Probability and Stochastic Processes (EL6303) November 6, 2018
NYU Tandon School of Engineering, Fall 2018
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Exercise 6 Solutions

1

(a)

$$E\{Y\} = E\{\prod_{i=1}^{n} X_i\} = E^n\{X\} = a^n$$

(b)

(i)

$$P(N=1) = p$$
$$P(N=2) = p(1-p)$$
$$P(N=n) = p(1-p)^{n-1} \rightarrow \text{geometric}$$

(ii)

$$E\{Z|N=n\} = a^n$$
$$E\{E\{Z|N\}\} = \sum_{n=1}^{\infty} p(1-p)^{n-1} a^n = ap \sum_{n=0}^{\infty} ((1-p)a)^n$$
$$= \frac{ap}{1-a+ap}$$

(iii)

$\sum_{n=0}^{\infty} ((1-p)a)^n$ is finite when $a < \frac{1}{1-p}$

2

(a)

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{for } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$E(X) = \int x\, f_X(x)\, dx = \int_{-1}^{1} x \cdot \frac{1}{2}\, dx = \underline{\underline{0}}$$

$$E(Y) = \int x^2 f_X(x)\, dx = \int_{-1}^{1} x^2 \frac{1}{2}\, dx = \left[\frac{x^3}{6}\right]_{-1}^{1} = \underline{\underline{\frac{1}{3}}}$$

(b)

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \int_{-1}^{1} x^3 \cdot \frac{1}{2} dx - 0 \cdot \frac{1}{3} = 0$$

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\ \mathrm{Var}(Y)}} = \underline{\underline{0}}.$$

(c)

$\underline{\underline{\text{No}}}$, they are not independent since Y is a function of X. BUT they are uncorrelated since the correlation coefficient is $\underline{\underline{0}}$.

(d)

$$\text{MMSE estimator is } h(X) = E[Y|X] = E[X^2|X] = X^2$$

$$\text{Then } e_1 = E\left[(Y - h(X))^2\right] = E\left[(X^2 - X^2)^2\right] = 0.$$

(e)

LMMSE estimator $h(X) = aX + b$ such that

$$a = \frac{Cov(X,Y)}{Var(X)} = 0, \quad b = E(Y) - aE(X) = E(Y) = \frac{1}{3}.$$

$$\Rightarrow h(X) = 1/3.$$

$$e_2 = E\left[\left(Y - \tfrac{1}{3}\right)^2\right] = \int_{-1}^{1} \left(x^2 - \tfrac{1}{3}\right)^2 \cdot \frac{1}{2}\, dx = \frac{4}{45}$$

3

(a)

$$
\begin{aligned}
Y_1 = aX_1 + bX_2 \Rightarrow \mathrm{Var}(Y_1) &= E\left((Y_1 - E(Y_1))^2\right) \\
&= E\left(Y_1^2\right) \\
&= a^2E(X_1^2) + b^2E(X_1^2) + 2abE(X_1X_2) \\
&= a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2
\end{aligned}
$$

(b)

$$
\begin{aligned}
Y_2 = cX_1 + dX_2 \Rightarrow E(Y_1Y_2) &= acE(X_1^2) + (bc+ad)E(X_1X_2) + bdE(X_2^2) \\
&= ac\sigma_1^2 + (bc+ad)\rho\sigma_1\sigma_2 + bd\sigma_2^2
\end{aligned}
$$

(c)

We have to find d such that $E(Y_1Y_2) = 0$. Using $\sigma_1^2 = \sigma_2^2 = \sigma^2$ and $a = b = c = 1$, we have the following:

$$
\begin{aligned}
E(Y_1Y_2) = \sigma^2 + (1+d)\rho\sigma^2 + d\sigma^2 = \sigma^2[(1+d)(1+\rho)] &= 0 \\
\Rightarrow 1 + d &= 0 \\
\Rightarrow d &= -1.
\end{aligned}
$$

(d)

$\rho = 0$ means that X_1 and X_2 are uncorrelated. Moreover, since they are jointly Gaussian, they are also independent. Sum of three independent Gaussian random variables is another Gaussian random variable, with the mean and variance computed as follows:

$$
\begin{aligned}
Y_1 = aX_1 + bX_2 + Z_1 \Rightarrow & E(Y_1) = E(aX_1 + bX_2 + Z_1) = 0, \\
\Rightarrow & \mathrm{Var}(Y_1) = a^2\sigma_1^2 + b^2\sigma_2^2 + 1.
\end{aligned}
$$

Therefore, Y_1 is $N(0, a^2\sigma_1^2 + b^2\sigma_2^2 + 1)$ with the following pdf:

$$
f_{Y_1}(y) = \frac{1}{\sqrt{2\pi(a^2\sigma_1^2 + b^2\sigma_2^2 + 1)}} e^{-\frac{y^2}{2\left(a^2\sigma_1^2 + b^2\sigma_2^2 + 1\right)}}
$$

