Lezione 14

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Ricordiamo che abbiamo dimostrato:

$\Sigma_0 : AX = 0$ sistema lineare omogeneo su K in n variabili.

$S_0 \subseteq K^n$ insieme delle soluzioni.

$$S_0 = \left\{ (y_1, \dots, y_n) \in K^n \;\middle|\; A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0 \right\}$$

- S_0 è un sottosp. vett. di K^n e $\dim S_0 = n - \text{rango}(A)$

Esercizio:

$$\Sigma_0 : \begin{cases} x_1 + x_2 - x_3 + 2x_4 + x_5 = 0 \\ 2x_1 + x_2 + x_3 + x_4 - x_5 = 0 \\ x_1 + 2x_3 - x_4 - 2x_5 = 0 \end{cases}$$

Determinare S_0 e una sua base

$$A = \begin{pmatrix} \boxed{1} & 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 1 & -1 \\ 1 & 0 & 2 & -1 & -2 \end{pmatrix} \quad \begin{matrix} \underline{e}^2 \to \underline{e}^2 - 2\underline{e}^1 \\ \underline{e}^3 \to \underline{e}^3 - \underline{e}^1 \end{matrix} \quad \begin{pmatrix} 1 & 1 & -1 & 2 & 1 \\ 0 & \boxed{-1} & 3 & -3 & -3 \\ 0 & -1 & 3 & -3 & -3 \end{pmatrix}$$

$$\underline{e}^3 \to \underline{e}^3 - \underline{e}^2 \quad \begin{pmatrix} \boxed{1} & 1 & -1 & 2 & 1 \\ 0 & \boxed{-1} & 3 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \underline{e}_2 \to -\underline{e}_2 \\ \underline{e}_1 \to \underline{e}_1 - \underline{e}_2 \end{matrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & -1 & -2 \\ 0 & 1 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{cases} x_1 = -2x_3 + x_4 + 2x_5 \\ x_2 = 3x_3 - 3x_4 - 3x_5 \end{cases}$$

$$S_0 = \{(-2x_3 + x_4 + 2x_5, 3x_3 - 3x_4 - 3x_5, x_3, x_4, x_5) \mid x_3, x_4, x_5\} \subseteq \mathbb{R}^5$$

$\dim S_0 = 3$

$$\| $$

$$(-2x_3, 3x_3, x_3, 0, 0) + (x_4, -3x_4, 0, x_4, 0) + (2x_5, -3x_5, 0, 0, x_5) =$$

$$= x_3(-2, 3, 1, 0, 0) + x_4(1, -3, 0, 1, 0) + x_5(2, -3, 0, 0, 1)$$

$\Rightarrow T = \{(-2, 3, 1, 0, 0), (1, -3, 0, 1, 0), (2, -3, 0, 0, 1)\}$ è un sist. di gen. di S_0

$|T| = 3 = \dim S_0$ $\Rightarrow$

$\Rightarrow$ T è una base di S_0

Oss:

$$\begin{pmatrix} -2 & 3 & 1 & 0 & 0 \\ 1 & -3 & 0 & 1 & 0 \\ 2 & -3 & 0 & 0 & 1 \end{pmatrix}$$

3 colonne lin. indip.

Esercizio

$$\Sigma_0 : \begin{cases} x_2 - x_3 + 2x_4 + x_5 = 0 \\ x_1 + 2x_2 + x_3 - x_4 + 2x_5 = 0 \\ x_1 + 3x_2 + x_4 + 3x_5 = 0 \\ 2x_1 + 3x_2 + 3x_3 + x_4 - x_5 = 0 \end{cases}$$

5 variabili; $K = \mathbb{R}$

Determinare S_0 e una sua base

$$A = \begin{pmatrix} 0 & 1 & -1 & 2 & 1 \\ 1 & 2 & 1 & -1 & 2 \\ 1 & 3 & 0 & 1 & 3 \\ 2 & 3 & 3 & 1 & -1 \end{pmatrix} \qquad \underline{e}^1 \longleftrightarrow \underline{e}^2 \qquad \begin{pmatrix} \boxed{1} & 2 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 & 1 \\ 1 & 3 & 0 & 1 & 3 \\ 2 & 3 & 3 & 1 & -1 \end{pmatrix}$$

