Probability and Stochastic Processes (EL6303) October 1, 2018
NYU Tandon School of Engineering, Fall 2018
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Exercise 3

1. You order food for delivery through Seamless. The time it takes for the food to arrive can be represented as a random variable T with pdf

$$f(t) = \begin{cases} \lambda e^{\lambda t} & 0 < t \leq t_1 \\ a & t_1 < t < t_2 \\ 0 & \text{else} \end{cases}$$

   (a) Find a in terms of λ, t_1, t_2 to ensure that the $f(t)$ represents a pdf.

   (b) The longer it takes for the food to arrive, the colder it will get, and its value for you will be lower. We describe the *value* by a random variable $V = h(T)$, such that

$$h(t) = \begin{cases} v_0 - t & 0 < t < \frac{t_1+t_2}{2} \\ 0 & \text{else} \end{cases}$$

   where v_0 is a constant indicating the initial value if the food were to arrive immediately. We assume $v_0 > \frac{t_1+t_2}{2}$.
   Find an expression for the cdf of V.

   (c) In class we showed that the exponential distribution satisfies the *memoryless* property, that is if X is an exponential random variable with pdf $f(x) = \lambda e^{\lambda x}, x > 0$, then

$$P(X > t + s | X > t) = P(X > s), t, s > 0.$$

   Does T satisfy the memoryless property? Prove your result.

2. Three types of jobs arrive at a computer server:

   - With probability p_1, a type 1 job arrives, requiring an exponentially distributed service time T_1 with parameter λ, i.e., $f_{T_1}(t_1) = \lambda e^{-\lambda t_1}$,
   - With probability p_2, a type 2 job arrives, requiring a uniformly distributed service time T_2 in the interval $[0, T]$,
   - With probability $p_3 = 1 - p_1 - p_2$, a type 3 job arrives, requiring a fixed service time $T_3 = K$, where $K < T$.

(a) Find $P(T_1 \leq K)$.

(b) Find $P(T_2 \leq K)$.

(c) Find the probability that service time of an arbitrary job, arriving according to the distribution (p_1, p_2, p_3), is less than or equal to K.

(d) If service time of a job is known to be less than or equal to K, what is the probability that it is a type 1 job?

3. Your smart phone data usage every month is represented by the random variable X (in GB) which has the following probability density function:

$$f_X(x) = 0.2\delta(x-1) + 0.1(u(x) - u(x-a)),$$

where

$$u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & \text{else} \end{cases}$$

Note that this is mixed discrete-continuous distribution. The probability mass at 1 GB represents binge watching your favorite Netflix show in the months Netflix releases new episodes, which happens with probability 0.2.

(a) Find a.

(b) Find the cumulative distribution function $F_X(x)$ of X. Sketch $F_X(x)$.

(c) Suppose TMobile charges \$30 for the first 2.5 GBs, and $30 + 5(X - 2.5))$ dollars for $X > 2.5$, where X is your data usage in GBs. Your total monthly cost is represented by $Y = g(X)$. Sketch $g(X)$.

(d) Find the cumulative distribution function $F_Y(y)$ of Y.

(e) Find the probability density function $f_Y(y)$ of Y.

(f) Suppose Verizon charges \$10 for each GB usage, that is the cost of Verizon, $h(X)$ is

$$h(X) = 10j, \; (j-1) \leq X < j, j = 1, 2, \; \ldots$$

where X is your data usage in GBs. If your goal is to minimize your median cost, which one would you choose, TMobile or Verizon? Explain.

Hint: Median value of a random variable X is smallest x for which the cumulative distribution function $F_X(x) = 0.5$

4. We model the lifetime of a smartphone as a random variable X (in years) with the following probability density function:

$$f_X(x) = \begin{cases} 0.5e^{-x/2}, & 0 \leq x < 2 \\ a & 2 \leq x < 4 \\ 0, & \text{else} \end{cases}$$

(a) Find a.

(b) Find the cumulative distribution function $F_X(x)$ of X. Sketch $F_X(x)$.

(c) You can buy insurance from AppleCare which will give you $g(X) = 150(4 - X)$ dollars if your phone fails in X years. Let $Y = g(X)$. Find the cumulative distribution function $F_Y(y)$ of Y.

(d) Find the probability density function $f_Y(y)$ of Y.

(e) You are willing to pay \$ C, for AppleCare where C is the median of Y. Find C. *Hint:* Median value of a random variable Y is smallest y for which the cumulative distribution function $F_Y(y) = 0.5$

(f) Suppose another insurance company GeekSquad gives you $h(X)$ dollars where

$$h(x) = \begin{cases} 500, & 0 \leq x < 1 \\ 200 & 1 \leq x < 4 \end{cases}$$

Let $Z = h(X)$. Is Z a continuous, discrete or mixed random variable? Find $P(Z \leq 300)$.

5. A function $h(x)$ is concave if for all x_1 and x_2 in its domain and $\lambda \in [0, 1]$

$$\lambda h(x_1) + (1 - \lambda)h(x_2) \leq h(\lambda x_1 + (1 - \lambda)x_2). \tag{1}$$

(a) Suppose $Y \sim Bern(\rho)$. Prove that for $h(.)$ concave

$$E(h(Y)) \leq h(E(Y)) \tag{2}$$

(b) Prove that $\ln(x)$, $x > 0$ is a concave function.

(c) A mobile user's satisfaction (or utility) of wireless services is usually represented by the function $u(R) = \ln(R)$, where R is the communication rate in Mbits/sec. Suppose $R \sim Unif(100, 500)$.

i) Find $E(R)$.

ii) Find $E(u(R))$.

iii) Find $u(E(R))$.

iv) Does inequality (2) hold? Comment.

Hint: You can use $\ln(100) \approx 4, \ln(300) \approx 5.7, \ln(500) \approx 6$.

6. Suppose X is a random variable with the probability density function $f_X(x) = e^{-(x+2)}, x > -2$. Let $Z = h(X)$ with

$$h(x) = \begin{cases} 0, & x < -1 \\ -1 - x, & -1 \leq x < 0 \\ x, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

(a) Find and sketch $f_Z(z)$, the probability density function of Z.

(b) Find and sketch $F_Z(z)$, the cumulative distribution function of Z.

(c) Find $E(Z)$.