# The computational complexity of deciding whether a finite algebra generates a minimal variety.

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## The Minimal Variety Problem

**Input:** *A finite algebra* $\mathbf{A}$ *of finite signature.*

**Problem:** *Decide if the variety generated by* $\mathbf{A}$ *is minimal.*

What is the computational complexity of this problem?

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In 1955, Dana Scott observed that there is a brute force algorithm to decide this problem.

## The Tarski's Finite Basis Problem

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What is the computational complexity of this problem?

Way too hard! McKenzie showed in 1993 that there is no algorithm for deciding this.

## The Finite Algebra Membership Problem For A Finite Algebra $\mathbf{B}$ of Finite Signature

**Input:** *A finite algebra* $\mathbf{A}$ *of the signature of* $\mathbf{B}$.

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In 1998, Zoltan Székely devised a seven-element algebra $\mathbf{S}$ to use for $\mathbf{B}$ for which this problem is NP-complete.

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In 2000, Cliff Bergman and Giora Słutzki found Kalicki's algorithm is in 2EXPTIME.

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In 2004, Marcel Jackson and Ralph McKenzie devised a finite semigroup $\mathbf{B}$ for which this problem is NP-hard.

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In 2009, Marcin Kozik devised a finite algebra $\mathbf{E}$ to use for $\mathbf{B}$ for which this problem in 2EXPTIME-complete

## The Congruence Distributive Variety Problem

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According to folklore (but probably Bjarni Jónsson is the folk mentioned), there is a brute force algorithm to decide this.

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In 2009, Ralph Freese and Matthew Valeriote proved that this problem, as well as several similar problems, is EXPTIME-complete.

## The Minimal Variety Problem

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What is the computational complexity of this problem?

# Outline

# Dana Scott's Brute Force Algorithm

Let $\mathbf{A}$ be a nontrivial finite algebra of finite signature. To decide whether $\mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{A}$ is a minimal variety

Step I Make a list $\mathbf{B}_0, \mathbf{B}_1, \ldots,$ up to isomorphism, of all the 2-generated algebras in $\mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{A}$.

Step II For each algebra $\mathbf{B}_i$ on the list decide whether $\mathbf{A} \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{B}_i$.

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# Kearnes and Szendrai Offer an Alternative

## The Kearnes-Szendrei Characterization

Let **C** be a finite strictly simple algebra and let $e$ be a minimal idempotent term operation on **C**. The following are equivalent.

(a) **C** generates a minimal variety.

(b) **C** is not Abelian or has a trivial subalgebra and for some positive natural number $n$, there exist binary terms $f_i$ and unary terms $g_i$ and $h_i$ for $0 \leq i \leq n$ such that all the following equations hold in **C**.

$$\circledast \begin{cases} x \approx f_0(x, eg_0(x)) \\ f_i(x, eh_i(x)) \approx f_{i+1}(x, eg_{i+1}(x)) \quad (0 \leq i \leq n-1) \\ f_n(x, eh_n(x)) \approx e(x) \end{cases}$$

(c) If $\mathbf{D} \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{A}$ and **D** is nontrivial and has no proper nontrivial subalgebras, then $\mathbf{D} \cong \mathbf{A}$.

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Step IV Construct a minimal idempotent term function $e$ of **C**

Step V Determine whether **C** is Abelian or has a trivial subalgebra.

Step VI Determine whether there are finite systems of unary and binary terms so that the equations $\circledast$ hold in **C**.

How hard can that be?
Apart for Step III, all this can be done in deterministic exponential time.

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## A Theorem of Keith and Ágnes, more or less

The Minimal Variety Problem can be settled in 2EXPTIME.

# Outline

**Theorem**

The Minimal Variety Problem is 2EXPTIME Complete.

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The proof reduces Marcin Kozik's finite algebra membership problem for Kozik's algebra **E** to the minimal variety problem.

# Kozik's Algebra $\mathbf{E}$

Marcin Kozik's uses a deterministic 2EXPTIME complete language and an alternating Turing machine that recognizes this language in EXPSPACE. Kozik, using methods inspired by Ralph McKenzie, devised an algebra $\mathbf{E}$ based on the alternating Turing machine.

