The order of principal congruences of a bounded lattice.
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Summary

We characterize the order of principal congruences of a bounded lattice as a bounded ordered set. We also state a number of open problems in this new field.

arxiv: 1309.6712

Background

Let A be a lattice (resp., join-semilattice with zero). We call A *representable* if there exist a lattice L such that A is isomorphic to the congruence lattice of L, in formula, $A \cong \operatorname{Con} L$ (resp., A is isomorphic to the join-semilattice with zero of compact congruences of L, in formula, $A \cong \operatorname{Con}_{\mathrm{c}} L$).

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Or equivalently: Characterize representable join-semilattices as distributive join-semilattice with zero.
This conjecture was refuted in F. Wehrung in 2007.

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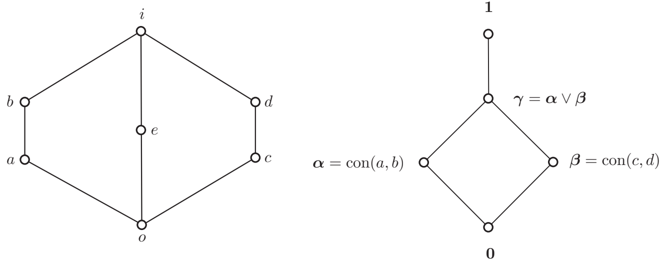
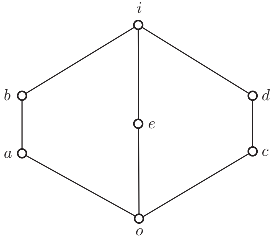
(b) $\operatorname{Con_c} L$ is the set of compact elements of $\operatorname{Con} L$, a lattice theoretic characterization of this subset.

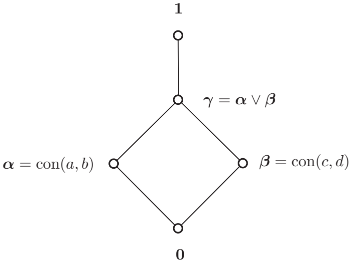
(c) $\operatorname{Princ} L$ is a directed subset of $\operatorname{Con_c} L$ containing the zero and join-generating $\operatorname{Con_c} L$; there is no lattice theoretic characterization of this subset.



This is the lattice N_7 and its congruence lattice $\mathsf{B}_2 + 1$. Note that $\operatorname{Princ} \mathsf{N}_7 = \operatorname{Con} \mathsf{N}_7 - \{\gamma\}$, while in the standard representation K of $\mathsf{B}_2 + 1$ as a congruence lattice (G. Grätzer and E. T. Schmidt, 1962), we have $\operatorname{Princ} K = \operatorname{Con} K$.

Principal congruences





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This example shows that $\operatorname{Princ} L$ has no lattice theoretic description in $\operatorname{Con} L$.

Theorem 1

For a bounded lattice L, the order $\operatorname{Princ} K$ is bounded. We now state the converse.

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Theorem

Let P be an order with zero and unit. Then there is a bounded lattice K such that

$$P \cong \operatorname{Princ} K.$$

If P is finite, we can construct K as a finite lattice.

Lattice Problem 1

Problem

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G. Czédli solved this problem for countable lattices
arXiv:1305.0965

Lattice Problem 2

Even more interesting would be to charaterize the pair $P = \operatorname{Princ} L$ in $S = \operatorname{Con}_{\mathrm{c}} L$ by the properties that P is a directed order with zero that join-generates S. We have to rephrase this so it does not require a solution of the congruence lattice characterization problem.

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Problem

Let S be a representable join-semilattice. Let $P \subseteq S$ be a directed order with zero and let P join-generate S. Under what conditions is there a lattice K such that $\operatorname{Con}_c K$ is isomorphic to S and under this isomorphism $\operatorname{Princ} K$ corresponds to P?

