Polymorphisms of Binary Treelike Structures

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Outline

- Fixed template constraint satisfaction problem: the homomorphism problem for finite relational structures.

$$\text{HOM}(\mathbb{A}) = \{\mathbb{X} \mid \mathbb{X} \to \mathbb{A}\}.$$

- We are interested in membership in the class CSP($\mathbb{A}$), a computational problem that obviously lies in the complexity class **NP**.
- Dichotomy Conjecture (Feder and Vardi): for a finite $\mathbb{A}$, either HOM($\mathbb{A}$) has polynomial time membership or it has **NP**-complete membership problem.

Particular cases already known to exhibit the dichotomy:

- Schaefer's dichotomy for 2-element templates;
- dichotomy for undirected graph templates due to Hell and Nešetřil
- 3-element templates (Bulatov);
- digraphs with no sources and sinks (Barto, Kozik and Niven); also some special classes of oriented trees (Barto, Bulin)
- tournaments (Jackson, Kowalski, Niven)
- templates in which every subset is a fundamental unary relation (Bulatov, also Barto).

- Feder and Vardi reduced the problem of proving the dichotomy conjecture to the particular case of digraph CSPs, and even to digraph CSPs whose template is a balanced digraph (a digraph on which there is a level function).
- Specifically, for every template $\mathbb{A}$ there is a balanced digraph $\mathbb{D}$ such that CSP($\mathbb{A}$) is polynomial time equivalent to CSP($\mathbb{D}$).
- Some of the precise structure of CSP($\mathbb{A}$) is necessarily altered in the transformation to CSP($\mathbb{D}$).

- Algebraic approach to the CSP dichotomy conjecture: associate polynomial time algorithms to Pol($\mathbb{A}$)
- complexity of CSP($\mathbb{A}$) is precisely (up to logspace reductions) determined by the polymorphisms of $\mathbb{A}$.

- Atserias (2006) revisited a construction from Feder and Vardi's original article to construct a tractable digraph CSP that is provably not solvable by the bounded width (local consistency check) algorithm.
- This construction relies heavily on finite model-theoretic machinery: quantifier preservation, cops-and-robber games (games that characterize width k), etc.

The path $\mathbb{N}$

$$\begin{tikzcd}
& \beta & & \delta \\
\alpha \arrow[ur] & & \gamma \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Theorem

Let $\mathbb{A}$ be a relational structure. There exists a digraph $\mathbb{D}_{\mathbb{A}}$ such that the following holds: let Σ be any linear idempotent set of identities such that each identity in Σ is either balanced or contains at most two variables. If the digraph $\mathbb{N}$ satisfies Σ, then $\mathbb{D}_{\mathbb{A}}$ satisfies Σ if and only if $\mathbb{A}$ satisfies Σ.
The digraph $\mathbb{D}_{\mathbb{A}}$ can be constructed in logspace with respect to the size of A.

Corollary

Let $\mathbb{A}$ be a CSP (HOM) template. Then each of the following hold equivalently on $\mathbb{A}$ and $\mathbb{D}_{\mathbb{A}}$.

- *Taylor polymorphism or equivalently weak near-unanimity (WNU) polymorphism or equivalently cyclic polymorphism* (*conjectured to be equivalent to being tractable if $\mathbb{A}$ is a core*);
- *Polymorphisms witnessing SD*($\wedge$) (*equivalent to bounded width*);
- *totally symmetric idempotent (TSI) polymorphisms of all arities* (*equivalent to width* 1);

Furthermore, $\mathrm{HOM}(\mathbb{A})$ *and* $\mathrm{HOM}(\mathbb{D}_{\mathbb{A}})$ *are logspace equivalent.*

τ-Trees

- Let τ be a finite relational vocabulary.
- $\mathbb{A}$ is a τ-tree if its Gaifman digraph is an oriented tree;
- Leaf: vertex element which is the leaf of the Gaifman digraph;
- Pendant vertex: block element adjacent to a leaf;
- Binary τ-tree: every block is of in-degree two (i.e. τ-relations are binary.)

