Epimorphisms in certain varieties of partially ordered semigroups

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Motivation

- Let U be a subsemigroup of a semigroup S. Then $d \in S$ is said to be in the *dominion* of U if for every pair of semigroup homomorphisms $f, g : S \longrightarrow T$ with $f\,|_U = g\,|_U$ we have $f(d) = g(d)$.

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- The set of all such elements of S is called the dominion of U is S, denoted by $dom_S(U)$. This is a subsemigroup of S.

- Clearly $U \subseteq dom_S(U)$.

- U is said to be *closed* in S if $dom_S(U) = U$. We term U *absolutely closed* if it is closed in all of its semigroup extensions.

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- Consequently, a variety of semigroups is absolutely closed iff its epimorphisms are surjective.

Motivation

In 1966 J.R. Isbell proved the following theorem.

Theorem

$d \in dom_S(U)$ iff there exists a system of equalities (called a zigzag)

$$\begin{aligned} d &= s_1 u_1 & u_1 &= v_1 t_1 \\ s_1 v_1 &= s_2 u_2 & u_2 t_1 &= v_2 t_2 \\ &\vdots & &\vdots \\ s_{n-1} v_{n-1} &= u_n & u_n t_{n-1} &= d \end{aligned}$$

with $s_1, \ldots, s_{n-1}; t_1, \ldots, t_{n-1} \in S$, $u_1, \ldots, u_n; v_1, \ldots, v_{n-1} \in U$.

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1. to present an analogous zigzag theorem for partially ordered semigroups, briefly posemigroups and
2. discuss closure properties and epimorphisms in certain varieties of posemigroups.

- A partially ordered semigroup, briefly *posemigroup*, is a pair $(S, \leq)$ comprising a semigroup S and a partial order $\leq$ (on S) that is *compatible* with the binary operation, i.e. for all $s_1, s_2, t_1, t_2 \in S$,

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$$(s_1 \leq t_1, s_2 \leq t_2) \Longrightarrow s_1 s_2 \leq t_1 t_2.$$

- A posemigroup *homomorphism* $f : (S, \leq_S) \longrightarrow (T, \leq_T)$ is a monotone semigroup homomorphism.

Definitions

- Let $(U, \leq_U)$ be a subposemigroup of a posemigroup $(S, \leq_S)$. Then the subposemigroup

  $$\{x \in S : \alpha, \beta : (S, \leq_S) \to (T, \leq_T), \alpha|_U = \beta|_U \implies \alpha(x) = \beta(x)\}$$

  is called the *dominion* of $(U, \leq_U)$ in $(S, \leq_S)$, where α and β are posemigroup homomorphisms.

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  is called the *dominion* of $(U, \leq_U)$ in $(S, \leq_S)$, where α and β are posemigroup homomorphisms.

- We shall denote this set by $\widehat{dom}_S(U)$.

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- $(U, \leq_U)$ will be called absolutely closed if it is closed in all of its posemigroup extensions.

Zigzag theorem for posemigroups

Theorem

$d \in \widehat{dom}_S(U)$ if and only if there exists a system of inequalities

$$
\begin{array}{ll}
d \leq s_1 u_1 & u_1 \leq v_1 t_1 \\
s_1 v_1 \leq s_2 u_2 & u_2 t_1 \leq v_2 t_2 \\
\vdots & \vdots \\
s_{n-1} v_{n-1} \leq u_n & u_n t_{n-1} \leq d \\
v_n \leq s_{n+1} u_{n+1} & d \leq v_n t_{n+1} \\
s_{n+1} v_{n+1} \leq s_{n+2} u_{n+2} & u_{n+1} t_{n+1} \leq v_{n+1} t_{n+2} \\
\vdots & \vdots \\
s_{n+m} v_{n+m} \leq d & u_{n+m} t_{n+m} \leq v_{n+m}
\end{array}
$$

where $s_i, t_i \in S$ and $u_i, v_i \in U$.

Remarks

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- So, given a subposemigroup $(U, \leq_U)$ of a posemigroup $(S, \leq_S)$, one may also consider the (algebraic) dominion $dom_S(U)$ of U in S.
- Because every equality in Isbell's zigzag can be treated as an inequality, one can easily write from a 'zigzag of equalities' a 'zigzag of inequalities'. We therefore have

$$d \in dom_S(U) \Longrightarrow d \in \widehat{dom}_S(U).$$

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$$d \in dom_S(U) \Longrightarrow d \in \widehat{dom}_S(U).$$

- More precisely: $U \subseteq dom_S(U) \subseteq \widehat{dom}_S(U) \subseteq S$.

- So the closure of $(U, \leq_U)$ in $(S, \leq_S)$ implies the closure of U in S.

