Probability and Stochastic Processes (EL6303), Section I — October 3, 2018
NYU Tandon School of Engineering, Fall 2018
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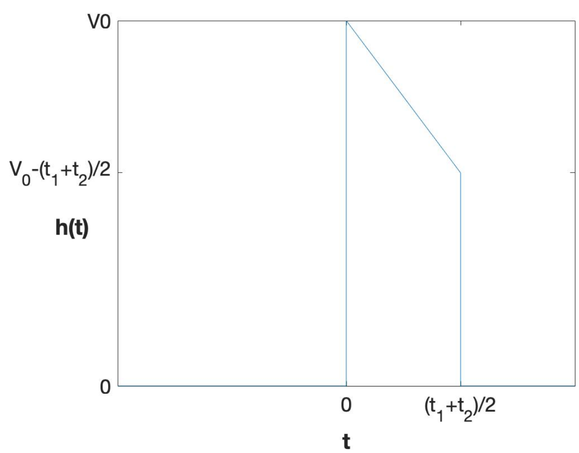
Exercise 3 Solutions

1. $f_T(t) = \begin{cases} \lambda e^{\lambda t} & 0 < t \leq t_1 \\ a & t_1 \leq t \leq t_2 \\ 0 & \text{else} \end{cases}$

(a)

$$\int_{-\infty}^{\infty} f_T(t)dt = 1$$
$$\int_0^{t_1} \lambda e^{\lambda t} dt + \int_{t_1}^{t_2} a dt = 1$$
$$e^{\lambda t} \Big|_0^{t_1} + at \Big|_{t_1}^{t_2} = 1$$
$$e^{\lambda t_1} - 1 + a(t_2 - t_1) = 1$$
$$a = \frac{2 - e^{\lambda t_1}}{t_2 - t_1}$$

(b) $V = h(t) = \begin{cases} v_0 - t & 0 < t \leq \dfrac{t_1 + t_2}{2} \\ 0 & \text{else} \end{cases}$



$$F_V(v) = \begin{cases} 0 & v < 0 \\ P(T \leq 0) + P\left(T \geq \dfrac{t_1 + t_2}{2}\right) & 0 < v < \left(V_0 - \dfrac{t_1 + t_2}{2}\right) \\ ? & \left(V_0 - \dfrac{t_1 + t_2}{2}\right) < v < V_0 \\ 1 & V_0 < v \end{cases}$$

From the pdf of t,

$$\begin{aligned} P(t < 0) &= 0 \\ P\left(T \geq \frac{t_1 + t_2}{2}\right) &= a\left(t_2 - \frac{t_1 + t_2}{2}\right) \\ &= a\left(\frac{t_2 - t_1}{2}\right) \end{aligned}$$

For $V_0 - \dfrac{t_1 + t_2}{2} < v < V_0$, let $t_r = V_0 - v$

Thus, $0 < t_r < \dfrac{t_1 + t_2}{2}$

For $\left(V_0 - \dfrac{t_1 + t_2}{2}\right) < v < V_0$,

$$\begin{aligned} P(V < v) &= P(t \notin [0, t_r]) \\ &= 1 - P(t \in [0, t_r]) \end{aligned}$$

Case 1: $0 < t_r < t_1$

$$\begin{aligned} P(t \in [0, t_r]) &= F_T(t_r) - F_T(0) \\ &= \int_0^{t_r} \lambda e^{\lambda t} dt - 0 && (F_T(0) = 0) \\ &= e^{\lambda t_r} - 1 \\ &= e^{\lambda(V_0 - V)} - 1 \end{aligned}$$

Thus, $P(V < v) = 1 - P(t \in [0, t_r]) = 2 - e^{\lambda(V_0 - V)}$

Case 2: $t_1 < t_r < \dfrac{t_1 + t_2}{2}$

$$\begin{aligned} P(t \in [0, t_r]) &= F_T(t_r) - F_T(0) \\ &= \int_0^{t_1} \lambda e^{\lambda t} dt + \int_{t_1}^{t_r} a dt - 0 && (F_T(0) = 0) \\ &= \left(e^{\lambda t_1} - 1\right) + a(t_r - t_1) \end{aligned}$$

