Finiteness properties of direct products

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FWF Der Wissenschaftsfonds.

Nice Boring Theorem

$\mathbf{A} \times \mathbf{B}$ satisfies $\mathcal{P}$ iff $\mathbf{A}$ and $\mathbf{B}$ satisfy $\mathcal{P}$.

Examples for property $\mathcal{P}$:
being **finitely generated, finitely presented, residually finite**,...

Example

1. **Groups:** $\mathbf{G} \times \mathbf{H}$ is finitely generated iff $\mathbf{G}, \mathbf{H}$ are finitely generated (same for finitely presented, residually finite).
2. **Semigroups:** $\langle \mathbb{N}, + \rangle$ is finitely generated but $\langle \mathbb{N}, + \rangle^2$ is not.

Problem

Which algebras and properties give Nice Boring Theorems?

For semigroups see Robertson, Ruškuc, Wiegold (1998), Gray, Ruškuc, (2009).

1. Finite generation

Lemma (Folklore)

If $\mathbf{A} \times \mathbf{B}$ is finitely generated, then $\mathbf{A}$, $\mathbf{B}$ are finitely generated.

Proof.

Being finitely generated is inherited by homomorphic images. $\square$

Theorem

Let $\mathbf{A}, \mathbf{B}$ in an idempotent variety $[t(x, \ldots, x) \approx x$ for all terms $t]$.
Then $\mathbf{A} \times \mathbf{B}$ is finitely generated iff $\mathbf{A}, \mathbf{B}$ are finitely generated.

Proof.

If X, Y generate $\mathbf{A}, \mathbf{B}$, then $X \times Y$ generates $\mathbf{A} \times \mathbf{B}$. $\square$

Remark

We have a NBT for lattices but not for their expansions:
$\mathbf{A} := \langle \mathbb{N}, \max, \min, x+1 \rangle$ is generated by 1, but $\mathbf{A}^2$ is not finitely generated.

Theorem (Geddes, PhD-thesis)

Let $\mathbf{A}, \mathbf{B}$ in a congruence permutable variety of finite signature $\mathcal{F}$. Then $\mathbf{A} \times \mathbf{B}$ is finitely generated iff $\mathbf{A}, \mathbf{B}$ are finitely generated.

Remark

Finite signature is necessary.
$\mathbf{A} := \langle \mathbb{Z}^{\mathbb{N}}, +, -, \text{ all constants} \rangle$ is generated by $\emptyset$ but $\mathbf{A}^2$ is **not** finitely generated.

Proof, $\Leftarrow$.

Let X, Y generate $\mathbf{A}, \mathbf{B}$. Fix $u \in A, v \in B$. Define

$$\begin{aligned} Z \quad := \quad & X \times \{v\} \cup \{u\} \times Y \cup \{(u,v)\} \cup \\ & \{(f^{\mathbf{A}}(u,\ldots,u), v) \mid f \in \mathcal{F}\} \cup \\ & \{(u, f^{\mathbf{B}}(v,\ldots,v)) \mid f \in \mathcal{F}\} \end{aligned}$$

Claim: $\forall a \in A$: $(a,v) \in \langle Z \rangle$

Have term s over $\mathcal{F}$ and $x_1,\ldots,x_k \in X$: $s^{\mathbf{A}}(x_1,\ldots,x_k) = a$.

Induct on length of s:

1. If s is a variable, then $a = x_i$ and $(a,v) \in Z$.
2. Assume $s = f(t_1,\ldots,t_n)$ for $f \in \mathcal{F}$, terms $t_1,\ldots,t_n$. For $a_i := t_i^{\mathbf{A}}(x_1,\ldots,x_k)$, we have $(a_i, v) \in \langle Z \rangle$.

$$\begin{array}{llll} (& f^{\mathbf{A}}(a_1,\ldots,a_n), & f^{\mathbf{B}}(v,\ldots,v) &) \in \langle Z \rangle \\ (& u, & f^{\mathbf{B}}(v,\ldots,v) &) \in Z \\ (& u, & v &) \in Z \end{array}$$

Applying the Mal'cev term in each row yields $(a,v) \in \langle Z \rangle$.

Proof, continued.

For all $a \in A, b \in B$

$$\begin{aligned} (a, v) &\in \langle Z \rangle \\ (u, v) &\in Z \\ (u, b) &\in \langle Z \rangle \end{aligned}$$

Applying the Mal'cev term in each row yields $(a, b) \in \langle Z \rangle$.

