Probability and Stochastic Processes (EL6303) November 12, 2018
NYU Tandon School of Engineering, Fall 2018
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Exercise 7

1. Suppose m horses run in a race, where horse i wins with probability $p(i)$, $i = 1, \ldots, m$. For every dollar you bet on horse i you get $o(i)$ dollars if that horse wins. You divide your total wealth in the horse race according to $b(i), i = 1, \ldots, m$, where $b(i)$ represents the proportion of the money you bet on horse i. Hence $\sum_{i=1}^{m} b(i) = 1$.

   For example, if horse 1 wins, your wealth is multiplied by $b(1)o(1)$. Note that the money you bet on any other horse will be lost.

   We assume that the horses run in n races, and that outcome of race j, denoted by $X_j \in \{1, \ldots, m\}$, is iid according to $\mathbf{p} = (p(1), \ldots, p(m))$. You use the same betting strategy $\mathbf{b} = (b(1), \ldots, b(m))$ at each race to reinvest your wealth and the return $\mathbf{o} = (o(1), \ldots, o(m))$ remains the same for each race. Then your wealth (relative to your initial investment) after n races is

$$S_n = \prod_{j=1}^{n} b(X_j)o(X_j).$$

   (a) Show that

$$\lim_{n\to\infty} \frac{1}{n} \log(S_n) = W(\mathbf{b}),$$

   where

$$W(\mathbf{b}) = \sum_{i=1}^{m} p(i)log(b(i)o(i)),$$

   the convergence is in probability and log is base 2. The term $W(\mathbf{b})$ is called the *doubling rate*. Clearly explain your steps.

   Note that the above result suggests that for large n,

$$S_n \approx 2^{nW(\mathbf{b})}. \tag{1}$$

   (b) Suppose $p(1) > 0$, and you set $b(1) = 0$, while $b(i) > 0$ for $i = 2, 3, \ldots, m$.

   i. Let N be the first race after which you lose all your money. Of course for all $j \geq N, S_j = 0$. Find the probability distribution of N assuming you bet indefinitely ($n \to \infty$).

ii. Find $P(S_n = 0)$. What happens as $n \to \infty$?

iii. Compute $W(\mathbf{b})$. Is your answer from part 1(b)ii consistent with equation (1)? Comment.

(c) For a race with two horses where $p(1) = p$, find the best betting strategy $\mathbf{b} = (b(1), b(2))$ that maximizes your doubling rate $W(\mathbf{b})$. Show your work.

2. Suppose that you send 7-letter tweets. You randomly and independently type one of 26 lowercase letters according to a uniform distribution.

(a) What is the probability that you type "covfefe"?

(b) Now you send multiple iid 7-letter tweets, each according to the distribution above. Let $Y_i = 1$ if your i'th tweet is "covfefe," $Y_i = 0$ otherwise. Find

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n Y_i}{n},$$

where the limit is in probability.

(c) Using your answer to part (b), find an approximate value for n such that after n tweets you expect to see about one "covfefe" tweet.

3. Consider an n-dimensional box with random variable X_i representing length of side i. We assume X_i are i.i.d, $i = 1, \dots, n$. The volume of the box is

$$V_n = \prod_{i=1}^n X_i.$$

We are interested in

$$L_n = V_n^{1/n}$$

for large n. Note that L_n is the side length of an n-dimensional cube with same volume V_n.

(a) Find A such that

$$\lim_{n\to\infty} \log_2(L_n) = A$$

where the convergence is *in probability.* Explain your answer.

Hint: Thinking about weak law of large numbers may be useful.

(b) Find an expression for $[E(V_n)]^{1/n}$ and find B where

$$\lim_{n\to\infty} [E(V_n)]^{1/n} = B.$$

(c) Suppose that the X_i's are Bernoulli(p) with

$$P(X = x) = \begin{cases} (1-p), & x = 1 \\ p, & x = 2 \end{cases}$$

i. Evaluate A from part (a).

ii. Evaluate B from part (b).

iii. Compare 2^A and B. Do you expect them to be equal? Explain.

4. Consider a sequence of random variables $X_1, X_2, \ldots$. We say that the sequence X_n converges to the random variable X *in the r'th mean* if

$$\lim_{n\to\infty} E(|X_n - X|^r) = 0.$$

For $r = 1$, this is also called *convergence in mean.*

For $r = 2$, this is also called *convergence in mean square.*

(a) Prove that if a sequence of random variables X_n converges to X in mean square, then X_n converges to X in mean.

Hint: Consider $Y_n = |X_n - X|$. Think about the variance of Y_n, and what it tells you about the relationship between the first and second moments of Y_n.

(b) Suppose X_n has the following probability distribution:

$$X_n = \begin{cases} \sqrt{n}, & \text{with probability } 1/n \\ 0, & \text{with probability } (1 - 1/n) \end{cases}$$

i. Prove that X_n converges in mean to $X = 0$.

ii. Does X_n converge in mean square to $X = 0$? Explain your answer.

(c) Does convergence in mean imply convergence in mean square? Explain.

5. Consider two sequences of random variables X_n and Y_n, $n = 1, 2, \ldots$ and a random variable X. We are given that

$$P(|X_n - X| \leq Y_n) = 1,$$

for all n. Also $E(Y_n) \to 0$ as $n \to \infty$.

(a) Find $\lim_{n\to\infty} E(|X_n - X|)$. Explain your steps.

(b) Prove that $X_n \to X$ in probability as $n \to \infty$.

6. In this problem, we develop a weak law of large numbers for a correlated sequence $X_1, X_2, ...$ of random variables. In particular, each X_i has expected value $E[X_i] = u$, and the random sequence has covariance function

$$C_X[m,k] = Cov[X_m, X_{m+k}] = \sigma^2 a^{|k|}$$

where a is a constant such that $|a| < 1$. For this correlated random sequence, we can define the sample mean of n samples as

$$M_n = \frac{X_1 + ... + X_n}{n}$$

(a) Show that for general $X_1, X_2, \ldots, X_n$, the variance of $W_n = X_1 + ... + X_n$ is

$$Var[W_n] = \sum_{i=1}^{n} Var[X_i] + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} Cov[X_i, X_j].$$

(b) Use part (a) to show that

$$Var[X_1 + ... + X_n] \leq n\sigma^2(\frac{1+a}{1-a}).$$

(c) Use the Chebyshev inequality to show that for any $c > 0$,

$$P[|M_n - u| \geq c] \leq \frac{\sigma^2(a+1)}{n(1-a)c^2}.$$

(d) Use part (b) to show that for any $c > 0$,

$$\lim_{n\to\infty} P[|M_n - u| \geq c] = 0.$$