \alpha_2 c_2 + \alpha_2^2 t + \dots + \alpha_m c_m + \alpha_m^2 t - b = 0$$

$$(\alpha_1^2 + \alpha_2^2 + \dots + \alpha_m^2)t = b - \alpha_1 c_1 - \alpha_2 c_2 - \dots - \alpha_m c_m$$

$$\bar{t} = \frac{b - \alpha_1 c_1 - \alpha_2 c_2 - \dots - \alpha_m c_m}{\alpha_1^2 + \alpha_2^2 + \dots + \alpha_m^2}$$

$$\bar{P}(c_1 + \alpha_1 \bar{t}, c_2 + \alpha_2 \bar{t}, \dots, c_m + \alpha_m \bar{t})$$

$$\overrightarrow{PP} (\alpha_1 \bar{t}, \alpha_2 \bar{t}, \dots, \alpha_m \bar{t})$$

$$\|\overrightarrow{PP}\| = \sqrt{\alpha_1^2 \bar{t}^2 + \alpha_2^2 \bar{t}^2 + \dots + \alpha_m^2 \bar{t}^2} = \sqrt{(\alpha_1^2 + \alpha_2^2 + \dots + \alpha_m^2)} \frac{(b - \alpha_1 c_1 - \alpha_2 c_2 - \dots - \alpha_m c_m)^2}{(\alpha_1^2 + \alpha_2^2 + \dots + \alpha_m^2)} =$$

$$= \frac{| \alpha_1 c_1 + \alpha_2 c_2 + \dots + \alpha_m c_m - b |}{\sqrt{\alpha_1^2 + \alpha_2^2 + \dots + \alpha_m^2}} = d(P, \bar{P}) = d(P, \mathcal{H})$$

$$\underline{\text{Esempio}} \quad m=2 \quad \mathcal{R}=(0, \mathcal{B})$$

$$\mathcal{H}: \quad 3x_1 + 2x_2 - 4 = 0 \quad P(1, -2)$$

$$d(P, \mathcal{H}) = \frac{|3-4-4|}{\sqrt{9+4}} = \frac{5}{\sqrt{13}}$$

$$\dim \mathcal{E} = m \quad \mathcal{R} = (0, \mathcal{B})$$

$$\mathcal{H}, \mathcal{H}' \text{ iperpiani paralleli: } \overrightarrow{H} = \overrightarrow{H'}$$

$$\mathcal{H}: \quad \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_m x_m - b = 0$$

$$\mathcal{H}': \quad \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_m x_m - b' = 0$$

$$\overrightarrow{PQ} = \overrightarrow{QR}$$

$$\overrightarrow{PQ} = \overrightarrow{QR}$$

$$\overrightarrow{PQ} = \overrightarrow{QR}$$

$$P(c_1, c_2, \dots, c_m) \in \mathcal{H}$$

$$Q(d_1, d_2, \dots, d_m) \in \mathcal{H}$$

$$d(P, \mathcal{H}') = \frac{| \alpha_1 c_1 + \alpha_2 c_2 + \dots + \alpha_m c_m - b' |}{\sqrt{\alpha_1^2 + \dots + \alpha_m^2}} = \frac{| b - b' |}{\sqrt{\alpha_1^2 + \dots + \alpha_m^2}}$$

$$P \in \mathcal{H} \implies \alpha_1 c_1 + \alpha_2 c_2 + \dots + \alpha_m c_m = b$$

$$d(Q, \mathcal{H}') = \frac{| \alpha_1 d_1 + \dots + \alpha_m d_m - b' |}{\sqrt{\alpha_1^2 + \dots + \alpha_m^2}} = \frac{| b - b' |}{\sqrt{\alpha_1^2 + \dots + \alpha_m^2}}$$

$$d(\mathcal{H}, \mathcal{H}') = d(P, \mathcal{H}') \quad \forall P \in \mathcal{H}$$

$$d(\mathcal{H}, R) \quad \forall R \in \mathcal{H}'$$

$$d(\mathcal{H}, \mathcal{H}') = d(P, \mathcal{H}') \quad \forall P \in \mathcal{H}$$

$$d(\mathcal{H}, R) \quad \forall R \in \mathcal{H}'$$

Esempio: $m=2$

$$\mathcal{L}: -x_1 + 3x_2 - 2 = 0$$

$$\mathcal{L}': 2x_1 - 6x_2 + 3 = 0$$

$$P(1,1) \in \mathcal{L}$$

$$d(\mathcal{L}, \mathcal{L}') = \frac{|-2+3|}{\sqrt{1+9}} = \frac{\sqrt{2}}{\sqrt{10}} = \frac{1}{\sqrt{10}}$$

