EL 6303, Probability and Stochastic Processes **Name:** ____________________

Fall 2016

Section A1, A2

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Final

Time Limit: 3 hours

Total: 100 points

- Closed book/closed notes/no electronics. Four formula sheets allowed.
- Please show your work in the **blue books**.
- Include your name and student ID on both this sheet and the blue books. Return both at the end of the exam.
- Good luck!

1. *(Total: 20 pts)* Consider a communication system where the transmitted information is represented as a random variable X with

$$P(X = 1) = P(X = -1) = 1/2.$$

The receiver observes Y such that

$$Y = X + Z,$$

where $Z \sim N(0, \sigma^2)$ is the additive Gaussian noise. We assume X and Z are independent.

(a) *(4 pts)* Find $f(y|x)$, the conditional pdf of Y given X for $x = 1$ and $x = -1$.

(b) *(4 pts)* Find $f(y)$, the pdf of Y.

(c) *(6 pts)* Given $Y = 0.5$, which input value has higher probability: $x = 1$ or $x = -1$? Prove your result.

Hint: For $y = 0.5$, compare $p(x|y)$, conditional pmf of X given Y, for $x = 1$ and $x = -1$. You do not need to calculate $p(x|y)$ exactly to do the comparison.

(d) *(6 pts)* Suppose receiver estimates the transmitted signal as $\hat{X}$ using the following rule

$$\hat{X} = \begin{cases} 1, & \text{if } Y \geq 0 \\ \text{-1}, & \text{if } Y < 0 \end{cases}$$

Find $P(\hat{X} \neq X \mid X = -1)$. Your answer should be in terms of the Q-function

$$Q(u) = \int_u^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz.$$

2. *(Total: 15 pts)* Consider $Q \sim \text{Unif}(0,1)$ and binary random variables $X_1, \dots X_n$ such that

$$P(X_i = j|Q = q) = \begin{cases} q, & j = 1 \\ 1 - q, & j = 0 \end{cases}$$

for $i = 1, \dots, n$. Also,

$$p(x_1, \dots x_n|q) = \prod_{i=1}^{n} p(x_i|q),$$

Hence X_i are conditionally i.i.d given Q.

(a) *(3 pts)* Find $P(X_n = 1)$.

(b) *(5 pts)* Find the joint pmf of $(X_1, \dots, X_n)$.

(c) *(7 pts)* Find $P(X_n = 1|X_{n-1} = x_{n-1}, \dots, X_1 = x_1)$.

Hint: You can use $\int_0^1 u^k(1-u)^{n-k}du = \frac{k!(n-k)!}{(n+1)!}$.

3. *(Total: 20 pts)* Consider a stochastic system which scales its input by $A(t)$, where $A(t)$ is a stochastic process. Hence

$$Y(t) = A(t)X(t),$$

where $X(t)$ is the system input, $Y(t)$ is the output. We assume $A(t)$ has mean $\mu_A(t)$ and autocorrelation $R_{AA}(t_1, t_2)$.

   (a) *(6 pts)* Suppose $X(t)$ is a deterministic signal.

      i. Find the mean of $Y(t)$.

      ii. Find the autocorrelation of $Y(t)$.

      iii. If $A(t)$ is WSS, is $Y(t)$ also WSS? Explain.

   (b) *(14 pts)* Suppose $X(t)$ is a stochastic process with mean $\mu_X(t)$ and autocorrelation $R_{XX}(t_1, t_2)$. We assume $X(t)$ and $A(t)$ are independent.

      i. Find the mean of $Y(t)$.

      ii. Find the autocorrelation of $Y(t)$.

      iii. Find the cross correlation $R_{XY}(t_1, t_2)$ between the input and the output.

      iv. If both $A(t)$ and $X(t)$ are WSS, is $Y(t)$ also WSS? Explain.

      v. If both $A(t)$ and $X(t)$ are WSS, are $X(t)$ and $Y(t)$ jointly WSS? Explain.

      vi. Suppose both $A(t)$ and $X(t)$ were Gaussian processes. Would $Y(t)$ be also Gaussian? Explain.

      vii. Suppose $X(t)$ is a white noise process. Would $Y(t)$ be also white noise? Explain.

