D.Clark et a

Sample Problem

Prior Method

Evolutionary Computation

The Bucket Algorithm

Idemprimali

Continui

Example

Reference

Reference

Separating Relations

Algebraic Terms through Simulated Biological Evolution

David M. Clark¹ Maarten Keijzer² Lee Spector³

¹Mathematics: SUNY, New Paltz, NY, USA
 ²Machine Learning: Chordiant, Inc., Amsterdam, NDL
 ³Cognitive Science: Hampshire College, MA, USA

October 6, 2013

(clarkd@newpaltz.edu)

A finite non-trivial groupoid **G** is **idemprimal** if every operation $h: G^k \to G$ that preserves the idempotents of **G** is a term operation.

D. Clark Ct (

Sample Problem

Prior Method

Evolutionary Computation

The Bucket Algorithm

Validity and Idemprimali

Continuit

Exampl

Thanks!

References

Separating Relations

Examp

Thank

Reference

Separating Relations A finite non-trivial groupoid \mathbf{G} is **idemprimal** if every operation $h: G^k \to G$ that preserves the idempotents of \mathbf{G} is a term operation.



A finite non-trivial groupoid \mathbf{G} is **idemprimal** if it has no non-trivial subalgebras, automorphisms or congruences and the ternary discriminator operation

$$t(a,b,c) := \begin{cases} a \text{ if } a = b \\ c \text{ if } a \neq b \end{cases}$$

is a term operation of \mathbf{G} .

Prior Method

Evolutionary Computation

The Bucker Algorithm

Validity and Idemprimalit

Continuit

Example

Thanks!

Reference

Separating Relations **Murskii's Theorem.** (1975) The proportion of n-element groupoids $\mathbf{G} = \langle \{0,1,2,\ldots,n-1\},* \rangle$ that are idemprimal approaches 1 as $n \to \infty$.

Separating

Murskii's Theorem. (1975) The proportion of n-element groupoids $\mathbf{G} = \langle \{0, 1, 2, \dots, n-1\}, * \rangle$ that are idemprimal approaches 1 as $n \to \infty$.

Problem:

Is this groupoid idemprimal?

$$Pi := \langle \{0, 1, 2, 3, 4\}, * \rangle$$

*	0	1	2	3	4
0	1	4	1	0	4
1	2	1	0	3	0
2	3	4	2	4	3
3	2	3	3	4	1
4	2	1	4	0 3 4 4 3	3

Sample Problem

Prior Method

Evolutionary Computation

Validity and

Continuit

Exampl

Reference

Reference

Separating Relations **Murskii's Theorem.** (1975) The proportion of n-element groupoids $\mathbf{G} = \langle \{0, 1, 2, \dots, n-1\}, * \rangle$ that are idemprimal approaches 1 as $n \to \infty$.

Problem:

Is this groupoid idemprimal?

$$\textbf{Pi} := \langle \{0,1,2,3,4\}, * \rangle$$

if and only if it has a discriminator term

*	0	1	2	3	4
0	1	4	1	0	4
1	2	1	0	3	0
2	3	4	2	4	3
3	2	3	3	4	1
4	2	4 1 4 3 1	4	3	3

Separating

Murskii's Theorem. (1975) The proportion of n-element groupoids $\mathbf{G} = \langle \{0, 1, 2, \dots, n-1\}, * \rangle$ that are idemprimal approaches 1 as $n \to \infty$.

Problem:

Is this groupoid idemprimal?

$$\textbf{Pi} := \langle \{0,1,2,3,4\}, * \rangle$$

if and only if it has a discriminator term

*	0	1	2	3	4
0	1	4 1 4 3 1	1	0	4
1	2	1	0	3	0
2	3	4	2	4	3
3	2	3	3	4	1
4	2	1	4	3	3

Term Generation Problem: Given a finite groupoid **G** and an operation $h: G^k \to G$ that is a term operation of **G**, find a term that represents h.

Prior Methods

Evolutionary

The Bucket Algorithm

Validity and

Continuit

Example

Thanks!

Reference

Separating Relations Let $\mathbf{G} = \langle \{0,1,2,3,4\}, * \rangle$ be a 5-element idemprimal groupoid. Two prior methods might generate a discriminator term.

Evolutionary Computation

The Bucket Algorithm

Idemprimali

Continuit

Zxampie

Reference

Separating Relations Let $\mathbf{G} = \langle \{0,1,2,3,4\}, * \rangle$ be a 5-element idemprimal groupoid. Two prior methods might generate a discriminator term.

■ UACalc will tell us if **G** is primal and, if so, will quickly produce a set of small terms from which we can recursively generate a discriminator term.

D.Clark et a

Sample Problem

Prior Methods

Evolutionary Computation

The Bucket Algorithm

Idemprimal

_ .

Reference

Separating Relations Let $\mathbf{G} = \langle \{0,1,2,3,4\}, * \rangle$ be a 5-element idemprimal groupoid. Two prior methods might generate a discriminator term.

I UACalc will tell us if **G** is primal and, if so, will quickly produce a set of small terms from which we can recursively generate a discriminator term. BUT, the generated term will have $\approx 10^{10}$ variables(!)

Evolutionary Computation

The Bucket Algorithm

ldemprimality

Examp

Thombs

Reference

Separating Relations Let $\mathbf{G} = \langle \{0, 1, 2, 3, 4\}, * \rangle$ be a 5-element idemprimal groupoid. Two prior methods might generate a discriminator term.

- I UACalc will tell us if ${\bf G}$ is primal and, if so, will quickly produce a set of small terms from which we can recursively generate a discriminator term. BUT, the generated term will have $\approx 10^{10}$ variables(!)
- 2 UACalc will construct the free algebra $\mathbf{F}_3(\mathbf{G})$ and find a near minimal length term for the discriminator.

Evolutionary Computation

The Bucket Algorithm

Continuity

Continuity

Thank

Reference

Separating Relations Let $\mathbf{G} = \langle \{0, 1, 2, 3, 4\}, * \rangle$ be a 5-element idemprimal groupoid. Two prior methods might generate a discriminator term.

