Probability and Stochastic Processes (EL6303), Section I — October 10, 2018
NYU Tandon School of Engineering, Fall 2018
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Exercise 4 Solutions

1. Solution of Q1.

   (a) From the figure,

$$f(z) = \frac{1}{a^2}z + \frac{1}{a}, \forall z \in [-a, 0)$$
$$f(z) = -\frac{1}{16}z + \frac{1}{4}, \forall z \in [0, 4]$$
$$f(z) = 0, \ \textit{for all other } z$$

   Verify: First $f(z) \geq 0, \forall z$. Second, the integral of $f(z)$ from $[-a, 4]$ is 1.

   (b)

$$E[Z] = \int_{-\infty}^{\infty} zf(z)dz = \int_{-a}^{0} z(\frac{1}{a^2}z + \frac{1}{a})dz + \int_{0}^{4} z(-\frac{1}{16}z + \frac{1}{4})dz = 2/3 - a/6.$$

   (c)

$$E[Z^2] = \int_{-\infty}^{\infty} z^2 f(z)dz = \int_{-a}^{0} z^2(\frac{1}{a^2}z + \frac{1}{a})dz + \int_{0}^{4} z^2(-\frac{1}{16}z + \frac{1}{4})dz = 4/3 + a^2/12.$$

$$Var[Z] = E[Z^2] - E[Z]^2 = 4/3 + a^2/12 - (2/3 - a/6)^2 = 8/9 + (2a)/9 + a^2/18 = 8/3.$$

   (d)

$$P(|Z| > 2) = P(Z > 2) + P(Z < 2) = 2\int_{2}^{4} (-\frac{1}{16}z + \frac{1}{4})dz = 0.25.$$

   (e)

$$P(|Z - E[Z]| \geq k) \leq \frac{Var[Z]}{k^2}$$

   Take $k = 2$, we have

$$P(|Z| \geq 2) \leq 2/3$$

   It is a useful bound because $0.25 < 2/3$.

2. Solution of Q2

(a) .

$$① \int_{-\infty}^{\infty} f_Y(y)dy = \int_c^4 \frac{1}{2(4-c)}\,dy + \int_4^8 \frac{1}{8}dy = \frac{1}{2(4-c)}\, y\Big]_c^4 + \frac{1}{8}\, y\Big]_4^8 = \frac{1}{2} + \frac{1}{2} = 1$$

$$② \quad f(y) \geqslant 0 \qquad \forall y$$

(b) .

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \frac{1}{2(4-c)}\,\frac{y^2}{2}\Big]_c^4 + \frac{1}{8}\,\frac{y^2}{2}\Big]_4^8$$

$$= \frac{16-c^2}{2(4-c)\times 2} + \frac{8^2-4^2}{16} = \frac{4+c}{4} + \frac{12}{4} = 4 + \frac{c}{4}$$

(c) .

$$E[y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \frac{1}{2(4-c)}\,\frac{y^3}{3}\Big]_c^4 + \frac{1}{8}\,\frac{y^3}{3}\Big]_4^8$$

$$= \frac{(4^3-c^3)}{6(4-c)} + \frac{8^3-4^3}{8\times 3} = \frac{4^2+4c+c^2}{2\times 3} + \frac{56}{3} = \frac{c^2+4c+128}{6}$$

(d) .

$$P(Y>4) = \int_4^8 f_Y(y)\,dy = \frac{1}{8}\, y\Big]_4^8 = \frac{1}{2}$$

(e) .

y is nonnegative and $4>0$

$$\begin{cases} P(Y>4) \leq \dfrac{E(Y)}{4} \\ E[Y] = 4 + \dfrac{c}{4} = 4 + \dfrac{1}{2} = \dfrac{9}{2} \end{cases} \Longrightarrow P(Y>4) \leq \frac{9}{8}$$

not useful,

$$P(Y>4) = 1 - P(Y \leq 4) = 1 - \underbrace{F_Y(4)}_{\text{always less than 1}} \leq 1 \qquad F_Y(4) \geq 0$$

not useful

3. Solution of Q3

a) we should show that

(I) $f(x) \geq 0$ for any x : this is satisfied for any $a>0$ which is true from assumption of problem

(II) $\int_{-\infty}^{+\infty} f(x)\, dx = 1$

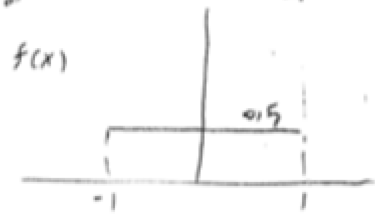
$$\int_{-\infty}^{+\infty} f(x)\, dx = \int_{-1}^{0} 0.5\, dx + \int_{0}^{\frac{1}{2a}} a\, dx = 0.5 + \frac{a}{2a} = 1 \checkmark$$

(I) & (II) $\Rightarrow$ $f(x)$ is a valid pdf.

b)
$$E[X] = \int_{-\infty}^{+\infty} x f(x)\, dx = \int_{-1}^{0} 0.5x\, dx + \int_{0}^{\frac{1}{2a}} ax\, dx$$
$$= \frac{1}{4} x^2 \Big|_{-1}^{0} + \frac{a}{2} x^2 \Big|_{0}^{\frac{1}{2a}}$$
$$= -\frac{1}{4} + \frac{a}{2} \frac{1}{(2a)^2} = \frac{1}{8a} - \frac{1}{4}$$

c) $E[x^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-1}^{1} \frac{x^2}{2}\,dx = \left.\frac{x^3}{6}\right|_{-1}^{1} = \frac{2}{6} = \boxed{\frac{1}{3}}$

$a = \frac{1}{2} \Rightarrow$ $f(x)$



d) $Var(x) = E[x^2] - (E[x])^2 = \frac{1}{3} - \left(\frac{1}{4} - \frac{1}{4}\right)^2 = \boxed{\frac{1}{3}}$

e) $P(|x| > \frac{1}{2}) = P\big(\underbrace{(x > \frac{1}{2}) \cup (x < -\frac{1}{2})}_{\text{mutually exclusive}}\big)$

$= P(x > \frac{1}{2}) + P(x < -\frac{1}{2})$

$= \int_{1/2}^{1} \frac{1}{2}\,dx + \int_{-1}^{-1/2} \frac{1}{2}\,dx = \frac{1}{4} + \frac{1}{4} = \boxed{\frac{1}{2}}$

f) CHEBYSHEV INEQUALITY $\quad P\{|x - \eta| \geq \varepsilon\} \leq \frac{\sigma^2}{\varepsilon^2}$

$\eta = E[x] = 0$

$\varepsilon = \frac{1}{2} \quad \Rightarrow \quad P\{|x| \geq \frac{1}{2}\} \leq \frac{4}{3}$

$\sigma^2 = \frac{1}{3}$

$\frac{4}{3} > 1 \Rightarrow$ not useful since probability cannot be greater than 1.

4. Solution of Q4

(a)

$$E[Y] = \sum_{n=0}^{\infty} n e^{-u} \frac{u^n}{n!} = e^{-u} u \sum_{n=0}^{\infty} \frac{u^n}{n!} = u,$$

because $\sum_{n=0}^{\infty} P(Y = n) = e^{-u} \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1$.

(b)

$$E[Y^2] = u^2 + u$$
$$Var[Y] = E[Y^2] - E[Y]^2 = u.$$

(c) i.

$$P(Y > 7u) = 1 - P(Y \leq 7u) = 1 - F_Y(7u) = 1 - \sum_{n=0}^{7u} e^{-u} \frac{u^n}{n!}$$

ii. The Markov equality

$$P(Y \geq a) \leq E[Y]/a$$

leads to

$$P(Y \geq 7u) \leq E[Y]/(7u) = 1/7.$$

(d)

$$E[(Y-2)^2] = E[Y^2] - 4E[Y] + 4 = u^2 - 3u + 4.$$

5. Solution of Q5

(a) $I_J \sim Bern(\gamma)$ $\quad P(I_J) = \begin{cases} 1, & \gamma \\ 0, & 1-\gamma \end{cases}$

