Probability and Stochastic Processes (EL6303) October 17, 2018
NYU Tandon School of Engineering, Fall 2017
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Exercise 5 Solutions

1. (a) i.

$$P(Y=j)=P(X=j,Z=0)+P(X=j-1,Z=1)=\frac{1}{M}p+\frac{1}{M}(1-p)=\frac{1}{M},$$

for $j=0,...,M-1$. Note that the subtraction is in modulo M which gives $(0-1)(\text{mod}M)=M-1$.

ii.

$$\begin{aligned}P(X=k|Y=j)&=\frac{P(X=k,Y=j)}{P(Y=j)}\\&=\begin{cases}\frac{P(X=k,Z=0)}{P(Y=j)}=\frac{P(X=k)P(Z=0)}{P(Y=j)}=\frac{p/M}{1/M}=p & \text{, if } k=j,\\ \frac{P(X=k,Z=1)}{P(Y=j)}=\frac{P(X=k)P(Z=1)}{P(Y=j)}=\frac{(1-p)/M}{1/M}=1-p & \text{, if } k=j-1 \text{ mod M},\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

(b) i.

$$P(Y=j)=\begin{cases}P(X=j,Z=0)=\frac{2p}{M}, & \text{if } j \text{ is even},\\ P(X=j-1,Z=1)=\frac{2(1-p)}{M}, & \text{if } j \text{ is odd}.\end{cases}$$

ii.

$$P(X=k|Y=j)=\frac{P(X=k,Y=j)}{P(Y=j)}.$$

If $Y=j$ is even, then we have:

$$P(X=k|Y=j)=\begin{cases}\frac{P(X=k,Z=0)}{P(Y=j)}=\frac{P(X=k)P(Z=0)}{P(Y=j)}=\frac{2p/M}{2p/M}=1 & \text{, if } k=j,\\ 0 & \text{otherwise.}\end{cases}$$

On the other hand, if $Y=j$ is odd, we have:

$$P(X=k|Y=j)=\begin{cases}\frac{P(X=k,Z=1)}{P(Y=j)}=\frac{P(X=k)P(Z=1)}{P(Y=j)}=\frac{2(1-p)/M}{2(1-p)/M}=1 & \text{, if } k=j-1,\\ 0 & \text{otherwise.}\end{cases}$$

Hence, observing $Y=j$ will enable the receiver to identify the transmitted value of X correctly with probability 1.

(c) The distribution in part (b) since it identifies the transmitted value with probability 1.

2. (a) D is uniformly distributed over {0, 1, 2 ,3 } so we have

$$P(D=0) = P(D=1) = P(D=2) = P(D=3) = \frac{1}{4}$$

$$E[D] = \sum_{i=0}^{3} i \times P(D=i) = \frac{0+1+2+3}{4} = \frac{6}{4} = \frac{3}{2}.$$

(b) $X = min(D_1, D_2)$ where D_1 and D_2 are independent and uniformly distributed over {0,1,2,3}

i. We can see that X can only take values in {0,1,2,3}.

$$\begin{aligned} P(X=0) &= P(D_1 = 0 + D_2 = 0) \\ &= P(D_1 = 0) + P(D_2 = 0) - P(D_1 = 0, D_2 = 0) \\ &= \frac{1}{4} + \frac{1}{4} - \frac{1}{16} \\ &= \frac{7}{16} \end{aligned}$$

$$\begin{aligned} P(X=1) &= P((D_1 = 1, D_2 \geq 1) + (D_1 \geq 1, D_2 = 1)) \\ &= P(D_1 = 1, D_2 \geq 1) + P(D_1 \geq 1, D_2 = 1) - P(D_1 = 1, D_2 = 1) \\ &= \frac{1}{4} \times \frac{3}{4} + \frac{3}{4} \times \frac{1}{4} - \frac{1}{4} \times \frac{1}{4} \\ &= \frac{5}{16} \end{aligned}$$

$$\begin{aligned} P(X=2) &= P((D_1 = 2, D_2 \geq 2) + (D_1 \geq 2, D_2 = 2)) \\ &= P(D_1 = 2, D_2 \geq 2) + P(D_1 \geq 2, D_2 = 2) - P(D_1 = 2, D_2 = 2) \\ &= \frac{1}{4} \times \frac{2}{4} + \frac{2}{4} \times \frac{1}{4} - \frac{1}{4} \times \frac{1}{4} \\ &= \frac{3}{16} \end{aligned}$$

