Reflexive Relations on Lattices

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Algebras of Compatible Binary Relations

If $\mathbf{A}$ is any algebra, then

$$\mathcal{R}_2(\mathbf{A}) = \langle \mathrm{Sub}(\mathbf{A}^2), \cap, \circ, \cdot^{\cup}, \Delta, \nabla \rangle$$

- $\cap$ - intersection
- $\circ$ - composition
- $\cdot^{\cup}$ - converse
- Δ - identity relation
- ∇ - universal relation

$\mathbf{Ref}(\mathbf{A})$ = the subalgebra of $\mathcal{R}_2(\mathbf{A})$ consisting of reflexive relations

Theorem (C. Bergman 1998)

Any finite algebra $\mathbf{A}$ *with a majority term is determined up to categorical equivalence by* $\mathcal{R}_2(\mathbf{A})$.

Problem

Characterize those algebras $\mathbf{R}$ which are isomorphic to $\mathcal{R}_2(\mathbf{A})$ for some finite algebra $\mathbf{A}$ with a majority term.

Theorem (1998 - "The Subalgebra Theorem")

Suppose that $\mathbf{A}$ *is a finite algebra with a majority term and that* $\mathbf{R}$ *is a subalgebra of* $\mathcal{R}_2(\mathbf{A})$. *There is an algebra* $\mathbf{B}$ *with the same universe as* $\mathbf{A}$ *so that* $\mathbf{R} = \mathcal{R}_2(\mathbf{B})$.

- The set of algebras isomorphic to $\mathcal{R}_2(\mathbf{A})$ for some finite $\mathbf{A}$ with a majority term is closed under subalgebras.
- This set is also closed under finite products.
- What about homomorphisms?

Tools: primitive positive formulas and results of G. Bergman on subalgebras of finite powers of majority algebras

Theorem (2009)

The following are equivalent for a finite algebra $\mathbf{R} = \langle R, \cap, \circ, \cdot^{\cup}, \Delta, \nabla \rangle$.

- *There is a finite algebra* $\mathbf{B}$ *with a Boolean lattice reduct so that* $\mathbf{R} \cong \mathbf{Ref}(\mathbf{B})$.
- $\langle R, \cap, \circ, \Delta, \nabla \rangle$ *is a bounded distributive lattice with involultion* $\cdot^{\cup}$.

Tools: Subalgebra Theorem and localization via idempotent unary polynomials. Not extendable to arbitrary lattices.

Theorem (K. Kaarle 2012)

The following are equivalent for a finite algebra $\mathbf{R} = \langle R, \cap, \circ, \cdot^{\cup}, \Delta, \nabla \rangle$.

1. *There is a finite algebra* $\mathbf{A}$ *with a lattice reduct so that* $\mathbf{R} \cong \mathbf{Ref}(\mathbf{A})$.
2. *The following hold:*
   - $\langle R, \cap \rangle$ *is a semilattice with least element* Δ *and greatest* ∇,
   - $(x^{\cup})^{\cup} = x$
   - $(x \cap y)^{\cup} = x^{\cup} \cap y^{\cup}$
   - $\langle R, \circ, \Delta \rangle$ *is a monoid with* 0*-element* ∇.
   - $(x \circ y)^{\cup} = y^{\cup} \circ x^{\cup}$.
   - $\circ$ *distributes over* $\cap$. $\leftarrow$
3. *There is a finite algebra* $\mathbf{A}$ *with a majority term so that* $\mathbf{R} \cong \mathbf{Ref}(\mathbf{A})$.

Tools: Subalgebra Theorem and the construction on the next slide.

Kaarle's Construction

1. Note that there is a join operation $\vee$ associated with $\mathbf{R}$.
2. For each $s \in R$, define this binary relation on R

$$\theta_s = \{\langle x, y\rangle : x \circ s \geq y \text{ and } y \circ s^{\cup} \geq x\}.$$

3. Prove that $s \to \theta_s$ is an injective homomorphism from $\mathbf{R}$ to $\mathbf{Ref}(\langle R, \cap, \vee\rangle)$.
4. Apply the Subalgebra Theorem.

