Probability and Stochastic Processes (EL6303) December 16, 2015
NYU Tandon School of Engineering, Fall 2015
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Final

Closed book, closed notes, no electronics, no calculators.
Only four formula sheets are allowed. Use the space below.

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1. *(~~8~~ points)* Answer the following TRUE or FALSE. X, Y denote random variables, X_n is a sequence of random variables, $X(t)$ denotes a stochastic process.

   ** Circle the correct answer. No proof needed. **

   (a) X and Y are independent $\Rightarrow$ X and Y are uncorrelated.
   TRUE FALSE

   (b) X and Y are uncorrelated $\Rightarrow$ X and Y are independent.
   TRUE FALSE

   (c) $X_n \to X$ in probability as $n \to \infty \Rightarrow X_n \to X$ with probability 1 as $n \to \infty$.
   TRUE FALSE

   (d) $X_n \to X$ with probability 1 as $n \to \infty \Rightarrow X_n \to X$ in probability as $n \to \infty$.
   TRUE FALSE

   (e) $X(t)$ is wide sense stationary $\Rightarrow$ $X(t)$ is strict sense stationary.
   TRUE FALSE

   (f) $X(t)$ is strict sense stationary $\Rightarrow$ $X(t)$ is wide sense stationary. and its second moment exists
   TRUE FALSE

   (g) $X(t)$ is mean-ergodic $\Rightarrow$ $X(t)$ is wide sense stationary.
   TRUE FALSE

   (h) $X(t)$ is wide sense stationary $\Rightarrow$ $X(t)$ is mean-ergodic.
   TRUE FALSE

   (i) $X(t)$ is wide sense stationary $\Rightarrow$ $X^3(t)$ is wide-sense stationary.
   TRUE FALSE

   (j) $X(t), Y(t)$ jointly wide sense stationary $\Rightarrow$ $X(t) + 5Y(t)$ is wide sense stationary.
   TRUE FALSE

2. *(15 points)* You are running late for EL 6303 final and decide to either take a taxi or Lyft. The waiting time for taxi is a random variable T_T, the waiting time for Lyft is a random variable T_L. We assume

$$T_T \sim \text{Uniform}(0, 10), T_L \sim \text{Uniform}(0, 5),$$

We also assume that once the taxi or Lyft car arrive, it takes them the same amount of time to get to school.

The price of the taxi is fixed at $P_T = 15$ dollars. Lyft price P_L, on the other hand, is a random variable. We assume P_L is independent of T_L and that

$$P_L \sim \text{Uniform}(14, 26).$$

You are concerned about getting to school as early as possible, but you also don't want to pay too much. Your *overall cost* is a weighted combination of these two objectives. Hence the overall cost for i is

$$C_i = aT_i + bP_i,$$

where $a, b \geq 0$ are constants and $i = T, L$ denotes taxi or Lyft.

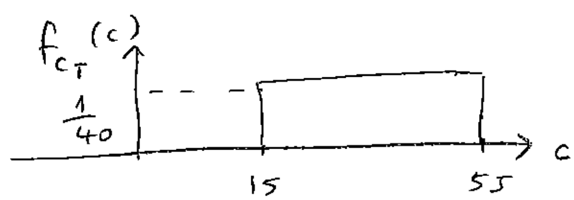
5 (a) Suppose $a = 4, b = 1$. *Find and sketch* the probability density function of C_T.

4 (b) For $a = 4, b = 1$, for which mode of transportation (taxi or Lyft) your *average* total $E(C_i)$ is smaller? Explain.

3 (c) If you didn't care about the time, but only wanted to minimize the price, how would you set a and b? Which one would you choose, taxi or Lyft? Explain.

