Probability and Stochastic Processes (EL6303) November 27, 2018
NYU Tandon School of Engineering, Fall 2018
Instructor: Dr. Elza Erkip

Exercise 8 Solutions

1

(a) i. $E(Y(t)) = E(A_1X(t-T_1)) + E(A_2X(t-T_2))$

$= \cancel{E(A_1)}^{0}E(X(t-T_1)) + \cancel{E(A_2)}^{0}E(X(t-T_2))$

$= 0$

(Assume X is real for simplicity) $\longrightarrow$ ii. $R_Y(t_1,t_2) = E\left[(A_1X(t_1-T_1)+A_2X(t_1-T_2))(A_1X(t_2-T_1)+A_2X(t_2-T_2))\right]$

$= E(A_1^2X(t_1-T_1)X(t_2-T_1)) + E(A_2^2X(t_1-T_2)X(t_2-T_2)) + \cancel{\text{cross terms}}^{0}$

$= E(A_1^2)E(X(t_1-T_1)X(t_2-T_1)) + E(A_2^2)E(X(t_1-T_2)X(t_2-T_2))$

$= \left[E(A_1^2)+E(A_2^2)\right]R_X(t_1-t_2)$

iii. $E(Y(t))$ is constant & $R_Y(t_1,t_2) = R_Y(t_1-t_2) \Rightarrow Y(t)$ is WSS.

(b) $X(t) = \cos(2\pi f_c t)$, $T_1 \sim Unif\left(0,\frac{1}{f_c}\right)$, $T_2 \sim Unif\left(0,\frac{2}{f_c}\right)$

i. $E(X(t)) = E(\cos(2\pi f_c t)) = \cos(2\pi f_c t) \leftarrow$ is a function of $t \Rightarrow X(t)$ is NOT WSS.

ii. $E(Y(t)) = \cancel{E(A_1)}^{0}E\left(\cos(2\pi f_c(t-T_1))\right) + \cancel{E(A_2)}^{0}E\left(\cos(2\pi f_c(t-T_2))\right)$

$= 0$

iii. $E(Y(t_1)Y(t_2)) = E(A_1^2)E\left(X(t_1-T_1)X(t_2-T_1)\right) + E(A_2^2)E\left(X(t_1-T_2)X(t_2-T_2)\right)$ (from part (a))

$= E(A_1^2)E\left[\cos(2\pi f_c(t_1-T_1))\cos(2\pi f_c(t_2-T_1))\right] + E(A_2^2)E\left[\cos(2\pi f_c(t_1-T_2))\cos(2\pi f_c(t_2-T_2))\right]$

$= \frac{E(A_1^2)}{2}\left[E\left(\cos(2\pi f_c(t_1-t_2))\right) + E\left(\cos(2\pi f_c(t_1+t_2-2T_1))\right)\right]$

$+ \frac{E(A_2^2)}{2}\left[E\left(\cos(2\pi f_c(t_1-t_2))\right) + E\left(\cos(2\pi f_c(t_1+t_2-2T_2))\right)\right]$

$$\bullet\ E\left(\cos(2\pi f_c(t_1+t_2-2T_1))\right) = \int_0^{1/f_c} f_c \cdot \cos\left(2\pi f_c(t_1+t_2-2T_1)\right) dT_1 = 0$$

$$\bullet\ E\left(\cos(2\pi f_c(t_1+t_2-2T_2))\right) = \int_0^{2/f_c} \frac{f_c}{2} \cos\left(2\pi f_c(t_1+t_2-2T_2)\right) dT_2 = 0$$

$$\Rightarrow R_y(t_1,t_2) = \frac{X(t_1-t_2)}{2}\left(E(A_1^2)+E(A_2^2)\right)$$

iv. $E(Y(t))$ is constant & $R_Y(t_1,t_2) = R_Y(t_1-t_2) \Rightarrow Y(t)$ is WSS.

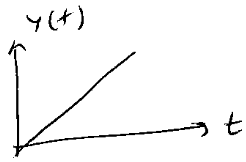
(a) $E\left(E(Y(t)\mid Z)\right) = \sum_{i=1}^{k} p_i\, s_i(t)$

(b) $R_Y(t_1, t_2) = E\left(Y(t_1)\, Y(t_2)\right)$

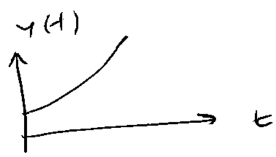
$= \sum_{i=1}^{k} p_i\, s_i(t_1)\, s_i(t_2)$

(c) No, mean depends on t
R_Y not a fn. of $t_1 - t_2$.

