Lecture XI
Stability of Linear Systems

- Linearization
- Definition of stability
- Necessary and sufficient conditions for stability

Mathematical Modeling

Finite-dimensional differential equations:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n$$

Equilibrium x_e if it satisfies $f(x_e) = 0.$

Without loss of generality, assume $x_e = 0.$

If not, consider $y = x - x_e$. Then,

$$\dot{y} = f\left(y + x_e\right)$$

has an equilibrium at the origin, i.e. $y_e = 0.$

Linearization

From nonlinear to linear systems:

$$\dot{x}_l = Ax_l, \quad x_l \in \mathbb{R}^n$$

where

$$A = \left. \frac{\partial f(x)}{\partial x} \right|_{x=0} \doteq \frac{\partial f}{\partial x}(0) \in \mathbb{R}^{n \times n}$$

Often, the eq. is called "first-order approximation", or *linearization*, of the original nonlinear equation around the equilibrium point $x_e = 0$.

Comment

The linearized model only represents a good (local!) approximation of the nonlinear model near the equilibrium of interest:

$$\ddot{\theta} = k_1 \sin\theta - k_2\dot{\theta} \quad \text{(Rotational Pendulum)}$$

Equilibria:

$$\begin{pmatrix} \dot{\theta} \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 2m\pi \end{pmatrix}, \quad \begin{pmatrix} \dot{\theta} \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ (2m+1)\pi \end{pmatrix}$$

where $m \in \mathbb{Z}$.

Comment (cont'd)

Around the first set of equilibria, the (local) linearized model of

$$\ddot{\theta} = k_1 \sin\theta - k_2\dot{\theta}$$

becomes:

$$(\text{S1}) \quad \ddot{\theta} = k_1\theta - k_2\dot{\theta}$$

However, around the second set of equilibria, the linearized model is totally different:

$$(\text{S2}) \quad \ddot{\theta} = -k_1\theta - k_2\dot{\theta}$$

Why Linearization Useful?

(Poincare-Lyapunov Theorem)

If the linearized system is stable, then the original nonlinear system is also stable.

Stability (Lyapunov, 1892)

We are only interested in "asymptotic stability".

Roughly speaking, we want to study the following two properties:

- continuity of the solution $x(t)$ w.r.t. $x(0)$:

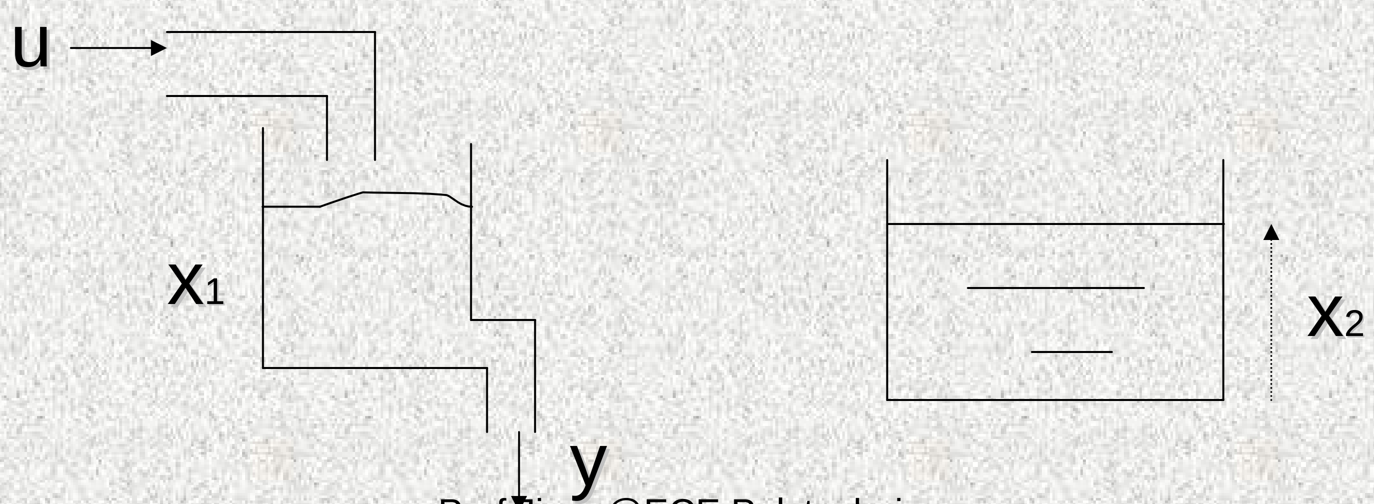
  $\left|x(0)\right| < \delta \quad \Rightarrow \quad \left|x(t)\right| < \varepsilon < \infty, \forall t.$