$$\frac{|2-6+3|}{\sqrt{4+36}} = \frac{1}{\sqrt{40}} = \frac{1}{2\sqrt{10}}$$

Esempio $\mathcal{E} = \mathbb{R}^2$ $\tilde{\mathcal{E}} = \mathbb{R}^2$ col prodotto scalare standard

$$R = (O=(2,-3), \mathcal{B} = ((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})))$$

$$u = (0,0) \quad u \equiv_R \phi_{\mathcal{B}}(\overrightarrow{u}) = \phi_{\mathcal{B}}((2,-3))$$

$$(\vec{\mathcal{E}}, \mathcal{E}, \pi) \quad \dim \mathcal{E} = 3 \quad R = (O, \mathcal{B})$$

$$\mathcal{H}: a_1x_1 + a_2x_2 + a_3x_3 - b = 0$$

$$\mathcal{L}: \begin{cases} a_1x_1 + a_2x_2 + a_3x_3 - \beta = 0 \\ a'_1x_1 + a'_2x_2 + a'_3x_3 - \beta' = 0 \end{cases}$$

$$\mathcal{L} \cap \mathcal{H} \neq \emptyset, \text{ allora } d(\mathcal{L}, \mathcal{H}) = 0$$

Altrimenti $\mathcal{L} \parallel \mathcal{H}$

$$\forall P \in \mathcal{L}, \quad d(P, \mathcal{H}) = d(P, \mathcal{L})$$

$$\text{Esercizio: } R = (O, \mathcal{B}) \quad \mathcal{H}: 2x_1 - x_2 + 2x_3 - 1 = 0$$

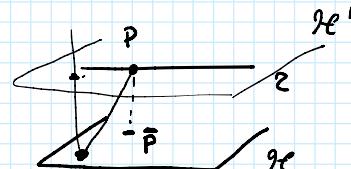
(i) determinare una retta \mathcal{L} : $\mathcal{L} \parallel \mathcal{H}$ e $P(0,2,1) \in \mathcal{L}$ ($P \notin \mathcal{H}$)

(ii) determinare le distanze tra \mathcal{L} e \mathcal{H} .

$$(i') \quad \vec{\mathcal{L}}: 2x_1 - x_2 + 2x_3 = 0 \quad u(1,0,-1)$$

$$\mathcal{L}: \begin{cases} x_1 = t \\ x_2 = 2 \\ x_3 = 1-t \end{cases}$$

$$(ii) \quad d(\mathcal{L}, \mathcal{H}) = d(P, \mathcal{H}) = \frac{|-2+2-1|}{\sqrt{1+1+4}} = \frac{1}{3}$$



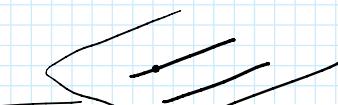
$$(\vec{\mathcal{E}}, \mathcal{E}, \pi) \quad \dim \mathcal{E} = 3 \quad R = (O, \mathcal{B})$$

$\mathcal{L}, \mathcal{L}'$ due rette

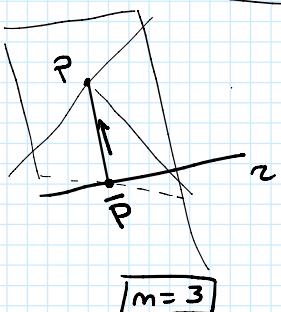
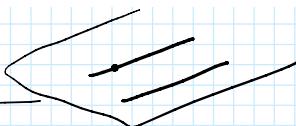
$$\bullet \quad \mathcal{L} \cap \mathcal{L}' \neq \emptyset \Rightarrow d(\mathcal{L}, \mathcal{L}') = 0$$

$$\bullet \quad \mathcal{L} \parallel \mathcal{L}' \Rightarrow d(\mathcal{L}, \mathcal{L}') = d(P, \mathcal{L}') \quad \forall P \in \mathcal{L}$$

$$\dim^\perp \mathcal{L} = 2$$



$$\bullet z \parallel z' \Rightarrow d(z, z') = d(P, z') \quad \forall P \in z$$



Esempio:

$$z: \begin{cases} x_1 = 3 + t \\ x_2 = -2 - t \\ x_3 = 1 + 2t \end{cases}$$

$$P(0, 0, 1) \notin z$$

$$u(1, -1, 2)$$

H piano ortogonale a z e passante per P

$$\bar{P} = H \cap z$$

$$d(P, z) = d(P, \bar{P})$$

$$z': z' \perp z \quad \& \quad |z' \cap z| = 1$$

$$\vec{z}' \subseteq \perp_{\vec{z}} \quad \Rightarrow \quad z' \subseteq \text{piano ortogonale a } z \text{ e passante per } P$$