3 (d) If you didn't care about the price, but only wanted to minimize your time, how would you set a and b? Which one would you choose, taxi or Lyft? Explain.

a) $C_T = aT_T + bP_T = 4T_T + 15$



$F_{C_T}(c) = P(C_T \leq c)$

$= P\left(T_T \leq \frac{c-15}{4}\right)$

$f_{C_T}(c) = \frac{1}{4} f_{T_T}\left(\frac{c-15}{4}\right)$

$C_T \sim \text{unif}(15, 55)$

b) $E(C_i) = a\,E(T_i) + b\,E(P_i)$

$E(T_T) = 5$

$E(P_T) = 15$

$E(T_L) = 2.5$

$E(P_L) = 20$

$\Rightarrow E(C_T) = 4 \times 5 + 15 = 35$

$E(C_L) = 4 \times 2.5 + 20 = 30$

C_L is smaller.

c) $a = 0,\ b = 1$

Choose Taxi

d) $a = 1,\ b = 0$

Choose Lyft

3. *(13 points)* Suppose that $N(t)$, the number of passengers arriving at Jay Street-Metrotech station to take the A train, can be modeled as a Poisson process with parameter β. That is

$$P(N(t) = n) = \frac{(\beta t)^n}{n!} e^{-\beta t}, t \geq 0, n = 0, 1, \ldots.$$

The waiting time T_A for the A train is independent of $N(t)$ and has an exponential distribution with parameter λ_A. That is the density of T_A is given by

$$f_{T_A}(t) = \lambda_A e^{-\lambda_A t}, t \geq 0.$$

Let Y be number of passengers getting on the A train from Jay Street-Metrotech station. Hence $Y = N(T_A)$.

5 (a) Find the expected value of Y in terms of β and λ_A

5 (b) Find the variance of Y in terms of β and λ_A

(c) Now suppose that all the $N(t)$ passengers can either take the A train or the C train, whichever comes first. The waiting time T_C for the C train is independent of $N(t)$ and T_A, and is exponential with parameter λ_C. Let Z be the number of passengers getting on the first arriving train from Jay Street-Metrotech station.

5 i. Find $E(Z)$.

3 ii. Find the relationship between λ_A and λ_C to have $E(Z) = E(Y)/3$.

(a) $E(Y) = E(E(Y | T_A))$

$= E(\underbrace{E(N(T_A) | T_A)}_{\beta T_A})$ mean of (Poisson)

$= \beta \, E(T_A)$

$= \dfrac{\beta}{\lambda_A}$ (mean of exponential)

b) $\mathrm{Var}(N(t)) = \beta t = E(N^2(t)) - \underbrace{(E(N(t)))^2}_{(\beta t)^2}$

$\Rightarrow E(N^2(t)) = \beta t + (\beta t)^2$

$E(Y^2) = E(\underbrace{E(Y^2 | T_A)})$

$E(N(T_A)^2 | T_A) = \beta T_A + \beta^2 T_A^2$

$= \beta E(T_A) + \beta^2 E(T_A^2) = \frac{\beta}{\lambda_A} + \beta^2 \left(\frac{1}{\lambda_A^2} + \frac{1}{\lambda_A^2} \right)$

$\mathrm{var}(T_A)$ $\quad E^2(T_A)$

$= \frac{\beta}{\lambda_A} + \frac{2\beta^2}{\lambda_A^2}$

$\mathrm{Var}(Y) = E(Y^2) - E^2(Y)$

$= \beta^2 E(T_A^2)$

$= \frac{\beta^2}{\lambda_A^2} + \frac{\beta}{\lambda_A}$

c) $T = \min(T_A, T_c)$

$P(T > t) = P(T_A > t)\, P(T_c > t)$

$= \left(\int_t^\infty \lambda_A e^{-\lambda_A \tau} d\tau \right) \left(\int_t^\infty \lambda_c e^{-\lambda_c \tau} d\tau \right)$

$-\lambda_A \tau = u$

$-\lambda_A d\tau = -du$

$\int_{-\infty}^{-\lambda_A t} e^u du = e^{-\lambda_A t}$

$$P(T > t) = e^{-(\lambda_A + \lambda_C)t}$$

$$F_T(t) = 1 - e^{-(\lambda_A + \lambda_C)t}$$

$$f_T(t) = (\lambda_A + \lambda_C) e^{-(\lambda_A + \lambda_C)} t \qquad Exp(\lambda_A + \lambda_C)$$