$$\begin{aligned} P(X=3) &= P(D_1 = 3, D_2 = 3) = \frac{1}{4} \times \frac{1}{4} \\ &= \frac{1}{16} \end{aligned}$$

ii. Calculating $E[X]$:

$$\begin{aligned}
E[X] &= \sum_{i=0}^{3} i \times P(X = i) \\
&= 0 \times \frac{7}{16} + 1 \times \frac{5}{16} + 2 \times \frac{3}{16} + 3 \times \frac{1}{16} \\
&= \frac{7}{8}
\end{aligned}$$

(c) $Y = \max(D_2, D_3)$ where D_2 and D_3 are independent and uniformly distributed over set {1,2}

i. Y can only take values 0 and 1.

$$\begin{aligned}
P(Y = 0) &= P(D_2 = 0, D_3 = 0)) \\
&= \frac{1}{2} \times \frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
P(Y = 1) &= 1 - P(Y = 0) \\
&= \frac{3}{4}
\end{aligned}$$

ii.

$$\begin{aligned}
E[Y] &= \sum_{i=0}^{1} i \times P(Y = i) \\
&= 0 \times \frac{1}{4} + 1 \times \frac{3}{4} \\
&= \frac{3}{4}
\end{aligned}$$

(d) We can see that $E[Y] = \frac{3}{4}$ is less than $E[X] = \frac{7}{8}$ and so the method in part (c) causes less expected delay.

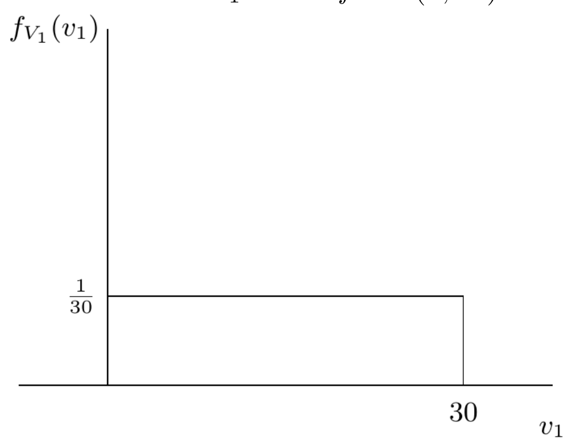
3. (a) $a = 2$ and $b = 1$, we have: $V_1 = 2 \times 15 - T_1 = 30 - T_1$

$$\begin{aligned}
P(30 - T_1 \leq v_1) &= P(30 - v_1 \leq T_1) \\
&= \frac{1}{30} \int_{30-v_1}^{\infty} U(t) - U(t-30)dt \\
&= \frac{1}{30} \int_{\max(30-v_1,0)}^{30} dt \\
&= \begin{cases} 0, & \text{if } v_1 < 0 \\ \frac{v_1}{30}, & \text{if } 0 \leq v_1 \leq 30 \\ 1 & \text{if } 30 < v_1, \end{cases}
\end{aligned}$$

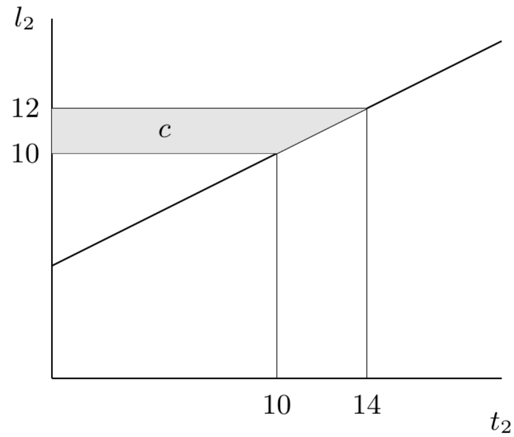
we know $f_{V_1}(v_1) = \frac{d}{dv_1} F_{V_1}(v_1)$. We have

$$f_{V_1}(v_1) = \begin{cases} 0, & \text{if } v_1 < 0 \\ \frac{1}{30}, & \text{if } 0 \geq v_1 \leq 30 \\ 0 & \text{if } 30 < v_1, \end{cases}$$

