Probability and Stochastic Processes (EL6303), Section I November 14, 2018
NYU Tandon School of Engineering, Fall 2018
Instructor: *Dr. Elza Erkip*

Exercise 7 Solutions

1. Solution of Q1.

1)

(a) $$\lim_{n\to\infty} \frac{1}{n} \log(S_n) = \lim \frac{1}{n} \sum_{j=1}^{n} \underbrace{\log(b(X_j)\,o(X_j))}_{Y(j)}$$

$Y(j)$'s are i.i.d since X_j's are i.i.d

$\Rightarrow$ using weak law of large numbers

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} Y_j = E[Y] = \sum_{i=1}^{m} \log(b(i)o(i))\,p(i)$$

with probability 1

(b)

i) $P(N=1) = p(1)$

$P(N=2) = p(1)(1-p(1))$ $\rightsquigarrow$ $\boxed{P(N=n) = (1-p(1))^{n-1}\, p(1)}$

ii) $P(S_n = 0) = 1 - P(S_n \neq 0) = 1 - (\underbrace{1-p(1)}_{<1})^n$

$n \to \infty \Rightarrow (1-p(1))^n \to 0 \Rightarrow \boxed{P(S_n = 0) \to 1}$

iii) $W(b) = \underbrace{p(1)\log(0)}_{=-\infty} + \sum_{i=2}^{m} p(i)\log(b(i)\,o(i))$

$\Rightarrow 2^{nW(b)} = 0 \Rightarrow$ it is consistent

(c)

$W(b) = p(1)\log(b(1)\,o(1)) + (1-p(1))\log((1-b(1))\,o(2))$

$$\frac{\partial W(b)}{\partial b(1)} = 0 \Rightarrow \frac{p(1)}{\ln 2 \; b(1)} - \frac{(1-p(1))}{\ln 2 \; (1-b(1))} = 0$$

$$\Rightarrow (1-b(1))\,p(1) - (1-p(1))\,b(1) = 0$$

$$\Rightarrow \boxed{\begin{aligned} b(1) &= p(1) \\ b(2) &= 1-p(1) \end{aligned}}$$

2. Solution of Q2.

(a) $p = \frac{1}{(26)^7}$

(b) $\frac{\sum Y_i}{n} \to E(Y_i)$ in prob. by WLLN

$$E(Y_i) = P(Y_i = 1) = \frac{1}{(26)^7}$$

(c) $\sum Y_i \approx \frac{n}{(26)^7} \approx 1$

$$\Rightarrow n \approx (26)^7$$

3. Solution of Q3.

(a) .

$\log L_n = \frac{1}{n} \log V_n = \frac{1}{n} \log \prod_i X_i = \frac{1}{n} \sum_i \log X_i$

Note $\log L_n = \frac{1}{n} \sum_i Y_i$ for $Y_i = \log X_i$ are iid RV's.

Thus by WLLN

$\lim_{n\to\infty} \log L_n = E Y_i$ in probability

$$\Rightarrow A = E \log X_1$$

(b) .

$$\left(E(Y_n)\right)^{1/n} = \left(E\prod_i X_i\right)^{1/n} \underset{\substack{\uparrow \\ \text{By} \\ \text{independence}}}{=} \left(\prod_i E X_i\right)^{1/n} \underset{\substack{\uparrow \\ \text{By} \\ \text{identicality}}}{=} \left((E X_1)^n\right)^{1/n}$$

$$= \underline{\underline{E X_1}}$$

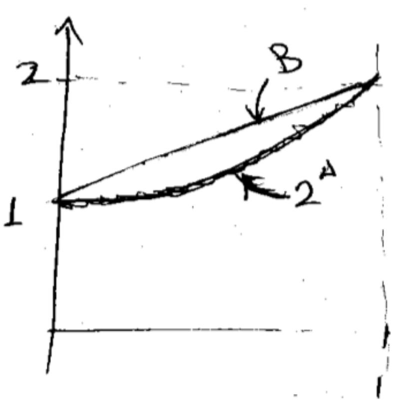
(c) .

