Probability and Stochastic Processes (EL6303) November 2, 2018
NYU Tandon School of Engineering, Fall 2018
Instructor: *Dr. Elza Erkip*

Midterm
Please write your name and Net-id in the blue book eg: Ojas Kanhere, ok671
Do NOT write your N# number

Closed book/closed notes. No electronics, no calculators.
One 8.5 ×11 inch sheet of notes allowed
Time: 2 hours 25 minutes
Total 100 + 7 points

1. *(20 + 7 points)*
Suppose a store sells two types of rectangular poster boards, whose sides have random lengths.

- Type 1 has first side length $X_1 \sim \text{unif}(1,2)$, second side length $X_2 \sim \text{unif}(3,5)$.
- Type 2 has first side length $Y_1 \sim \text{unif}(2,3)$, second side length $Y_2 \sim \text{unif}(1,3)$.

We assume X_1, X_2, Y_1, and Y_2 are independent.

(a) *(10 points)* Find the distribution of the area $A_1 = X_1 X_2$.

(b) *(4 points)* Find $E(A_1)$.

(c) Suppose you buy the type 1 board with probability $\frac{1}{2}$, type 2 board with probability $\frac{1}{2}$. Let Z_1 be the first side length of the board you buy, Z_2 be the second side length.

i. *(5 points)* Find the joint distribution of Z_1 and Z_2.

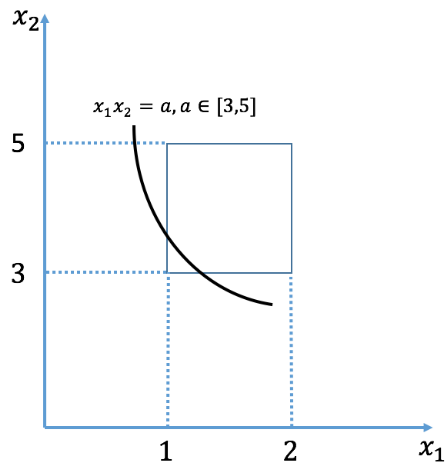
ii. *(4 points)* Are Z_1 and Z_2 independent? Explain.

iii. *(4 points)* Find the expected area of the board you buy $A_{buy} = E(Z_1 Z_2)$.

Solution:

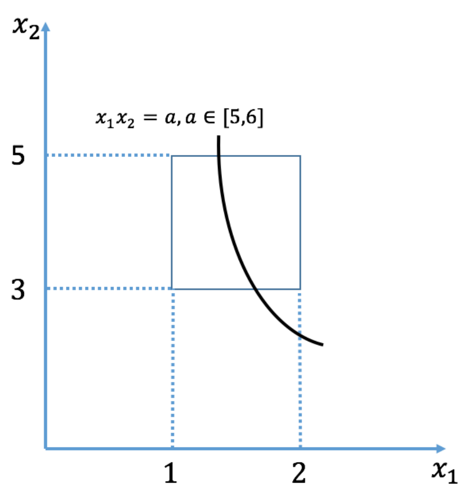
(a) Since X_1 and X_2 are independent, their joint probability density $f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = 1/2, x_1 \in [1,2], x_2 \in [3,5]$.
For $a \in [3,5]$,

$$F_{A_1}(a) = P(A_1 \leq a) = \int_3^a \int_1^{a/x_2} 0.5 dx_1 dx_2 = 0.5 \int_3^a (a/x_2 - 1) dx_2.$$



For $a \in [5, 6]$,

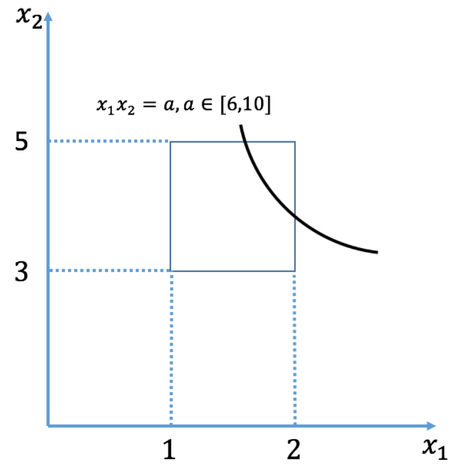
$$F_{A_1}(a) = 0.5 \int_3^5 \int_1^{a/x_2} dx_1 dx_2 = 0.5 \int_3^5 (a/x_2 - 1) dx_2.$$