- I UACalc will tell us if $\bf G$ is primal and, if so, will quickly produce a set of small terms from which we can recursively generate a discriminator term. BUT, the generated term will have $\approx 10^{10}$ variables(!)
- 2 UACalc will construct the free algebra $\mathbf{F}_3(\mathbf{G})$ and find a near minimal length term for the discriminator. BUT, if \mathbf{G} has 2 idempotents, $\mathbf{F}_3(\mathbf{G})$ will have 5^{123} elements.

Sample Problem

Prior Methods

Evolutionary Computation

The Bucket Algorithm

Continuity

Continuity

Thanks

Reference

Reference

Separating Relations Let $\mathbf{G} = \langle \{0, 1, 2, 3, 4\}, * \rangle$ be a 5-element idemprimal groupoid. Two prior methods might generate a discriminator term.

- I UACalc will tell us if ${\bf G}$ is primal and, if so, will quickly produce a set of small terms from which we can recursively generate a discriminator term. BUT, the generated term will have $\approx 10^{10}$ variables(!)
- 2 UACalc will construct the free algebra $\mathbf{F}_3(\mathbf{G})$ and find a near minimal length term for the discriminator. BUT, if \mathbf{G} has 2 idempotents, $\mathbf{F}_3(\mathbf{G})$ will have 5^{123} elements. Checking 10^6 terms per second, the expected time to find a discriminator term will be $\approx 10^{72}$ years(!!)

O.Clark et a

Problem

Prior Method

Evolutionary Computation

The Bucket Algorithm

Validity and

Continuit

Example

Thanks!

Deference

Separating

Evolutionary computation finds solutions to problems by simulating biological evolution.



Idemprima

Zxampi

Reference

Separating Relations **Evolutionary computation** finds solutions to problems by simulating biological evolution.

The **fitness** of a term $t(\vec{x})$ to represent $h: G^k \to G$ is

$$\mathsf{fit}\,(t(\vec{x})) := |\{\vec{b} \in G^k : t(\vec{b}) = h(\vec{b})\}|.$$

Idemprimalit

Examp

Thanks!

Reference

Separating Relations **Evolutionary computation** finds solutions to problems by simulating biological evolution.

The **fitness** of a term $t(\vec{x})$ to represent $h: G^k \to G$ is

$$fit(t(\vec{x})) := |\{\vec{b} \in G^k : t(\vec{b}) = h(\vec{b})\}|.$$

The Bucket Algorithm. Evolves two finite, dynamically changing sets of terms:

Reference

Separating Relations **Evolutionary computation** finds solutions to problems by simulating biological evolution.

The **fitness** of a term $t(\vec{x})$ to represent $h: G^k \to G$ is

$$fit(t(\vec{x})) := |\{\vec{b} \in G^k : t(\vec{b}) = h(\vec{b})\}|.$$

The Bucket Algorithm. Evolves two finite, dynamically changing sets of terms:

 \mathbb{M} , a set of **male terms** $m(\vec{x})$;

Validity and Idemprimality

Examp

THATIKS

Reference

Separating Relations **Evolutionary computation** finds solutions to problems by simulating biological evolution.

The **fitness** of a term $t(\vec{x})$ to represent $h: G^k \to G$ is

$$fit (t(\vec{x})) := |\{\vec{b} \in G^k : t(\vec{b}) = h(\vec{b})\}|.$$

The Bucket Algorithm. Evolves two finite, dynamically changing sets of terms:

 \mathbb{M} , a set of **male terms** $m(\vec{x})$;

 \mathbb{F} , a set of **female terms** $f(\vec{x}, \lozenge)$, each containing a single occurrence of the new variable \lozenge (the **lozenge**).

Sample Problem

Prior Methods

Evolutionary Computation

The Bucke Algorithm

Validity and

Continuity

Example

Thanks!

Deference

Separating Relations

$$m(\vec{x}) + f(\vec{x}, \lozenge) \implies f(\vec{x}, m(\vec{x}))$$

father mother child

D.Clark et a

Sample Problem

Prior Methods

Evolutionary Computation

The Bucket Algorithm

Validity and Idemprimality

Continuity

Example

Thanks

Reference

Separating Relations

$$m(\vec{x}) + f(\vec{x}, \diamondsuit) \implies f(\vec{x}, m(\vec{x}))$$

father mother child

The **fitness** of a male term $m(\vec{x})$ is

$$\mathsf{Fit}\,(\mathit{m}(\vec{x})) := \mathit{Max}\{\mathsf{fit}\,(\mathit{f}(\vec{x},\mathit{m}(\vec{x})) \mid \mathit{f}(\vec{x}) \in \mathbb{F}\}.$$

D.Clark et a

Sample Problem

Prior Method

Evolutionary Computation

The Bucket Algorithm

Idemprima

Continuity

Example

Thanks

References

Separating Relations

$$m(\vec{x}) + f(\vec{x}, \diamondsuit) \implies f(\vec{x}, m(\vec{x}))$$

father mother child

The **fitness** of a male term $m(\vec{x})$ is

$$\mathsf{Fit}\,(\mathit{m}(\vec{x})) := \mathit{Max}\{\mathsf{fit}\,(\mathit{f}(\vec{x},\mathit{m}(\vec{x})) \mid \mathit{f}(\vec{x}) \in \mathbb{F}\}.$$

The **fitness** of a female term $f(\vec{x}, \lozenge)$ is

$$\mathsf{Fit}\,(f(\vec{x},\Diamond)) := \mathit{Max}\{\mathsf{fit}\,(f(\vec{x},\mathit{m}(\vec{x})) \mid \mathit{m}(\vec{x}) \in \mathbb{M}\}.$$

D.Clark et al

Sample Problem

Prior Methods

Evolutionary Computation

The Bucket Algorithm

Validity and

Continuity

Example

Thanks!

