Lezione 24

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Esercizi:

- V, $\dim V = 3$ $\quad \mathbb{R}$ $\quad \mathcal{B} = (e_1, e_2, e_3)$ base di V

$T: V \longrightarrow V$ $\quad A = M_{\mathcal{B}}(T) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$|A - \lambda I_3| = \left| \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & -\lambda \end{pmatrix} \right| = (1-\lambda)^2(-\lambda) = 0 \iff \lambda_1 = 1 \quad o \quad \lambda_2 = 0$

$ma(1) = 2, \quad ma(0) = 1$

U_0: $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underline{0}$ $\quad \begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$ $\quad \begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \end{cases}$ $\quad S_0 = \{(-x_3, -x_3, x_3) \mid x_3 \in \mathbb{R}\} =$
$= \mathcal{L}((-1,-1,1)) = \phi_{\mathcal{B}}(U_0)$

$(-x_3, -x_3, x_3) = x_3(-1,-1,1)$

$U_0 = \mathcal{L}(-e_1 - e_2 + e_3) = \operatorname{Ker}(T)$

U_1: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ $\quad \{ x_3 = 0$ $\quad S_0 = \{(x_1, x_2, 0) \mid \overset{1}{x_1}, \overset{0}{\underset{1}{x_2}} \in \mathbb{R}\} =$
$= \mathcal{L}((1,0,0), (0,1,0)) = \phi_{\mathcal{B}}(U_1)$

$(x_1, x_2, 0) = x_1(1,0,0) + x_2(0,1,0)$

autovettori della matrice

$U_1 = \mathcal{L}(e_1, e_2)$

autovettori dell'endomorfismo

$\overline{\mathcal{B}} = \{-e_1 - e_2 + e_3, e_1, e_2\}$ base spettrale

Una matrice che diagonalizza $\quad Q = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = M_{\overline{\mathcal{B}}\mathcal{B}}(id_V)$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = Q^{-1} A Q$$

T è diagonalizzabile, A è diagonalizzabile.

- $T: V \longrightarrow V$ $\quad \mathcal{B} = (e_1, e_2, e_3)$ $\quad \mathbb{R}$

$A = M_{\mathcal{B}}(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ $\quad ?\, u: \phi_{\mathcal{B}}(u) = (0,1,0)$?
$u = 0 \cdot e_1 + 1 \cdot e_2 + 0 \cdot e_3$

$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = \lambda^2(1-\lambda) = 0 \iff \lambda_1 = 0 \quad o \quad \lambda_2 = 1$

$ma(0) = 2 \qquad ma(1) = 1$

U_0: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ $\quad \begin{cases} x_1 = 0 \\ x_3 = 0 \end{cases}$ $\quad S_0 = \{(0, x_2, 0) \mid x_2 \in \mathbb{R}\} = \mathcal{L}((0,1,0))$
$= \phi_{\mathcal{B}}(U_0) \Rightarrow U_0 = \operatorname{Ker}(T) =$
$= \mathcal{L}(e_2)$

U_1: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ $\quad \begin{cases} x_2 = x_3 \\ x_1 = x_3 \end{cases}$ $\quad S_0 = \{(x_3, x_3, x_3) \mid x_3 \in \mathbb{R}\} = \phi_{\mathcal{B}}(U_1)$
$\quad = \mathcal{L}((1,1,1)) \Rightarrow$
$\Rightarrow U_1 = \mathcal{L}(e_1 + e_2 + e_3)$

Non c'è base spettrale.

T e A non sono diagonalizzabili: $\quad \dim(U_0 \oplus U_1) = \dim U_0 + \dim U_1 =$
$= 1 + 1 = 2 \neq 3$

Non c'è base spettrale.

