Independence of multi-term commutators and centralizers

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University of Waterloo

Algebras and Lattices in Hawai'i
U. Hawai'i at Manoa, May 23, 2018

Once upon a time. . .

In early 1977, Bill Lampe announced:

Theorem

For every signature, there exists an algebraic lattice which cannot be represented as the congruence lattice of an algebra in that signature.

In particular:

> *Not every algebraic lattice can be represented as the congruence lattice of an algebra with one binary operation.*

This refuted a conjecture from the 1960s.

The key lemma in the proof (from Freese, Lampe & Taylor, 1979) is:

Bill's Key Lemma

If Con $\mathbf{A}$ *satisfies*

$$\begin{gathered}\forall \textit{ compact } \alpha,\ \exists \textit{ compact } \beta,\gamma \textit{ such that} \\ \beta\vee\gamma\geq\alpha \textit{ and } \beta\wedge\alpha=\gamma\wedge\alpha=0,\end{gathered} \tag{$*$}$$

then the terms of $\mathbf{A}$ *satisfy*

$$\begin{gathered}\forall\ (n+m)\textit{-ary term } t(\mathbf{x},\mathbf{y}),\ \forall\ \mathbf{a},\mathbf{b}\in A^n,\ \forall\ \mathbf{c},\mathbf{d}\in A^m, \\ (t(\mathbf{a},\mathbf{c})=t(\mathbf{a},\mathbf{d})) \implies (t(\mathbf{b},\mathbf{c})=t(\mathbf{b},\mathbf{d})).\end{gathered} \tag{$**$}$$

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then the terms of $\mathbf{A}$ *satisfy*

$$\begin{array}{c} \forall \ (n+m)\text{-ary term } t(\mathbf{x}, \mathbf{y}), \forall \ \mathbf{a}, \mathbf{b} \in A^n, \forall \ \mathbf{c}, \mathbf{d} \in A^m, \\ (t(\mathbf{a}, \mathbf{c}) = t(\mathbf{a}, \mathbf{d})) \implies (t(\mathbf{b}, \mathbf{c}) = t(\mathbf{b}, \mathbf{d})). \end{array} \tag{$**$}$$

This came to be called the **Term Condition** (TC).

Under Freese & McKenzie and then Hobby & McKenzie, the Term Condition came to be the foundation for commutator theory.

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In particular, $\mathbf{A}$ satisfies TC $\iff$ $\mathbf{A}$ is abelian.

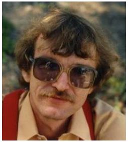
Bill's Key Lemma (Restated)

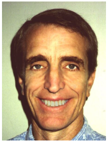
There exist algebraic lattices L with the property that

Every algebra representing L (as a congruence lattice) is abelian.

Nashville, April 1997

Pawel Idziak, Bill, and I share a house.





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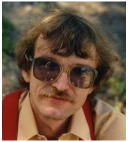

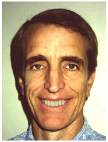

Alcohol was consumed.

Nashville, April 1997

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One weekend, Keith Kearnes came up for a visit. (Pawel was away?)

Factoid: Earlier, Keith and Ágnes Szendrei had circulated their manuscript,

"The Relationship between two commutators."

(**Question:** There are <u>two</u> commutators??)

Matrices, modules, and TC

Given an algebra $\mathbf{A}$, its **matrices** are 2×2 arrays

$$\begin{bmatrix} t(\mathbf{a},\mathbf{c}) & t(\mathbf{a},\mathbf{d}) \\ t(\mathbf{b},\mathbf{c}) & t(\mathbf{b},\mathbf{d}) \end{bmatrix}$$

with t an $(n+m)$-ary term and $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ tuples from A.

Write as $\begin{bmatrix} r & s \\ u & v \end{bmatrix}$.

Heuristic: sign the entries to get $\begin{bmatrix} +r & -s \\ -u & +v \end{bmatrix}$.

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Write as $\begin{bmatrix} r & s \\ u & v \end{bmatrix}$. Heuristic: sign the entries to get $\begin{bmatrix} +r & -s \\ -u & +v \end{bmatrix}$.

The point: In any module, all matrices satisfy $\boxed{r - s + v - u = 0.}$

Hence modules satisfy

$\boxed{\forall \text{ matrix}, \quad r = s \implies u = v.}$ (This is precisely the TC.)

