ECE-GY 9243/ME-GY 7973 Spring 2019

Optimal and Learning Control for Robotics

Homework 2

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Exercise 1 [Dynamic Programming]

a) If doing DP,

- 1) For every possible final state x_3 compute the final cost as 1;

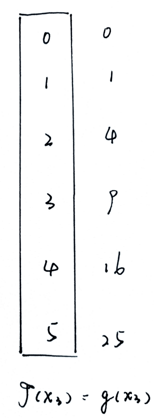

Figure 1

- 2) For every possible state x_2 at stage 2 compute minimum cost to go as 2;

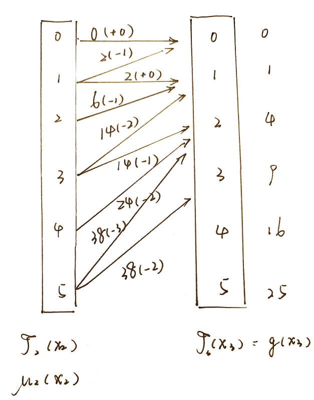

Figure 2

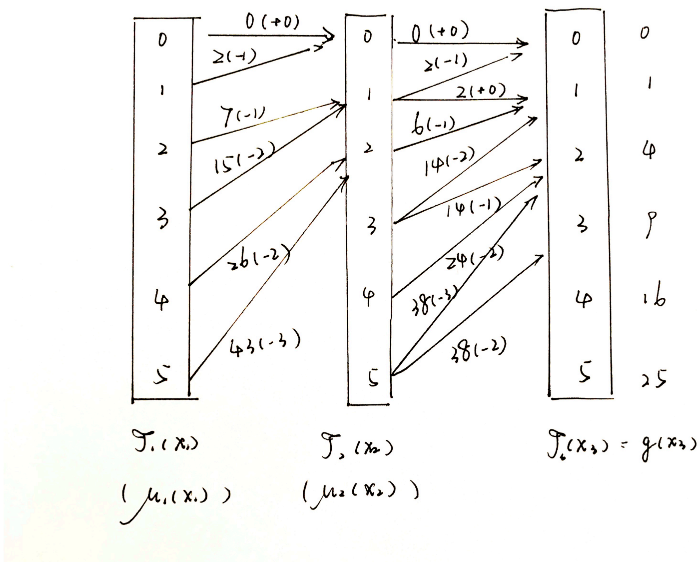

Figure 3

- 3) For every possible state x_1 at stage 2 compute minimum cost to go as 3;
- 4) For state x_0 at stage 0, compute the minimum cost to go as 4

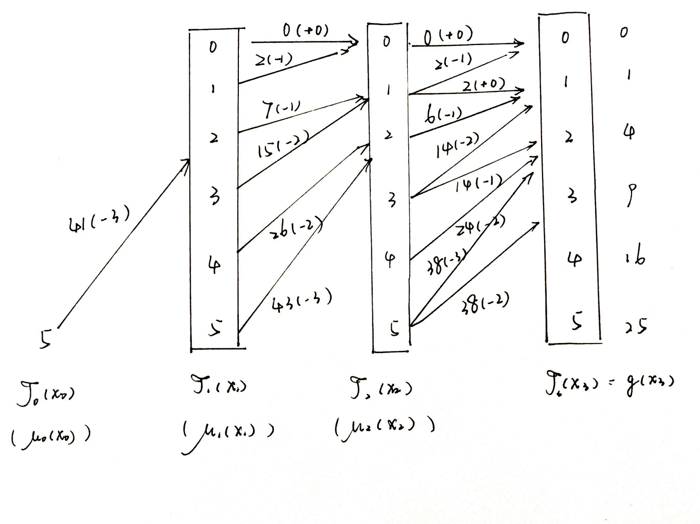

Figure 4

Correction: (* Sorry I was making a mistake, the cost from $x_1 = 5$ to $x_2 = 2$ is 40, though it does not affect the final results)

Instead of doing DP we can formulate it as a finite horizon discrete-time Riccatti equation,

$$
\begin{aligned}
&\min\, x_N^\top Q_N x_N + \sum_{k=0}^{N=3} x_k^\top Q_k x_k + u_k^\top R_k u_k \\
&s.t.\ x_n = A_n x_n + B_n u_n
\end{aligned}
\tag{1}
$$

where $Q_n = R_n = A_n = B_n = 1\ \forall n \in \{0,1,2,3\}$, so apply the solution to Riccatti equation as follows:

$$
\begin{aligned}
P_N &= Q_N = 1 \\
K_n &= -(B_n^\top P_{n+1} B_n + R_n)^{-1} B_n^\top P_{n+1} A_n \\
Pn &= Q_n + A_n^\top P_{n+1} A_n + A_n^\top P_{n+1} B_n K_n \\
u_n^* &= K_n x_n \\
J^* &= x_0^\top P_0 x_0
\end{aligned}
$$

