Spectra in sub-signatures of RA

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Definitions

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A *proper relation algebra* is a sequence

$$\mathfrak{A} = \langle A, +, \cdot, -, ;, \breve{}, 0, 1, 1', \leq \rangle$$

where A is a set of binary relations on some set U. *Representable* relation algebras are isomorphic to a proper relation algebras.

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A *relation algebra* is an algebra with the same signature as an RRA, where A is not necessarily a set of binary relations.

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What happens if we weaken the requirements for representability? Is there another signature for which the class of representable algebras is more well-behaved?

Weaker definitions of representability

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If $\Omega = \langle +, \cdot, -, ;, \breve{}, 0, 1, 1', \leq \rangle$, then the Ω-representable relation algebras are exactly the members of WRRA.

Theorem - Jónsson/Tarski

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If $\Omega = \langle +, \cdot, -, ;, \breve{}, 0, 1, 1', \leq \rangle$ then *every* relation algebra is Ω-representable.

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The *point algebra* is the RA with atoms $r, \breve{r}, 1'$ determined by the equation $r;r = r$ (along with the RA axioms).

Definition

The *pentagonal algebra* is the symmetric RA with atoms $r, b, 1'$ determined by the equations $r;r = 1' + b$, $b;b = 1' + r$, and $r;b = b;r = r + b$.

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The pentagonal algebra is only representable on a set of order 5, but whenever $\langle +, ;, \breve{} \rangle \subseteq \Omega \subseteq \langle +, ;, \breve{}, 0, 1, 1', \leq \rangle$, $\mathrm{Spec}_{\Omega}(\mathfrak{A}) = \{3, 5\}$.

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The pentagonal algebra is weakly representable on 5^n points for $n \geq 1$.