i. $A = E \log X = (1-p)\log 1 + p\log 2$

$$= \underline{\underline{p}}$$

ii. $\left(E Y_n\right)^{1/n} = E X = (1-p)\cdot 1 + p\cdot 2 = \underline{\underline{1+p}}$.

iii. $2^A = 2^p$, $B = 1+p$



No, 2^A is less than or equal to B, with equality only at $p \in \{0, 1\}$

4. Solution of Q4.

1. $Var(Y_n) = E(|Y_n|^2) - E^2(|Y_n|)$

$Var(Y_n) \geq 0 \longrightarrow E(|Y_n|^2) \geq E^2(|Y_n|) \Longrightarrow E(|X_n - X|^2) \geq E^2(|X_n - X|)$

if $\lim_{n \to \infty} E(|X_n - X|^2) = 0 \Longrightarrow$ since $E^2(|X_n - X|) \geq 0 \Longrightarrow \lim_{n \to \infty} E(|X_n - X|) = 0$

2. (a) $E(X_n) = [\sqrt{n}(\frac{1}{n}) + 0(1 - \frac{1}{n})] = \frac{1}{\sqrt{n}} \Longrightarrow \lim_{n \to \infty} E(X_n) = 0 \longrightarrow$

$\lim_{n \to \infty} E(|X_n - 0|) = 0 \Longrightarrow$ it converges to $X = 0$

(b) No

$E(|X_n - 0|^2) = E(|X_n|^2) = n(\frac{1}{n}) + 0(1 - \frac{1}{n}) = 1$

$\Longrightarrow \lim_{n \to \infty} E[(X_n - 0)^2] \neq 0 \Longrightarrow$ Does not converge

3. No. The sequence in problem 2 is a counter example.

note that as in the solution of problem 1:

$E(|X_n - X|^2) \geq \underbrace{E(|X_n - X|)}_{0} \xRightarrow{\text{implies}} \boxed{\lim_{n \to \infty} E(|X_n - X|^2) \geq 0}$

$\downarrow$

does not require it to be zero

5. Solution of Q5.

(a) Define a new sequence of random variables $Z_n = |X_n - X|$, suppose that Z_n is bounded above by a sequence M_n that does not relay on Z_n, i.e., $Z_n \leq M_n, n = 1, 2, ...,$ then

$$\begin{aligned} E(Z_n|Y_n) &= \int_{Z_n \leq Y_n} Z_n f_{Z_n|Y_n} dZ_n + \int_{Z_n > Y_n} Z_n f_{Z_n|Y_n} dZ_n \\ &\leq \int_{Z_n \leq Y_n} Y_n f_{Z_n|Y_n} dZ_n + \int_{Z_n > Y_n} M_n f_{Z_n|Y_n} dZ_n \\ &= Y_n \int_{Z_n \leq Y_n} f_{Z_n|Y_n} dZ_n + M_n \int_{Z_n > Y_n} f_{Z_n|Y_n} dZ_n \\ &= Y_n P(Z_n \leq Y_n) + M_n P(Z_n > Y_n) = Y_n \end{aligned}$$

By the tower rule, and then take the limit,

$$\lim_{n\to\infty} E(Z_n) = \lim_{n\to\infty} E(E(Z_n|Y_n)) = \lim_{n\to\infty} E(Y_n) = 0.$$

(b) Use Markov equality and take the limit,

$$\lim_{n\to\infty} P(Z_n \geq \epsilon) \leq \lim_{n\to\infty} \frac{E(Z_n)}{\epsilon} = 0$$

Therefore, $\lim_{n\to\infty} P(Z_n > \epsilon) = 0, \forall \epsilon > 0$, we prove that X_n converge to X in probability.

6. Solution of Q6.

(a) .