- attractiveness: $x(t) \to 0$ as $t \to \infty$.

Example of
neutral vs. asymptotic stability

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$



Lyapunov's First Theorem (1892)

- If the linearized model $\dot{x}_l = Ax_l$ is (asymptotically) stable, then the original nonlinear system $\dot{x} = f(x)$ is also (asymptotically) stable at $x_e = 0$.

- If $\dot{x}_l = Ax_l$ is unstable (i.e. not stable), then $\dot{x} = f(x)$ is also unstable.

Simple Examples

(1) $\dot{x} = -x + 2x^2 \doteq f(x)$

$$\dot{x}_l = -x_l \doteq A x_l$$

are both (asymptotically) stable at the origin.

(2) $\dot{x} = \sin x \doteq f(x)$

$$\dot{x}_l = x_l \doteq A x_l$$

are both unstable at the origin.

A Necessary and Sufficient Condition for Stability

Consider the linear time-invariant system

$$\dot{x} = Ax, \quad x(0) = x_o \in \mathbb{R}^n.$$

It is (asymptotically) stable if and only if A is Hurwitz, i.e. all its eigenvalues have negative real part.

Proof : Using the Jordan canonical form.

Lyapunov Matrix Equation

If A is a Hurwitz matrix, then the solution to

$$A^T P + PA = -I$$

is symmetric and positive definite. Indeed,

$$P = \int_0^\infty e^{A^T t} e^{At} dt.$$

Sketch of Proof

- The solution of $\dot{X} = A^T X + XA,\ X(0) = C$

  is: $X(t) = e^{A^T t} C e^{At}$.

- Integrating both sides from 0 to ∞ leads to:

$$-C = A^T \left(\int_0^\infty X(s)\, ds \right) + \left(\int_0^\infty X(s)\, ds \right) A$$

$$Thus,\ \text{when } C = I,\ P = \int_0^\infty X(s)\, ds := \int_0^\infty e^{A^T t} e^{At} dt.$$

Another Proof of Stability

Now, let's prove the stability of

$$\dot{x} = Ax, \quad x(0) = x_o$$

where A is Hurwitz.

Consider the function $V(x) = x^T P x$.

Differentiating $V(x(t))$ with respect to time yields

$$\dot{V} = x^T(t)\left(A^T P + PA\right)x(t) \le -x^T(t)x(t)$$

$$\le -x^T(t)Px(t)/\lambda_{\max}(P) \doteq -\mu V$$

Another Proof of Stability (cont'd)

From the fact

$$\dot{V} \le -\mu V, \quad \mu \doteq 1/\lambda_{\max}(P) > 0,$$

we have

$$V(t) \le e^{-\mu t} V(0)$$

So, $V(t) = x^T(t) P x(t)$, *and* thus $x(t)$,

converge to 0 at an exponential rate.

$V = x^T P x$ is often called a Lyapunov function.

Test for stability of *A*

Let P be determined by the matrix equation

$$A^T P + PA = -I.$$

Then, A is a stable matrix (Hurwitz) iff P is positive definite.

Proof

$\Rightarrow$: $P = \int_0^{\infty} e^{A^T t} e^{At} dt$ is a positive definite matrix.

$\Leftarrow$: Assume P is positive definite. Let $x(t)$

be solution to $\dot{x} = Ax,\ x(0) = x_o$.

Then, direct computation gives

$$\frac{d}{dt}\left(x^T(t)Px(t)\right) = -x^T(t)x(t).$$

Integrating both sides from 0 to t_1 implies:

Proof (cont'd)

Integrating both sides from 0 to t_1 implies

$$\int_0^{t_1} \left\| x(t) \right\|^2 dt = x^T(0)Px(0) - x^T(t_1)Px(t_1)$$

$\leq x^T(0)Px(0)$ because P positive definite

$< \infty, \quad \forall t_1 > 0.$

So, for **any** $x(0)$, $x(t) \to 0$, leading to stability of A, as wished.

Extension:

Discrete-Time Equations & Systems

- Solutions of an inhomogeneous linear equation

  $x(k+1) = A(k)x(k) + f(k),$

  with given $x(0) = x_o \in \mathbb{R}^n.$

- Stability of linear difference equations

  $x(k+1) = Ax(k), \quad x(k) \in \mathbb{R}^n$

Solutions of Discrete-Time Equations

Solutions of an inhomogeneous linear equation

$$x(k+1) = A(k)x(k) + f(k)$$

with given $x(k_0) = x_o \in \mathbb{R}^n$.

Clearly, it holds

$$
\begin{aligned}
&x(k)\\
&= A(k-1)x(k-1) + f(k-1)\\
&= A(k-1)A(k-2)x(k-2) + A(k-1)f(k-2) + f(k-1)\\
&\ \vdots\\
&= \underbrace{A(k-1)A(k-2)\cdots A(j)}_{\Phi(k,j)}\,x(j)\\
&+\sum_{l=j}^{k-1}\underbrace{A(k-1)A(k-2)\cdots A(l+1)}_{\Phi(k,l+1)}\,f(l), \quad \Phi(k,k) \triangleq I
\end{aligned}
$$

So, the general solution with $x(k_0) = x_o$ is:

$$x(k) = \Phi\left(k, k_0\right) x_o + \sum_{l=k_0}^{k-1} \Phi\left(k, l+1\right) f(l)$$

where $\Phi\left(k, k_0\right)$ is called "transition matrix".

Comment

Unlike the continuous-time case, the discrete-time transition matrix

$$\Phi\left(k, j\right) = \begin{cases} A\left(k-1\right) A(k-2)\cdots A(j), & \forall k \geq j+1 \\ I, & k = j \end{cases}$$

may *not* be invertible! Here is such a simple example:

$$x(k+1) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x(k)$$

Stability of Discrete-Time Linear Systems

A linear system taking the discrete-time form

$$x(k+1) = Ax(k)$$

is (asymptotically) stable if and only if all eigenvalues of A have magnitude strictly less than unity.

Example from Economics

Notations:

$$y(k) \;=\; \text{national income in year } k;$$

$$c(k) \;=\; \text{consumer expenditure;}$$

$$i(k) \;=\; \text{private investment;}$$

$$g(k) \;=\; \text{government expenditure.}$$

Example from Economics

A simple classical model in economics:

$$y(k) \;=\; c(k)+i(k)+g(k),$$

$$c(k+1) \;=\; \alpha y(k), \quad 0<\alpha<1,$$

$$i(k+1) \;=\; \beta\left[c(k+1)-c(k)\right], \quad \beta>0.$$

Example from Economics

$$x(k+1) = \underbrace{\begin{pmatrix} \alpha & \alpha \\ \beta(\alpha-1) & \beta\alpha \end{pmatrix}}_{A} x(k) + \underbrace{\begin{bmatrix} \alpha \\ \beta\alpha \end{bmatrix}}_{B} g(k),$$

$$y(k) = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{C} x(k) + g(k)$$

where

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \triangleq \begin{bmatrix} c(k) \\ i(k) \end{bmatrix}, \; g(k) = \text{input}, \; y(k) = \text{output}.$$

Exercise

- Compute the transition matrix of the economic model.
- Study the stability of the economic model.

Homework 10

Consider $\dot{x} = Ax + g(x),\ x(0) = x_o \in \mathbb{R}^n,$ where

- A is a stable matrix.
- $\|g(x)\| / \|x\| \to 0,$ as $\|x\| \to 0.$
- $\|x_o\|$ is sufficiently small.

Can you try to prove that the solution $x(t)$ of the nonlinear equation converges to 0, as $t \to \infty$?