Probability and Stochastic Processes (EL6303)
NYU Tandon School of Engineering, Fall 2015
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October 28, 2015

Midterm

Closed book, closed notes, no electronics, no calculators.
Only two formula sheets are allowed. Use the space below.
Long questions: 70 points
Multiple choice and true/false: 30 points.

1. *(10 points)* Consider random variables (U, V) with joint probability density function

$$f_{UV}(u,v) = \begin{cases} K, & 0 < u < 1, 0 < v < 1 \\ K, & -1 < u < 0, -1 < v < 0 \\ 0, & \text{else} \end{cases}$$

Let $Z = UV, W = V$.

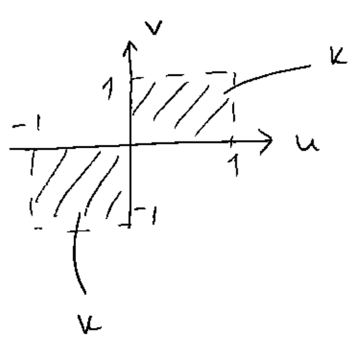
(a) Find K.

(b) Are U and V independent? Explain.

(c) Find the set $\{(z,w) : f_{ZW}(z,w) > 0\}$ where $f_{ZW}(z,w)$ is the joint probability density function of (Z,W).

(d) Find $f_{ZW}(z,w)$ the joint probability density function of (Z,W).

(a)



$\iint f_{uv}(u,v)\,du\,dv = 2K = 1$

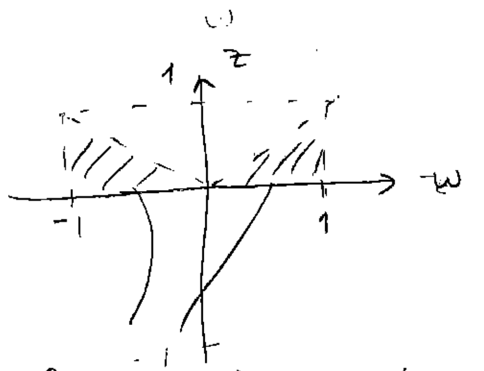
$\Rightarrow \boxed{K = 1/2}$

(b) No, because $f_{uv}(u,v)=0$ for ex at $(-1/2, 1/2)$ even though $f_u(u) \neq 0$

(c) $z = uv \rightarrow$ takes values in $(0, |w|)$

$w = v \rightarrow$ takes values in $(-1, 1)$

$$\{(z,w) : f_{zw}(z,w) > 0\} = \{-1 < w < 1,\ 0 < z < |w|\}$$



$f_{zw}(z,w) > 0$ here

(d) $J = \det \begin{bmatrix} v & u \\ 0 & 1 \end{bmatrix} = v = w$

For $\left.\begin{array}{l} z = uv \\ w = v \end{array}\right\}$ $\begin{array}{l} u = z/w \\ v = w \end{array}$ Only one soln for (z,w) in the shaded region in part (c).

$$f_{zw}(z,w) = \frac{1}{|w|} f_{uv}(z/w, w)$$

$$= \begin{cases} \dfrac{1}{2|w|} & -1 < w < 1,\ 0 < z < |w| \\ 0 & \text{else} \end{cases}$$

2. *(10 points)* Suppose random variables X and Y are jointly normal $N(0,0,\sigma_1^2,\sigma_2^2,\rho)$. Let

$$Z = dX + cY, W = cX + dY,$$

for some constants c, d.

(a) Find $E(Z^2)$ and $E(ZW)$.

(b) For $\sigma_1 = \sigma_2 = 1$, and $d = 1$, find c so that Z and W are independent. Explain.

$N(0,0,\sigma_1^2,\sigma_2^2,\rho)$, $\quad Z = dX + cY$, $\quad W = cX + dY$.