(b) $X = \sum_{J=1}^{n} I_J$

(c) X takes on values $\{0, 1, \ldots, n\}$

$X \sim Binomial(n, \gamma)$

$P(X = i) = \binom{n}{i} \gamma^i (1-\gamma)^{n-i}$

(d) (i) $E(X) = n\gamma$

$\text{var}(X) = n\gamma(1-\gamma)$

$$P(|X - n\gamma| \geq n\gamma) \leq \frac{n\gamma(1-\gamma)}{n^2\gamma^2} \quad \text{by Cheb. ineq}$$

$$= \frac{1-\gamma}{n\gamma}$$

$$P(|X - n\gamma| \geq n\gamma) = P(X \geq 2n\gamma) \quad (X \text{ is nonnegative})$$

$$= 1 - P(0 \leq X \leq 2n\gamma)$$

$$\Rightarrow P(0 \leq X \leq 2n\gamma) \geq 1 - \frac{1-\gamma}{n\gamma}$$

(ii) $P(X \geq 2n\gamma) \leq \frac{n\gamma}{2n\gamma} = \frac{1}{2}$

$$\Rightarrow P(0 \leq X \leq 2n\gamma) \geq \frac{1}{2}$$

(iii) $\gamma = 0.01$, $n = 100$

Cheb. bound $1 - \frac{0.99}{1} = 0.01$

Markov bound $\frac{1}{2}$ tighter

6. Solution of Q6

(a) "Poisson random variable X with parameter λ" means that:

$$P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

$$\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}=1 \Rightarrow \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}=e^{\lambda} \Rightarrow \frac{d}{d\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}=\frac{d}{d\lambda}e^{\lambda}$$

$$\Rightarrow \sum_{k=0}^{\infty} k\frac{\lambda^{k-1}}{k!}=e^{\lambda} \Rightarrow \sum_{k=0}^{\infty} k\frac{\lambda^k}{k!}=\lambda e^{\lambda} \quad (*)$$

$$\Rightarrow \sum_{k=0}^{\infty} ke^{-\lambda}\frac{\lambda^k}{k!}=\lambda \Rightarrow E\{X\} = \sum_{k=0}^{\infty} ke^{-\lambda}\frac{\lambda^k}{k!} = \lambda$$

$$\sigma^2 = E\{(X-\eta)^2\} = E\{X^2\}-\eta^2 = E\{X^2\}-\lambda^2$$

$$E\{X^2\} = \sum_{k=0}^{\infty} k^2 e^{-\lambda}\frac{\lambda^k}{k!} = ?$$

From (*), we have

$$\sum_{k=0}^{\infty} k\frac{\lambda^k}{k!}=\lambda e^{\lambda} \Rightarrow \frac{d}{d\lambda}\sum_{k=0}^{\infty} k\frac{\lambda^k}{k!}=\frac{d}{d\lambda}\lambda e^{\lambda}=e^{\lambda}+\lambda e^{\lambda}$$

$$\Rightarrow \sum_{k=0}^{\infty} k^2\frac{\lambda^{k-1}}{k!}=(1+\lambda)e^{\lambda} \Rightarrow \sum_{k=0}^{\infty} k^2 e^{-\lambda}\frac{\lambda^k}{k!}=(1+\lambda)\lambda$$

Thus, we have

$$\sigma^2 = E\{X^2\} - \lambda^2 = (1+\lambda)\lambda - \lambda^2 = \lambda$$

Chebyshev's Inequality tells that $P\{|X-\eta| \geq \varepsilon\} \leq \frac{\sigma^2}{\varepsilon^2}$.

$$\Rightarrow P\{|X-\eta| < \varepsilon\} > 1 - \frac{\sigma^2}{\varepsilon^2}.$$

Let $\varepsilon = \lambda$ in Poisson, we have $P\{|X-\lambda| < \lambda\} > 1 - \frac{\lambda}{\lambda^2} = \frac{\lambda-1}{\lambda}$.

$$|X-\lambda| < \lambda \Rightarrow -\lambda < X - \lambda < \lambda \Rightarrow 0 < X < 2\lambda$$

Therefore, $P(0 < X < 2\lambda) > \frac{\lambda-1}{\lambda}$

(b) $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^\lambda \xrightarrow{\frac{d}{d\lambda}} \sum_{k=0}^{\infty} k \frac{\lambda^{k-1}}{k!} = e^\lambda$

$$\xrightarrow{\frac{d}{d\lambda}} \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^{k-2}}{k!} = e^\lambda$$

$$\Rightarrow E\{X(X-1)\} = \lambda^2$$

$$\sum_{k=0}^{\infty} k(k-1) \frac{\lambda^{k-2}}{k!} = e^\lambda \xrightarrow{\frac{d}{d\lambda}} \sum_{k=0}^{\infty} k(k-1)(k-2) \frac{\lambda^{k-3}}{k!} = e^\lambda$$

$$\Rightarrow E[X(X-1)(X-2)] = \lambda^3.$$