Deference

Separating Relations

A Basic Bucket System

D.Clark et a

Sample Problem

Prior Methods

Evolutionary

The Bucket Algorithm

Validity and

Thank

Reference

Separating

A Basic Bucket System

M	F
\mathbb{M}_{n^k} ?	? \[\mathbb{F}_{n^k}
:	<u>:</u>
$\mathbb{M}_1 \left[(x_4x_1)x_3^2 \right]$	\Diamond $x_2\Diamond$ \mathbb{F}_1
M_0 x_0x_2 x_3	$x_2(x_3\lozenge)$ \mathbb{F}_0

D.Clark et al

Sample Problem

Prior Methods

Evolutionary

The Bucket Algorithm

Validity and

Continuit

Example

T1. . . . 1 . . 1

D (

Separating Relations

Initialize buckets.

.Clark et al

Sample Problem

Prior Methods

Evolutionary

The Bucket Algorithm

Algorithm

C----i---i--

Evample

-. .

Reference

Separating Relations

Initialize buckets.

M	F
\mathbb{M}_{n^k}	$oxed{\mathbb{F}_{n^k}}$
:	:
M_1	$oxed{\mathbb{F}_1}$
\mathbb{M}_0 x_0x_2 x_3	$igspace$ $igspace{\mathbb{F}_0}$

D.Clark et al

Sample Problem

Prior Methods

Evolutionary Computation

The Bucket Algorithm

Validity and

Continuity

Example

Thanks

Deference

Separating Relations

Calculate fitness values.

.Clark et al

Sample Problem

Prior Methods

Evolutionary Computation

The Bucket Algorithm

Idemprima

Continuity

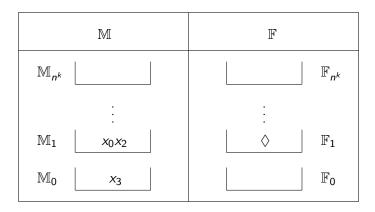
Lxampi

Inanks

Reference

Separating Relations

Calculate fitness values.



D.Clark et a

Sample

Prior Methods

Evolutionary Computation

The Bucket Algorithm

Validity and

ontinuity

Example

Thanks!

Deference

Separating Relations Review configuration sometime later.

D.Clark et a

Sample Problem

Prior Methods

Evolutionary

The Bucket Algorithm

Idemprima

Continuity

Examp

Thank

Reference

Separating Relations

Review configuration sometime later.

M	F
\mathbb{M}_{n^k}	$oxed{\mathbb{F}_{n^k}}$
<u>:</u>	<u>:</u>
\mathbb{M}_1 x_0x_2	$x_2(\lozenge x_4)$ \mathbb{F}_1
$M_0 x_5, x_3, x_1$	$oxed{\mathbb{F}_0}$

D.Clark et a

Sample Problem

Prior Methods

E. d. C.

Computation

Validity and Idemprimality

ontinuity

Example

Thanks!

References

Separating Relations Construct a new female term $x_2((x_7 \lozenge) x_4)$ from x_7 and $x_2(\lozenge x_4)$.

Choose a new term, say x_7 , and a female term, say $x_2(\lozenge x_4)$.

Validity and Idemprimality

Sample Problem

Prior Methods

Evolutionary Computation

The Bucket Algorithm

Validity and Idemprimality

Continuit

Exampl

i nanks:

Reference

Separating Relations Choose a new term, say x_7 , and a female term, say $x_2(\lozenge x_4)$. Construct a new female term $x_2((x_7 \lozenge) x_4)$ from x_7 and $x_2(\lozenge x_4)$.

We would add $x_2((x_7 \diamondsuit) x_4)$ to F_0 with the goal of finding $m(\vec{x})$ such that $x_2((x_7 m(\vec{x})) x_4)^{\mathbf{G}} = h$.

Prior Methods

Evolutionary Computation

The Bucket Algorithm

Validity and Idemprimality

Continuit

Exampl

Thanks

Reference

Separating Relations Choose a new term, say x_7 , and a female term, say $x_2(\lozenge x_4)$.

Construct a new female term $x_2((x_7 \lozenge) x_4)$ from x_7 and $x_2(\lozenge x_4)$.

We would add $x_2((x_7 \lozenge) x_4)$ to F_0 with the goal of finding $m(\vec{x})$ such that $x_2((x_7 m(\vec{x})) x_4)^G = h$.

Is there such a term $m(\vec{x})$?

Reference

Separating Relations Choose a new term, say x_7 , and a female term, say $x_2(\lozenge x_4)$.

Construct a new female term $x_2((x_7 \lozenge) x_4)$ from x_7 and $x_2(\lozenge x_4)$.

We would add $x_2((x_7 \diamondsuit) x_4)$ to F_0 with the goal of finding $m(\vec{x})$ such that $x_2((x_7 m(\vec{x})) x_4)^G = h$.

Is there such a term $m(\vec{x})$? If so, then $x_2((x_7 \lozenge)x_4)$ must be **valid** with respect to h in the sense that

$$(\forall \vec{a} \in G^k)(\exists b \in G) : a_2((a_7b)a_4) = h(\vec{a}),$$

namely, $b = m(\vec{a})$.

Reference

Separating Relations Choose a new term, say x_7 , and a female term, say $x_2(\Diamond x_4)$.

Construct a new female term $x_2((x_7 \lozenge) x_4)$ from x_7 and $x_2(\lozenge x_4)$.

We would add $x_2((x_7 \lozenge) x_4)$ to F_0 with the goal of finding $m(\vec{x})$ such that $x_2((x_7 m(\vec{x})) x_4)^{\mathbf{G}} = h$.

Is there such a term $m(\vec{x})$? If so, then $x_2((x_7 \lozenge)x_4)$ must be **valid** with respect to h in the sense that

$$(\forall \vec{a} \in G^k)(\exists b \in G) : a_2((a_7b)a_4) = h(\vec{a}),$$

namely, $b = m(\vec{a})$.

