

Ricordiamo che abbiamo dimostrato:

$\Sigma_0 : Ax = 0$  sistema lineare omogeneo in  $K^m$  in  $n$  variabili.

$S_0 \subseteq K^m$  insieme delle soluzioni.

$$S_0 = \left\{ (y_1, \dots, y_m) \in K^m \mid A \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = 0 \right\}$$

- $S_0$  è un sottospazio vett. di  $K^m$  e  $\dim S_0 = m - \text{rang}(A)$

Esercizio:

$$\Sigma_0 : \begin{cases} x_1 + x_2 - x_3 + 2x_4 + x_5 = 0 \\ 2x_1 + x_2 + x_3 + x_4 - x_5 = 0 \\ x_1 + 2x_3 - x_4 - 2x_5 = 0 \end{cases}$$

Determina  $S_0$  e una sua base

$$A = \begin{pmatrix} \boxed{1} & 1 & -1 & 2 & 1 \\ \underline{\boxed{2}} & 1 & 1 & 1 & -1 \\ \underline{\boxed{1}} & 0 & 2 & -1 & -2 \end{pmatrix} \quad \begin{array}{l} \underline{x}_1 \rightarrow \underline{x}_1 - 2\underline{x}_2 \\ \underline{x}_3 \rightarrow \underline{x}_3 - \underline{x}_2 \end{array} \quad \begin{pmatrix} 1 & 1 & -1 & 2 & 1 \\ 0 & \boxed{-1} & 3 & -3 & -3 \\ 0 & \underline{-1} & 3 & -3 & -3 \end{pmatrix}$$

$$\begin{array}{l} \underline{x}_3 \rightarrow \underline{x}_3 - \underline{x}_2 \\ \underline{x}_1 \rightarrow -\underline{x}_2 \end{array} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & -1 & -2 \\ 0 & 1 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \left\{ \begin{array}{l} x_1 = -2x_3 + x_4 + 2x_5 \\ x_2 = 3x_3 - 3x_4 - 3x_5 \end{array} \right.$$

$$S_0 = \{(-2x_3 + x_4 + 2x_5, 3x_3 - 3x_4 - 3x_5, x_3, x_4, x_5) \mid x_3, x_4, x_5\} \subseteq \mathbb{R}^5$$

||

$$\dim S_0 = 3$$

$$(-2x_3, 3x_3, x_3, 0, 0) + (x_4, -3x_4, 0, x_4, 0) + (2x_5, -3x_5, 0, 0, x_5) =$$

$$= x_3 (-2, 3, 1, 0, 0) + x_4 (1, -3, 0, 1, 0) + x_5 (2, -3, 0, 0, 1)$$

$$\Rightarrow T = \{(-2, 3, 1, 0, 0), (1, -3, 0, 1, 0), (2, -3, 0, 0, 1)\} \text{ è un ns. d. gen. di } S_0$$

$$|T| = 3 = \dim S_0$$

$$\Rightarrow T \text{ è una base di } S_0$$

$$\text{Oss: } \begin{pmatrix} -2 & 3 & 1 & 0 & 0 \\ 1 & -3 & 0 & 1 & 0 \\ 2 & -3 & 0 & 0 & 1 \end{pmatrix}$$

3 colonne lin. indip.

Esercizio

$$\Sigma_0 : \begin{cases} x_2 - x_3 + 2x_4 + x_5 = 0 \\ x_4 + 2x_5 + x_3 - x_4 + 2x_5 = 0 \\ x_1 + 3x_2 + x_4 + 3x_5 = 0 \\ 2x_1 + 3x_2 + 3x_3 + x_4 - x_5 = 0 \end{cases}$$