we get: $P_3 = 1$, $K_2 = \frac{1}{2}$, $P_2 = \frac{3}{2}$, $K_1 = -\frac{3}{5}$, $P_1 = \frac{8}{5}$, $K_1 = -\frac{8}{13}$, with a little approximation the state sequence should be $x_0 = 5, x_1 = 2, x_2 = 1, x_3 = 0$ or $x_0 = 5, x_1 = 2, x_2 = 1, x_3 = 1$, while the policy squence shouldbe $u_0 = -3, u_1 = -1, u_2 = -1$ or $u_0 = -3, u_1 = -1, u_2 = 0$, the optimal cost value $J^* = 41$.

b) While $x_3 = 5$ has been a constraint, the cost of the final state is fixed as 25, processing the same steps in a) we get the optimal cost $J^* = 76$, the optimal sequences are $\{x_0 = 5, u_0 = -3, x_1 = 3, u_1 = 0,\ x_2 = 1, u_2 = 3, x_3 = 5\}$ or $\{x_0 = 5, u_0 = -2, x_1 = 3, u_1 = 0, x_2 = 3, u_2 = 2, x_3 = 5\}$

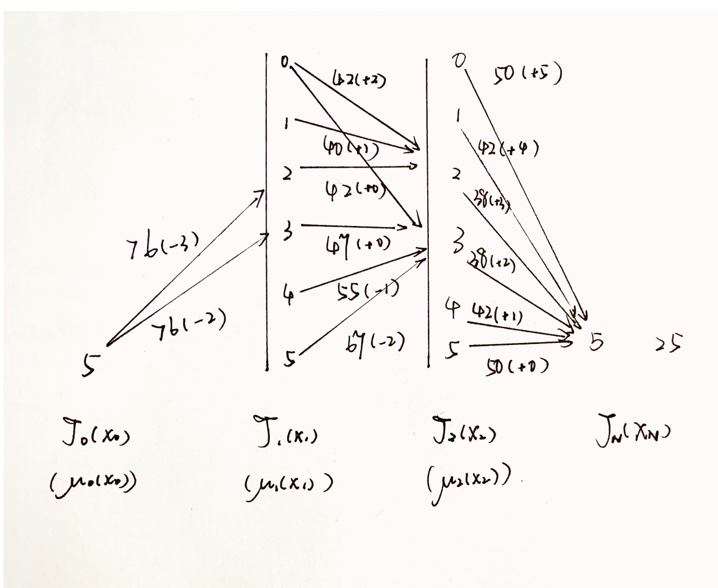

Figure 5

c) if there's a stochastic term w_n, the DP process will be minimizing the expectation at each time step, as 6 shows. 7 shows the possible paths, note that the cost of each path is not equal, the optimal one is $x_0 = 5, x_1 = 1, x_2 = 0, x_3 = 0$, which we cannot ensure.

