Probability and Stochastic Processes (EL6303) November 2, 2018
NYU Tandon School of Engineering, Fall 2018
Instructor: *Dr. Elza Erkip*

Midterm
Please write your name and Net-id in the blue book eg: Ojas Kanhere, ok671
Do NOT write your N# number

Closed book/closed notes. No electronics, no calculators.
One 8.5 ×11 inch sheet of notes allowed
Time: 2 hours 25 minutes
Total 100 points

1. *(20 points)*
Suppose a store sells two types of rectangular poster boards, whose sides have random lengths.

- Type 1 has first side length $X_1 \sim \text{unif}(1, 2)$, second side length $X_2 \sim \text{unif}(3, 5)$.
- Type 2 has first side length $Y_1 \sim \text{unif}(2, 3)$, second side length $Y_2 \sim \text{unif}(1, 3)$.

We assume X_1, X_2, Y_1, and Y_2 are independent.

(a) Find the distribution of the area $A_1 = X_1X_2$.
(b) Find $E(A_1)$.
(c) Suppose you buy the type 1 board with probability $\frac{1}{2}$, type 2 board with probability $\frac{1}{2}$. Let Z_1 be the first side length of the board you buy, Z_2 be the second side length.
  i. Find the joint distribution of Z_1 and Z_2.
  ii. Are Z_1 and Z_2 independent? Explain.
  iii. Find the expected are of the board you buy $A_{buy} = E(Z_1Z_2)$.

2. *(20 points)*
   Suppose we have two independent discrete random variables X and Y with

$$X \sim \text{unif}\{1, 2, \ldots, k\},$$
$$Y \sim \text{unif}\{\frac{k}{2} + 1, \ldots, \frac{3k}{2}\}.$$

   We assume k is a positive even integer.
   Let

$$Z = \min(X, Y)$$
$$W = \max(X, Y)$$

   (a) Find the probability mass function of Z.
   (b) Find the probability mass function of W.
   (c) Find the joint probability mass function of (Z, W).
   (d) Let $U = W + Z$. Find the probability mass function of U, when $k = 4$.

3. *(20 points)*

There are two kinds of phones in the market: a cheap one with lifetimes $X \sim Exp(\lambda_1)$ and an expensive one with lifetime $Y \sim Exp(\lambda_2)$. We have $\lambda_1 \geq \lambda_2$. You decide to buy a cheap phone, and when it breaks, another cheap phone. We assume the lifetime of the cheap phones X_1, X_2 are independent and also independent of Y. Your mom buys the expensive phone.

(a) Find the probability density function of the total lifetime of your phones $X_1 + X_2$.

(b) Find the joint distribution of $X_1 + X_2$ and Y.

(c) Compare the total expected lifetime of your phones with the expected lifetime of your mom's phone.

(d) What is the probability that your mom's phone outlasts the total lifetime of your two phones?

Hint:

$X \sim Exp(\lambda)$ means $f_X(x) = \lambda e^{-\lambda x}$

Also, $\int_0^\infty x e^{-\lambda x} dx = \dfrac{1}{\lambda^2}$

4. *(20 points)*

Suppose $B_1, B_2 \ldots$ are independent binary digits such that for all i

$$p(B_i = 0) = p(B_i = 1) = \frac{1}{2}$$

Let $W = 0.B_1 \cdots B_n$ be a number in the interval $[0, 1)$ whose binary expansion has the most significant bit B_1 and least significant bit B_n.
Hence, $W = \sum_{i=1}^{n} B_i \; 2^{-i}$ in decimal.
Let $H = \sum_{i=1}^{n} B_i$. H is called the "Hamming weight of W".

(a) Find $E(W)$.

(b) Let $X \sim \text{unif}[0, 1]$. Find $E(X)$.

(c) As $n \to \infty$, compare (a) and (b). Comment.

(d) Find $P(B_1 = 1 | H = k)$.

(e) Find $P(H = i | W < 2^{-(n-2)})$ for $i = 1, \ldots, n$.

5. *(20 points)*

A taxi driver is waiting for passengers at the airport. Suppose passengers arrive every minute where $n = 1, 2, \ldots$ denotes time in minutes.

At any given time instant, the number of passengers arriving is represented by a discrete random variable X such that $P(X = i) = p_i \quad i = 0, 1, 2$. Note that at most two passengers arrive at any time. Assume that the number of passengers arriving at a given time instant is independent of the number of passengers arriving at any other time instant.

(a) The taxi driver wishes to take only one passenger. Let T be the waiting time. Find the probability mass function of T.

(b) Now the taxi driver decides that he will leave as soon as he has at least two passengers. Note that if the first non-zero arrival consists of one passenger, the driver waits for at least one more passenger to arrive. Let U be the waiting time. Find the probability mass function of U.

(c) We assume the taxi driver loses c dollars per minute while he is waiting.

   i. If he only takes one passenger, as in (a), how much should the passenger pay so that the driver's expected earning is positive?

   ii. When the driver decides to take at least two passengers, as in (b), he charges s dollars per passenger. Find s such that the driver's expected earning is positive. To solve this part, assume $p_0 = 0$

Hint:

$$\sum_{i=1}^{\infty} ip^i = \frac{p}{(1-p)^2}$$

$$\sum_{i=1}^{\infty} i^2 p^i = \frac{p(p+1)}{(1-p)^3}$$