Probability and Stochastic Processes (EL6303) December 15, 2017
NYU Tandon School of Engineering, Fall 2017
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Final

Closed book/closed notes. No electronics, no calculators. Two formula sheets allowed.
Total: 100 points
Time: 3 hours

1. *(Total: 18 points)* Consider a random variable θ that is uniformly distributed in the interval $(0,1]$. Suppose for a given θ, $X_1, \ldots, X_n$ are iid according to Geometric distribution with parameter θ. That is:

$$P(X_k = i|\theta) = (1-\theta)^{i-1}\theta, i = 1, 2, \ldots,$$

for all $k = 1, \ldots, n$.

(a) *(3 pts)* Find the pmf of X_1, that is $P(X_1 = i), i = 1, 2, \ldots$.

(b) *(4 pts)* Find the joint pmf of (X_1, X_2). Are X_1 and X_2 independent?

(c) *(8 pts)*

i. Find $P(X_1, \ldots, X_n|\theta)$.

ii. Find $\theta \in (0,1]$ that maximizes $P(X_1, \ldots, X_n|\theta)$. This is called *the Maximum Likelihood (ML)* estimate of θ.

(d) *(3 points)* The *Maximum-a-Posteriori (MAP)* estimate of θ is the one that maximizes $P(\theta|X_1, \ldots, X_n)$. For this question would the MAP estimate and the ML estimate be the same or different? Explain your answer. *You do not need to compute the MAP estimate to answer this question.*

Hint: You can use $\int_0^1 u^k (1-u)^{n-k} du = \frac{k!(n-k)!}{(n+1)!}$.

2. *(Total: 11 points)* Suppose that you send 7-letter tweets. You randomly and independently type one of 26 lowercase letters according to a uniform distribution.

   (a) *(3 points)* What is the probability that you type "covfefe"?

   (b) *(5 points)* Now you send multiple iid 7-letter tweets, each according to the distribution above. Let $Y_i = 1$ if your i'th tweet is "covfefe," $Y_i = 0$ otherwise. Find

   $$\lim_{n\to\infty} \frac{\sum_{i=1}^n Y_i}{n},$$

   where the limit is in probability.

   (c) *(3 points)* Using your answer to part (b), find an approximate value for n such that after n tweets you expect to see about one "covfefe" tweet.

3. *(Total: 20 points)* Consider a discrete random variable Z that takes on values in $\{1, \ldots, k\}$ with probabilities $p_1, \ldots, p_k$. Also consider k deterministic signals $s_i(t), i = 1, \ldots, k$. We define the stochastic process $Y(t)$ as

$$Y(t) = s_Z(t).$$

Hence $Y(t) = s_i(t)$ when $Z = i$.

(a) *(3 points)* Find $E(Y(t))$.

(b) *(3 points)* Find $R_Y(t_1, t_2)$.

(c) *(2 points)* Is $Y(t)$ WSS? Explain.

(d) *(3 points)* For $k = 2$, $s_1(t) = t, s_2(t) = t^2$, draw different realizations of $Y(t)$.

(e) *(9 points)* Now suppose we have a sequence of iid random variables $Z_1, Z_2, \ldots$ distributed according to $(p_1, \ldots, p_k)$. We define the stochastic process $U(t), t \geq 0$ as

$$U(t) = s_{Zj}(t), (j-1)T < t \leq jT,$$

where T is constant.

i. Find $R_U(t_1, t_2)$ for (t_1, t_2) such that $(j-1)T < t_1 \leq jT, (j-1)T < t_2 \leq jT$ for some j.

ii. Find $R_U(t_1, t_2)$ for (t_1, t_2) such that $(j-1)T < t_1 \leq jT, (l-1)T < t_2 \leq lT$ where $j \neq l$.

iii. For $k = 2$, $s_1(t) = t, s_2(t) = t^2$, draw different realizations of $U(t)$.

4. *(Total: 16 points)* Let $X_n, n = 1, 2, \ldots$ denote an iid sequence of Gaussian random variables with zero mean and unit variance. We define Y_n as the *weighted moving average* of two consecutive values of X_n as follows:

$$Y_n = aX_n + bX_{n-1}, n = 1, 2, \ldots.$$

We assume $X_0 = 0$.

(a) *(2 points)* Find $R_X(i, j)$.