Thus,

$$\begin{aligned} P(V < v) &= 1 - P(t \in [0, t_r]) \\ &= 2 - e^{\lambda t_1} + a(t_1 - t_r) \\ &= a(t_2 - t_1) + a(t_1 - (V_0 - v)) \\ &= a(t_2 + v - V_0) \end{aligned}$$

To summarize,

$$F_V(v) = \begin{cases} 0 & v < 0 \\ \dfrac{a(t_2 - t_1)}{2} & 0 < v < \left(V_0 - \dfrac{t_1 + t_2}{2}\right) \\ a(t_2 + v - V_0) & \left(V_0 - \dfrac{t_1 + t_2}{2}\right) < v < V_0 - t_1 \\ 2 - e^{\lambda(V_0 - v)} & V_0 - t_1 < v < V_0 \\ 1 & v > V_0 \end{cases}$$

(c) $P(X > t + s | X > t) = \dfrac{P\left((X > t + s) \cap (X > t)\right)}{P(X > t)} = \dfrac{P(X > t + s)}{P(X > t)}$

Let $t = t_1$ & $s = t_2 - t_1$

We check if $\dfrac{P(T > t_2)}{P(T > t_1)}$ is equal to $P(T > t_2 - t_1)$

$P(T > t_2) = 0, \text{but } P(T > t_2 - t_1) \neq 0$

Hence, it is not memoryless

(a)

$$P(T_1 \leq K) = \int_0^K f_{T_1}(t_1)dt_1 = \int_0^K \lambda e^{-\lambda t_1} dt_1 = -e^{-\lambda t_1}]_{t_1=0}^{t_1=K} = 1 - e^{-\lambda K}.$$

(b)

$$P(T_2 \leq K) = \int_0^K \frac{1}{T} dt_1 = \frac{K}{T}.$$

(c) Using the law of total probability, we have:

$$P(S.T \leq K) = \sum_{i=1}^{3} P(S.T \leq K|\text{type } i) \cdot P(\text{type } i) = p_1(1 - e^{-\lambda K}) + p_2\frac{K}{T} + p_3.$$

Where $S.T$ is service time.

(d) Using the Bayes' rule, we have:

$$P(\text{type } 1|S.T \leq K) = \frac{P(S.T \leq K|\text{type } 1)P(\text{type } 1)}{P(S.T \leq K)} = \frac{p_1(1 - e^{-\lambda K})}{p_1(1 - e^{-\lambda K}) + p_2\frac{K}{S.T} + p_3}$$

3

(a)

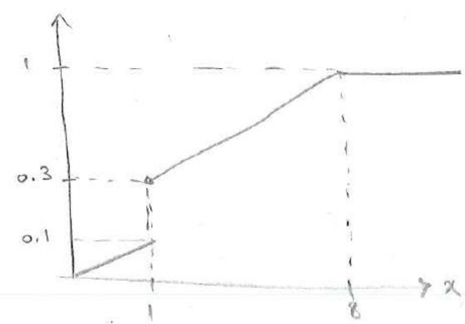
$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Longrightarrow \quad 0.2\int_{-\infty}^{\infty} \delta(x-1)\,dx + 0.1\int_{-\infty}^{\infty}\left[u(x) - u(x-a)\right]dx = 1 \Longrightarrow$$

