

Lezione 18

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$$(\vec{E}, \mathcal{E}, \pi)$$

$$\pi: \mathcal{E} \times \mathcal{E} \xrightarrow{(P, Q)} \vec{\mathcal{E}}$$

Ricordiamo: $\mathcal{H} \subseteq \mathcal{E}$ è sottospazio euclideo (o affine) se

$$(i) \quad \vec{\mathcal{H}} = \pi(\mathcal{H} \times \mathcal{H}) = \{ \vec{PQ} \mid P, Q \in \mathcal{H} \} \text{ è sottosp. vett. di } \vec{\mathcal{E}}$$

$$(ii) \quad \forall A \in \mathcal{H}, \forall a \in \vec{\mathcal{H}}, \text{ l'unico punto } X \in \mathcal{E} \text{ tali che } \vec{AX} = a \text{ appartiene ad } \mathcal{H}.$$

$\forall P_0 \in \mathcal{E}, \forall U \subseteq \vec{\mathcal{E}}$ sottosp. vett. poniamo considerare:

$$(P_0, U) \stackrel{\text{def.}}{=} \{ Q \in \mathcal{E} \mid \vec{P_0Q} \in U \} \quad \text{varietà-lineare}$$

$$\stackrel{"}{=} P_0 + U$$

- Ogni varietà-lineare è un sottosp. euclideo (o affine)

DIM.: Th: (P_0, U) soddisfa le condizioni (i) e (ii) (FACOLTATIVA)

- $P_0: \vec{P_0P_0} = 0 \in U \Rightarrow P_0 \in (P_0, U)$ per def. di (P_0, U)
- Vediamo che (P_0, U) soddisfa la (ii):
 $A \in (P_0, U), a \in U$ Sappiamo che $\exists! X \in \mathcal{E}: \vec{P_0X} = a \in U \Rightarrow X \in (P_0, U)$
- Resta da dimostrare che $U = \pi((P_0, U) \times (P_0, U))$
 " $\exists! P, Q \in (P_0, U) \Rightarrow \vec{P_0P}, \vec{P_0Q} \in U \Rightarrow -\vec{P_0P}, \vec{P_0Q} \in U \Rightarrow \vec{P_0P}, \vec{P_0Q} \in U$
 \downarrow
 U è sottosp. vett.

$$\text{Allora: } \vec{PQ} = \vec{P_0P} + \vec{P_0Q} \in U$$

$$\stackrel{\text{"}}{\in} \pi(P, Q)$$

$$\stackrel{\leq}{=} u \in U, P_0 \in (P_0, U) \text{ e vale (ii)} \Rightarrow \exists! X \in (P_0, U): \vec{P_0X} = u \in U$$

$$\pi(P, X)$$

- Ogni sottosp. euclideo (o affine) è una varietà-lineare:

DIM.: $(\vec{\mathcal{E}}, \mathcal{E}, \pi_1)$ Th: $\forall P_0 \in \mathcal{E}, \mathcal{H} = (P_0, \vec{\mathcal{E}})$ (FACOLTATIVA)

$$\stackrel{\leq}{=} P \in \mathcal{H}, \vec{\mathcal{E}} = \pi(\mathcal{E} \times \mathcal{E}) = \{ \vec{PQ} \mid P, Q \in \mathcal{E} \}$$

$$P_0 \in \mathcal{E} \quad \Rightarrow \quad \vec{P_0P} \in \vec{\mathcal{E}} \Rightarrow P \in (P_0, \vec{\mathcal{E}})$$

$$\stackrel{\exists!}{=} P \in (P_0, \vec{\mathcal{E}}) \stackrel{a}{\Rightarrow} \vec{P_0P} \in \vec{\mathcal{E}} \stackrel{(ii)}{\Rightarrow} \exists! X \in \mathcal{E}: \vec{P_0X} = a$$

$$\stackrel{P_0 \in \mathcal{E}}{\Rightarrow} \Rightarrow P = X \in \mathcal{H}.$$

Ricordiamo di abbiemo come rappresentare i sottosp. euclidi (o affini) di spaz. euclidi di dimensione finita, fissato un nf. contenente.