Idemprimality Theorem. If **G** is idemprimal, then there is a term $m(\vec{x})$ such that $f(\vec{x}, m(\vec{x}))^{\mathbf{G}} = h$ if and only if $f(\vec{x}, \diamondsuit)$ is valid with respect to h.

D.Clark et al

Sample Problem

Prior Methods

Evolutionary

The Bucket

Validity and Idemprimality

Continuity

Example

Thanks!

Deference

Separating Relations If $x_2((x_7 \lozenge) x_4)$ is valid, add her to \mathbb{F}_0 .

D.Clark et a

Sample

Prior Methods

Evolutionary

The Bucket

Validity and Idemprimality

Continuity

- .

Thank

Reference

Separating Relations

If $x_2((x_7 \lozenge) x_4)$ is valid, add her to \mathbb{F}_0 .

M	F
\mathbb{M}_{n^k}	$oxed{\mathbb{F}_{n^k}}$
<u>:</u>	<u>:</u>
\mathbb{M}_1 x_0x_2	$x_2(\lozenge x_4)$ \mathbb{F}_1
\mathbb{M}_0 x_5, x_3, x_1	$[x_2((x_1\lozenge)x_4)]$ \mathbb{F}_0

Sample Problem

Prior Method

Evolutionary

The Bucket

Validity and Idemprimalit

Continuity

Example

Thanks

Deference

Separating Relations

D.Clark et a

Sample Problem

Prior Methods

Evolutionary Computation

The Bucket Algorithm

Validity and Idemprimalit

Continuity

Example

Thanks

Reference

Separating Relations

A finite groupoid **G** is **term continuous** if

$$|\{f(\vec{x}, m(\vec{x}))^{\mathbf{G}} : m(\vec{x}) \text{ a term}\}| \to 1$$
 as the depth of \lozenge in $f(\vec{x}, \lozenge) \to \infty$.

D.Clark et a

Sample Problen

Prior Methods

Evolutionary Computation

The Bucket Algorithm

Validity an Idemprima

Continuity

Exampl

Reference

Separating Relations

A finite groupoid **G** is **term continuous** if

$$|\{f(\vec{x},m(\vec{x}))^{\mathbf{G}}:m(\vec{x}) \text{ a term}\}| \to 1$$
 as the depth of \Diamond in $f(\vec{x},\Diamond) \to \infty$.

Continuity Theorem. A finite groupoid **G** is term continuous if it is asymptotically complete and has no subgroupoid with a separating relation.

D.Clark et a

Sample Problen

Prior Methods

Evolutionary Computation

The Bucket Algorithm

Validity and

Continuity

Example

Reference

Separating Relations

$$\Diamond \to x_0 \Diamond \to x_0 (\Diamond x_3) \to x_0 ((\Diamond x_2^2) x_3)
\to x_0 ((((x_4 x_1) \Diamond) x_2^2) x_3)
\to x_0 ((((x_4 x_1) (\Diamond (x_5 x_1^2))) x_2^2) x_3)
\to x_0 ((((x_4 x_1) ((\Diamond (x_2^2 x_0^2)) (x_5 x_1^2))) x_2^2) x_3) \to \dots$$

A finite groupoid **G** is **term continuous** if

$$|\{f(\vec{x},m(\vec{x}))^{\mathbf{G}}:m(\vec{x}) \text{ a term}\}| \to 1$$
 as the depth of \Diamond in $f(\vec{x},\Diamond) \to \infty$.

Continuity Theorem. A finite groupoid **G** is term continuous if it is asymptotically complete and has no subgroupoid with a separating relation. (IJAC, Vol. 23, No. 5 (2013) 1175-1205.)

Conjecture 1. Almost every finite groupoid is asymptotically complete.

Problem Problem

Prior Methods

Evolutionary

The Bucket

Validity and Idemprimality

Continuity

Example

Thanks!

Reference

Separating

D.Clark et a

Sample Problem

Prior Methods

Evolutionary Computation

The Bucket Algorithm

Idemprimal

Continuity

Example

Thanks!

Reference

Separating

Conjecture 1. Almost every finite groupoid is asymptotically complete.

Conjecture 2. Almost every finite groupoid has no subgroupoid with a separating relation.

Sample Problem

Prior Methods

Evolutionary Computation

Algorithm

Validity and

Idemprimality

Continuity

Example

Thanks!

Reference

Separating Relations **Conjecture 1.** Almost every finite groupoid is asymptotically complete.

Conjecture 2. Almost every finite groupoid has no subgroupoid with a separating relation.

Example. Asked to find a discriminator term for the 5-element groupoid **Pi**, the Bucket Algorithm produced the following output.

D.Clark et a

Sample Problem

Prior Method

Evolutionary Computation

Algorithm

Validity and

Idemprimality Continuity

Examp

Thanks

Reference

C ------

Separating Relations **Conjecture 1.** Almost every finite groupoid is asymptotically complete.

Conjecture 2. Almost every finite groupoid has no subgroupoid with a separating relation.

Example. Asked to find a discriminator term for the 5-element groupoid **Pi**, the Bucket Algorithm produced the following output.

Length:

Continuity

Conjecture 1. Almost every finite groupoid is asymptotically complete.

Conjecture 2. Almost every finite groupoid has no subgroupoid with a separating relation.

Example. Asked to find a discriminator term for the 5-element groupoid Pi, the Bucket Algorithm produced the following output.

Length: 793 variables ($< 10^{10}$ variables)

Evolutionary Computation

Algorithm

Validity and

Continuity

Example

Thanks

Reference

Separating

Conjecture 1. Almost every finite groupoid is asymptotically complete.

Conjecture 2. Almost every finite groupoid has no subgroupoid with a separating relation.

Example. Asked to find a discriminator term for the 5-element groupoid **Pi**, the Bucket Algorithm produced the following output.

Length: 793 variables ($< 10^{10}$ variables)

Time:

D.Clark et a

Sample Problem

Prior Methods

Evolutionary Computation

Algorithm

Validity and

Continuity

Example

Thanks!

