Probability and Stochastic Processes (EL6303) November 30, 2018
NYU Tandon School of Engineering, Fall 2018
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Quiz 3 Solution

1. *(10 points)*

   (a) *(3 points)*

$$
\begin{aligned}
E(U^2) &= E(aY_2 - Y_1)^2 \\
&= E(Y_1^2) + a^2E(Y_2)^2 - 2aE(Y_1Y_2) \\
&= 4(a^2 + 1 + a)
\end{aligned}
$$

$\frac{\mathrm{d}E(U^2)}{\mathrm{d}a} = 2a + 1 = 0 \implies a = -1/2$

   (b) *(3 points)*
$U = aY_2 - Y_1 = -1/2Y_2 - Y_1$
Y_1, Y_2 are jointly Gaussian $\implies$ U is Gaussian
Thus, (U, Y_2) are jointly Gaussian.

$$
\begin{aligned}
E(U) &= 0 \\
Var(U) &= 3
\end{aligned}
$$

Thus,

$$
\begin{aligned}
U &\sim \mathcal{N}(0, 3). \\
Y_2 &\sim \mathcal{N}(0, 4)
\end{aligned}
$$

Now let us calculate the covariance between U and Y_2.

$$
\begin{aligned}
E(UY_2) &= E((-1/2Y_2 - Y_1)(Y_2)) \\
&= -\frac{1}{2} \cdot 4 + 2 \\
&= 0
\end{aligned}
$$

U, Y_2 are uncorrelated, Jointly Gaussian $\implies$ (U, Y_2) are independent.
Thus,

$$(Y_2, U) \sim \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \quad \begin{bmatrix}4 & 0\\0 & 3\end{bmatrix}\right)$$

(c) *(4 points)*

$$\tilde{Y}_1 = a_2Y_2 + a_3Y_3.$$

$$\begin{aligned}E[(\tilde{Y}_1 - Y_1)^2] &= E((Y_1 - (a_2Y_2 + a_3Y_3))^2)\\ &= E(Y_1^2) + a_2^2E(Y_2^2) + a_3^2E(Y_3^2) - 2a_2E(Y_1Y_2)\\ &- 2a_3E(Y_1Y_3) + a_2a_3E(Y_2Y_3)\end{aligned}$$

$$\begin{aligned}\frac{\partial E[(\tilde{Y}_1 - Y_1)^2]}{\partial a_2} &= (2a_2)(4) - 2(-2) = 0\\ \implies a_2 &= -\frac{1}{2}\end{aligned}$$

$$\begin{aligned}\frac{\partial E[(\tilde{Y}_1 - Y_1)^2]}{\partial a_3} &= (2a_3)(4) - 2(-1) = 0\\ \implies a_2 &= -\frac{1}{4}\end{aligned}$$

2. *(10 points)*

(a) *(3 points)*
$E(z_i) = 1/3$
By the Weak Law of Large Numbers $Y_n \to E(z) = \frac{1}{3}$ in probability.

$$\begin{aligned}&\implies P\left(|Y_n - \frac{1}{3}| > \epsilon\right) \to 0 && \text{as } n \to \infty\\ &\implies P\left(|Y_n - \frac{1}{3}| < \epsilon\right) \to 1 && \text{as } n \to \infty\\ &\implies P\left(-\epsilon + \frac{1}{3} < Y_n < \epsilon + \frac{1}{3}\right) \to 1 && \text{as } n \to \infty\end{aligned}$$

(b) *(5 points)*

$$\begin{aligned} Y_n = \frac{1}{3} &\implies \frac{1}{n}\sum_{i=1}^{n} Z_i = \frac{1}{3} \\ &\implies \sum_{i=1}^{n} Z_i = \frac{n}{3} \end{aligned}$$

If n is not a multiple of 3, $P\left(Y_n = \frac{1}{3}\right) = 0$
Suppose n is a multiple of 3.
Let n = 3k, k being a large positive integer.

$$\begin{aligned} P\left(Y_n = \frac{1}{3}\right) &= P\left(\sum_{i=1}^{3k} Z_i = k\right) \\ &= \binom{3k}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{n-k} \\ &= \frac{(3k)!}{k!(2k)!}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{2k} \\ &\approx \frac{\sqrt{2\pi}\frac{(3k)^{3k+\frac{1}{2}}}{e^{3k}}}{\sqrt{2\pi}\frac{k^{k+\frac{1}{2}}}{e^k}\sqrt{2\pi}\frac{(2k)^{2k+\frac{1}{2}}}{e^{2k}}}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{2k} \\ &\to 0 \text{ as } k \to \infty, n \to \infty \end{aligned}$$

(c) *(2 points)*
While

$$\begin{aligned} &\epsilon > 0 \text{ leads to } P\left(-\epsilon + \frac{1}{3} \le Y_n \le \epsilon + \frac{1}{3}\right) \to 1 \\ &\epsilon = 0 \text{ leads to the same probability being } 0 \end{aligned}$$