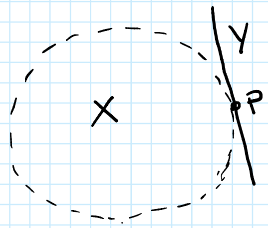
Lezione 20

venerdì 21 maggio 2021 11:00

$(\vec{\mathcal{E}}, \mathcal{E}, \pi)$ $\qquad \dim \mathcal{E} = n$ $\qquad \mathcal{R} = (O, \mathcal{B})$

$P, Q \in \mathcal{E}, \qquad d(P,Q) \overset{\text{def.}}{=} \|\vec{PQ}\|$

$X, Y \subseteq \mathcal{E}, \qquad d(X,Y) \overset{\text{def.}}{=} \inf\{d(P,Q) \mid P \in X, Q \in Y\}$



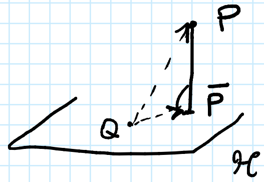
$P \notin X$

$d(X,Y) = 0$

Siano $P \in \mathcal{E}$ e $\mathcal{H}$ un iperpiano

Se $P \in \mathcal{H}$, allora $d(P,\mathcal{H}) = d(P,P) = 0$

Se $P \notin \mathcal{H}$, allora:



$d(P,\mathcal{H}) = d(P,\bar{P})$

Esempio: $\quad n = 4 \qquad \mathcal{R} = (O,\mathcal{B})$

$\mathcal{H}: x_1 - 3x_2 - x_3 + 2x_4 - 1 = 0 \qquad w(1,-3,-1,2) \perp \mathcal{H}, \quad {}^{\perp}\vec{\mathcal{H}} = \mathcal{L}(w)$

$P(0,1,2,0)$

$$r: \begin{cases} x_1 = t \\ x_2 = 1 - 3t \\ x_3 = 2 - t \\ x_4 = 2t \end{cases} \qquad T(\underset{2/5}{t}, 1-3\underset{2/5}{t}, 2-\underset{2/5}{t}, 2\underset{2/5}{t})$$

$r \cap \mathcal{H}: \quad t - 3 + 9t - 2 + t + 4t - 1 = 0 \qquad 15t - 6 = 0 \Rightarrow \bar{t} = \frac{6}{15} = \frac{2}{5}$

$\bar{P}\left(\frac{2}{5}, -\frac{1}{5}, \frac{8}{5}, \frac{4}{5}\right)$

$$d(P,\mathcal{H}) = d(P,\bar{P}) = \|\vec{P\bar{P}}\| = \sqrt{\frac{4}{25} + \frac{36}{25} + \frac{4}{25} + \frac{16}{25}} = \sqrt{\frac{60}{25}} = \frac{\sqrt{60}}{5}$$

$\vec{P\bar{P}}\left(\frac{2}{5}, -\frac{6}{5}, -\frac{2}{5}, \frac{4}{5}\right)$

In generale:

$\mathcal{H}: a_1x_1 + a_2x_2 + \dots + a_nx_n - b = 0 \qquad P(c_1, c_2, \dots, c_n) \qquad (a_1, a_2, \dots, a_n) \neq \underline{0}$

$w(a_1, a_2, \dots, a_n) \qquad {}^{\perp}\vec{\mathcal{H}} = \mathcal{L}(w)$

$$r: \begin{cases} x_1 = c_1 + a_1 t \\ x_2 = c_2 + a_2 t \\ \vdots \\ x_n = c_n + a_n t \end{cases} \qquad r \perp \mathcal{H} \quad \text{e} \quad P \in r$$