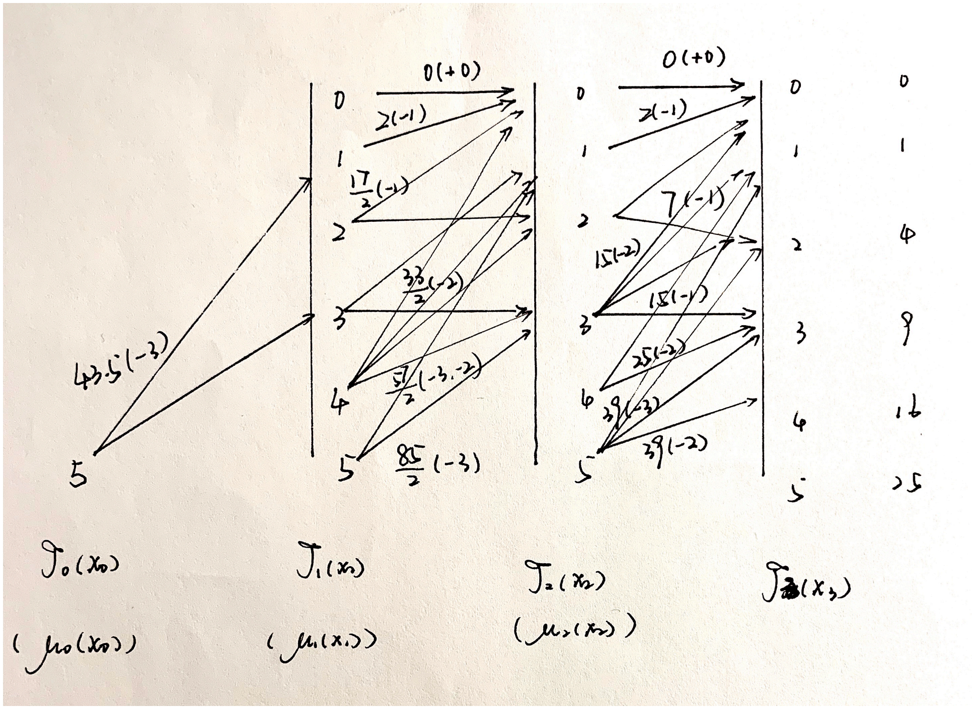

Figure 6

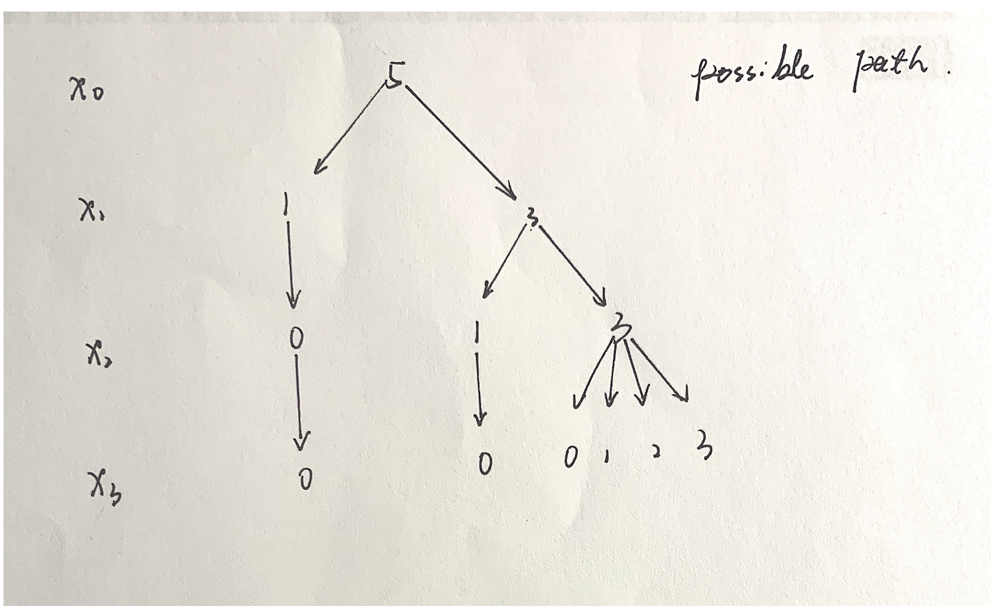

Figure 7

The expected optimal cost $J^* = 43.25$, the optimal policy is $u_0 = -3$, $u_1(1) = -1$, $u_1(3) = -2$, $u_2(0) = 0$, $u_2(1) = -1$, $u_2(3) = -2$ *or* -1.

Correction: (*Again mistakes were made when $x_0 = 5$ applying control ($u_0 = -3$) the cost is 43.25 and when $x_1 = 2$ applying $u_1 = -2$ the cost is minimum 8)

Exercise 2 [Controllability]

- When $u_n = 0$, the systems of form $x_{n+1} = Ax_n$ are stable if.f all eigenvalues of A have magnitude strictly less than unity.

  for a) and b) the eigenvalues of A are $(1.61803399, -0.61803399, 1.5)$;

  for c) the eigenvalues of A are $(0.5, -0.5, 0.5)$;

  for d) the eigenvalues of A are (0.70380158, 0.24007567, −0.44387725).

  Therefore c) and d) are stable systems when $u_n = 0$.

- The systems of form $x_{n+1} = Ax_n + Bu_n$ are controllable if.f the matrix

  $$S = [B\ AB\ A^2B\ \cdots\ A^{k-1}B] \tag{2}$$

  is full row rank (where k is the size of vector x_n)

  for a)

  $$S = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

  for b)

  $$S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1.5 & 2.25 \\ 1 & 0 & 1 \end{bmatrix}$$

  for c)

  $$S = \begin{bmatrix} 1 & 1 & 0.75 \\ 0 & -1 & 0 \\ 1 & 0.5 & 0.25 \end{bmatrix}$$

  for d)

  $$S = \begin{bmatrix} 0 & 0.5 & 0 \\ 1 & -0.5 & 0.25 \\ 0 & 0 & -0.05 \end{bmatrix}$$

  Therefore a) is not controllable b), c), d) are controllable.

- If a system is controllable, then the LQR optimal control makes the system stable, so for b), c) and d) we can find a law u_n to stablize the systems.

Exercise 3 [Linear Quadratic Regulators]

As answered in Jupyter notebook file *Linear Quadratic Regulators.ipynb*

Exercise 4 [Cart-Pole Model]

As answered in Jupyter notebook file $Cart - Pole\ Model.ipynb$

Exercise 5 [Direct Transcription]

As answered in Jupyter notebook file $Direct\ Transcription\ Methods.ipynb$