Probability and Stochastic Processes (EL6303) — October 9, 2018
NYU Tandon School of Engineering, Fall 2018
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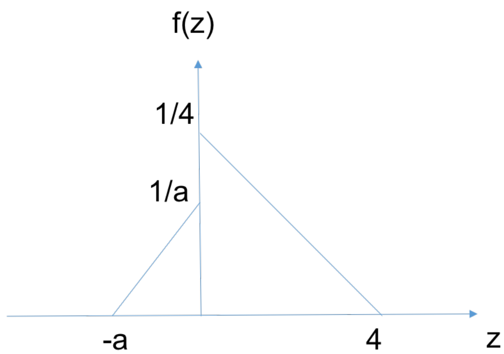
Exercise 4

1. Noise in a communication system is represented by a random variable Z. The probability density function (pdf) $f(z)$ of Z is shown below:



   (a) Confirm that $f(z)$ is a pdf for any value of a.

   (b) Find the mean of Z as a function of a.

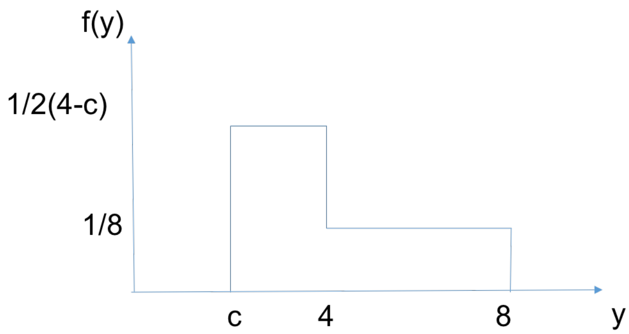
   For the rest of the question, assume $a = 4$.

   (c) Find the variance of Z.

   (d) Find $P(|Z| > 2)$.

   (e) Using Chebyshev inequality, find a bound for $P(|Z| > 2)$. Is the bound useful in this case? Answer by comparing with part (d).

2. The total time it takes to get to school is represented by a random variable Y. The probability density function (pdf) $f(y)$ of Y is shown below (each unit of Y represents 10 minutes):



(a) Confirm that $f(y)$ is a pdf for any $c < 4$.

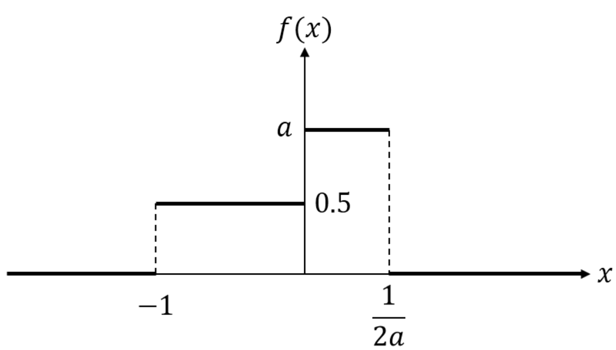
(b) Find the mean of Y as a function of c.

(c) Find the second moment of Y as a function of c.

(d) Find $P(Y > 4)$.

(e) Assume $c = 2$. Using Markov inequality, find a bound for $P(Y > 4)$. Is the bound useful in this case? Answer by comparing with part (d).

3. The probability density function (pdf) of random variable X is shown below.



(a) Confirm that $f(x)$ is a valid pdf for every $a > 0$.

(b) Find the mean of X as a function of a.

for the rest of the question assume $a = \frac{1}{2}$.

(c) Find the second moment of X.

(d) Find the variance of X.

(e) Find $P(|X| > \frac{1}{2})$.

(f) Use Chebyshev's inequality to find an upper-bound for $P(|X| > \frac{1}{2})$. Is this bound useful? Explain.

4. The number of calls Y arriving at a call center in a day can be modeled as having Poisson distribution with parameter μ. Hence

$$P(Y = n) = e^{-\mu}\frac{\mu^n}{n!}, n = 0, 1, \ldots.$$

   (a) Find the expected value of Y.
   (b) Find the variance of Y.
   (c) Suppose that the call center is only able to handle up to 7μ number of calls, and temporarily shuts down if the number of calls exceeds this threshold.
      i. Write an expression (in terms of Y) for the probability of temporary shut down.
      ii. Find a (non-trivial) upper bound to the probability of temporary shut down.
   (d) Let $u(Y) = (Y - 2)^2$. Find $E(u(Y))$.

5. You are in charge of testing in an electronics manufacturing facility. A particular type of capacitor is *defective* with probability γ. Let $I_j = 1$ if capacitor j is defective, 0 otherwise. You test n such capacitors and note down X, the number of capacitors that are defective. We assume each capacitor fails (is defective) independently.

   (a) Find the probability mass function of I_j for $j = 1, \ldots, n$.
   (b) Find an expression for X in terms of $I_j, j = 1, \ldots, n$.
   (c) Find the probability mass function of X. What values does X take?
   (d) Your boss would like to make sure that the total number of capacitors that are defective is small. You say you can only guarantee this probabilistically. To accomplish this you suggest finding a lower bound for

$$P(0 \leq X \leq 2n\gamma).$$

      i. Find a lower bound using Chebychev inequality.
      ii. Find a lower bound using Markov inequality.
      iii. For $\gamma = 0.01, n = 100$ evaluate both bounds. Which one is tighter?

   Here $\eta = E(X), \sigma^2 = \mathrm{Var}(X), \alpha$ a positive constant.

6. Suppose X is a Poisson(λ) random variable. Show that:

   (a) $P(0 < X < 2\lambda) > \frac{\lambda-1}{\lambda}$.
   (b) $E(X(X - 1)) = \lambda^2$.
   (c) $E(X(X - 1)(X - 2)) = \lambda^3$.