we can see that $V_1 \sim Uniform(0, 30)$



(b) $a = 2$ and $b = 1$, we have $V_2 = 2L_2 - T_2$



We want the probability of being in the shaded region which is equal to :

$$
\begin{aligned}
P(V_2 > 10) &= P(2L_2 - T_2 > 10) \\
&= \int\int_{t,l\in c} f_{L_2}(l) f_{T_2}(t) dl dt \\
&= \text{area of shaded region} \times \frac{1}{20} \times \frac{1}{2} \\
&= \frac{(10+14)\times(12-10)}{2} \times \frac{1}{20} \times \frac{1}{2} \\
&= \frac{3}{5}
\end{aligned}
$$

(c) $a = 2$ and $b = 1$, we have

i.

$$
\begin{aligned}
E[V_1] &= E[30 - T_1] \\
&= 30 - E[T_1] = 30 - 15 \\
&= 15
\end{aligned}
$$

ii.

$$
\begin{aligned}
E[V_2] &= E[2L_2 - T_2] \\
&= 2 \times E[L_2] - E[T_2] = 2 \times 11 - 10 \\
&= 12
\end{aligned}
$$

iii. Since $E[V_1] > E[V_2]$, you order from Restaurant 1, Elza's Diner.

(d) We don't care about the time it takes to get the food so $b = 0$ and a can be any positive number. We have,

$$E[V_1] = 15 \times a$$
$$E[V_2] = 11 \times a$$

we see that $E[V_1]$ is bigger so we will order from Elza's Diner to have better quality.

4

a)

$$F_W(w) = \Pr(W \le w) = \Pr(t - b\sqrt{v} \le w) = \Pr(t \le b\sqrt{v} + w)$$

$$\left\{ \begin{array}{l} -\infty < t \le b\sqrt{v} + w \\ 0 \le v < \infty \end{array} \right\} \longrightarrow F_W(w) = \int_0^\infty \int_{-\infty}^{w+b\sqrt{v}} f_{TV}(t, v)\, dt\, dv$$

b)

$$f_W(w) = \frac{dF_W(w)}{dw} = \frac{d}{dw}\left(\int_0^\infty \int_{-\infty}^{w+b\sqrt{v}} f_{TV}(t, v)\, dt\, dv \right)$$

$$= \int_0^\infty \frac{d}{dw}\left(\int_{-\infty}^{w+b\sqrt{v}} f_{TV}(t, v)\, dt \right) dv$$

$$= \int_0^\infty f_{TV}(w + b\sqrt{v}, v)\, dv$$

c)

independency: $f_{TV}(t,v) = f_T(t)\, f_V(v) = \begin{cases} \frac{1}{8000} & -40 \le T \le 40,\ 0 \le V \le 100 \\ 0 & \text{o.w.} \end{cases}$

$$P(W < -30) = P(W \le -30) \overset{(a)}{=} F_W(-30) = \int_0^\infty \int_{-40}^{-30+4\sqrt{v}} f_{TV}(t,v)\, dt\, dv$$

$(-30 \le -30 + 4\sqrt{v} \le 10)$

$$= \int_0^{100} \int_{-40}^{-30+b\sqrt{v}} \frac{1}{8000}\, dt\, dv = \int_0^{100} \frac{1}{8000}(10 + 4\sqrt{v})\, dv$$

$$= \frac{1}{8000}\int_0^{100} dv + \frac{4}{8000}\int_0^{100} v^{1/2}\, dv$$

$$= \frac{1}{8} + \frac{4}{8000}\left[\frac{v^{3/2}}{3/2}\right]_0^{100}) =$$

$$= \frac{1}{3} + \frac{1}{2000} \times \frac{2}{3} \times 1000 = \frac{1}{3} + \frac{1}{8} = \frac{11}{24}$$

5. (a)

$$F_B(b) = pr(B \le b) = pr\left(\frac{w}{h^2} \le b\right) \overset{H^2 \ge 0}{=\!=} pr(w < bh^2)$$

$$\left\{ \begin{array}{l} 0 \le w \le bh^2 \\ 0 \le h < \infty \end{array} \right\} \to F_B(b) = \int_0^\infty \int_0^{bh^2} f_{WH}(w,h)\, dw\, dh$$

