Probability and Stochastic Processes (EL6303) September 25, 2018
NYU Tandon School of Engineering, Fall 2018
Instructor: Dr. Elza Erkip

Exercise 1 Solutions

Q1

Let us define the following events:
D: The event that both the items are defective.
A: The selected box is A.
B: The selected box is B.

a) $P(D) = P(D \cap S) = P\big(D \cap (A \cup B)\big) = P(D \cap A) + P(D \cap B) = P(D|A)P(A) + P(D|B)P(B) = \frac{P_A^2 + P_B^2}{2}$

b) $P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{P(D|A)P(A)}{P(D)} = \frac{P_A^2}{P_A^2 + P_B^2}$

Q2

(a) An outcome specifies whether the connection speed is high (h), medium (m), or low (l) speed, and whether the signal is a mouse click (c) or a tweet (t). The sample space is

$$S = \{ht, hc, mt, mc, lt, lc\}. \tag{1}$$

(b) The event that the wi-fi connection is medium speed is $A_1 = \{mt, mc\}$.

(c) The event that a signal is a mouse click is $A_2 = \{hc, mc, lc\}$.

(d) The event that a connection is either high speed or low speed is $A_3 = \{ht, hc, lt, lc\}$.

(e) Since $A_1 \cap A_2 = \{mc\}$ and is not empty, A_1, A_2, and A_3 are not mutually exclusive.

Q3

In the second method, let K = the number of tests for each 10-people group. Then K takes only 1 or 11. For each group, the average number of tests equals:

$$
\begin{aligned}
&1\times P(K=1)+11\times P(K=11)\\
&\quad =P(\text{All 10 people are negative})+11P(\text{Not all 10 people are negative})\\
&\quad =0.9^{10}+11(1-0.9^{10})\approx 7.513
\end{aligned}
$$

Then, for 100 people, the average number of tests equals approximately

$10\times 7.513\approx 75$. That is by second method we can save about 25 tests.

Q4: Remark: The key to this problem is to realize that w_i is deterministic when the weighing strategy is given.

(a) Because of the order is given, we have $w_5 = 1, w_4 = 2, w_3 = 3, w_4 = 4$. Note that $w_5 = 4$ because after 4 weighings if you haven't encountered the heavy box, you don't need another weighing to conclude that the last one is heavy. Therefore,

$$\sum_{i=1}^{5} p_i w_i = 4 * 0.4 + 4 * 0.2 + 3 * 0.15 + 2 * 0.15 + 0.1 = 3.25.$$

(b) Yes. Weight the boxes in the order of decreasing probability, i.e., the order should be 1,2,3,4,5.

$$\sum_{i=1}^{5} p_i w_i = 4 * 0.1 + 4 * 0.15 + 3 * 0.15 + 2 * 0.2 + 0.4 = 2.25.$$

(c) Yes.

- If box 1 is heavy, then you find out after the first weighing, $w_1 = 1$.
- If box 2 is heavy, then the first step tells us that box 1 has regular weight. After the second step, we find that either box 2 or box 3 is the heavy one. We just need another weighing in step 3 to find out that box 2 is the heavy one $w_2 = 3$.
- If box 3 is heavy, similar to the above, we need 3 weighings $w_3 = 3$.
- If box 4 is heavy, we do steps 1,2, and 4, thus $w_4 = 3$.
- If box 5 is heavy, we do steps 1,2 and 4 and realize neither of boxes $\{1, 2, 3, 4\}$ are heavy. So box 5 must be the heavy one, thus $w_5 = 3$.

$$\sum_{i=1}^{5} p_i w_i = 1 * 0.4 + 3 * 0.2 + 3 * 0.15 + 3 * 0.15 + 3 * 0.1 = 2.2.$$

Q5

(a)

$$
\begin{aligned}
P(\text{flu}|\text{negative}) &= \frac{P(\text{negative}|\text{flu})P(\text{flu})}{P(\text{negative})} \\
&= \frac{P(\text{negative}|\text{flu})P(\text{flu})}{P(\text{negative}|\text{flu})P(\text{flu}) + P(\text{negative}|\text{Noflu})P(\text{Noflu})} \\
&= \frac{(1-q)f}{(1-q)f + 1(1-f)} = \frac{(1-q)f}{1-fq}.
\end{aligned}
$$

(b)

(i)

$$
\begin{aligned}
P(\text{flu}|\text{both negative}) &= \frac{P(\text{both negative, flu})}{P(\text{both negative})} \\
&= \frac{P(\text{first test negative, second test negative}|\text{flu})P(\text{flu})}{P(\text{both negative})}
\end{aligned}
$$

Use the conditionally independent condition:

$$
\begin{aligned}
&P(\text{first test negative, second test negative}|\text{flu}) \\
&= P(\text{first test negative}|\text{flu})P(\text{second test negative}|\text{flu}) = (1-q)^2.
\end{aligned}
$$

The probability of both tests are negative via the total law of probability is

$$
\begin{aligned}
&P(\text{both negative}) = P(\text{both negative}|\text{flu})P(\text{flu}) \\
&+ P(\text{both negative}|\text{Noflu})P(\text{Noflu}) = (1-q)^2 f + (1-f)
\end{aligned}
$$

Thus, the final result is

$$
P(\text{flu}|\text{both negative}) = \frac{(1-q)^2 f}{(1-q)^2 f + (1-f)}.
$$

(ii) The only way you could have a positive test is if you have the flu. Hence you have the flu with probability 1.