5 variables;  $k = \mathbb{R}$

Determinar  $\mathcal{S}_0$  e una su base

$$A = \begin{pmatrix} 0 & 1 & -1 & 2 & 1 \\ 1 & 2 & 1 & -1 & 2 \\ 1 & 3 & 0 & 1 & 3 \\ 2 & 3 & 3 & 1 & -1 \end{pmatrix}$$

$$\underline{Q}^1 \leftrightarrow \underline{Q}^2$$

$$\begin{pmatrix} \boxed{1} & 2 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 & 1 \\ \textcircled{1} & 3 & 0 & 1 & 3 \\ \textcircled{2} & 3 & 3 & 1 & -1 \end{pmatrix}$$

$$\begin{array}{l} \underline{Q}^3 \rightarrow \underline{Q}^3 - \underline{Q}^1 \\ \underline{Q}^4 \rightarrow \underline{Q}^4 - 2\underline{Q}^1 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & \boxed{1} & -1 & 2 & 1 \\ 0 & \textcircled{1} & -1 & 2 & 1 \\ 0 & -1 & 1 & 3 & -5 \end{pmatrix}$$

$$\begin{array}{l} \underline{Q}^3 \rightarrow \underline{Q}^3 - \underline{Q}^2 \\ \underline{Q}^4 \rightarrow \underline{Q}^4 + \underline{Q}^2 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & -4 \end{pmatrix} \xrightarrow{\text{R3}}$$

$$\underline{Q}^4 \rightarrow \underline{Q}^3$$

$$\begin{pmatrix} \textcircled{1} & 2 & 1 & -1 & 2 \\ 0 & \boxed{1} & -1 & 2 & 1 \\ 0 & 0 & 0 & \boxed{5} & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{Q}^3 \rightarrow \frac{1}{5} \underline{Q}^3$$

$$\begin{pmatrix} 1 & 2 & 1 & \textcircled{-1} & 2 \\ 0 & 1 & -1 & \textcircled{2} & 1 \\ 0 & 0 & 0 & \boxed{1} & -\frac{4}{5} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{Q}^2 \rightarrow \underline{Q}^1 - 2\underline{Q}^3$$

$$\underline{Q}^2 \rightarrow \underline{Q}^2 + \underline{Q}^3$$

$$\begin{pmatrix} \textcircled{1} & \textcircled{2} & 1 & 0 & \frac{6}{5} \\ 0 & \boxed{1} & -1 & 0 & \frac{13}{5} \\ 0 & 0 & 0 & 1 & -\frac{6}{5} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{Q}^1 \rightarrow \underline{Q}^1 - 2\underline{Q}^2$$

$$\begin{pmatrix} \boxed{1} & 0 & 3 & 0 & -4 \\ 0 & \boxed{1} & -1 & 0 & \frac{13}{5} \\ 0 & 0 & \boxed{1} & -\frac{6}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{6}{5} - 2 \frac{13}{5} = -\frac{20}{5} = -4$$

$$\dim \mathcal{S}_0 = 2$$

$$\begin{cases} x_4 = -3x_3 + 4x_5 \\ x_2 = x_3 - \frac{13}{5}x_5 \\ x_4 = \frac{6}{5}x_5 \end{cases}$$

$$\mathcal{S}_0 = \{(-3x_3 + 4x_5, x_3 - \frac{13}{5}x_5, x_3, \frac{6}{5}x_5, x_5) \mid x_3, x_5 \in \mathbb{R}\}$$

11

$$(-3x_3, x_3, x_3, 0, 0) + (4x_5, -\frac{13}{5}x_5, 0, \frac{6}{5}x_5, x_5) =$$

$$= x_3(-3, 1, 1, 0, 0) + x_5(4, -\frac{13}{5}, 0, \frac{6}{5}, 1)$$

$$\Rightarrow \mathcal{S}_0 = \{(-3, 1, 1, 0, 0), (4, -\frac{13}{5}, 0, \frac{6}{5}, 1)\}$$

$$\text{base} = \{(-3, 1, 1, 0, 0), (4, -\frac{13}{5}, 0, \frac{6}{5}, 1)\}$$

$$\begin{pmatrix} -3 & 1 & \textcircled{1} & 0 & 0 \\ 4 & -\frac{13}{5} & \textcircled{0} & \frac{6}{5} & 1 \end{pmatrix}$$

R

$$\left( \begin{array}{cc} 1 & -\frac{3}{5} \\ 0 & \frac{4}{5} \end{array} \right)$$

2 colonne lin. indip.

Teorema. Sei  $\mathcal{W} \subseteq K^m$  sottosp. vett. numerico.

Essere  $\Sigma_0 : Ax = 0$  tale che  $\mathcal{S}_0 = \mathcal{W}$ .