$$\begin{aligned} \underline{e}^3 &\to \underline{e}^3 - \underline{e}^1 \\ \underline{e}^4 &\to \underline{e}^4 - 2\underline{e}^1 \end{aligned} \qquad \begin{pmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & \boxed{1} & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & -1 & 1 & 3 & -5 \end{pmatrix} \qquad \begin{aligned} \underline{e}^3 &\to \underline{e}^3 - \underline{e}^2 \\ \underline{e}^4 &\to \underline{e}^4 + \underline{e}^2 \end{aligned}$$

$$\begin{pmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & -4 \end{pmatrix} \qquad \underline{e}^4 \longrightarrow \underline{e}^3 \qquad \begin{pmatrix} \boxed{1} & 2 & 1 & -1 & 2 \\ 0 & \boxed{1} & -1 & 2 & 1 \\ 0 & 0 & 0 & \boxed{5} & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \underline{e}^3 \to \frac{1}{5}\underline{e}^3$$

$$\begin{pmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & \boxed{1} & -4/5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} \underline{e}^2 &\to \underline{e}^2 - 2\underline{e}^3 \\ \underline{e}^1 &\to \underline{e}^1 + \underline{e}^3 \end{aligned} \qquad \begin{pmatrix} 1 & 2 & 1 & 0 & 6/5 \\ 0 & \boxed{1} & -1 & 0 & 13/5 \\ 0 & 0 & 0 & 1 & -4/5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{e}^1 \longrightarrow \underline{e}^1 - 2\underline{e}^2 \qquad \begin{array}{c} \begin{array}{ccccc} & & x_3 & & x_4 \end{array} \\ \begin{pmatrix} \boxed{1} & 0 & 3 & 0 & -4 \\ 0 & \boxed{1} & -1 & 0 & 13/5 \\ 0 & 0 & 0 & \boxed{1} & -4/5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\frac{6}{5} - \frac{26}{5} = \frac{-20}{5} = -4$$

$\dim S_0 = 2$

$$\begin{cases} x_1 = -3x_3 + 4x_5 \\ x_2 = x_3 - \frac{13}{5} x_5 \\ x_4 = \frac{4}{5} x_5 \end{cases}$$

$$S_0 = \left\{ \left(-3x_3 + 4x_5,\ x_3 - \tfrac{13}{5}x_5,\ x_3,\ \tfrac{4}{5}x_5,\ x_5\right) \mid x_3, x_5 \in \mathbb{R} \right\}$$

$$\parallel$$

$$(-3x_3, x_3, x_3, 0, 0) + (4x_5, -\tfrac{13}{5}x_5, 0, \tfrac{4}{5}x_5, x_5) =$$

$$= x_3(-3, 1, 1, 0, 0) + x_5(4, -\tfrac{13}{5}, 0, \tfrac{4}{5}, 1)$$

$$\Rightarrow \quad S_0 = \mathcal{L}\left((-3,1,1,0,0), (4, -\tfrac{13}{5}, 0, \tfrac{4}{5}, 1)\right)$$

$$\text{base} = \left\{(-3, 1, 1, 0, 0), (4, -\tfrac{13}{5}, 0, \tfrac{4}{5}, 1)\right\} \qquad \begin{pmatrix} -3 & 1 & 1 & 0 & 0 \\ 4 & -13/5 & 0 & 4/5 & 1 \end{pmatrix}$$

$\left(4 \quad -\frac{13}{5} \quad 0 \quad \frac{4}{5} \quad 1 \right)$

2 colonne lin. indip.

Teorema. Sia $W \subseteq K^m$ sottosp. vett. numerico.
Esiste $\Sigma_0 : AX = 0$ tale che $S_0 = W$.

DIM Sia $B = (w_1, \dots, w_h)$ una base ordinata di W, $\dim W = h$.

