Algebraic Terms through Simulated Biological Evolution

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$$\Updownarrow$$

A finite non-trivial groupoid **G** is **idemprimal** if it has no non-trivial subalgebras, automorphisms or congruences and the ternary discriminator operation

$$t(a, b, c) := \begin{cases} a \text{ if } a = b \\ c \text{ if } a \neq b \end{cases}$$

is a term operation of **G**.

Murskiĭ's Theorem. (1975) *The proportion of n-element groupoids* $\mathbf{G} = \langle \{0, 1, 2, \ldots, n-1\}, *\rangle$ *that are idemprimal approaches 1 as* $n \to \infty$.

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Problem:

Is this groupoid idemprimal?

$$\mathbf{Pi} := \langle\{0, 1, 2, 3, 4\}, *\rangle$$

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Term Generation Problem: Given a finite groupoid $\mathbf{G}$ and an operation $h : G^k \to G$ that is a term operation of $\mathbf{G}$, find a term that represents h.

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   BUT, if $\mathbf{G}$ has 2 idempotents, $\mathbf{F}_3(\mathbf{G})$ will have 5^{123} elements. Checking 10^6 terms per second, the expected time to find a discriminator term will be $\approx 10^{72}$ years(!!)

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$\mathbb{M}$, a set of **male terms** $m(\vec{x})$;

$\mathbb{F}$, a set of **female terms** $f(\vec{x}, \lozenge)$, each containing a single occurrence of the new variable $\lozenge$ (the **lozenge**).

$$\begin{array}{ccccc} m(\vec{x}) & + & f(\vec{x},\Diamond) & \Longrightarrow & f(\vec{x},m(\vec{x})) \\ \textit{father} & & \textit{mother} & & \textit{child} \end{array}$$

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A Basic Bucket System

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$\mathbb{M}$		$\mathbb{F}$	
$\mathbb{M}_{n^k}$	?	?	$\mathbb{F}_{n^k}$
	$\vdots$	$\vdots$	
$\mathbb{M}_1$	$(x_4 x_1) x_3^2$	$\Diamond \quad x_2 \Diamond$	$\mathbb{F}_1$
$\mathbb{M}_0$	$x_0 x_2 \quad x_3$	$x_2(x_3 \Diamond)$	$\mathbb{F}_0$

Initialize buckets.

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$\mathbb{M}$			$\mathbb{F}$
$\mathbb{M}_{n^k}$			$\mathbb{F}_{n^k}$
	$\vdots$	$\vdots$	
$\mathbb{M}_1$			$\mathbb{F}_1$
$\mathbb{M}_0$	$x_0 x_2 \quad x_3$	$\Diamond$	$\mathbb{F}_0$

Calculate fitness values.

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$\mathbb{M}_{n^k}$			$\mathbb{F}_{n^k}$
	$\vdots$	$\vdots$	
$\mathbb{M}_1$	$x_0 x_2$	$\Diamond$	$\mathbb{F}_1$
$\mathbb{M}_0$	x_3		$\mathbb{F}_0$

Review configuration sometime later.

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$\mathbb{M}$		$\mathbb{F}$	
$\mathbb{M}_{n^k}$			$\mathbb{F}_{n^k}$
	$\vdots$	$\vdots$	
$\mathbb{M}_1$	x_0x_2	$x_2(\Diamond x_4)$	$\mathbb{F}_1$
$\mathbb{M}_0$	x_5, x_3, x_1		$\mathbb{F}_0$

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We would add $x_2((x_7\Diamond)x_4)$ to F_0 with the goal of finding $m(\vec{x})$ such that $x_2((x_7 m(\vec{x}))x_4)^{\mathbf{G}} = h$.

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Is there such a term $m(\vec{x})$? If so, then $x_2((x_7\Diamond)x_4)$ must be **valid** with respect to h in the sense that

$$(\forall \vec{a} \in G^k)(\exists b \in G) : a_2((a_7 b)a_4) = h(\vec{a}),$$

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Idemprimality Theorem. *If* $\mathbf{G}$ *is idemprimal, then there is a term* $m(\vec{x})$ *such that* $f(\vec{x}, m(\vec{x}))^{\mathbf{G}} = h$ *if and only if* $f(\vec{x}, \Diamond)$ *is valid with respect to* h*.*

If $x_2((x_7\Diamond)x_4)$ is valid, add her to $\mathbb{F}_0$.