T e A non sono diagonalizzabili: $\dim(U_0 \oplus U_1) = \dim U_0 + \dim U_1 = 1+1 = 2 \neq 3$

- $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ $\quad \mathcal{B} = ((1,0,0),(0,1,0),(0,0,1))$ $\quad \mathbb{R}$

$$A = M_{\mathcal{B}}(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = (1-\lambda)[-\lambda(1-\lambda)+1] =$$

$$= (1-\lambda)[-\lambda+\lambda^2+1] = 0 \Leftrightarrow \lambda_1 = 1 \quad \text{opp} \quad \lambda^2-\lambda+1=0$$

$ma(1) = 1$ $\qquad \lambda = \frac{1\mp\sqrt{1-4}}{2} \in \mathbb{C}$

$$U_1: \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \begin{cases} x_3 = 0 \\ x_1 + x_2 = 0 \end{cases} \qquad S_0 = \{(-x_2, x_2, 0) \mid x_2 \in \mathbb{R}\} =$$

$$= \mathcal{L}((-1,1,0)) = \phi_{\mathcal{B}}(U_1) = U_1$$

è l'identità.

T e A non sono diagonalizzabili.

- $T: V \longrightarrow V$ $\quad \mathcal{B} = (u_1, u_2, u_3)$ $\quad \mathbb{R}$

$$A = M_{\mathcal{B}}(T) = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

$$|A-\lambda I_3| = \begin{vmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 2 & 4-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & 4-\lambda \\ 2 & 2 \end{vmatrix} =$$

$$= (4-\lambda)[(4-\lambda)^2-4] - 2[2(4-\lambda)-4] + 2[4-2(4-\lambda)] =$$

$$= (4-\lambda)[16-8\lambda+\lambda^2-4] - 2[8-2\lambda-4] + 2[4-8+2\lambda] =$$

$$= (4-\lambda)[12-8\lambda+\lambda^2] - 2[4-2\lambda] + 2[-4+2\lambda] =$$

$$= 48 - 32\lambda + 4\lambda^2 - 12\lambda + 8\lambda^2 - \lambda^3 - 8 + 4\lambda - 8 + 4\lambda =$$

$$= -\lambda^3 + 12\lambda^2 - 36\lambda + 32 \quad = 0$$

$1 \quad -1 + 12 - 36 + 32 \neq 0$

$-1 \quad 1 + 12 + 36 + 32 \neq 0$

Teorema di Ruffini

$2 \quad -8 + 48 - 72 + 32 = 0 \Rightarrow (\lambda-2)$ divide il polinomio caratteristico

$$\begin{array}{r|l} -\lambda^3+12\lambda^2-36\lambda+32 & \lambda-2 \\ \hline +\lambda^3-2\lambda^2 & -\lambda^2+10\lambda-16 \\ 10\lambda^2-36\lambda+32 & \\ -10\lambda^2+20\lambda & \\ -16\lambda+32 & \\ +16\lambda-32 & \\ 0 & \end{array}$$

$$|A-\lambda I| = (\lambda-2)(-\lambda^2+10\lambda-16) =$$

$$= (2-\lambda)(\lambda^2-10\lambda+16) =$$

$$\lambda = 5 \mp \sqrt{25-16} = 5 \mp \sqrt{9} = 5 \mp 3 = \begin{cases} \lambda_1 = 8 \\ \lambda_2 = 2 \end{cases}$$

$$= (2-\lambda)(\lambda-2)(\lambda-8) = -(\lambda-2)^2(\lambda-8)$$

$ma(2) = 2, \; ma(8) = 1$

$$U_2: \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \{ x_1 + x_2 + x_3 = 0 \qquad S_0 = \{(-x_2-x_3, x_2, x_3) \mid \overset{1}{x_2}, \overset{0}{x_3}\; \overset{0}{}\overset{1}{} \in \mathbb{R}\} =$$

$$= \mathcal{L}((-1,1,0),(-1,0,1)) = \phi_{\mathcal{B}}(U_2)$$

$U_2: \begin{pmatrix} 2&2&2\\ 2&2&2\\ 2&2&2 \end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}=0 \qquad \{ x_1+x_2+x_3=0 \qquad S_0=\{(-x_2-x_3, x_2, x_3) \mid x_2, x_3\in\mathbb{R}\}=$