$w_1 = (a_1^1, a_1^2, \dots, a_1^m) \in W \subseteq K^m$

$\vdots$

$w_h = (a_h^1, a_h^2, \dots, a_h^m) \in W \subseteq K^m$

$W = \mathcal{L}(B)$

$u \in (x_1, \dots, x_m) \in K^m$

$$u \in \underset{\mathcal{L}(B)}{\overset{\shortparallel}{W}} \iff \text{rango} \begin{pmatrix} a_1^1 & \cdots & a_h^1 & x_1 \\ a_1^2 & & a_h^2 & x_2 \\ \vdots & & \vdots & \vdots \\ a_1^m & \cdots & a_h^m & x_m \end{pmatrix} = h \iff$$

$B = \underbrace{\begin{matrix} a_1^1 & \cdots & a_h^1 \\ \vdots & & \vdots \\ a_1^m & \cdots & a_h^m \end{matrix}}_{\text{rango } h}$

$\iff$

PRIMO MODO (riduzione di Gauss):

una matrice $\overline{B}$ ridotta a gradini ottenuta a partire da B ha h pivot

$$\overline{B} = \begin{pmatrix} \underline{p_1^1} & \cdots & \cdots & b_1^1 x_1 + \cdots + b_m^1 x_m \\ 0 & \underline{p_2^2} & \cdots & b_1^2 x_1 + \cdots + b_m^2 x_m \\ 0 & 0 & & \vdots \\ 0 & 0 & \underline{p_h^h} & b_1^h x_1 + \cdots + b_m^h x_m \\ \vdots & \vdots & 0 & b_1^{h+1} x_1 + \cdots + b_m^{h+1} x_m \\ & & \vdots & \vdots \\ 0 & 0 & 0 & b_1^m x_1 + \cdots + b_m^m x_m \end{pmatrix}$$

rango $(\overline{B}) = h$ $\iff$
pivot di $\overline{B} = h$

$$\iff \Sigma_0 \begin{cases} b_1^{h+1} x_1 + \cdots + b_m^{h+1} x_m = 0 \\ \vdots \\ b_1^m x_1 + \cdots + b_m^m x_m = 0 \end{cases}$$

è il sist. lineare che cercavamo
$S_0 = W$

$\iff$

SECONDO MODO (teorema degli orlati)

$$\text{rango} \begin{pmatrix} a_1^1 & \cdots & a_h^1 & x_1 \\ a_1^2 & & a_h^2 & x_2 \\ \vdots & & \vdots & \vdots \\ a_1^m & \cdots & a_h^m & x_m \end{pmatrix} = h$$

$B = \underbrace{\begin{matrix} a_1^1 & \cdots & a_h^1 \\ \vdots & & \vdots \\ a_1^m & \cdots & a_h^m \end{matrix}}_{\text{rango } h}$

Per il Teorema degli orlati; esiste un minore M di B di ordine h con determinante non nullo.

A meno di uno scambio di righe, possiamo supporre che

$$M = \begin{pmatrix} a^1_1 & \cdots & a^1_h \\ a^2_1 & \cdots & a^2_h \\ \vdots & & \\ a^h_1 & \cdots & a^h_h \end{pmatrix} \qquad |M| \neq 0$$

prime h righe

Allora, $\operatorname{rango}(B) = h \iff$ tutti gli $n-h$ orlati di M hanno determinante nullo

$$M_1, \dots, M_{n-h}$$

$$\Sigma_0 : \begin{cases} |M_1| = 0 \\ \vdots \\ |M_h| = 0 \end{cases} \qquad S_0 = W. \qquad \square$$

Def. Il sistema Σ_0 ottenuto nella dimostrazione del precedente teorema si dice RAPPRESENTAZIONE (CARTESIANA) di W.

Esempio:

$W \subseteq \mathbb{R}^3 \qquad W = \mathcal{L}((2,1,-2)) \qquad \dim W = 1$

$u = (x_1, x_2, x_3) \in \mathbb{R}^3, \qquad u \in W \iff \operatorname{rango} \begin{pmatrix} \boxed{2} & x_1 \\ 1 & x_2 \\ -2 & x_3 \end{pmatrix} = 1$

PRIMO MODO: $\quad \underline{e}^2 \to \underline{e}^2 - \frac{1}{2}\underline{e}^1 \qquad \underline{e}^3 \to \underline{e}^3 + \underline{e}^1$

$$\begin{pmatrix} 2 & x_1 \\ 0 & x_2 - \frac{1}{2}x_1 \\ 0 & x_3 + x_1 \end{pmatrix}$$

$$W : \begin{cases} x_2 - \frac{1}{2}x_1 = 0 \\ x_3 + x_1 = 0 \end{cases}$$

SECONDO MODO: $\quad M = (2) \qquad M_1 = \begin{pmatrix} 2 & x_1 \\ 1 & x_2 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 2 & x_1 \\ -2 & x_3 \end{pmatrix}$

