Maltsev Conditions on the Feder-Vardi Reduction to Bipartite Graphs with Constants

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Plan

- Willard's Lemma about bipartite graphs and two sorted structures
- The Feder-Vardi construction.
- Identities that aren't preserved
- Identities that are preserved

Definition

Let $\mathbb{G} = (G, E)$ be a bipartite graph with bipartition $G = A \cup B$. That is, $E \subseteq A \times B \cup B \times A$ and $A \cap B = \varnothing$. Define $\overrightarrow{\mathbb{G}}$ to be the two sorted structure $(A, B; \overrightarrow{E})$ where $\overrightarrow{E} = E \cap A \times B$.

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$$\begin{array}{c}
\begin{array}{cc} f^A & f^B \\ \downarrow & \downarrow \end{array} \\
(a_1, b_1) \ \in \overrightarrow{E} \\
(a_2, b_2) \ \in \overrightarrow{E} \\
\vdots \\
(a_n, b_n) \ \in \overrightarrow{E} \\
\hline
(f^A(\overline{a}), f^B(\overline{b})) \in \overrightarrow{E}
\end{array}$$

Lemma (Willard)

Let Σ be a Maltsev condition satisfied by the two element connected graph, and $\mathbb{G}$ be a bipartite graph. Then $\mathbb{G}$ satisfies Σ if and only if $\overrightarrow{\mathbb{G}}$ does.

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- For a relational structure to satisfy an identity means that it has polymorphisms that do.
- For a two sorted structure to satisfy an identity, such as $s \approx t$ means that it has polymorphisms (s^A, s^B) and (t^A, t^B) satisfying $s^A \approx t^A$ and $s^B \approx t^B$.

The Feder-Vardi Construction

Let $\mathbb{A} = (A, \mathcal{R})$ be a relational structure. Let $\mathcal{R} = \{R_1, \ldots, R_m\}$, and k be the maximum arity of the R_i. The Feder-Vardi graph is constructed as follows:

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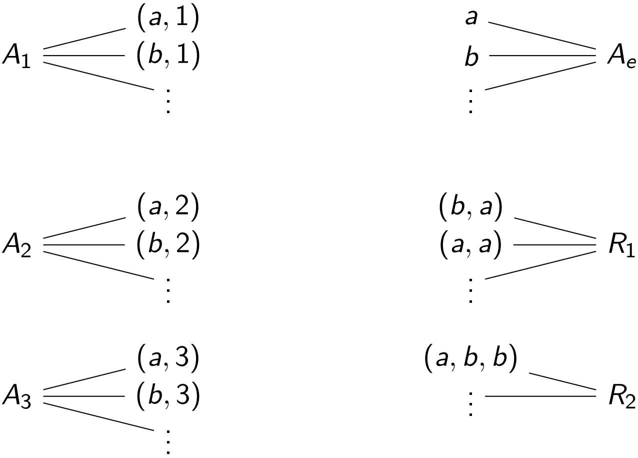
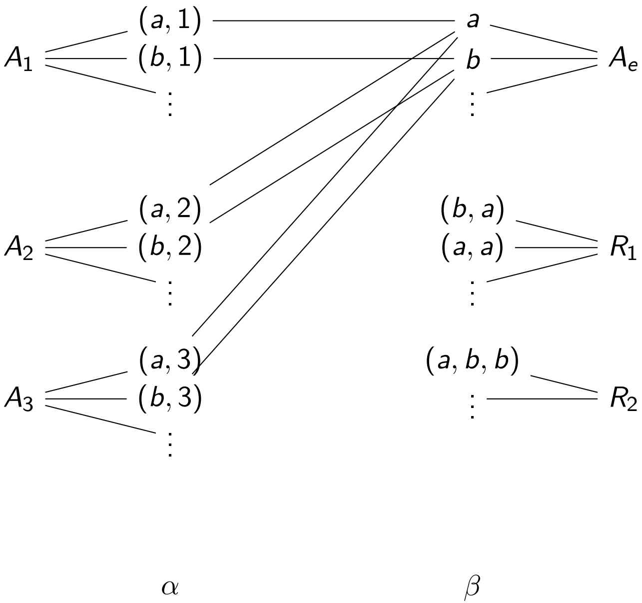
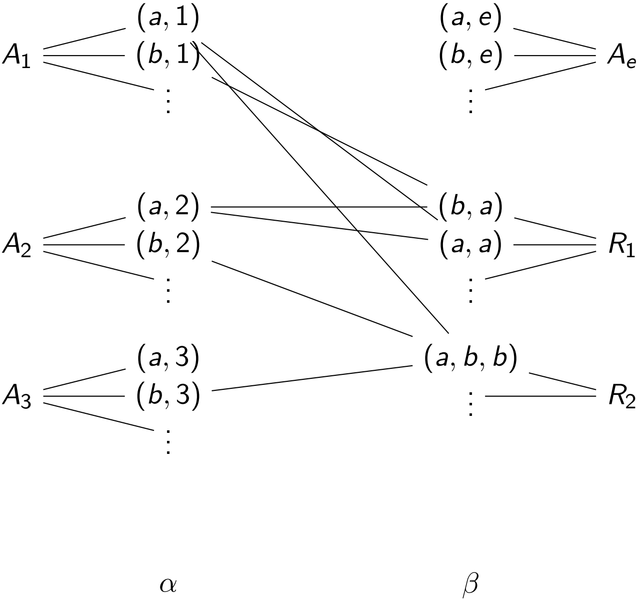
- There is a vertex, α, adjacent to each tuple in each relation, and the elements of A.
- There is a vertex, β, adjacent to (a, i) for each $a \in A$ and $i = 1, \ldots, k$. It is also adjacent to α.