For $a \in [6, 10]$,

$$F_{A_1}(a) = 1 - 0.5 \int_{a/2}^5 \int_{a/x_2}^2 dx_1 dx_2 = 1 - 0.5 \int_{a/2}^5 (2 - a/x_2) dx_2.$$

Combining, the CDF of A_1 can be written as follows



$$F_{A_1}(a) = \begin{cases} 0, & \text{for } a < 3 \\ \frac{a}{2}\ln\frac{a}{3} - \frac{(a-3)}{2}, & \text{for } 3 \le a \le 5 \\ \frac{a}{2}\ln\frac{5}{3} - 1, & \text{for } 5 \le a \le 6 \\ \frac{a}{2} + \frac{a}{2}\ln\frac{10}{a} - 4, & \text{for } 6 \le a \le 10 \\ 1, & \text{for } a > 10 \end{cases}$$

(b)

$$E(A_1) = E(X_1)E(X_2) = 6,$$

where we have used the independence of X_1 and X_2.

(c) i. Note that

$$f_{Z_1,Z_2}(z_1,z_2) = P(\text{Type 1})f(z_1,z_2|\text{ Type 1}) + P(\text{Type 2})f(z_1,z_2|\text{Type 2})$$

Since for Type 1, side lengths are independent,

$$\begin{aligned} f(z_1,z_2|\text{ Type 1}) &= f(z_1|\text{ Type 1})f(z_2|\text{ Type 1}) \\ &= 1/(2-1)\cdot 1/(5-3) = 1/2, \forall z_1 \in [1,2], z_2 \in [3,5] \end{aligned}$$

Similarly,

$$\begin{aligned} f(z_1,z_2|\text{ Type 2}) &= f(z_1|\text{Type 2})f(z_2|\text{Type 2}) \\ &= 1/(3-2)\cdot 1/(3-1) = 1/2, \forall z_1 \in [2,3], z_2 \in [1,3]. \end{aligned}$$

Therefore, we have

$$f_{Z_1,Z_2}(z_1,z_2) = \begin{cases} 1/4, & \text{for } z_1 \in [1,2], z_2 \in [3,5] \\ 1/4, & \text{for } z_1 \in [2,3], z_2 \in [1,3] \\ 0, & \text{else} \end{cases}$$

ii.

$$f_{Z_1}(z_1) = 1/2, \forall z_1 \in [1,3]$$
$$f_{Z_2}(z_2) = 1/4, \forall z_2 \in [1,5]$$

Thus,

$$f_{Z_1,Z_2}(z_1,z_2) \neq f_{Z_1}(z_1) f_{Z_2}(z_2)$$

Note that when we know the value of Z_1, by looking at its interval, we know which type of board was picked, and hence we know the range of Z_2.

iii.

$$\begin{aligned}
E(Z_1Z_2) &= E(Z_1Z_2|\text{Type 1})P(\text{Type 1}) + E(Z_1Z_2|\text{Type 2})P(\text{Type 2}) \\
&= E(X_1X_2)P(\text{Type 1}) + E(Y_1Y_2)P(\text{Type 2}) \\
&= E(X_1)E(X_2)P(\text{Type 1}) + E(Y_1)E(Y_2)P(\text{Type 2}) \\
&= 3 + 5/2 \\
&= 11/2.
\end{aligned}$$

2. *(20 points)*
Suppose we have two independent discrete random variables X and Y with

$$X \sim \text{unif}\{1, 2, \ldots, k\},$$
$$Y \sim \text{unif}\{\frac{k}{2} + 1, \ldots, \frac{3k}{2}\}.$$

We assume k is a positive even integer.
Let

$$Z = \min(X, Y)$$
$$W = \max(X, Y)$$

(a) *(5 points)* Find the probability mass function of Z.

(b) *(5 points)* Find the probability mass function of W.

(c) *(5 points)* Find the joint probability mass function of (Z, W).

(d) *(5 points)* Let $U = W + Z$. Find the probability mass function of U, when $k = 4$.