$$(\vec{E}, \mathcal{E}, \pi) \quad \dim \mathcal{E} = n \quad R = (O, B)$$

\mathcal{H} sottosp. euclideo, $\dim \mathcal{H} = h$

$$((P_0, \vec{\mathcal{E}})) \quad Q \equiv_R (x_1, \dots, x_m) \in \mathcal{E} \quad , \quad P_0 \equiv_R (x_1, \dots, x_m)$$

$$Q \in \mathcal{H} \Leftrightarrow \vec{P}_0 Q \in \vec{\mathcal{E}} \Leftrightarrow \phi_B(\vec{P}_0 Q) \in \phi_B(\vec{\mathcal{E}}) : \underline{AX = 0} \quad \text{range}(A) = m - b.$$

$$(x_1 - a_1, \dots, x_m - a_m)$$

$$\Leftrightarrow A \begin{pmatrix} x_1 - a_1 \\ \vdots \\ x_m - a_m \end{pmatrix} = 0 \Leftrightarrow A \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \underbrace{A \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}}_{\underline{b}}$$

$\Sigma: AX = b$ ha come soluzione le m-uple delle coordinate di tutti i vettori punti di $\vec{\mathcal{E}}$.

$\Sigma_0: AX = 0$ rappresenta $\vec{\mathcal{E}}$ in B

Sia $B_{\vec{\mathcal{E}}} = \{v_{k1}, \dots, v_{kh}\}$ una base di $\vec{\mathcal{E}}: \vec{\mathcal{E}} = \mathbb{L}(v_{k1}, \dots, v_{kh})$

$$\phi_B(v_{ki}) = (a_{k1}^i, \dots, a_{km}^i)$$

:

$$\phi_B(v_{hi}) = (a_{h1}^i, \dots, a_{hm}^i)$$

$$Q \in \mathcal{H} \Leftrightarrow \vec{P}_0 Q \in \vec{\mathcal{E}} \Leftrightarrow \phi_B(\vec{P}_0 Q) \in \phi_B(\vec{\mathcal{E}}) = \mathbb{L}(\phi_B(v_{k1}), \dots, \phi_B(v_{kh}))$$

$$\Leftrightarrow \exists t_1, \dots, t_h \in K: (x_1 - a_1, \dots, x_m - a_m) = t_1(a_{11}^i, \dots, a_{1m}^i) + \dots + t_h(a_{h1}^i, \dots, a_{hm}^i)$$

$$\left\{ \begin{array}{l} x_1 = a_1 + t_1 a_{11}^i + \dots + t_h a_{1h}^i \\ \vdots \\ x_m = a_m + t_1 a_{m1}^i + \dots + t_h a_{mh}^i \end{array} \right.$$

$$t_1, \dots, t_h \in K$$

rappresentazione parametrica di \mathcal{E} in R



$\mathcal{E}, \mathcal{E}'$ sottosp. euclidi (o affini)

$\mathcal{E} \neq \mathcal{E}'$ sono paralleli $\Rightarrow \vec{\mathcal{E}} \subseteq \vec{\mathcal{E}}'$ opp. $\vec{\mathcal{E}}' \subseteq \vec{\mathcal{E}}$

Esercizio: (E, E, π) $\dim E = 3$ $R = (0, B)$ $Q \in R(x_1, x_2, x_3)$

Rappresentazione in R : il piano \mathcal{E} passante per $P_0(3, 4, 2)$ e di direttrice

$$\vec{\mathcal{E}} = \mathbb{L}(v_1(1, 2, 0), v_2(2, 1, 1))$$

$$\mathcal{E}: \left\{ \begin{array}{l} x_1 = 3 + t_1 1 + t_2 2 \\ x_2 = 4 + t_1 2 + t_2 1 \\ x_3 = 2 + t_1 0 + t_2 1 \end{array} \right. \quad \left\{ \begin{array}{l} x_1 = 3 + t_1 + 2t_2 \\ x_2 = 4 + 2t_1 + t_2 \\ x_3 = 2 + t_2 \end{array} \right.$$