So Z generates $\mathbf{A} \times \mathbf{B}$. $\square$

2. Finite presentations

Definition

$\mathbf{A}$ in a variety $\mathcal{V}$ is **finitely presented** if

$$\mathbf{A} \cong \mathbf{F}_{\mathcal{V}}(x_1, \dots, x_k)/\mathrm{Cg}\left((r_1, s_1), \dots, (r_n, s_n)\right)$$

for some $k, n \in \mathbb{N}$ and $(r_1, s_1), \dots, (r_n, s_n) \in \mathbf{F}_{\mathcal{V}}(x_1, \dots, x_k)^2$.

In particular, free algebras over finite sets are finitely presented.

Theorem

Let $\mathcal{V}$ be the variety of loops with signature $(\cdot, \backslash, /, 1)$.
Then $\mathbf{F}_{\mathcal{V}}(x) \times \mathbf{F}_{\mathcal{V}}(x)$ is not finitely presented.

Theorem

Let $\mathcal{V}$ be the variety of lattices, $\mathbf{2} := \langle \{0, 1\}, \wedge, \vee \rangle$.
Then $\mathbf{F}_{\mathcal{V}}(x_1, x_2, x_3) \times \mathbf{2}$ is not finitely presented.

Proof, $\mathbf{A} \in \mathcal{V}$ is not finitely presented.

1. Find X finite and an onto homomorphism $h \colon \mathbf{F}_{\mathcal{V}}(X) \to \mathbf{A}$.
2. Suppose $\ker h$ is generated by some $(r_1, s_1), \dots, (r_n, s_n)$.
3. Find $u, v \in \mathbf{F}_{\mathcal{V}}(X)$ such that $h(u) = h(v)$ in $\mathbf{A}$ but

$$u \not\equiv v \text{ in } \mathbf{F}_{\mathcal{V}}(X)/\mathrm{Cg}\,((r_1, s_1), \dots, (r_n, s_n)).$$

   Contradiction.

For the word problem in 3. we use

- for loops: Evans' confluent rewriting systems (1951).
- for lattices: Dean's solution of the word problem (1964). $\square$

3. Residually finite

Definition

$\mathbf{A}$ is **residually finite** if for any distinct $a, b \in A$ there exist $\rho \in \mathrm{Con}(\mathbf{A})$ such that A/ρ is finite and $a \not\equiv b \mod \rho$.

Lemma (Folklore)

If $\mathbf{A}, \mathbf{B}$ are residually finite, then $\mathbf{A} \times \mathbf{B}$ is residually finite.

Proof.

If $\alpha \in \mathrm{Con}(\mathbf{A})$ separates a_1, a_2, then $\alpha \times 1_B \in \mathrm{Con}(\mathbf{A} \times \mathbf{B})$ separates $(a_1, b_1), (a_2, b_2)$. □

The converse holds for example

- if $\mathbf{A}, \mathbf{B}$ embed into $\mathbf{A} \times \mathbf{B}$,
  NBT for algebras with idempotents (groups, monoids, lattices)
- if $\mathrm{Con}(\mathbf{A} \times \mathbf{B}) = \mathrm{Con}(\mathbf{A}) \times \mathrm{Con}(\mathbf{B})$.
  NBT for congruence distributive varieties

Theorem

Let $\mathbf{A}, \mathbf{B}$ in a congruence modular variety.
Then $\mathbf{A} \times \mathbf{B}$ is residually finite iff $\mathbf{A}, \mathbf{B}$ are residually finite.

Proof, $\Rightarrow$.

Let $a_1, a_2 \in A$ be distinct, fix $b \in B$.
Have $\rho \in \mathrm{Con}(\mathbf{A} \times \mathbf{B})$ of finite index and $(a_1, b) \not\equiv_\rho (a_2, b)$. Show

$$\sigma := \{(u, v) \in A^2 \mid \exists z \in B \colon (u, z) \equiv_\rho (v, z)\}$$

- is a congruence on $\mathbf{A}$,
- has finite index, and
- separates a_1, a_2

using commutators and a difference term. $\square$

Problems

Problem

When is a subdirect product of finitely generated lattices finitely generated?

Problem

Characterize the finitely presented loops, lattices, ... $\mathbf{A}, \mathbf{B}$ such that $\mathbf{A} \times \mathbf{B}$ is finitely presented.

Problem

Is the following decidable:
Given finitely presented semigroups $\mathbf{A} := \langle X \mid R \rangle, \mathbf{B} := \langle Y \mid S \rangle$.
Is $\mathbf{A} \times \mathbf{B}$ finitely presented?

Problem

Does $\mathbf{A} \times \mathbf{B}$ residually finite imply that $\mathbf{A}, \mathbf{B}$ are residually finite in varieties with difference term?

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