Probability and Stochastic Processes (EL6303), Section A September 10, 2018
NYU Tandon School of Engineering, Fall 2018
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Diagnostic Quiz

1. If $A \cap B = \emptyset$, $P(A) \neq 0, P(B) \neq 0, P(C) \neq 0$, then state whether the following are true or false. Explain.

   (a) $P(A|B) = P(A)$.

   (b) $P(A \cap B^c) = P(A)$.

   (c) $P(A \cup B) < P(A) + P(B)$.

   (d) $P(A^c \cap B^c) = P(A^c)P(B^c)$.

   (e) $P(A \cup B|C) = P(A|C) + P(B|C)$.

Solution 1

(a) **False**

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = 0 \neq P(A).$$

(b) **True**
$P(A) = P(A \cap (B^c \cup B)) = P(A \cap B^c) + P(A \cap B) = P(A \cap B^c)$.

(c) **False**
$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B)$.

(d) **False**
$P(A^c \cap B^c) = 1 - P((A^c \cap B^c)^c) = 1 - P(A \cup B) = 1 - (P(A) + P(B))$. Also we know $P(A^c)P(B^c) = (1 - P(A))(1 - P(B)) = 1 - (P(A) + P(B)) + P(A)P(B)$. Thus, it is true only when $P(A)P(B) = 0$ which is not possible for $P(A) \neq 0$ and $P(B) \neq 0$. Thus, the statement $P(A^c \cap B^c) \neq P(A^c)P(B^c)$ means that A^c and B^c are dependent.

(e) **True**

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\begin{aligned}
P(A \cup B|C) &= \frac{P((A \cup B) \cap C)}{P(C)} \\
&= \frac{P((A \cap C) \cup (B \cap C))}{P(C)} \\
&= \frac{P(A \cap C) + P(B \cap C)}{P(C)} \\
&= P(A|C) + P(B|C).
\end{aligned}
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2. Consider a communication channel that flips each transmitted bit with probability α. In order to protect transmitted information, we repeat the same bit multiple times. For example, to communicate a 0, we send 000; to communicate a 1, we send 111. At the receiver if we receive more 0's than 1's, we decide the original bit was a 0; otherwise we decide the original bit was a 1.

   (a) Suppose we would like to communicate 0. Find probability that we receive 101.

   (b) If probability of communicating a 0 and 1 are both 0.5, find the probability that the receiver makes an error.

Solution 2

(a)

$$P(101|send0) = \alpha * (1-\alpha) * \alpha.$$

(b) Because of the law of the total probability, i.e.,

$$P(error) = P(error|send\ 0)P(send\ 0) + P(error|send\ 1)P(send\ 1)$$

The receiver makes an error if the channel flips two or three times, which is called event A. Then, $P(A) = \binom{3}{2} * \alpha * (1-\alpha) * \alpha + \binom{3}{3} * \alpha * \alpha * \alpha = 3\alpha^2 - 2\alpha^3$. Thus, the final probability of error is

$$P(error) = (3\alpha^2 - 2\alpha^3) * 0.5 + (3\alpha^2 - 2\alpha^3) * 0.5 = 3\alpha^2 - 2\alpha^3.$$

3. (a) Suppose two random variables X and Y are Gaussian and uncorrelated. Are X and Y independent? Explain.

   (b) Consider two independent random variables X and Y. Are $\cos(X)$ and Y^3 independent? Explain.

   (c) Suppose we pick a number uniformly in the unit interval $(0, 1)$. What is the probability that we pick a rational number? Explain.

Solution 3

(a) Yes, X and Y are independent. In general, when two random variables are uncorrelated, they need not be independent. However, when two random variables are jointly Gaussian, they are independent.

(b) Yes. Continuous functions of independent random variables are independent.

(c) The probability that we pick a rational number is 0. This is because the number of irrational numbers is far more than the number of rational numbers. Put more formally, the *measure* of rational numbers in the unit interval $(0, 1)$ is 0. Thus, the probability of picking a rational number is 0.