DIM Sei  $B = (x_1, \dots, x_h)$  una base ordinata di  $\mathcal{W}$ ,  $\dim \mathcal{W} = h$ .

$$x_1 = (a_1^1, a_1^2, \dots, a_1^m) \in \mathcal{W} \subseteq K^m$$

⋮

$$x_h = (a_h^1, a_h^2, \dots, a_h^m) \in \mathcal{W} \subseteq K^m$$

$$\mathcal{W} = \mathcal{L}(B)$$

$$u \in (x_1, \dots, x_h) \in K^m$$

$$u \in \mathcal{W} \iff \text{range}$$

$$\mathcal{L}(B)$$

$$B = \begin{pmatrix} a_1^1 & \cdots & a_1^h & x_1 \\ a_2^1 & \cdots & a_2^h & x_2 \\ \vdots & \ddots & \vdots & \vdots \\ a_h^1 & \cdots & a_h^h & x_m \end{pmatrix} = h \iff$$

↔

PRIMO MODO (riduzione di Gauss) :

una matrice  $\bar{B}$  ridotta a gradini ottenuta a partire da  $B$  ha  $h$  pivot

$$\bar{B} = \begin{pmatrix} p_1^1 & \cdots & \cdots & b_1^1 x_1 + \cdots + b_m^1 x_m \\ 0 & p_2^2 & \cdots & b_1^2 x_1 + \cdots + b_m^2 x_m \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & p_h^h \\ \vdots & & & \vdots \\ 0 & 0 & 0 & b_1^{h+1} x_1 + \cdots + b_m^{h+1} x_m \\ & & & \vdots \\ & & & b_1^m x_1 + \cdots + b_m^m x_m \end{pmatrix}$$

$$\text{range}(\bar{B}) = h \iff \# \text{pivot di } \bar{B} = h$$

$$\iff \sum_0 \begin{cases} b_1^{h+1} x_1 + \cdots + b_m^{h+1} x_m = 0 \\ \vdots \\ b_1^m x_1 + \cdots + b_m^m x_m = 0 \end{cases} \quad \text{e il rest. lineare da cercavamo} \\ \mathcal{S}_0 = \mathcal{W}$$

↔

SECONDO MODO (teorema degli ordini)

$$\text{range} \begin{pmatrix} a_1^1 & \cdots & a_1^h & x_1 \\ a_2^1 & \cdots & a_2^h & x_2 \\ \vdots & \ddots & \vdots & \vdots \\ a_h^1 & \cdots & a_h^h & x_m \end{pmatrix} = h$$

Per il Teorema degli ordini; esiste un minor  $M$  di  $B$  di ordine  $h$  con determinante non nullo.

A meno di uno scambio di righe, poniamo supponere che

$$M = \begin{pmatrix} \underline{\alpha}_1^1 & \dots & \underline{\alpha}_h^1 \\ \underline{\alpha}_1^2 & \dots & \underline{\alpha}_h^2 \\ \vdots & & \vdots \\ \underline{\alpha}_1^n & \dots & \underline{\alpha}_h^n \end{pmatrix} \quad |M| \neq 0$$

primo in righe

Allora,  $\text{rang}(B) = h \Leftrightarrow$  tutti gli  $m-h$  ordini di  $M$  hanno determinante nullo

$$M_1, \dots, M_{m-h}$$

$$\Sigma_0: \begin{cases} |M_1| = 0 \\ \vdots \\ |M_h| = 0 \end{cases} \quad g_0 = \mathcal{W}. \quad \square$$

Def. Se sisteme  $\Sigma_0$  ottenuto nella dimostrazione del precedente teorema si dice RAPPRESENTAZIONE (CARTESIANA) di  $\mathcal{W}$ .