$|M_1| = 2x_2 - x_1 \qquad |M_2| = 2x_3 + 2x_1$

$$W : \begin{cases} 2x_2 - x_1 = 0 \\ 2x_3 + 2x_1 = 0 \end{cases}$$

Esercizio: $\quad U \subseteq \mathbb{R}^3 \qquad U = \mathcal{L}((1,0,2),(2,1,3))$

$$\begin{pmatrix} 1 & 2 & x_1 \\ 0 & 1 & x_2 \\ 2 & 3 & x_3 \end{pmatrix} \quad \underline{e}^3 \to \underline{e}^3 - 2\underline{e}^1 \quad \begin{pmatrix} 1 & 2 & x_1 \\ 0 & 1 & x_2 \\ 0 & -1 & x_3 - 2x_1 \end{pmatrix} \quad \underline{e}^3 \to \underline{e}^3 + \underline{e}^2$$

$$\begin{pmatrix} 1 & 2 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & x_3 - 2x_4 + x_2 \end{pmatrix} \qquad U: \left\{ x_3 - 2x_4 + x_2 = 0 \right.$$

Applicare anche il SECONDO MODO.

<u>Esercizio</u> $W \subseteq \mathbb{R}^4$ $\qquad W = \mathcal{L}((1,0,1,1),(2,1,2,3),(1,1,\cancel{1},2))$

$$\begin{pmatrix} \boxed{\begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix}} & 1 & x_1 \\ & 1 & x_2 \\ 1 \quad 2 & 1 & x_3 \\ 1 \quad 3 & 2 & x_4 \end{pmatrix} \qquad \begin{matrix} \underline{e}^3 \to \underline{e}^3 - \underline{e}^1 \\ \underline{e}^4 \to \underline{e}^4 - \underline{e}^1 \end{matrix} \qquad \begin{pmatrix} 1 & 2 & 1 & x_1 \\ 0 & \boxed{1} & 1 & x_2 \\ 0 & 0 & 0 & x_3 - x_1 \\ 0 & 1 & 1 & x_4 - x_1 \end{pmatrix}$$

$$\underline{e}^4 \to \underline{e}^4 - \underline{e}^2 \qquad \begin{pmatrix} \boxed{1} & 2 & 1 & x_1 \\ 0 & \boxed{1} & 1 & x_2 \\ 0 & 0 & 0 & x_3 - x_1 \\ 0 & 0 & 0 & x_4 - x_1 - x_2 \end{pmatrix} \qquad W: \begin{cases} x_3 - x_1 = 0 \\ x_4 - x_1 - x_2 = 0 \end{cases}$$

$\dim U = 2$

<u>Esercizio</u> $W \subseteq \mathbb{R}^5$ $\qquad W = \mathcal{L}((2,1,0,3,1),(2,-1,1,4,0))$ $\qquad \dim W = 2$

PRIMO MODO

$$\begin{pmatrix} 2 & 2 & x_1 \\ 1 & -1 & x_2 \\ 0 & 1 & x_3 \\ 3 & 4 & x_4 \\ 1 & 0 & x_5 \end{pmatrix} \qquad \begin{matrix} \underline{e}^2 \to \underline{e}^2 - \frac{1}{2}\underline{e}^1 \\ \underline{e}^4 \to \underline{e}^4 - \frac{3}{2}\underline{e}^1 \\ \underline{e}^5 \to \underline{e}^5 - \frac{1}{2}\underline{e}^1 \end{matrix} \qquad \begin{pmatrix} 2 & 2 & x_1 \\ 0 & \boxed{-2} & x_2 - \frac{1}{2}x_1 \\ 0 & 1 & x_3 \\ 0 & 1 & x_4 - \frac{3}{2}x_1 \\ 0 & -1 & x_5 - \frac{1}{2}x_1 \end{pmatrix}$$