$=\mathcal{L}((-1,1,0),(-1,0,1)) = \phi_{\mathcal{B}}(U_2)$

$$U_2=\mathcal{L}(-u_1+u_2, -u_1+u_3)$$

$U_8: \begin{pmatrix} \boxed{-4}&2&2\\ 2&-4&2\\ 2&2&-4 \end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}=0 \qquad\qquad 0=-6x_2+6x_3$

$\underline{e}^1 \to \underline{e}^2 \quad \begin{pmatrix} 2&-4&2\\ -4&2&2\\ 2&2&-4\end{pmatrix} \qquad \begin{matrix}\underline{e}^2\to\underline{e}^2+2\underline{e}^1\\ \underline{e}^3\to\underline{e}^3-\underline{e}^1\end{matrix} \quad \begin{pmatrix} 2&-4&2\\ 0&-6&6\\ 0&6&-6\end{pmatrix} \qquad \underline{e}^3\to\underline{e}^3-\underline{e}^2 \begin{pmatrix} 2&-4&2\\ 0&-6&6\\ 0&0&0\end{pmatrix}$

$\begin{cases} x_1-2x_2+x_3=0\\ x_2=x_3\end{cases} \qquad \begin{cases} x_1-x_2=0\\ x_2=x_3\end{cases} \qquad \begin{cases} x_1=x_2\\ x_2=x_3\end{cases} \qquad S_0=\{(x_2,x_2,x_2)\mid x_2\in\mathbb{R}\}=$

$=\mathcal{L}((1,1,1)) = \phi_{\mathcal{B}}(U_8)$

$\Rightarrow U_8=\mathcal{L}(u_1+u_2+u_3)$

$\overline{\mathcal{B}}=\{-u_1+u_2, -u_1+u_3, u_1+u_2+u_3\}$ base spettrale

$$Q=\begin{pmatrix} -1&-1&1\\ 1&0&1\\ 0&1&1\end{pmatrix} \qquad Q^{-1}AQ=\begin{pmatrix} 2&0&0\\ 0&2&0\\ 0&0&8\end{pmatrix} = M_{\overline{\mathcal{B}}}(T)=\overline{A}$$

$\quad 2 \quad 2 \quad 8 \qquad = M_{\overline{\mathcal{B}}\mathcal{B}}(id_V)$

Ricordiamo: $u\in V$, $\quad u\equiv_{\overline{\mathcal{B}}}(\bar{x}_1,\dots,\bar{x}_n) \qquad u\equiv_{\mathcal{B}}(x_1,\dots,x_n)$

$T(u)\equiv_{\overline{\mathcal{B}}}(\bar{y}_1,\dots,\bar{y}_n) \qquad T(u)\equiv_{\mathcal{B}}(y_1,\dots,y_n)$

$A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix} \qquad \overline{A}\begin{pmatrix}\bar{x}_1\\ \vdots\\ \bar{x}_n\end{pmatrix}=\begin{pmatrix}\bar{y}_1\\ \vdots\\ \bar{y}_n\end{pmatrix}$

$Q\begin{pmatrix}\bar{x}_1\\ \vdots\\ \bar{x}_n\end{pmatrix}=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} \qquad AQ\begin{pmatrix}\bar{x}_1\\ \vdots\\ \bar{x}_n\end{pmatrix}=Q\begin{pmatrix}\bar{y}_1\\ \vdots\\ \bar{y}_n\end{pmatrix} \Rightarrow \qquad \Rightarrow \overline{A}=Q^{-1}AQ.$

$Q\begin{pmatrix}\bar{y}_1\\ \vdots\\ \bar{y}_n\end{pmatrix}=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix} \qquad \Rightarrow Q^{-1}AQ\begin{pmatrix}\bar{x}_1\\ \vdots\\ \bar{x}_n\end{pmatrix}=\begin{pmatrix}\bar{y}_1\\ \vdots\\ \bar{y}_n\end{pmatrix}$

$$E=\begin{pmatrix} 1&-1&-1\\ 1&0&1\\ 1&1&0\end{pmatrix} \qquad E^{-1}AE=\begin{pmatrix} 8&0&0\\ 0&2&0\\ 0&0&2\end{pmatrix}$$