Esempio:

$$\mathcal{W} \subseteq \mathbb{R}^3 \quad \mathcal{W} = \mathcal{L}((2, 1, -2)) \quad \dim \mathcal{W} = 1$$

$$u = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad u \in \mathcal{W} \Leftrightarrow \text{rang} \begin{pmatrix} \underline{\alpha}_1 & x_1 \\ \underline{\alpha}_2 & x_2 \\ \underline{\alpha}_3 & x_3 \end{pmatrix} = 1$$

$$\text{PRIMO MODO:} \quad \begin{array}{l} \underline{\alpha}_2^2 \rightarrow \underline{\alpha}_2^2 - \frac{1}{2} \underline{\alpha}_1^2 \\ \underline{\alpha}_3^3 \rightarrow \underline{\alpha}_3^3 + \underline{\alpha}_1^3 \end{array} \quad \left( \begin{array}{cc} 2 & x_1 \\ 0 & \left\{ \begin{array}{l} x_2 - \frac{1}{2} x_1 \\ x_3 + x_1 \end{array} \right. \\ 0 & x_2 \end{array} \right)$$

$$\mathcal{W}: \begin{cases} x_2 - \frac{1}{2} x_1 = 0 \\ x_3 + x_1 = 0 \end{cases}$$

$$\text{SECONDO MODO:} \quad M = (2) \quad M_1 = \begin{pmatrix} 2 & x_1 \\ 0 & x_2 \end{pmatrix} \quad M_2 = \begin{pmatrix} 2 & x_1 \\ 0 & x_3 \end{pmatrix}$$

$$|M_1| = 2x_2 - x_1 \quad |M_2| = 2x_3 + 2x_1$$

$$\mathcal{W}: \begin{cases} 2x_2 - x_1 = 0 \\ 2x_3 + 2x_1 = 0 \end{cases}$$

$$\text{Esercizio:} \quad U \subseteq \mathbb{R}^3 \quad U = \mathcal{L}((1, 0, 2), (2, 1, 3))$$

$$\begin{pmatrix} 1 & 2 & x_1 \\ 0 & 1 & x_2 \\ 2 & 3 & x_3 \end{pmatrix} \quad \underline{\alpha}_3^3 \rightarrow \underline{\alpha}_3^3 - 2 \underline{\alpha}_1^3 \quad \left( \begin{array}{ccc} 1 & 2 & x_1 \\ 0 & 1 & x_2 \\ 0 & -1 & x_3 - 2x_1 \end{array} \right) \quad \underline{\alpha}_3^3 \rightarrow \underline{\alpha}_3^3 + \underline{\alpha}_1^3$$

$$\left( \begin{array}{ccc|c} 1 & 2 & x_2 \\ 0 & 1 & x_2 \\ 0 & 0 & x_3 - 2x_4 + x_2 \end{array} \right) \quad U: \begin{cases} x_3 - 2x_4 + x_2 = 0 \\ \text{Applicare anche il SECONDO MODO.} \end{cases}$$

Esercizio  $V \subseteq \mathbb{R}^4$   $V = \{(1, 0, 1, 1), (2, 1, 2, 3), (1, 1, 1, 2)\}$

$$\left( \begin{array}{cc|cc} 1 & 2 & 1 & x_4 \\ 0 & 1 & 1 & x_2 \\ 1 & 2 & 1 & x_3 \\ 1 & 3 & 2 & x_4 \end{array} \right) \quad \begin{array}{l} \underline{\underline{0}}^3 \rightarrow \underline{\underline{0}}^3 - \underline{\underline{0}}^1 \\ \underline{\underline{0}}^4 \rightarrow \underline{\underline{0}}^4 - \underline{\underline{0}}^2 \end{array} \quad \left( \begin{array}{cc|cc} 1 & 2 & 1 & x_2 \\ 0 & 1 & 1 & x_2 \\ 0 & 0 & 0 & x_3 - x_2 \\ 0 & 0 & 1 & x_4 - x_2 \end{array} \right)$$

$$\underline{\underline{0}}^4 \rightarrow \underline{\underline{0}}^4 - \underline{\underline{0}}^2 \quad \left( \begin{array}{cc|cc} 1 & 2 & 1 & x_1 \\ 0 & 1 & 1 & x_2 \\ 0 & 0 & 0 & x_3 - x_2 \\ 0 & 0 & 0 & x_4 - x_2 - x_2 \end{array} \right) \quad V: \begin{cases} x_3 - x_2 = 0 \\ x_4 - x_2 - x_2 = 0 \end{cases}$$

$$\dim V = 2$$

Esercizio  $V \subseteq \mathbb{R}^5$   $V = \{(2, 1, 0, 3, 1), (2, -1, 1, 4, 0)\}$   $\dim V = 2$