Reference

Separating Relations **Conjecture 1.** Almost every finite groupoid is asymptotically complete.

Conjecture 2. Almost every finite groupoid has no subgroupoid with a separating relation.

Example. Asked to find a discriminator term for the 5-element groupoid **Pi**, the Bucket Algorithm produced the following output.

Length: 793 variables ($< 10^{10}$ variables)

Time: 18 minutes ($< 10^{72}$ years)

Algebraic

Terms through t(x,y,z) =Simulated Biological

Evolution

Example

D.Clark et a

Sample Problem

Prior Method

Evolutionary Computation

The Bucket Algorithm

Continuity

Example

T1.....1

Reference

Separating Relations t(x,y,z) =

zxxyz*x**yy*xx**y*zx*yz**yx*y**xzx*y*z*x*yx*z**xxzx****zx*zx***z*z**xyz**xyxy*****y*xx*xxxx**zzzzx *z***xx**z**zzzxz*x*xxzzz**zxy***yz*y*xx*y****xz***z*y*y**zx*xz***yx**z**zzzzz*y*zy***zz*xz****xx *z**zzzxz*x*xxzzz**zxy***yz*yxz**zz*x***xx*y****xz***z*y*y**zx*xz****z**z*yx*z**z*yz*z**z x*zz**y*x*x**yxx*y*xz***z*zzzx**x*z*yy**xxzx****x**zzzzxxy*yx*xzz*xxx**y****y****xy*x*z*xz****xx**z* **yz*y*x*y*yz***zzyx**yz*x*x*y**xz**z***xxyy****xx*z*zzzzz*xzyyy**z**z**y**yxzxxy****z***xz**z**xy* *xz*****xy*x**zzxxz*****yvx*yz**xy*z**z***x**xyxy***z*y*xzx*y*yx*z*xyxy***z*y**x*z*x**x*xx** x**x***z*zv***yv*v*z*v**yz***yz*x*x*xz*x**y*z**z*y**zyxyy**xyzy***yz**xz*xz*xyzy***zyx yz**yx*zx****x**z*y*y****yxyzzy*zz**z**xxzx*****xy*xx**x*x**x**xy***zyz*x*xxyx*xx*xzzx*xy**zzz**y*y* y***z********yx*y*z**yxxz*zxyy***z*yzz***yxz**zx*xz***zzz***zy*z****x*zx*zx*zx*zx*zx*yx*y***zx*zx y*z**x*yy*zyyzyy*yy*x*******x*y**y**zzx*x***yyxz*xy***z*zyy*yy*x*x*y*xx**xxy***zz*xx**zy*x*x*xx zx*****x*x*zxzz**x*x*yy*y*xxxyx****zy**yzzy**x**zyxx*xy**yyz**y*x*xzyx*y*z*xy****xxxy***xxxy** ****yyzxy***zx*xz******xx*z*z*z*y*yy*y*yz*x*y*y***y*zy*yzy*****zxxzy***yyx***z*xy***yxz***z*zz* x*x***zx*xzxx*zxz****z***x*xx**zz****

D.Clark et a

Sample Problem

Prior Method

Evolutionary Computation

The Bucket Algorithm

Validity and Idemprimalit

Continuit

Exampl

Thanks!

Reference

Separating Relations

Thanks for staying to the end of the conference!

— DC

D.Clark et a

Sample Problem

Prior Method

Evolutionary Computation

The Bucket Algorithm

Continuity

Continuity

Thank

References

Separating Relations

References

- D. Clark, Evolution of algebraic terms 1: term to term operation continuity, *International Journal of Algebra and Computation*, Vol. 23, No. 5 (2013) 1175-1205.
- D. Clark, B. Davey, J.Pitkethly, D.Rifqui, Flat unars: the primal, the semi-primal and the dualisable, *Algebra Universalis* Vol 63, No. 4 (2010), 303-329.
- 3 D. Clark, M. Keijzer and L. Spector, Evolution of algebraic terms 2: evolutionary algorithms, in preparation.
- 4 L. Spector, D. Clark, B. Barr, J. Klein and I. Lindsay, Genetic programming for finite algebras, *Genetic and Evolutionary Computation Conference* (GECCO) 2008

 Proceedings, Atlanta GA (July 2008), (ACM), ISBN: 978-1-60558-130-9, 1291-1298. [Won first place in 2008 ACM Hummie Competition.]

Clark et al

Problem

Prior Methods

Evolutionary

Algorithm
Validity and

Continuity

Examp

Thanks

References

Separating Relations D.Clark et a

Sample Problem

Prior Methods

Evolutionary Computation

The Bucket Algorithm

Idemprimality

Evama

Thankel

Reference

Separating Relations **Definition.** A binary relation σ on G is a **separating relation** if, for all $a, b, c \in G$,

- $\mathbf{1} \ \sigma \neq \emptyset$,
- $(a,b) \in \sigma \text{ implies } (b,a) \in \sigma$,
- $(a,a) \notin \sigma$,
- 4 $(a,b) \in \sigma$ implies $(ac,bc) \in \sigma$ and $(ca,cb) \in \sigma$.

Reference

Separating Relations **Definition.** A binary relation σ on G is a **separating relation** if, for all $a, b, c \in G$,

- $\mathbf{1} \ \sigma \neq \emptyset$,
- $(a,b) \in \sigma$ implies $(b,a) \in \sigma$,
- $(a,a) \notin \sigma$,
- **4** $(a,b) \in \sigma$ implies $(ac,bc) \in \sigma$ and $(ca,cb) \in \sigma$.

For example, if ${\bf G}$ is a non-trivial quasigroup, then \neq is a separating relation on ${\bf G}$.

Separating Relations **Definition.** A binary relation σ on G is a **separating relation** if, for all $a, b, c \in G$,

- $1 \sigma \neq \emptyset$,
- $(a,b) \in \sigma$ implies $(b,a) \in \sigma$,
- $(a,a) \notin \sigma$,
- 4 $(a,b) \in \sigma$ implies $(ac,bc) \in \sigma$ and $(ca,cb) \in \sigma$.