Lattice Problem 3

For a lattice L, let us define a *valuation* v on $\operatorname{Con_c} L$ as follows: for a compact congruence $\boldsymbol{\alpha}$ of L, let $v(\boldsymbol{\alpha})$ be the smallest integer n such that the congruence $\boldsymbol{\alpha}$ is the join of n principal congruences. A valuation v has some obvious properties, for instance, $v(\mathbf{0}) = 0$ and $v(\boldsymbol{\alpha} \vee \boldsymbol{\beta}) \leq v(\boldsymbol{\alpha}) + v(\boldsymbol{\beta})$. Note the connection with $\operatorname{Princ} L$:

$$\operatorname{Princ} L = \{\, \boldsymbol{\alpha} \in \operatorname{Con_c} L \mid v(\boldsymbol{\alpha}) \leq 1 \,\}.$$

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Problem

Let S be a representable join-semilattice. Let v map S to the natural numbers. Under what conditions is there an isomorphism φ of S with $\mathrm{Con_c}\, K$ for some lattice K so that under φ the map v corresponds to the valuation on $\mathrm{Con_c}\, K$?

Lattice Problem 4

Let D be a finite distributive lattice. In G. Grätzer and E. T. Schmidt 1962, we represent D as the congruence lattice of a finite lattice K in which *all congruences are principal* (that is, $\operatorname{Con} K = \operatorname{Princ} K$).

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Problem

Let D be a finite distributive lattice. Let Q be a subset of D satisfying $\{0, 1\} \cup \operatorname{Ji} D \subseteq Q \subseteq D$. When is there a finite lattice K such that $\operatorname{Con} K$ is isomorphic to D and under this isomorphism $\operatorname{Princ} K$ corresponds to Q?

Example:
Let D be the eight-element Boolean lattice. Let Q be a subset of D containing 0 and 1 and the three atoms (the join-irreducible elements).

Lemma

If there is a finite lattice K such that $\operatorname{Con} K$ is isomorphic to D and under this isomorphism $\operatorname{Princ} K$ corresponds to Q, then Q has seven or eight elements.

Lattice Problem 5

In particular, let $Q = \mathrm{Con}\, L$.

Problem

Let $\mathbf{K}$ *be a class of lattices with the property that every finite distributive lattice* D *can be represented as the congruence lattice of some finite lattice in* $\mathbf{K}$*. Under what conditions on* $\mathbf{K}$ *is it true that every every finite distributive lattice* D *can be represented as the congruence lattice of some finite lattice* L *in* $\mathbf{K}$ *with the additional property:* $\mathrm{Con}\, L = \mathrm{Princ}\, L$.

Theorem 2

G. Grätzer and E. T. Schmidt, *An extension theorem for planar semimodular lattices.* Periodica Mathematica Hungarica. arXiv: 1304.7489

Theorem

Every finite distributive lattice D can be represented as the congruence lattice of a finite, planar, semimodular lattice K with the property that all congruences are principal.

In fact, K is constructed as a "rectangular lattice".

Problem 6

In the finite variant of the valuation problem, we need an additional property.

Problem

Let S be a finite distributive lattice. Let v be a map of D to the natural numbers satisfying $v(0) = 0$, $v(1) = 1$, and $v(a \vee b) \leq v(a) + v(b)$ for $a, b \in D$. When is there an isomorphism φ of D with $\mathrm{Con}\, K$ for some finite lattice K such that under φ the map v corresponds to the valuation on $\mathrm{Con}\, K$?

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Remember Theorem 1:

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Remember Problems 2 and 3:

Problem

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Problem

Let S be a representable join-semilattice. Let v map S to the natural numbers. Under what conditions is there an isomorphism φ of S with $\operatorname{Con_c} K$ for some lattice K so that under φ the map v corresponds to the valuation on $\operatorname{Con_c} K$?

Problem 9

In E. T. Schmidt 1962 (see also G. Grätzer and E. T. Schmidt 2003), for a finite distributive lattice D, a countable modular lattice M is constructed with $\operatorname{Con} M \cong D$.

Problem

In Theorem 1, for a finite P, can we construct a countable modular lattice K?

Some of these problems seem to be of interest for algebras other than lattices as well.