PRIMO  
MODO

$$\left( \begin{array}{ccccc} 2 & 1 & 0 & 3 & 1 \\ 1 & -1 & 1 & 4 & 0 \\ 0 & 1 & 1 & 3 & 1 \\ 3 & 4 & x_4 & x_5 \\ 2 & 0 & x_5 & x_5 \end{array} \right) \quad \begin{array}{l} \underline{\underline{0}}^2 \rightarrow \underline{\underline{0}}^2 - \frac{1}{2}\underline{\underline{0}}^1 \underline{\underline{0}}^2 \\ \underline{\underline{0}}^4 \rightarrow \underline{\underline{0}}^4 - \frac{3}{2}\underline{\underline{0}}^2 \underline{\underline{0}}^2 \\ \underline{\underline{0}}^5 \rightarrow \underline{\underline{0}}^5 - \frac{1}{2}\underline{\underline{0}}^2 \underline{\underline{0}}^2 \end{array} \quad \left( \begin{array}{ccccc} 2 & 1 & 0 & x_4 & x_5 \\ 0 & -2 & 1 & x_2 - \frac{3}{2}x_2 & x_4 \\ 0 & 1 & 1 & x_3 & x_5 - \frac{3}{2}x_2 \\ 0 & 1 & x_4 - \frac{3}{2}x_2 & x_4 & x_5 - \frac{3}{2}x_2 \\ 0 & -1 & x_5 - \frac{1}{2}x_2 & x_2 + \frac{1}{2}x_2 & x_5 \end{array} \right)$$

$$\begin{array}{l} \underline{\underline{0}}^3 \rightarrow \underline{\underline{0}}^3 + \frac{1}{2}\underline{\underline{0}}^2 \underline{\underline{0}}^2 \\ \underline{\underline{0}}^4 \rightarrow \underline{\underline{0}}^4 + \frac{1}{2}\underline{\underline{0}}^2 \underline{\underline{0}}^2 \\ \underline{\underline{0}}^5 \rightarrow \underline{\underline{0}}^5 - \frac{1}{2}\underline{\underline{0}}^2 \underline{\underline{0}}^2 \end{array} \quad \left( \begin{array}{ccccc} 0 & 1 & 0 & x_1 & x_2 \\ 0 & 1 & -2 & x_2 - \frac{3}{2}x_2 & x_4 \\ 0 & 0 & 0 & x_3 + \frac{1}{2}x_2 - \frac{1}{2}x_2 & x_5 \\ 0 & 0 & 0 & x_4 - \frac{3}{2}x_2 + \frac{1}{2}x_2 - \frac{1}{2}x_2 & x_5 \\ 0 & 0 & 0 & x_5 - \frac{1}{2}x_2 - \frac{1}{2}x_2 + \frac{1}{2}x_2 & x_5 \end{array} \right)$$

$$V: \begin{cases} -\frac{1}{2}x_1 + \frac{1}{2}x_2 + x_3 = 0 \\ -\frac{7}{2}x_2 + \frac{1}{2}x_2 + x_4 = 0 \\ -\frac{1}{2}x_4 - \frac{1}{2}x_2 + x_5 = 0 \end{cases}$$

SECONDO  
MODO

rank  $\left( \begin{array}{ccccc|cc} 2 & 1 & 0 & 3 & 1 & 3 & 1 \\ 2 & -1 & 1 & 4 & 0 & 4 & 0 \\ x_1 & x_2 & x_3 & x_4 & x_5 & & \end{array} \right) = 2$   $M = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \Rightarrow |M| = 1 \neq 0$

$$M_1 = \begin{pmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ x_1 & x_2 & x_3 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 1 & 4 \\ x_2 & x_3 & x_4 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ x_2 & x_3 & x_5 \end{pmatrix}$$

$$W: \begin{cases} |M_1| = -2x_3 + x_4 - 2x_2 - 2x_3 = 0 \\ |M_2| = x_4 - 3x_3 - 3x_2 - 4x_3 = 0 \\ |M_3| = x_5 - x_3 - x_2 = 0 \end{cases}$$

$(V, +, \cdot)$   $k$ ,  $\dim V = n$ ,  $B = (e_1, \dots, e_n)$  basi ordinate di  $V$

$W \subseteq V$  sottosp. vett.,  $\dim W = h$

$$\phi_B: V \longrightarrow k^m$$

$$u \mapsto (x_1, \dots, x_m) : u = x_1 e_1 + \dots + x_n e_n$$

$$\phi_B(W) \subseteq k^m, \quad \dim(\phi_B(W)) = \dim W = h$$

sottosp. vett.