Actually, this is a special case of a much more sophisticated construction.

An infinite extension.

Only need to extend the Subalgebra Theorem and apply Kaarle's Construction.

Primitive Positive Definitions

Theorem (Finite – Bodnarchuk, Kalužnin, Kotov, Romov – 1969)

Suppose that R is a set of binary relations on a finite set A. There is an algebra $\mathbf{A}$ with universe A so that $\mathrm{Sub}(\mathbf{A}^2) = R$ if and only if R is closed under primitive positive definitions.

Theorem (Infinite)

Suppose that R is a set of binary relations on a (possibly infinite) set A. There is an algebra $\mathbf{A}$ with universe A so that $\mathrm{Sub}(\mathbf{A}^2) = R$ if and only if

- *R is closed under primitive positive definitions allowing infinite quantification and conjunction and*
- *As a lattice R is algebraic.*

Theorem (G. Bergman 1977)

Suppose $\mathbf{A}$ *is an algebra with a majority term.*

- *The subalgebras of finite powers of* $\mathbf{A}$ *are uniquely determined by their binary projections.*
- *The systems of binary projections of subalgebras of finite powers of* $\mathbf{A}$ *are easily recognizable.*

Theorem (G. Bergman 1977)

Suppose $\mathbf{L}$ *is a complete lattice.*

- *The complete sublattices of arbitrary powers of* $\mathbf{L}$ *are uniquely determined by their binary projections.*
- *The systems of binary projections of complete sublattices of arbitrary powers of* $\mathbf{L}$ *are easily recognizable.*

Theorem (Subalgebra Theorem)

Suppose that $\mathbf{A}$ *is a finite algebra with a majority term and that* $\mathbf{R}$ *is a subalgebra of* $\mathcal{R}_2(\mathbf{A})$. *There is an algebra* $\mathbf{B}$ *with the same universe as* $\mathbf{A}$ *so that* $\mathbf{R} = \mathcal{R}_2(\mathbf{B})$.

Theorem (Infinite Variant for Lattices)

Suppose that $\mathbf{L}$ *is a complete lattice and that* $\mathbf{R}$ *is a subalgebra of* $\mathcal{R}_2(\mathbf{L})$. *If*

- *Every member of* $\mathbf{R}$ *is a complete sublattice of* $\mathbf{L}^2$ *and*
- *As a lattice* $\mathbf{R}$ *is algebraic*

then there is an algebra $\mathbf{B}$ *with the same universe as* $\mathbf{L}$ *so that* $\mathbf{R} = \mathcal{R}_2(\mathbf{B})$.

Theorem

The following are equivalent for a countable algebra $\mathbf{R} = \langle R, \cap, \circ, \cdot^{\cup}, \Delta, \nabla \rangle$.

- *There is an algebra* $\mathbf{A}$ *with a complete lattice reduct* $\mathbf{L}$ *so that* $\mathbf{R} \cong \mathbf{Ref}(\mathbf{A})$ *and so that every relation in* $\mathbf{Ref}(\mathbf{A})$ *is a complete sublattice of* $\mathbf{L}^2$.
- *The following hold:*
  - $\langle R, \cap \rangle$ *is a semilattice with least element* Δ *and greatest* ∇,
  - $(x^{\cup})^{\cup} = x$
  - $(\wedge x_i)^{\cup} = \wedge (x_i^{\cup})$ $\leftarrow$
  - $\langle R, \circ, \Delta \rangle$ *is a monoid with* 0*-element* ∇.
  - $(x \circ y)^{\cup} = y^{\cup} \circ x^{\cup}$.
  - $\circ$ *distributes over* **arbitrary** $\cap$. $\leftarrow$
  - *As a lattice* $\mathbf{R}$ *is algebraic.* $\leftarrow$

Thanks