Universality and Q-Universality in Varieties of Quasi-Stone Algebras

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Abstract

The lattice $L_V(\mathbf{QS})$ of subvarieties of the variety $\mathbf{QS}$ of quasi-Stone algebras ordered by inclusion is an $\omega+1$ chain.

It is shown that the lattice $L_Q(\mathbf{Q_{0,1}})$ of subquasivarieties of the variety $\mathbf{Q_{0,1}}$ is a 4-element chain (where $\mathbf{Q_{0,1}}$ is the variety of height 3 in $L_V(\mathbf{QS})$), $L_Q(\mathbf{Q_{2,0}})$ is a finite non-modular lattice (where $\mathbf{Q_{2,0}}$ is the variety of height 4), $L_Q(\mathbf{Q_{3,0}})$ is still a finite lattice (where $\mathbf{Q_{3,0}}$ is the variety of height 7), whilst $L_Q(\mathbf{Q_{2,1}})$ is a countably infinite lattice of finite breadth, thereby satisfying a non-trivial lattice identity, and is locally finite (where $\mathbf{Q_{2,1}}$ is the variety of height 8). In the process, the critical algebras in $\mathbf{Q_{2,1}}$ are completely determined.

It is further shown that $L_Q(\mathbf{Q_{1,2}})$ is finite-to-finite relatively universal (in the sense of Hedrlín and Pultr), hence, it is uncountable and does not have finite breadth (where $\mathbf{Q_{1,2}}$ is the variety of height 9). Furthermore, it is shown that $L_Q(\mathbf{Q_{1,2}})$ is not Q-universal (in the sense of Sapir), thereby showing false a long-standing conjecture that every finite-to-finite relatively universal variety is Q-universal.

Finally, it is shown that the variety $\mathbf{Q_{2,2}}$ (of height 13) is finite-to-finite universal and, hence, Q-universal. It follows, for example, that the lattice $L_Q(\mathbf{Q_{2,2}})$ has a free lattice on a countably infinite set of generators as a sublattice (thereby failing every non-trivial lattice identity).

No proper subvariety of $\mathbf{Q_{1,2}}$ is finite-to-finite relatively universal to any of its proper subvarieties, nor is any proper subvariety of $\mathbf{Q_{2,2}}$ universal. [1]

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