Lezione 18

venerdì 14 maggio 2021 10:54

$(\vec{\mathcal{E}}, \mathcal{E}, \pi)$ $\qquad \pi: \mathcal{E} \times \mathcal{E} \longrightarrow \vec{\mathcal{E}}$, $(P,Q) \rightsquigarrow \vec{PQ}$

Ricordiamo: $\mathcal{H} \subseteq \mathcal{E}$ è sottospazio euclideo (o affine) se

(i) $\vec{\mathcal{H}} = \pi(\mathcal{H} \times \mathcal{H}) = \{ \vec{PQ} \mid P, Q \in \mathcal{H} \}$ è sottosp. vett. di $\vec{\mathcal{E}}$

(ii) $\forall A \in \mathcal{H}, \forall a \in \vec{\mathcal{H}}$, l'unico punto $X \in \mathcal{E}$ tale che $\vec{AX} = a$ appartiene ad $\mathcal{H}$.

$\forall P_0 \in \mathcal{E}$, $\forall U \subseteq \vec{\mathcal{E}}$ sottosp. vett. possiamo considerare:

$(P_0, U) \overset{def.}{=} \{ Q \in \mathcal{E} \mid \vec{P_0Q} \in U \}$ $\qquad$ Varietà lineare

$\overset{\shortparallel}{P_0 + U}$

- Ogni varietà lineare è un sottosp. euclideo (o affine)

<u>DIM</u>: Th: (P_0, U) soddisfa le condizioni (i) e (ii) $\qquad$ (FACOLTATIVA)

- P_0: $\vec{P_0P_0} = \underline{0} \in U \Rightarrow P_0 \in (P_0, U)$
- Vediamo che (P_0, U) soddisfa la (ii)

$A \in (P_0, U)$, $a \in U$ $\quad$ Sappiamo che $\exists!\, X \in \mathcal{E}$: $\vec{P_0X} = a \in U \xRightarrow{\text{per def. di } (P_0,U)} X \in (P_0, U)$

- Resta da dimostrare che $U = \pi((P_0,U) \times (P_0,U))$

"$\supseteq$" $P, Q \in (P_0, U) \Rightarrow \vec{P_0P}, \vec{P_0Q} \in U \underset{U \text{ è sottosp. vett.}}{\Rightarrow} -\vec{P_0P}, \vec{P_0Q} \in U \Rightarrow \vec{PP_0}, \vec{P_0Q} \in U$

Allora: $\underset{\pi((P,Q))}{\overset{}{\vec{PQ}}} = \vec{PP_0} + \vec{P_0Q} \in U$

"$\subseteq$" $u \in U$, $P_0 \in (P_0, U)$ e vale (ii) $\Rightarrow \exists!\, X \in (P_0, U)$: $\underset{\pi((P,X))}{\vec{P_0X}} = u \in U$

- Ogni sottosp. euclideo (o affine) è una varietà lineare:

<u>DIM</u>: $(\vec{\mathcal{H}}, \mathcal{H}, \pi_1)$ $\quad$ Th: $\forall P_0 \in \mathcal{H}$, $\mathcal{H} = (P_0, \vec{\mathcal{H}})$ $\qquad$ (FACOLTATIVA)

"$\subseteq$" $\left.\begin{array}{l} P \in \mathcal{H} \\ P_0 \in \mathcal{H} \end{array}\right\} \Rightarrow \vec{P_0P} \in \vec{\mathcal{H}} \Rightarrow P \in (P_0, \vec{\mathcal{H}})$ $\qquad \vec{\mathcal{H}} = \pi(\mathcal{H} \times \mathcal{H}) = \{ \vec{PQ} \mid P, Q \in \mathcal{H} \}$

"$\supseteq$" $P \in (P_0, \vec{\mathcal{H}}) \Rightarrow \underset{a}{\vec{P_0P}} \in \vec{\mathcal{H}}$, $P_0 \in \mathcal{H}$ $\xRightarrow{(ii)} \exists!\, X \in \mathcal{H}$: $\vec{P_0X} = a$

$\Rightarrow P = X \in \mathcal{H}$.

Ricordiamo che abbiamo come rappresentare i sottosp. euclidei (o affini) di spaz. euclidei di dimensione finita, fissato un rif. cartesiano.