(b)

$$f_B(b) = \frac{dF_B(b)}{db} = \frac{d}{db}\left(\int_0^\infty \int_0^{bH^2} f_{WH}(w,h)\, dw\, dh\right)$$

$$= \int_0^\infty \left(\frac{d}{db} \int_0^{bH^2} f_{WH}(w,h)\, dw\right) dh$$

$$= \int_0^\infty \underbrace{h^2}\, f_{WH}\underbrace{(bh^2, h)}\, dh$$

(c)

independency: $f_{WH}(w,h) = f_W(w)\, f_H(h) = \begin{cases} \frac{1}{75} & 25 \le w \le 100,\ 1 \le h \le 2 \\ 0 & o.w. \end{cases}$

$$pr(B \le 25) = F_B(25) \overset{(a)}{=} \int_0^\infty \int_0^{25h^2} f_{WH}(w,h)\, dw\, dh$$

$$= \int_1^2 \int_{25}^{25h^2} \frac{1}{75}\, dw\, dh$$

$(25 \le 25h^2 \le 100)$

$$= \int_1^2 \frac{25}{75}(h^2 - 1) = \frac{1}{3}\left(\frac{h^3}{3} - h\right)\Big|_1^2 = \frac{4}{9}$$

$$pr(B > 25) = 1 - P(B \le 25) = 1 - \frac{4}{9} = \frac{5}{9}$$

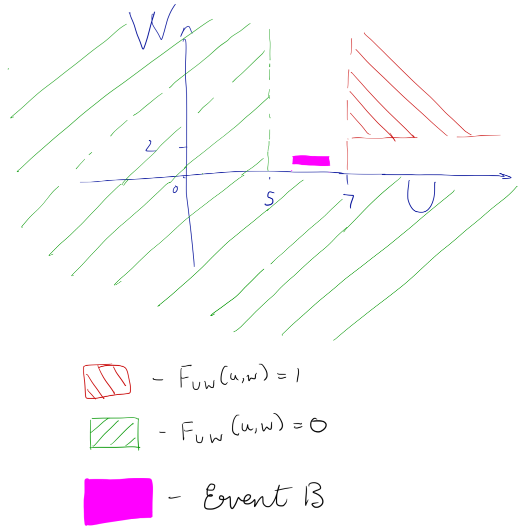


Figure 1

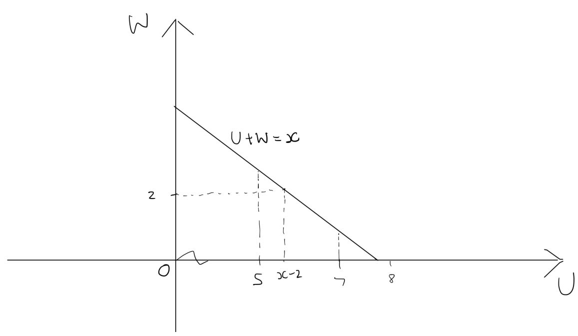


Figure 2: $7 \leq x \leq 8$

6. (a) Refer to Fig. 1

   (b) $P(B) = F_{U,W}(6.5, 1) - F_{U,W}(5.5, 1) - F_{U,W}(6.5, 0.5) + F_{U,W}(5.5, 0.5)$

   (c) Since $0 \leq W \leq 2$ and $5 \leq U \leq 7$

$$\Pr(\min(U+W, 8) \leq x) = \begin{cases} 1 & x \geq 8 \\ \Pr(U+W \leq x) & 5 \leq x < 8 \\ 0 & x \leq 5 \end{cases}$$

First consider the case when $7 \leq x < 8$.