Problem

Can we characterize the order $\operatorname{Princ} \mathfrak{A}$ *for an algebra* $\mathfrak{A}$ *as an order with zero?*

Problem 9

Problem

For an algebra $\mathfrak{A}$, how is the assumption that the unit congruence $\mathbf{1}$ is compact reflected in the order $\operatorname{Princ}\mathfrak{A}$?

Problem 10

Problem

Let $\mathfrak{A}$ *be an algebra and let* $\operatorname{Princ}\mathfrak{A} \subseteq Q \subseteq \operatorname{Con}_{\mathsf{c}}\mathfrak{A}$*. Does there exist an algebra* $\mathfrak{B}$ *such that* $\operatorname{Con}\mathfrak{A} \cong \operatorname{Con}\mathfrak{B}$ *and under this isomorphism* Q *corresponds to* $\operatorname{Princ}\mathfrak{B}$*?*

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Extend the concept of valuation to algebras in general. State and solve Problem 3 for algebras.

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Problem 12

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Can we sharpen the result of G. Grätzer and E. T. Schmidt 1960: every algebra $\mathfrak{A}$ *has a congruence-preserving extension* $\mathfrak{B}$ *such that* $\operatorname{Con}\mathfrak{A} \cong \operatorname{Con}\mathfrak{B}$ *and* $\operatorname{Princ}\mathfrak{B} = \operatorname{Con_c}\mathfrak{B}$.

Problem 12

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Can we sharpen the result of G. Grätzer and E. T. Schmidt 1960: every algebra $\mathfrak{A}$ has a congruence-preserving extension $\mathfrak{B}$ such that $\operatorname{Con}\mathfrak{A} \cong \operatorname{Con}\mathfrak{B}$ *and* $\operatorname{Princ}\mathfrak{B} = \operatorname{Con}_{\mathrm{c}}\mathfrak{B}$.

I do not even know whether every algebra $\mathfrak{A}$ has a proper congruence-preserving extension $\mathfrak{B}$.

Proof by Picture

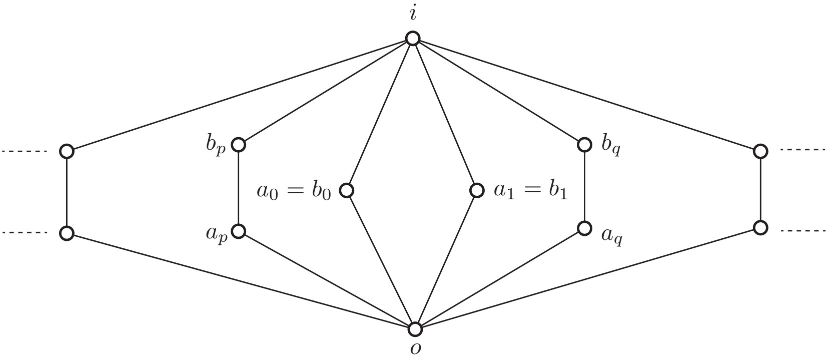
For a bounded order Q, let Q^{-} denote the order Q with the bounds removed. Let P be the order in Theorem 1. Let 0 and 1 denote the zero and unit of P, respectively. We denote by P^{d} those elements of P^{-} that are not comparable to any other element of P^{-}, that is,

$$P^{\mathrm{d}} = \{\, x \in P^{-} \mid x \parallel y \text{ for all } y \in P^{-},\ y \neq x \,\}.$$

Proof by Picture: The Lattice F

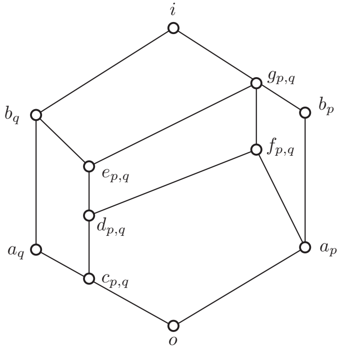
We first construct the lattice F consisting of the elements o, i and the elements a_p, b_p for every $p \in P$, where $a_p \neq b_p$ for every $p \in P^-$ and $a_0 = b_0$, $a_1 = b_1$.

