Probability and Stochastic Processes (EL6303) December 7, 2018
NYU Tandon School of Engineering, Fall 2018
Instructor: *Dr. Elza Erkip*

Name: ______________________________
Net ID: ______________________________

Quiz 4 Solution
Please write your name and net-id eg: Ojas Kanhere, ok671
Do NOT write your N# number
Closed book/closed notes. No electronics, no calculators.
Total 20 points
Time: 40 minutes

1. *(10 points)* Let $U(t)$ and $V(t)$ be independent, WSS random processes with zero means and the same autocorrelation function $R(\tau) = R_U(\tau) = R_V(\tau)$. Let $Z(t)$ be the random process defined by $Z(t) = U(t)\cos t + V(t)\sin t$.

   (a) Find the mean of $Z(t)$, $E(Z(t))$.

   (b) Find the autocorrelation function of $Z(t)$, $R_Z(t, t+\tau) = E(Z(t)Z(t+\tau))$.

   (c) Is $Z(t)$ is a WSS random process? Explain.

   (d) Suppose $U(t)$ and $V(t)$ are independent SSS random processes. Is $Z(t)$ a SSS random process? Explain.

   Hint:

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

Solution of Q1:

(a)

$$E[Z(t)] = E[U(t)\cos t + V(t)\sin t] = E[U(t)]\cos t + E[V(t)]\sin t = 0.$$

(b)

$$\begin{aligned}
E[Z(t)Z(t+\tau)] &= E[(U(t)\cos t + V(t)\sin t)(U(t+\tau)\cos(t+\tau) + V(t+\tau)\sin(t+\tau))] \\
&= E[U(t)U(t+\tau)]\cos t\cos(t+\tau) + E[U(t)V(t+\tau)]\cos t\sin(t+\tau) \\
&+ E[V(t)U(t+\tau)]\sin t\cos(t+\tau) + E[V(t)V(t+\tau)]\sin t\sin(t+\tau) \\
&= R(\tau)[\cos t\cos(t+\tau) + \sin t\sin(t+\tau)] \\
&= R(\tau)\cos(\tau)
\end{aligned}$$

(c) WSS, because the mean does not depend on t and the autocorrelation function is a function of τ only.

(d) Not necessarily SSS. For example, suppose $U(t) = U, V(t) = V$ where U and V are two independent random variables. While $U(t)$ and $V(t)$ are SSS, $Z(t)$ will not be.

2. *(10 points)* Let $N(t)$ be a Poisson process with rate $\lambda > 0$. Hence

$$P(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, n = 0, 1, \ldots.$$

Let X_1 be the time of the first arrival, X_2 be the time of the second arrival. Suppose you are told that there is exactly one arrival in the interval $[0, t_0]$.

(a) Show that the two events "there is exactly one arrival in the interval $[0, t_0]$" and "$N(t_0) = 1$" are the same.

(b) Find the conditional pdf of X_1 given that there is exactly one arrival in the interval $[0, t_0]$.

(c) Find the conditional pdf of X_2 given that there is exactly one arrival in the interval $[0, t_0]$.

Solution of Q2:

(a) First, Poisson process has $N(0) = 0$ and is defined for $t \geq 0$. Second, event $N(t_0) = 1$ means that there is exactly one arrival up till time t_0. Thus, it is equivalent to say that 'there is exactly one arrival in the interval $[0, t_0]$.'

(b) First, find conditional CDF of X_1:

$$P(X_1 \leq t | N(t_0) = 1) = \frac{P(X_1 \leq t, N(t_0) = 1)}{P(N(t_0) = 1)}$$

Case 1: If $t \geq t_0$, $P(X_1 \leq t, N(t_0) = 1) = 1$.
Case 2: If $t \in [0, t_0]$,

$$\begin{aligned}
&P(X_1 \leq t, N(t_0) = 1) \\
&= P(N(t) = 1, N(t_0) = 1) \\
&= P(N(t_0) - N(t) = 0, N(t) - N(0) = 1) \\
&= P(N(t_0) - N(t) = 0) \times P(N(t) - N(0) = 1) \\
&= e^{-\lambda(t_0 - t)} \times e^{-\lambda t}(\lambda t) \\
&= e^{-\lambda t_0}(\lambda t)
\end{aligned}$$

In the above, we have used that a Poisson process with parameter λ satisfies the following properties:

- $X(0) = 0$.
- $X(t_2) - X(t_1) \sim \text{Poisson}(\lambda(t_2 - t_1))$
- For $t_1 < t_2 < t_3 < t_4, X(t_2) - X(t_1)$ and $X(t_4) - X(t_3)$ are independent.

Also

$$P(N(t_0) = 1) = e^{-\lambda t_0}(\lambda t_0).$$

Then

$$\begin{aligned}
P(X_1 \leq t | N(t_0) = 1) &= \frac{e^{-\lambda t_0}(\lambda t)}{e^{-\lambda t_0}(\lambda t_0)} \\
&= \frac{t}{t_0}
\end{aligned}$$

Taking derivative of the CDF with respect to t we get

$$f_{X_1}(t | N(t_0) = 1) = 1/t_0, t \in [0, t_0]$$

Note that X_1 is uniform in the interval $[0, t_0]$ when we are given that there is exactly one arrival in the interval $[0, t_0]$.

(c) First, note that

$$1 - F_{X_2}(t|N(t_0) = 1) = P(X_2 \geq t|N(t_0) = 1) = \frac{P(X_2 \geq t, N(t_0) = 1)}{P(N(t_0) = 1)}$$

Case 1: If $t \leq t_0$, $P(X_2 \geq t, N(t_0) = 1) = 0$.
Case 2: If $t > t_0$,

$$\begin{aligned}
&P(X_2 \geq t, N(t_0) = 1) \\
&= P(N(t) = 1, N(t_0) = 1) \\
&= P(N(t) - N(t_0) = 0, N(t_0) - N(0) = 1) \\
&= P(N(t) - N(t_0) = 0)P(N(t_0) = 1)
\end{aligned}$$

Therefore,

$$P(X_1 \geq t|N(t_0) = 1) = P(N(t) - N(t_0) = 0) = e^{-\lambda(t-t_0)}.$$

Hence

$$f_{X_2}(t|N(t_0) = 1) = \lambda e^{-\lambda(t-t_0)}, t > t_0.$$