A Syntactic Approach to the Complexity of Linear Idempotent Mal'cev Conditions

Jonah Horowitz

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Theorem (Freese, Valeriote 2009)

Let $\mathbf{A}$ be a finite idempotent algebra. Then $\mathbf{A}$ supports a majority term operation if and only if for every $0, 1, 2, 3, 4, 5 \in A$ there are $6, 7, 8 \in A$ such that

$$\left(\begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 6 \\ 7 \\ 8 \end{pmatrix}\right) \in Cg_{\mathbf{A}^3}\left(\begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}\right) \wedge Cg_{\mathbf{A}^3}\left(\begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}\right) \text{ and}$$

$$\left(\begin{pmatrix} 6 \\ 7 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}\right) \in Cg_{\mathbf{A}^3}\left(\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}\right) \wedge Cg_{\mathbf{A}^3}\left(\begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}\right).$$

p is a Mal'cev operation on A if for every $x, y \in A$

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Theorem (Freese, Valeriote 2009)

Let $\mathbf{A}$ be a finite idempotent algebra. Then $\mathbf{A}$ supports a Mal'cev term operation if and only if for every $0, 1, 2, 3 \in A$ it is the case that

$$(\binom{1}{3}, \binom{0}{2}) \in Cg_{\mathbf{A}^2}(\binom{0}{2}, \binom{0}{3}) \circ Cg_{\mathbf{A}^2}(\binom{0}{3}, \binom{1}{3}).$$

Corollary (Freese, Valeriote 2009)

Whether or not a finite idempotent algebra possesses a majority term can be determined in polynomial time.

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Whether or not a finite idempotent algebra possesses a Mal'cev term can be determined in polynomial time.

Let p be a Mal'cev operation.
Suppose that p_1 is almost a Mal'cev operation, but for a few $x, y \in A$ $p_1(y, y, x) \neq x$.
Suppose that p_2 is almost a Mal'cev operation, but for a few $x, y \in A$ $p_2(x, y, y) \neq x$.

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Then we can define

$$p'(x, y, z) := p(p_1(x, y, z), p_1(y, y, z), z)$$

$$p''(x, y, z) := p(x, p_2(x, y, y), p_2(x, y, z))$$

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So what?

Suppose that for every $a, b, c, d \in A$ there is an idempotent p such that

$$p(a, b, b) = a \text{ and}$$

$$p(d, d, c) = c.$$

Call p a local Mal'cev operation on a, b, c, d.

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Call p a local Mal'cev operation on a, b, c, d.
Pick $e, f \in A$ and let q be a local Mal'cev operation on $a, b, e, p(f, f, e)$ and let q' be a local Mal'cev operation on $e, p(e, f, f), c, d$.

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$$p'(x, y, z) := q(p(x, y, z), p(y, y, z), z)$$

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Let **A** be a finite idempotent algebra. Then **A** supports a Mal'cev term operation if and only if for every $0, 1, 2, 3 \in A$ it is the case that

$$\binom{0}{2} \in Sg_{\mathbf{A}}\{\binom{0}{3}, \binom{1}{3}, \binom{1}{2}\}.$$

based on Berman, Idziak, Marcovič, McKenzie, Valeriote, Willard 2010

Let Γ be a set of columns of x's and y's (of height n) and let **A** be a finite algebra. If

$$\begin{pmatrix} x \\ \vdots \\ x \end{pmatrix} \in Sg_{\mathbf{F}_{V(\mathbf{A})}\{x,y\}}\Gamma$$

then call a term which witnesses this fact a Γ-special cube term.

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then call a term which witnesses this fact a Γ-special cube term.

So a majority term is a $\{\begin{pmatrix} y \\ x \\ x \end{pmatrix}, \begin{pmatrix} x \\ y \\ x \end{pmatrix}, \begin{pmatrix} x \\ x \\ y \end{pmatrix}\}$-special cube term,

and a Mal'cev term is a $\{\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} y \\ y \end{pmatrix}, \begin{pmatrix} y \\ x \end{pmatrix}\}$-special cube term.

Let Γ be a set of columns of x's and y's (of height n), let $a, b \in A$, let $i < n$ and let p be an operation on A whose variables are indexed by Γ. Define $\gamma_i : A^2 \to A^\Gamma$ to be the function where

$$\gamma_i(a,b)(C) := \begin{cases} a & \text{if the } i\text{th element of } C \text{ is } x \\ b & \text{otherwise} \end{cases}$$

Say that p is a local Γ-special cube operation on (a, b, i) if $p(\gamma_i(a,b)) = a$. Given any subset S of $A^2 \times \{0, \ldots, n-1\}$, say that p is a local Γ-special cube operation on S if p is a local Γ-special cube operation on (a, b, i) for every $(a, b, i) \in S$.

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Theorem

Let Γ be (almost) an order ideal in the nth power of the semilattice $x < y$ and let **A** be a finite idempotent algebra. Then **A** supports a Γ-special cube term if and only if there is a local Γ-special cube operation on S for every $S \subseteq A^2 \times \{0, \ldots, |\Gamma| - 1\}$ with $|S| = |\Gamma|$.

Fix $\Gamma = (C_0, \ldots, C_{n-1})$ and A.
Assume that for some $k \geq n$ A supports local Γ-special cube operations on all sets of size k. It suffices to show for an arbitrary $S \subseteq A^2 \times \{0, \ldots, n-1\}$ with $|S| = k + 1$, that A supports a local Γ-special cube operation on S.
Choose $(a, b, i) \in S$ with $|S \cap (A^2 \times \{i\})| > 1$ and define

$$T := S \setminus \{(a, b, i)\}$$

$$R := S \setminus (A^2 \times \{i\}) \cup \{(a, p_T(\gamma_i(a, b)), i)\}$$

where p_T is the local Γ-special cube operation on T. For each $j < n$ define

$$z_j(\overline{x}) := \begin{cases} x_j & \text{if } C_j(i) = x \\ p_T(\overline{x}) & \text{if } C_j \text{ has exactly one } y\text{, at } i \\ p_T(\gamma_i(x_{q_j}, x_j)) & \text{otherwise} \end{cases}$$

where C_{q_j} is the column covered by C_j such that they differ only in position i. Then define

$$p_S(\overline{x}) := p_R(\overline{z}(\overline{x})).$$

Corollary

Given $k \geq 3$, it is checkable in polynomial time whether or not a finite idempotent algebra supports a k-ary near unanimity term.
Given $k \geq 2$, it is checkable in polynomial time whether or not a finite idempotent algebra supports a k-edge term.

Note that McKenzie independantly arrived at similar results to this corollary for near unanimity terms and Mal'cev terms using different methods.

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The interesting case: if $i = 0$, define

$$T := S \setminus \{(a,b,0)\}$$

$$Q := \{(d,c,1) | (c,d,2) \in T\} \cup \{(d,c,2) | (c,d,1) \in T\} \cup \{(a, p_T(a,b,a), 0)\}$$

$$R := S \setminus (A^2 \times \{0\}) \text{ and define}$$

$$p_S(x,y,z) = p_R(x, p_Q(x, p_T(x,y,z), z), z)$$

Theorem (Valeriote, 2013)

A similar construction works for congruence n-permutability.

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