The lattice F:



Proof by Picture: The Lattice K

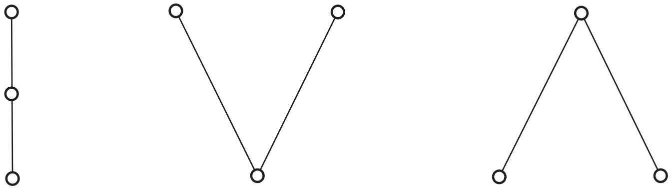
We are going to construct the lattice K (of Theorem 1) as an extension of F. For $p \prec q$, between the edges $[a_p, b_p]$ and $[a_q, b_q]$ we insert the lattice $S = S(p, q)$:



The principal congruence of K representing $p \in P^-$ will be $\mathrm{con}(a_p, b_p)$.

Proof by Picture: The Orders C, V, and H

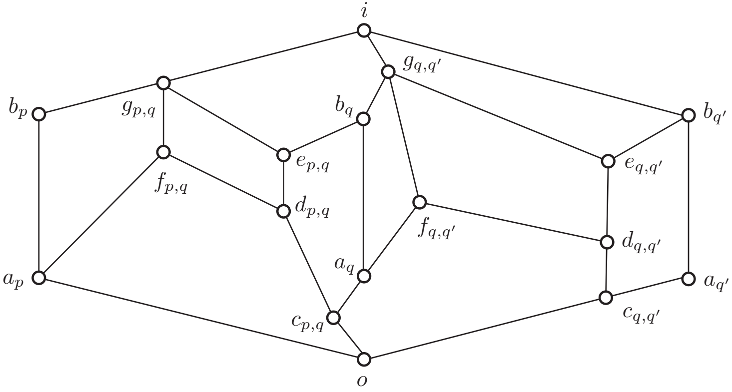
For $x \in S(p, q)$ and $y \in S(p', q')$, $p \prec q$, $p' \prec q'$ we have to find $x \vee y$ and $x \wedge y$.
If $\{p, q\} \cap \{p', q'\} = \varnothing$, then x and y are complimentary.
If $\{p, q\} \cap \{p', q'\} \neq \varnothing$, then $\{p, q\} \cup \{p', q'\}$ form a three element order C, V, or H:



Proof by Picture

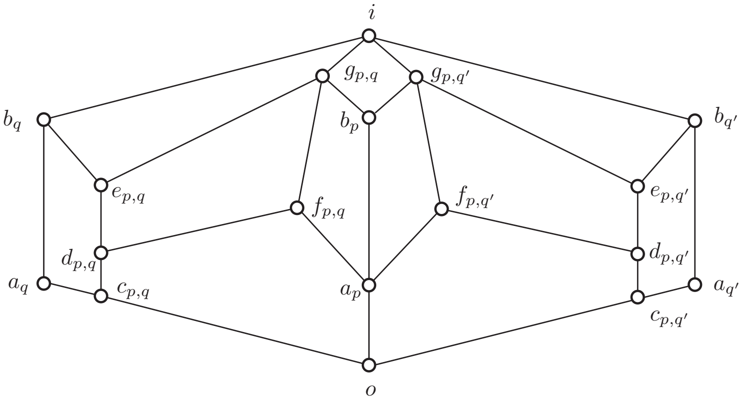
We form $x \vee y$ and $x \wedge y$ in the appropriate lattices, $S_{\mathrm{C}} = S(p < q,\ q < q')$, $S_{\mathrm{V}} = S(p < q,\ p < q')$ with $q \neq q'$, and $S_{\mathrm{H}} = S(p < q,\ p' < q)$ with $p \neq p'$.

The lattice $S_C = S(p < q,\ q < q')$:



Proof by Picture: The Lattice S_V

The lattice $S_V = S(p < q,\ p < q')$ with $q \neq q'$:



The lattice $S_{\mathsf{H}} = S(p < q,\ p' < q)$ with $p \neq p'$: