(1) Given:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$$

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda-1 & 0 & 1 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-1 \end{bmatrix} = 0$$

$$(\lambda-1)^2(\lambda-2) = 0$$

$\lambda_1 = 1$ (multiplicity 2)

$\lambda_2 = 2$ (multiplicity 1)

We know $(\lambda_i I - A)^g P = 0$

For $\lambda = 1$,

$$(I-A)^2 P = 0$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} P = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$P_2(\lambda_1 = 1) = \{ p^1 \mid p^1 = \mathrm{col}(\alpha, 0, \gamma) \}$$

For $\lambda = 2$

$$(2I - A)P = 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$P_1(\lambda_2 = 2) = \{ p^2 \mid p^2 = \mathrm{col}(0, \beta, 0) \}$$

Hence, $p^1 = \begin{bmatrix} \alpha \\ 0 \\ \gamma \end{bmatrix}$, $p^2 = \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix}$

(2)

i) Given:

$$X = \begin{bmatrix} \sqrt{2} \\ -9 \\ 84 \end{bmatrix}$$

Express X as unique representations of principal vectors found in problem (1),

From problem (1),

$$p^1 = \begin{bmatrix} \alpha \\ 0 \\ \gamma \end{bmatrix}, \quad p^2 = \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix}$$

$$X = \sqrt{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 9\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 84\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

ii) Given:

$$X = \begin{bmatrix} 0 \\ 9.3 \\ 0 \end{bmatrix}$$

Similarly: $X = 9.3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, where $p^2 = \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix}$ and $\beta = 1$

3) $A = \begin{pmatrix} \lambda & \lambda & \lambda \\ 0 & \lambda & \lambda \\ 0 & 0 & \lambda \end{pmatrix} \Rightarrow \lambda I - A = \begin{pmatrix} 0 & -\lambda & -\lambda \\ 0 & 0 & -\lambda \\ 0 & 0 & 0 \end{pmatrix}$

$$\text{null}(P_1) = \langle 1, 0, 0 \rangle$$

$$(\lambda I - A)^2 = \begin{pmatrix} 0 & -\lambda & -\lambda \\ 0 & 0 & -\lambda \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\lambda & -\lambda \\ 0 & 0 & -\lambda \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \lambda^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{null}(P_2) = \langle 1, 0, 0 \rangle ; \langle 0, 1, 0 \rangle$$

$$(\lambda I - A)^3 = \begin{pmatrix} 0 & 0 & \lambda^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\lambda & -\lambda \\ 0 & 0 & -\lambda \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{null}(P_3) = \langle 1, 0, 0 \rangle ; \langle 0, 1, 0 \rangle ; \langle 0, 0, 1 \rangle$$

$$v^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v^2 = (A - \lambda I) v^1 = \begin{pmatrix} 0 & \lambda & \lambda \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ 0 \end{pmatrix}$$

$$v^3 = (A - \lambda I) v^2 = \begin{pmatrix} 0 & \lambda & \lambda \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \lambda \\ \lambda \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda^2 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow P = \begin{pmatrix} 0 & \lambda & \lambda^2 \\ 0 & \lambda & 0 \\ 1 & 0 & 0 \end{pmatrix} ; P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{\lambda} & 0 \\ \frac{1}{\lambda^2} & -\frac{1}{\lambda^2} & 0 \end{pmatrix}$$

$$\Rightarrow P^{-1} A P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{\lambda} & 0 \\ \frac{1}{\lambda^2} & \frac{-1}{\lambda^2} & 0 \end{pmatrix}\begin{pmatrix} \lambda & \lambda & \lambda \\ 0 & \lambda & \lambda \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} 0 & \lambda & \lambda^2 \\ 0 & \lambda & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix}$$