Possiamo rappresentare  $\phi_B(W)$ :  $\Sigma: Ax = 0$

RAPPRESENTAZIONE CARTESIANA  
di  $\phi_B(W)$

si dice RAPPRESENTAZIONE CARTESIANA  
di  $W$  rispetto alle basi ordinate  $B$   
(riferimento)

Esempio:  $V = \mathbb{R}[x] \leq 2$   $B = (1, x, x^2)$

$$W = \mathcal{L}(1-x)$$

Rappresentazione  $W$  in  $B$ :

$$\begin{aligned} \phi_B: V &\longrightarrow \mathbb{R}^3 \\ a_0 + a_1 x + a_2 x^2 &\mapsto (a_0, a_1, a_2) \end{aligned}$$

$$\phi_B(W) = \mathcal{L}(\phi_B(1-x)) = \mathcal{L}((1, -1, 0))$$

$$\begin{pmatrix} 1 & x_1 \\ -1 & x_2 \\ 0 & x_3 \end{pmatrix} \xrightarrow{x_2 \rightarrow x_2 + x_1} \begin{pmatrix} 1 & x_1 \\ 0 & x_2 + x_1 \\ 0 & x_3 \end{pmatrix}$$

$$W: \begin{cases} x_2 + x_1 = 0 \\ x_3 = 0 \end{cases} \quad \begin{aligned} g_0 &= \phi_B(W) \\ W &= \phi_B^{-1}(g_0) \end{aligned}$$

Col teorema degli ordini:

$$M = (1) \quad |M| = 1$$

$$M_1 = \begin{pmatrix} 1 & x_1 \\ -1 & x_2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & x_1 \\ 0 & x_3 \end{pmatrix}$$

$$\begin{cases} |M_2| = x_2 + x_4 = 0 \\ |M_3| = x_3 = 0 \end{cases}$$

Rappresentazioni parametriche:

$\mathcal{W} \subseteq \mathbb{K}^m$  sotto spazio vettoriale numerico,  $\dim \mathcal{W} = h$

Come nella dimostrazione del teorema:

$$\begin{aligned} \mathcal{B} &= (x_1, \dots, x_h), \quad x_1 = (\alpha_1^1, \alpha_1^2, \dots, \alpha_1^m) \in \mathcal{W} \subseteq \mathbb{K}^m \\ \text{base di } \mathcal{W} &\vdots \\ &\vdots \\ x_h = (\alpha_h^1, \alpha_h^2, \dots, \alpha_h^m) \in \mathcal{W} \subseteq \mathbb{K}^m \end{aligned} \quad \mathcal{W} = \mathcal{L}(\mathcal{B})$$

$$u = (x_1, \dots, x_h) \in \mathbb{K}^m$$

$$u \in \mathcal{W} \Leftrightarrow u \in \mathcal{L}(x_1, \dots, x_h) \Leftrightarrow \exists t_1, \dots, t_h \in \mathbb{K} :$$

$$\begin{aligned} (x_1, \dots, x_h) &= t_1 (\alpha_1^1, \alpha_1^2, \dots, \alpha_1^m) + \dots + t_h (\alpha_h^1, \alpha_h^2, \dots, \alpha_h^m) = \\ &= (\alpha_1^1 t_1 + \dots + \alpha_1^h t_h, \alpha_2^1 t_1 + \dots + \alpha_2^h t_h, \dots, \alpha_m^1 t_1 + \dots + \alpha_m^h t_h) \\ \begin{cases} x_1 = \alpha_1^1 t_1 + \dots + \alpha_1^h t_h \\ x_2 = \alpha_2^1 t_1 + \dots + \alpha_2^h t_h \\ \vdots \\ x_m = \alpha_m^1 t_1 + \dots + \alpha_m^h t_h \end{cases} & \text{RAPPRESENTAZIONE PARAMETRICA DI } \mathcal{W}. \end{aligned}$$