$T(c_1+a_1t, c_2+a_2t, \dots, c_n+a_nt)$

$$a_1(c_1+a_1t)+a_2(c_2+a_2t)+\dots+a_n(c_n+a_nt)-b=0$$

$$a_1c_1+a_1^2t+a_2c_2+a_2^2t+\dots+a_nc_n+a_n^2t-b=0$$

$$(a_1^2+a_2^2+\dots+a_n^2)t = b-a_1c_1-a_2c_2-\dots-a_nc_n$$

$$\bar t = \frac{b-a_1c_1-a_2c_2-\dots-a_nc_n}{a_1^2+a_2^2+\dots+a_n^2}$$

$\bar P(c_1+a_1\bar t, c_2+a_2\bar t, \dots, c_n+a_n\bar t)$

$\overrightarrow{P\bar P}(a_1\bar t, a_2\bar t, \dots, a_n\bar t)$

$$\|\overrightarrow{P\bar P}\| = \sqrt{a_1^2\bar t^2+a_2^2\bar t^2+\dots+a_n^2 t^2} = \sqrt{\cancel{(a_1^2+a_2^2+\dots+a_n^2)}\,\frac{(b-a_1c_1-a_2c_2-\dots-a_nc_n)^2}{(a_1^2+a_2^2+\dots+a_n^2)^{\cancel{2}}}} =$$

$$= \frac{|a_1c_1+a_2c_2+\dots+a_nc_n-b|}{\sqrt{a_1^2+a_2^2+\dots+a_n^2}} = d(P,\bar P) = d(P,\mathcal H)$$

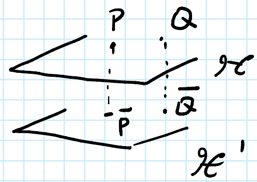
<u>Esempio</u> $n=2$ $\quad R=(O,\mathcal B)$

$\mathcal H: 3x_1+2x_2-4=0$ $\quad P(1,-2)$

$$d(P,\mathcal H) = \frac{|3-4-4|}{\sqrt{9+4}} = \frac{5}{\sqrt{13}}.$$

$\dim \mathcal E = n$ $\quad R=(O,\mathcal B)$

$\underset{n-1}{\mathcal H}, \underset{n-1}{\mathcal H'}$ iperpiani paralleli: $\vec{\mathcal H} = \vec{\mathcal H'}$



$\mathcal H: a_1x_1+a_2x_2+\dots+a_nx_n-b=0$

$\mathcal H': a_1x_1+a_2x_2+\dots+a_nx_n-b'=0$

$\|P\bar P\| \overset{?}{=} \|Q\bar Q\|$

$P(c_1,c_2,\dots,c_n) \in \mathcal H$

$Q(d_1,d_2,\dots,d_n) \in \mathcal H$

$$d(P,\mathcal H') = \frac{|a_1c_1+a_2c_2+\dots+a_nc_n-b'|}{\sqrt{a_1^2+\dots+a_n^2}} = \frac{|b-b'|}{\sqrt{a_1^2+\dots+a_n^2}}$$

$P\in\mathcal H \Rightarrow a_1c_1+a_2x_2+\dots+a_nc_n = b$

$$d(Q,\mathcal H') = \frac{|\overbrace{a_1d_1+\dots+a_nd_n}^{b}-b'|}{\sqrt{a_1^2+\dots+a_n^2}} = \frac{|b-b'|}{\sqrt{a_1^2+\dots+a_n^2}}$$

$d(\mathcal H,\mathcal H') = d(P,\mathcal H')$ $\quad \forall P\in\mathcal H$

$\qquad\qquad\quad \| $

$\qquad\quad d(\mathcal H, R)$ $\quad \forall R\in\mathcal H'$

$d(\mathcal{H},\mathcal{H}') = d(P,\mathcal{H}') \qquad \forall P \in \mathcal{H}$

$\qquad\qquad\quad \shortparallel$

$\qquad\quad d(\mathcal{H},R) \qquad \forall R \in \mathcal{H}'$

Esempio: $n=2$

$r: -x_1 + 3x_2 - 2 = 0 \qquad \vec{r}: -x_1 + 3x_2 = 0 \qquad \vec{r} = \vec{r}\,' \Rightarrow r \,//\, r'$

$r': 2x_1 - 6x_2 + 3 = 0 \qquad \vec{r}\,': 2x_1 - 6x_2 = 0$

$\qquad\qquad\qquad\qquad\qquad r': -x_1 + 3x_2 - \frac{3}{2} = 0$

$P(1,1) \in r$

$$d(r,r') = \frac{|-2+\frac{3}{2}|}{\sqrt{1+9}} = \frac{\frac{1}{2}}{\sqrt{10}} = \frac{1}{2\sqrt{10}}$$

$$\shortparallel \quad \frac{|2-6+3|}{\sqrt{4+36}} = \frac{1}{\sqrt{40}} = \frac{1}{2\sqrt{10}}$$