For example, if $\bf G$ is a non-trivial quasigroup, then \neq is a separating relation on $\bf G$.

Lemma. If a groupoid has a separating relation, then it is not term continuous.

Test for Separating Relations.

Separating Relations

Referenc

Separating Relations

Test for Separating Relations.

 $\textbf{Pi}:=\langle\{0,1,2,3,4\},*\rangle$

*	0	1	2	3	4
0	1	4	1	0	4
1	2	1	0	3	0
2	3	4	2	4	3
3	2	3	3	4	1
4	1 2 3 2 2	1	4	3	3

*	0	1	2	3	4
0	Х				
1		Х			
2			Х		
2				Х	
4					Х

Continuity

Exampl

D . C

Reference

Separating Relations

Test for Separating Relations.

 $\textbf{Pi} := \langle \{0,1,2,3,4\}, * \rangle$

*	0	1	2	3	4
0	1	4	1	0	4
1	2	1	0	3	0
2	3	4	2	4	3
3	2	3	3	4	1
4	2	1	4	0 3 4 4 3	3

*	0	1	2	3	4
0	Х				
1		Х			
2			Х		
3				Х	
4					Х

Idemprimal

Example

Reference

Separating Relations

Test for Separating Relations.

 $\textbf{Pi}:=\langle\{0,1,2,3,4\},*\rangle$

*	0	1	2	3	4
0	1	4	1	0	4
1	2	1	0	3	0
2	3	4	2	4	3
3	2	3	3	4	1
4	2	1	4	0 3 4 4 3	3

*	0	1	2	3	4
0	Х				
1		Х			
2			Х		
2 3 4				Х	X
4					X

Reference

Separating Relations

Test for Separating Relations.

 $\textbf{Pi} := \langle \{0,1,2,3,4\}, * \rangle$

*	0	1	2	3	4
0	1	4	1	0	4
1	2	1	0	3	0
2	3	4	2	4	3
3	2	3	3	4	1
4	1 2 3 2 2	1	4	3	3

*	0	1	2	3	4
0	Х				
1		Χ			
2			Χ		
0 1 2 3 4				Х	Х
4				Х	Х

Evolutionary Computation

The Bucket Algorithm

Continuit

Example

Deference

Separating Relations

Test for Separating Relations.

 $\textbf{Pi}:=\langle\{0,1,2,3,4\},*\rangle$

*	0	1	1 0 2 3 4	3	4
0	1	4	1	0	4
1	2	1	0	3	0
2	3	4	2	4	3
3	2	3	3	4	1
4	2	1	4	3	3

*	0	1	2	3	4
0	Х		Х		Х
1		Х		Х	Х
2	х		Χ	Χ	Х
3		Х	Χ	Х	Х
4	×	Х	X X X	X	X

Reference

Separating Relations

Test for Separating Relations.

 $\textbf{Pi}:=\langle\{0,1,2,3,4\},*\rangle$

*	0	1	2	3	4
0	1	4	1	0	4
1	2	1	0	3	0
2	3	4	2	4	3
3	2	3	3	4	1
4	2	1	4	0 3 4 4 3	3

Evolutionary Computation

Algorithm

Continuit

Thank

Reference

Separating Relations

Test for Separating Relations.

Pi := $\langle \{0, 1, 2, 3, 4\}, * \rangle$

Not in any separating relation.

*	0	1	2	3	4
0	1	4	1	0	4
1	2	1	0	3	0
2	3	4	2	4	3
3	2	3	3	4	1
4	2	1	1 0 2 3 4	3	3

*	0	1	2	3	4
0	Х	Х	Х	Х	Х
1	X	Χ	Χ	Χ	Χ
2	х	Χ	Χ	Χ	Х
3	х	Χ	Χ	Χ	Х
4	X	Χ	Χ	Χ	Χ

Conclusion:

Pi has no subalgebra with a separating relation.

Prior Method

Evolutionary

The Bucket Algorithm

Validity and Idemprimalit

Continuit

Example

Thanks

Deference

Separating Relations Let $\vec{d} \in G^k$ and let $a \in G$. For $H \in \mathbb{N}$ define

 $\beta_{\vec{d},a}(H) := \operatorname{Prob}\langle t(\vec{d}) = a \mid t \text{ is a term of height } \leq H \rangle.$

D.Clark et a

Sample Problen

Prior Methods

Evolutionary Computation

The Bucket Algorithm

Continuity

Example

Thanks

Reference

Separating Relations Let $\vec{d} \in G^k$ and let $a \in G$. For $H \in \mathbb{N}$ define

$$\beta_{\vec{d},a}(H) := \operatorname{Prob}\langle t(\vec{d}) = a \mid t \text{ is a term of height } \leq H \rangle.$$

Definition. The groupoid **G** is **asymptoticly complete** if, for each $k \in \mathbb{N}$, each $\vec{d} \in \mathbf{G}^k$ and each $a \in G$, the sequence $\beta_{\vec{d},a}$ either has constant value 0 or is eventually bounded away from 0.

Algebraic Terms through Simulated

Biological Evolution

Test for Asymptotic Completeness.

Separating Relations

Test for Asymptotic Completeness.

Pi distributions with $k \in \mathbb{N}$ and $\vec{d} \in Pi^k$.

$$H \quad \beta_{\vec{d},0}(H) \quad \beta_{\vec{d},1}(H) \quad \beta_{\vec{d},2}(H) \quad \beta_{\vec{d},3}(H) \quad \beta_{\vec{d},4}(H)$$

1

2

,

4

8

6

32

64

500

Evolutionary

The Bucket Algorithm

Validity and

Continuit

Example

Thanks!

References

Separating Relations

Test for Asymptotic Completeness.

Pi distributions with k = 10 and $\vec{d} = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$.

Test for Asymptotic Completeness.

Pi distributions with k = 10 and $\vec{d} = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$.

