Probability and Stochastic Processes (EL6303) October 19, 2018
NYU Tandon School of Engineering, Fall 2018
Instructor: *Dr. Elza Erkip*

Quiz 2 Solutions

1. (a) When $a \geq 0$
   $\mathrm{P}((X \geq a)) = \int_a^\infty f_X(x)\ \mathrm{dx}$
   $= \int_a^\infty \lambda e^{-\lambda x}\ \mathrm{dx},$
   $= e^{-\lambda a}$
   When $a < 0$, $\mathrm{P}(X \geq a) = 1$

   (b)

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\ \mathrm{dx}$$
$$= \int_0^\infty x\lambda e^{-\lambda x}\ \mathrm{dx}$$
$$= \frac{1}{\lambda}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\mathrm{dx}$$
$$= \int_0^\infty x^2\lambda e^{-\lambda x}\ \mathrm{dx}$$
$$= \frac{2}{\lambda^2}$$

   Thus, $Var(X) = E(X^2) - (E(X))^2 = \dfrac{1}{\lambda^2}$

   (c) By Markov's inequality, for a positive random variable, when $a \geq 0$
   $P(X \geq a) \leq \dfrac{E(X)}{a}$
   Thus, for the exponential random variable,
   $P(X \geq a) \leq \dfrac{1/\lambda}{a} = \dfrac{1}{\lambda a} \quad a \geq 0.$

   (d) By the Chebyshev's inequality, for a random variable Y with mean μ and variance σ^2 ,when $a \geq 0$

$$P(|Y - \mu| \geq a) \leq \frac{\sigma^2}{a^2} = \frac{E(|Y - \mu|^2)}{a^2}$$

Set $X = |Y - \mu|$ to get
$P(X \geq a) \leq \dfrac{E(X^2)}{a^2} = \dfrac{2}{\lambda^2 a^2} \quad a \geq 0$

(e) The bound in part (c) is tighter than part (d) when $\dfrac{1}{\lambda a} < \dfrac{2}{\lambda^2 a^2}$
i.e., when $\lambda a < 2$

(f)

$$\begin{aligned}
\Phi_X(s) &= E(e^{sx}) \\
&= \int_0^\infty e^{sx} \lambda e^{-\lambda x} \text{ dx} \\
&= \lambda \int_0^\infty e^{(s-\lambda)x} \\
&= \frac{\lambda}{\lambda - s}, s < \lambda
\end{aligned}$$

Note that the moment generating function is undefined for $s \geq \lambda$

(g) When $a \geq 0$

$$\begin{aligned}
P(X \geq a) &= E\left(I(X \geq a)\right) \\
&\leq E\left(e^{s(X-a)}\right) \quad s \geq 0 \\
&= e^{-sa} E\left(e^{sX}\right) \\
&= e^{-sa} E\left(\Phi_X(s)\right) \quad \text{when } s \geq 0 \text{ and when } \Phi_X(s) \text{ is well defined}
\end{aligned}$$

Since this inequality is true for all $s \geq 0$ when $\Phi_X(s)$ is well defined,

$$\text{P}(X \geq a) \leq \min_{\substack{s \geq 0 \\ \Phi_X(s) \text{ well defined}}} [e^{-sa} \Phi_X(s)].$$

(h) When X $\sim$ Exponential(λ),

$$P(X \geq a) \leq \min_{\substack{s \geq 0 \\ \Phi_X(s) \text{ well defined}}} \left[e^{-sa} \Phi_X(s)\right]$$
$$P(X \geq a) \leq \min_{\substack{s \geq 0 \\ s < \lambda}} \left[e^{-sa} \frac{\lambda}{\lambda - s}\right]$$

The minima of can be found by setting the derivative of
$e^{-sa} \dfrac{\lambda}{\lambda - s}$ with respect to s to be 0.

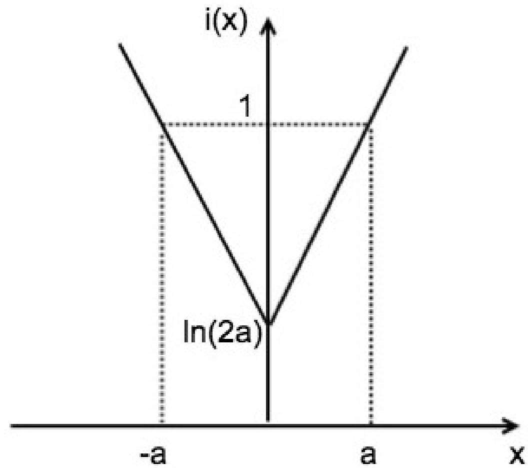
The minima occurs at $s = \lambda - \frac{1}{a}$.
Thus,
$P(X \geq a) \leq (a\lambda) \cdot e^{1-a\lambda}$

(i) When $a = 1$ and $\lambda = 1$,
As per the bound in (c):
$P(X \geq a) \leq \frac{1}{\lambda a} = 1$
As per the bound in (d):
$P(X \geq a) \leq \frac{2}{\lambda^2 a^2} = 2$
As per the bound in (h):
$P(X \geq a) \leq (a\lambda) \cdot e^{1-a\lambda} = 1$

Bounds (c) and (h) are equally strong. Technically they are stronger than the bound in (d), however none of the inequalities tell us anything useful as by definition, $P(X \geq a) \leq 1$ for all random variables X.

2. (a)

$$i(x) = -ln(\frac{1}{2a}e^{-\frac{|x|}{a}}) = -ln\frac{1}{2a} - ln(e^{-\frac{|x|}{a}})$$
$$= \frac{|x|}{a} + ln(2a), a > 0.$$



Case 1: $i < ln(2a)$:

$$F_I(i) = P(I \leq i) = 0.$$
$$f_I(i) = 0, i < ln(2a)$$

Case 2: $i \geq ln(2a)$:

$$F_I(i) = P(I \leq i) = P(\frac{|x|}{a} + ln(2a) \leq i) = P(|x| \leq a(i - ln(2a)))$$
$$= \int_{-a(i-ln(2a))}^{a(i-ln(2a))} f_X(x)dx = \int_{-a(i-ln(2a))}^{a(i-ln(2a))} \frac{1}{2a}e^{-\frac{|x|}{a}}dx = 2\int_0^{a(i-ln(2a))} \frac{1}{2a}e^{-\frac{x}{a}}dx. = 1 - 2ae^{-i}.$$

From the CDF, we take derivative to obtain the PDF

$$f_I(i) = \frac{dF_I(i)}{di} = 2ae^{-i}, i \geq ln(2a)$$

(b) Since both $i(x)$ and the $f_X(x)$ are even functions, we have

$$h(X) \triangleq E(i(X)) = \int_{-\infty}^{\infty} i(x) f_X(x) dx$$
$$= 2\int_0^{\infty} (\frac{|x|}{a} + ln(2a))(\frac{1}{2a}e^{-\frac{|x|}{a}})dx$$
$$= \frac{1}{a^2}\int_0^{\infty} xe^{-x/a}dx + \frac{ln(2a)}{a}\int_0^{\infty} e^{-x/a}dx = 1 + ln(2a)$$

(c) When $0 < a < e^{-1}/2$, we have $1 + ln(2a) < 0$.
When $a \geq e^{-1}/2$, we have $1 + ln(2a) \geq 0$.
Thus, it is **not** always non-negative.