(a) $\rho = \dfrac{E\{XY\}}{\sigma_1 \sigma_2}$

$$E(Z^2) = E\{(dX+cY)^2\} = E\{d^2X^2 + 2dcXY + c^2Y^2\}$$
$$= d^2\sigma_1^2 + 2dc\rho\sigma_1\sigma_2 + c^2\sigma_2^2$$
$$E(ZW) = E\{(dX+cY)(cX+dY)\}$$
$$= dc\sigma_1^2 + (d^2+c^2)E\{XY\} + dc\sigma_2^2$$
$$= dc(\sigma_1^2+\sigma_2^2) + (d^2+c^2)\rho\sigma_1\sigma_2$$

(b) $E\{ZW\} = 2dc + (d^2+c^2)\rho = \rho c^2 + 2c + \rho = 0$

$$\Rightarrow \quad c = \frac{-1 \pm \sqrt{1-\rho^2}}{\rho}$$

3. *(10 points)* Suppose Y is a random variable with the probability density function $f_Y(y) = e^{-(y+2)}, y > -2$. Let $U = h(Y)$ with

$$h(y) = \begin{cases} 0, & y < -1 \\ -1-y, & -1 \le y < 0 \\ y, & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases}$$

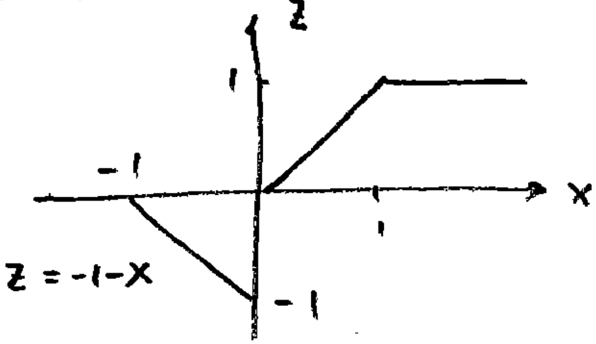
(a) Find and sketch $f_U(u)$, the probability density function of U.

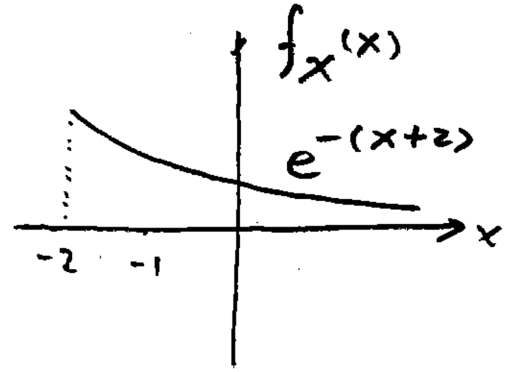
(b) Find and sketch $F_U(u)$, the cumulative distribution function of U.

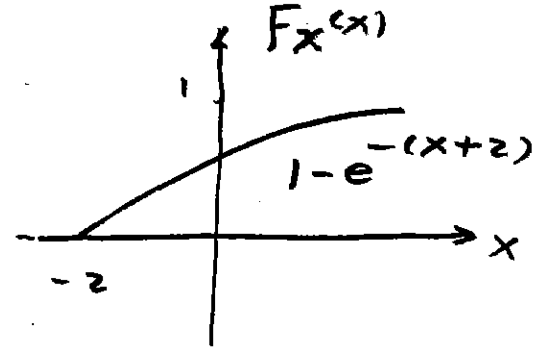
(c) Find $E(U)$.

(odd) 1

1.







For $-1<z<0$, $f_z(z)=f_x(-1-z) = e^{-(-1-z+2)} = e^{-(1-z)}$.

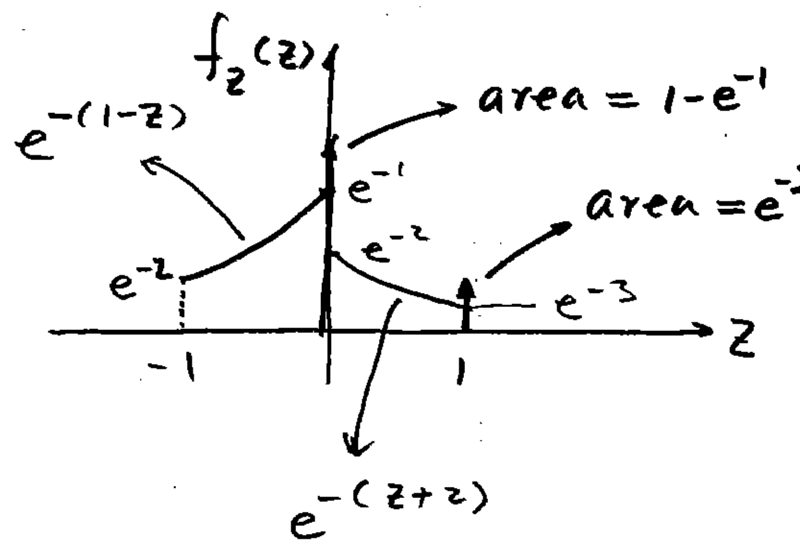
For $z=0$, $f_z(z) = F_X(-1) \overset{\delta(z)}{=} (1-e^{-(x+2)})\Big|_{x=-1} \cdot \delta(z) = (1-e^{-1})\,\delta(z)$.