A_1	$(a,1)$ $(b,1)$ $\vdots$	a b $\vdots$	A_e
A_2	$(a,2)$ $(b,2)$ $\vdots$	(b,a) (a,a) $\vdots$	R_1
A_3	$(a,3)$ $(b,3)$ $\vdots$	(a,b,b) $\vdots$	R_2
	α	β	



α β


(a, 1)
(b, 1)
a
b
A1
Ae
(a, 2)
(b, 2)
A2
(b, a)
(a, a)
R1
(a, 3)
(b, 3)
A3
(a, b, b)
R2
α
β



A_1
$(a, 1)$
$(b, 1)$
A_2
$(a, 2)$
$(b, 2)$
A_3
$(a, 3)$
$(b, 3)$
(a, e)
(b, e)
A_e
(b, a)
(a, a)
R_1
(a, b, b)
R_2
α
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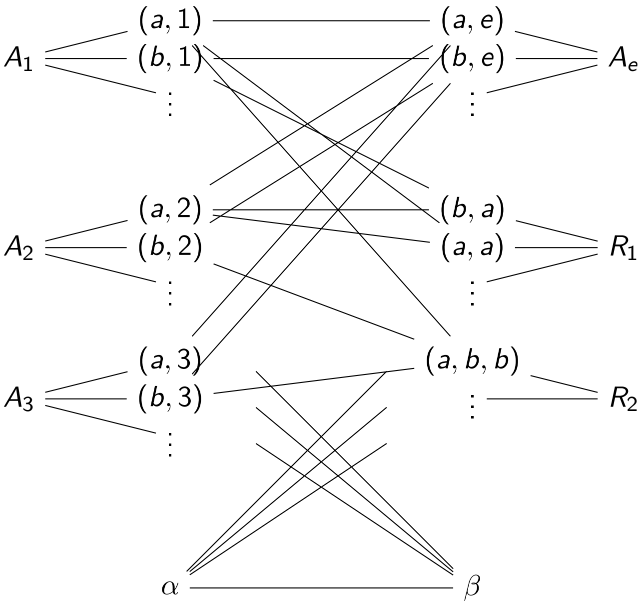

$(a,1)$
$(b,1)$
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(a,e)
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A_e
$(a,2)$
$(b,2)$
A_2
(b,a)
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3. $\mathbb{A}$ pp-interprets in $(\mathbb{G}_{\mathbb{A}})^c$, but not the other way around.
4. (3) implies that any idempotent Maltsev condition satisfied by $\mathbb{G}_{\mathbb{A}}$ is satisfied by $\mathbb{A}$. So the interesting questions about preservation of idempotent Maltsev conditions are of the form “if $\mathbb{A}$ satisfies Σ, does $\mathbb{G}_{\mathbb{A}}$?

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Definition

Let $\mathbb{G} = (G, E)$ be a graph. For $a \in G$, define $N(a) = \{b \in G : (a, b) \in E\}$.

Lemma

Let $\mathbb{G}$ be a graph containing a 6 cycle, $u - a - c - v - d - b$ satisfying

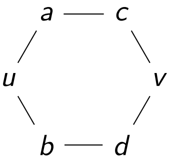
1. $N(u) \cap N(c) = \{a\}$
2. $N(a) \cap N(v) = \{c\}$.

Then $\mathbb{G}$ has no NU-polymorphism, no edge polymorphism, and no Maltsev polymorphism.

Proof of Lemma

For Maltsev.

Let m be an idempotent polymorphism of $\mathbb{G}$ satisfying $m(x, y, y) \approx x$. We will show that $m(a, a, b) = a$.



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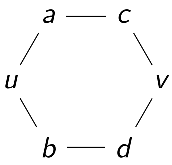

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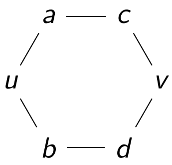
- By assumption, $m(a, b, b) = a$ and $m(v, b, b) = v$.
- $m(c, u, d)$ is adjacent to both $m(a, b, b)$ and (v, b, b) since m is a polymorphism.



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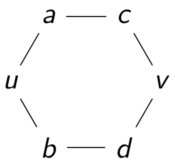
- This forces $m(c, u, d) = c$.



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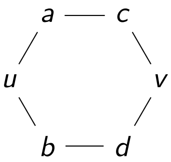

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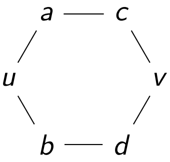
- This forces $m(c, u, d) = c$.
- $m(a, a, b)$ is adjacent to $m(c, u, d) = c$ and $m(u, u, u) = u$.
- Therefore, $m(a, a, b) = a$.



What about the Feder-Vardi graphs?

Theorem

Let $\mathbb{A} = (A, \mathcal{R})$ be a relational structure with $|A| \geq 2$, and at least one nonempty relation with positive arity. Then the Feder-Vardi graph, $\mathbb{G}_{\mathbb{A}}$ of $\mathbb{A}$ has no NU-polymorphism, no edge polymorphism, and no Maltsev polymorphism.

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Let $a, b \in A$ with $a \neq b$. Then $A_1 - (a, 1) - a - A_e - b - (b, 1)$ satisfies the conditions of the lemma. □

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Darn! No matter what polymorphism $\mathbb{A}$ satisfies, its Feder-Vardi graph can't possibly have any of these nice polymorphisms. Is anything preserved?

Some Maltsev conditions do survive

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- In fact, any identity of the form $f(\text{variables}) \approx g(\text{variables})$ is satisfied by one iff it is satisfied by the other. This includes an unpublished Maltsev condition due to Kozik, Krokhin, Valeroite, and Willard for omitting types $\mathbf{1}$ and $\mathbf{2}$.

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Our proofs of these results rely on the result stated about two-sorted structures.

Thank you!

References and Acknowledgements

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