(c)

(i) Since the test is positive only if you have the flu, the strategy is to repeat the test until you get a positive or you reach the n test limit.

(ii) You are not sure whether you have the flu if all n tests are negative. The probability that you have flu and all n tests are negative is $(1-q)^n$. The probability that you do not have flu and all n tests are negative is 1. Hence by law of total probability

$$P(\text{not sure}) = P(\text{not sure}|\text{flu})P(\text{flu}) + P(\text{not sure}|\text{no flu})P(\text{no flu})$$
$$= (1-q)^n f + 1(1-f)$$

Q6

T_0 ——"0" is transmitted, T_1 ——"1" is transmitted,

R_0 ——"0" is received, R_1 ——"1" is received.

$$P(R_0) = P(R_0 \mid T_0)P(T_0) + P(R_0 \mid T_1)P(T_1) = (0.7)(0.6) + (0.2)(0.4) = 0.5$$
$$P(R_1) = P(R_1 \mid T_0)P(T_0) + P(R_1 \mid T_1)P(T_1) = (0.3)(0.6) + (0.8)(0.4) = 0.5$$

Or using matrix: $(0.6,\ 0.4)\begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} = (0.42+0.08,\ 0.18+0.32) = (0.5,\ 0.5)$

Using the total probability, we get
$$P(\text{error}) = P(R_1 \mid T_0)P(T_0) + P(R_0 \mid T_1)P(T_1) = 0.3\times 0.6 + 0.2\times 0.4 = 0.26$$

Q7

a)

$$\begin{aligned}\mathbb{P}[\text{0 is received}] =& \mathbb{P}[\text{0 is transmitted, 0 is received}] + \mathbb{P}[\text{1 is transmitted, 0 is received}] \\ =& \mathbb{P}[\text{0 is transmitted}] \cdot \mathbb{P}[\text{0 is received} \mid \text{0 is transmitted }] + \\ & \mathbb{P}[\text{1 is transmitted}] \cdot \mathbb{P}[\text{0 is received} \mid \text{1 is transmitted}] \\ =& q \cdot 1 + (1-q) \cdot p = q + p(1-q)\end{aligned}$$

b)

$$\begin{aligned}\mathbb{P}[\text{1 is received}] =& \mathbb{P}[\text{0 is transmitted, 1 is received}] + \mathbb{P}[\text{1 is transmitted, 1 is received}] \\ =& \mathbb{P}[\text{0 is transmitted}] \cdot \mathbb{P}[\text{1 is received} \mid \text{0 is transmitted }] + \\ & \mathbb{P}[\text{1 is transmitted}] \cdot \mathbb{P}[\text{1 is received} \mid \text{1 is transmitted}] \\ =& q \cdot 0 + (1-q) \cdot (1-p) = (1-q)(1-p)\end{aligned}$$

c)

$$\begin{aligned}\mathbb{P}[\text{error}] =& \mathbb{P}[\text{0 is transmitted, 1 is received}] + \mathbb{P}[\text{1 is transmitted, 0 is received}] \\ =& \mathbb{P}[\text{0 is transmitted}] \cdot \mathbb{P}[\text{1 is received} \mid \text{0 is transmitted }] + \\ & \mathbb{P}[\text{1 is transmitted}] \cdot \mathbb{P}[\text{0 is received} \mid \text{1 is transmitted}] \\ =& q \cdot 0 + (1-q)p = (1-q)p\end{aligned}$$

d)

$$\begin{aligned}\mathbb{P}[\text{1 is transmitted} \mid \text{0 is received}] =& \frac{\mathbb{P}[\text{1 is transmitted,0 is received}]}{\mathbb{P}[\text{0 is received}]} \\ =& \frac{p(1-q)}{q+p(1-q)} = \frac{0.8p}{0.2+0.8p}\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}[\text{0 is transmitted} \mid \text{0 is received}] =& \frac{\mathbb{P}[\text{0 is transmitted,0 is received}]}{\mathbb{P}[\text{0 is received}]} \\ =& \frac{q}{q+p(1-q)} = \frac{0.2}{0.2+0.8p}\end{aligned}$$

note the following

$$\mathbb{P}[\text{1 is transmitted} \mid \text{0 is received}] > \mathbb{P}[\text{0 is transmitted} \mid \text{0 is received}]$$

only when $p > 0.25$. Therefore the best estimate (BE) given 0 is received when $q = 0.2$ is

$$BE(p) = \begin{cases} 1 & \text{if } p > 0.25, \\ 0 & \text{else} \end{cases}$$