4

(a)

$$T = \begin{cases} T_1 & \text{with probability } P_1 \\ T_2 & \text{with probability } P_2 \\ T_3 & \text{with probability } P_3 \end{cases}$$

$$\begin{aligned} E[T] &= E[E[T|i]] \\ &= p_1 E[T|i=1] + p_2 E[T|i=2] + p_3 E[T|i=3] \\ &= p_1 E[T_1] + p_2 E[T_2] + p_3 E[T_3] \end{aligned}$$

(b)

$$T_1 \sim Uniform(0,2) \longrightarrow E[T_1] = 1$$
$$T_2 \sim Uniform(0,3) \longrightarrow E[T_2] = 1.5$$
$$T_3 \sim Uniform(0,4) \longrightarrow E[T_3] = 2$$

$$E[T] = 0.2 \times 1 + 0.3 \times 1.5 + 0.5 \times 2 = 1.65$$

5. (a)

$$\begin{aligned}
P(X_1 = i) &= \int P(X_1 = i|\theta)P(\theta)d\theta \\
&= \int_0^1 (1-\theta)^{i-1}(\theta)d\theta \\
&= \frac{(i-1)!}{(i+1)!} \\
&= \frac{1}{i(i+1)}
\end{aligned}$$

(b)

$$\begin{aligned}
P(X_1 = i, X_2 = j) &= \int P(X_1 = i, X_2 = j|\theta)P(\theta)d\theta \\
&= \int_0^1 (1-\theta)^{i-1}\ \theta\ (1-\theta)^{j-1}\ \theta d\theta \\
&= \frac{2!(i+j-2)!}{(i+j+1)!} \\
&= \frac{2}{(i+j-1)(i+j)(i+j+1)}
\end{aligned}$$

Thus, X_1 and X_2 are not independent.

(c)

$$\begin{aligned}
P(X_1, X_2 \ldots X_n|\theta) &= \Pi_{i=1}^{n} P(X_i|\theta) \\
&= (1-\theta)^{(\sum x_i)-n}\ \theta^n
\end{aligned}$$

(d) $\max(P(X_1, X_2 \ldots X_n|\theta) = \max((\sum x_i) - n))\,(\log(1-\theta)) + n\log(\theta)$
Differentiating with respect to θ,
$-\frac{\sum x_i - n}{1-\theta} + \frac{n}{\theta} = 0$
$\hat{\theta} = \frac{n}{\sum x_i}$

(e) $P(\theta|X_1, X_2 \ldots X_n) = \dfrac{P(X_1, X_2 \ldots X_n|\theta)P(\theta)}{P(X_1, X_2 \ldots X_n)}$

Since $P(\theta)$ is equal to 1 (a constant) and $P(X_1, X_2 \ldots X_n)$ does not depend on θ, the MAP estimate is equal to the ML estimate

6

(a) First, we calculate the marginal PDF for $0 \le y \le 1$:

$$f_Y(y) = \int_0^y 2(y+x)\,dx = 2xy + x^2\Big|_{x=0}^{x=y} = 3y^2. \tag{1}$$

This implies the conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{2}{3y} + \frac{2x}{3y^2} & 0 \le x \le y, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) The minimum mean square error estimate of X given $Y = y$ is

$$\hat{x}_M(y) = \mathrm{E}\left[X|Y=y\right] = \int_0^y \left(\frac{2x}{3y} + \frac{2x^2}{3y^2}\right) dx = 5y/9. \tag{3}$$

Thus the MMSE estimator of X given Y is $\hat{X}_M(Y) = 5Y/9$.

(c) To obtain the conditional PDF $f_{Y|X}(y|x)$, we need the marginal PDF $f_X(x)$. For $0 \le x \le 1$,

$$f_X(x) = \int_x^1 2(y+x)\,dy = y^2 + 2xy\Big|_{y=x}^{y=1} = 1 + 2x - 3x^2. \tag{4}$$

For $0 \le x \le 1$, the conditional PDF of Y given X is

$$f_{Y|X}(y|x) = \begin{cases} \frac{2(y+x)}{1+2x-3x^2} & x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(d) The MMSE estimate of Y given $X = x$ is

$$\begin{aligned} \hat{y}_M(x) = \mathrm{E}\left[Y|X=x\right] &= \int_x^1 \frac{2y^2 + 2xy}{1+2x-3x^2}\,dy \\ &= \frac{2y^3/3 + xy^2}{1+2x-3x^2}\Bigg|_{y=x}^{y=1} \\ &= \frac{2+3x-5x^3}{3+6x-9x^2}. \end{aligned}$$