Dualizable Algebras

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CU Boulder CU Boulder/U Szeged

AMS Sectional Meeting
Louisville, KY, October 5–6, 2013

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For each $\mathbb{B}$, the function

$$\begin{array}{rccc}
e_{\mathbb{B}}: & \mathbb{B} & \longrightarrow & \mathbb{B}^{\partial\partial} = \mathrm{Hom}(\mathbb{B}^{\partial}, \mathbf{A}) \\
 & b & \longmapsto & (\chi \mapsto \chi(b))
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is a 1–1 algebra homomorphism.

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Definition. $\mathbb{A}$ is *dualized* by $\mathbf{A}$ if $e_{\mathbb{B}}$ is onto for all $\mathbb{B}$.

$\mathbb{A}$ is *dualizable* if it is dualized by some $\mathbf{A}$.

Two Galois Connections

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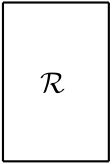
Let $\mathbb{A}$ be a finite algebra, let $\mathcal{R}$ be a set of relations on A.



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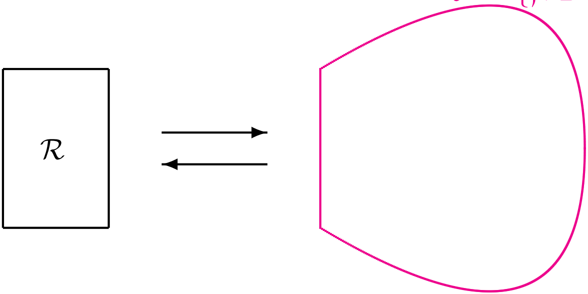
$$\mathcal{F} = \{\mathfrak{f} \colon \mathbb{B}^{\partial} \xrightarrow{\text{cont}} A \mid \mathbb{B} \in \mathsf{SP}(\mathbb{A})\}$$



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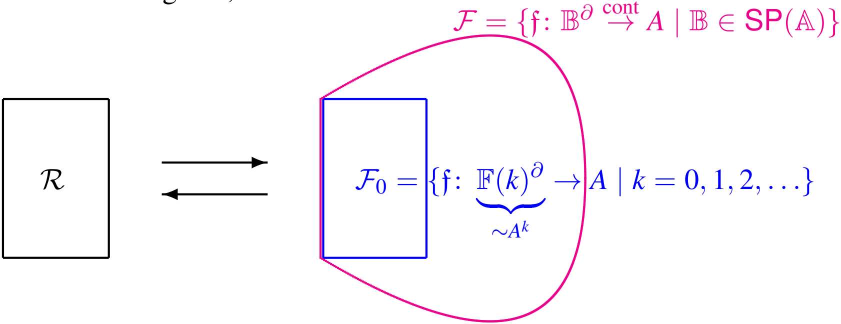

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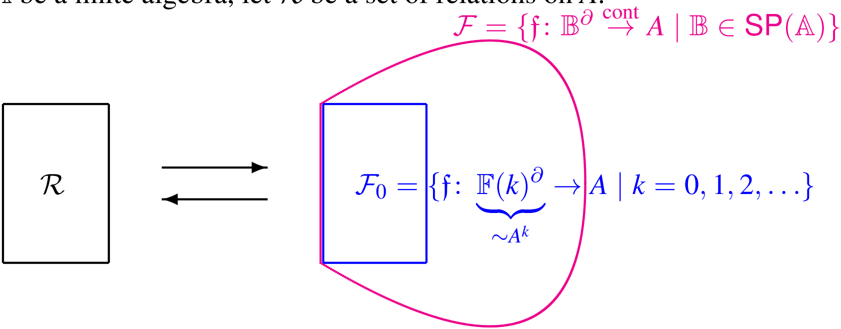

The compatibility of a function with a relation determines a Galois connection between $\mathcal{R}$ and $\mathcal{F}_0$, and between $\mathcal{R}$ and $\mathcal{F}$.

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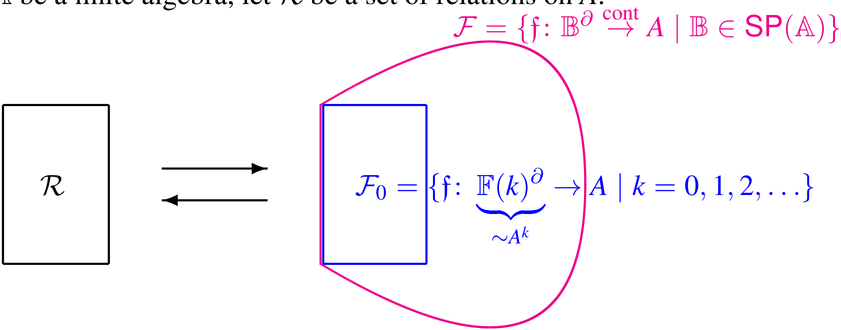

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$$\mathcal{R} \models_{\mathrm{c}} \rho \quad \text{if any } \mathfrak{f} \in \mathcal{F}_0 \text{ preserving } \mathcal{R} \text{ also preserves } \rho,$$
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[Z, DHP]
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Theorem. [Willard, Zádori]
Assume that $\mathcal{R}$ is a finite set of compatible relations of $\mathbb{A}$.
If $\mathcal{R} \models_{\mathrm{d}} \rho$ for every compatible relation ρ of $\mathbb{A}$, then $\mathbb{A}$ is dualizable.
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Definition. Call $\mathbb{A}$ *finitely related* if there is a finite set $\mathcal{R}$ of compatible relations of $\mathbb{A}$ such that $\mathcal{R} \models_{\mathrm{c}} \rho$ for all compatible relations ρ of $\mathbb{A}$.

