Probability and Stochastic Processes (EL6303) October 16, 2018
NYU Tandon School of Engineering, Fall 2018
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Exercise 5

1. Consider the following communication system. Transmitter sends X, which takes values from the set $\{0, \ldots, M-1\}, M > 2$ and the receiver observes a noisy version, Y such that

$$Y = (X + Z)(\text{mod } M),$$

where mod M represents modulo M operation. We assume Z is independent of X and has Bernoulli(p) distribution with $P(Z = 0) = p$.

   (a) Suppose $X \sim \text{Uniform}\{0, \ldots, M-1\}$.

      i. Find the probability mass function (pmf) of Y.

      ii. Find the conditional pmf of X given Y, that is $P(X = k|Y = j), k, j = 0, \ldots, M-1$.

   (b) Suppose M is even and X is uniformly distributed over all *even* numbers between 0 and $M-1$, that is $P(X = i) = 2/M$ if i even, and zero otherwise .

      i. Find the probability mass function (pmf) of Y.

      ii. Find the conditional pmf of X given Y, that is $P(X = k|Y = j), k, j = 0, \ldots, M-1$.

   (c) Considering your answers for the previous parts, which distribution of X allows you to better identify the transmitted value from the received one?

2. (a) Suppose we have a communication link which introduces delay, represented by the *discrete* random variable D. We assume that D is uniformly distributed over $\{0, 1, 2, 3\}$. Find the mean of D.

   (b) Now suppose we have access to two links, and you decide to send your file over both links at the same time. This way, you can observe a delay $X = \min(D_1, D_2)$, where D_1 and D_2 represent the random delays on links 1 and 2 respectively. We assume D_1 and D_2 are independent and uniformly distributed over $\{0, 1, 2, 3\}$.

      i. Find the probability mass function (pmf) of X.

      ii. Find the mean of X.

(c) Alternatively, you can split your file across the two links. This way, each link has a lower load, and less delay. To model this, we now assume that link delays are represented as D_3 and D_4, which are independent and uniformly distributed over $\{0, 1\}$. But in this case, you have to wait for each part of your file to arrive, so your delay is $Y = \max(D_3, D_4)$.

   i. Find the pmf of Y.

   ii. Find the mean of Y.

(d) Which method results in lower expected delay?

3. You do not have time to cook since you are studying for your midterm. You can order food from Elza's Diner (restaurant 1) or Erkip Burger (restaurant 2). The time it takes for the Diner food to arrive is represented by a random variable T_1, for the Burger food to arrive by a random variable T_2. We assume

$$T_1 \sim \text{Uniform}(0, 30), T_2 \sim \text{Uniform}(0, 20).$$

The longer it takes for the food to arrive, the colder it will get, and its value for you will be lower. We describe the *value* by a random variable

$$V_i = aL_i - bT_i,$$

where L_i is a random variable indicating how much you typically like the food from restaurant $i = 1, 2$, and a and b are constants, $a, b \geq 0$. We assume $L_1 = 15$ and $L_2 \sim \text{Uniform}(10, 12)$. Also, L_2 and T_2 are independent.

(a) Suppose $a = 2, b = 1$. *Find and sketch* the probability density function of V_1.

(b) For $a = 2, b = 1$, find $P(V_2 > 10)$.

(c) For $a = 2, b = 1$,

   i. Find $E(V_1)$.

   ii. Find $E(V_2)$.

   iii. Which restaurant would you order from if your goal is to maximize your mean value? Explain.

(d) If you didn't care about when you get the food, but only wanted to maximize how much you enjoy the food, how would you set a and b? Which restaurant would you choose? Explain.

4. When the weather is windy, it feels colder. The windchill takes into account how your body feels as a result of the wind and can be approximated by W

$$W = T - b\sqrt{V},$$

where T denotes the temperature (in centigrade), V denotes the wind speed (in m/s) and b is a constant.

Suppose that (T, V) has the joint pdf $f_{TV}(t, v)$. Note that V is a non-negative random variable, whereas T and W can be positive or negative.

(a) Find the cdf $F_W(w)$ of W in terms of $f_{TV}(t, v)$.

(b) Find the pdf $f_W(w)$ of W in terms of $f_{TV}(t, v)$.

(c) We will model temperature and wind speed as $T \sim \text{Uniform}(-40, 40)$, $V \sim \text{Uniform}(0, 100)$ and T and V as independent. Also, we have $b = 4$. Find $P(W < -30)$.

5. The Body Mass Index (BMI) of a person is calculated as

$$B = \frac{W}{H^2},$$

where W denotes the weight (in kg) and H denotes height (m).

Suppose that (W, H) have the joint pdf $f_{WH}(w, h)$. Note that both W and H are non-negative random variables.

(a) Find the cdf $F_B(b)$ of B in terms of $f_{WH}(w, h)$.

(b) Find the pdf $f_B(b)$ of B in terms of $f_{WH}(w, h)$.

(c) We will model adult weight and height as $W \sim \text{Uniform}(25, 100)$, $H \sim \text{Uniform}(1, 2)$ and W and H as independent. Find $P(B > 25)$.

6. Let U be a random variable that denotes your arrival time at the shopping mall. We know that $P(5\ pm \leq U \leq 7\ pm) = 1$. Let W be another random variable that denotes the time (in hours) you spend at the mall. We know that $P(0 \leq W \leq 2) = 1$. Let $F_{UW}(u, w)$ denote the joint CDF of (u, w), and $f_{UW}(u, w)$ denote the joint PDF.

(a) Sketch the region where we have $F_{UW}(u, w) = 0$. Also sketch the region where we have $F_{UW}(u, w) = 1$.

(b) Suppose B is the event that you arrive between $5:30\ pm$ and $6:30\ pm$ and spend between 0.5 hours and 1 hour shopping. Find $P(B)$ in terms of $F_{UW}(u, w)$.

(c) Now suppose the mall closes at 8 pm. Your departure time from the mall is $X = \min(U + W, 8)$. Find $f_X(x)$, the PDF of X in terms of $f_{UW}(u, w)$.

(d) Assuming that U and W are independent, for $U \sim unif(5, 7)$, $W \sim unif(0, 2)$, find $P(X \geq 8)$.