$$\text{Esempio: } U = \mathcal{L}((-1, 0, 2), (2, 1, 3)) \subseteq \mathbb{R}^3$$

$$\begin{cases} x_1 = t_1 + 2t_2 \\ x_2 = t_2 \\ x_3 = 2t_1 + 3t_2 \end{cases}$$

$$\text{Esempio: } V = \mathbb{R}[x] \leq 2 \quad \mathcal{B} = (1, x, x^2)$$

$$\mathcal{W} = \mathcal{L}(1-x) \quad \Phi_{\mathcal{B}}(1-x) = (1, -1, 0)$$

$$\begin{cases} x_1 = t \\ x_2 = -t \\ x_3 = 0 \end{cases} \quad \begin{array}{l} \text{RAPPRESENTAZIONE PARAMETRICA DI } \mathcal{W} \\ \text{rispetto a } \mathcal{B} \\ (\text{in } \mathcal{B}) \end{array}$$

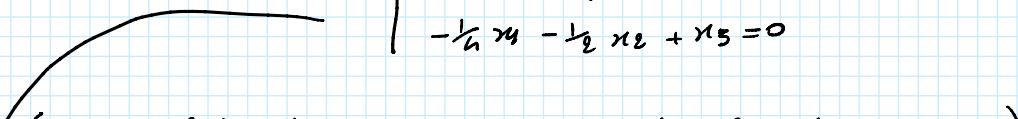
Oss

$\Downarrow \quad \leftarrow$  "eliminiamo il parametro"

$$\begin{cases} t = x_1 \\ x_2 = -x_1 \\ x_3 = 0 \end{cases} \quad \leftarrow \text{RAPPRESENTAZIONE CARTESIANA (non parametrica)}$$

VICEVERSA:

$$\mathcal{W}: \quad \begin{cases} -\frac{1}{2}x_1 + \frac{1}{2}x_2 + x_3 = 0 \\ -\frac{1}{2}x_1 + \frac{1}{2}x_2 + x_4 = 0 \\ -\frac{1}{2}x_1 - \frac{1}{2}x_2 + x_5 = 0 \end{cases}$$



$$1 - \frac{1}{2}x_4 - \frac{1}{2}x_2 + x_5 = 0$$

$$\left\{ \begin{array}{l} -x_1 + 2x_2 + 4x_3 = 0 \\ -7x_1 + 2x_2 + 6x_4 = 0 \\ -x_1 - 2x_2 + 4x_5 = 0 \end{array} \right.$$

$$\left( \begin{array}{ccccc} -1 & 2 & 4 & 0 & 0 \\ -7 & 2 & 0 & 6 & 0 \\ -1 & -2 & 0 & 0 & 4 \end{array} \right)$$

$$\left\{ \begin{array}{l} x_3 = \frac{1}{2}x_1 - \frac{1}{2}x_2 \\ x_4 = \frac{3}{2}x_1 - \frac{1}{2}x_2 \\ x_5 = \frac{1}{2}x_1 + \frac{1}{2}x_2 \end{array} \right.$$

$$\mathcal{S}_0 = \{(x_1, x_2, \frac{1}{2}x_1 - \frac{1}{2}x_2, \frac{3}{2}x_1 - \frac{1}{2}x_2, \frac{1}{2}x_1 + \frac{1}{2}x_2) \mid x_1, x_2 \in \mathbb{R}\}$$

$$\left\{ \begin{array}{l} x_1 = t_1 \\ x_2 = t_2 \\ x_3 = \frac{1}{2}t_1 - \frac{1}{2}t_2 \\ x_4 = \frac{3}{2}t_1 - \frac{1}{2}t_2 \\ x_5 = \frac{1}{2}t_1 + \frac{1}{2}t_2 \end{array} \right. \quad t_1, t_2 \in \mathbb{R}$$

Esercizio:

$$\mathcal{W} = \mathcal{L}((1, 2, 0, 1), (0, 0, 1, 2)) \subseteq \mathbb{R}^4$$

$$U = \mathcal{L}((1, 0, 1, 2), (1, 2, 1, 3))$$

Determinare  $\mathcal{W} \cap U$ .