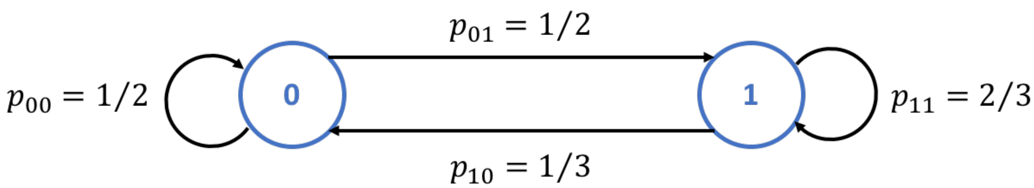
(b) *(2 points)* Find $E(Y_n)$.

(c) *(5 points)* Find $R_Y(i, j)$.

(d) *(2 points)* Is Y_n WSS? Explain.

(e) *(5 points)* For $a = b = 1$, ind the joint pdf of (Y_1, Y_2, Y_3).

5. *(Total: 20 points)* The oil sensor in your car displays $Q = 0$ if your car does not need oil change, $Q = 1$ if it does. Let Q_t denote the oil sensor display at time $t = 1, 2, \ldots$. We assume $Q_1, Q_2, \ldots$ are random and can be represented as a Markov chain whose state transition diagram is shown below.



The pmf of Q_t is denoted by $\mu_t = (P(Q_t) = 0, P(Q_t = 1))$.

(a) *(4 points)* For $\mu_1 = (1, 0)$, find μ_4.

(b) *(3 points)* For $\mu_1 = (1, 0)$, find the probability that Q_3 and Q_4 are different.

(c) *(3 points)* Find $P(Q_{t+2} = 1 | Q_t = 0)$. Are Q_{t+2} and Q_t independent? Explain.

(d) *(3 points)* Find the stationary distribution of the above Markov chain. *You will not get credit if you do not show your work.*

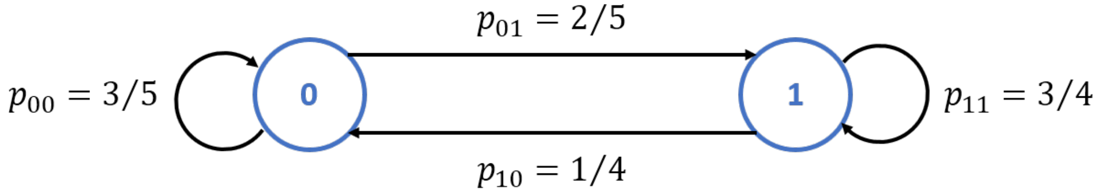
(e) *(7 points)* Suppose now that the display is broken and is sometimes stuck at 0. To model this, let $M_t = 1$ if the display works at time t, and $M_t = 0$ otherwise. Hence the display will show

$$Y_t = M_t Q_t,$$

where Q_t is the correct display as above. We assume that the M and Q processes are independent.

i. For $M_1, M_2, \ldots$ iid with $P(M_1 = 1) = p$, and $Q_1 = 1$, find $P(Y_4 = 1)$.

ii. Suppose the process M_t follows another Markov chain shown below. For $P(M_1 = 1) = p$, and $Q_1 = 1$, find $P(Y_2 = 0)$.



6. *(Total: 15 points)* Let $X(t)$ denote the total number of e-mails that arrive in your mailbox up to time $t, t \geq 0$. We assume $X(t)$ is a Poisson process with parameter λ. Recall that a Poisson process with parameter λ satisfies the following properties:

   - $X(0) = 0$.
   - $X(t_2) - X(t_1) \sim \text{Poisson}(\lambda(t_2 - t_1))$
   - For $t_1 < t_2 < t_3 < t_4, X(t_2) - X(t_1)$ and $X(t_4) - X(t_3)$ are independent.

   Now suppose you delete each e-mail you get with probability p. We assume you delete e-mails independently from one another and from $X(t)$. The total number of e-mails deleted up to time t are denoted by $D(t)$, and the e-mails that are still in your inbox are denoted by $Y(t)$. Clearly $X(t) = D(t) + Y(t)$.

   (a) *(5 points)* Find the probability mass function of $D(t)$.
   Hint: $D(t) = k$ suggests that $X(t) = n$ and k e-mails were deleted for $n = k, k+1, \ldots$.

   (b) *(5 points)* Argue that $D(t)$ is also a Poisson process. Find the parameter of this process.

   (c) *(5 points)* Are $D(t)$ and $Y(t)$ independent? Prove your result.

   Hint: Poisson(γ) pmf is $\frac{\gamma^n e^{-\gamma}}{n!}, n = 0, 1, \ldots$.