EL 6303, Probability and Stochastic Processes

Fall 2016

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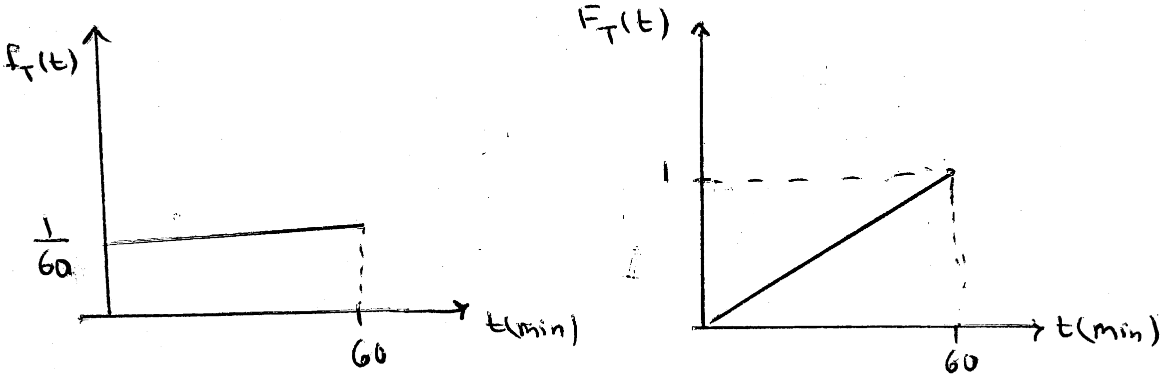
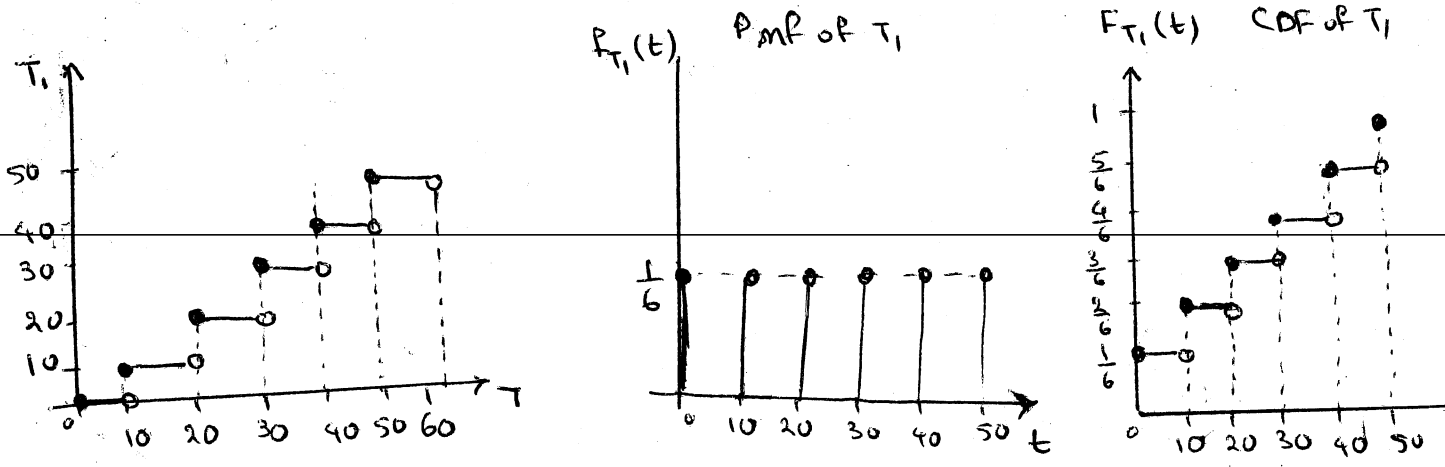
Solutions to Midterm

1)