$$\begin{matrix} \underline{e}^3 \to \underline{e}^3 + \frac{1}{2}\underline{e}^2 \\ \underline{e}^4 \to \underline{e}^4 + \frac{1}{2}\underline{e}^2 \\ \underline{e}^5 \to \underline{e}^5 - \frac{1}{2}\underline{e}^2 \end{matrix} \qquad \begin{pmatrix} \boxed{2} & 2 & x_1 \\ 0 & \boxed{-2} & x_2 - \frac{1}{2}x_1 \\ 0 & 0 & x_3 + \frac{1}{2}x_2 - \frac{1}{4}x_1 \\ 0 & 0 & x_4 - \frac{3}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{4}x_1 \\ 0 & 0 & x_5 - \frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{4}x_1 \end{pmatrix}$$

$$W: \begin{cases} -\frac{1}{4}x_1 + \frac{1}{2}x_2 + x_3 = 0 \\ -\frac{7}{4}x_1 + \frac{1}{2}x_2 + x_4 = 0 \\ -\frac{1}{4}x_1 - \frac{1}{2}x_2 + x_5 = 0 \end{cases}$$

SECONDO MODO

$$\text{rango} \begin{pmatrix} 2 & \boxed{\begin{matrix} 1 & 0 \\ -1 & 1 \end{matrix}} & 3 & 1 \\ 2 & & 4 & 0 \\ x_1 & x_2 \quad x_3 & x_4 & x_5 \end{pmatrix} = 2 \qquad M = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \Rightarrow |M| = 1 \neq 0$$

$$M_1 = \begin{pmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ x_1 & x_2 & x_3 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 1 & 4 \\ x_2 & x_3 & x_4 \end{pmatrix} \qquad M_3 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ x_2 & x_3 & x_5 \end{pmatrix}$$

$$W: \begin{cases} |M_1| = -2x_3 + x_1 - 2x_2 - 2x_3 = 0 \\ |M_2| = x_4 - 3x_3 - 3x_2 - 4x_3 = 0 \\ |M_3| = x_5 - x_3 - x_2 = 0 \end{cases}$$

$(V, +, \cdot)$ K, $\dim V = n$, $\mathcal{B} = (e_1, \dots, e_n)$ base ordinata di V

$W \subseteq V$ sottosp. vett., $\dim W = h$

$$\phi_{\mathcal{B}} : V \longrightarrow K^n$$
$$u \rightsquigarrow (x_1, \dots, x_n) \ : \ u = x_1 e_1 + \dots + x_n e_n$$

$\phi_{\mathcal{B}}(W) \subseteq K^n$, $\dim(\phi_{\mathcal{B}}(W)) = \dim W = h$

sottosp. vett.

Possiamo rappresentare $\phi_{\mathcal{B}}(W)$: $\Sigma_0 : AX = 0$

RAPPRESENTAZIONE CARTESIANA di $\phi_{\mathcal{B}}(W)$

si dice RAPPRESENTAZIONE CARTESIANA di W rispetto alla base ordinata $\mathcal{B}$ (riferimento)

Esempio: $V = \mathbb{R}[x]_{\leq 2}$ $\mathcal{B} = (1, x, x^2)$

$W = \mathcal{L}(1-x)$ Rappresentare W in $\mathcal{B}$:

$$\phi_{\mathcal{B}} : V \longrightarrow \mathbb{R}^3$$
$$a_0 + a_1 x + a_2 x^2 \longrightarrow (a_0, a_1, a_2)$$

$$\phi_{\mathcal{B}}(W) = \mathcal{L}(\phi_{\mathcal{B}}(1-x)) = \mathcal{L}((1,-1,0))$$

$$\begin{pmatrix} \boxed{1} & x_1 \\ -1 & x_2 \\ 0 & x_3 \end{pmatrix} \quad \underline{a}^2 \to \underline{a}^2 + \underline{a}^1 \quad \begin{pmatrix} 1 & x_1 \\ 0 & x_2 + x_1 \\ 0 & x_3 \end{pmatrix}$$

$$W: \begin{cases} x_2 + x_1 = 0 \\ x_3 = 0 \end{cases} \qquad S_0 = \phi_{\mathcal{B}}(W)$$

$$W = \phi_{\mathcal{B}}^{-1}(S_0)$$

Col teorema degl. orlati:

$M = (1)$ $|M| = 1$

$M_1 = \begin{pmatrix} 1 & x_1 \\ -1 & x_2 \end{pmatrix}$, $M_2 = \begin{pmatrix} 1 & x_1 \\ 0 & x_3 \end{pmatrix}$

$$\begin{cases} |M_2| = x_2 + x_4 = 0 \\ |M_3| = x_3 = 0 \end{cases}$$

Rappresentazioni parametriche:

