Probability and Stochastic Processes (EL6303) December 16, 2015
NYU Tandon School of Engineering, Fall 2015
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Final

Closed book, closed notes, no electronics, no calculators.
Only four formula sheets are allowed. Use the provided space to wrote your answers.

1. *(10 points)* Answer the following TRUE or FALSE. X, Y denote random variables, X_n is a sequence of random variables, $X(t)$ denotes a stochastic process.

   ** Circle the correct answer. Briefly explain.**

   (a) X and Y are independent $\Rightarrow$ X and Y are uncorrelated.

   TRUE FALSE

   (b) X and Y are uncorrelated $\Rightarrow$ X and Y are independent.

   TRUE FALSE

   (c) $X_n \to X$ in probability as $n \to \infty \Rightarrow X_n \to X$ with probability 1 as $n \to \infty$.

   TRUE FALSE

   (d) $X_n \to X$ with probability 1 as $n \to \infty \Rightarrow X_n \to X$ in probability as $n \to \infty$.

   TRUE FALSE

   (e) $X(t)$ is wide sense stationary $\Rightarrow$ $X(t)$ is strict sense stationary.

   TRUE FALSE

   (f) $X(t)$ is strict sense stationary and has second moment $\Rightarrow$ $X(t)$ is wide sense stationary.

   TRUE FALSE

   (g) $X(t)$ is mean-ergodic $\Rightarrow$ $X(t)$ is wide sense stationary.

   TRUE FALSE

   (h) $X(t)$ is wide sense stationary $\Rightarrow$ $X(t)$ is mean-ergodic.

   TRUE FALSE

   (i) $X(t)$ is wide sense stationary $\Rightarrow$ $X^3(t)$ is wide-sense stationary.

   TRUE FALSE

   (j) $X(t), Y(t)$ jointly wide sense stationary $\Rightarrow$ $X(t) + 5Y(t)$ is wide sense stationary.

   TRUE FALSE

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2. *(15 points)* You are running late for EL 6303 final and decide to either take a taxi or Lyft. The waiting time for taxi is a random variable T_T, the waiting time for Lyft is a random variable T_L. We assume

$$T_T \sim \text{Uniform}(0, 10), T_L \sim \text{Uniform}(0, 5),$$

We also assume that once the taxi or Lyft car arrive, it takes them the same amount of time to get to school.

The price of the taxi is fixed at $P_T = 15$ dollars. Lyft price P_L, on the other hand, is a random variable. We assume P_L is independent of T_L and that

$$P_L \sim \text{Uniform}(14, 26).$$

You are concerned about getting to school as early as possible, but you also don't want to pay too much. Your *overall cost* is a weighted combination of these two objectives. Hence the overall cost for i is

$$C_i = aT_i + bP_i,$$

where $a, b \geq 0$ are constants and $i = T, L$ denotes taxi or Lyft.

(a) Suppose $a = 4, b = 1$. *Find and sketch* the probability density function of C_T.

(b) For $a = 4, b = 1$, for which mode of transportation (taxi or Lyft) your *average* total $E(C_i)$ is smaller? Explain.

(c) If you didn't care about the time, but only wanted to minimize the price, how would you set a and b? Which one would you choose, taxi or Lyft? Explain.

(d) If you didn't care about the price, but only wanted to minimize your time, how would you set a and b? Which one would you choose, taxi or Lyft? Explain.

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3. *(18 points)* Suppose that $N(t)$, the number of passengers arriving at Jay Street-Metrotech station to take the A train, can be modeled as a Poisson process with parameter β. That is

$$P(N(t) = n) = \frac{(\beta t)^n}{n!} e^{-\beta t}, t \geq 0, n = 0, 1, \ldots.$$

The waiting time T_A for the A train is independent of $N(t)$ and has an exponential distribution with parameter λ_A. That is the density of T_A is given by

$$f_{T_A}(t) = \lambda_A e^{-\lambda_A t}, t \geq 0.$$

Let $Y = N(T_A)$ be number of passengers getting on the A train from Jay Street-Metrotech station.

(a) Find the expected value of Y in terms of β and λ_A.

(b) Find the variance of Y in terms of β and λ_A.

(c) Now suppose that all the arriving passengers can either take the A train or the C train, whichever comes first. The waiting time T_C for the C train is independent of $N(t)$ and T_A, and is exponential with parameter λ_C. Let Z be the number of passengers getting on the first arriving train from Jay Street-Metrotech station.

   i. Find $E(Z)$.

   ii. Find the relationship between λ_A and λ_C to have $E(Z) = E(Y)/3$.