$$H: x_1 - x_2 + 2x_3 + k = 0$$

$$0 - 0 + 2 + k = 0 \quad \Rightarrow \quad k = -2$$

$$\bar{P} = z \cap H$$

$$T(3+t, -2-t, 1+2t) \in H \Rightarrow$$

$$3+t + 2 + t + 1 + 4t - 2 = 0 \Rightarrow 6t + 5 = 0 \Rightarrow t = -\frac{5}{6}$$

$$\bar{P}\left(3 - \frac{5}{6}, -2 + \frac{5}{6}, 1 - \frac{10}{6}\right)$$

$$d(P, z) = d(P, \bar{P})$$

• z e z' sghembe

Teorema delle comune perpendicolare:

z e z' rette sghembe. Esiste un'unica retta s ch è ortogonale sia a z sia a z' ed è incidente sia a z sia a z' :

$\exists!$ retta $s: s \perp z \text{ e } s \perp z'$

$$P = s \cap z \neq \emptyset \quad \& \quad P \in s \cap z' \neq \emptyset$$

Inoltre:

$$d(z, z') = d(P, P')$$

DIM

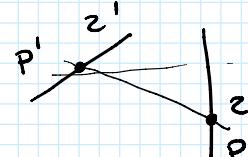
$$R = (0, B) \quad B \text{ ortonormale}$$

$$z: \begin{cases} x = x_0 + lt \\ y = y_0 + mt \\ z = z_0 + nt \end{cases}$$

$$u(l, m, n)$$

$$z': \begin{cases} x = x'_0 + l't' \\ y = y'_0 + m't' \\ z = z'_0 + n't' \end{cases}$$

$$u'(l', m', n')$$



$$P(x_0 + lt, y_0 + mt, z_0 + nt) \in z$$

$$P'(x'_0 + l't', y'_0 + m't', z'_0 + n't') \in z'$$

$$\begin{aligned} \text{Imponiamo che: } & \overrightarrow{PP'} \perp z \Leftrightarrow \langle \overrightarrow{PP'}, u \rangle = 0 \quad (*) \\ & \overrightarrow{PP'} \perp z' \Leftrightarrow \langle \overrightarrow{PP'}, u' \rangle = 0 \end{aligned}$$

$$\overrightarrow{PP'} / \cdot \quad \text{out} \quad \dots \quad \text{in} \quad \perp \quad \dots \quad \perp \quad \dots \quad \perp \quad \dots \quad \perp$$

$$\Leftrightarrow \overrightarrow{PP'} \perp z' \Leftrightarrow \langle \overrightarrow{PP'}, u' \rangle = 0$$

$$\overrightarrow{PP'}(x_0' + l'z' - x_0 - lz, y_0' + m'z' - y_0 - mz, z_0' + n'z' - z_0 - nz)$$

$$\begin{aligned} &\stackrel{(*)}{\Leftrightarrow} \begin{cases} (x_0' + l'z' - x_0 - lz)l + (y_0' + m'z' - y_0 - mz)m + (z_0' + n'z' - z_0 - nz)n = 0 \\ (x_0' + l'z' - x_0 - lz)l' + (y_0' + m'z' - y_0 - mz)m' + (z_0' + n'z' - z_0 - nz)n' = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} (l'l + m'm + n'n)z' + (-l^2 - m^2 - n^2)t + \dots = 0 \\ (l'^2 + m'^2 + n'^2)t' + (-ll' - mm' - nn')t + \dots = 0 \\ \|u'\|^2 - \langle u, u' \rangle \end{cases} \end{aligned}$$

$$\det \begin{pmatrix} \langle u, u' \rangle & -\|u\|^2 \\ \|u'\|^2 & -\langle u, u' \rangle \end{pmatrix} = -\langle u, u' \rangle^2 + \|u\|^2 \|u'\|^2$$

Se $\det(A) \neq 0$, per il Teorema di Cramer $\exists! (t|t')$ che soddisfa il sistema,
per cui $\exists! (P'|P)$ che soddisfa il Teorema

Se $\det(A) = 0$, allora $\langle u, u' \rangle^2 = \|u\|^2 \|u'\|^2$, ovvero $|\langle u, u' \rangle| = \|u\| \|u'\|$
come la disegualanza di Schwarz vale come uguaglianza
e questo è possibile se e solo se $\{u, u'\}$ è lin. dipendente: $u \parallel u'$ \Rightarrow ASSURDO.