Esempio $\mathcal{E} = \mathbb{R}^2 \qquad \vec{\mathcal{E}} = \mathbb{R}^2$ col prodotto scalare standard

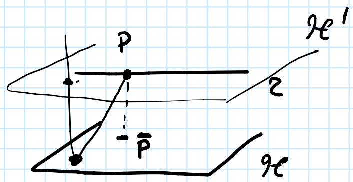
$$\mathcal{R} = \left(O=(2,-3),\ \mathcal{B}=\left(\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right)\right)\right)$$

$u=(0,0) \qquad u \equiv_{\mathcal{R}} \phi_{\mathcal{B}}(\overrightarrow{uO}) = \phi_{\mathcal{B}}(\underline{(2,-3)})$

$(\vec{\mathcal{E}}, \mathcal{E}, \pi) \qquad \dim \mathcal{E} = 3 \qquad \mathcal{R} = (O, \mathcal{B})$

$\mathcal{H}: a_1x_1 + a_2x_2 + a_3x_3 - b = 0$

$$r: \begin{cases} \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 - \beta = 0 \\ \alpha_1'x_1 + \alpha_2'x_2 + \alpha_3'x_3 - \beta' = 0 \end{cases}$$



$r \cap \mathcal{H} \neq \emptyset$, allora $d(r,\mathcal{H}) = 0$

Altrimenti $r \,//\, \mathcal{H}$

$\forall P \in r, \quad d(P,\mathcal{H}) = d(r,\mathcal{H})$

Esercizio: $\mathcal{R} = (O,\mathcal{B}) \qquad \mathcal{H}: 2x_1 - x_2 + 2x_3 - 1 = 0$

(i) determinare una retta r: $r \,//\, \mathcal{H}$ e $P(0,2,1) \in r \qquad (P \notin \mathcal{H})$

(ii) determinare la distanza tra r e $\mathcal{H}$.

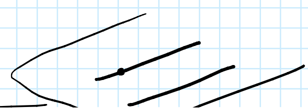
(i) $\vec{\mathcal{H}}: 2x_1 - x_2 + 2x_3 = 0 \qquad u(1,0,-1) \qquad r: \begin{cases} x_1 = t \\ x_2 = 2 \\ x_3 = 1-t \end{cases}$

(ii) $$d(r,\mathcal{H}) = d(P,\mathcal{H}) = \frac{|-2+2-1|}{\sqrt{4+1+4}} = \frac{1}{3}$$

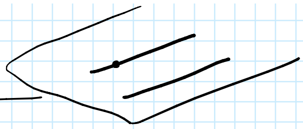
$(\vec{\mathcal{E}}, \mathcal{E}, \pi) \qquad \dim \mathcal{E} = \underline{3} \qquad \mathcal{R} = (O,\mathcal{B})$

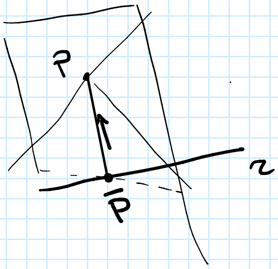
r, r' due rette $\qquad \dim^{\perp} \vec{r} = 2$

- $r \cap r' \neq \emptyset \Rightarrow d(r,r') = 0$
- $r \,//\, r' \Rightarrow d(r,r') = d(P,r') \ \forall P \in r$



- $r // r' \Rightarrow d(r,r') = d(P,r')\ \forall P \in r$





$\mathcal{H}$ piano ortogonale a r e passante per P

$\overline{P} = \mathcal{H} \cap r$

$d(P,r) = d(P,\overline{P})$.

Esempio: $\boxed{n=3}$

$$r: \begin{cases} x_1 = 3 + t \\ x_2 = -2 - t \\ x_3 = 1 + 2t \end{cases} \qquad P(0,0,1) \notin r \qquad u(1,-1,2)$$

r': $\quad r' \perp r$ e $|r' \cap r| = 1$

$\Updownarrow$

$\vec{r'} \subseteq {}^{\perp}\vec{r} \quad \Rightarrow \quad r' \subseteq$ piano ortogonale a r e passante per P

$\mathcal{H}: \ x_1 - x_2 + 2x_3 + k = 0 \qquad\qquad \mathcal{H}: x_1 - x_2 + 2x_3 - 2 = 0$