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$$
\begin{aligned}
\Diamond \to x_0\Diamond \to x_0(\Diamond x_3) &\to x_0((\Diamond x_2^2)x_3) \\
&\to x_0((((x_4x_1)\Diamond)x_2^2)x_3) \\
&\to x_0((((x_4x_1)(\Diamond(x_5x_1^2)))x_2^2)x_3) \\
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A finite groupoid **G** is **term continuous** if

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|\{f(\vec{x}, m(\vec{x}))^{\mathbf{G}} : m(\vec{x}) \text{ a term}\}| \to 1
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Continuity Theorem. *A finite groupoid* $\mathbf{G}$ *is term continuous if it is asymptotically complete and has no subgroupoid with a separating relation.*

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Continuity Theorem. *A finite groupoid* $\mathbf{G}$ *is term continuous if it is asymptotically complete and has no subgroupoid with a separating relation.* (IJAC, Vol. 23, No. 5 (2013) 1175-1205.)

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Example. *Asked to find a discriminator term for the 5-element groupoid* **Pi***, the Bucket Algorithm produced the following output.*

Length: 793 variables ($< 10^{10}$ variables)
Time: 18 minutes ($< 10^{72}$ years)

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zxxyz*x**yy*xx**y*zx*yz**yx*y**xzx*y*z*x**yx*z**xxzx****zx*zx***z*z***y*zx*xyxy*****y**x***xx**zzzzx
*z***xx**z**zzzxz*x*xxzzz**zxy***yz*y*xx*y*****xz***z*y*y**zx*xz****yx**z**zzzzz*y*zy***zz*xz****xx*
*z**zzzxz*x*xxzzz**zxy***yz*yxz**zz*x***xx*y*****xz***z*y*y**zx*xz****xx**z**zzyy*z***z*y*y*zyx*z**x
x*zz**y*x*x**yxx*y*xz***z*zzzx**x*z*yy**xxzx****x***zzzxxy*yx*xzz*xxx**y****y****xy*x*z*xz****xx**z*
****yxxx*xx***yyz*zz****y***xz*x*xz*xy***x**zzyz**yz*x*x*y**xz**z***zyz*xxz***x***zx**zxx*xx*z*yy***
yz*y*x*y*yz*zzyx**yz*x*x*y**xz**z***xxyy****xx*z*zzzxz*xzyyy**z**z**y**yxzxxy****z***xz**z***xy*
****yx*y*yz***zzzx**y****yy*y*z*zzxxx*xy***zx****xx*yx**x*x**yx**z***xx*****xx*xz**x*yxzy*zz*yx**yy
*xz*****xy*x**zzxxz*****yyx*yz**xy*z**z****x**xyxy***z*y*xzx*y*yx*z*xyxy***z*y***z*x*zxy**x*xz*xx**
x**x****z*zy***yy*y*y*z*y**yz***yz*yxz*x**xz*x**y*z**z*y**zyxyy**xyzy***yz**xz*xyzy***yz***zyx**x*z
yz**yx*zx*****x***z*y*y****yxyzzy*zz**z**xxzx*****xy*xx**x*x****xy***zyz*x*xxyx*xx*xzzx*xy**zz**y*y*
y***z*******zyx*y*z**yxxz*zxyy***z*yzz***yxz**zx*xz****zzz***zy*z*****x*zx*z****zzz*yx*y***zx*zx**z
y*z**x*yy*zyyzyy*yy**x*******x*y**y**zzx*x**yyxz*xy***z*zyy*yy*x*x*y*xx**xxy***zz*xx**zy*x****x*xz*
zx******x***zxzz**x**x*yy*y*xxxyx****zy**yzzy**x**zyxx*xy**yyz**y****x*zyx*y*z*xy*****xxxy***xz*z*y
****yyzxy***zx*xz******xx*z*z*zz*y*yy*y*yz*x*y*y***y*zy*yzy******zxxzy***yyx***z*xy****yxz***z*z*zx*
zz*****yx*x*yy*xy*xzxy*zy***zz*z*zz*y*z*y*z*zyxy**x*zz*z*x**zyzy*****yyyz***yx*x**y***zz*xx*yzy***zx
*zz**xyx******yx*zzyx*zy**z******yxy*z******yxx******zxxxz*zzx**zx*z*zyzzy*y**xy*y*y***y*****z**z**
x*x***zx*xzxzx*zxz****z***x*xx**zz*****

Thanks for staying to the end of the conference!

— DC

References

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Definition. A binary relation σ on G is a **separating relation** if, for all $a, b, c \in G$,

1. $\sigma \neq \varnothing$,
2. $(a, b) \in \sigma$ implies $(b, a) \in \sigma$,
3. $(a, a) \notin \sigma$,
4. $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$.

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For example, if $\mathbf{G}$ is a non-trivial quasigroup, then $\neq$ is a separating relation on $\mathbf{G}$.

Lemma. *If a groupoid has a separating relation, then it is not term continuous.*

Test for Separating Relations.