$$0.2 + 0.1(a) = 1 \Longrightarrow \boxed{a = 8}$$

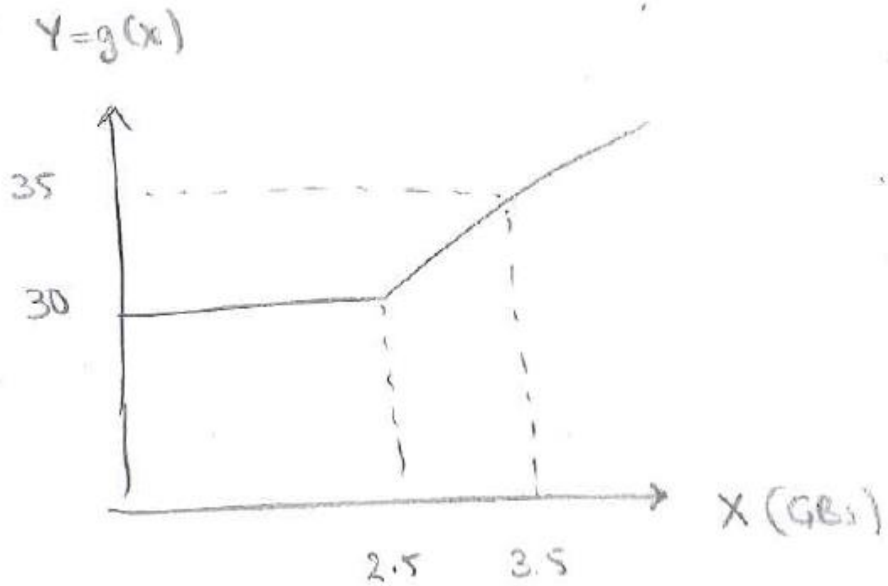
(b)

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0 & x < 0 \\ 0.1x & 0 \leq x < 1 \\ 0.1x + 0.2 & 1 \leq x < 8 \\ 1 & 8 \leq x \end{cases}$$

or $F_X(x) = 0.2\,u(x-1) + 0.1\,x\,(u(x) - u(x-8))$



(c)



(d)

if $y < 30 \implies F_Y(y) = 0$

if $y = 30 \implies F_Y(y) = P_X(0 \leq x < 2.5) = F_X(2.5) - F_X(0) = 0.45$

if $30 < y < 57.5 \implies F_Y(y) = P(Y \leq 30) + P_X(30 < Y < y)$

$$= 0.45 + P_X\left(2.5 < X < \frac{y-30}{5} + 2.5\right)$$

$$= 0.45 + F_X\left(\frac{y-30}{5} + 2.5\right) - F_X(2.5)$$

$$= 0.45 + \frac{1}{10}\left(\frac{y-30}{5}\right) = \frac{y}{50} - 0.15$$

if $y \geq 57.5 \implies F_Y(y) = 1$

$$F_Y(y) = \begin{cases} 0 & y < 30 \\ \frac{y}{50} - 0.15 & 30 \leq y < 57.5 \\ 1 & 57.5 \leq y \end{cases}$$

(e)

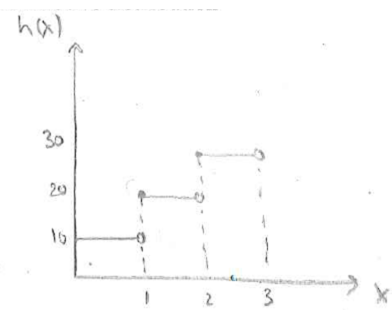
$$f_Y(y) = \frac{dF_Y(y)}{dy}$$

$$f_Y(y) = \begin{cases} 0.45 & y = 30 \\ \frac{1}{50} & 30 < y < 57.5 \\ 0 & \text{else} \end{cases}$$

or $f_Y(y) = 0.45\,\delta(y-30) + \frac{1}{50}\left[u(y-30) - u(y-57.5)\right]$

check: $\int_{-\infty}^{\infty} f_Y(y)\,dy = 0.45 + \frac{1}{50}\int_{30}^{57.5} dy = 1$

(f)



$Z = 10j \quad , \quad j-1 = \lfloor X \rfloor \Longrightarrow j = \lfloor X \rfloor + 1$ $\qquad \lfloor X \rfloor \leq a \Longrightarrow X < a+1$

$F_Z(z) = P_Z(Z \leq z) = P_X\left(10(\lfloor X \rfloor + 1) \leq z\right) = P_X\left(\lfloor X \rfloor \leq \underbrace{\frac{z}{10} - 1}_{\text{integer}}\right)$

$= P_X\left(X < \frac{z}{10}\right) = F_X\left(\frac{z}{10}\right) - \cancel{P_X\left(z = 10\right)} = F_X\left(\frac{z}{10}\right)$