Solution:

(a) If $z = 1, 2, \cdots, \frac{k}{2}$, then

$$P(Z = z) = P(X = z) = 1/k.$$

If $z = \frac{k}{2} + 1, \cdots, k$, then

$$\begin{aligned} P(Z = z) &= P(X = z, Y > z) + P(Y = z, X > z) + P(X = z, Y = z) \\ &= P(X = z)P(Y \geq z + 1) + P(Y = z)P(X \geq z + 1) + P(X = z, Y = z) \\ &= \frac{1}{k}\frac{3k/2 - z}{k} + \frac{1}{k}\frac{k - z}{k} + \frac{1}{k^2} \\ &= \frac{5k/2 - 2z + 1}{k^2}. \end{aligned}$$

For all other z, $P(Z = z) = 0$.

(b) Similar to part (a), we have

$$P(W = w) = \begin{cases} 0, & \text{for } w \leq k/2 \\ \frac{2w - k/2 - 1}{k^2}, & \text{for } \frac{k}{2} + 1 \leq w \leq k \\ 1/k, & \text{for } k < w \leq 3k/2 \end{cases}$$

(c) Case 1: If $1 \le z \le \frac{k}{2}$ and $k < w \le \frac{3k}{2}$, then

$$P(Z = z, W = w) = P(X = z, Y = w) = 1/k \cdot 1/k = \frac{1}{k^2}.$$

Case 2: If $1 \le z \le \frac{k}{2}$ and $\frac{k}{2} + 1 \le w \le k$, then

$$P(Z = z, W = w) = P(X = z, Y = w) = 1/k \cdot 1/k = \frac{1}{k^2}.$$

Case 3: If $\frac{k}{2} + 1 \le z \le k$ and $k < w \le \frac{3k}{2}$, then

$$P(Z = z, W = w) = P(X = z, Y = w) = 1/k \cdot 1/k = \frac{1}{k^2}.$$

Case 4: If $\frac{k}{2} + 1 \le z \le k$ and $\frac{k}{2} + 1 \le w \le k$, then

$$P(Z = z, W = w) = 0 \text{ if } z > w$$
$$P(Z = z, W = w) = P(X = z, Y = w) = \frac{1}{k^2} \text{ if } z = w$$
$$P(Z = z, W = w) = P(X = z, Y = w) = \frac{1}{k^2} \text{ if } z < w$$

(d) $k = 4$,

$$P(U = u) = P(X + Y = u) = \begin{cases} 1/16, & \text{for } u = 4 \\ 1/8, & \text{for } u = 5 \\ 3/16, & \text{for } u = 6 \\ 1/4, & \text{for } u = 7 \\ 3/16, & \text{for } u = 8 \\ 1/8, & \text{for } u = 9 \\ 1/16, & \text{for } u = 10 \end{cases}$$

3. *(20 points)*

There are two kinds of phones in the market: a cheap one with lifetimes $X \sim Exp(\lambda_1)$ and an expensive one with lifetime $Y \sim Exp(\lambda_2)$. We have $\lambda_1 \geq \lambda_2$. You decide to buy a cheap phone, and when it breaks, another cheap phone. We assume the lifetime of the cheap phones X_1, X_2 are independent and also independent of Y. Your mom buys the expensive phone.

(a) *(6 points)* Find the probability density function of the total lifetime of your phones $X_1 + X_2$.

(b) *(4 points)* Find the joint distribution of $X_1 + X_2$ and Y.

(c) *(4 points)* Compare the total expected lifetime of your phones with the expected lifetime of your mom's phone.

(d) *(6 points)* What is the probability that your mom's phone outlasts the total lifetime of your two phones?

Hint:

$X \sim Exp(\lambda)$ means $f_X(x) = \lambda e^{-\lambda x}$

Also, $\int_0^\infty x e^{-\lambda x} dx = \dfrac{1}{\lambda^2}$

Solution:

(a) Let $U = X_1 + X_2$. The pdf of the sum of two independent random variables is the convolution of their pdfs.

$$f_U(u) = (f_{X_1} \star f_{X_2})(u) = \int_{-\infty}^{\infty} f_{X_1}(u-z) f_{X_2}(z) dz$$
$$= \int_0^u \lambda_1 e^{-\lambda_1(u-z)} \lambda_1 e^{-\lambda_1 z} dz = \lambda_1^2 u e^{-\lambda_1 u}, u \geq 0.$$

