Probability and Stochastic Processes (EL6303) September 24, 2018
NYU Tandon School of Engineering, Fall 2018
Instructor: *Dr. Elza Erkip*

Exercise 2

1. A coin with $P(H) = p, P(T) = q$ is tossed n times.

   (a) Find the probability that the number of heads is even as a function of p and q.

   (b) Check your answer to part a) when $p = 1$ and $q = 0$. Does it make sense? Explain.

   (c) Check your answer to part a) when $.Don \to \infty$. Does it make sense? Explain.

   Hint: You can use the following equalities:

$$\begin{aligned} (q+p)^n &= \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}, \\ (q-p)^n &= \sum_{k=0}^{n} \binom{n}{k} (-p)^k q^{n-k}. \end{aligned}$$

2. Consider the following probability density function for the random variable X.

$$f_X(x) = (\alpha x + \beta)(U(x-a) - U(x-2))$$

   where $U(x)$ is the unit step function

$$U(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

   We know the following:

   - $P(X \leq 0) = 0$ and $P(X < \epsilon) > 0$ for any $\epsilon > 0$.
   - $P(0 \leq X \leq 1) = 5/16$.

   Answer the following questions:

   (a) Find α, β and a.

   (b) Sketch $f_X(x)$.

(c) Find $P(X > 0.5)$.

(d) Find $F_X(x)$, the cumulative distribution function of X.

(e) Sketch $F_X(x)$.

3. Every morning Hillary arrives at school between 8 am and 9 am. Her arrival time (in minutes) is denoted by continuous random variable T and follows a uniform distibution. Her friend Donald measures Hillary's arrival time with three different watches.

   (a) Sketch the PDF and CDF of T.

   (b) The first watch Donald uses has a resolution of 10 minutes, that is it only shows times that are a multiple of 10. The time measured by this watch is T_1. Sketch T_1 as a function of T. Then sketch the PDF and the CDF of T_1.

   (c) The second watch Donald uses stops working from 8:30 am to 8:40 am and then starts working normal. Hence from 8:40 am onwards, it is 10 minutes behind. The time measured by this watch is T_2. Sketch T_2 as a function of T. Then sketch the PDF and the CDF of T_2.

   (d) The third watch Donald uses skips 10 minutes at 8:30 am to 8:40 am. Hence from 8:30 am onwards it is 10 minutes ahead. The time measured by this watch is T_3. Sketch T_3 as a function of T. Then sketch the PDF and the CDF of T_3.

   Note: **Please correctly label all axes!** For discrete or mixed random variables you can use the impulse function to show the point masses.

4. The temperature in a field is modeled as a random variable $X \sim N(\mu, \sigma^2)$ where $\mu = 50°\text{F}$ and variance $\sigma^2 = 100$. When the temperature goes below 20°F or above 85°F, it is considered dangerous for the crops.

   (a) Find the probability that the crops are in danger.

   A cheap sensor measures the clipped temperature according to the function $h(x)$:

$$h(x) = \begin{cases} 25, & x \leq 25 \\ x & 25 < x < 75 \\ 75 & x \geq 75 \end{cases}$$

   Let $Y = h(X)$ be the temperature measured by the sensor.

   (b) Is Y a continuous, discrete or mixed random variable? Explain.

   (c) Explain how you would use the sensor output Y to detect the danger.

   (d) What is the probability of false alarm when the sensor is used? That is, find

$$P(\text{sensor announces danger}|\text{no danger}).$$

(e) What is the probability that the sensor misses the danger? That is, find

$$P(\text{sensor does not announce danger}|\text{danger}).$$

Note: Your answers should be expressed in terms of the Q function, which is

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{u^2/2} du.$$

5. Consider a communication system in which packets are transmitted over a noisy channel. A packet is either received correctly or transmission fails. If a packet is received correctly, the receiver sends an acknowledgment (ACK) to the transmitter and we are done. Otherwise, the transmitter re-sends the same packet until it receives an ACK. Each transmission independently fails with probability p. There can be up to m retransmissions. If after m retransmissions there is no ACK, the packet is **dropped**; otherwise we have **successful delivery**.

   (a) Find the probability of successful delivery.

   (b) For $p = 0.25$, what is the smallest m to ensure that probability of a dropped packet is not more than 0.01?

   (c) Now suppose the same packet is sent to N independent receivers. For each receiver transmissions fail with probability p. The transmitter stops when it receives ACKs from *all* N receivers. There can be up to m retransmissions. Find the probability of successful delivery to all N receivers.

   (d) Compare part (c) with part (a). Which success probability is higher? What happens to your answer in part (c) when $N \to \infty$?

   (e) Consider the N receiver scenario as in part (c). But now the transmitter declares success when it receives at least one ACK *from any receiver* after at most m retransmissions. Find the probability of success. Compare with part (a). What happens to your answer when $N \to \infty$?

6. Suppose the random variable N_t represents the number of e-mails you receive up to time t. We assume N_t has Poisson distribution with parameter μt, that is

$$P(N_t = k) = P(\text{k emails up to time t}) = \mathrm{e}^{-\mu \mathrm{t}} \frac{(\mu \mathrm{t})^{\mathrm{k}}}{\mathrm{k}!}, \quad \mathrm{k} = 0, 1, 2, \ldots$$

   We define T_1 as the arrival time of the first e-mail.

   (a) Suppose you know $T_1 > t$. What does this imply about N_t?

   (b) Find the cumulative distribution function (cdf) of T_1.

   (c) Find the probability density function (pdf) of T_1.