For $0<z<1$, $f_z(z) = f_X(z) = e^{-(z+2)}$.

For $z=1$, $f_z(z) = (1-F_X(1))\,\delta(z-1) = (1-1+e^{-(x+2)})\,\delta(z-1)$

$= e^{-3}\delta(z-1)$

$$f_z(z) = \begin{cases} e^{-(1-z)}, & -1<z<0 \\ (1-e^{-1})\,\delta(z), & \text{at } z=0 \\ e^{-(z+2)}, & 0<z<1 \\ e^{-3}\delta(z-1), & \text{at } z=1 \\ 0, & \text{else} \end{cases}$$



For $-1 < z < 0$, $F_Z(z) = P(Z \le z) = P((Z<-1) \cup (-1 \le Z \le z))$

$= P(-1 \le -1 - X \le z) = P(-1 - z \le X < 0)$

$= F_X(0) - F_X(-1-z) = 1 - e^{-(0+2)} - 1 + e^{-(-1-z+2)}$

$= e^{-(1-z)} - e^{-2}$

Or $F_Z(z) = \int_{-\infty}^{z} f_Z(\alpha)\,d\alpha = \int_{-\infty}^{-1} f_Z(\alpha)\,d\alpha + \int_{-1}^{z} e^{-(1-\alpha)}\,d\alpha$

$= (e^{-1})\, e^{\alpha}\Big|_{-1}^{z} = e^{-1+z} - e^{-2}$

For $0 < z < 1$, $F_Z(z) = P(Z \le z) = P(-1 < Z < 0,\ Z = 0,\ 0 \le Z < z)$

$= P(-1 < Z < 0) + P(Z=0) + P(0 < X < z)$

$= (e^{-(1-z)} - e^{-2})_{-1}^{0} + (1 - e^{-1}) + P(0 < X < z) = 1 - e^{-(z+2)}$

Or use upper part of Z, get $F_Z(z) = P(Z \le z) =$

$= P(X \le z) = F_X(z) = 1 - e^{-(z+2)}$

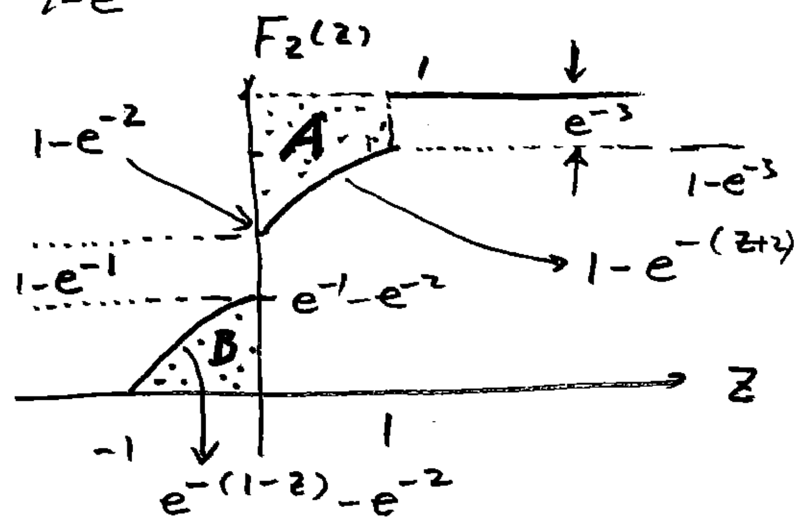
$$F_Z(z) = \begin{cases} 0, & z < -1 \\ e^{-(1-z)} - e^{-2}, & -1 \le z < 0 \\ 1 - e^{-(z+2)}, & 0 \le z < 1 \\ 1, & z \ge 1. \end{cases}$$

(3) $E(Z) =$

$\int_{-1}^{0} z e^{-(1-z)}\,dz$

$+ \int_{0}^{1} z\, e^{-(z+2)}\,dz + e^{-3}$

$= -e^{-1} - e^{-3} + 3e^{-2}$



Or $E(Z) = A - B = \int_{0}^{1} (1 - (1 - e^{-(z+2)}))\,dz$

$- \int_{-1}^{0} (e^{-(1-z)} - e^{-2})\,dz = -e^{-1} - e^{-3} + 3e^{-2}$

4. *(10 points)* Suppose $Y_1, \ldots, Y_n$ are independent random variables that have the same probability density function $f_Y(Y)$. Let

$$Z = \max(Y_1, \ldots, Y_n).$$

(a) Find the cumulative distribution function $F_Z(z)$ of Z in terms of $f_Y(y)$.