4. *(12 points)* Consider two sequences of random variables X_n and Y_n, $n = 1, 2, \ldots$ and a random variable X. We are given that

$$P(|X_n - X| \leq Y_n) = 1,$$

for all n. Also $E(Y_n) \to 0$ as $n \to \infty$.

(a) Find $\lim_{n\to\infty} E(|X_n - X|)$. Explain your steps.

(b) Prove that $X_n \to X$ in probability as $n \to \infty$.

5. *(15 points)* Consider the stochastic process $X(t) = A\cos(\pi t) + B\sin(\pi t)$. Here A and B are independent random variables such that

$$A = \begin{cases} -1, & \text{with probability } 3/4 \\ 3, & \text{with probability } 1/4 \end{cases}$$

$$B = \begin{cases} 1, & \text{with probability } 3/4 \\ -3, & \text{with probability } 1/4 \end{cases}$$

(a) Find $E(X(t))$.

(b) Find $R_{XX}(t_1, t_2)$.

(c) Is $X(t)$ WSS? Explain.

(d) Find the distribution of $X(0)$.

(e) Find the distribution of $X(0.25)$.

(f) Is $X(t)$ SSS? Explain.

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6. *(15 points)* Consider a zero mean WSS stochastic process $X(t)$ with power spectral density

$$S_{XX}(w) = \begin{cases} N_0/2, & -B \leq w \leq B \\ 0, & \text{else} \end{cases}$$

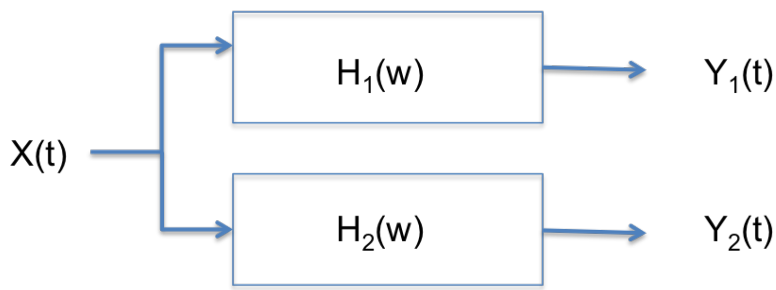
We consider two LTI systems with transfer functions $\mathcal{F}(h_1(t)) = H_1(w)$ and $\mathcal{F}(h_2(t)) = H_2(w)$ such that

$$H_1(w) = \begin{cases} K_1, & w \in (-0.1B, 0.1B) \cup (-B, -0.9B) \cup (0.9B, B) \\ 0, & \text{else} \end{cases}$$

$$H_2(w) = \begin{cases} -K_2, & w \in (-a, a) \cup (-0.8B, -0.7B) \cup (0.7B, 0.8B) \\ 0, & \text{else} \end{cases}$$

where K_1, $K_2 > 0$ and $0 \leq a \leq 0.1B$. Here $h_i(t)$ is the impulse response of system i and $\mathcal{F}$ denotes Fourier transform.

Now consider $Y_1(t)$ and $Y_2(t)$ below:



(a) Find $E(Y_1^2(3))$.

(b) Find $E(Y_2^2(-5))$.

(c) Find an expression for the cross-correlation function of $Y_1(t)$ and $Y_2(t)$, $R_{Y_1,Y_2}(t_1, t_2) = E(Y_1(t_1)Y_2(t_2))$.

(d) For $B = 100$ rad/sec, find the largest set of a's for which $Y_1(t)$ and $Y_2(t + \pi)$ are uncorrelated.

7. *(15 points)* Suppose $N(t)$ is a WSS Gaussian white noise stochastic process with $E(N(t)) = 0$ and $R_{NN}(\tau) = q\delta(\tau)$. Consider two (deterministic) pulses $p_1(t)$ and $p_2(t)$ such that

- $p_i(t) \neq 0$ only when $0 \leq t \leq T$, $i = 1, 2$.
- $\int_0^T p_i^2(t)dt = 1$, $i = 1, 2$.
- $\int_0^T p_1(t)p_2(t)dt = 0$, that is $p_1(t)$ and $p_2(t)$ are orthogonal.

Consider

$$N_i(t) = \int_0^t N(u)p_i(u)du, i = 1, 2, 0 \leq t \leq T.$$

(a) Find $E(N_1(t))$.

(b) Find $E(N_1^2(t))$.

(c) Find $E(N_1(t)N_2(t))$.

(d) Now we set $t = T$. Find the joint distribution of $N_1(T)$ and $N_2(T)$.

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