EL 6303, Probability and Stochastic Processes **Name:** ____________________

Fall 2016

Section A1, A2

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Midterm

Time Limit: 2.5 hours

Please show your work. Good luck!

1. *(Total: 20 pts)* Every morning Hillary arrives at school between 8 am and 9 am. Her arrival time (in minutes) is denoted by continuous random variable T and follows a uniform distibution. Her friend Donald measures Hillary's arrival time with three different watches.

   (a) *(5 pts)* Sketch the PDF and CDF of T. Find $E(T)$.

   (b) *(5 pts)* The first watch Donald uses has a resolution of 10 minutes, that is it only shows times that are a multiple of 10. The time measured by this watch is T_1. Sketch T_1 as a function of T. Then sketch the PDF and the CDF of T_1.

   (c) *(5 pts)* The second watch Donald uses stops working from 8:30 am to 8:40 am and then starts working normal. Hence from 8:40 am onwards, it is 10 minutes behind. The time measured by this watch is T_2. Sketch T_2 as a function of T. Then sketch the PDF and the CDF of T_2.

   (d) *(5 pts)* The third watch Donald uses skips 10 minutes at 8:30 am to 8:40 am. Hence from 8:30 am onwards it is 10 minutes ahead. The time measured by this watch is T_3. Sketch T_3 as a function of T. Then sketch the PDF and the CDF of T_3.

   Note: **Please correctly label all axes!** For discrete or mixed random variables you can use the impulse function to show the point masses.

2. *(Total: 20 pts)* You have two coins, 1 and 2. Coin 1 has bias α, that is $P(Heads) = \alpha$. Coin 2 has bias β. At $t = 1$ you pick one coin randomly, resulting in random variable X_1. We have $P(X_1 = 1) = p$, that is you pick coin 1 with probability p. You flip the coin you picked. If it shows $Tails$, you use the same coin at time $t = 2$. Otherwise, you switch coins. You continue the same procedure for $t = 2, 3, \ldots$ resulting in $X_2, X_3, \ldots$ where X_i denotes the coin used at time i.

   (a) *(4 pts)* Find the conditional PMF of X_2 given X_1, that is $P(X_2 = i | X_1 = j), i, j = 1, 2$.

   (b) *(4 pts)* Find the marginal PMF of X_2.

   (c) *(4 pts)* How would you choose p so that X_1 and X_2 have the same marginal distribution?

   (d) *(4 pts)* Find the joint PMF of (X_1, X_2, X_3).

   (e) *(4 pts)* For $p = 1, \beta = 1 - \alpha$, compare marginal distributions of X_2 and X_3. What about the marginal distributions of $X_t, t = 4, \ldots$?

3. *(Total:20 pts)* A function $h(x)$ is concave if for all x_1 and x_2 in its domain and $\lambda \in [0, 1]$

$$\lambda h(x_1) + (1 - \lambda)h(x_2) \le h(\lambda x_1 + (1 - \lambda)x_2).$$

   (a) *(5 pts)* Suppose $Y \sim Bern(\rho)$. Prove that for $h(.)$ concave

$$E(h(Y)) \le h(E(Y)) \tag{1}$$

   (b) *(3 pts)* Prove that $\ln(x)$, $x > 0$ is a concave function.

   (c) A mobile user's satisfaction (or utility) of wireless services is usually represented by the function $u(R) = \ln(R)$, where R is the communication rate in Mbits/sec. Suppose $R \sim Unif(100, 500)$.

      i) *(4 pts)* Find $E(R)$.

      ii) *(4 pts)* Find $E(u(R))$.

      iii) *(3 pts)* Find $u(E(R))$.

      iv) *(1 pts)* Does inequality (1) hold? Comment.

   Hint: You can use $\ln(100) \approx 4, \ln(300) \approx 5.7, \ln(500) \approx 6$.

4. *(Total: 20 pts)* Let X be a random variable that denotes your arrival time (in hours) at the shopping mall. We know that $P(6\ pm \leq X \leq 8\ pm) = 1$. Let Y be another random variable that denotes the time (in hours) you spend at the mall. We know that $P(0 \leq Y \leq 2) = 1$. Let $F_{XY}(x, y)$ denote the joint CDF of (x, y), and $f_{XY}(x, y)$ denote the joint PDF.

   (a) *(3 pts)* Sketch the region where we have $F_{XY}(x, y) = 0$. Also sketch the region where we have $F_{XY}(x, y) = 1$.

   (b) *(3 pts)* Suppose A is the event that you arrive between $6:30\ pm$ and $7:30\ pm$ and spend between 0.5 hours and 1 hour shopping. Find $P(A)$ in terms of $F_{XY}(x, y)$.

   (c) *(8 pts)* Now suppose the mall closes at 9 *pm*. Your departure time from the mall is $Z = \min(X + Y, 9)$. Find $f_Z(z)$, the PDF of Z in terms of $f_{XY}(x, y)$.

   (d) *(6 pts)* Assuming that X and Y are independent, for $X \sim unif(6, 8)$, $Y \sim unif(0, 2)$, find $P(Z \geq 9)$.

5. *(Total: 20 pts)* Number of packets arriving at a router in an hour is given by N, where N has Poisson(μ) distribution. Hence

$$P(N = n) = e^{-\mu}\frac{\mu^n}{n!}, n = 0, 1, \dots$$

Packet headers contain two destination addresses, destination 1 or 2. Suppose each packet is addressed to destination 1 with probability p, independent of other packets and independent of N. Let X denote the number of packets for destination 1 and Y denote the number of packets for destination 2.

(a) *(4 pts)* Find $P(X = k|N = n)$ and $P(Y = k|N = n)$, $k = 0, 1, \dots, n$.

(b) *(5 pts)* Find the PMFs of X and Y.

(c) *(4 pts)* Find $P(X = k, Y = j|N = n)$. What are the possible values of k and j?

(d) *(5 pts)* Find the joint PMF of X and Y.

(e) *(2 pts)* Are X and Y independent? Comment.