$$Q \in \mathcal{H} \Leftrightarrow \text{rang} \begin{pmatrix} 1 & 2 & x_1 - 3 \\ 0 & 1 & x_2 - 4 \\ 0 & 0 & x_3 - 2 \end{pmatrix} = 2$$

$$\text{GAUSS: } \underline{x_2} \rightarrow \underline{x_2} - 2\underline{x_1} \quad \left(\begin{array}{ccc} 1 & 2 & x_1 - 3 \\ 0 & -3 & x_2 - 4 - 2x_1 + 6 \\ 0 & 1 & x_3 - 2 \end{array} \right) \quad \underline{x_3} \rightarrow \underline{x_3} + \frac{1}{3} \underline{x_2}$$

$$\left(\begin{array}{ccc} 1 & 2 & x_1 - 3 \\ 0 & -3 & -2x_1 + x_2 + 2 \\ 0 & 1 & x_3 - 2 \end{array} \right)$$

$$\begin{pmatrix} 1 & 2 & x_4 - 3 \\ 0 & -3 & -2x_4 + x_2 + 2 \\ 0 & 0 & x_3 - 2 - \frac{2}{3}x_4 + \frac{1}{3}x_2 + \frac{8}{3} \end{pmatrix}$$

$$H: -\frac{2}{3}x_4 + \frac{1}{3}x_2 + x_3 - \frac{4}{3} = 0$$

$$\vec{H}: -\frac{2}{3}x_4 + \frac{1}{3}x_2 + x_3 = 0$$

Teorema degli ordini: $H = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ $|H| = 2 \neq 0$

$$Q \in H \Leftrightarrow \begin{vmatrix} 1 & 2 & x_4 - 3 \\ 2 & 1 & x_2 - 4 \\ 0 & 1 & x_3 - 2 \end{vmatrix} = 0 \Leftrightarrow x_3 - 2 + 2(x_4 - 3) - x_2 + 4 - 4x_3 + 8 = 0 \Leftrightarrow 2x_4 - x_2 - 3x_3 + 4 = 0$$

Determinare il piano parallelo a H e passante per $A(-2, 1, 5)$

$$\vec{H}': 2x_4 - x_2 - 3x_3 = 0 \quad \vec{H}'': 2x_4 - x_2 - 3x_3 + d = 0$$

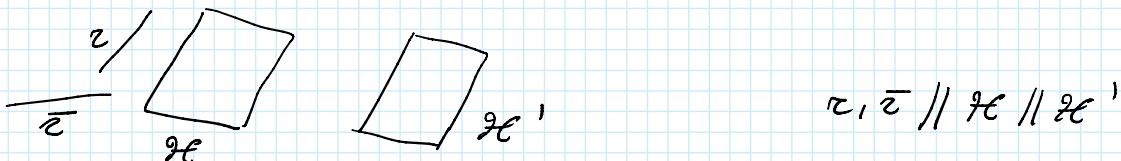
$$A \in \vec{H}' \Rightarrow 2(-2) - 1 - 3 \cdot 5 + d = 0 \Rightarrow d = 20$$

$$H': 2x_4 - x_2 - 3x_3 + 20 = 0$$

Determinare una retta r parallela a H e passante per $B(3, 1, 0)$

$$r: \begin{cases} x_4 = 3 + t_1 \\ x_2 = 1 + 2t_1 \\ x_3 = 0 + 0 \end{cases}$$

$$r: \begin{cases} t_1 = x_4 - 3 \\ x_2 = 1 + 2x_4 - 6 \\ x_3 = 0 \end{cases} \quad \begin{cases} x_2 - 2x_4 + 5 = 0 \\ x_3 = 0 \end{cases}$$



$$\bar{r}: \begin{cases} x_4 = 3 + 2t_1 \\ x_2 = 1 + t_1 \\ x_3 = 0 + t_1 \end{cases}$$

$$r \not\parallel \bar{r}$$

$$r, \bar{r} \parallel H \parallel H'$$

(\vec{E}, E, π) H, H' sottsp. incidenti (o affini)

H e H' sono incidenti se $H \cap H' \neq \emptyset$

H e H' sono sghembi se $H \cap H' = \emptyset$ e $H \not\parallel H'$.

