Finite Taylor Algebras, Pointing Terms and Cubed Elements

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Outline

- CSP Dichotomy on small domains
- Bounded-Width iff $SD(\wedge)$
  - New characterization for f.g $SD(\wedge)$ idempotent varieties (Barto-Kozik)
- f.g Taylor varieties (idempotent)

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- $|A| = 2$ (Schaefer[1978]-Jeavons[1998])
- $|A| = 3$ (Bulatov [2006])
- $|A| = 4$ (Markovic, McKenzie, et al [?])
- $|A| < 4$ relies heavily on Post[1941]
- NO G-set(polymorphism algebra is Taylor) $\Rightarrow$ every 2-element subalgebra has

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- $|A| < 4$ relies heavily on Post[1941]
- NO G-set(polymorphism algebra is Taylor) $\Rightarrow$ every 2-element subalgebra has
  - constant operation
  - semilattice
  - majority
  - affine

3-element Taylor Algebras

Let $\mathbf{A} = \{a, b, c\}$ idempotent and $\theta = \{[a, b], [c]\} \in \operatorname{Con} \mathbf{A}$.

- $\mathbf{A}/\theta$ is affine
  - $[a, b]$ affine
  - $[a, b]$ semilatt
  - $[a, b]$ majority
- $\mathbf{A}/\theta$ has semilatt, or majority.
  - $[a, b]$ affine
  - $[a, b]$ semilatt
  - $[a, b]$ majority
- $\mathbf{A}$ simple: reduce relations to 2-element domains, other characterizations of subpowers
  - non-singleton subalgebras - Post[1941]
  - strictly simple - Szendrei[1990]

With a Different Lens

Absorption, small generating sets, algebraic characterization of consistency($BW \Leftrightarrow SD(\wedge)$)

- $\mathbf{A}/\theta$ is affine
  - $[a, b]$ affine (few subpowers)
  - $\{a\} \triangleleft [a, b]$ but $\{b\} \not\triangleleft [a, b]$ (Malcev on Top - Maroti)
  - $\{a\}, \{b\} \triangleleft [a, b]$(few subpowers)
- $\mathbf{A}/\theta$ absorption: $\{c\} \triangleleft \mathbf{A}$ or $[a, b] \triangleleft \mathbf{A}$
  - $[a, b]$ affine (Markovic,McKenzie - Semilatt over malcev) or (few subpowers)
  - bounded-width
- $\mathbf{A}$ simple
  - absorption - (2,3)-consistency output nicely decomposes (Prague Strategy) - induction smaller domains
  - no absorption - in SD(∧) variety (2,3)-consistency output nicely decomposes (Prague Strategy) - smaller domains

Larger Domains

Larger Domains

- Cannot rely on "complete" classification of clones
- But "some" classification of finitely related clones
  - few subpowers (many contributors)
  - SD($\wedge$)???
- Strictly simple (Szendrei[1990])- arbitrary finite domains
- Absorption - works well inductively

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- Strictly simple (Szendrei[1990])- arbitrary finite domains
- Absorption - works well inductively
  - new algorithm conservative CSP (Barto[2010])
  - Absorption/no Absorption $\rightarrow$ decompose Prague strategies(SD($\wedge$) $\Rightarrow$ BW)
  - $a \prec_m b$ iff $m(\bar{x})$ is a cube term for (a, b) evaluating to a
    - $a \prec_m x,\ \forall x \in \mathbf{B} \triangleleft \mathbf{A} \Rightarrow a \prec_{m\star r} x,\ \forall x \in \mathbf{A}$

Pointed elements

Definition

A term $p(x_1,\dots,x_n)$ *points* to a if $\exists a_1,\dots,a_n$ such that $\forall i \leq n$,

$$p(a_1,\dots,a_{i-1},x,a_i,\dots,a_n) = a$$

The element a is *pointed* and the term $t(\bar{x})$ is a *pointing term*.

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Lemma

(Barto, Kozik) Let $\mathbf{A}$ *be a finite simple algebra with no proper absorbing subalgebra in a SD(∧) variety. Then for every* $a \in A$ *there exists a term which points to* a.

Let $\mathrm{Pt}\,\mathbf{A}$ denote the set of pointed elements.

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- $B \triangleleft A \Rightarrow \mathrm{Pt}\,\mathbf{B} \subseteq \mathrm{Pt}\,\mathbf{A}$

Hereditary Characterization

Theorem

(B,K) A finite idempotent algebra $\mathbf{A}$ *generates a SD*($\wedge$) *variety iff every* $\mathbf{B} \leq \mathbf{A}$ *has pointed elements.*

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Lemma

(B,K) Let $\mathbf{A}$ *be finite idempotent Taylor algebra. If* $\mathbf{A}$ *is without proper absorption, then* $\exists t(x_1, \ldots, x_m)$ *s.t.* $\forall a, b \in A$, $i \leq m$, $\exists a_1, \ldots, a_m$ *s.t.*

$$t(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_m) = b$$

So if $p(\bar{x})$ points to some $a \in A$, then $t * p$ satisfies

$$\bigwedge_{1 \leq i \leq mn} (\forall b)(\exists a_1, \ldots, a_{mn})[t * p(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_{mn}) = \{b\}]$$

on pilgrimage

Definition

An element $a \in A$ is *cubed* if $\exists t(x_1, \dots, x_n)$, $(a_1, \dots, a_n) \in A^n$, subsets $C_1, \dots, C_k$ of $[n]$ with $\bigcup C_i = [n]$ s.t.

$$t(b_1, \dots, b_n) = a \quad \text{whenever}$$

- $|\{i : a_i \neq b_i\}| = C_j$ and $|\{b_i : i \in C_j\}| = 1$ for some j

The tuple $\bar{a} = (a_1, \dots, a_n)$ is called the *base* of a, the collection $C_1, \dots, C_k$ an *index cover*, and t is called a *pointing cube-term*.

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- pointing term = pointing cube-term where index cover is collection of singletons.
- The index cover $C_1, \ldots, C_k$ can be pair-wise incomparable under inclusion.

Subalgebra of cubed elements

Let $\mathrm{CPt}\,\mathbf{A}$ be the set of cubed elements.

Lemma

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be algebras and* $\phi \in \hom(\mathbf{A}, \mathbf{B})$. *The following hold.*

- $\mathrm{CPt}\,\mathbf{A} \leq \mathbf{A}$
- $\phi(\mathrm{CPt}\,\mathbf{A}) \leq \mathrm{CPt}\,\phi(\mathbf{A})$
- $\mathrm{Pt}\,\mathbf{A} \leq \mathrm{CPt}\,\mathbf{A}$
- $\mathbf{B} \triangleleft \mathbf{A} \Rightarrow \mathrm{CPt}\,\mathbf{B} \leq \mathrm{CPt}\,\mathbf{A}$

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- Strongly abelian algebras cannot have cubed elements

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Theorem

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What is this good for?

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So if $\mathbf{A}$ is without proper absorption, then $\exists s(x_1, \ldots, x_n)$ s.t.

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- Modifying Bulatov with Prague strategies
  - walking absorption around small potatoes of prague strategy produces
    - new absorption
    - new smaller subalgebras
    - constraint relations of very restrictive kinds
- An unexpected problem........

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  - Failure to anticipate time demands of teaching 150+ Calculus class