$(\vec{\mathcal{E}}, \mathcal{E}, \pi)$ $\quad \dim \mathcal{E} = n$ $\qquad \mathcal{R} = (O, \mathcal{B})$

$\mathcal{H}$ sottosp. euclideo, $\dim \mathcal{H} = h$

$\overset{\shortparallel}{(P_0, \vec{\mathcal{H}})}$ $\qquad Q \equiv_{\mathcal{R}} (x_1, \dots, x_n) \in \mathcal{E}$, $\quad P_0 \equiv_{\mathcal{R}} (a_1, \dots, a_n)$

$Q\in\mathcal{H} \Leftrightarrow \overrightarrow{P_0Q}\in\overrightarrow{\mathcal{H}} \Leftrightarrow \underset{\substack{\parallel \\ (x_1-a_1,\dots,x_n-a_1)}}{\phi_B(\overrightarrow{P_0Q})} \in \phi_B(\overrightarrow{\mathcal{H}}) : \underline{AX=0}$ $\quad$ rango$(A)=n-h$.

$$\Leftrightarrow \quad A\begin{pmatrix} x_1-a_1 \\ \vdots \\ x_n-a_n\end{pmatrix}=0 \quad \Leftrightarrow \quad A\begin{pmatrix} x_1 \\ \vdots \\ x_n\end{pmatrix} = \underbrace{A\begin{pmatrix} a_1 \\ \vdots \\ a_n\end{pmatrix}}_{\substack{\parallel \\ \underline{b}}}$$

$\Sigma : AX=\underline{b}$ ha come soluzioni le n-uple delle coordinate di tutti e soli i punti di $\mathcal{H}$.

$\Sigma_0 : AX=0$ rappresenta $\overrightarrow{\mathcal{H}}$ in B

Sia $B_{\overrightarrow{\mathcal{H}}}=\{w_1,\dots,w_h\}$ una base di $\overrightarrow{\mathcal{H}}$: $\overrightarrow{\mathcal{H}}=\mathcal{L}(w_1,\dots,w_h)$

$\phi_B(w_1)=(a^1_1,\dots,a^n_1)$

$\vdots$

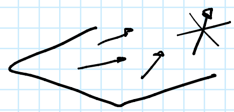
$\phi_B(w_h)=(a^1_h,\dots,a^n_h)$

$Q\in\mathcal{H} \Leftrightarrow \overrightarrow{P_0Q}\in\overrightarrow{\mathcal{H}} \Leftrightarrow \phi_B(\overrightarrow{P_0Q}) \in \phi_B(\overrightarrow{\mathcal{H}})=\mathcal{L}(\phi_B(w_1),\dots,\phi_B(w_h))$

$\Leftrightarrow \exists\, t_1,\dots,t_h\in K : (x_1-a_1,\dots,x_n-a_n)=t_1(a^1_1,\dots,a^n_1)+\dots+t_h(a^1_h,\dots,a^n_h)$

$$\begin{cases} x_1 = a_1+t_1a^1_1+\dots+t_ha^1_h \\ \vdots \\ x_n=a_n+t_1a^n_1+\dots+t_ha^n_h\end{cases}$$

$t_1,\dots,t_h\in K$
rappresentazione parametrica di $\mathcal{H}$ in R



$\mathcal{H}, \mathcal{H}'$ sottosp. euclidei (o affini)

$\mathcal{H}$ e $\mathcal{H}'$ sono paralleli $\Leftrightarrow$ $\overrightarrow{\mathcal{H}}\subseteq\overrightarrow{\mathcal{H}'}$ opp. $\overrightarrow{\mathcal{H}'}\subseteq\overrightarrow{\mathcal{H}}$

Esercizio: $(\vec{\mathcal{E}},\mathcal{E},\pi)$ $\quad$ dim $\mathcal{E}=3$ $\quad$ $R=(O,B)$ $\quad$ $Q\equiv_R(x_1,x_2,x_3)$

Rappresentare in R il piano $\mathcal{H}$ passante per $P_0(3,4,2)$ e di giacitura

$\overrightarrow{\mathcal{H}}=\mathcal{L}(w_1(1,2,0),\ w_2(2,1,1))$

$$\mathcal{H}: \begin{cases} x_1=3+t_1\,1+t_2\,2 \\ x_2=4+t_1\,2+t_2\,1 \\ x_3=2+t_1\,0+t_2\,1\end{cases} \qquad \begin{cases} x_1=3+t_1+2t_2 \\ x_2=4+2t_1+t_2 \\ x_3=2+t_2\end{cases}$$

$$Q\in\mathcal{H} \Leftrightarrow \text{rango}\begin{pmatrix} 1 & 2 & x_1-3 \\ 2 & 1 & x_2-4 \\ 0 & 1 & x_3-2\end{pmatrix}=2$$

GAUSS: $\underline{a}_2\to\underline{a}_2-2\underline{a}_1$
$$\begin{pmatrix} 1 & 2 & x_1-3 \\ 0 & -3 & x_2-4-2x_1+6 \\ 0 & 1 & x_3-2\end{pmatrix}$$
$\underline{a}^3\to\underline{a}^3+\frac{1}{3}\underline{a}^2$

$$\begin{pmatrix} 1 & 2 & x_1-3 \\ 0 & -3 & -2x_1+x_2+2 \\ \dots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & x_1-3 \\ 0 & -3 & -2x_1+x_2+2 \\ 0 & 0 & x_3-2-\frac{2}{3}x_1+\frac{1}{3}x_2+\frac{2}{3} \end{pmatrix}$$

$\mathcal{H}: -\frac{2}{3}x_1+\frac{1}{3}x_2+x_3-\frac{4}{3}=0$ $\qquad$ $\overrightarrow{\mathcal{H}}: -\frac{2}{3}x_1+\frac{1}{3}x_2+x_3=0$

Teorema degli orlati: $\quad M=\begin{pmatrix} 2 & 1 \\ 0 & 1\end{pmatrix} \quad |M|=2\neq 0$

$$Q\in\mathcal{H} \iff \begin{vmatrix} 1 & 2 & x_1-3 \\ 2 & 1 & x_2-4 \\ 0 & 1 & x_3-2 \end{vmatrix}=0 \iff x_3-2+0+2(x_1-3)-x_2+4-4x_3+8=0$$
$$\iff 2x_1-x_2-3x_3+4=0$$

Determinare il piano $\mathcal{H}'$ parallelo a $\mathcal{H}$ e passante per $A(-2,1,5)$

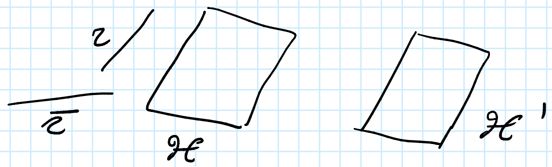
$\overrightarrow{\mathcal{H}'}: 2x_1-x_2-3x_3=0$ $\qquad$ $\mathcal{H}': 2x_1-x_2-3x_3+d=0$

$A\in\mathcal{H}' \Rightarrow 2(-2)-1-3\cdot 5+d=0 \Rightarrow d=20$

$\mathcal{H}': 2x_1-x_2-3x_3+20=0$

Determinare una retta r parallela a $\mathcal{H}$ e passante per $B(3,1,0)$

$$r: \begin{cases} x_1 = 3+t_1 \\ x_2 = 1+2t_1 \\ x_3 = 0+0 \end{cases} \qquad r: \begin{cases} t_1 = x_1-3 \\ x_2 = 1+2x_1-6 \\ x_3=0 \end{cases} \qquad \begin{cases} x_2-2x_1+5=0 \\ x_3=0 \end{cases}$$



$r, \bar{r} \parallel \mathcal{H} \parallel \mathcal{H}'$

$$\bar{r}: \begin{cases} x_1 = 3+2t_1 \\ x_2 = 1+1t_1 \\ x_3 = 0+t_1 \end{cases} \qquad r \nparallel \bar{r}$$

$(\vec{\mathcal{E}},\mathcal{E},\pi)$ $\qquad$ $\mathcal{H}, \mathcal{H}'$ sottosp. euclidei (o affini)

$\mathcal{H}$ e $\mathcal{H}'$ sono incidenti se $\mathcal{H}\cap\mathcal{H}'\neq\emptyset$

$\mathcal{H}$ e $\mathcal{H}'$ sono sghembi se $\mathcal{H}\cap\mathcal{H}'=\emptyset$ e $\mathcal{H}\nparallel\mathcal{H}'$.

Esercizio: $\dim\mathcal{E}=n$ $\qquad$ $\mathcal{R}=(O,\mathcal{B})$

Siano $P(c_1,\dots,c_n)$ e $Q(d_1,\dots,d_n)$ due punti distinti.

Determinare la retta passante per P e Q. $\qquad \overrightarrow{PQ}\neq \underline{0}$

$P,Q\in r \qquad \vec{r}\ni\overrightarrow{PQ}$

$\vec{r}=\mathcal{L}(\overrightarrow{PQ}(d_1-c_1,\dots,d_n-c_n))$

$$r: \begin{cases} x_1 = d_1+(d_1-c_1)t \\ x_2 = d_2+(d_2-c_2)t \\ \vdots \\ x_n = d_n+(d_n-c_n)t \end{cases}$$

$\dim\mathcal{E}=2 \qquad P(-3,4) \qquad Q(0,1) \qquad r: \begin{cases} x_1 = 0+3t \\ x_2 = 1-3t \end{cases} \qquad \vec{r}=\mathcal{L}(\overrightarrow{PQ}(3,-3))$

$\dim \mathcal{E} = 2 \qquad P(-3,4) \qquad Q(0,1)$

$$r: \begin{cases} x_1 = 0 + 3t \\ x_2 = 1 - 3t \end{cases} \qquad \vec{r} = \mathcal{L}(\overrightarrow{PQ}(3,-3))$$

$$\bar{r}: \begin{cases} x_1 = 0 + t \\ x_2 = 1 - t \end{cases} \qquad \vec{\bar{r}} = \mathcal{L}(u(1,-1))$$

$$\Downarrow$$

$$r = \bar{r}$$

$$\bar{r}: \begin{cases} t = x_1 \\ x_2 = 1 - x_1 \end{cases}$$

$$\bar{r}: x_1 + x_2 - 1 = 0$$

$(\vec{\mathcal{E}}, \mathcal{E}, \pi) \qquad \dim \mathcal{E} = 2 \qquad \mathcal{R} = (O, \mathcal{B})$

condizioni di incidenza tra rette in un piano

$r: ax + by + c = 0 \qquad \vec{r}: ax + by = 0$

$r': a'x + b'y + c' = 0 \qquad \vec{r}': a'x + b'y = 0$

$r = r'$ opp. $r \neq r'$

$$r \cap r': \begin{cases} ax + by + c = 0 \\ a'x + b'y + c' = 0 \end{cases} \qquad C = \left(\begin{array}{cc|c} a & b & c \\ a' & b' & c' \end{array}\right)$$

(A = the first two columns)

$1 \leq \text{rango}(A) \leq \text{rango}(C) \leq 2$

1	1	$\Rightarrow r = r'$
1	2	$\Rightarrow r \cap r' = \emptyset$ e $\vec{r} = \vec{r}'$
2	2	$\Rightarrow \lvert r \cap r' \rvert = 1$

$$\vec{r} \cap \vec{r}': \begin{cases} ax + by = 0 \\ a'x + b'y = 0 \end{cases}$$

$(\vec{\mathcal{E}}, \mathcal{E}, \pi) \qquad \dim \mathcal{E} = 3 \qquad \mathcal{R} = (O, \mathcal{B})$

condizioni di incidenza tra due rette

$$r: \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \end{cases}$$

$$r': \begin{cases} \alpha x + \beta y + \gamma z + \delta = 0 \\ \alpha' x + \beta' y + \gamma' z + \delta' = 0 \end{cases}$$

$$r \cap r': \begin{cases} ax + by \cdots + d = 0 \\ a'x + \cdots + d' = 0 \\ \alpha x + \cdots + \delta = 0 \\ \alpha' x + \cdots + \delta' = 0 \end{cases}$$

$$C = \left(\begin{array}{ccc|c} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \end{array}\right) \qquad \dim \vec{r} \cap \vec{r}' = \underline{0}$$

(A = the first three columns)

$2 \leq \text{rango}(A) \leq \text{rango}(C) \leq 4$

2	2	$\Rightarrow r = r'$
2	3	$\Rightarrow r \cap r' = \emptyset$ e $\vec{r} = \vec{r}'$
3	3	$\Rightarrow \lvert r \cap r' \rvert = 1$
3	4	$\Rightarrow r \cap r' = \emptyset$ e $\vec{r} \cap \vec{r}' = \underline{0}$

r e r' sono sghembe

Condizioni di incidenza tra una retta e un piano:

$$r: \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \end{cases} \qquad \mathcal{H}: \alpha x + \beta y + \gamma z + \delta = 0$$

$$r \cap \mathcal{H}: \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \\ \alpha x + \beta y + \gamma z + \delta = 0 \end{cases} \qquad C = \left(\begin{array}{ccc|c} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \end{array}\right)$$

$$r \cap \mathcal{H}: \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \\ \alpha x + \beta y + \gamma z + \delta = 0 \end{cases} \qquad C = \underbrace{\left(\begin{array}{ccc|c} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \end{array}\right)}_{A}$$

$$2 \le \text{rango}(A) \le \text{rango}(C) \le 3$$

2	2	$\Rightarrow r \subseteq \mathcal{H}$
2	3	$\Rightarrow \vec{r} \cap \vec{\mathcal{H}} = \vec{r}$ e $r \cap \mathcal{H} = \emptyset$ quindi $r \parallel \mathcal{H}$ e $r \cap \mathcal{H} = \emptyset$
3	3	$\Rightarrow \lvert r \cap \mathcal{H} \rvert = 1$ ← Teor. di Cramer

$$\vec{r} \cap \vec{\mathcal{H}}: \begin{cases} ax + by + cz = 0 \\ a'x + b'y + c'z = 0 \\ \cancel{\alpha x + \beta y + \gamma z = 0} \end{cases}$$

Condizioni di incidenza tra due piani: ESERCIZIO.

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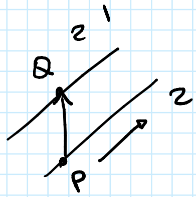
dim $\mathcal{E} = 3$ $\qquad \mathcal{R} = (O, \mathcal{B})$

due rette parallele oppure incidenti sono complanari; ovvero esiste un piano che contiene entrambe

$$r: \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \end{cases}$$

$$r': \begin{cases} \alpha x + \beta y + \gamma z + \delta = 0 \\ \alpha' x + \beta' y + \gamma' z + \delta' = 0 \end{cases}$$



- $r \parallel r'$

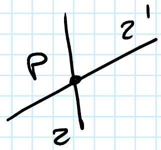
$\vec{r} = \vec{r}' = \mathcal{L}(u)$

$$\mathcal{H} = (P, \mathcal{L}(u, \vec{PQ})) \supseteq (P, \mathcal{L}(u)) = r$$
$$\supseteq (Q, \mathcal{L}(u)) = r'$$

- $r \cap r' = P$

$\vec{r} = \mathcal{L}(u)$

$\vec{r}' = \mathcal{L}(v)$ $\neq$



$$\mathcal{H} = (P, \mathcal{L}(u, v))$$

$$r: \begin{cases} x + y = 1 \\ y - z = 0 \end{cases} \qquad r': \begin{cases} x + 2y = 2 \\ 3y - z = 1 \end{cases} \qquad C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 2 \\ 0 & 3 & -1 & 1 \end{pmatrix}$$

$$|C| = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 3 & -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 3 & -1 & 1 \end{vmatrix} = -6 + 2 + 2 + (-1) - 1 + 3 = 2 \neq 0$$

$$r'': \begin{cases} x = 2 + t \\ y = -1 + t \\ z = -1 - t \end{cases} \qquad \vec{r}'' = \mathcal{L}(w(1,1,-1)) \qquad r \nparallel r'' \qquad r \cap r'' = P(2,-1,-1)$$

$$\vec{r}:\begin{cases} x+y=0 \\ y-z=0 \end{cases} \qquad S_0=\{(-y,y,y)\mid y\in\mathbb{R}\}$$

$$\vec{r}=\mathcal{L}(u(-1,1,1))$$

$$\mathcal{H}:\begin{cases} x=2+t_1-t_2 \\ y=-1+t_1+t_2 \\ z=-1-t_1+t_2 \end{cases} \qquad \left|\begin{pmatrix} 1 & -1 & x-2 \\ 1 & 1 & y+1 \\ -1 & 1 & z+1 \end{pmatrix}\right|=0.$$