(i) $$E(Z) = \frac{\beta}{\lambda_A + \lambda_C}$$

(ii) $$\frac{\cancel{\beta}}{\lambda_A + \lambda_C} = \frac{1}{3} \frac{\cancel{\beta}}{\lambda_A} \Rightarrow 3\lambda_A = \lambda_A + \lambda_C$$

$$\Rightarrow \boxed{2\lambda_A = \lambda_C}$$

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4. (~~10~~ *points*) Consider two sequences of random variables X_n and Y_n, $n = 1, 2, \ldots$ and a random variable X. We are given that

$$P(|X_n - X| \leq Y_n) = 1,$$

for all n. Also $E(Y_n) \to 0$ as $n \to \infty$.

6 (a) Find $\lim_{n\to\infty} E(|X_n - X|)$. Explain your steps.

6 (b) Prove that $X_n \to X$ in probability as $n \to \infty$.

(a) Let $A_n = \{ |X_n - X| \leq Y \}$

$P(A_n) = 1$

$$\underset{\downarrow \atop 0}{E(|X_n - X|)} = E(|X_n - X| \mid A_n) \underbrace{P(A_n)}_{1} + E(|X_n - X| \mid A_n^c) \cancel{P(A_n^c)}^{0}$$

$$\leq E(Y_n \mid A_n)$$

$$= E(Y_n)$$

$$\downarrow$$

$$0$$

DONE.

(b) $P(|X_n - X| > \delta) \leq \dfrac{E(|X_n - X|)}{\delta}$ by markov ~~ineq~~

$\to 0$ for any $\delta > 0$

$\Rightarrow X_n \to X$ in prob.

5. *(15 points)* Consider the stochastic process $X(t) = A\cos(\pi t) + B\sin(\pi t)$. Here A and B are independent random variables such that

$$A = \begin{cases} -1, & \text{with probability } 3/4 \\ 3, & \text{with probability } 1/4 \end{cases}$$

$$B = \begin{cases} 1, & \text{with probability } 3/4 \\ -3, & \text{with probability } 1/4 \end{cases}$$

(a) Find $E(X(t))$.

(b) Find $R_{XX}(t_1, t_2)$.

(c) Is $X(t)$ WSS? Explain.

(d) Find the ~~joint~~ distribution of $X(0)$ ~~and X(0.25)~~.

(e) Find the ~~joint~~ distribution of $X(0.25)$ ~~and X(0.5)~~.

(f) Is $X(t)$ SSS? Explain.

a) $E(X(t)) = \underbrace{E(A)}_{0}\cos(\pi t) + \underbrace{E(B)}_{0}\sin(\pi t)$

$= 0$

b) $R_{XX}(t_1, t_2) = E(X(t_1)X(t_2))$

$= E\big((A\cos(\pi t_1) + B\sin(\pi t_1))(A\cos(\pi t_2) + B\sin(\pi t_2))\big)$

$= E(A^2)\cos(\pi t_1)\cos(\pi t_2) + E(B^2)\sin(\pi t_1)\sin(\pi t_2)$

$= \frac{5}{2}\cos(\pi(t_1 - t_2))$

c) Yes. mean indep of t, $R_{XX}(t_1, t_2)$ fn of $t_1 - t_2$.

d) $X(0) = A = \begin{cases} -1 & \text{w.p. } 3/4 \\ 3 & \text{w.p. } 1/4 \end{cases}$

e) $X(0.25) = \dfrac{A+B}{\sqrt{2}} = \begin{cases} -4/\sqrt{2} & \text{w.p. } 3/16 \\ 0 & \text{w.p. } 10/16 \\ 4/\sqrt{2} & \text{w.p. } 3/16 \end{cases}$

(f) Not SSS. $X(t)$ depends on t.

6. *(15 points)* Consider a zero mean WSS stochastic process $X(t)$ with power spectral density

$$S_{XX}(w) = \begin{cases} N_0/2, & -B \leq w \leq B \\ 0, & \text{else} \end{cases}$$

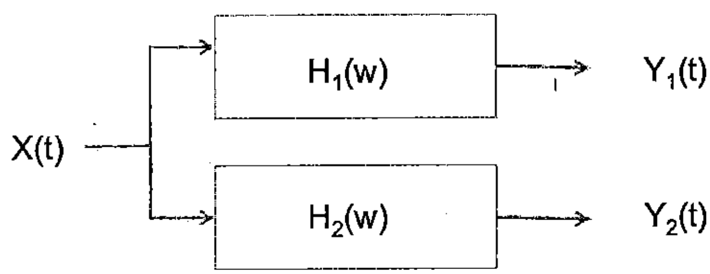
We consider two LTI systems with transfer functions $\mathcal{F}(h_1(t)) = H_1(w)$ and $\mathcal{F}(h_2(t)) = H_2(w)$ such that

$$H_1(w) = \begin{cases} K_1, & w \in (-0.1B, 0.1B) \cup (-B, -0.9B) \cup (0.9B, B) \\ 0, & \text{else} \end{cases}$$

$$H_2(w) = \begin{cases} -K_2, & w \in (-a, a) \cup (-0.8B, -0.7B) \cup (0.7B, 0.8B) \\ 0, & \text{else} \end{cases}$$

where K_1, $K_2 > 0$ and $0 \leq a \leq 0.1B$. Here $h_i(t)$ is the impulse response of system i and $\mathcal{F}$ denotes Fourier transform.

Now consider $Y_1(t)$ and $Y_2(t)$ below:



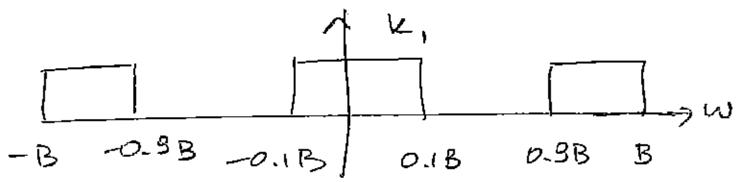
(a) Find $E(Y_1^2(3))$.

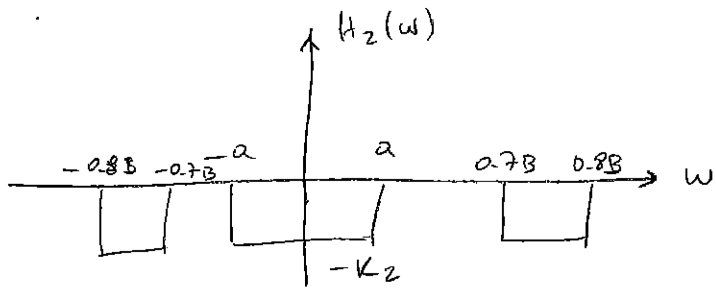
(b) Find $E(Y_2^2(-5))$.

(c) Find an expression for the cross-correlation function of $Y_1(t)$ and $Y_2(t)$, $R_{Y_1,Y_2}(t_1, t_2) = E(Y_1(t_1)Y_2(t_2))$.

(d) For $B = 100$ rad/sec, find the largest set of a's for which $Y_1(t)$ and $Y_2(t + \pi)$ are uncorrelated.

$H_1(\omega)$





(a) $S_{Y_1}(\omega) = S_X(\omega)\,|H_1(\omega)|^2$

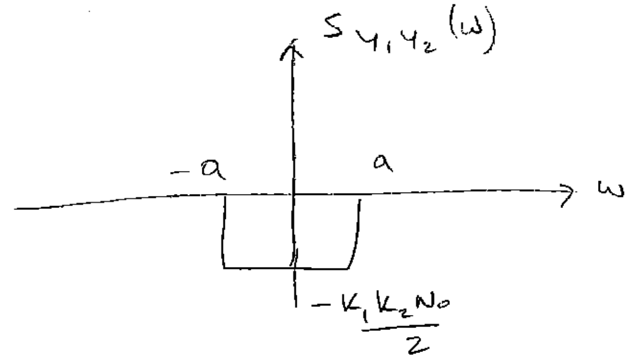
$$E(Y_1^2(t)) = R_{Y_1}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{Y_1}(\omega)\,d\omega$$

$$= \frac{K_1^2 N_0}{4\pi}\, 0.4B \qquad \text{for any } t$$

(b) $S_{Y_2}(\omega) = S_X(\omega)\,|H_2(\omega)|^2$

$$E(Y_2^2(t)) = \frac{K_2^2 N_0}{4\pi}\left(0.2B + 2a\right)$$

(c) $S_{Y_1Y_2}(\omega) = S_X(\omega)\, H_1(\omega)\, \underbrace{H_2^*(\omega)}_{H_2(\omega)}$



$$S_{Y_1Y_2}(\omega) = \begin{cases} -\dfrac{K_1K_2N_0}{2} & |\omega| < a \\ 0 & |\omega| > a \end{cases}$$

$$\Rightarrow R_{Y_1Y_2}(\underset{\tau = t_1 - t_2}{\tau}) = \frac{\sin a\tau}{\pi\tau}$$

(d) $R_{Y_1Y_2}(\pi) = 0 \quad \Rightarrow \quad \sin(a\pi) = 0$

$a = 0, 1, \ldots, 10$

integer

7. *(15 points)* Suppose $N(t)$ is a WSS Gaussian white noise stochastic process with $E(N(t)) = 0$ and $R_{NN}(\tau) = q\delta(\tau)$. Consider two (deterministic) pulses $p_1(t)$ and $p_2(t)$ such that

   - $p_i(t) \neq 0$ only when $0 \leq t \leq T$, $i = 1, 2$.
   - $\int_0^T p_i^2(t)dt = 1$, $i = 1, 2$.
   - $\int_0^T p_1(t)p_2(t)dt = 0$, that is $p_1(t)$ and $p_2(t)$ are orthogonal.

   Consider
   $$N_i(t) = \int_0^t N(u)p_i(u)du, i = 1, 2, 0 \leq t \leq T.$$

   (a) Find $E(N_1(t))$.

   (b) Find $E(N_1^2(t))$.

   (c) Find $E(N_1(t)N_2(t))$.

   (d) Are $N_1(t)$ and $N_2(t)$ uncorrelated for an arbitrary $0 \leq t \leq T$? Are they independent? Explain.

   (e) Now we set $t = T$ ~~and assume $\mu = 0$~~. Find the joint distribution of $N_1(T)$ and $N_2(T)$.

(a) $$E N_1(t) = \int_0^t \underbrace{E(N(u))}_{0} p_1(u)\,du$$

$$= 0$$

(b) $$E(N_1^2(t)) = \int_0^t \int_0^t \underbrace{E(N(u)N(v))}_{q\delta(u-v)} p_1(u)p_1(v)\,dv\,du$$

$$= q\int_0^t p_1^2(u)\,du$$

(c) $$E\big(N_1(t)\,N_2(t)\big) = \int_0^t\!\!\int_0^t \underbrace{E\big(N(u)N(v)\big)}_{q\delta(u-v)}\, \rho_1(u)\rho_2(v)\,dv\,du$$

$$= q \int_0^t \rho_1(u)\rho_2(u)\,du$$

(d) ~~In general~~ $\cancel{\int_0^t \rho_1(u)\rho_2(u)\,du \neq 0}$

~~$\Rightarrow N_1(t)$ and $N_2(t)$ are not uncorr.~~

~~are not indep.~~

(d) $$E\big(N_1(T)\,N_2(T)\big) = q \int_0^T \rho_1(u)\rho_2(u)\,du = 0$$

$\Rightarrow N_1(T),\ N_2(T)$ uncorr and <u>indep</u> since Gaussian

$$\begin{pmatrix} N_1(T) \\ N_2(T) \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}\right)$$

or $N_1(T),\ N_2(T)$ iid $\sim N(0, q)$