(b) Find the probability density function $f_Z(z)$ of Z in terms of $f_Y(y)$.

(c) For $Y \sim$ Exponential(1), with $f_Y(y) = e^{-y}, y > 0$.
Find $P(Z \geq 1)$.

(a) Cdf of Y_i $F_Y(y) = P(Y \leq y) = \int_{-\infty}^{y} f_Y(u)\,du$

$$F_Z(z) = P(Z \leq z) = P(\max(Y_1, \ldots Y_n) \leq z)$$
$$= P(Y_i \leq z \quad i = 1, 2, \ldots, n)$$
$$= \prod_{i=1}^{n} P(Y_i \leq z)$$
$$= [F_Y(z)]^n$$

(b) $f_Z(z) = \dfrac{d F_Z(z)}{dz} = n[F_Y(z)]^{n-1} f_Y(z)$

(b) $P(Z \geq 1) = 1 - P(Z < 1) = F_Z(1)$
$$= [F_Y(1)]^n$$

$$F_Y(1) = \int_0^1 e^{-u}\,du = 1 - e^{-1}$$

$$\Rightarrow P(Z \geq 1) = 1 - \left(1 - \tfrac{1}{e}\right)^n$$

6. *(15 points)* Consider $X = B\cos(w_c t + \Theta)$. Throughout this question we will consider a *fixed* t.

   (a) Suppose w_c and Θ are constants and B is a Rayleigh random variable with pdf

   $$f_B(b) = be^{-b^2/2}, b \geq 0.$$

   Find $E(X^2)$.

   (b) Now suppose B and w_c are constants and Θ is a uniform random variable in the interval $(-\pi, \pi)$. Find $E(X)$.

   (c) Now suppose B and Θ are constants and w_c is a uniform random variable in the interval $(w_0 - w_1, w_0 + w_1)$.
   Evaluate $E(X)$ for $\Theta = 0$, $w_1 = \pi/2$ Hz, $t = 1$.

(a) w_c, θ, t fixed $\Rightarrow$ $\cos(w_c t + \theta)$ fixed

$$E(X^2) = E(B^2)\cos^2(w_c t + \theta)$$

$$E(B^2) = \int_0^\infty b^2 f_B(b)\,db$$

$$= \int_0^\infty b^3 e^{-b^2/2}\,db \qquad u = \frac{b^2}{2} \quad du = b\,db$$

$$= \int_0^\infty 2u\, e^{-u}\,du$$

$$= 2\underbrace{\int_0^\infty u e^{-u}\,du}_{\text{mean of } \exp(1) \text{ r.v } = 1}$$

$$= 2$$

(b) $E(X) = B\,E(\cos(\omega_c t+\theta))$

$$= B\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega_c t+\theta)\,d\theta$$

$$= \frac{B}{2\pi} \sin(\omega_c t+\theta)\Big|_{-\pi}^{\pi}$$

$$= \frac{B}{2\pi}\Big[\underbrace{\sin(\omega_c t+\pi)}_{\sin(\omega_c t)} - \underbrace{\sin(\omega_c t-\pi)}_{\sin(\omega_c t)}\Big]$$

$$= 0$$

(c) $E(X) = B\,E(\cos(\omega_c t+\theta))$ $\quad t=1, \theta=0$

$$= B\,E(\cos(\omega_c))$$

$$= \frac{B}{2\omega_1}\int_{\omega_0-\omega_1}^{\omega_0+\omega_1} \cos(\omega_c)\,d\omega_c$$

$$= \frac{B}{2\omega_1} \sin(\omega_c)\Big|_{\omega_0-\omega_1}^{\omega_0+\omega_1}$$

$$= \frac{B}{2\omega_1}\left[\sin(\omega_0+\omega_1) - \sin(\omega_0-\omega_1)\right]$$

$$= \frac{B}{\cancel{2}\omega_1}\left(\cancel{2}\sin(\omega_1)\cos(\omega_0)\right)$$ (See formulas

$\omega_1 = \pi/2$
$\Rightarrow \sin(\omega_1) = 1$ $\quad = \frac{2B}{\pi}\cos(\omega_0)$