With each appropriate word $w$, Kozik constructs an algebra $\mathbf{S}_w$, which is not much more complicated that the word $w$, and shows that the problem

> **Input:** *An appropriate word $w$.*
>
> **Problem:** *Decide if $\mathbf{S}_w \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{E}$ .*

is 2EXPTIME complete.

# A Plan for Using Kozik's Theorem

If Kozik's algebra $\mathbf{E}$ generated a minimal variety, almost all our troubles would be over. Unfortunately, this seems not to be the case. To overcome this difficulty we will expand $\mathbf{E}$ to $\mathbf{E}^\circ$ to force $\mathbf{E}^\circ$ to generate a minimal variety while managing to retain the complexity of the corresponding finite algebra membership problem.

## Plan of the Proof

We will prove that for any word $w$

$$\mathbf{S}_w \in \mathrm{H}\,\mathrm{S}\,\mathrm{P}\,\mathbf{E}$$

if and only if

$\mathbf{S}_w^{\circ}$ generates a minimal variety.

## Plan of the Proof

We do this in two stages:

(1) $\mathbf{S}_w \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{E} \iff \mathbf{S}_w^\circ \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{E}^\circ$

(2) $\mathbf{S}_w^\circ \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{E}^\circ \iff \mathbf{S}_w^\circ$ generates a minimal variety.

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(2) $\mathbf{S}_w^\circ \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{E}^\circ \iff \mathbf{S}_w^\circ$ generates a minimal variety.

For (1) the right-to-left direction follows by ignoring all operations added in the expansion.

Item (2) is an immediate consequence of the fact (?) that $\mathbf{E}^\circ$ generates a minimal variety.

## Plan of the Proof

So we need to see only two things:

Step 1. $\mathbf{E}^\circ$ generates a minimal variety.

Step 2. $\mathbf{S}_w \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{E} \Longrightarrow \mathbf{S}_w^\circ \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{E}^\circ$

# A Little Bit about Kozik's **E**

Kozik's algebra **E** is complicated since it is devised from an arbitrary alternating Turing machine and must reflect the complexity of such machines.

The algebra **E** does have an element $\perp$ named by as constant symbol, which is a sink in the sense that if $\perp$ is an input of any basic operation of **E**, then the output is also $\perp$.

# Expanding $\mathbf{E}$ to $\mathbf{E}^\circ$

We obtain $\mathbf{E}^\circ$ from $\mathbf{E}$ by expanding the signature in two ways:

1. For each element $e \in E$ other than $\perp$, we add a constant symbol $c_e$ to name $e$.
2. For each element $e \in E$ other than $\perp$, we add a two-place operation symbol $Q_e$ so that

$$Q_e^{\mathbf{E}^\circ}(a, b) = \begin{cases} b & \text{if } e = a \\ \perp & \text{otherwise} \end{cases}$$

We call these two-place operations Pigozzi operations.

# Step 1: Listen to Don Pigozzi

## $\mathbf{E}^\circ$ generates a minimal variety

It is enough to show that $\mathbf{E}^\circ$ can be embedded into every nontrivial algebra $\mathbf{B} \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{E}^\circ$ via the map that sends each element of $E^\circ$ to the element of $\mathbf{B}$ named by the corresponding constant symbol. The only real issue is to show that this map is one-to-one. It is the Pigozzi operations that save the day.

We duplicate here Don Pigozzi's reasoning. First, observe that the following equations are true in $\mathbf{E}^\circ$:

$$(\star) \quad \begin{cases} Q_a c_a y \approx y & \text{for all proper elements } a \in E \\ Q_a c_b y \approx \bot & \text{for all proper elements } a, b \in E \text{ with } a \neq b \\ Q_a \bot y \approx \bot & \text{for all proper elements } a \in E \end{cases}$$

Now let $\mathbf{B}$ be any nontrivial algebra belonging to the variety generated by $\mathbf{E}^\circ$. Let $h : E \to B$ be defined by

$$h(a) = \begin{cases} c_a^{\mathbf{B}} & \text{if } a \text{ is a proper element of } E \\ \perp^{\mathbf{B}} & \text{if } a \text{ is } \perp^{\mathbf{E}} \end{cases}$$

Because $\mathbf{B}$ is in the variety generated by $\mathbf{E}^\circ$ and because every proper element of $E$ is named by a constant symbol, we find that $h$ is a homomorphism. So we only have to argue that $h$ is one-to-one.