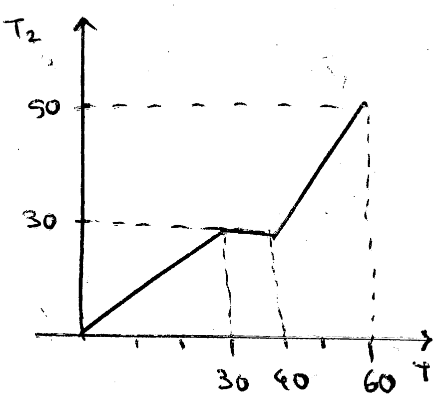
(a) $E(T) = 30$



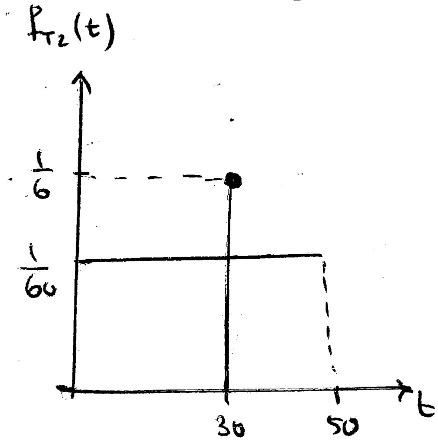
(b)



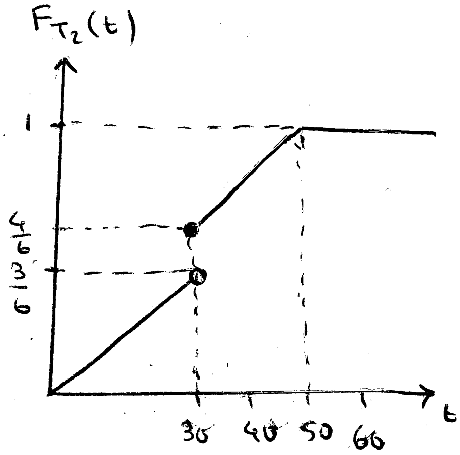
(c)



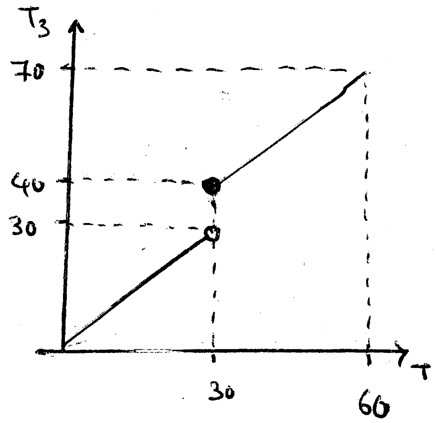
PDF of T_2



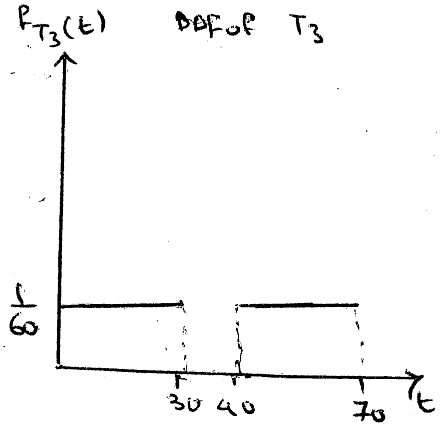
CDF of T_2



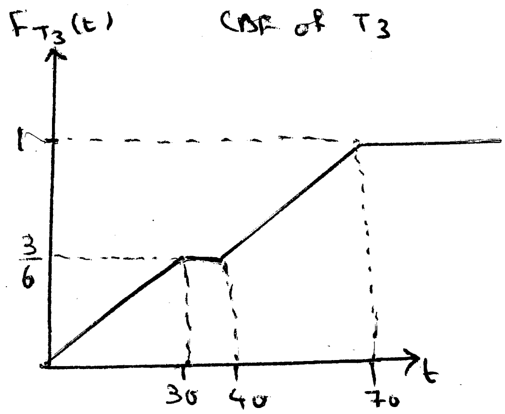
(d)



PDF of T_3



CDF of T_3



(2) a)

$$P(X_2=i \mid X_1=j) = \begin{cases} 1-\alpha & i=1,\ j=1 \\ \beta & i=1,\ j=2 \\ \alpha & i=2,\ j=1 \\ 1-\beta & i=2,\ j=2 \end{cases}.$$

b)
$$P(X_2=i) = \sum_j P(X_2=i \mid X_1=j) \cdot P(X_1=j)$$

$$= P(X_2=i \mid X_1=1) \cdot P(X_1=1) + P(X_2=i \mid X_1=2) \cdot P(X_1=2)$$

$$= \begin{cases} (1-\alpha)p + \beta(1-p), & \text{if } i=1 \\ \alpha p + (1-\beta)(1-p), & \text{if } i=2 \end{cases}$$

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c) $P(X_1=1) = P(X_2=1)$

$$\Rightarrow p = (1-\alpha)p + \beta(1-p) \Rightarrow \boxed{p = \frac{\beta}{\alpha+\beta}}$$

d)

$$P(X_1=i, X_2=j, X_3=k) = \begin{cases} p\cdot(1-\alpha)(1-\alpha) & 1 \quad 1 \quad 1 \\ p(1-\alpha)\,\alpha & 1 \quad 1 \quad 2 \\ p\,\alpha\,\beta & 1 \quad 2 \quad 1 \\ p\,\alpha(1-\beta) & 1 \quad 2 \quad 2 \\ (1-p)\,\beta\,(1-\alpha) & 2 \quad 1 \quad 1 \\ (1-p)\,\beta\,\alpha & 2 \quad 1 \quad 2 \\ (1-p)(1-\beta)\,\beta & 2 \quad 2 \quad 1 \\ (1-p)(1-\beta)(1-\beta) & 2 \quad 2 \quad 2 \end{cases} \qquad \begin{array}{ccc} i, & j, & k \end{array}$$

e) For $p=1$, $\beta = 1-\alpha$, from part (b)

$$P(X_2=i) = \begin{cases} \alpha & \text{if } i=2 \\ 1-\alpha & \text{if } i=1 \end{cases}$$

Similarly, ~~by~~ marginalizing over the joint PMF from part (d)

$$P(X_3=k) = \sum_{i=1,2}\sum_{j=1,2} P(X_1=i, X_2=j, X_3=k)$$

$$= \begin{cases} (1-\alpha)^2 + \alpha(1-\alpha) & \text{if } k=1 \\ \alpha(1-\alpha) + \alpha^2 & \text{if } k=2 \end{cases}$$

$$= \begin{cases} \alpha & \text{if } k=2 \\ 1-\alpha & \text{if } k=1 \end{cases} \Rightarrow$$

Thus X_2 and X_3 have the same PMF.

Note that for any $t = 2, \ldots$

$$P(X_t = 1) = P(X_{t-1} = 1) \cdot (1-\alpha) + P(X_{t-1} = 2) \cdot (1-\alpha)$$

and therefore for $t = 4$:

$$P(X_4 = 1) = (1-\alpha)^2 + \alpha(1-\alpha) = 1-\alpha.$$

For general t by repeating the same argument

$$P(X_t = i) = 1 - \alpha.$$

Question 3

a) $Y = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$

$\mathbb{E}(Y) = 1 \times p + 0 \times (1-p) = p$

$\mathbb{E}(h(Y)) = h(1) \times p + h(0) \times (1-p) = p h(1) + (1-p) h(0)$

In concavity inequality let $x_1 = 1$, $x_2 = 0$ and $\lambda = p \in [0, 1]$:

$$\underbrace{p h(1) + (1-p) h(0)}_{E(h(Y))} \leq \underbrace{h(p \times 1 + (1-p) \times 0) = h(p)}_{h(E(Y))}$$

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b) From calculus we know that: $\frac{d^2}{dx^2} f(x) \leq 0 \Rightarrow f(x)$ is conca

solution 1:

$$\frac{d^2}{dx^2} \ln(x) = \frac{-1}{x^2} < 0 \rightarrow \ln x \text{ is concave}$$

solution 2: Using the definition we should show that:

$\forall x_1, x_0, \lambda \in [0\ 1]$: $\lambda \ln(x_1) + (1-\lambda) \ln(x_2) \leq \ln(\lambda x_1 + (1-\lambda) x_2)$

$$\Leftrightarrow \ln(x_1^{\lambda}) + \ln(x_2^{1-\lambda}) \leq \ln(\lambda x_1 + (1-\lambda) x_2)$$

$$\Leftrightarrow \ln(x_1^{\lambda} x_2^{1-\lambda}) \leq \ln(\lambda x_1 + (1-\lambda) x_2)$$

$$\Leftrightarrow x_1^{\lambda} x_2^{1-\lambda} \leq \lambda x_1 + (1-\lambda) x_2 \quad (*)$$

for $\lambda = \frac{1}{2}$: $\sqrt{x_1 x_2} \leq \frac{x_1 + x_2}{2}$ which is correct based on the geometric arithmatic mean inequality. Using that inequality one can show that $(*)$ holds for any $\lambda \in [0\ 1]$

■

c)

i) $\mathbb{E}(R) = \int_{-\infty}^{+\infty} r f_R(r)\, dr = \int_{100}^{500} \frac{r}{400} dr = \frac{1}{400} \left. \frac{r^2}{2} \right]_{100}^{500} = 300$

ii) $\mathbb{E}(u(R)) = \int_{-\infty}^{+\infty} u(r) f_R(r) dr = \int_{100}^{500} \frac{\ln(r)}{400} dr = \frac{1}{400} \left. (r\ln r - r) \right]_{100}^{500}$

$= \frac{1}{400} (500 \ln 500 - 500 - 100 \ln 100 + 100$

$= \frac{5}{4} \ln 500 - \frac{5}{4} - \frac{1}{4} \ln 100 + \frac{1}{4} \overset{\text{(Hint)}}{=\!=} \frac{22}{4} = 5.5$

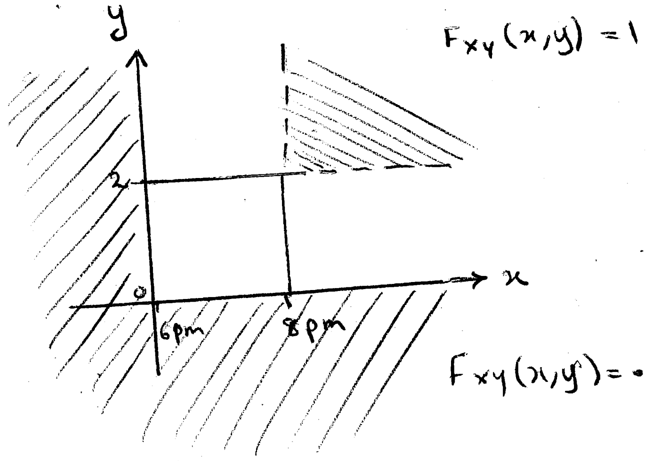
iii) $u(\mathbb{E}(R)) = \ln(300) \overset{\text{(Hint)}}{=\!=} 5.7$

iv) Yes, it holds because $5.5 \leq 5.7$

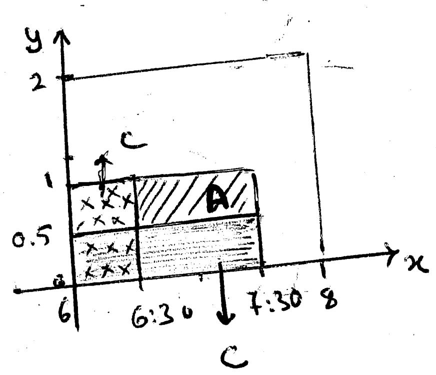
$\ln x$ is concave

4)

(a)



(b)



$P(A) = P(X \leq 7{:}30\text{ pm and } Y \leq 1) - P(B) - P(C) + P(A \cap C)$

$= F_{XY}(7{:}30\text{pm}, 1) - F_{xy}(6{:}30\text{pm}, 1) - F_{XY}(7{:}30\text{ pm}, 0.5) + F_{xy}(6{:}30\text{pm}, 0.5)$

(c) (I) for $z > 9$ or $z < 6$ $\Longrightarrow F_z(z) = 0$

(II) for $6 \leq z < 9$ assign $U = X + Y$, $V = Y$

$f_z(z) = f_u(z) = \int f_{uv}(z, v)\, dv$

Jacobian Matrix is $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Longrightarrow \det(J) = 1$

Therefore, $f_{uv}(z, v) = f_{xy}(z - v, u)$ since $X = U - V$, $V = Y$

So $f_U(z) = \int_0^z f_{XY}(z-v, v)\,dv$. But $f_{XY}(z-v, v) = 0$ for $\underline{\underline{z-v>8}}$ or $\underline{\underline{z-v<6}}$. Thus,

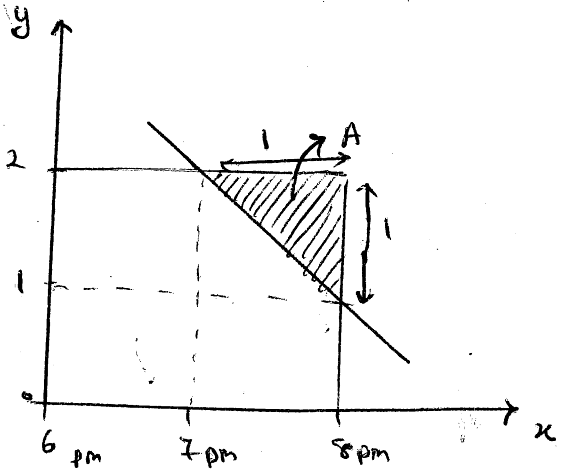
$$f_U(z) = \int_{\max(z-8,0)}^{\min(z-6,2)} f_{XY}(z-v, v)\,dv = \begin{cases} \int_0^{z-6} f_{XY}(z-v, v)\,dv & \text{for } z \le 8 \\ \int_{z-8}^{2} f_{XY}(z-v, v)\,dv & \text{for } z > 8 \end{cases}$$

(III) at 9, z has a point mass with weight $P(z=9) = P(X+Y \geq 9)$

$$P(X+Y \geq 9) = \int_1^2 \int_{9-v}^{8} f_{XY}(u,v)\,du\,dv. \text{ Therefore;}$$

$$f_Z(z) = \delta(z-9) \int_1^2 \int_{9-v}^{8} f_{XY}(u,v)\,du\,dv + \begin{cases} \int_0^{z-6} f_{XY}(z-v, v)\,dv & \text{for } 6 \le z \le 8 \\ \int_{z-8}^{2} f_{XY}(z-v, v)\,dv & \text{for } 8 < z < 9 \\ 0 & \text{otherwise} \end{cases}$$

(d) $$f_{XY}(x,y) = \begin{cases} f_X(x) \cdot f_Y(y) = \frac{1}{4} & 6 \le x \le 8 \text{ and } 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$



$$P(z \geq 9) = P(z=9) = P(X+Y \geq 9)$$

$$= \iint_A f_{XY}(x,y)\,dx\,dy$$

$$= \iint_A \frac{1}{4}\,dx\,dy = \frac{1}{4}\,\text{Area}(A) = \frac{1}{8}$$

5 (a) $$P(X=k \mid N=n) = \binom{n}{k} p^k (1-p)^{n-k} \qquad k=0,1,\dots,n$$

$$P(Y=k \mid N=n) = \binom{n}{k} p^{n-k} (1-p)^{k} \qquad k=0,1,\dots,n$$

(b) $$P(X=k) = \sum_{n=k}^{\infty} P(X=k \mid N=n)\, P(N=n)$$ Note: $k \le n$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{e^{-\mu}\mu^n}{n!}$$

$$= p^k e^{-\mu} \sum_{n=k}^{\infty} \frac{\cancel{n!}}{k!(n-k)!} \frac{(\mu(1-p))^{n-k}\mu^k}{\cancel{n!}}$$

$$= \frac{(\mu p)^k e^{-\mu}}{k!} \underbrace{\sum_{n=k}^{\infty} \frac{[\mu(1-p)]^{n-k}}{(n-k)!}}_{\underbrace{\sum_{i=0}^{\infty} \frac{[\mu(1-p)]^{i}}{i!}}_{e^{\mu(1-p)}}}$$

$$= e^{-\mu p} \frac{(\mu p)^k}{k!}$$

$\Rightarrow X \sim \text{Poisson}(\mu p)$

similarly $Y \sim \text{Poisson}(\mu(1-p))$

(c) $P(X=k, Y=J \mid N=n) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & k=0,1,\ldots,n \\ & J=n-k \\ 0 & \text{else} \end{cases}$

(d) $P(X=k, Y=J) = P(X=k, Y=J \mid N=k+J)\, P(N=k+J)$

no other value of N is valid

$$= \binom{k+J}{k} p^k (1-p)^J \cdot \frac{e^{-\mu} \mu^{(k+J)}}{(k+J)!}$$

$$= \frac{(\mu p)^k e^{-\mu p}}{k!} \quad \frac{[\mu(1-p)]^J e^{-\mu(1-p)}}{J!}$$

(e) YES. X and Y are independent since

$P(X=k, Y=J) = P(X=k)\, P(Y=J)$