Probability and Stochastic Processes (EL6303) October 12, 2018
NYU Tandon School of Engineering, Fall 2018
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Quiz 1 Solutions

1. (a)

$$\begin{aligned}
\mathrm{P}(e) &= \mathrm{P}(e \text{ received } |\text{`0' transmitted}) \cdot \mathrm{P}(\text{`0' transmitted}) \\
&+ \mathrm{P}(e \text{ received}|\text{`1' transmitted }) \cdot \mathrm{P}(\text{`1' transmitted}) \\
&= p\alpha + q\beta
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{P}(\text{`0' transmitted}|e \text{ received}) &= \frac{\mathrm{P}(e \text{ received}|\text{ `0' transmitted}) \cdot \mathrm{P}(\text{`0' transmitted})}{\mathrm{P}(e \text{ received})} \\
&= \frac{p\alpha}{p\alpha + q\beta}
\end{aligned}$$

(c) Since ‘0’ is received, the only possible symbol that could have been sent is 0 (the communication channel never flips the bits)
Thus, P(‘0’ transmitted | ‘0’ received) = 1

(d) The only time you are not certain about the channel inputs is when e is received. When a ‘0’ is received, you know for sure that a ‘0’ was sent and when a ‘1’ is received, you know for sure that a ‘1’ was sent. Thus,
P(not certain) = P(e) = $p\alpha + q\beta$

(e) i. The communication channel does not flip the bits. Out of the three channel outputs, since one is 0, the only possible channel input is 000. Thus,
P(‘000’ transmitted | ‘ee0’ received) = 1.

ii.

$$\begin{aligned}
\mathrm{P}(\text{`}eee\text{' received}) =& \mathrm{P}(\text{`}eee\text{' received }|\text{`000' transmitted}) \cdot \mathrm{P}(\text{`000' transmitted}) \\
&+ \mathrm{P}(\text{`}eee\text{' received }|\text{`111' transmitted}) \cdot \mathrm{P}(\text{`111' transmitted})
\end{aligned}$$

$$
\begin{aligned}
&\mathrm{P}(\text{`000' transmitted } |\text{`}eee\text{' received})\\
=&\frac{\mathrm{P}(\text{`}eee\text{' received } |\text{`000' transmitted})\cdot \mathrm{P}(\text{`000' transmitted})}{\mathrm{P}(\text{`}eee\text{' received})}\\
=&\frac{\alpha p^3}{\alpha p^3+\beta q^3}\\
=&\frac{1}{2} \qquad\qquad (\alpha=\beta=\frac{1}{2}, p=q)
\end{aligned}
$$

iii. Even if one of the channel outputs is a '0' or a '1', we know for sure that a '0' or a '1' was sent, respectively. Thus, the probability that we will not be able to determine the channel input with certainty is,
P(not certain) = P('eee' received) $= \alpha p^3 + \beta q^3 = p^3$

2. (a) We need to compute the probability of a total number of $X = x$ shots. Thus, first $x-1$ shots are not successful and the shot x is successful.

$$\Pr(X=x) = (1-p)^{x-1}p,\ \ x = 1, 2, \cdots$$

(b) We need to compute the probability of a total number of $Y = y$ shots. Thus, the last shot should be successful and there should be $k-1$ successful shots among the previous $y-1$ shots.

$$\Pr(Y=y) = \binom{y-1}{k-1}(1-p)^{y-k}p^{k-1}p,\ \ y = k, k+1, \cdots$$

(c) i. If you fail at the first shot, you have to succeed for the remaining three. We define A as the probability of three success out out four and A_i as the event that you make three success out of four and fail at the i'th shot. Thus A_i are mutual exclusive and their union is A. Thus, we have

$$
\begin{aligned}
\Pr(A_1) &= (1-p)pq^2\\
\Pr(A_2) &= p(1-q)pq\\
\Pr(A_3) &= pq(1-q)p\\
\Pr(A_4) &= pq^2(1-q)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\Pr(A) &= \sum_i \Pr(A_i)\\
&= 2p^2(1-q)q + (1-p)pq^2 + p(1-q)q^2\\
&= 2p^2q + 2pq^2 - 3p^2q^2 - pq^3
\end{aligned}
$$

ii. We need to compute the probability of a total number of $Z = z$ shots. You stop as soon as you make two successful shots. So shot z must be a success, and you have only one success among the previous $z-1$ shots.
We define B_i as the event that you are successful at shot $i \in [1, z-1]$ as well as the shot z. For $z = 2$,

$$\Pr(Z = z) = pq.$$

For $z \geq 3$, we have

$$Pr(B_{z-1}) = (1-p)^{z-2}pq,$$

and for $i = 1, 2, \dots, z-2$ we have

$$\Pr(B_i) = (1-p)^{z-3}(1-q)pq.$$

Combining,

$$P(Z = z) = \sum_{i=1}^{z-1} \Pr(B_i) = (1-p)^{z-3}pq((z-2)(1-q) + (1-p)).$$