(d) $Z=1$



$Z=2$



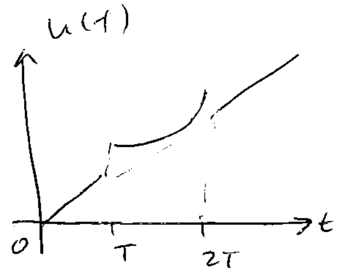
(e) (i) $R_u(t_1, t_2) = \sum_{i=1}^{n} p_i\; s_i(t_1)\, s_i(t_2)$ ↗ same as (b)

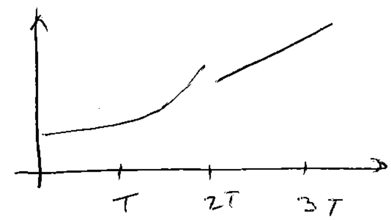
since both t_1 and t_2 experience Z_J

(ii) $R_u(t_1, t_2) = E\left(E\left(u(t_1)\, u(t_2) \mid Z_J\, Z_\ell\right)\right)$

$= \sum_{i,J} p_i\, p_k\, s_i(t)\, s_k(J)$

(iii)





These were drawn for $s_1 = t$,
$s_2 = e^t$

3

(a) $R_x(i,j) = E(X_i X_j)$

$= E(X_i)E(X_j)$

$= 0$

(b) $E(Y_n) = aE(X_n) + bE(X_{n-1})$

$= 0$

(c) $R_y(i,j) = E(Y_i Y_j)$

$= E((aX_i + bX_{i-1})(aX_j + bX_{j-1}))$

If $i = j$: $R_y(i,j) = a^2 + b^2$

$i = j-1$: $R_y(i,j) = ab$

$i = j+1$: $R_y(i,j) = ab$

otherwise $R_y(i,j) = 0$

(d) WSS since mean $= 0$

autocorr only depends on $i-j$

(e) $Y_1 = X_1$

$Y_2 = X_2 + X_1$

$Y_3 = X_3 + X_2$

$\Rightarrow$ $x_1 = y_1$

$x_2 = y_2 - y_1$

$x_3 = y_3 - y_2 + y_1$

$$J = \det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = 1$$

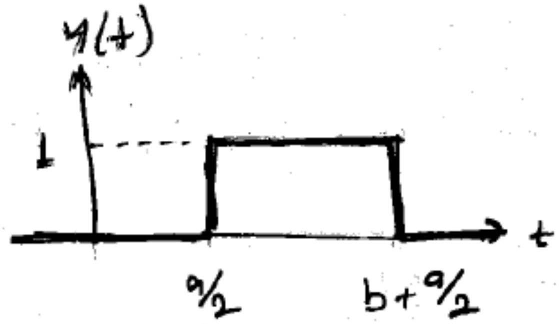
$$\Rightarrow f(y_1, y_2, y_3) = \frac{1}{(2\pi)^{3/2}} e^{-y_1^2/2} e^{-(y_2-y_1)^2/2} e^{-(y_3-y_2+y_1)^2/2}$$

4

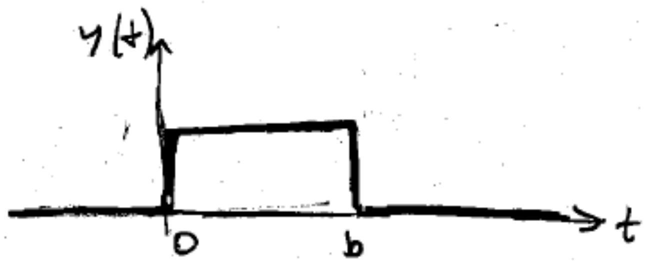
(a)

i.

For $T = a/2$



For $T = 0$



ii.

$$\eta_Y(t) = E\,X(t-T) = \int_0^a x(t-\tau)\cdot\frac{1}{a}\,d\tau$$

$$= \begin{cases} t/a, & \text{if } 0<t<a,b \\ 1, & \text{if } a<t<b \\ b/a, & \text{if } b<t<a \\ \frac{a+b-t}{a}, & \text{if } a,b<t<a+b \\ 0, & \text{otherwise.} \end{cases}$$

iii.

$$R_y(t_1, t_2) = E\left(X(t_1 - T)\, X(t_2 - T)\right)$$

Note for $t_1 + b < t_2 \Rightarrow t_1 - T + b < t_2 - T$

and therefore either $X(t_1 - T)$ or $X(t_2 - T)$

is 0. $\Rightarrow X(t_1 - T)\, X(t_2 - T) = 0$

$$\Rightarrow R_y(t_1, t_2) = E\,0 = \underline{\underline{0}}\,.$$

iv.