Н	$eta_{\vec{d},0}(H)$	$eta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$eta_{ec{d},4}(H)$
1	0	0	1	0	0
2	0	0	1	0	0
4	0	0	1	0	0
6	0	0	1	0	0
8	0	0	1	0	0
10	0	0	1	0	0
16	0	0	1	0	0
500	0	0	1	0	0

Validity and Idemprimali

Continuit

Exampl

Thanks!

References

Separating Relations

Test for Asymptotic Completeness.

Pi distributions with k = 10 and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3).$

$$H \quad \beta_{\vec{d},0}(H) \quad \beta_{\vec{d},1}(H) \quad \beta_{\vec{d},2}(H) \quad \beta_{\vec{d},3}(H) \quad \beta_{\vec{d},4}(H)$$

Test for Asymptotic Completeness.

Pi distributions with k = 10 and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3).$

 $H = \beta_{\vec{d},0}(H) = \beta_{\vec{d},1}(H) = \beta_{\vec{d},2}(H) = \beta_{\vec{d},3}(H) = \beta_{\vec{d},4}(H)$

0.2 0.1 0.1

0.5

Continuity

Exampi

Reference

Separating Relations

Test for Asymptotic Completeness.

Pi distributions with k = 10 and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3).$

$$H \quad \beta_{\vec{d},0}(H) \quad \beta_{\vec{d},1}(H) \quad \beta_{\vec{d},2}(H) \quad \beta_{\vec{d},3}(H) \quad \beta_{\vec{d},4}(H)$$

0.2 0.1 0.1

.1 0.5

2 0.127273 0.127273 0.145455 0.254545 0.345455

Separating Relations

Test for Asymptotic Completeness.

Pi distributions with k = 10 and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3)$.

$$H \quad \beta_{\vec{d},0}(H) \quad \beta_{\vec{d},1}(H) \quad \beta_{\vec{d},2}(H) \quad \beta_{\vec{d},3}(H) \quad \beta_{\vec{d},4}(H)$$

1 0.2 0.1 0.1 0.5 0.1

 $2 \quad 0.127273 \quad 0.127273 \quad 0.145455 \quad 0.254545 \quad 0.345455$

4 0.098817 0.182526 0.088076 0.368477 0.262102

Test for Asymptotic Completeness.

Pi distributions with k = 10 and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3).$

0.127273 0.1272730.145455 0.254545 0.345455

0.098817 0.182526 0.088076 0.368477 0.262102

8 0.103371 0.091358 0.365539 0.195561 0.244171

Separating Relations

Test for Asymptotic Completeness.

Pi distributions with k = 10 and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3).$

2 0.127273 0.127273 0.145455 0.254545 0.345455

4 0.098817 0.182526 0.088076 0.368477 0.262102

8 0.103371 0.195561 0.091358 0.365539 0.244171

16 0.103372 0.195275 0.091609 0.365434 0.244311

Separating

Test for Asymptotic Completeness.

Pi distributions with k = 10 and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3)$.

Η	$eta_{\vec{d},0}(H)$	$eta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.2	0.1	0.1	0.5	0.1
2	0.127273	0.127273	0.145455	0.254545	0.345455
4	0.098817	0.182526	0.088076	0.368477	0.262102
8	0.103371	0.195561	0.091358	0.365539	0.244171
16	0.103372	0.195275	0.091609	0.365434	0.244311
32	0.103372	0.195275	0.091609	0.365434	0.244310

D.Clark et a

Sample Problem

Prior Method

Computation

Algorithm

Continuity

- .

Thanks

Reference

Separating Relations

Test for Asymptotic Completeness.

Pi distributions with k = 10 and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3)$.

Н	$eta_{ec{d},0}(H)$	$eta_{\vec{d},1}(H)$	$eta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.2	0.1	0.1	0.5	0.1
2	0.127273	0.127273	0.145455	0.254545	0.345455
4	0.098817	0.182526	0.088076	0.368477	0.262102
8	0.103371	0.195561	0.091358	0.365539	0.244171
16	0.103372	0.195275	0.091609	0.365434	0.244311
32	0.103372	0.195275	0.091609	0.365434	0.244310
64	0.103372	0 195275	0.091609	0 365434	0 244310

Test for Asymptotic Completeness.

Pi distributions with k = 10 and $\vec{d} = (3, 4, 3, 0, 2, 3, 3, 0, 1, 3).$

Н	$eta_{\vec{d},0}(H)$	$eta_{\vec{d},1}(H)$	$eta_{\vec{d},2}(H)$	$eta_{ec{d},3}(H)$	$eta_{ec{d},4}(H)$
1	0.2	0.1	0.1	0.5	0.1
2	0.127273	0.127273	0.145455	0.254545	0.345455
4	0.098817	0.182526	0.088076	0.368477	0.262102
8	0.103371	0.195561	0.091358	0.365539	0.244171
16	0.103372	0.195275	0.091609	0.365434	0.244311
32	0.103372	0.195275	0.091609	0.365434	0.244310
64	0.103372	0.195275	0.091609	0.365434	0.244310
500	0.103372	0.195275	0.091609	0.365434	0.244310

Test for Asymptotic Completeness.

Н	$\beta_{\vec{d},0}(H)$	$eta_{\vec{d},1}(H)$	$eta_{\vec{d},2}(H)$	$eta_{ec{d},3}(H)$	$eta_{\vec{d},4}(H)$	
1	0	0	1	0	0	
2	0	0	1	0	0	
4	0	0	1	0	0	
6	0	0	1	0	0	
8	0	0	1	0	0	
10	0	0	1	0	0	
16	0	0	1	0	0	
500	0	0	1	0	0	

Validity and Idemprimal

Continuit

Exampl

Thanks!

Reference

Separating Relations

Test for Asymptotic Completeness.

$$H \quad \beta_{\vec{d},0}(H) \quad \beta_{\vec{d},1}(H) \quad \beta_{\vec{d},2}(H) \quad \beta_{\vec{d},3}(H) \quad \beta_{\vec{d},4}(H)$$

Test for Asymptotic Completeness.

Pi distributions with k = 10 and $\vec{d} = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$.

Separating Relations

Test for Asymptotic Completeness.

Pi distributions with k = 10 and $\vec{d} = (2, 2, 2, 2, 2, 2, 4, 2, 2, 2)$.

$$H = \beta_{\vec{d},0}(H) = \beta_{\vec{d},1}(H) = \beta_{\vec{d},2}(H) = \beta_{\vec{d},3}(H) = \beta_{\vec{d},4}(H)$$

1 0.0 0.0 0.9

0.0

0.1

0.000000 0.000000 0.810001

0.099999

00000

Exampl

Thanks!

Reference

Separating Relations

Test for Asymptotic Completeness.

Pi distributions with k = 10 and $\vec{d} = (2, 2, 2, 2, 2, 2, 4, 2, 2, 2)$.

$$H \quad \beta_{\vec{d},0}(H) \quad \beta_{\vec{d},1}(H) \quad \beta_{\vec{d},2}(H) \quad \beta_{\vec{d},3}(H) \quad \beta_{\vec{d},4}(H)$$

1 0.0 0.0 0.9 0.0 0.1

2 0.000000 0.000000 0.810001 0.099999 0.090000

4 0.007380 0.029583 0.430369 0.283600 0.248968

Examp

.

Reference

Separating Relations

Test for Asymptotic Completeness.

Pi distributions with k = 10 and $\vec{d} = (2, 2, 2, 2, 2, 2, 4, 2, 2, 2)$.

$$H \qquad \beta_{\vec{d},0}(H) \qquad \beta_{\vec{d},1}(H) \qquad \beta_{\vec{d},2}(H) \qquad \beta_{\vec{d},3}(H) \qquad \beta_{\vec{d},4}(H)$$

1 0.0 0.0 0.9 0.0 0.1

2 0.000000 0.000000 0.810001 0.099999 0.090000

 $4 \quad 0.007380 \quad 0.029583 \quad 0.430369 \quad 0.283600 \quad 0.248968$

8 0.095683 0.211801 0.070920 0.382629 0.238968

16

0.103370

Reference

Separating Relations

Test for Asymptotic Completeness.

0.195268

Pi distributions with k = 10 and $\vec{d} = (2, 2, 2, 2, 2, 2, 4, 2, 2, 2)$.

Η	$eta_{ec{d},0}(H)$	$eta_{ec{d},1}(H)$	$eta_{\vec{d},2}(H)$	$eta_{ec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.0	0.0	0.9	0.0	0.1
2	0.000000	0.000000	0.810001	0.099999	0.090000
4	0.007380	0.029583	0.430369	0.283600	0.248968
8	0.095683	0.211801	0.070920	0.382629	0.238968

0.091610

0.365436

Separating Relations

Test for Asymptotic Completeness.

Н	$eta_{\vec{d},0}(H)$	$eta_{\vec{d},1}(H)$	$eta_{\vec{d},2}(H)$	$eta_{\vec{d},3}(H)$	$eta_{ec{d},4}(H)$
1	0.0	0.0	0.9	0.0	0.1
2	0.000000	0.000000	0.810001	0.099999	0.090000
4	0.007380	0.029583	0.430369	0.283600	0.248968
8	0.095683	0.211801	0.070920	0.382629	0.238968
16	0.103370	0.195268	0.091610	0.365436	0.244316
32	0.103372	0.195275	0.091609	0.365434	0.244310

D.Clark et a

Sample Problem

Prior Method

Computation Computation

Algorithm

Continuity

Continuity

Thank

Reference

Separating

Test for Asymptotic Completeness.

Η	$eta_{ec{d},0}(H)$	$eta_{ec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$eta_{\vec{d},4}(H)$
1	0.0	0.0	0.9	0.0	0.1
2	0.000000	0.000000	0.810001	0.099999	0.090000
4	0.007380	0.029583	0.430369	0.283600	0.248968
8	0.095683	0.211801	0.070920	0.382629	0.238968
16	0.103370	0.195268	0.091610	0.365436	0.244316
32	0.103372	0.195275	0.091609	0.365434	0.244310
64	0.103372	0 195275	0.091609	0 365434	0.244310

D.Clark et a

Sample Problem

Prior Method

Evolutionary Computation

The Bucket Algorithm

Continuity

Examp

Reference

Reference

Test for Asymptotic Completeness.

Н	$eta_{\vec{d},0}(H)$	$eta_{\vec{d},1}(H)$	$eta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$eta_{ec{d},4}(H)$
1	0.0	0.0	0.9	0.0	0.1
2	0.000000	0.000000	0.810001	0.099999	0.090000
4	0.007380	0.029583	0.430369	0.283600	0.248968
8	0.095683	0.211801	0.070920	0.382629	0.238968
16	0.103370	0.195268	0.091610	0.365436	0.244316
32	0.103372	0.195275	0.091609	0.365434	0.244310
64	0.103372	0.195275	0.091609	0.365434	0.244310
500	0.103372	0.195275	0.091609	0.365434	0.244310

Test for Asymptotic Completeness.

Pi distributions with k = 10 and $\vec{d} = (2, 2, 2, 2, 2, 2, 4, 2, 2, 2)$.

Sample Problem

Prior Method

Computation Computation

The Bucket Algorithm

Continuity

Example

References

Separating

Н	$\beta_{\vec{d},0}(H)$	$eta_{ec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
1	0.0	0.0	0.9	0.0	0.1
2	0.000000	0.000000	0.810001	0.099999	0.090000
4	0.007380	0.029583	0.430369	0.283600	0.248968
8	0.095683	0.211801	0.070920	0.382629	0.238968
16	0.103370	0.195268	0.091610	0.365436	0.244316
32	0.103372	0.195275	0.091609	0.365434	0.244310
64	0.103372	0.195275	0.091609	0.365434	0.244310
500	0.103372	0.195275	0.091609	0.365434	0.244310

Experimental evidence that Pi is asymptoticly complete.