Esercizio: $\dim E = m$ $R = (O, \mathcal{B})$

Siano $P(c_1, \dots, c_m)$ e $Q(d_1, \dots, d_m)$ due punti distinti.

Determinare la retta passante per P e Q . $\vec{PQ} \neq 0$

$$P, Q \in r \quad \vec{r} \ni \vec{PQ}$$

$$\vec{r} = \mathcal{L}(\vec{PQ}, (d_1 - c_1, \dots, d_m - c_m))$$

$$r: \begin{cases} x_1 = d_1 + (d_1 - c_1)t \\ x_2 = d_2 + (d_2 - c_2)t \\ \vdots \\ x_m = d_m + (d_m - c_m)t \end{cases}$$

$$\dim E = 2 \quad P(-3, 4) \quad Q(0, 1)$$

$$r: \begin{cases} x_1 = 0 + 3t \\ x_2 = 1 - 3t \end{cases}$$

$$\vec{r} = \mathcal{L}(\vec{PQ}, (3, -3))$$

$$\dim E = 2 \quad P(-3, 4) \quad Q(0, 1)$$

$$r: \begin{cases} x_1 = 0 + 3t \\ x_2 = 1 - 3t \end{cases}$$

$$\vec{e} = \vec{PQ} / \|\vec{PQ}\| (3, -3)$$

$$\vec{r}: \begin{cases} t = x_1 \\ x_2 = t - 1 \end{cases}$$

$$r: x_1 + x_2 - 1 = 0$$

$$\vec{r}: \begin{cases} x_1 = 0 + t \\ x_2 = 1 - t \end{cases}$$

$$\vec{e} = \vec{u} / \|\vec{u}\| (1, -1)$$

$$\downarrow \\ r = \vec{e}$$

$$(\vec{E}, E, \pi) \quad \dim E = 2$$

$$r: ax + by + c = 0$$

$$r': a'x + b'y + c' = 0$$

$$r = r' \text{ opp. } r \neq r'$$

$$r \cap r': \begin{cases} ax + by + c = 0 \\ a'x + b'y + c' = 0 \end{cases}$$

$$R = (0, \infty)$$

$$\vec{r}: ax + by = 0$$

$$\vec{r}': a'x + b'y = 0$$

condizioni di incidenza
tra rette in un piano

$$C = \underbrace{\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}}_A \quad 1 \leq \text{rang}(A) \leq \text{rang}(C) \leq 2$$

$$\begin{matrix} 1 & 1 \\ 1 & 2 \end{matrix} \Rightarrow r = r' \\ \Rightarrow r \cap r' = \emptyset$$

$$\vec{r} \cap \vec{r}': \begin{cases} ax + by = 0 \\ a'x + b'y = 0 \end{cases}$$

$$\vec{r} = \vec{r}'$$

$$\begin{matrix} 2 & 2 \end{matrix} \Rightarrow |r \cap r'| = 1$$

$$(\vec{E}, E, \pi) \quad \dim E = 3 \quad R = (0, \infty)$$

condizioni di incidenza tra due rette

$$r: \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \end{cases}$$

$$r': \begin{cases} \alpha x + \beta y + \gamma z + \delta = 0 \\ \alpha'x + \beta'y + \gamma'z + \delta' = 0 \end{cases}$$

$$r \cap r': \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \\ \alpha x + \beta y + \gamma z + \delta = 0 \\ \alpha'x + \beta'y + \gamma'z + \delta' = 0 \end{cases}$$

$$C = \underbrace{\begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \end{pmatrix}}_A$$

$$\dim \vec{r} \cap \vec{r}' = 0$$

$$2 \leq \text{rang}(A) \leq \text{rang}(C) \leq 4$$

$$2 \quad 2 \quad \Rightarrow \quad r = r'$$

$$2 \quad 3 \quad \Rightarrow \quad r \cap r' = \emptyset \quad \text{e} \quad \vec{r} = \vec{r}'$$

$$3 \quad 3 \quad \Rightarrow \quad |r \cap r'| = 1$$

$$3 \quad 4 \quad \Rightarrow \quad r \cap r' = \emptyset \quad \text{e} \quad \vec{r} \cap \vec{r}' = 0$$