2-Term Condition

Given $\begin{bmatrix} r_1 & s_1 \\ u_1 & v_1 \end{bmatrix}$, $\begin{bmatrix} r_2 & s_2 \\ u_2 & v_2 \end{bmatrix}$ and their signed versions $\begin{bmatrix} +r_i & -s_i \\ -u_i & +v_i \end{bmatrix}$,

observe that in any module, if

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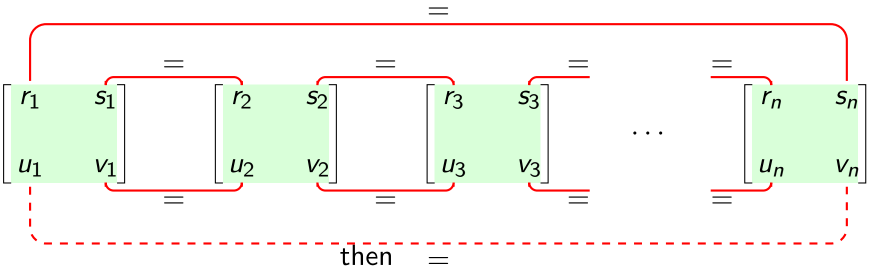
Definition (Kiss, 1992)

This implication (on all pairs of matrices) is the **2-Term Condition** (2TC).

ω-cycle Term Condition

Given $n \geq 2$, the n-**cycle Term Condition** (nTC) is the following:

For any n matrices $\begin{bmatrix} r_1 & s_1 \\ u_1 & v_1 \end{bmatrix}, \begin{bmatrix} r_2 & s_2 \\ u_2 & v_2 \end{bmatrix}, \ldots, \begin{bmatrix} r_n & s_n \\ u_n & v_n \end{bmatrix}$, if



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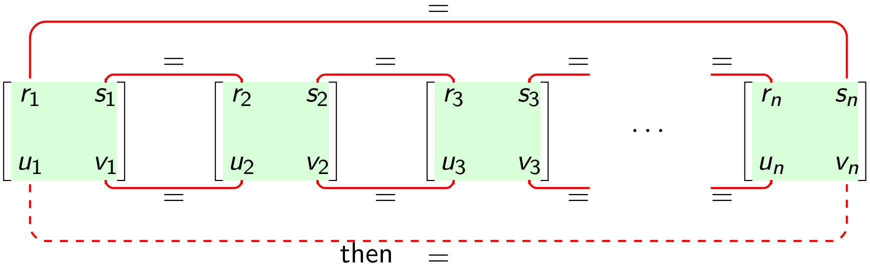

Definition (Lipparini, 1996, unpubl.)

The ω-**cycle Term Condition** (ωTC) is $\bigwedge_n n$TC.

Linear Term Condition

Most generally,

Definition (Quackenbush, 1985)

The **Linear Term Condition** (ℓTC) is the following assertion:

> For all $n \geq 1$, for any n matrices, and for any perfect matching between the main-diagonal entries and the counter-diagonal entries of these matrices: <u>if</u> all but one of the pairs in the matching is an equality, <u>then</u> so is the remaining pair.

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Take-away:

$$\text{“I am a module”} \implies \ell\text{TC} \implies \omega\text{TC} \implies \cdots \implies n\text{TC} \implies$$
$$\implies \cdots \implies 3\text{TC} \implies 2\text{TC} \implies \text{TC}.$$

Also: ℓTC $\iff$ “quasi-affine” (subreduct of poly-equiv to a module).

Commutators

1. Each condition ($*$TC) gives rise to a **centralizer relation**

$$C_*(\alpha, \beta; \delta).$$

(Restrict to matrices with $\mathbf{a}\alpha\mathbf{b}$ and $\mathbf{c}\beta\mathbf{d}$; relativize $=$ to δ.)

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In general,
$$[\alpha, \beta] \leq [\alpha, \beta]_2 \leq \cdots \leq [\alpha, \beta]_n \leq \cdots \leq [\alpha, \beta]_\omega \leq [\alpha, \beta]_\ell \leq \alpha \wedge \beta.$$
$$(\text{Abelian} \iff [1, 1] = 0. \quad \text{Quasi-affine} \iff [1, 1]_\ell = 0.)$$

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(Abelian $\iff [1, 1] = 0.$ Quasi-affine $\iff [1, 1]_\ell = 0.$)

Keith and Ági's paper was about $[-,-]$ and $[-,-]_\ell$.

So back to our story...

Bill, Keith and I are in Nashville in April 1997.

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Bill's Key Lemma

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Every algebra representing L (as a congruence lattice) is abelian.

Trying to make conversation, I ask:

Question 1

Is it possible to improve Bill's old result, by showing that there exist algebraic lattices L such that

Every algebra representing L (as a congruence lattice) is quasi-affine ?

The importance of hygiene

Bill takes a shower.

Upon exiting the shower, Bill announces ‘No.’

Theorem 1

For every algebraic lattice there exists a representing algebra that is not quasi-affine.

The proof

"It's easy," says Bill. Given an algebraic lattice L:

1. Represent L by a closure system of equivalence relations on a set B.
2. Add a violation of $[1,1]_\ell = 0$ via two partial binary operations.
3. Also name every element by a constant (to make $\operatorname{Aut} \mathbf{B}$ trivial).
4. Then use the Grätzer-Schmidt-style construction from Bill's "Independence II" paper, 1972.
5. In particular, use Bill's fine analysis of "free extension" to conclude that nothing collapses.

It was easy.

We wrote it up anyway.

Our paper

NOTHING FORCES QUASI-AFFINENESS

KEITH, BILL, AND ROSS

In general algebra, notions of abelianness have been around for a long time. The oldest is the Term Condition (TC); variants are the Two-Term Condition (2-TC), the Cycle Term Condition (ω-TC), and the Linear Term Condition (LTC). These are listed in order of increasing strength; thus an algebra satisfying LTC satisfies the other term conditions as well. An algebra satisfies LTC if and only if it is quasi-affine, that is, is a subalgebra of a polynomial reduct of a module over some ring. [History, references]

A celebrated result of the oldest author proves the existence of algebraic lattices L with the property that any algebra having L as its congruence lattice (up to isomorphism) must satisfy TC. In this paper we show that this cannot be said of any of the other three term conditions mentioned above. That is, we prove:

Theorem 0.1. *For every algebraic lattice L with $|L| > 1$ there exists an algebra $\mathbf{A}$ such that $\operatorname{Con}\mathbf{A} \cong L$ and $\mathbf{A}$ does not satisfy 2-TC.*

Before proceeding, we need some definitions.

Definition 0.2. An algebra $\mathbf{A}$ satisfies the *Two Term Condition* (2-TC) if for all $n, m > 0$, all $\bar{a}, \bar{b} \in A^n$ and $\bar{c}, \bar{d} \in A^m$, and any $(n+m)$-ary term operations $s, t \in \operatorname{Clo}_{n+m}\mathbf{A}$,

$$\left.\begin{array}{rcl} s(\bar{a},\bar{c}) & = & t(\bar{a},\bar{d}) \\ s(\bar{a},\bar{d}) & = & t(\bar{a},\bar{c}) \\ s(\bar{b},\bar{d}) & = & t(\bar{b},\bar{c}) \end{array}\right\} \quad \text{imply} \quad s(\bar{b},\bar{c}) = t(\bar{b},\bar{d}).$$

Definition 0.3. Let $\mathcal{H}$ be a closure system on the set B; i.e., $\mathcal{H}$ is a set of equivalence relations on B, and $\mathcal{H}$ is closed under arbitrary intersections. If $X \subseteq B^2$ then $\operatorname{Cg}_{\mathcal{H}}(X)$ denotes the least member of $\mathcal{H}$ which includes X. A member of $\mathcal{H}$ is *$\mathcal{H}$-principal* if it is of the form $\operatorname{Cg}_{\mathcal{H}}(a,b)$ for some $a, b \in B$.

We assume throughout this paper that in a partial algebra, the domain of every operation is nonempty.

Definition 0.4. A closure system $\mathcal{H}$ on B is called a *unary-algebraic closure system for B* if either of the following equivalent conditions are met (see [6, Lemma 5]):

(1) $\mathcal{H} = \operatorname{Con}\mathbf{B}$ for some partial algebra with universe B.
(2) $0_B \in \mathcal{H}$ and $\mathcal{H} = \{\theta \in \operatorname{Eqv}(B) : \theta = \bigcup_{a\theta b} \operatorname{Cg}_{\mathcal{H}}(a,b)\}$.

Definition 0.5. Let $\mathbf{B} = \langle B, F\rangle$ be a partial algebra with universe B, and let $\mathcal{H}$ be a unary-algebraic closure system for B. $\mathbf{B}$ and $\mathcal{H}$ form a *pointed two-dimensional pair* if

(1) $\mathcal{H} \subseteq \operatorname{Con}\mathbf{B}$.
(2) Every compact element of the lattice $\langle \mathcal{H}, \subseteq\rangle$ is $\mathcal{H}$-principal.

Date: May 15, 1998.
1991 *Mathematics Subject Classification.* XXXXX.
Key words and phrases. XXXXX.

(3) For each fundamental $f \in F$ of arity $n > 0$:
(a) For each $\bar{a} \in B^n \setminus \operatorname{Dom}(f)$ there is a unique smallest $\theta \in \mathcal{H}$ such that $(a_1/\theta \times \cdots \times a_n/\theta) \cap \operatorname{Dom}(f) \neq \emptyset$.
(b) If $\bar{a}, \bar{b} \in \operatorname{Dom}(f)$ then $(a_i, b_i) \in \operatorname{Cg}_{\mathcal{H}}(f(\bar{a}), f(\bar{b}))$ for $i = 1, \dots, n$.
(4) Every element of B is named by a nullary operation in F.

Proof of Theorem 0.1. Let L be the given algebraic lattice with $|L| > 1$. Let $\mathcal{C} = \langle C, \vee\rangle$ be the semilattice with zero of compact elements of L. For each ideal I of $\mathcal{C}$ let $\theta_I = 0_C \cup I^2$, and put $\mathcal{H} = \{\theta_I : I \text{ an ideal of } \mathcal{C}\}$. It is known that $\mathcal{H}$ is a unary-algebraic closure system, $\langle \mathcal{H}, \subseteq\rangle$ is isomorphic to L, and every compact element of $\langle \mathcal{H}, \subseteq\rangle$ is $\mathcal{H}$-principal. We will use this fact as our starting point.

Let a, b, c, d, u, v, w be distinct objects not in C and put $U = \{a, b, c, d, u, v, w\}$ and $B = C \cup U$. Define two binary partial operations, s and t, on B as follows:

(1) $\operatorname{Dom}(s) = \{(a,c), (a,d), (b,d)\}$ while $\operatorname{Dom}(t) = \{(a,c), (a,d), (b,c)\}$.
(2) $s(a,c) = t(a,d) = u$, $s(a,d) = t(a,c) = v$, and $s(b,d) = t(b,c) = w$.

Let F consist of $\{s, t\}$ together with a set of nullary operations naming each element of B, and put $\mathbf{B} = \langle B, F\rangle$.

Next, fix a nonzero element p of C. For each ideal I of $\mathcal{C}$ define

$$\theta_I^* = \begin{cases} 0_C \cup (I \cup U)^2 & \text{if } p \in I \\ 0_C \cup I^2 & \text{otherwise.} \end{cases}$$

Let $\mathcal{H}^* = \{\theta_I^* : I \text{ an ideal of } \mathcal{C}\}$. It is easily checked that $\mathbf{B}$ and $\mathcal{H}^*$ form a pointed two-dimensional pair and $\langle \mathcal{H}^*, \subseteq\rangle \cong \langle \mathcal{H}, \subseteq\rangle$ ($\cong L$). Now open [5] to page 270; it can be seen by inspection that any pointed two-dimensional pair in our sense is a two-dimensional pair in the sense of [5] (since $\operatorname{Aut}(\mathbf{B})$ is trivial). Thus the construction in the proof of [5, Theorem 1] can be carried out with $\mathbf{B}_0 = \mathbf{B}$ and $\mathcal{H}_0 = \mathcal{H}^*$ as the starting point. The result is a (total) algebra $\mathbf{A}$ having $\mathbf{B}$ as a subreduct and such that $\operatorname{Con}\mathbf{A} \cong L$. In fact, using the terminology of [4, 5], $\mathbf{A}$ has the partial algebra $\mathbf{B}[F]$ as a subreduct, where $\mathbf{B}[F]$ is $\mathbf{B}$ "freely extended by F." One property of $\mathbf{B}[F]$ is the following (see [4, p. 102]): if $(x,y) \in B^2 \setminus \operatorname{Dom}(s^{\mathbf{B}})$ while $(x', y') \in B^2 \setminus \operatorname{Dom}(t^{\mathbf{B}})$, then $s(x,y)$ and $t(x',y')$ are defined in $\mathbf{B}[F]$ and $s(x,y) \neq t(x',y')$. Thus $\mathbf{A}$ fails to satisfy 2-TC. $\square$

Acknowledgment. The first and third authors would like to thank the second author for certain habits of personal hygiene.