$\underline{\underline{\text{No}}}$. Since $\eta_y(t)$ is not a constant (i.e. it changes as t changes), $y(t)$ is $\underline{\underline{\text{not}}}$ a WSS process.

(b)

i.

$$\eta_y(t) = E\left(X(t-T)\right) = E\left(E\left(X(t-T) \mid T\right)\right)$$

$$= E\left(\eta_x(t-T)\right) = E\,\eta_x = \eta_x\,.$$

ii.

$$R_y(t_1,t_2) = E\left(X(t_1-T)X(t_2-T)\right)$$
$$= E\left(E\left(X(t_1-T)X(t_2-T) \mid T\right)\right)$$
$$= E\left(R_X(t_1-T-t_2+T)\right) = E\ R_X(t_1-t_2)$$
$$= R_X(t_1-t_2).$$

iii.

Yes. Since $E\,Y(t) = \eta_X$ is a constant and $R_Y(t_1,t_2) = R_X(t_1-t_2)$ is only a function of t_1-t_2.

5

Assume all signals are real for simplicity

(a) i. $E(Y(t)) = E(A(t)X(t)) = X(t)\cdot\mu_A(t)$

ii. $R_{YY}(t_1,t_2) = E(A(t_1)X(t_1)A(t_2)X(t_2)) = X(t_1)X(t_2)R_{AA}(t_1,t_2)$

iii. No. Even if $R_{AA}(t_1,t_2) = R_{AA}(\tau)$, $X(t_1)$ & $X(t_2)$ still depend on t_1 & t_2, hence $Y(t)$ is not WSS.

(b) i. $E(Y(t)) = E(A(t)X(t)) = E(A(t))\,E(X(t)) = \mu_A(t)\mu_X(t)$

ii. $R_{YY}(t_1,t_2) = E(A(t_1)X(t_1)A(t_2)X(t_2)) = E(A(t_1)A(t_2))\,E(X(t_1)X(t_2)) = R_{AA}(t_1,t_2)R_{XX}(t_1,t_2)$

iii. $R_{XY}(t_1,t_2) = E(X(t_1)Y(t_2)) = E(X(t_1)X(t_2))\,E(A(t_2)) = R_{XX}(t_1,t_2)\mu_A(t_2)$

iv. $E(Y(t)) = \mu_A(t)\mu_X(t) = \mu_A\mu_X \leftarrow$ Constant. ✓

$R_{YY}(t_1,t_2) = R_{AA}(t_1,t_2)R_{XX}(t_1,t_2) = R_{AA}(\tau)R_{XX}(\tau)$, where $\tau = t_1 - t_2 \leftarrow$ Depends only on τ ✓ $\}$ WSS ✓

v. $R_{XY}(t_1,t_2) = R_{XX}(t_1,t_2)\mu_A(t_2) = R_{XX}(\tau)\mu_A \leftarrow$ Depends only on $\tau \Rightarrow X(t)$ & $Y(t)$ are jointly WSS.

vi. No. Multiplication of Gaussian processes will not result in a Gaussian process.

vii. $R_{YY}(t_1,t_2) = R_{AA}(t_1,t_2)\cdot \underbrace{q(t_1)\delta(t_1-t_2)}_{\text{Since } X(t) \text{ is white noise.}} = \begin{cases} 0 & \text{if } t_1 \neq t_2 \\ R_{AA}(\underset{t_1,t_1}{\bullet})\cdot q(t_1) & \text{if } t_1 = t_2 \end{cases} \Rightarrow$ $R_{YY}(t_1,t_2)$ is non zero only if $t_1 = t_2$, thus $Y(t)$ is white noise.

6

$f_\Lambda(\lambda) = \alpha e^{-\alpha\lambda}$ for $\lambda > 0$

$N(t)$ conditioned on $\Lambda = \lambda$ is poisson process with rate λ.

(a) $P(N(t) = n \mid \Lambda = \lambda) = \boxed{\dfrac{(\lambda t)^n e^{-\lambda t}}{n!}}$ $\quad t > 0$

(b) $P(N(t) = n) = \displaystyle\int_{\lambda=0}^{\infty} P(N(t) = n \mid \Lambda = \lambda)\, f_\Lambda(\lambda)\, d\lambda$

$$= \int_0^\infty \frac{(\lambda t)^n e^{-\lambda t}}{n!} \alpha e^{-\alpha\lambda} d\lambda = \frac{\alpha t^n}{n!} \int_0^\infty \lambda^n e^{-\lambda(\alpha+t)} d\lambda$$

$$\Rightarrow P(N(t) = n) = \frac{\alpha t^n}{\cancel{n!}} \frac{\cancel{n!}}{(\alpha+t)^{n+1}} = \boxed{\frac{\alpha t^n}{(\alpha+t)^{n+1}}} \quad \underline{t > 0}$$

