Probability and Stochastic Processes (EL6303) October 28, 2015
NYU Tandon School of Engineering, Fall 2015
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Midterm

Closed book, closed notes, no electronics, no calculators.
Only two formula sheets are allowed. Use the space below.
Long questions: 70 points, multiple choice and true/false: 30 points.

1. *(10 points)* Suppose X is a random variable with the probability density function $f_X(x) = e^{-(x+2)}, x > -2$. Let $Z = h(X)$ with

$$h(x) = \begin{cases} 0, & x < -1 \\ -1-x, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

   (a) Find and sketch $f_Z(z)$, the probability density function of Z.

   (b) Find and sketch $F_Z(z)$, the cumulative distribution function of Z.

   (c) Find $E(Z)$.

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2. *(10 points)* Suppose random variables X and Y are jointly normal $N(0, 0, \sigma_1^2, \sigma_2^2, \rho)$. Let

$$Z = aX + bY, W = bX + aY,$$

for some constants a, b.

(a) Find $E(Z^2)$ and $E(ZW)$.

(b) For $\sigma_1 = \sigma_2 = 1$, and $a = 1$, find b so that Z and W are independent. Explain.

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3. *(10 points)* Consider random variables (X, Y) with joint probability density function

$$f_{XY}(x,y) = \begin{cases} c, & 0 < x < 1, 0 < y < 1 \\ c, & -1 < x < 0, -1 < y < 0 \\ 0, & \text{else} \end{cases}$$

Let $Z = XY, W = Y$.

(a) Find c.

(b) Are X and Y independent? Explain.

(c) Find the set $\{(z,w) : f_{ZW}(z,w) > 0\}$ where $f_{ZW}(z,w)$ is the joint probability density function of (Z,W).

(d) Find $f_{ZW}(z,w)$ the joint probability density function of (Z,W).

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4. *(10 points)* Suppose $X_1, \ldots, X_n$ are independent random variables that have the same probability density function $f_X(x)$. Let

$$Y = \max(X_1, \ldots, X_n).$$

(a) Find the cumulative distribution function $F_y(y)$ of Y in terms of $f_X(x)$.

(b) Find the probability density function $f_y(y)$ of Y in terms of $f_X(x)$.

(c) For $X \sim \text{Exponential}(1)$, with $f_X(x) = e^{-x}, x > 0$.
Find $P(Y \geq 1)$.

5. *(15 points)*

You are in charge of testing in an electronics manufacturing facility. A particular type of capacitor is *defective* with probability β. Let $I_j = 1$ if capacitor j is defective, 0 otherwise. You test n such capacitors and note down X, the number of capacitors that are defective. We assume each capacitor fails (is defective) independently.

(a) Find the probability mass function of I_j for $j = 1, \ldots, n$.

(b) Find an expression for X in terms of $I_j, j = 1, \ldots, n$.

(c) Find the probability mass function of X. What values does X take?

(d) Your boss would like to make sure that the total number of capacitors that are defective is small. You say you can only guarantee this probabilistically. To accomplish this you suggest finding a lower bound for

$$P(0 \leq X \leq 2n\beta).$$

i. Find a lower bound using Chebychev inequality.

ii. Find a lower bound using Markov inequality.

iii. For $\beta = 0.01, n = 100$ evaluate both bounds. Which one is tighter?

Hint: Chebychev inequality: $P(|X - \eta| \geq \epsilon) \leq \sigma^2/\epsilon^2$,

Markov inequality: $P(X \geq \alpha) \leq \eta/\alpha$, X is a non-negative random variable.

Here $\eta = E(X), \sigma^2 = \text{Var}(X), \alpha$ a positive constant.

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6. *(15 points)* Consider $X = A\cos(w_c t + \Theta)$. Throughout this question we will consider a *fixed* t.

   (a) Suppose w_c and Θ are constants and A is a Rayleigh random variable with pdf

   $$f_A(a) = ae^{-a^2/2}, a \geq 0.$$

   Find $E(X^2)$.

   (b) Now suppose A and w_c are constants and Θ is a uniform random variable in the interval $(-\pi, \pi)$. Find $E(X)$.

   (c) Now suppose A and Θ are constants and w_c is a uniform random variable in the interval $(w_0 - w_1, w_0 + w_1)$.
   Evaluate $E(X)$ for $\Theta = 0$, $w_1 = \pi/2$ Hz, $t = 1$.

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