$\quad 8 \quad 2 \quad 2$

- $A_k=\begin{pmatrix} 1&1&0\\ 0&1&k\\ 0&k&1\end{pmatrix} \qquad k\in\mathbb{R} \qquad$ Per quali valori di k la matrice A_k è diagonalizzabile?

$$|A_k-\lambda I_3| = \begin{vmatrix} 1-\lambda&1&0\\ 0&1-\lambda&k\\ 0&k&1-\lambda\end{vmatrix} = (1-\lambda)\left[(1-\lambda)^2-k^2\right] = (1-\lambda)\left[1-2\lambda+\lambda^2-k^2\right]=0 \Leftrightarrow$$

$\Leftrightarrow \lambda=1 \quad$ opp. $\quad \lambda^2-2\lambda+1-k^2=0$

$\lambda = 1\mp\sqrt{1-1+k^2} = 1\mp|k| = 1\mp k$

$\lambda_1=1, \quad \lambda_2=1-k, \quad \lambda_3=1+k$

Se $\underline{k=0}$, $\lambda=1$ $ma(1)=3$

$U_1: \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$ $\dim U_1 = 2 \neq 3$ non è diag.

$1-k = 1+k \iff 1-2k = 1 \iff k=0$

Se $\underline{k\neq 0}$, abbiamo 3 autovalori e due a due distinti
diagonalizzabile.

- $\dim \mathcal{E} = 3$, $R=(O,\mathcal{B})$ ortonormale $\quad r: \begin{cases} x+y-z+2=0 \\ 2x-y+z=0 \end{cases}$

(i) Determinare il piano π ortogonale a r e passante per il punto $A(1,0,1)$.

$\vec{r}: \begin{cases} x+y-z=0 \\ 2x-y+z=0 \end{cases}$ $\begin{pmatrix} \boxed{1} & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$ $\underline{a}^2 \to \underline{a}^2 - 2\underline{a}^1$ $\begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & 3 \end{pmatrix}$ $\begin{cases} x_1 = 0 \\ x_2 = x_3 \end{cases}$

$\vec{r} = \mathcal{L}(u(0,1,1))$

$\pi: x_2 + x_3 + d = 0$ $\quad A \in \pi \Rightarrow 0+1+d=0 \Rightarrow d=-1$

$\pi: x_2 + x_3 - 1 = 0$ $\quad w = u(0,1,1)$

(ii) Determinare due differenti piani ortogonali a π e passanti per $B(1,1,0)$

$\mathcal{H}: ax_1 + bx_2 + cx_3 + d = 0$ $\quad \mathcal{H} \perp \pi \iff (a,b,c)(0,1,1)=0 \iff$
$\iff b+c=0 \iff b=-c$

$(1,0,0)$ $\quad (0,1,-1)$

$\mathcal{H}': x_1 + k' = 0$ $\quad B \in \mathcal{H}' \Rightarrow 1+k'=0 \Rightarrow k'=-1 \Rightarrow \mathcal{H}': x_1 - 1 = 0$

$\mathcal{H}'': x_2 - x_3 + k'' = 0$ $\quad B \in \mathcal{H}'' \Rightarrow 1-0+k''=0 \Rightarrow k''=-1 \Rightarrow \mathcal{H}'': x_2 - x_3 - 1 = 0$

(iii) Determinare il piano parallelo a r, ortogonale a $\pi': -2x+y-z+1=0$
e passante per $C(1,2,-1)$

(iv) Data la retta $s: \begin{cases} x = t \\ y = 1-t \\ z = 3+t \end{cases}$ dire se s e r sono sghembe e se sono ortogonali.