Probability and Stochastic Processes (EL6303) September 15, 2018
NYU Tandon School of Engineering, Fall 2018
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Exercise 1

1. Let P_A represent the probability that a randomly selected item from box A is defective; similarly, P_B represents the probability that a randomly selected item from box B is defective. Boxes A and B are identical. Two items are randomly selected from one of the boxes.

   (a) What is the probability that both the items are defective?

   (b) Suppose both the items are found to be defective; what is the probability that they came from box A?

2. A hypothetical WiFi transmission can take place at any of three speeds depending on the condition of the radio channel between a laptop and an access point. The speeds are high (h) at 54 Mb/s, medium (m) at 11 Mb/s, and low (l) at 1 Mb/s. A user of the WiFi connection can transmit a short signal corresponding to a mouse click (c), or a long signal corresponding to a tweet (t). Consider the experiment of monitoring WiFi signals and observing the transmission speed and the length. An observation is a two-letter word, for example, a high-speed, mouse-click transmission is hm.

   (a) What is the sample space of the experiment?

   (b) Let A1 be the event medium speed connection. What are the outcomes in A1?

   (c) Let A2 be the event mouse click. What are the outcomes in A2?

   (d) Let A3 be the event high speed connection or low speed connection. What are the outcomes in A3?

   (e) Are A1, A2, and A3 mutually exclusive?

   (f) Are A1, A2, and A3 collectively exhaustive?

3. Suppose 100 people are waiting for a blood test for a kind of disease. The probability that for one person the test result is positive equals 0.1. Two test methods are as follows:

   - Method 1: Test each person one by one. Then we have to do 100 tests.

- Method 2: Divide 100 people into 10 groups with 10 people in each group. Then, mix 10 peoples blood and test that mixture. If the result is negative, then everyone in this group is negative. If the result is positive, then we test these 10 peoples blood one by one. Now, the number of tests becomes 1+10=11.

Question: On the average, what is the number of the tests we need to do by the second method?

4. Suppose we have 5 boxes, exactly one of which is known to be heavier. The remaining 4 have the same weight, 40 lb each. We have a bathroom scale to weigh the boxes. We denote the probability of i'th box being heavy as p_i. It is known that $(p_1, \ldots, p_5) = (0.4, 0.2, 0.15, 0.15, 0.1)$. You will use the scale to find the heavy box.

   You are interested in the *average number of weighings* defined as

   $$\sum_{i=1}^{5} p_i w_i,$$

   where w_i is the number of weighings required if the heavy one is box i. We will consider different strategies to weigh the boxes.

   (a) Suppose you weigh one box at a time in the order $5, 4, \ldots, 1$. You stop when you can identify the heavy box. What is the average number of weighings? Note that after 4 weighings if you haven't encountered the heavy box, you don't need another weighing to conclude that the last one is heavy.

   (b) Is there a better order of weighings that minimizes the average number of weighings? Explain your rationale and find the average number of weighings for your proposed order.

   (c) Now you get smarter and decide to weigh a subset of boxes at a time. You weigh different subsets till you can identify the heavy box. Each weighing tells you whether the heavy box is in the subset you weighed or is one of the remaining boxes. Consider the following sequence of weighings:

   - Weigh box 1.
   - If weight(box 1) = 40 lb, then weigh boxes 2 and 3 together.
   - If weight(box 2, box 3) $>$ 80 lb, weigh box 2.
   - If weight(box 2, box 3) = 80 lb, weigh box 4.

   Can you identify the heavy box? Explain. If yes, find the average number of weightings.

5. Flu season is on its way. It is expected that the probability of catching a flu this year will be f. Luckily, a simple blood test can tell you whether you have the flu virus or

not, so you can take Tamiflu to prevent a full-blown flu from developing. The test is accurate if you do not have the flu; it always is negative. However, if you have the flu, the test is positive with probability q where q is large but not equal to 1.

(a) What is the probability that you have the flu even though the test is negative?

(b) To increase accuracy, you can do the above blood test multiple times. It is known that the test outcomes are conditionally independent given the state of you flu.

Suppose you do the test twice.

i. What is the probability that you have the flu even though both tests are negative?

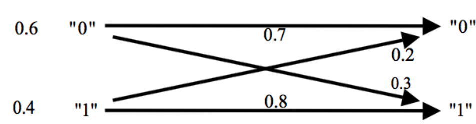
ii. What is the probability that you have the flu if the first test is negative, but the second test is positive?

(c) You repeat the blood test multiple times. Your goal is to stop as soon as you know for sure whether you have the flu or not. If you still don't know after n tests, you stop.

i. Devise a strategy to achieve your goal. Explain.

ii. Find the probability that after n tests, you do not know for sure whether you have the flu or not.

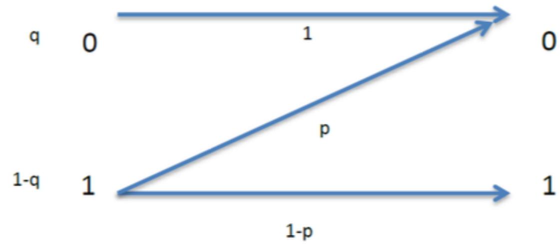
6. A digital signal 1 or 0 is transmitted through a noisy channel, the received data may be different from the signal sent out. Suppose the transmitter sends out 0 with probability 0.6, and 1 with probability 0.4. When 0 is transmitted, the receiver receives 0 with probability 0.7, and1 with probability 0.3. When 1 is transmitted, the receiver receives 0 with probability 0.2, and 1 with probability 0.8.



(a) Find the probability that data 0 is received.

(b) Find the probability of error (i.e. the probability that 1 is received when 0 was transmitted or 0 is received when1 was transmitted).

7. Consider the following communication channel, also known as the Z-channel: The channel input is either a 0 or a 1. A 0 is transmitted with probability q, a 1 is transmitted with probability $1 - q$, where $0 \leq q \leq 1$. When a 0 is transmitted, with probability 1 it is received as a 0. However, when a 1 is transmitted, the probability of receiving a 1 is $1 - p$, the probability of receiving a 0 is p, where $0 \leq p \leq 1$.



(a) Find P(0 received). Your answer should be in terms of p and q.

(b) Find P(1 received). Your answer should be in terms of p and q.

(c) Find P(error) $=$ P(transmitted and received symbols do not agree). Your answer should be in terms of p and q.

(d) For this part assume $q = 0.2$. Suppose you receive a 0. Your best estimate of the transmitted symbol can be found by finding the symbol (0 or 1) that maximizes P(symbol transmitted$|$0 received) as a function of p. Compute what your best estimate of the transmitted symbol would be.