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Remarks.

- There exist dualizable algebras that are not finitely related. [Pitkethly]
- Most algebras that are known to be dualizable are finitely related.

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(1) $\mathbb{A}$ *is* (a) *finitely related* & (b) $\mathcal{V}(\mathbb{A})$ *is CD.*
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(i) *$\mathbb{A}$ has a cube term ($\Rightarrow \mathcal{V}(\mathbb{A})$ is CM), and*

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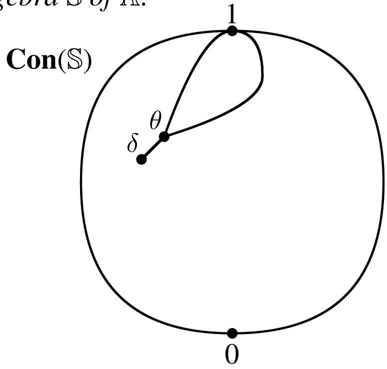

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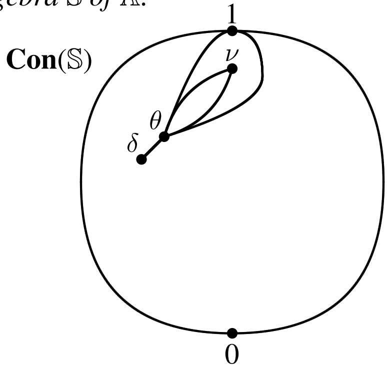

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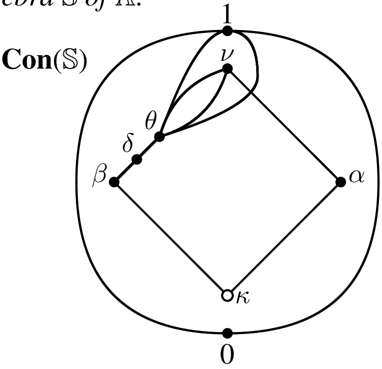

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$\exists \kappa \in \mathcal{Q}\text{-}\mathbf{Con}(\mathbb{A})$

$\exists \beta \leq \delta$

$\exists \alpha$ with $[\alpha,\alpha] \leq \kappa$ s.t.

$\alpha \vee \beta = \nu$

$\alpha \wedge \beta = \kappa$

Applications

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1. [Davey–Werner]
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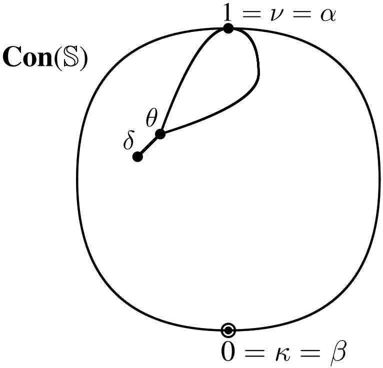

2. [NEW]
If $\mathbb{A}$ is a module,
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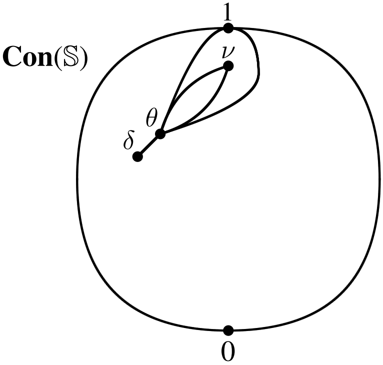
Applications

3. [Nickodemus]
If $\mathbb{A}$ is a group in which all Sylow subgroups are abelian, then $\mathbb{A}$ is dualizable.



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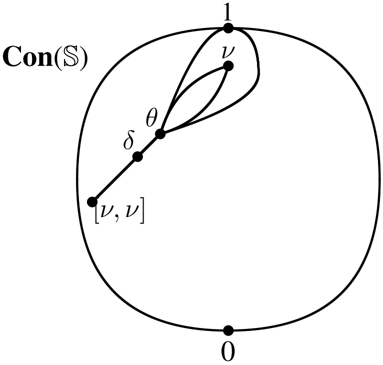

