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Homework 1.

1. proof: $f(n) = \Theta(g(n))$ implies $\exists\ c_1, c_2, n_0 > 0$
   $\forall n \geq n_0 \quad f(n) \in [c_2 g(n),\ c_1 g(n)]$
   Which implies $g(n) \in [\frac{1}{c_1} f(n),\ \frac{1}{c_2} g(n)]$ Q.E.D.
   vice versa.

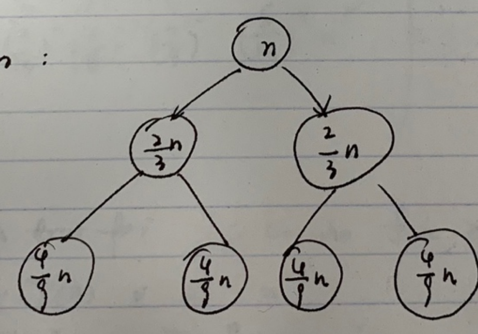
2.

	A	B	O	o	Ω	ω	Θ
a.	$\log^k n$	n^{ϵ}	✓	✓	X	X	X
b.	n^k	c^n	✓	✓	X	X	X
c.	$\sqrt{n}$	$n^{\sin n}$	X	X	X	X	X
d.	2^n	$2^{n/2}$	X	X	✓	✓	X
e.	$n^{\lg c}$	$c^{\lg n}$	✓	X	✓	X	✓
f.	$\lg(n!)$	$\lg(n^n)$	✓	X	✓	X	✓

3. a.

$1 = n^{\frac{1}{\lg n}} \leq \lg(\lg^* n) \leq \lg^*(\lg n) = \lg^* n \leq 2^{\lg^* n}$
$\leq \ln(\ln n) \leq \sqrt{\lg n} \leq \lg n \leq \lg^2 n \leq 2^{\sqrt{2\lg n}} \leq \sqrt{2}^{\lg n} \leq 2^{\lg n} = n$
$\leq n\lg n = \lg(n!) \leq n^2 = 4^{\lg n} \leq n^3 \leq n^{\lg(\lg n)} = \lg n^{\lg n} = (\lg n)!$
$\leq (\frac{3}{2})^n \leq 2^n \leq e^n \leq n \cdot 2^n \leq n! \leq (n+1)! \leq 2^{2^n} \leq 2^{2^{n+1}}$

4. Sol: iteration :



	$T(n)$	$(\frac{2}{3})^0$
$\frac{4}{3}n$	$T(\frac{2}{3}n)$	$(\frac{2}{3})^1$
$\frac{16}{9}n$	$T(\frac{4}{9}n)$	$(\frac{2}{3})^2$

$T(n) = \cancel{2^{\log_{3/2} n}\, T(1)} + \sum_{i=0}^{k} (\frac{2}{3})^i \cdot n \cdot 2^i$

$(\frac{2}{3})^k \cdot n = 1$

$k = \log_{\frac{3}{2}}(n)$

$= \cancel{2^{\log_{3/2} n}\, T(1)} + \frac{1 - (\frac{4}{3})^{k+1}}{-\frac{1}{3}} \cdot n$

$= \cancel{2^{\log_{3/2} n}\, T(1)} + 3\left((\frac{4}{3})^{\log_{\frac{3}{2}} n + 1} - 1\right) n$

$T(n) = \cancel{n^{\log_{3/2}2}}\ 3(n^{\log_{3/2}2-1} - 1)\cdot n = \Theta(n^{\log_{3/2}2})$

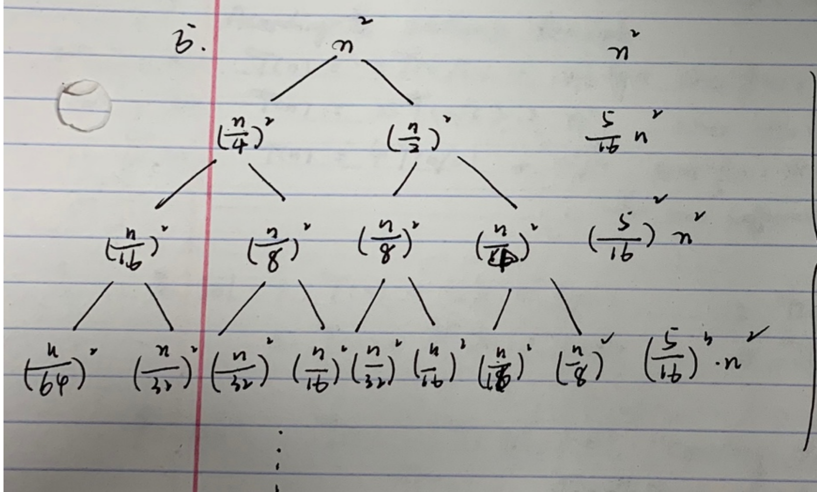
Substitution: $T(k) = \Theta(k^{\log_{3/2}2} - k)$ $\exists\, c_1, c_2, n_0$.