The closure properties

The following result tells that the converse of the above statement is also true.

- **Proposition** A subposemigroup $(U, \leq_U)$ is closed in a posemigroup $(S, \leq_S)$ if and only if U is such in S as a semigroup.

The following result tells that the converse of the above statement is also true.

- **Proposition** A subposemigroup $(U, \leq_U)$ is closed in a posemigroup $(S, \leq_S)$ if and only if U is such in S as a semigroup.

- **Corollary** A posemigroup $(U, \leq)$ is absolutely closed if and only if it is such as a semigroup within the class of semigroups that qualify as posemigroup extensions of $(U, \leq)$, for certain compatible partial orders.

Absolutely closed varieties (**unordered context**)

Recall that a semigroup S is called a right group if every equation $ax = b$ with $a, b \in S$ has a unique solution in S.

Left groups are defined similarly

Theorem (Higgins)

The absolutely closed varieties of semigroups are exactly the varieties consisting entirely of semilattices of groups, or entirely of right groups or entirely of left groups.

Definition

- A class of posemigroups is called a *variety* (of posemigroups) if it is closed under taking products (endowed with componentwise order), homomorphic images (where of course we are only considering the monotone semigroup homomorphisms) and subposemigroups.

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- It is also possible to describe posemigroup varieties alternatively with the help of inequalities using a Birkhoff type characterization.

- **Example** The class of all bounded posemigroups is a variety of posemigroups defined by $\{(xy)z = x(yz)\} \cup \{x \leq a \ (b \leq x)\}$. Also, if $\mathcal{V}$ is a class of posemigroups that forms a variety of semigroups (if the orders are disregarded), then the subclass $\mathcal{V}'$ of all bounded posemigroups in $\mathcal{V}$ is a variety of posemigroups defined by the identities of $\mathcal{V}$ and the inequality(s) $x \leq a \ (b \leq x)$, provided $\mathcal{V}'$ is non-empty.

Absolutely closed varieties (**ordered context**)

From the above corollary, viz,

- **Corollary** A posemigroup $(U, \leq)$ is absolutely closed if and only if it is such as a semigroup within the class of semigroups that qualify as posemigroup extensions of $(U, \leq)$, for certain compatible partial orders,

  and Higgins' theorem we obtian the following result.

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- **Corollary** Any variety $\mathcal{V}$ of posemigroups obtained by endowing a subclass of a class of semigroups of Higgins' theorem with compatible orders will be absolutely closed (in order theoretic sense).

Absolutely closed varieties (**ordered context**)

- **Example** The class of all semilattices with natural orders forms an absolutely closed variety of posemigroups. The assertion also remains true if we substitute all the orders by their duals.

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- **Example** The classes of bounded posemigroups that are right groups or left groups or semilattices of groups are absolutely closed (in ordered context).

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- **Example** The classes of bounded posemigroups that are right groups or left groups or semilattices of groups are absolutely closed (in ordered context).

- **Question** Are there any (order theoretic) varieties of absolutely closed posemigroups other then those of the above corollary? Especially are there any order theoretic varieties 'bigger' than those of Higgins'.

Epimorphisms of posemigroups

- A posemigroup homomorphism $f : (S, \leq_S) \longrightarrow (T, \leq_T)$ is termed an *epimorphism* if it is right cancellative in the usual sense of category theory, i.e. for any pair of posemigroup homomorphisms $g, h : T \longrightarrow W$, $g \circ f = h \circ f \Longrightarrow g = h$.

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- And of course for posemigroups also, $f : (S, \leq_S) \longrightarrow (T, \leq_T)$ is an epimorphism iff $\widehat{dom}_T(\mathrm{Im}\, f) = (T, \leq_T)$.

Epimorphisms of posemigroups

- **Proposition** Let $\mathcal{V}$ be a variety of absolutely closed semigroups. Let $\mathcal{V}'$ be the variety of posemigroups obtained by equipping members of $\mathcal{V}$ with all or some of their compatible orders. Then a posemigroup homomorphism f is epi in $\mathcal{V}$ iff it is such in $\mathcal{V}'$.

Epimorphisms of posemigroups

Question Given that $f : S \longrightarrow T$ is not epi in a class $\mathcal{C}$ of semigroups (where $\mathcal{C}$ is different from the varieties of Higgins' theorem), how (and when) can we find $\leq_S$ and $\leq_T$ so that $f : (S, \leq_S) \longrightarrow (T, \leq_T)$ is an epimorphism of posemigroups?

Acknowledgement and References

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[1] Sohail Nasir: Zigzag theorem for partially ordered monoids. To appear in Comm. Algebra.

[2] Sohail Nasir: Absolute closure for pomonoids. Submitted.

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THANK YOU