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With minor adjustments, the same argument shows:

Theorem 2

For every algebraic lattice L there exists an algebra $\mathbf{A}$ such that $\operatorname{Con}\mathbf{A} \cong L$ and $[\alpha,\beta]_2 = \alpha \wedge \beta$ (hence $[\alpha,\beta]_\ell = \alpha \wedge \beta$) for all α, β.

Perhaps we could have written it up and submitted it to the Mailbox.

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Question 2

Suppose L is an algebraic lattice and $F(x,y)$ is an arbitrary binary operation on L satisfying the "obvious" minimal requirements for being a (symmetric) commutator operation. Can we construct an algebra $\mathbf{A}$ with

$$\langle L, \vee, \wedge, F\rangle \quad \cong \quad \langle \mathrm{Con}\,\mathbf{A}, \vee, \cap, [-,-]_\ell\rangle \ ?$$

$$\text{Or with} \quad \langle L, \vee, \wedge, F\rangle \quad \cong \quad \langle \mathrm{Con}\,\mathbf{A}, \vee, \cap, [-,-]_2\rangle \ ?$$

June 1998

We believed we had answered the question "Yes."

We should have written this up.

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Question 3

Suppose L is an algebraic lattice and $F_2, F_3, \ldots, F_n, \ldots, F_\omega, F_\ell$ are binary operations on L satisfying the "obvious" minimal requirements for being n-cycle ($n \geq 2$), ω-cycle, and linear commutator operations. Can we construct an algebra $\mathbf{A}$ with

$$\langle L, \vee, \wedge, F_2, \ldots, F_\omega, F_\ell \rangle \quad \cong \quad \langle \mathrm{Con}\,\mathbf{A}, \vee, \cap, [-,-]_2, \ldots, [-,-]_\omega, [-,-]_\ell \rangle \text{ ?}$$

I.e., are these commutators "essentially" independent of Con $\mathbf{A}$ <u>and</u> each other?

Szeged, August 1998

Drinking cappuccinos and beer under umbrellas in sweltering Szeged,

Keith, Bill and I devised beautiful arguments proving:

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Theorem 3

Yes: these commutators are essentially independent.

And the "obvious" minimal requirements are

1. *Each* $[-,-]_*$ *is submultiplicative and compactly determined.*
2. $[-,-]_2$ *and* $[-,-]_\ell$ *are symmetric.*
3. $[-,-]_n \leq [-,-]_{n+1}$ *for all n;* $\bigvee_n [-,-]_n = [-,-]_\omega \leq [-,-]_\ell$.
4. $[\alpha \wedge \beta, \alpha \vee \beta]_\ell \leq [\alpha, \beta]_\ell$.

Proofs required new ingredients

- A finer analysis of the the Grätzer-Schmidt construction.
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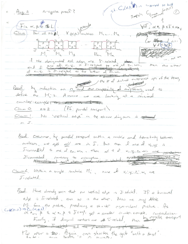
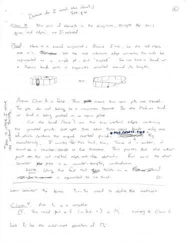
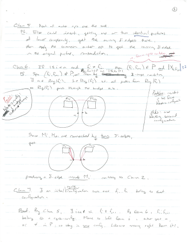
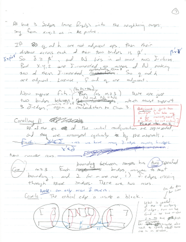

...

Proofs required new ingredients

- A finer analysis of the the Grätzer-Schmidt construction.
- Complicated arguments establishing $C_*(\alpha, \beta; \delta)$ where desired.

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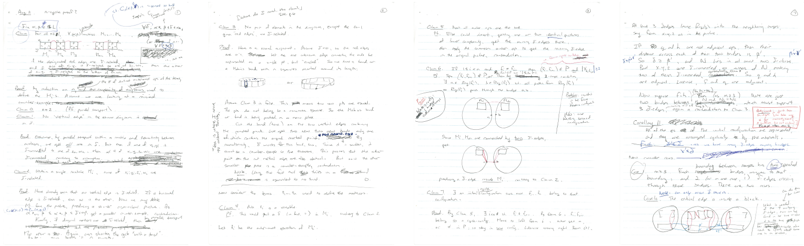

. . .

We should have typed it all up.

But instead, we got greedy.

Centralizers

Representing commutators is child's play.

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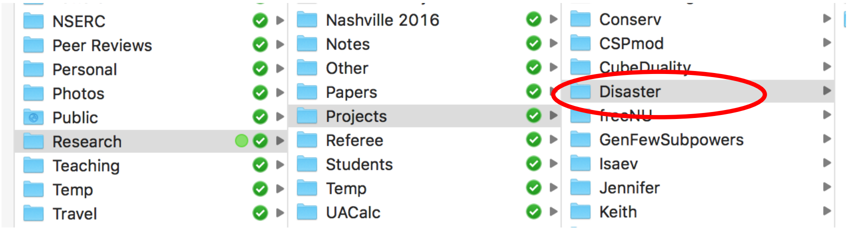
Question 4

Suppose L is an algebraic lattice and $R_2, R_3, \ldots, R_n, \ldots, R_\omega, R_\ell$ are 3-ary relations on L satisfying the "obvious" minimal requirements for being n-cycle ($n \geq 2$), ω-cycle, and linear centralizer relations. Can we construct an algebra $\mathbf{A}$ with

$$\langle L, \vee, \wedge, R_2, \ldots, R_\omega, R_\ell \rangle \quad \cong \quad \langle \mathrm{Con}\,\mathbf{A}, \vee, \cap, C_2, \ldots, C_\omega, C_\ell \rangle\ ?$$

A Disaster is born

Somewhere around this time we started calling this project “Disaster.”



And we got a mascot:

Major Disaster (DC Comics Inc.)

Summer 2000

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Theorem 4

We can represent C_ω and a single n-cycle centralizer C_{n_0}, subject to the "obvious" minimal requirements:

- *Each C_* is "compactly determined" (in a precise way).*
- $C_\omega(-,-;-)$ *imples* $C_{n_0}(-,-;-)$.
- $C_{n_0}(\alpha,\beta;-)$ *iff* $C_{n_0}(\beta,\alpha;-)$ *if* $n_0 = 2$.
- $C_\omega(-,\beta;\delta)$ *implies* $C_\omega(-,\beta\vee\delta;\delta)$.

Summer 2002

Now we can represent three centralizers: $C_{n_0}, \quad C_\omega, \quad C_\ell.$

Further "obvious" requirements:

- $C_\ell(-,-;-)$ implies $C_\omega(-,-;-)$.
- $C_\ell(\alpha,\beta;-)$ iff $C_\ell(\beta,\alpha;-)$.
- $C_\ell(\alpha,\beta;-)$ implies $C_\ell(\alpha\wedge\beta,\alpha\vee\beta;-)$.
- $C_\ell(-,\beta;\delta)$ implies $C_\ell(-,\beta\vee\delta;\delta)$.

October 2004

Further progress: $C_{n_0},\ C_{n_0+1},\ C_{n_0+2},\ \ldots,\ C_{2n_0-1},\ C_\omega,\ C_\ell.$

But here we were stuck.

October 2004

Further progress: $C_{n_0},\ C_{n_0+1},\ C_{n_0+2},\ \ldots,\ C_{2n_0-1},\ C_\omega,\ C_\ell$.

But here we were stuck.

The Grätzer-Schmidt construction always produces an algebra **A** whose congruences are 4-permutable.

Lemma

If **A** *is* 4*-permutable, then for all n,* $C_{2n}(-,\beta;\delta)$ *implies* $C_n(-,\beta\vee\delta;\delta)$.

But for general **A**, *this implication does not hold.*

So our methods <u>cannot</u> represent all possible cycle centralizer sequences $\langle C_2,\ C_3,\ \ldots,\ C_n,\ \ldots\rangle$.

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— OR —

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Mahalo!