$c_2(k^{\log_{3/2}2} - k) \le T(k) \le c_1(k^{\log_{3/2}2} - k)$ $\forall k \ge n_0$.

$$\begin{aligned} T(n) &= 2T(\tfrac{2}{3}n) + n \\ &\le 2c_1\left((\tfrac{2}{3})^{\log_{3/2}2}\cdot n^{\log_{3/2}2} - \tfrac{2}{3}\cdot n\right) + n \\ &= c_1\cdot n^{\log_{3/2}2} - (\tfrac{4}{3}c_1 - 1)n. \\ &= c_1\left[n^{\log_{3/2}2} - (\tfrac{4}{3} - \tfrac{1}{c_1})n\right] \\ &\le c_1\left[n^{\log_{3/2}2} - n\right] \quad \text{if } \tfrac{4}{3} - \tfrac{1}{c_1} \ge 1 \Rightarrow c_1 \ge 3 \end{aligned}$$

for "$\ge$" Apply the same process. thus verified.

5.



$\cancel{(\tfrac{5}{16})^k n} =$

$(\tfrac{1}{4})^k\cdot n = 1$ $k = \log_4 n$.

The left bottom will stop

$$k \quad T(n) = \sum_{i=0}^{k} (\tfrac{5}{16})^i \cdot n^2 = \frac{1 - (\tfrac{5}{16})^{\log_4 n + 1}}{\tfrac{11}{16}}\cdot n^2 = \Theta(n^2)$$

Suppose this is true for $\exists\, c_1\, c_2\, n_0$ $\forall k \ge n_0$

$c_2\cdot n^2 \le T(k) \le c_1\cdot n^2$

$$\begin{aligned} T(n) &= T(\tfrac{n}{4}) + T(\tfrac{n}{2}) + n^2 \\ &\le c_1\tfrac{n^2}{16} + c_1\tfrac{n^2}{4} + n^2 \le (\tfrac{5}{16}c_1 + 1)\cdot n^2 \\ &\le c_1\cdot n^2 \quad \text{if } \tfrac{5}{16}c_1 + 1 \le c_1 \quad c_1 \ge \tfrac{16}{11} \end{aligned}$$

Apply the similar methods for $\Omega(\ )$. thus verified.

6. $T(n') = 9T(n'^{\frac{1}{6}}) + \log^2 n'$ $\quad \frac{1}{6}n = \lg n'^{\frac{1}{6}}$ $\quad n' = 2^n$

$\Rightarrow T(2^n) = 9T(2^{\frac{1}{6}n}) + \log_2^2 2^n$

$= 9T(2^{\frac{1}{6}n}) + n^2\log_2 2$

$= 9T(2^{\frac{1}{6}n}) + n^2$

$(\frac{1}{6})^k \cdot n = 1 \qquad k = \log_6 n.$

$T(2^n) = \sum_{i=0}^{k} 9^i \cdot n^2 (\frac{1}{36})^i$

$= \sum_{i=0}^{k} (\frac{1}{4})^i \cdot n^2 = \frac{4}{3}[1 - (\frac{1}{4})^{\log_6 n + 1}] \cdot n^2$

$= \Theta(n^2)$

$T(n') = \Theta(\lg^2 n')$

7. According to Master's Principle

a) $T(n) = 2T(n/3) + n^{\frac{1}{2}}\lg n$ since $f(n) = O(n^{\log_3 2 - \varepsilon})$ $T(n) = \Theta(n^{\log_3 2})$

b) $T(n) = 25T(n/5) + n^2$ since $f(n) = \Theta(n^{\log_5 25})$ $T(n) = \Theta(n^2\log n)$

c) $T(n) = 4T(n/2) + n^2\sqrt{n}$ since $f(n) = \Omega(n^{2+\frac{1}{2}})$ and $4(\frac{n}{2})^2\sqrt{\frac{n}{2}} < c \cdot n^2\sqrt{n}$

for some sufficiently large n $T(n) = \Theta(f(n))$

$= \Theta(n^2\sqrt{n})$

8. Sol: 1. $T(n) \le c \cdot k = \Theta(1)$ $\qquad$ 2. $T(n) = \max(ck_1, ck_2) = \Theta(1)$

3. $T(n) = N \cdot \Theta(1) = \Theta(N)$ $\qquad$ 4. $T(n) = N \cdot M \cdot \Theta(1) = O(N^2)$

9. sol: $T(n) = T(n/2) + 1$ according to Master's Principle:

$T(n) = \Theta(\log n)$ since its worst case.

$T(n) = O(\log n)$