Tmobile: $F_Y(y_m) = 0.5 \Longrightarrow \frac{y_m}{50} - 0.15 = 0.5 \Longrightarrow y_m = 50(0.65) = 32.5$ \$

Veriton $F_Z(z_m) = 0.5 \Longrightarrow F_X\left(\frac{z_m}{10}\right) = 0.5 \Longrightarrow 0.1\left(\frac{z_m}{10}\right) + 0.2 = 0.5 \Longrightarrow z_m = 30$ \$ ✓

4

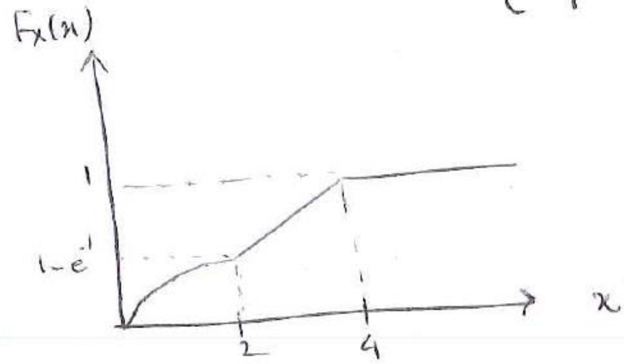
(a)

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \frac{1}{2}\int_0^2 e^{-x/2}\,dx + 2a = -e^{-x/2}\Big]_0^2 + 2a = -e^{-1}+1+2a$$

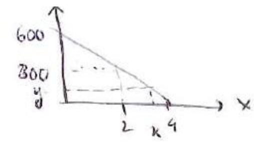
$$-e^{-1}+1+2a = 1 \Longrightarrow \boxed{a = \frac{1}{2}e^{-1}}$$

(b)

$$F_X(x) = \int_{-\infty}^{x} F_X(t)\,dt = \begin{cases} 0 & x<0 \\ 1-e^{-x/2} & 0\leq x<2 \\ 1-2e^{-1}+\frac{x}{2e} & 2\leq x<4 \\ 1 & 4\leq x \end{cases}$$



(c)



if $y < 0 \Longrightarrow F_Y(y) = 0$

if $0 \leq y < 300 \Longrightarrow 2 < x \leq 4$

$$F_Y(y) = P_Y(Y \leq y) = P_X\left(4 - \frac{y}{150} \leq x < 4\right) = F_X(4) - F_X\left(4 - \frac{y}{150}\right)$$

$$= 1 - \left(1 - \frac{2}{e} + \frac{1}{2e}\left(4 - \frac{y}{150}\right)\right) = \frac{y}{300e}$$

if $300 \leq y < 600 \Longrightarrow 0 < x \leq 2$

$$F_Y(y) = P_Y(Y \leq y) = P_X\left(4 - \frac{y}{150} \leq x < 2\right) + P_X(X \geq 2)$$

$$= F_X(2) - F_X\left(4 - \frac{y}{150}\right) + 1 - F_X(2) = 1 - F_X\left(4 - \frac{y}{150}\right) = 1 - \left(1 - e^{-\frac{1}{2}\left(4 - \frac{y}{150}\right)}\right)$$

$$= \exp\left(\frac{y}{300} - 2\right)$$

if $Y \geq 600 \Longrightarrow F_Y(y) = 1$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{y}{300e} & 0 \leq y < 300 \\ \exp\left(\frac{y}{300} - 2\right) & 300 \leq y < 600 \\ 1 & 600 \leq y \end{cases}$$

(d)

$$f_Y(y) = \begin{cases} \frac{1}{300e} & 0 \leq y < 300 \\ \frac{1}{300} \exp\left(\frac{y}{300} - 2\right) & 300 \leq y \leq 600 \\ 0 & \text{else} \end{cases}$$

