Probability and Stochastic Processes (EL6303), Section I — September 26, 2018
NYU Tandon School of Engineering, Fall 2018
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Exercise 2 Solutions

1. Solution of Q1

a) If k is the number of heads, then

$$P(\text{even}) = P(k=0) + P(k=2) + \ldots$$
$$= q^n + \binom{n}{2} p^2 q^{n-2} + \binom{n}{4} p^4 q^{n-4} + \ldots$$

But

$$1 = (q+p)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = \binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \ldots$$
$$(q-p)^n = \sum_{k=0}^{n} \binom{n}{k} (-p)^k q^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k p^k q^{n-k}$$
$$= \binom{n}{0} p^0 q^n - \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} - \ldots$$

Adding, we obtain $1 + (q-p)^n = 2P(\text{even})$. Therefore,
$P(\text{even}) = \{1 + (q-p)^n\}/2$

b) if $p = 1$ and $q = 0$ then according to part a) we have $P(even) = 0.5(1 + (-1)^n)$ which makes sense because it is 1 when n is even and it is 0 when n is odd.

c) If 0< $p, q < 1$ then $P(even)$ converges to 0.5 according to part a) which makes sense. If $q = 1$ then $P(even)$ converges to zero which makes sense. When $p = 1$, $P(even)$ does not have a limit but it still makes sense according to part b).

2. Solution of Q2

(a) Since $P(X \le 0) = \int_{-\infty}^{0} f_X(x)dx = 0$ and $f_X(x) \ge 0, \forall x \in R$, $f_X(x) = 0, \forall x \le 0$. Since $U(x-2) = 0$ when $x = \epsilon \in [0, 2)$, if $a > 0$, there must exist a ϵ sufficiently small such that $P(X < \epsilon) = 0$. Hence, $a < 0$ contradicts $f_X(x) = 0, \forall x \le 0$. Thus, $a = 0$.

$$P(0 \le X \le 1) = \int_0^1 f_X(x)dx = \int_0^1 (\alpha x + \beta)dx = 0.5\alpha + \beta = 5/16.$$

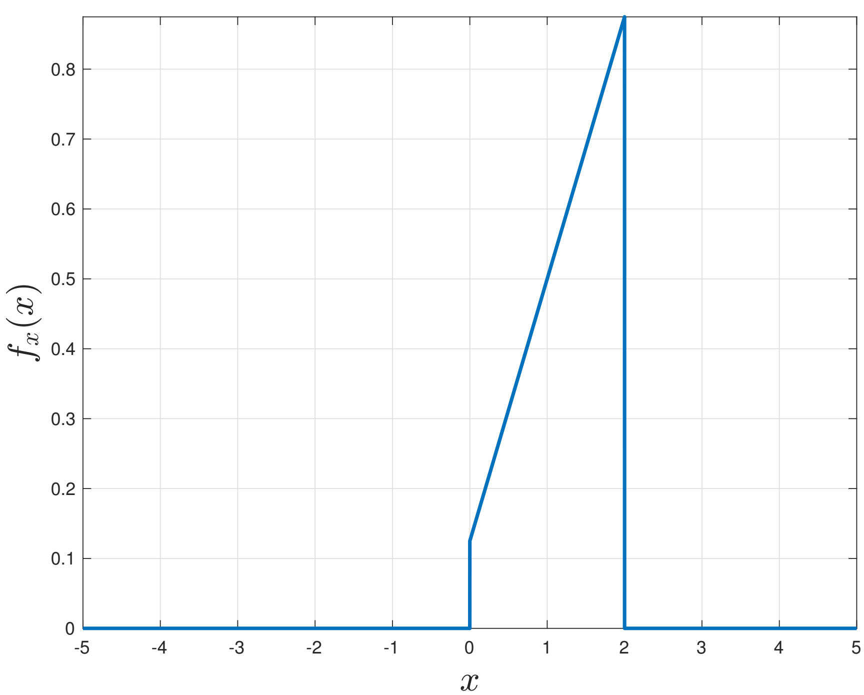
Also

$$P(X \ge 0) = \int_0^2 f_X(x)dx + \int_2^\infty f_X(x)dx$$
$$= \int_0^2 (\alpha x + \beta)dx + \int_2^\infty (\alpha x + \beta) * (1-1)dx = 2(\alpha + \beta) = 1$$

Thus, we have

$$\alpha = 3/8, \beta = 1/8$$

(b) See attached figure.



