A multi-sorted construction in finitely decidable varieties

Matthew Smedberg

Vanderbilt University
Department of Mathematics

5 October 2013

Decidability and Finite Decidability

We say that that a variety $\mathcal{V}$ is said to be *decidable* if $\mathrm{Th}(\mathcal{V})$ is a computable set of sentences, and likewise *finitely decidable* if $\mathrm{Th}_{\mathrm{fin}}(\mathcal{V}) = \mathrm{Th}(\mathcal{V}_{\mathrm{fin}})$ is a computable set.

Locally finite *decidable* varieties were studied by McKenzie and Valeriote in the 80s; they showed that every such variety is a varietal product of

- a strongly abelian subvariety,
- an affine subvariety, and
- a discriminator subvariety.

Since every finitely generated discriminator variety is decidable, and an explicit characterization exists for decidability in strongly abelian varieties (more on this in a moment), it only remained to determine which affine varieties are decidable.

Open Problem 1

For which finite rings $\mathbf{R}$ *is the variety of all* $\mathbf{R}$*-modules decidable? Finitely decidable?*

No decomposition for FD varieties

In 1997, Idziak characterized locally finite FD varieties having modular congruence lattices (which, by TCT and a nice result of Valeriote and Willard, are exactly those varieties omitting type 1), up to Problem 1.

Idziak's proof essentially gives a recipe for taking a FD variety $\mathcal{V}$ admitting only type 3, another FD variety $\mathcal{W}$ of modules, and producing a FD supervariety $\mathcal{V} \subset \mathcal{V} \star \mathcal{W}$ such that

- if $\mathbf{A} \in \mathcal{V} \star \mathcal{W}$ omits type 2, then $\mathbf{A} \in \mathcal{V}$;
- if $\mathbf{A}/\mathrm{Rad}(\mathbf{A})$ (which must lie in $\mathcal{V}$) is subdirectly irreducible, every $\mathrm{Rad}(\mathbf{A})$-block is a module from $\mathcal{W}$;
- there exist algebras in $\mathcal{V} \star \mathcal{W}$ whose congruence lattice is the ordered sum of a nontrivial solvable interval and a nontrivial chain of boolean type covers.

Going in the reverse direction, Idziak gave sufficient conditions that guarantee, for modular variety $\mathcal{V}$ and $\mathbf{A} \in \mathcal{V}$, that the poset of meet-irreducible $\tau \in \mathrm{Con}(\mathbf{A})$ with boolean upper cover forms a tree $\mathbb{T}$ with root $\top_A$; that each $\tau/[\tau,\tau]$-block is a module over a ring uniformly definable from the variety; and that $\mathbf{A}$ could be isomorphically recovered from the family

$$\langle \mathbf{A}/[\tau,\tau] \colon \tau \in \mathbb{T} \rangle$$

in a uniform way. It follows that deciding the theory of $\mathcal{V}$ is no harder than deciding $\mathcal{V}_3$ and the theory of modules over the indicated ring.

Question

Can we do something similar in the case where a variety $\mathcal{V}$ omits type 2: that is, find a variety $\mathcal{W}$ of strongly abelian algebras such that we can associate blocks of strongly abelian congruences in $\mathcal{V}$ to algebras in $\mathcal{W}$ well enough to determine the structure of algebras in $\mathcal{V}$?

Back to strongly abelian algebras

Lemma

Let $\mathbf{A}$ *be a finite, strongly abelian algebra, and let* k *be the largest number such that* $\mathbf{A}$ *has an idempotent term operation* $t(x_1, \ldots, x_k)$ *that depends on all its variables. Then we can find a term* $d(x_1, \ldots, x_k)$ *and a decomposition*

$$A = A_1 \times \cdots \times A_k$$

such that

$$d(a_1, \ldots, a_k) = d \begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_k^1 \\ a_1^2 & a_2^2 & \cdots & a_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^k & a_2^k & \cdots & a_k^k \end{pmatrix} = \begin{matrix} a_1^1 \\ a_2^2 \\ \vdots \\ a_k^k \end{matrix}$$

Suppose $d(x_1, \ldots, x_k)$ acts as a decomposition term throughout the strongly abelian variety $\mathcal{V}$. Valeriote and McKenzie define a k-sorted companion variety $\mathcal{V}[d]$ by turning each n-ary fundamental operation of $\mathcal{V}$ into k operations of arity $k \cdot n$.

Theorem 1

- *$\mathcal{V}$ is essentially k-ary iff $\mathcal{V}[d]$ is essentially unary.*
- *If $\mathcal{V}[d]$ is not essentially unary, then $\mathcal{V}[d]$ and $\mathcal{V}$ are both undecidable and finitely undecidable.*

Question

Can we do something similar in the case where we cannot decompose an algebra into a product of strongly abelian and totally nonabelian factors?

