On Epimorphisms of Ordered Algebras

Nasir Sohail[1] and Boža Tasić[2]
[1] Department of Mathematics, Wilfrid Laurier University, Waterloo, ON Canada
[2] Ted Rogers School of Business Management, Ryerson University, Toronto, ON Canada

Given a type $\mathcal{F}$, an ordered $\mathcal{F}$-algebra is a triple $\mathbf{A} = (A, \mathcal{F}^{\mathbf{A}}, \leq_A)$, where $(A, \mathcal{F}^{\mathbf{A}})$ is an $\mathcal{F}$-algebra and $(A, \leq_A)$ is a partially ordered set (briefly, poset), such that every $f^{\mathbf{A}} \in \mathcal{F}^{\mathbf{A}}$ is a monotone function, i.e., if $f \in \mathcal{F}_k$ and $a_1, b_1, \ldots, a_k, b_k \in A$, then

$$(a_1 \leq_A b_1 \wedge \cdots \wedge a_k \leq_A b_k) \Rightarrow f^{\mathbf{A}}(a_1, \ldots, a_k) \leq_A f^{\mathbf{A}}(b_1, \ldots, b_k).$$

The initial account of ordered universal algebras, to our best knowledge, was provided by Bloom in [1]; though the monograph [2] by Fuchs, written in the early 1960's, gave an outline of the theory of ordered groups, rings, fields and semigroups. It was recently proved in [3], that epimorphisms (briefly, epis) are preserved by the forgetful functor from the category of partially ordered monoids (briefly, pomonoids) to the category of (unordered) monoids. In this presentation, we shall present our recent work that pursued this line of research in the context of ordered universal algebras. Accordingly, it was proved that epis are surjective in certain varieties (categories) of all ordered $\mathcal{F}$-algebras.

Given an ordered subalgebra $\mathbf{B}$ of an ordered $\mathcal{F}$-algebra $\mathbf{A}$, the dominion of $\mathbf{B}$ in $\mathbf{A}$ is the ordered subalgebra with the universe,

$$\mathrm{dom}_{\mathbf{A}}\mathbf{B} = \{d \in A \mid f(d) = g(d)\ \forall f, g : \mathbf{A} \longrightarrow \mathbf{C} \text{ with } f|_{\mathbf{B}} = g|_{\mathbf{B}}\}.$$

We say that $\mathbf{B}$ is closed in $\mathbf{A}$ if $\mathrm{dom}_{\mathbf{A}}\mathbf{B} = \mathbf{B}$. If $\mathcal{V}$ is a variety of ordered algebras then $\mathbf{B} \in \mathcal{V}$ is said to be absolutely closed if it is closed in every extension in $\mathcal{V}$. We call $\mathcal{V}$ absolutely closed if every $\mathbf{B} \in \mathcal{V}$ is absolutely closed. It can be easily observed that epis are surjective in $\mathcal{V}$ if and only if $\mathcal{V}$ is absolutely closed if and only if $\mathcal{V}$ has the special amalgamation property. The latter means if $\phi_i : \mathbf{C} \longrightarrow \mathbf{A}_i$, $i \in \{1, 2\}$, are embeddings of ordered algebras in $\mathcal{V}$, such that $\mathbf{A}_1$ is isomorphic to $\mathbf{A}_2$ via, say $\nu : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$, with $\nu \circ \phi_1 = \phi_2$, then the canonical embeddings $\eta_i : \mathbf{A}_i \longrightarrow \mathbf{A}_1 *_{\mathbf{C}} \mathbf{A}_2$ (the pushout) agree only on $\mathbf{C}$.

We prove that pushouts exist in the varieties of ordered algebras defined through inequalities between constants, and that η_i agree on $\mathbf{C}$ only (as mentioned above). It follows as consequence that epis in the variety of all (unordered) algebras of type $\mathcal{F}$ are also surjective. Our approach is of combinatorial nature, where we first define ordered term algebras and then manipulate different operations on terms.

References

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