Absorbing Subuniverses

Absorbing subuniverse $C \trianglelefteq \mathbb{A}$: there exists a polymorphism t such that

$$\begin{aligned}
t(A, C, C, \ldots, C) &\subseteq C \\
t(C, A, C, \ldots, C) &\subseteq C \\
&\vdots \\
t(C, C, \ldots, C, A) &\subseteq C
\end{aligned}$$

Theorem

(Barto-Kozik) Let $\mathbb{A}$ *be a binary* τ*-tree with a w.n.u. polymorphism* w*. Then, in* $\mathbb{D}_{\mathbb{A}}$*, either the set of vertices or the set of blocks contains a singleton absorbing subuniverse with respect to* w*.*

Presence of Commutative Binary Polymorphisms

Theorem

Let $\mathbb{A}$ be a binary τ-tree. Then, the following are equivalent:

1. $\mathbb{A}$ *admits a commutative binary polymorphism.*
2. $\mathbb{A}$ *admits TSIs.*
3. $\mathbb{A}$ *has tree duality.*

The extended X-property of Hell, Nešetril, and Zhu yields a commutative binary polymorphism on any oriented tree, and their result follows from this theorem.

- The proof is essentially inductive on the size of the τ-tree.
- Using absorption, one can find a congruence α of the Gaifman digraph of $\mathbb{A}$, so that the TSIs can be lifted from the quotient digraph to the Gaifman digraph.
- The TSIs can be extended from the Gaifman digraph to $\mathbb{D}_{\mathbb{A}}$, and then to $\mathbb{A}$.

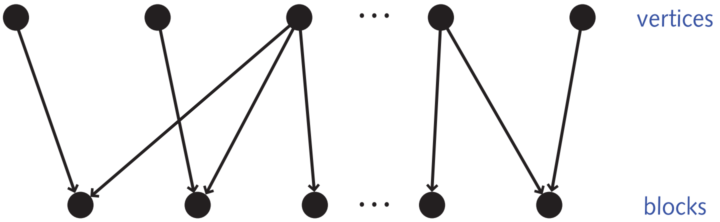

Figure: Gaifman digraph of a binary τ-tree

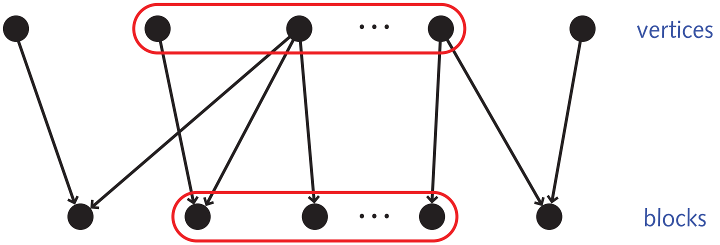

Figure: Gaifman digraph of a binary τ-tree with maximal proper absorbing subuniverses

Leap From Binary to Higher Arity WNUs

Theorem

Let $\mathbb{A}$ be a finite oriented tree. Then, if $\mathbb{A}$ admits an n-ary w.n.u., it admits w.n.u's of all arities at least n. Consequently, if $\mathbb{A}$ admits a w.n.u. polymorphism, it is of bounded width.

- We reduce the proof to a specific subclass of oriented trees: $\mathbb{D}_{\mathbb{A}}$.
- if $\mathbb{A}$ has an *n*-ary w.n.u., so does $\mathbb{D}_{\mathbb{A}}$ and the Gaifman digraph.
- Given an *n*-ary w.n.u., we can arrange things in such a way that, when applied to an *n*-tuple of elements which are not all on the same level, it is always a "projection onto the first element on the highest level".
- The proof is again inductive; basic cases are covered by the results of Barto and Bulin.
- Finding a congruence α no longer works (lack of commutativity), but we can still locate suitable maximal proper absorbing subuniverses on both levels of the Gaifman digraph.
- After that, the $(n+1)$-ary w.n.u. is constructed on the Gaifman digraph, and then extended to $\mathbb{D}_{\mathbb{A}}$.