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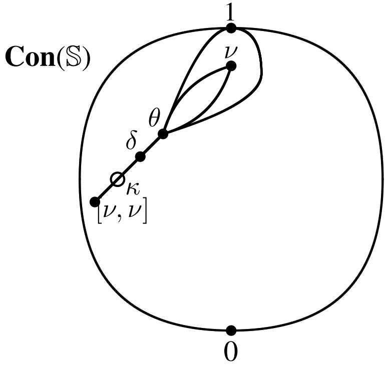

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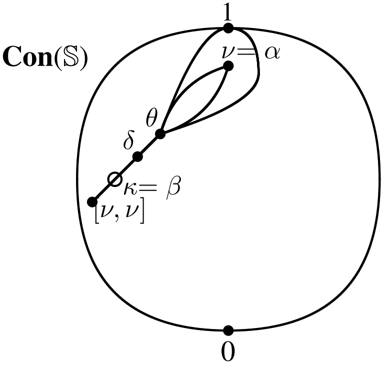

Applications

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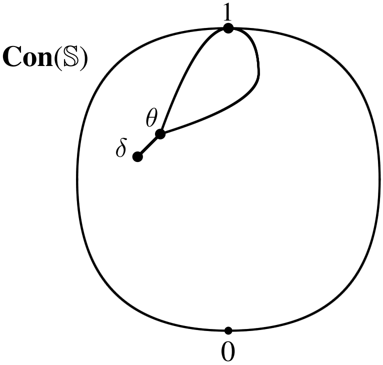
If $\mathbb{A}$ is a ring such that in each subring the Jacobson radical squares to 0, then $\mathbb{A}$ is dualizable.



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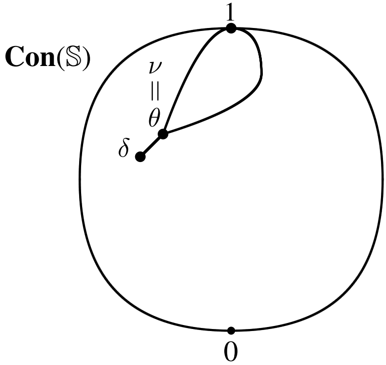

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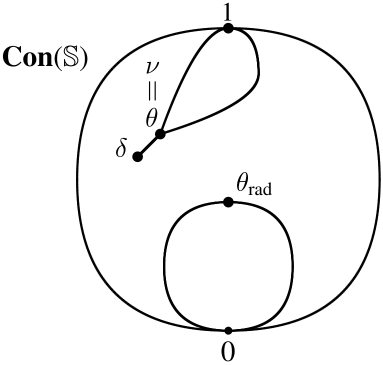

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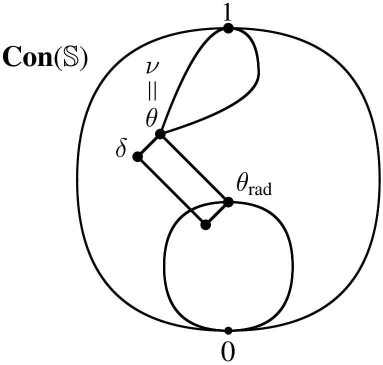

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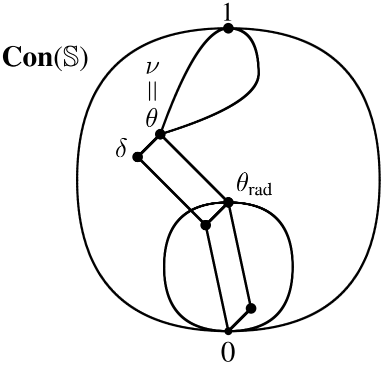

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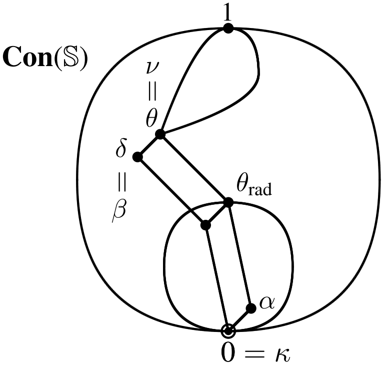

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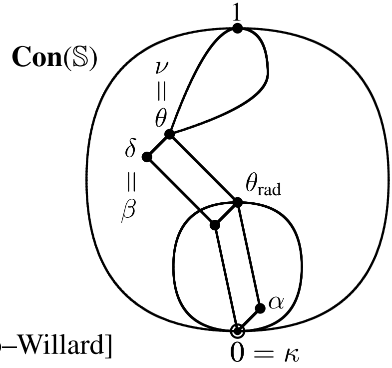

([Clark–Idziak–Sabourin–Szabó–Willard] for commutative rings.)

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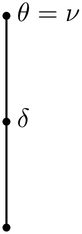

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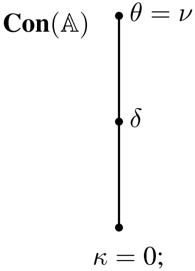

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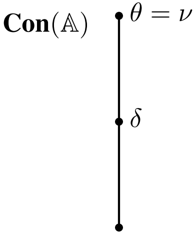

$\kappa = 0; \alpha, \beta \leq \delta,$

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