$W \subseteq K^n$ sottosp. vett. numerico, $\dim W = h$

Come nella dimostrazione del teorema:

$\mathcal{B} = (w_1, \dots, w_h)$, base di W

$w_1 = (a^1_1, a^2_1, \dots, a^n_1) \in W \subseteq K^n$

$\vdots$

$w_h = (a^1_h, a^2_h, \dots, a^n_h) \in W \subseteq K^n$

$W = \mathcal{L}(\mathcal{B})$

$u = (x_1, \dots, x_n) \in K^n$

$u \in W \iff u \in \mathcal{L}(w_1, \dots, w_h) \iff \exists\, t_1, \dots, t_h \in K :$

$$(x_1, \dots, x_n) = t_1(a^1_1, a^2_1, \dots, a^n_1) + \dots + t_h(a^1_h, a^2_h, \dots, a^n_h) =$$
$$= (a^1_1 t_1 + \dots + a^1_h t_h,\ a^2_1 t_1 + \dots + a^2_h t_h,\ \dots,\ a^n_1 t_1 + \dots + a^n_h t_h)$$

$$\begin{cases} x_1 = a^1_1 t_1 + \dots + a^1_h t_h \\ x_2 = a^2_1 t_1 + \dots + a^2_h t_h \\ \vdots \\ x_n = a^n_1 t_1 + \dots + a^n_h t_h \end{cases}$$

RAPPRESENTAZIONE PARAMETRICA di W.

Esempio: $U = \mathcal{L}((1,0,2), (2,1,3)) \subseteq \mathbb{R}^3$

$$\begin{cases} x_1 = t_1 + 2t_2 \\ x_2 = t_2 \\ x_3 = 2t_1 + 3t_2 \end{cases}$$

Esempio: $V = \mathbb{R}[x]_{\leq 2}$ $\quad \mathcal{B} = (1, x, x^2)$

$W = \mathcal{L}(1 - x)$ $\quad \phi_{\mathcal{B}}(1 - x) = (1, -1, 0)$

$$\begin{cases} x_1 = t \\ x_2 = -t \\ x_3 = 0 \end{cases}$$

RAPPRESENTAZIONE PARAMETRICA di W rispetto a $\mathcal{B}$ (in $\mathcal{B}$)

OSS

$\Downarrow$ ← "eliminiamo il parametro"

$$\begin{cases} t = x_1 \\ x_2 = -x_1 \\ x_3 = 0 \end{cases}$$

← RAPPR. CARTESIANA (non parametrica)

VICEVERSA:

$$W: \begin{cases} -\frac{1}{4} x_1 + \frac{1}{2} x_2 + x_3 = 0 \\ -\frac{3}{4} x_1 + \frac{1}{2} x_2 + x_4 = 0 \\ -\frac{1}{4} x_1 - \frac{1}{2} x_2 + x_5 = 0 \end{cases}$$

$$\left| -\tfrac{1}{4}x_1 - \tfrac{1}{2}x_2 + x_5 = 0\right.$$

$$\begin{cases} -x_1 + 2x_2 + 4x_3 = 0 \\ -7x_1 + 2x_2 + 4x_4 = 0 \\ -x_1 - 2x_2 + 4x_5 = 0 \end{cases} \qquad \begin{pmatrix} -1 & 2 & 4 & 0 & 0 \\ -7 & 2 & 0 & 4 & 0 \\ -1 & -2 & 0 & 0 & 4 \end{pmatrix}$$

$$\begin{cases} x_3 = \frac{1}{4}x_1 - \frac{1}{2}x_2 \\ x_4 = \frac{7}{4}x_1 - \frac{1}{2}x_2 \\ x_5 = \frac{1}{4}x_1 + \frac{1}{2}x_2 \end{cases}$$

$$S_0 = \{(x_1, x_2, \tfrac{1}{4}x_1 - \tfrac{1}{2}x_2, \tfrac{7}{4}x_1 - \tfrac{1}{2}x_2, \tfrac{1}{4}x_1 + \tfrac{1}{2}x_2) \mid x_1, x_2 \in \mathbb{R}\}$$

$$\begin{cases} x_1 = t_1 \\ x_2 = t_2 \\ x_3 = \frac{1}{4}t_1 - \frac{1}{2}t_2 \\ x_4 = \frac{7}{4}t_1 - \frac{1}{2}t_2 \\ x_5 = \frac{1}{4}t_1 + \frac{1}{2}t_2 \end{cases} \qquad t_1, t_2 \in \mathbb{R}$$