$0 - 0 + 2 + k = 0 \quad \Rightarrow \quad k = -2$

$\overline{P} = r \cap \mathcal{H} \qquad T(3+t, -2-t, 1+2t) \in \mathcal{H} \Rightarrow$

$\Rightarrow 3 + t + 2 + t + \cancel{2} + 4t - \cancel{2} = 0 \Rightarrow 6t + 5 = 0 \Rightarrow t = -5/6$

$\overline{P}\left(3 - \frac{5}{6}, -2 + \frac{5}{6}, 1 - \frac{10}{6}\right)$

$d(P,r) = d(P,\overline{P})$

- r e r' sghembe

Teorema della comune perpendicolare:

r e r' rette sghembe. Esiste un'unica retta s che è ortogonale sia a r sia a r' ed è incidente sia r sia r':

$\exists!$ retta s: $\quad s \perp r$ e $s \perp r'$

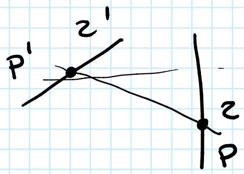
$P = s \cap r \neq \emptyset$ e $P' = s \cap r' \neq \emptyset$.

Inoltre: $d(r,r') = d(P,P')$

<u>DIM</u> $\quad \mathcal{R} = (O, \mathcal{B})$ $\mathcal{B}$ ortonormale

$$r: \begin{cases} x = x_0 + lt \\ y = y_0 + mt \\ z = z_0 + nt \end{cases} \qquad r': \begin{cases} x = x_0' + l't' \\ y = y_0' + m't' \\ z = z_0' + n't' \end{cases}$$

$u(l,m,n) \qquad u'(l',m',n')$



$P(x_0 + lt, y_0 + mt, z_0 + nt) \in r$

$P'(x_0' + l't', y_0' + m't', z_0' + n't') \in r'$

Imponiamo che: $\quad \overrightarrow{PP'} \perp r \iff \langle \overrightarrow{PP'}, u \rangle = 0 \quad \overset{(*)}{\iff}$

e $\quad \overrightarrow{PP'} \perp r' \iff \langle \overrightarrow{PP'}, u' \rangle = 0$

e $\quad PP' \perp r' \iff \langle PP', u' \rangle = 0$

$\overrightarrow{PP'}(x_0' + l't' - x_0 - lt,\ y_0' + m't' - y_0 - mt,\ z_0' + n't' - z_0 - nt)$

$$\overset{(*)}{\iff} \begin{cases} (x_0' + l't' - x_0 - lt)\,l + (y_0' + m't' - y_0 - mt)\,m + (z_0' + n't' - z_0 - nt)\,n = 0 \\ (x_0' + l't' - x_0 - lt)\,l' + (y_0' + m't' - y_0 - mt)\,m' + (z_0' + n't' - z_0 - nt)\,n' = 0 \end{cases}$$

$$\iff \begin{cases} \underbrace{(l'l + m'm + n'n)}_{=\langle u,u'\rangle}\,t' + \underbrace{(-l^2 - m^2 - n^2)}_{=-\|u\|^2}\,t + \dots\dots = 0 \\ \underbrace{(l'^2 + m'^2 + n'^2)}_{=\|u'\|^2}\,t' + \underbrace{(-ll' - mm' - nn')}_{-\langle u,u'\rangle}\,t + \dots\dots = 0 \end{cases}$$

$$\det \underbrace{\begin{pmatrix} \langle u,u' \rangle & -\|u\|^2 \\ \|u'\|^2 & -\langle u,u' \rangle \end{pmatrix}}_{A=} = -\langle u,u' \rangle^2 + \|u\|^2\,\|u'\|^2$$

se $\det(A) \neq 0$, per il Teorema di Cramer $\exists!\,(t',t)$ che soddisfa il sistema, perciò $\exists!\,(P',P)$ che soddisfa il Teorema

se $\det(A) = 0$, allora $\langle u,u' \rangle^2 = \|u\|^2\,\|u'\|^2$, ossia $|\langle u,u' \rangle| = \|u\|\,\|u'\|$

ossia la disuguaglianza di Schwarz vale come uguaglianza

e questo è possibile se e solo se $\{u,u'\}$ è lin. dipendente: $u \parallel u' \rightsquigarrow$ ASSURDO.