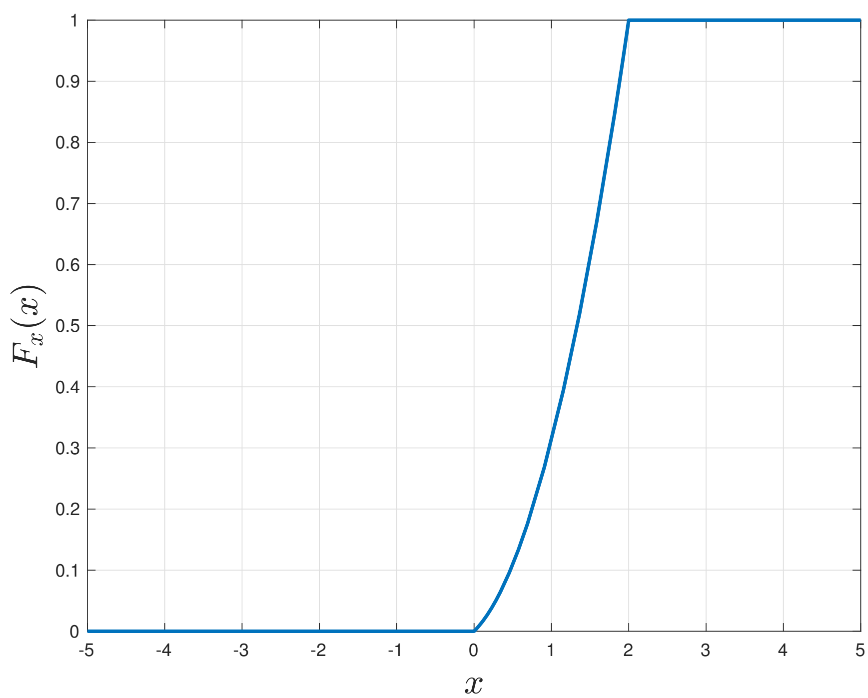
(c)

$$P(X \geq 0.5) = \int_{0.5}^{2} f_X(x)dx + \int_{2}^{\infty} f_X(x)dx = \int_{0.5}^{2} (\alpha x + \beta)dx = 57/64.$$

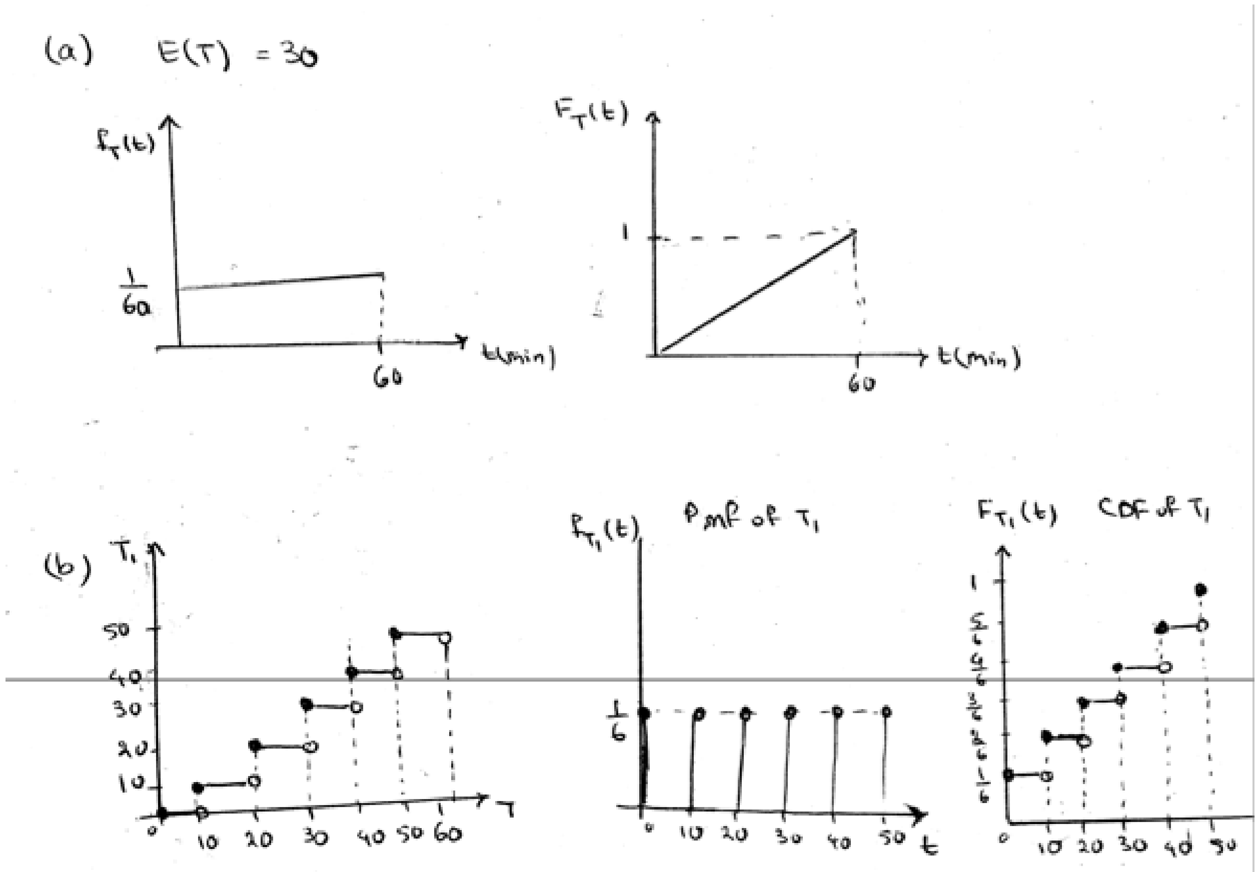
(d)

$$F_x(x) = \begin{cases} 0 & x \leq 0 \\ \frac{3}{16}x^2 + \frac{1}{8}x & 0 < x < 2 \\ 1 & x \geq 2 \end{cases}$$

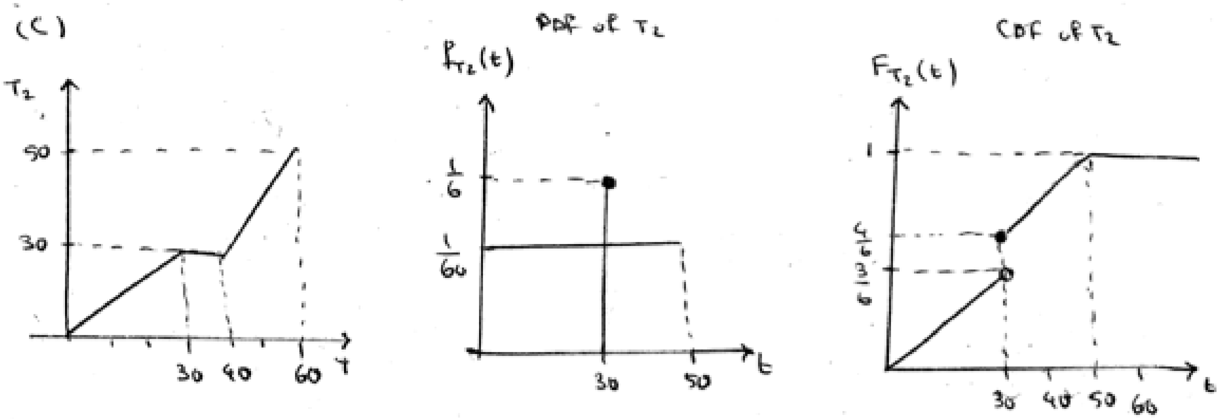
(e) See attached figure.



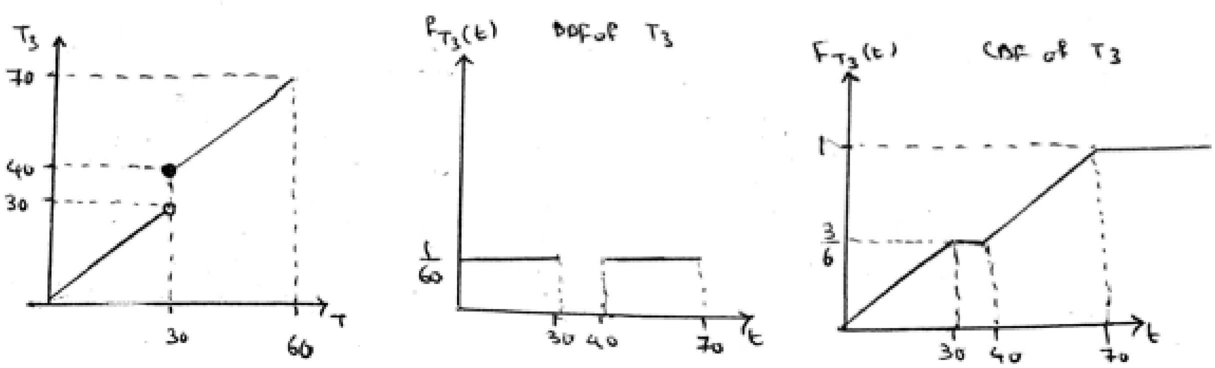
3. Solution of Q3



(c)


PDF of T2
CDF of T2
T_2
50
30
30 40 60
$f_{T_2}(t)$
$\frac{1}{6}$
$\frac{1}{60}$
30 50
$F_{T_2}(t)$
1
$\frac{4}{6}$
$\frac{3}{6}$
30 40 50 60

(d)


T_3
70
40
30
30 60
$f_{T_3}(t)$ PDF of T_3
$\frac{1}{60}$
30 40 70
$F_{T_3}(t)$ CDF of T_3
1
$\frac{3}{6}$
30 40 70

4. Solution of Q4

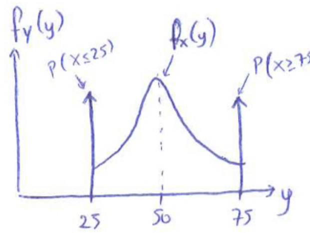
(a) $P(\text{Danger}) = P(X<20) + P(X>85)$

$$= P\left(\frac{X-\mu}{\sigma} < \frac{20-50}{10}\right) + P\left(\frac{X-\mu}{\sigma} > \frac{85-50}{10}\right)$$

$$= P\left(\frac{X-\mu}{\sigma} < -3\right) + P\left(\frac{X-\mu}{\sigma} > 3.5\right)$$

$$= Q(3) + Q(3.5)$$

(b)



Mixed.

(c) Estimate the probability of danger as: $P\binom{\text{Announce}}{\text{Danger}} = P(Y=25) + P(Y=75)$

$\Rightarrow$ Announce danger if $Y=25$ or $Y=75$.