$$\mathcal{W}: Ax = 0, \quad U: A^T x = 0$$

$$U \cap \mathcal{W}: \begin{cases} Ax = 0 \\ A^T x = 0 \end{cases}$$

$$\left( \begin{array}{cccc} 1 & 0 & x_2 & \\ 2 & 0 & x_1 & \\ 0 & 2 & x_3 & \\ 1 & 2 & x_4 & \end{array} \right) \xrightarrow{\begin{array}{l} \underline{x}_1 \rightarrow \underline{x}_1 - 2\underline{x}_2 \\ \underline{x}_4 \rightarrow \underline{x}_4 - \underline{x}_2 \end{array}} \left( \begin{array}{cccc} 1 & 0 & x_2 & \\ 0 & 0 & x_1 - 2x_2 & \\ 0 & 1 & x_3 & \\ 0 & 2 & x_4 - x_2 & \end{array} \right) \xrightarrow{\underline{x}_2 \leftrightarrow \underline{x}_3}$$

$$\left( \begin{array}{cccc} 1 & 0 & x_2 & \\ 0 & 1 & x_3 & \\ 0 & 0 & x_1 - 2x_2 & \\ 0 & 2 & x_4 - x_2 & \end{array} \right) \xrightarrow{\begin{array}{l} \underline{x}_1 \rightarrow \underline{x}_1 - 2\underline{x}_2 \\ \underline{x}_3 \rightarrow \underline{x}_3 - 2\underline{x}_2 \end{array}} \left( \begin{array}{cccc} 1 & 0 & x_2 & \\ 0 & 1 & x_3 & \\ 0 & 0 & x_1 - 2x_2 & \\ 0 & 0 & x_4 - x_2 - 2x_3 & \end{array} \right)$$

$$\mathcal{W}: \begin{cases} x_2 - 2x_4 = 0 \\ x_4 - x_2 - 2x_3 = 0 \end{cases}$$

$$\left( \begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ \hline 1 & 2 & 1 & 3 \\ x_1 & x_2 & x_3 & x_4 \end{array} \right)$$

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \quad |M| = 2 \neq 0$$

$$M_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ x_1 & x_2 & x_3 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ x_1 & x_2 & x_4 \end{pmatrix}$$

$$U \quad \begin{cases} |M_1| = 2x_3 + x_2 - 2x_1 - x_4 = 0 \end{cases}$$

$$|M_2| = 2x_4 + x_2 - 2x_1 - 3x_3 = -2x_1 - 2x_2 + 2x_4 = 0$$

$$W \cap U : \left\{ \begin{array}{l} x_2 - 2x_4 = 0 \\ x_4 - x_1 - 2x_3 = 0 \\ 2x_3 - 2x_4 = 0 \\ -2x_4 - 2x_2 + 2x_4 = 0 \end{array} \right) W$$

$$W : \left\{ \begin{array}{l} x_2 - 2x_4 = 0 \\ x_4 - x_1 - 2x_3 = 0 \end{array} \right.$$

OPPURE:

$$U : \left\{ \begin{array}{l} x_1 = t_1 + t_2 \\ x_2 = 2t_2 \\ x_3 = t_1 - t_2 \\ x_4 = t_1 + 3t_2 \end{array} \right.$$

$$u \in U \Leftrightarrow \exists t_1, t_2 \in \mathbb{R} : u = (t_1 + t_2, 2t_2, t_1 - t_2, \underline{t_1 + 3t_2})$$

$$u \in U \cap W \Leftrightarrow \left\{ \begin{array}{l} 2t_2 - 2(t_1 + t_2) = 0 \\ t_1 + 3t_2 - t_1 - t_2 - 2(t_1 + t_2) = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} \cancel{2t_2} - 2t_1 - \cancel{2t_2} = 0 \Rightarrow t_1 = 0 \\ \cancel{3t_2} - \cancel{t_1} - \cancel{2t_2} = 0 \Rightarrow 0 = 0 \end{array} \right.$$

$$u \in U \cap W \Leftrightarrow \exists t_2 \in \mathbb{R} : u = (\cancel{t_1}, \cancel{2t_1}, \cancel{t_1}, \cancel{3t_1}) \in \mathcal{L}((1, 2, 1, 3)) -$$

" "  
  $t_2(1, 2, 1, 3)$