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Pi $:= \langle \{0, 1, 2, 3, 4\}, *\rangle$

$*$	0	1	2	3	4
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Not in any separating relation.

$*$	0	1	2	3	4
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0	x				
1		x			
2			x		
3				x	x
4					x

Test for Separating Relations.

Pi $:= \langle \{0,1,2,3,4\}, *\rangle$

$*$	0	1	2	3	4
0	1	4	1	0	4
1	2	1	0	3	0
2	3	4	2	4	3
3	2	3	3	4	1
4	2	1	4	3	3

Not in any separating relation.

$*$	0	1	2	3	4
0	x				
1		x			
2			x		
3				x	x
4				x	x

Test for Separating Relations.

Pi $:= \langle \{0,1,2,3,4\}, *\rangle$

$*$	0	1	2	3	4
0	1	4	1	0	4
1	2	1	0	3	0
2	3	4	2	4	3
3	2	3	3	4	1
4	2	1	4	3	3

Not in any separating relation.

$*$	0	1	2	3	4
0	x		x		x
1		x		x	x
2	x		x	x	x
3		x	x	x	x
4	x	x	x	x	x

Test for Separating Relations.

Pi $:= \langle \{0,1,2,3,4\}, *\rangle$

$*$	0	1	2	3	4
0	1	4	1	0	4
1	2	1	0	3	0
2	3	4	2	4	3
3	2	3	3	4	1
4	2	1	4	3	3

Not in any separating relation.

$*$	0	1	2	3	4
0	x	x	x	x	x
1	x	x	x	x	x
2	x	x	x	x	x
3	x	x	x	x	x
4	x	x	x	x	x

Test for Separating Relations.

Pi $:= \langle\{0,1,2,3,4\},*\rangle$

$*$	0	1	2	3	4
0	1	4	1	0	4
1	2	1	0	3	0
2	3	4	2	4	3
3	2	3	3	4	1
4	2	1	4	3	3

Not in any separating relation.

$*$	0	1	2	3	4
0	x	x	x	x	x
1	x	x	x	x	x
2	x	x	x	x	x
3	x	x	x	x	x
4	x	x	x	x	x

Conclusion:
Pi has no subalgebra with a separating relation.

Let $\vec{d} \in G^k$ and let $a \in G$. For $H \in \mathbb{N}$ define

$$\beta_{\vec{d},a}(H) := \mathrm{Prob}\langle t(\vec{d}) = a \mid t \text{ is a term of height } \leq H\rangle.$$

Let $\vec{d} \in G^k$ and let $a \in G$. For $H \in \mathbb{N}$ define

$$\beta_{\vec{d},a}(H) := \text{Prob}\langle t(\vec{d}) = a \mid t \text{ is a term of height } \leq H\rangle.$$

Definition. The groupoid $\mathbf{G}$ is **asymptoticly complete** if, for each $k \in \mathbb{N}$, each $\vec{d} \in \mathbf{G}^k$ and each $a \in G$, the sequence $\beta_{\vec{d},a}$ either has constant value 0 or is eventually bounded away from 0.

Test for Asymptotic Completeness.

Test for Asymptotic Completeness.

Pi distributions with $k \in \mathbb{N}$ and $\vec{d} \in Pi^k$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1					
2					
4					
8					
6					
32					
64					
500					

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$.

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0	0	1	0	0
2	0	0	1	0	0
4	0	0	1	0	0
6	0	0	1	0	0
8	0	0	1	0	0
10	0	0	1	0	0
16	0	0	1	0	0
500	0	0	1	0	0

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.2	0.1	0.1	0.5	0.1

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.2	0.1	0.1	0.5	0.1
2	0.127273	0.127273	0.145455	0.254545	0.345455

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.2	0.1	0.1	0.5	0.1
2	0.127273	0.127273	0.145455	0.254545	0.345455
4	0.098817	0.182526	0.088076	0.368477	0.262102

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.2	0.1	0.1	0.5	0.1
2	0.127273	0.127273	0.145455	0.254545	0.345455
4	0.098817	0.182526	0.088076	0.368477	0.262102
8	0.103371	0.195561	0.091358	0.365539	0.244171

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.2	0.1	0.1	0.5	0.1
2	0.127273	0.127273	0.145455	0.254545	0.345455
4	0.098817	0.182526	0.088076	0.368477	0.262102
8	0.103371	0.195561	0.091358	0.365539	0.244171
16	0.103372	0.195275	0.091609	0.365434	0.244311

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (3,4,3,0,2,3,3,0,1,3)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.2	0.1	0.1	0.5	0.1
2	0.127273	0.127273	0.145455	0.254545	0.345455
4	0.098817	0.182526	0.088076	0.368477	0.262102
8	0.103371	0.195561	0.091358	0.365539	0.244171
16	0.103372	0.195275	0.091609	0.365434	0.244311
32	0.103372	0.195275	0.091609	0.365434	0.244310