To see that $h$ is one-to-one suppose $a, b \in E$ with $a \neq b$. It does no harm to assume that $a \neq \bot$. From the last two equations in $(\star)$ we have in $\mathbf{E}^\circ$

$$Q_a dy \approx \bot$$

where $d$ is either the constant symbol $c_b$ or the constant symbol $\bot$. So this equation must hold in $\mathbf{B}$ as well. We also see from the first equation in $(\star)$ that

$$Q_a c_a y \approx y$$

holds in $\mathbf{B}$.

Now let $e$ be any proper element of $B$. So we have

$$Q_a^{\mathbf{B}}(d^{\mathbf{B}}, e) = \bot \neq e = Q_a(c_a^{\mathbf{B}}, e).$$

It follows that $h(b) = d^{\mathbf{B}} \neq c_a^{\mathbf{B}} = h(a)$. So $h$ is one-to-one as desired. As a consequence, $\mathbf{E}^\circ$ generates a minimal variety.

# A Little Bit About $\mathbf{S}_w^\circ$

To handle the finite algebra membership problem, we devise $\mathbf{S}_w^\circ$ by a kind of amalgamation of $\mathbf{E}^\circ$ with $\mathbf{S}_w$. The two parts of $\mathbf{S}_w^\circ$ have just the sinking element $\perp$ in common. The Pigozzi operations are given their natural extensions to the whole of $\mathbf{S}_w^\circ$. With one technical exception, the output of any of the other operations when given inputs from both of the parts will be the sinking element $\perp$.

# Step 2: $\mathbf{S}_w \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{E} \Longrightarrow \mathbf{S}_w^\circ \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{E}^\circ$

Under the hypothesis that $\mathbf{S}_w \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathbf{E}$, Marcin Kozik finds a natural number $t$, a subalgebra $\mathbf{B}$ of $\mathbf{E}^t$, and a homomorphism $\varphi : \mathbf{B} \twoheadrightarrow \mathbf{S}_w$ with particularly nice features.

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We modify Kozik's ideas to obtain a slightly larger natural number $t^\circ$, a subalgebra $\mathbf{B}^\circ$ of $(\mathbf{E}^\circ)^{t^\circ}$ and a homomorphism $\varphi^\circ : \mathbf{B}^\circ \twoheadrightarrow \mathbf{S}_w^\circ$.

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The main difficulty arises from the fact all the constant $t^\circ$-tuples must be included in $B^\circ$ since these elements are the ones that are named by the new constant symbols. Showing that the presence of these extra elements has no adverse impact of Kozik's complicated line of reasoning is were the chief work of our proof is located.

## The Minmal Variety Problem For Groupoids

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### Conjecture

This problem is complete for deterministic 2EXPTIME.

## The Affine Complete Variety Problem

**Input:** *A finite algebra* $\mathbf{A}$ *of finite signature.*

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What is the computational complexity of this problem?

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**Problem:** *Decide if the variety generated by* $\mathbf{A}$ *is affine complete.*

What is the computational complexity of this problem?

In 2002, Kalle Kaarli and Alden Pixley gave a not quite brute force algorithm to decide this problem.

## THE AFFINE COMPLETE VARIETY PROBLEM

**Input:** *A finite algebra* $\mathbf{A}$ *of finite signature.*

**Problem:** *Decide if the variety generated by* $\mathbf{A}$ *is affine complete.*

What is the computational complexity of this problem?

It should be a homework problem for Ralph Freese and Matthew Valeriote to show that this problem is actually EXPTIME-complete.