Probability and Stochastic Processes (EL6303) — November 27, 2018
NYU Tandon School of Engineering, Fall 2018
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Exercise 8

1. In a multipath wireless channel, the transmitted signal $X(t)$ can be subject to random changes in amplitude and random delays. The received signal in such a channel is given by

$$Y(t) = A_1 X(t - T_1) + A_2 X(t - T_2),$$

where A_1, A_2, T_1, T_2 are independent random variables. We assume $E(A_1) = E(A_2) = 0$.

   (a) Suppose $X(t)$ is a WSS process with mean η and autocorrelation function $R_X(\tau)$. The process $X(t)$ is independent of A_1, A_2, T_1, T_2.

      i. Find $E(Y(t))$.
      ii. Find $R_Y(t_1, t_2)$.
      iii. Is $Y(t)$ WSS? Explain.

   (b) Now suppose $X(t) = \cos(2\pi f_c t)$ where f_c is a constant. Also, suppose $T_1 \sim \text{Unif}(0, 1/f_c)$ and $T_2 \sim \text{Unif}(0, 2/f_c)$

      i. Is $X(t)$ WSS? Explain.
      ii. Find $E(Y(t))$.
      iii. Find $R_Y(t_1, t_2)$.
      iv. Is $Y(t)$ WSS? Explain.

   Hint 1: Recall $E(Z) = E(E(Z|W))$ where (Z, W) are random variables.

   Hint 2: $\cos(a)\cos(b) = \frac{1}{2}(\cos(a - b) + \cos(a + b))$

2. Consider a discrete random variable Z that takes on values in $\{1, \dots, k\}$ with probabilities $p_1, \dots, p_k$. Also consider k deterministic signals $s_i(t), i = 1, \dots, k$. We define the stochastic process $Y(t)$ as

$$Y(t) = s_Z(t).$$

   Hence $Y(t) = s_i(t)$ when $Z = i$.

   (a) Find $E(Y(t))$.

(b) Find $R_Y(t_1, t_2)$.

(c) Is $Y(t)$ WSS? Explain.

(d) For $k = 2$, $s_1(t) = t, s_2(t) = t^2$, draw different realizations of $Y(t)$.

(e) Now suppose we have a sequence of iid random variables $Z_1, Z_2, \dots$ distributed according to $(p_1, \dots, p_k)$. We define the stochastic process $U(t), t \geq 0$ as

$$U(t) = s_{Zj}(t), (j-1)T < t \leq jT,$$

where T is constant.

i. Find $R_U(t_1, t_2)$ for (t_1, t_2) such that $(j-1)T < t_1 \leq jT, (j-1)T < t_2 \leq jT$ for some j.

ii. Find $R_U(t_1, t_2)$ for (t_1, t_2) such that $(j-1)T < t_1 \leq jT, (l-1)T < t_2 \leq lT$ where $j \neq l$.

iii. For $k = 2$, $s_1(t) = t, s_2(t) = t^2$, draw different realizations of $U(t)$.

3. Let $X_n, n = 1, 2, \dots$ denote an iid sequence of Gaussian random variables with zero mean and unit variance. We define Y_n as the *weighted moving average* of two consecutive values of X_n as follows:

$$Y_n = aX_n + bX_{n-1}, n = 1, 2, \dots.$$

We assume $X_0 = 0$.

(a) Find $R_X(i, j)$.

(b) Find $E(Y_n)$.

(c) Find $R_Y(i, j)$.

(d) Is Y_n WSS? Explain.

(e) For $a = b = 1$, ind the joint pdf of (Y_1, Y_2, Y_3).

4. In a wireless channel, transmitted signals can be subject to random delays. If $X(t)$ denotes the transmitted signal, the received signal $Y(t)$ can be represented as

$$Y(t) = X(t - T).$$

We will assume that the delay T is a uniform random variable in the interval $(0, a)$.

(a) Suppose $X(t)$ is a deterministic signal, given by

$$X(t) = \begin{cases} 1, & 0 \leq t < b \\ 0, & \text{else} \end{cases}$$

i. Sketch two possible realizations of $Y(t)$. Carefully label both axes.
ii. Find $\eta_Y(t) = E(Y(t))$.
iii. Find $R_Y(t_1, t_2) = E(Y(t_1)Y(t_2))$ for $t_1 + b < t_2$.
iv. Is $Y(t)$ WSS? Explain.

(b) Now suppose $X(t)$ is a WSS process with mean $\eta_X = E(X(t))$ and autocorrelation function $R_X(\tau) = E(X(t+\tau)X(t))$. We assume T is independent of $X(t)$.

i. Find $\eta_Y(t) = E(Y(t))$.
Hint: Recall $E(Z) = E(E(Z|W))$ where (Z, W) are random variables.
ii. Find $R_Y(t_1, t_2) = E(Y(t_1)Y(t_2))$.
iii. Is $Y(t)$ WSS? Explain.

5. Consider a stochastic system which scales its input by $A(t)$, where $A(t)$ is a stochastic process. Hence

$$Y(t) = A(t)X(t),$$

where $X(t)$ is the system input, $Y(t)$ is the output. We assume $A(t)$ has mean $\mu_A(t)$ and autocorrelation $R_{AA}(t_1, t_2)$.