Esercizio:

$W = \mathcal{L}((1,2,0,1),(0,0,1,2)) \subseteq \mathbb{R}^4$

$U = \mathcal{L}((1,0,1,1),(1,2,1,3))$

Determinare $W \cap U$.

$$W: AX = 0, \qquad U: A'X = 0 \qquad U \cap W: \begin{cases} AX = 0 \\ A'X = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 0 & x_1 \\ 2 & 0 & x_2 \\ 0 & 1 & x_3 \\ 1 & 2 & x_4 \end{pmatrix} \quad \begin{matrix} \underline{R}^2 \to \underline{R}^2 - 2\underline{R}^1 \\ \underline{R}^4 \to \underline{R}^4 - \underline{R}^1 \end{matrix} \quad \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 0 & x_2 - 2x_1 \\ 0 & 1 & x_3 \\ 0 & 2 & x_4 - x_1 \end{pmatrix} \quad \underline{R}^2 \leftrightarrow \underline{R}^3$$

$$\begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & x_2 - 2x_1 \\ 0 & 2 & x_4 - x_1 \end{pmatrix} \quad \underline{R}^4 \to \underline{R}^4 - 2\underline{R}^2 \quad \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & x_2 - 2x_1 \\ 0 & 0 & x_4 - x_1 - 2x_3 \end{pmatrix}$$

$$W: \begin{cases} x_2 - 2x_1 = 0 \\ \underset{3}{x_4} - \underset{1}{x_1} - 2\underset{1}{x_3} = 0 \end{cases}$$

$$\begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ 1 & 2 \end{matrix}} & \begin{matrix} 1 & 1 \\ 1 & 3 \end{matrix} \\ \begin{matrix} x_1 & x_2 \end{matrix} & \begin{matrix} x_3 & x_4 \end{matrix} \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \qquad |M| = 2 \neq 0$$

$$M_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ x_1 & x_2 & x_3 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ x_1 & x_2 & x_4 \end{pmatrix}$$

$$U \begin{cases} |M_1| = 2x_3 + \cancel{x_2} - 2x_1 - \cancel{x_2} = 0 \\ |M_2| = 2x_4 + x_2 - 2x_1 - 3x_2 = -2x_1 - 2x_2 + 2x_4 = 0 \end{cases}$$

$$W \cap U: \begin{cases} x_2 - 2x_1 = 0 \\ x_4 - x_1 - 2x_3 = 0 \\ 2x_3 - 2x_4 = 0 \\ -2x_1 - 2x_2 + 2x_4 = 0 \end{cases} \quad W$$

$$W: \begin{cases} x_2 - 2x_1 = 0 \\ x_4 - x_1 - 2x_3 = 0 \end{cases}$$

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$$U: \begin{cases} x_1 = t_1 + t_2 \\ x_2 = 2t_2 \\ x_3 = t_1 + t_2 \\ x_4 = t_1 + 3t_2 \end{cases}$$

$$u \in U \iff \exists\, t_1, t_2 \in \mathbb{R}: \ u = (t_1 + t_2,\ 2t_2,\ t_1 + t_2,\ \underline{t_1 + 3t_2})$$

$$u \in U \cap W \iff \begin{cases} 2t_2 - 2(t_1 + t_2) = 0 \\ t_1 + 3t_2 - t_1 - t_2 - 2(t_1 + t_2) = 0 \end{cases}$$

$$\begin{cases} \cancel{2t_2} - 2t_1 - \cancel{2t_2} = 0 \implies t_1 = 0 \\ \underline{\cancel{3t_2} - \cancel{t_2} - \cancel{2t_2} = 0} \implies 0 = 0 \end{cases}$$

$$u \in U \cap W \iff \exists\, t_2 \in \mathbb{R}: \ u = (t_2, 2t, t_2, 3t_2) \in \mathcal{L}((1, 2, 1, 3)) -$$

$$= t_2(1, 2, 1, 3)$$