r e r' sono negli stessi

Condizioni di incidenza tra una retta e un piano:

$$r: \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \end{cases}$$

$$dC: \alpha x + \beta y + \gamma z + \delta = 0$$

$$r \cap dC: \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \\ \alpha x + \beta y + \gamma z + \delta = 0 \end{cases}$$

$$C = \underbrace{\begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \end{pmatrix}}$$

$$z \cap \mathcal{H} : \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \\ \alpha x + \beta y + \gamma z + \delta = 0 \end{cases} \quad C = \left(\begin{array}{ccc|cc} a & b & c & a' & d \\ a' & b' & c' & d' & \\ \alpha & \beta & \gamma & & \end{array} \right) \quad \text{``A''}$$

$$2 \leq \text{range}(A) \leq \text{range}(C) \leq 3$$

$$\begin{matrix} 2 & 2 \\ 2 & 3 \end{matrix} \Rightarrow z \subseteq \mathcal{H} \quad \Rightarrow \vec{z} \cap \vec{\mathcal{H}} = \vec{z} \quad z \cap \mathcal{H} = \emptyset$$

quindi $z \parallel \mathcal{H}$ e $z \cap \mathcal{H} = \emptyset$

$$\begin{matrix} 3 & 3 \end{matrix} \Rightarrow |z \cap \mathcal{H}| = 1$$

\nwarrow Teor. di Cramer

Condizioni di incidenza tra due piani: ESERCIZIO.

$$\dim E = 3$$

$$R = (O, B)$$

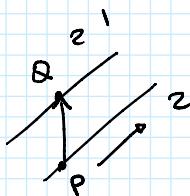
due rette parallele oppure incidenti sono complanari; oppure esiste un piano che contiene entrambe

$$z: \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \end{cases}$$

$$z': \begin{cases} \alpha x + \beta y + \gamma z + \delta = 0 \\ \alpha'x + \beta'y + \gamma'z + \delta' = 0 \end{cases}$$

$$\bullet z \parallel z'$$

$$\vec{z} = \vec{z}' = \mathcal{L}(u)$$

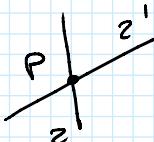


$$\begin{aligned} \mathcal{H} = (P, \mathcal{L}(u, \vec{PQ})) &\supseteq (P, \mathcal{L}(u)) = z \\ &\supseteq (Q, \mathcal{L}(u)) = z' \end{aligned}$$

$$\bullet z \cap z' = P$$

$$\vec{z} = \mathcal{L}(u)$$

$$\vec{z}' = \mathcal{L}(v)$$



$$\mathcal{H} = (P, \mathcal{L}(u, v))$$

$$z: \begin{cases} x + y = 1 \\ y - z = 0 \end{cases}$$

$$z': \begin{cases} x + 2y = 2 \\ 3y - z = 1 \end{cases}$$

$$C = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 2 \\ 0 & 3 & -1 & 1 \end{array} \right)$$

$$|C| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{vmatrix} = -6 + 2 + 2 + (-1) - 1 + 3 = 2 \neq 0$$

$$z'': \begin{cases} x = 2 + t \\ y = -1 + t \\ z = -1 - t \end{cases}$$

$$\vec{z}'' = \mathcal{L}(w(1, 1, -1)) \quad z \nparallel z'' \quad z \cap z'' = P(2, -1, -1)$$

$$\vec{v}: \begin{cases} x+y=0 \\ y-z=0 \end{cases} \quad S_0 = \{(-y, y, y) \mid y \in \mathbb{R}\}$$

$$\vec{v} = \lambda(u(-1, 1, 1))$$

$$H: \begin{cases} x = 2 + t_1 - t_2 \\ y = -1 + t_1 + t_2 \\ z = -1 - t_1 + t_2 \end{cases} \quad \left| \begin{pmatrix} 1 & -1 & x-2 \\ 1 & 1 & y+1 \\ -1 & 1 & z+1 \end{pmatrix} \right| = 0.$$