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.2	0.1	0.1	0.5	0.1
2	0.127273	0.127273	0.145455	0.254545	0.345455
4	0.098817	0.182526	0.088076	0.368477	0.262102
8	0.103371	0.195561	0.091358	0.365539	0.244171
16	0.103372	0.195275	0.091609	0.365434	0.244311
32	0.103372	0.195275	0.091609	0.365434	0.244310
64	0.103372	0.195275	0.091609	0.365434	0.244310

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.2	0.1	0.1	0.5	0.1
2	0.127273	0.127273	0.145455	0.254545	0.345455
4	0.098817	0.182526	0.088076	0.368477	0.262102
8	0.103371	0.195561	0.091358	0.365539	0.244171
16	0.103372	0.195275	0.091609	0.365434	0.244311
32	0.103372	0.195275	0.091609	0.365434	0.244310
64	0.103372	0.195275	0.091609	0.365434	0.244310
500	0.103372	0.195275	0.091609	0.365434	0.244310

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0	0	1	0	0
2	0	0	1	0	0
4	0	0	1	0	0
6	0	0	1	0	0
8	0	0	1	0	0
10	0	0	1	0	0
16	0	0	1	0	0
500	0	0	1	0	0

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (2, 2, 2, 2, 2, 2, 4, 2, 2, 2)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (2, 2, 2, 2, 2, 2, 4, 2, 2, 2)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.0	0.0	0.9	0.0	0.1

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (2, 2, 2, 2, 2, 2, 4, 2, 2, 2)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.0	0.0	0.9	0.0	0.1
2	0.000000	0.000000	0.810001	0.099999	0.090000

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (2, 2, 2, 2, 2, 2, 4, 2, 2, 2)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.0	0.0	0.9	0.0	0.1
2	0.000000	0.000000	0.810001	0.099999	0.090000
4	0.007380	0.029583	0.430369	0.283600	0.248968

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (2, 2, 2, 2, 2, 2, 4, 2, 2, 2)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.0	0.0	0.9	0.0	0.1
2	0.000000	0.000000	0.810001	0.099999	0.090000
4	0.007380	0.029583	0.430369	0.283600	0.248968
8	0.095683	0.211801	0.070920	0.382629	0.238968

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (2, 2, 2, 2, 2, 2, 4, 2, 2, 2)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.0	0.0	0.9	0.0	0.1
2	0.000000	0.000000	0.810001	0.099999	0.090000
4	0.007380	0.029583	0.430369	0.283600	0.248968
8	0.095683	0.211801	0.070920	0.382629	0.238968
16	0.103370	0.195268	0.091610	0.365436	0.244316

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (2, 2, 2, 2, 2, 2, 4, 2, 2, 2)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.0	0.0	0.9	0.0	0.1
2	0.000000	0.000000	0.810001	0.099999	0.090000
4	0.007380	0.029583	0.430369	0.283600	0.248968
8	0.095683	0.211801	0.070920	0.382629	0.238968
16	0.103370	0.195268	0.091610	0.365436	0.244316
32	0.103372	0.195275	0.091609	0.365434	0.244310

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (2, 2, 2, 2, 2, 2, 4, 2, 2, 2)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.0	0.0	0.9	0.0	0.1
2	0.000000	0.000000	0.810001	0.099999	0.090000
4	0.007380	0.029583	0.430369	0.283600	0.248968
8	0.095683	0.211801	0.070920	0.382629	0.238968
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32	0.103372	0.195275	0.091609	0.365434	0.244310
64	0.103372	0.195275	0.091609	0.365434	0.244310

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (2,2,2,2,2,2,4,2,2,2)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.0	0.0	0.9	0.0	0.1
2	0.000000	0.000000	0.810001	0.099999	0.090000
4	0.007380	0.029583	0.430369	0.283600	0.248968
8	0.095683	0.211801	0.070920	0.382629	0.238968
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32	0.103372	0.195275	0.091609	0.365434	0.244310
64	0.103372	0.195275	0.091609	0.365434	0.244310
500	0.103372	0.195275	0.091609	0.365434	0.244310

Test for Asymptotic Completeness.

Pi distributions with $k = 10$ and $\vec{d} = (2,2,2,2,2,2,4,2,2,2)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.0	0.0	0.9	0.0	0.1
2	0.000000	0.000000	0.810001	0.099999	0.090000
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32	0.103372	0.195275	0.091609	0.365434	0.244310
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500	0.103372	0.195275	0.091609	0.365434	0.244310

Experimental evidence that **Pi** is asymptoticly complete.