(d) • $P(\text{Announce danger}) = P(Y=25,$ [redacted] $) + P(Y=75) = P\left(\frac{X-\mu}{\sigma} < \frac{25-50}{10}\right) + P\left(\frac{X-\mu}{\sigma} > \frac{75-50}{\sigma}\right) = 2Q(2.5)$

• $P(\text{Danger}) = Q(3) + Q(3.5)$

• $P(\text{Announce danger} \mid \text{Danger}) = 1$

• $P(\text{No Danger}) = 1 - Q(3) - Q(3.5)$

$$\Rightarrow \underbrace{P(\text{Announce danger})}_{2Q(2.5)} = \underbrace{P(\text{Announce danger} \mid \text{Danger})}_{1} \underbrace{P(\text{Danger})}_{Q(3)+Q(3.5)} + \underbrace{P(\text{Announce danger} \mid \text{No danger})}_{?} \underbrace{P(\text{No danger})}_{1-Q(3)-Q(3.5)}$$

$$\Rightarrow P(\text{Announce danger} \mid \text{No danger}) = \frac{2Q(2.5) - Q(3) - Q(3.5)}{1 - Q(3) - Q(3.5)}$$

(e) $P(\text{Does not announce danger} \mid \text{Danger}) = P(25 < Y < 75 \mid X<20 \text{ or } X>85)$

$$= 0.$$

5. Solution of Q5

(a)

Note: In this solution, I assume that the total number of transmissions is 'm', at most. If we assume '$m+1$' for the maximum number of transmissions, we should change all m's to $m+1$ in this solution.

S_i: Success in ith transmission F_i: Failure in ith transmission

$$\begin{cases} p(\text{successful delivery}) = P(S_1) + P(S_2 | F_1) + P(S_3 | F_1, F_2) + \cdots + P(S_m | F_1, \cdots, F_{m-1}) \\ \overset{\text{independency}}{=} P(S_1) + P(S_2)P(F_1) + P(S_3)P(F_1)P(F_2) + \cdots + P(S_m)P(F_1)\cdots P(F_{m-1}) \\ = (1-p) + (1-p)p + (1-p)p^2 + \cdots + (1-p)p^{m-1} = (1-p)\dfrac{p^m - 1}{p-1} = \boxed{1-p^m} \end{cases}$$

(b)

$$p(\text{failure}) = 1 - P(\text{success}) = p^m$$

$$p^m \leq 0.01 \rightarrow \frac{1}{4^m} \leq \frac{1}{100} \rightarrow 4^m \geq 100 \rightarrow \boxed{m \geq 4}$$

(c)

$$P(\text{success for each receiver}) = 1 - p^m$$

$$p(\text{success for all receivers}) \overset{\text{independency}}{=} \prod_{i=1}^{N} P(\text{success for receiver } i)$$

$$= (1-p^m)^N$$

(d)

$0 \le p \le 1 \rightarrow 0 \le p^m \le 1 \rightarrow 0 \le 1 - p^m \le 1 \rightarrow (1-p^m)^N \le 1-p^m$

$\lim_{N \to \infty} (1-p^m)^N = 0$

Obviosly, when we have more than one receiver and every receiver must get the packet to be successful the probabilty of success is less than one-receiver case. when $N \to \infty$ then $p(\text{success})$ approaches to zero.

(e)

your answer when $N \to \infty$.

success $\triangleq$ at least one of the receivers gets the packet successfully.

$p(\text{success}) = 1 - p(\text{failure}) = 1 - p(\text{none of the receivers can get the packet})$

$= 1 - (\underbrace{p^m \times p^m \times \cdots \times p^m}_{N \text{ times}}) = 1 - p^{mN}$

$1 - p^m < 1 - p^{mN}$

$\lim_{N \to \infty} 1 - p^{mN} = 1$ $\rightarrow$ this means that if $N \to \infty$ with probability 1 at least one of the receivers would get the packet successfully.

6. Solution of Q6

Solution:

(a) $T_1 > t$ means that no emails has arrived up to time t, which means $N_t = 0$.

(b)
$$F_{T_1}(t) \triangleq P(T_1 \leq t) = P(N_t \neq 0) = 1 - P(N_t = 0) = 1 - e^{-\mu t}$$

(c)
$$f_{T_1}(t) = \frac{dF_{T_1}(t)}{dt} = \mu e^{-\mu t}.$$