Finite Monk algebras and equational bases defining RRA over wRRA

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This is joint work with Jacob Manske (Epic Systems Corporation) and Robin Hirsch (University College London).

A *relation algebra* is an abstract algebra $\langle A, +, \cdot, ^{-}, ;, ^{\cup}, 1' \rangle$ satisfying several equational axioms.

An algebra $\underline{A}$ is *representable* if there is an embedding $\underline{A} \to \langle P(E), \cup, \cap, ^{c}, |, ^{-1}, Id \rangle$, where E is some non-empty equivalence relation. The class RRA of representable algebras is a non-finitely based variety.

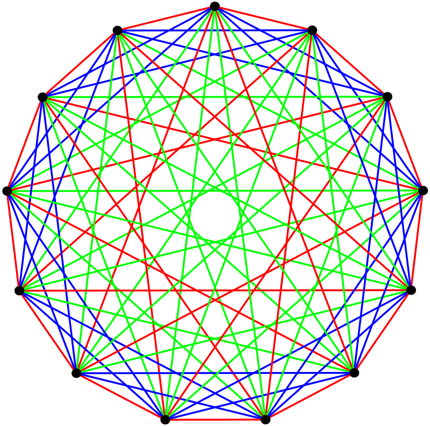
A *Monk algebra* (or Maddux algebra or Ramsey algebra) is a finite symmetric algebra with atoms $1', a_1, \dots, a_n$ so that

$$a_i ; a_i = \overline{a_i}$$

and for $i \neq j$,

$$a_i ; a_j = 0' = \overline{1'}.$$

equivalent characterization: the only forbidden cycles of atoms (triangles) are the 1-cycles $a_i a_i a_i$ (monochromatic triangles)



Is the Monk algebra $M(n)$ with n diversity atoms representable?

- Comer: YES for $n = 2, 3, 4, 5$ (late 80's)
- Maddux: YES for $n = 6, 7$ ($\sim$ 2010)
- A., Manske: YES for $9 \leq n \leq 300$ (except $n = 13, 292$)

(All cyclic group representations)

Look at primes $N = nk + 1$ with k even, find a generator x of $\mathbb{Z}_N^{\times}$, and construct the partition

$$X_0 = \left\{ x^0, x^n, x^{2n}, \ldots, x^{(k-1)n} \right\}$$

and $X_i = x \cdot X_{i-1},$ for $i = 1, 2, \ldots, n-1$.

Check that $X_i + X_i = \mathbb{Z}_N \setminus X_i$ and $X_i + X_j = \mathbb{Z}_N \setminus \{0\}$

A *weak representation* is an embedding into $\langle P(E), \cup, \cap, ^c, |, ^{-1}, Id \rangle$ that *need not preserve* $\cup$ or ^{c}.

Let wRRA denote the class of weakly representable algebras.

- wRRA is a variety (Pesci 2009)
- wRRA is not finitely based (Hodkinson-Mikulas 2000)
- RRA is not finitely based over wRRA (Andreka 1994)

RRA

$\Big|\ \Sigma'$

wRRA

$\Big|\ \Sigma$

RA

Theorem (A., Hirsch 2013)

Let Σ be an equational basis defining RRA over wRRA. Then $\forall N \in \mathbb{Z}^+$ there is an equation $\varepsilon \in \Sigma$ with more than N distinct variables.

Idea: Construct arbitrarily large weakly representable but not representable algebras whose "small" subalgebras are all representable.

Proof sketch:

- Suppose all equations in Σ have at most N distinct variables.
- Consider $M(n)$, $n > 2^{N+2}$.
- Split one atom into $k = R_n(3)$ parts to ensure non-representability, get $M(n,k)$.
- $M(n,k)$ is weakly representable (1-pt extension)
- Any subalgebra S generated by N or fewer elements must have an atom a that is above two or more "unsplit" atoms. Hence a is flexible.
- S is representable over a countable set.
- Hence $S \models \varepsilon$ for all such S.
- Therefore $M(n,k) \models \varepsilon$, and so Σ cannot define RRA over wRRA.

Future work: Does wRRA have a finite-variable basis? Presumably not.

Lyndon algebras from projective lines may decide the question.

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