4. *(Total: 15 pts)* Energy harvesting technology allows for storage of energy derived from external sources (e.g., solar power, thermal energy, ...) into batteries. Consider a battery that is used for completing tasks, and is charged by harvesting energy from the environment. The battery performs as follows:

   - At each unit of time, a task is requested with probability q and the battery consumes 1 Joule in order to complete the requested task. If the battery is empty, no task can be done, even if it is requested.
   - At each unit of time, 1 Joule of energy is harvested and stored in the battery with probability p, independently from whether a task is requested or not.

   The battery has a capacity of 2 Joules, so it cannot store more than 2 Joules, even if it harvests energy. The process $X_1, X_2, \ldots$ represents the total energy stored in the battery (in Joules) as a function of time. This process can be represented as a Markov chain with states $\{0, 1, 2\}$.

   (a) *(5 pts)* Draw the state transition diagram of this Markov chain.

   (b) *(2 pts)* Find the probability transition matrix $\mathbf{P}$.

   (c) *(6 pts)* Suppose $P(X_1 = 1) = 0.3, P(X_1 = 2) = 0.7$.

      i. What is the probability that the battery will be full at time $n = 2$, that is $P(X_2 = 2)$?

      ii. What is the probability that the battery will be empty at time $n = 2$?

      iii. Find $P(X_3 = 1)$.

   (d) *(2 pts)* Find $P(X_{i+2} = 0 | X_i = 2)$, that is the probability that the battery goes from full to empty in two time slots.

5. *(20 pts)* We consider a stochastic process $N(t)$ that counts the number of particles arriving at a Geiger counter. Suppose Λ is an exponential random variable, namely $f_\Lambda(\lambda) = \alpha e^{-\alpha\lambda}$ for $\lambda > 0$. Conditional on $\Lambda = \lambda$, $N(t)$ is a Poisson process with rate λ.

   (a) *(3 pts)* Find $P(N(t) = n|\Lambda = \lambda)$ for some $t > 0$.

   (b) *(6 pts)* Using the conditional probability you found in (a), find the marginal distribution of $N(t)$, $P(N(t) = n)$.
   Hint: $\int_0^\infty e^{-kx}x^n \,\mathrm{d}x = \frac{n!}{k^{n+1}}$

   (c) *(6 pts)* Let T_1 denote the time of the arrival of the first particle. Find $F_{T_1}(t_1)$, the cdf of T_1.

   (d) *(5 pts)* Is $N(t)$ an independent increment process? Prove your result.

6. *(Total: 10 pts)* Suppose $W(t)$ represents a Wiener process with parameter σ^2. Recall properties of the Wiener process (Brownian motion):

   I. $W(0) = 0$.

   II. $W(t_1) - W(t_2) \sim N(0, \sigma^2(t_1 - t_2)), t_1 > t_2$.

   III. $W(t)$ has independent increments, that is if (t_1, t_2) and (t_3, t_4) are non-overlapping intervals, $W(t_1) - W(t_2)$ and $W(t_3) - W(t_4)$ are independent.

   Let $X(t) = \frac{1}{\sqrt{c}} W(ct)$ for some constant $c > 0$. We will show that $X(t)$ is also a Wiener process.

   (a) *(1 pts)* Show that $X(t)$ satisfies property I.

   (b) *(3 pts)* Show that $X(t)$ satisfies property II.

   (c) *(4 pts)* Show that $X(t)$ satisfies property III.

   (d) *(2 pts)* Find the parameter of this Wiener process.

   Note: This is called the *self-similarity* property of the Wiener process.