(a) Suppose $X(t)$ is a deterministic signal.

i. Find the mean of $Y(t)$.
ii. Find the autocorrelation of $Y(t)$.
iii. If $A(t)$ is WSS, is $Y(t)$ also WSS? Explain.

(b) Suppose $X(t)$ is a stochastic process with mean $\mu_X(t)$ and autocorrelation $R_{XX}(t_1, t_2)$. We assume $X(t)$ and $A(t)$ are independent.

i. Find the mean of $Y(t)$.
ii. Find the autocorrelation of $Y(t)$.
iii. Find the cross correlation $R_{XY}(t_1, t_2)$ between the input and the output.
iv. If both $A(t)$ and $X(t)$ are WSS, is $Y(t)$ also WSS? Explain.
v. If both $A(t)$ and $X(t)$ are WSS, are $X(t)$ and $Y(t)$ jointly WSS? Explain.
vi. Suppose both $A(t)$ and $X(t)$ were Gaussian processes. Would $Y(t)$ be also Gaussian? Explain.
vii. Suppose $X(t)$ is a white noise process. Would $Y(t)$ be also white noise? Explain.

6. We consider a stochastic process $N(t)$ that counts the number of particles arriving at a Geiger counter. Suppose Λ is an exponential random variable, namely $f_\Lambda(\lambda) = \alpha e^{-\alpha\lambda}$ for $\lambda > 0$. Conditional on $\Lambda = \lambda$, $N(t)$ is a Poisson process with rate λ.

(a) Find $P(N(t) = n|\Lambda = \lambda)$ for some $t > 0$.

(b) Using the conditional probability you found in (a), find the marginal distribution of $N(t)$, $P(N(t) = n)$.
Hint: $\int_0^\infty e^{-kx}x^n \, \mathrm{d}x = \frac{n!}{k^{n+1}}$

(c) Let T_1 denote the time of the arrival of the first particle. Find $F_{T_1}(t_1)$, the cdf of T_1.

(d) Is $N(t)$ an independent increment process? Prove your result.

7. Two stochastic processes $X(t)$ and $Y(t)$ are called *jointly wide-sense stationary (WSS)* if

- $X(t)$ is WSS.
- $Y(t)$ is WSS.
- The cross-covariance function $R_{X,Y}(t, t + \tau)$ is only a function of τ for all t, τ. Hence we can write $R_{X,Y}(\tau)$.

Suppose $X(t)$ and $Y(t)$ are jointly WSS. Assume $X(t)$ has mean μ_X and auto-covariance function $R_X(\tau)$. Similarly $Y(t)$ has mean μ_Y and auto-covariance function $R_Y(\tau)$ and the cross-covariance function is $R_{X,Y}(\tau)$.

For the questions below, express all quantities in their simplest form and, when possible, in terms of $\mu_X, \mu_Y, R_X(\tau), R_Y(\tau), R_{X,Y}(\tau)$.

(a) Let $Z(t) = X(t) + Y(t)$.
   i. Find the mean of $Z(t)$.
   ii. Find the auto-correlation function $R_Z(t, t + \tau)$ of $Z(t)$.
   iii. Is $Z(t)$ WSS? Explain.

(b) Now let $W(t) = X(t)Y(t)$.
   i. Find the mean of $W(t)$.
   ii. Find the auto-correlation function $R_W(t, t + \tau)$ of $W(t)$.
   iii. Is $W(t)$ WSS? Explain.

(c) For independent $X(t)$ and $Y(t)$, repeat part (7b).

8. Consider the stochastic process $X(t) = W_1 sin(2\pi ft) + W_2 cos(2\pi ft)$. Let $\mathbf{W} = (W_1, W_2)$. We assume W_1 and W_2 are jointly normal with mean vector $\eta = E(\mathbf{W}) = (0, 0)$ and covariance matrix $\mathbf{C}$

$$\mathbf{C} = E[(\mathbf{W} - \eta)^t(\mathbf{W} - \eta)] = \begin{pmatrix} \sigma^2 & \rho\sigma \\ \rho\sigma & 1 \end{pmatrix}.$$

For the questions below, you may find the following trigonometric formulas useful:

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b),$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b).$$

(a) Find $E(X(t))$.

(b) Find $R_X(t_1, t_2)$.

(c) Find σ^2 and ρ such that $X(t)$ is WSS. Do you need W_1 and W_2 to be independent for $X(t)$ to be WSS? Explain your answer.

(d) For the σ^2 and ρ found in part (c), is $X(t)$ SSS? Explain.

9. Consider the stochastic process $X(t) = A\cos(\pi t) + B\sin(\pi t)$. Here A and B are independent random variables such that

$$A = \begin{cases} -1, & \text{with probability } 3/4 \\ 3, & \text{with probability } 1/4 \end{cases}$$

$$B = \begin{cases} 1, & \text{with probability } 3/4 \\ -3, & \text{with probability } 1/4 \end{cases}$$

(a) Find $E(X(t))$.

(b) Find $R_{XX}(t_1, t_2)$.

(c) Is $X(t)$ WSS? Explain.

(d) Find the distribution of $X(0)$.

(e) Find the distribution of $X(0.25)$.

(f) Is $X(t)$ SSS? Explain.