(b) Since X_1, X_2, Y are independent, U is also independent of Y. Thus,

$$f_{UY}(u, y) = f_U(u) f_Y(y) = \lambda_1^2 u e^{-\lambda_1 u} \lambda_2 e^{-\lambda_2 y}.$$

(c)

$$E(U) = E(X_1) + E(X_2) = 2/\lambda_1.$$
$$E(Y) = \int_0^\infty \lambda_2 y e^{-\lambda_2 y} dy = 1/\lambda_2.$$

Thus $E(U) \geq E(Y)$ if $2/\lambda_1 \geq 1/\lambda_2$.

(d)

$$
\begin{aligned}
P(Y > U) &= \int_0^\infty \int_0^y f_{UY}(u,y)dudy \\
&= \int_{-\infty}^\infty \int_u^\infty f_{UY}(u,y)dydu \\
&= \int_{-\infty}^\infty \int_u^\infty \lambda_1^2 u e^{-\lambda_1 u} \lambda_2 e^{-\lambda_2 y} dydu \\
&= \lambda_1^2 \int_0^\infty u e^{-(\lambda_1+\lambda_2)u} du \\
&= \frac{\lambda_1^2}{(\lambda_1+\lambda_2)^2}
\end{aligned}
$$

4. *(20 points)*
Suppose $B_1, B_2 \ldots$ are independent binary digits such that for all i

$$p(B_i = 0) = p(B_i = 1) = \frac{1}{2}$$

Let $W = 0.B_1 \cdots B_n$ be a number in the interval $[0, 1)$ whose binary expansion has the most significant bit B_1 and least significant bit B_n.
Hence, $W = \sum_{i=1}^{n} B_i \; 2^{-i}$ in decimal.
Let $H = \sum_{i=1}^{n} B_i$. H is called the "Hamming weight of W".

(a) *(5 points)* Find $E(W)$.

(b) *(3 points)* Let $X \sim \text{unif}[0, 1]$. Find $E(X)$.

(c) *(2 points)* As $n \to \infty$, compare (a) and (b). Comment.

(d) *(5 points)* Find $P(B_1 = 1|H = k)$.

(e) *(5 points)* Find $P(H = i|W < 2^{-(n-2)})$ for $i = 1, \ldots, n$.

Solution:

(a)

$$E(W) = E(\sum_{i=1}^{n} B_i \; 2^{-i}) = \sum_{i=1}^{n} E(B_i) \; 2^{-i} = \sum_{i=1}^{n} 1/2 \; 2^{-i} = \frac{1 - 2^{-n}}{2}$$

(b)

$$E(X) = 1/2.$$

(c)

$$\lim_{n \to \infty} \frac{1 - 2^{-n}}{2} = 1/2 = E(X).$$

As $n \to \infty$, the representation W well approximates binary expansion of X.

(d) First, $H = k$ means that there are k bits of 1, and other $n - k$ bits are 0. Thus, $P(H = k) = \binom{n}{k}(0.5)^n$.
Also, $P(H = k, B_1 = 1) = P(H = k|B_1 = 1)P(B_1 = 1) = \binom{n-1}{k-1}(0.5)^{n-1}$.
Therefore,

$$P(B_1 = 1|H = k) = \frac{P(H = k, B_1 = 1)}{P(H = k)} = k/n.$$

(e)

$$W = \sum_{i=1}^{n} B_i \ 2^{-i}$$

Then, the condition $W < 2^{-(n-2)}$ means that the first $n-2$ digits cannot be 1, i.e., $B_i = 0, \forall i = 1, 2, \cdots, n-2$. The last two digits can be either 0 or 1.

Therefore,

$$\begin{aligned}
P(H = 0|W < 2^{-(n-2)}) &= P(\text{last two digits are 0}) = 1/4 \\
P(H = 1|W < 2^{-(n-2)}) &= P(\text{last two digits are 0,1 or 1,0}) = 1/2 \\
P(H = 2|W < 2^{-(n-2)}) &= P(\text{last two digits are 1}) = 1/4 \\
P(H = i|W < 2^{-(n-2)}) &= 0, \forall i \geq 3
\end{aligned}$$

5. *(20 points)*

A taxi driver is waiting for passengers at the airport. Suppose passengers arrive every minute where $n = 1, 2, \ldots$ denotes time in minutes.