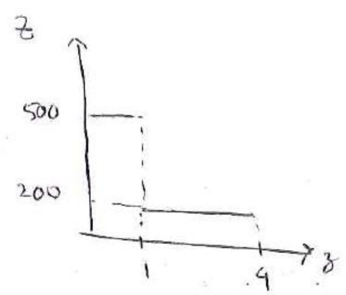
(e)

$$F_Y(c) = \frac{1}{2} \Longrightarrow \begin{cases} \frac{c}{300e} = \frac{1}{2} \Longrightarrow c = 150e > 300 \quad \times \\ \exp\left(\frac{c}{300} - 2\right) = \frac{1}{2} \Longrightarrow \frac{c}{300} - 2 = \ln\frac{1}{2} \Longrightarrow c = 300(2 - \ln 2) \end{cases}$$

$$\boxed{c = 600 - 300\ln 2}$$

(f)

Z is discrete



$$P(Z \leq 300) = P(1 \leq X \leq 4)$$
$$= P(1 \leq X < 2) + P(2 \leq X \leq 4)$$
$$= 1 - P(0 \leq X < 1) = 1 - (1 - e^{-1/2}) = e^{-1/2}$$

5

a) $Y = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$

$\mathbb{E}(Y) = 1 \times p + 0 \times (1-p) = p$

$\mathbb{E}(h(Y)) = h(1) \times p + h(0) \times (1-p) = p\,h(1) + (1-p)\,h(0)$

In concavity inequality let $x_1 = 1$, $x_2 = 0$ and $\lambda = p \in [0, 1]$:

$$\underbrace{p\,h(1) + (1-p)\,h(0)}_{E(h(Y))} \le \underbrace{h(p \times 1 + (1-p) \times 0) = h(p)}_{h(E(Y))} \qquad \blacksquare$$

b) solution 1 : From calculus we know that : $\frac{d^2}{dx^2} f(x) \le 0 \Rightarrow f(x)$ is conca

$$\frac{d^2}{dx^2} \ln(x) = \frac{-1}{x^2} < 0 \rightarrow \ln x \text{ is concave}$$

solution 2 : Using the definition we should show that :

$\forall x_1, x_2, \lambda \in [0\ 1] : \lambda \ln(x_1) + (1-\lambda) \ln(x_2) \le \ln(\lambda x_1 + (1-\lambda) x_2)$

$$\Leftrightarrow \ln(x_1^{\lambda}) + \ln(x_2^{1-\lambda}) \le \ln(\lambda x_1 + (1-\lambda) x_2)$$

$$\Leftrightarrow \ln(x_1^{\lambda} x_2^{1-\lambda}) \le \ln(\lambda x_1 + (1-\lambda) x_2)$$

$$\Leftrightarrow x_1^{\lambda} x_2^{1-\lambda} \le \lambda x_1 + (1-\lambda) x_2 \quad (*)$$

for $\lambda = \frac{1}{2}$: $\sqrt{x_1 x_2} \le \frac{x_1 + x_2}{2}$ which is correct based on the geometric arithmatic mean inequality. Using that inequality one can show that $(*)$ holds for any $\lambda \in [0\ 1]$ $\blacksquare$

c)

i) $\mathbb{E}(R) = \int_{-\infty}^{+\infty} r f_R(r)\, dr = \int_{100}^{500} \frac{r}{400}\, dr = \frac{1}{400} \left.\frac{r^2}{2}\right]_{100}^{500} = 300$

ii) $\mathbb{E}(u(R)) = \int_{-\infty}^{+\infty} u(r) f_R(r) dr = \int_{100}^{500} \frac{\ln(r)}{400} dr = \frac{1}{400} \left.(r\ln r - r)\right]_{100}^{500}$

$= \frac{1}{400}(500 \ln 500 - 500 - 100 \ln 100 + 100$

$= \frac{5}{4} \ln 500 - \frac{5}{4} - \frac{1}{4}\ln 100 + \frac{1}{4} \overset{(\text{Hint})}{=\!=} \frac{22}{4} = 5.5$