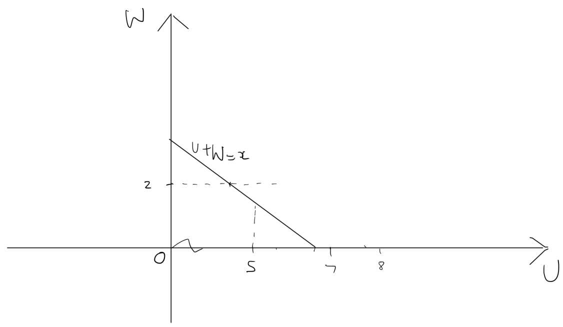


Figure 3: $5 \leq x \leq 7$

As seen in Fig. 2,

$$\begin{aligned}
\Pr(U+W \leq x) &= \int_5^{x-2}\int_0^2 f_{U,W}(u,w)dwdu + \int_{x-2}^{7}\int_0^{x-u} f_{U,W}(u,w)dwdu \\
f_X(x) &= \frac{\partial}{\partial x}\left(\int_5^{x-2}\int_0^2 f_{U,W}(u,w)dwdu + \int_{x-2}^{7}\int_0^{x-u} f_{U,W}(u,w)dwdu\right) \\
&= \frac{\partial}{\partial x}\left(\int_5^{x-2}\int_0^2 f_{U,W}(u,w)dwdu\right) + \frac{\partial}{\partial x}\left(\int_{x-2}^{7}\int_0^{x-u} f_{U,W}(u,w)dwdu\right) \\
&= \left(\int_0^2 f_{U,W}(x-2,w)dw\right) \\
&+ \left(-\int_0^{x-(x-2)} f_{U,W}(x-2,w)dw + \int_{x-2}^{7} f_{U,W}(u,x-u)du\right) \\
&= \int_{x-2}^{7} f_{U,W}(u,x-u)du
\end{aligned}$$

Now, consider the case when $5 \leq x \leq 7$. As seen in Fig. 3,

$$\begin{aligned}
\Pr(U+W \leq x) &= \int_5^{x}\int_0^{x-u} f_{U,W}(u,w)dwdu \\
f_X(x) &= \frac{\partial}{\partial x}\left(\int_5^{x}\int_0^{x-u} f_{U,W}(u,w)dwdu\right) \\
&= \int_0^{x-(x)} f_{U,W}(u,w)dw + \int_5^{x} f_{U,W}(u,x-u)du \\
&= \int_5^{x} f_{U,W}(u,x-u)du
\end{aligned}$$

Now, Pr(X=8) = Pr(X$\leq$ 8) - Pr(X$\leq$ 8^-)
= 1 - $\lim_{x\to 8^-} \int_{x-2}^{7} f_{U,W}(u,x-u)du$

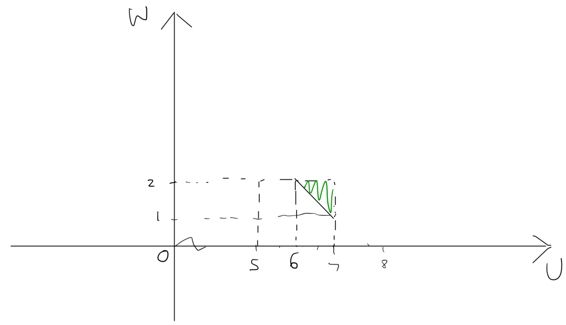


Figure 4: The shaded area is the region where $U + W > 8$

$= 1$ - $\int_6^7 f_{U,W}(u, 8-u)du$

Thus,

$$f_X(x) = \begin{cases} 1 & x > 8 \\ \int_{x-2}^{7} f_{U,W}(u, x-u) & 7 \leq x < 8 \\ \int_5^x f_{U,W}(u, x-u)du & 5 \leq x \leq 7 \\ 0 & x \leq 5 \end{cases}$$

"with a dirac delta at x=8", i.e.

$$\begin{aligned} f_X(x) = u(x-8) &+ \left(1 - \int_6^7 f_{U,W}(u, 8-u)du\right) \cdot \delta(x-8) \\ &+ \left(\int_{x-2}^{7} f_{U,W}(u, x-u)\right) \cdot (u(x-8) - u(x-7)) \\ &+ \left(\int_5^x f_{U,W}(u, x-u)du\right) \cdot (u(x-7) - u(x-5)) \end{aligned}$$

(d)

$$\begin{aligned} \Pr(X \geq 8) &= \Pr(X = 8) \\ &= Pr(U + W) \geq 8) \end{aligned}$$

Since U and W are independent and uniform,

$\Pr(U + W) \geq 8) = \dfrac{\text{Area of shaded region in Fig. 4}}{\text{Total Feasible area of } U \text{and } W} = \dfrac{1}{8}$