At any given time instant, the number of passengers arriving is represented by a discrete random variable X such that $P(X = i) = p_i \quad i = 0, 1, 2$. Note that at most two passengers arrive at any time. Assume that the number of passengers arriving at a given time instant is independent of the number of passengers arriving at any other time instant.

(a) *(5 points)* The taxi driver wishes to take only one passenger. Let T be the waiting time. Find the probability mass function of T.

(b) *(5 points)* Now the taxi driver decides that he will leave as soon as he has at least two passengers. Note that if the first non-zero arrival consists of one passenger, the driver waits for at least one more passenger to arrive. Let U be the waiting time. Find the probability mass function of U.

(c) We assume the taxi driver loses c dollars per minute while he is waiting.

i. *(5 points)* If he only takes one passenger, as in (a), how much should the passenger pay so that the driver's expected earning is positive?

ii. *(5 points)* When the driver decides to take at least two passengers, as in (b), he charges s dollars per passenger. Find s such that the driver's expected earning is positive. To solve this part, assume $p_0 = 0$

Hint:

$$\sum_{i=1}^{\infty} ip^i = \frac{p}{(1-p)^2}$$
$$\sum_{i=1}^{\infty} i^2p^i = \frac{p(p+1)}{(1-p)^3}$$

Solution:

(a)

$$\begin{aligned} P(T = k) &= P(\text{one passenger arrives at } n = k, \text{none arrives at } n = 1, 2, 3 \ldots k-1) \\ &= (1-p_1)^{k-1}(p_1) \qquad k = 1, 2, \ldots \end{aligned}$$

(b)

$$\begin{aligned} P(U = 1) &= P(\text{two passengers arrive at } n = 1) \\ &= p_2 \end{aligned}$$

$$
\begin{aligned}
P(U=k) &= P(\text{two passengers arrive at } n=k,\ \text{none arrive at } n=1,2,3\ldots k-1) \\
&\quad + P(\text{one passenger arrives at some time before } n=k, \\
&\quad \text{one or two passengers arrive at } n=k) \\
&= p_0^{k-1}p_2 + \binom{k-1}{1} p_0^{k-2}p_1(p_1+p_2) \\
&= p_0^{k-1}p_2 + (k-1)p_0^{k-2}p_1(p_1+p_2)
\end{aligned}
$$

(c) i.

$$
\begin{aligned}
E(T) &= \sum_{i=1}^{\infty} i(1-p_1)^{i-1}(p_1) \\
&= \frac{p_1}{1-p_1}\sum_{i=1}^{\infty} i(1-p_1)^i \\
&= \frac{p_1}{1-p_1}\frac{1-p_1}{(1-(1-p_1))^2} \\
&= \frac{1}{p_1}
\end{aligned}
$$

The driver loses c dollars a minute while waiting. If the passenger pays s dollars, his total earnings are $s-cT$. His expected earnings are $s-cE(T)$ For his expected earnings to be positive,

$$
\begin{aligned}
& s - cE(T) \geq 0 \\
& s \geq cE(T) \\
& \ = \frac{c}{p_1}
\end{aligned}
$$

ii.

$$
\begin{aligned}
E(U) &= \sum_{i=1}^{\infty} iP(U=i) \\
&= 1\cdot p_2 + 2\cdot p_1 \qquad\qquad (p_0=0,\ p_0+p_1+p_2=1)
\end{aligned}
$$

The driver loses c dollars a minute while waiting. If each passenger pays s dollars, his total expected earnings are $2s(p_2+p_1^2)+3s(p_1p_2)-cE(U)$. Note that the driver has two passengers with probability $p_2+p_1^2$ and three passengers with probability p_1p_2. For his expected earnings to be positive, we need

$$2s(p_2+p_1^2)+3s(p_1p_2)-cE(U) \geq 0.$$

Hence, we need

$$
\begin{aligned}
s &\geq \frac{cE(U)}{2(p_2 + p_1^2) + 3(p_1 p_2)} \\
&= \frac{c(p_2 + 2p_1)}{2(p_2 + p_1^2) + 3(p_1 p_2)}
\end{aligned}
$$