From the definition of the variance, we can write $\text{Var}[W_n] = \text{E}[(W_n - \text{E}[W_n])^2]$. For convenience, let μ_i denote $\text{E}[X_i]$. Since $W_n = \sum_{i=1}^{n} X_n$ and $\text{E}[W_n] = \sum_{i=1}^{n} \mu_i$, we can write

$$\text{Var}[W_n] = \text{E}\left[\left(\sum_{i=1}^{n}(X_i - \mu_i)\right)^2\right] = \text{E}\left[\sum_{i=1}^{n}(X_i - \mu_i)\sum_{j=1}^{n}(X_j - \mu_j)\right] \tag{1}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\text{Cov}\left[X_i, X_j\right]. \tag{2}$$

In terms of the random vector $\mathbf{X} = \begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix}'$, we see that $\text{Var}[W_n]$ is the sum of all the elements of the covariance matrix $\mathbf{C_X}$. Recognizing that $\text{Cov}[X_i, X_i] = \text{Var}[X]$ and $\text{Cov}[X_i, X_j] = \text{Cov}[X_j, X_i]$, we place the diagonal terms of $\mathbf{C_X}$ in one sum and the off-diagonal terms (which occur in pairs) in another sum to arrive at the formula in the theorem.

(b) .

$$\text{Var}[X_1 + \cdots + X_n] = \sum_{i=1}^{n}\text{Var}[X_i] + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\text{Cov}\left[X_i, X_j\right] \tag{1}$$

Note that $\text{Var}[X_i] = \sigma^2$ and for $j > i$, $\text{Cov}[X_i, X_j] = \sigma^2 a^{j-i}$. This implies

$$\begin{aligned}
\text{Var}[X_1 + \cdots + X_n] &= n\sigma^2 + 2\sigma^2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a^{j-i} \\
&= n\sigma^2 + 2\sigma^2\sum_{i=1}^{n-1}\left(a + a^2 + \cdots + a^{n-i}\right) \\
&= n\sigma^2 + \frac{2a\sigma^2}{1-a}\sum_{i=1}^{n-1}(1 - a^{n-i}). \qquad (2)
\end{aligned}$$

With some more algebra, we obtain

$$\begin{aligned}
&\text{Var}[X_1 + \cdots + X_n] \\
&\quad = n\sigma^2 + \frac{2a\sigma^2}{1-a}(n-1) - \frac{2a\sigma^2}{1-a}\left(a + a^2 + \cdots + a^{n-1}\right) \\
&\quad = \left(\frac{n(1+a)\sigma^2}{1-a}\right) - \frac{2a\sigma^2}{1-a} - 2\sigma^2\left(\frac{a}{1-a}\right)^2(1 - a^{n-1}). \qquad (3)
\end{aligned}$$

Since $a/(1-a)$ and $1 - a^{n-1}$ are both nonnegative,

$$\text{Var}[X_1 + \cdots + X_n] \le n\sigma^2\left(\frac{1+a}{1-a}\right). \tag{4}$$

(c) .

Since the expected value of a sum equals the sum of the expected values,

$$\mathsf{E}\left[M(X_1, \ldots, X_n)\right] = \frac{\mathsf{E}\left[X_1\right] + \cdots + \mathsf{E}\left[X_n\right]}{n} = \mu. \tag{5}$$

The variance of $M(X_1, \ldots, X_n)$ is

$$\begin{aligned} \mathsf{Var}[M(X_1, \ldots, X_n)] &= \frac{\mathsf{Var}[X_1 + \cdots + X_n]}{n^2} \\ &\leq \frac{\sigma^2(1+a)}{n(1-a)}. \end{aligned} \tag{6}$$

Applying the Chebyshev inequality to $M(X_1, \ldots, X_n)$ yields

$$\begin{aligned} \mathsf{P}\left[|M(X_1, \ldots, X_n) - \mu| \geq c\right] &\leq \frac{\mathsf{Var}[M(X_1, \ldots, X_n)]}{c^2} \\ &\leq \frac{\sigma^2(1+a)}{n(1-a)c^2}. \end{aligned} \tag{7}$$

(d) .

Taking the limit as n approaches infinity of the bound derived in part (b) yields

$$\lim_{n\to\infty} \mathsf{P}\left[|M(X_1, \ldots, X_n) - \mu| \geq c\right] \leq \lim_{n\to\infty} \frac{\sigma^2(1+a)}{n(1-a)c^2} = 0. \tag{8}$$

Thus

$$\lim_{n\to\infty} \mathsf{P}\left[|M(X_1, \ldots, X_n) - \mu| \geq c\right] = 0. \tag{9}$$