(c) $F_{T_1}(t_1) = P(T_1 \le t_1) = 1 - P(N(t_1) = 0) = 1 - \dfrac{\alpha}{\alpha + t_1}$

$$\Rightarrow \boxed{F_{T_1}(t_1) = \frac{t_1}{\alpha + t_1}}$$

(d)

$\because \quad (t_1, t_2) \cap (t_3, t_4) = \phi$

$$P(N(t_2)-N(t_1)=n \mid N(t_4)-N(t_3)=k) = \frac{P(N(t_2)-N(t_1)=n,\ N(t_4)-N(t_3)=k)}{P(N(t_4)-N(t_3)=k)}$$

$$= \frac{\int_0^\infty P(N(t_2)-N(t_1)=n,\ N(t_4)-N(t_3)=k \mid \Lambda=\lambda) f_\Lambda(\lambda)\,d\lambda}{\int_0^\infty P(N(t_4)-N(t_3)=k \mid \Lambda=\lambda) f_\Lambda(\lambda)\,d\lambda}$$

$$= \frac{\int_0^\infty P(N(t_2)-N(t_1)=n \mid \Lambda=\lambda)\, P(N(t_4)-N(t_3)=k \mid \Lambda=\lambda) f_\Lambda(\lambda)\,d\lambda}{\int_0^\infty P(N(t_4)-N(t_3)=k \mid \Lambda=\lambda) f_\Lambda(\lambda)\,d\lambda}$$

$$= \frac{\int_0^\infty \frac{e^{-\lambda(t_2-t_1)}\lambda^n (t_2-t_1)^n}{n!} \times \frac{e^{-\lambda(t_4-t_3)}\lambda^k (t_4-t_3)^k}{k!} \times \alpha e^{-\alpha\lambda}\,d\lambda}{\int_0^\infty \frac{e^{-\lambda(t_4-t_3)}\lambda^k (t_4-t_3)^k}{k!} \times \alpha e^{-\alpha\lambda}\,d\lambda}$$

$$= \frac{\cancel{\alpha}\,\frac{(t_2-t_1)^n}{n!} \times \cancel{\frac{(t_4-t_3)^k}{k!}} \int_0^\infty e^{-\lambda(t_2-t_1+t_4-t_3)} \lambda^{n+k} e^{-\alpha\lambda}\,d\lambda}{\cancel{\alpha}\,\cancel{\frac{(t_4-t_3)^k}{k!}} \int_0^\infty e^{-\lambda(t_4-t_3)} \lambda^k e^{-\alpha\lambda}\,d\lambda}$$

$$= \frac{(t_2-t_1)^n}{n!} \frac{\int_0^\infty e^{-\lambda(t_2-t_1+t_4-t_3)} \lambda^{n+k} e^{-\alpha\lambda}\,d\lambda}{\int_0^\infty e^{-\lambda(t_4-t_3)} \lambda^k e^{-\alpha\lambda}\,d\lambda} = A$$

on the other hand we have:

$$P(N(t_2)-N(t_1)=n) = \int_0^\infty P(N(t_2)-N(t_1)=n \mid \lambda) f_\Lambda(\lambda)\,d\lambda$$

$$= \frac{\alpha (t_2-t_1)^n}{n!} \int_0^\infty e^{-\lambda(t_2-t_1)} \lambda^n e^{-\alpha\lambda}\,d\lambda = B$$

independent

$A \neq B \Rightarrow N(t)$ is not independent increment.

Another way is to give a counter-example.

lets assume two intervals $(0\ \ 1)$, $(2\ \ \infty)$.

if we know that no arrival occurs in the second interval, it means that $\lambda=0$, which informs us that we also have no arrival in the first interval. Therefore, the number of arrivals are not independent in this case.

7

(a)

i.

$$E\{z(t)\} = E\{x(t)+y(t)\} = E\{x(t)\} + E\{y(t)\} = \mu_x + \mu_y$$

ii.

$$E\{(x(t)+y(t))(x^*(t+\tau)+y^*(t+\tau))\} = E\{x(t)x^*(t+\tau)\} +$$
$$E\{x(t)y^*(t+\tau)\} + E\{y(t)x^*(t+\tau)\} + E\{y(t)y^*(t+\tau)\}$$
$$= R_x(\tau) + R_{xy}(\tau) + R^*_{xy}(-\tau) + R_y(\tau)$$

iii.