Unifying the constructions

Let **A** be a finite algebra in a variety $\mathcal{V}$ which satisfies all the known necessary conditions for finite decidability, and suppose $\mathrm{typ}\{\mathcal{V}\} = \{1, 3\}$. Then the meet-irreducible congruences $\tau \in \mathrm{Con}(\mathbf{A})$ with boolean upper cover are arranged in a tree $\mathbb{T}$ with $\top_A$ at the root; for each $\tau \in \mathbb{T}$, the least $\zeta \in \mathrm{Con}(\mathbf{A})$ such that τ is solvable over ζ is in fact $[\tau, \tau]$; and **A** is a subdirect product of the $\mathbf{A}/[\tau, \tau]$.

Theorem 2 (S.)

Replacing $\mathbf{A}$ *by one of the* $\mathbf{A}/[\tau,\tau]$, *we know that* $\mathrm{Rad}(\mathbf{A})$ *is a strongly abelian congruence, and meet-irreducible with boolean monolith. Say* $\mathrm{Rad}(\mathbf{A})$ *has* m *blocks. For each block* B_i, *let* $\mathbf{A}_i$ *be the algebra induced on* B_i *by all terms (not polynomials) which respect it.* $\mathbf{A}_i$ *is a strongly abelian algebra, with* k_i*-ary decomposition term* d_i.

Then if any term of $\mathbf{A}_i$ *depends on more than* k_i *variables, then* $\mathcal{V}$ *is hereditarily finitely undecidable.*

Proof by construction

To prove it, we define a multi-sorted first-order language $L^\flat$. The sorts are

$$\{\langle i,j\rangle \colon 1 \le i \le m,\ 1 \le j \le k_i\}$$

and for each basic operation $f(x_1,\ldots,x_n)$ of $\mathbf{A}$ and any blocks

$$f : B_{i_1} \times \cdots \times B_{i_n} \to B_{i_0}$$

we include k_{i_0} basic operations

$$f^j_{i_1,\ldots,i_n}\begin{pmatrix} \langle i_1,1\rangle & \langle i_2,1\rangle & \cdots & \langle i_n,1\rangle \\ \langle i_1,2\rangle & \langle i_2,2\rangle & \cdots & \langle i_n,2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle i_1,k_{i_1}\rangle & \langle i_2,k_{i_2}\rangle & \cdots & \langle i_n,k_{i_n}\rangle \end{pmatrix} \to \langle i_0,j\rangle$$

Proof by construction

Then if d_i decomposes B_i into $B_{i,1} \times B_{i,2} \times \cdots \times B_{i,k_i}$, we define an $L^\flat$-structure $\mathbf{A}^\flat$, interpreting sort $\langle i, j \rangle$ by $B_{i,j}$, and evaluating the functions in the only reasonable way.

Lemma

$\mathbf{A}^\flat$ *is strongly abelian.*

(Recall that $\mathbf{A}$ was not even abelian!)

Proof by construction

Lemma

Call $\mathbf{C} \in \mathcal{V}$ *flattable if this construction sends it to a sorted algebra* $\mathbf{C}^\flat \in \mathcal{V}^\flat = \mathrm{HSP}(\mathbf{A}^\flat)$. *Let* $\mathbf{C}, \mathbf{C}_1, \mathbf{C}_2$ *be flattable. Then*

- $\mathbf{C}^\flat$ *is strongly abelian, and* $\mathbf{C}/\mathrm{Rad}(\mathbf{C})$ *is canonically isomorphic to* $\mathbf{A}/\mathrm{Rad}(\mathbf{A})$. *Call the canonical projection* $\pi_{\mathbf{C}}$.
- *Subalgebras of* $\mathbf{C}^\flat$ *are precisely the flat images of subalgebras of* $\mathbf{C}$ *which have nonempty intersection with every* $\mathrm{Rad}(\mathbf{C})$*-block.*
- *Homomorphic images of* $\mathbf{C}^\flat$ *are precisely the flat images of homomorphic images of* $\mathbf{C}$ *by solvable congruences.*
- *The product* $\mathbf{C}_1^\flat \times \mathbf{C}_2^\flat$ *is the flat image of*

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \pi_{\mathbf{C}_1}(x) = \pi_{\mathbf{C}_2}(y) \right\} \leq \mathbf{C}_1 \times \mathbf{C}_2$$

Proof by construction

The variety generated by $\mathbf{A}^\flat$, in other words, interprets into $\mathcal{V}$. Notice that the trivial algebra in this variety is obtained by flatting $\mathbf{A}/\mathrm{Rad}(\mathbf{A})$.

All that remains is now to show that Valeriote's proof of Theorem 1 goes through.

It does.

"When proving something is undecidable, use local structure. When proving something is decidable, you have to use global structure."

– Pawel Idziak

Open Problem 2

Are the algebras $\mathbf{A}/[\tau, \tau]$ *sufficiently regular to be manageable, while simultaneously encoding enough information to recover* $\mathbf{A}$ *in some uniform, first-order way?*

Open Problem 3

What about varieties that admit both type 1 and 2? Even in the abelian case, we're pretty much stumped so far.