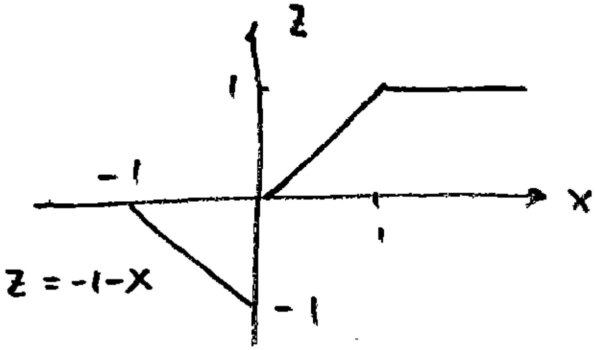
iii) $u(\mathbb{E}(R)) = \ln(300) \overset{(\text{Hint})}{=\!=} 5.7$

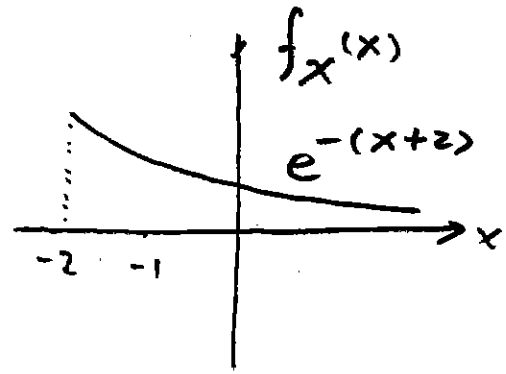
iv) Yes, it holds because $5.5 \leq 5.7$

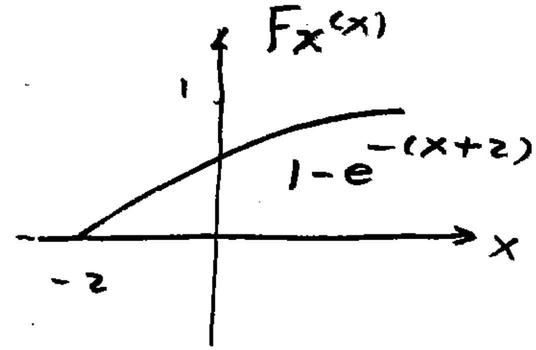
$\ln x$ is concave

6

(a)







For $-1<z<0$, $f_Z(z) = f_X(-1-z) = e^{-(-1-z+2)} = e^{-(1-z)}$.

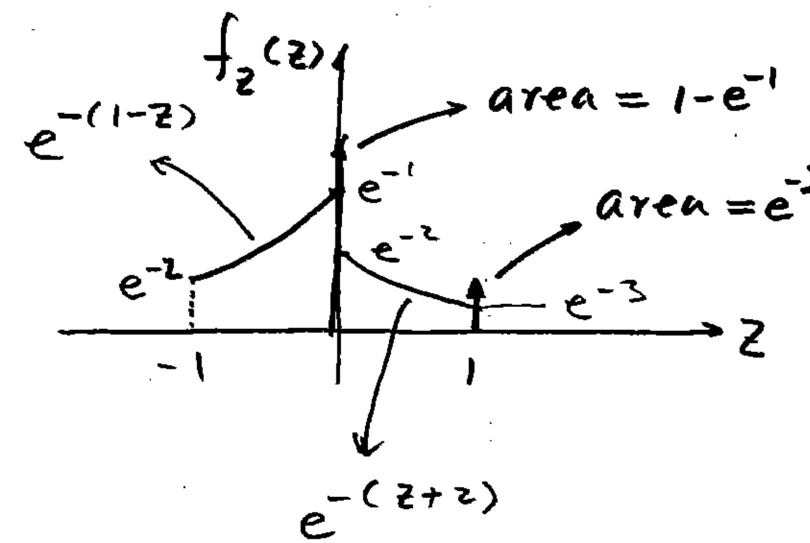
For $z=0$, $f_Z(z) = F_X(-1) \overset{\delta(z)}{=} (1-e^{-(x+2)})\big|_{x=-1} \cdot \delta(z) = (1-e^{-1})\,\delta(z)$.

For $0<z<1$, $f_Z(z) = f_X(z) = e^{-(z+2)}$.