Yes, it does not depend on *t* and mean is constant.

(b)

i.

$$E\{w(t)\} = E\{x(t)y(t)\} = R_{xy}(0)$$

ii.

$$E\{x(t)y(t)x^*(t+\tau)y^*(t+\tau)\} = E\{w(t)w^*(t+\tau)\}$$

iii.

No, since x(t) and y(t) are not independent or Gaussian

(c)

i.

$$E\{w(t)\} = E\{x(t)\}E\{y(t)\} = \mu_x\mu_y$$

ii.

$$E\{x(t)y(t)x^*(t+\tau)y^*(t+\tau)\} = E\{x(t)x^*(t+\tau)\}E\{y(t)y^*(t+\tau)\}$$
$$= R_x(\tau)R_y(\tau)$$

iii.

Yes, it only depends on τ and mean is constant.

8

(a)

$$E(X(t)) = E[W_1] \sin 2\pi f t + E[W_2] \cos 2\pi f t = 0$$

(b)

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = E[W_1 W_1] \sin(2\pi f t_1)\sin(2\pi f t_2)$$
$$+ E[W_1 W_2] \sin(2\pi f t_1)\cos(2\pi f t_2)$$
$$+ E[W_2 W_1] \cos(2\pi f t_1)\sin(2\pi f t_2)$$
$$+ E[W_2 W_2] \cos(2\pi f t_1)\cos(2\pi f t_2)$$

$$\rightarrow R_X(t_1, t_2) = \sigma^2 \sin(2\pi f t_1)\sin(2\pi f t_2) + \rho\sigma \sin(2\pi f(t_1 + t_2)) + \cos 2\pi f t_1 \cos 2\pi f t_2$$

(c)

$$\begin{cases} \sigma^2 = 1 \\ \rho = 0 \end{cases} \Longrightarrow R_X(\tau) = \cos(2\pi f \tau)$$

Yes they need to be uncorrelated, meaning $\rho\sigma = 0$ so $R_X(t_1, t_2) = R_X(t_1 - t_2)$

(d) Yes, if a normal process is WSS, it is also SSS.

9

(a)

$$\begin{aligned}E(X(t)) &= E(A\cos(\pi t) + B\sin(\pi t))\\ &= E(A)\cos(\pi t) + E(B)\sin(\pi t)\\ &= (-1\cdot 3/4 + 3\cdot 1/4)\cos(\pi t) + (1\cdot 3/4 - 3\cdot 1/4)\sin(\pi t)\\ &= 0\end{aligned}$$

(b)

$$\begin{aligned}R_{XX}(t_1,t_2) &= E(X(t_1)X(t_2))\\ &= E((A\cos(\pi t_1) + B\sin(\pi t_1))\cdot(A\cos(\pi t_2) + B\sin(\pi t_2)))\\ &= \cos(\pi t_1)\cos(\pi t_2)E(A^2) + \sin(\pi t_1)\sin(\pi t_2)E(B^2) + (\cos(\pi t_1)\sin(\pi t_2)\\ &+ \cos(\pi t_2)\sin(\pi t_1))E(AB)\end{aligned}$$

Since A and B are independent, $E(AB) = E(A)E(B) = 0$.
Also,

$$\begin{aligned}E(A^2) &= 1\cdot 3/4 + 3\cdot 1/4 = 3/2\\ E(B^2) &= 1\cdot 3/4 + 3\cdot 1/4 = 3/2\end{aligned}$$

Thus,

$$\begin{aligned}R_{XX}(t_1,t_2) &= 3/2(\cos(\pi t_1)\cos(\pi t_2) + \sin(\pi t_1)\sin(\pi t_2))\\ &= 3/2(\cos(\pi(t_1 - t_2)))\\ &= 3/2(\cos(\pi(\tau)))\end{aligned}$$

(c) Since the mean of $X(t)$ is constant and $R_{XX}(t_1,t_2)$ depends only on τ, $X(t)$ is WSS.

(d) $X(0) = A$
Thus,

$$X(0) = \begin{cases} -1, & \text{with probability } 3/4\\ 3, & \text{with probability } 1/4\end{cases}$$

(e) $X(0.25) = \dfrac{A+B}{\sqrt{2}}$

$$X(0.25) = \begin{cases} \dfrac{-4}{\sqrt{2}}, & \text{with probability } 3/16\\ 0, & \text{with probability } 5/8\\ \dfrac{4}{\sqrt{2}}, & \text{with probability } 3/16\end{cases}$$

(f) $X(0)$ and $X(0.25)$are not equal in distribution. Thus, X(t) is not SSS.