For $z=1$, $f_Z(z) = (1-F_X(1))\,\delta(z-1) = (1-1+e^{-(x+2)})\,\delta(z-1)$

$$= e^{-3}\delta(z-1)$$

$$f_Z(z) = \begin{cases} e^{-(1-z)}, & -1<z<0 \\ (1-e^{-1})\,\delta(z), & \text{at } z=0 \\ e^{-(z+2)}, & 0<z<1 \\ e^{-3}\delta(z-1), & \text{at } z=1 \\ 0, & \text{else} \end{cases}$$



(b)

For $-1 < z < 0$, $F_Z(z) = P(Z \le z) = P((Z < -1) \cup (-1 \le Z \le z))$

$= P(-1 \le -1 - X \le z) = P(-1 - z \le x < 0)$

$= F_X(0) - F_X(-1-z) = 1 - e^{-(0+2)} - 1 + e^{-(-1-z+2)}$

$= e^{-(1-z)} - e^{-2}$

Or $F_Z(z) = \int_{-\infty}^{z} f_Z(\alpha)\,d\alpha = \int_{-\infty}^{-1} f_Z(\alpha)\,d\alpha + \int_{-1}^{z} e^{-(1-\alpha)}\,d\alpha$

$= (e^{-1})\, e^{\alpha}\Big|_{-1}^{z} = e^{-1+z} - e^{-2}$

For $0 < z < 1$, $F_Z(z) = P(Z \le z) = P(-1 < Z < 0,\ Z = 0,\ 0 \le Z < z)$

$= P(-1 < Z < 0) + P(Z = 0) + P(0 < X < z)$

$= (e^{-(1-z)} - e^{-2})_{-1}^{0} + (1 - e^{-1}) + P(0 < X < z) = 1 - e^{-(z+2)}$

Or use upper part of z, get $F_Z(z) = P(Z \le z) =$

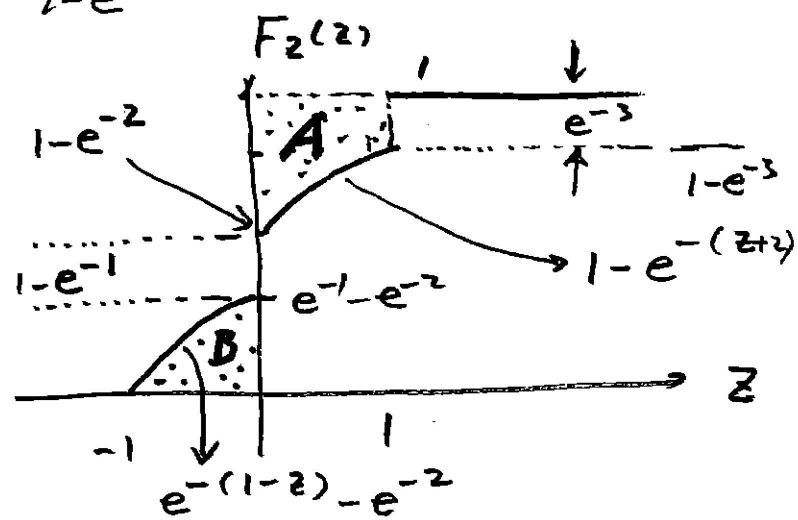
$= P(X \le z) = F_X(z) = 1 - e^{-(z+2)}$

$$F_Z(z) = \begin{cases} 0, & z < -1 \\ e^{-(1-z)} - e^{-2}, & -1 \le z < 0 \\ 1 - e^{-(z+2)}, & 0 \le z < 1 \\ 1, & z \geqslant 1. \end{cases}$$

(c) $E(Z) =$

$\int_{-1}^{0} z e^{-(1-z)}\,dz + \int_{0}^{1} z\, e^{-(z+2)}\,dz + e^{-3}$

$= -e^{-1} - e^{-3} + 3e^{-2}$



Or $E(Z) = A - B = \int_{0}^{1} (1 - (1 - e^{-(z+2)}))\,dz$

$- \int_{-1}^{0} (e^{-(1-z)} - e^{-2})\,dz = -e^{-1} - e^{-3} + 3e^{-2}$