Interval Enforceable Properties of Finite Groups

William DeMeo

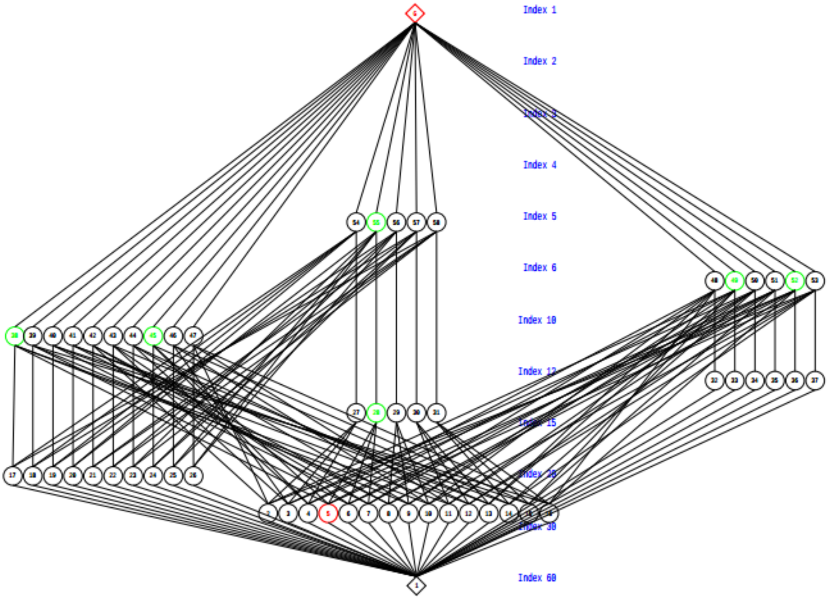
williamdemeo@gmail.com

University of South Carolina

AMS Southeastern Sectional Meeting

October 5, 2013

University of Louisville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Historically, much work has focused on:

- inferring properties of a group G from the structure of its lattice of subgroups $\mathrm{Sub}(G)$;
- inferring lattice theoretical properties of $\mathrm{Sub}(G)$ from properties of G.

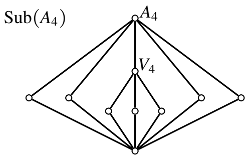
For some groups, $\mathrm{Sub}(G)$ determines G up to isomorphism.

EXAMPLES

The Klein 4-group, V_4.

The alternating groups, A_n $(n \geqslant 4)$.

Every finite nonabelian simple group.



For other groups, $\mathrm{Sub}(G)$ is isomorphic to the subgroup lattices of all groups in an infinite class of nonisomorphic groups.

EXAMPLES

$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of prime order.

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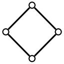
$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of order pq.

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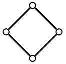
$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of order pq.

At the other extreme, there are finite lattices that are not subgroup lattices.

Example?

We are interested in the local structure of subgroup lattices, that is, the possible ***intervals***

$$[\![H, K]\!] := \{X \mid H \leqslant X \leqslant K\} \leqslant \mathrm{Sub}(G)$$

where $H \leqslant K \leqslant G$.

We restrict our attention to ***upper intervals***, where $K = G$, and ask two questions:

1. *What intervals $[\![H, G]\!]$ are possible?*
2. *What properties of a group G can be inferred from the shape of an upper interval in $\mathrm{Sub}(G)$?*

1. What intervals $[\![H, G]\!]$ are possible?

There is a remarkable theorem relating this question to the *finite lattice representation problem* (FLRP).

Theorem (Pálfy and Pudlák(1980))

The following statements are equivalent:

(A) *Every finite lattice is isomorphic to the congruence lattice of a finite algebra.*

(B) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

2. What properties of G can be inferred from $[\![H,G]\!]$?

A group theoretical property $\mathcal{P}$ is

- ***interval enforceable*** (IE) provided there exists a lattice L such that

  if $G \in \mathfrak{G}$ and $L \cong [\![H,G]\!]$, then G has property $\mathcal{P}$.

- ***core-free interval enforceable*** (cf-IE) provided $\exists L$ st

  if $G \in \mathfrak{G}$, $L \cong [\![H,G]\!]$, H core-free, then G has property $\mathcal{P}$.

- ***minimal interval enforceable*** (min-IE) provided $\exists L$ st
  if $G \in \mathfrak{G}$, $L \cong [\![H,G]\!]$, and if G has minimal order (wrt $L \cong [\![H,G]\!]$), then G has property $\mathcal{P}$.

Nonsolvability

There exist finite lattices that cannot occur as upper intervals in the subgroup lattices of finite *solvable* groups.

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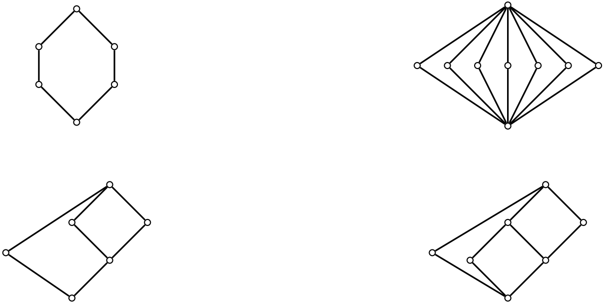
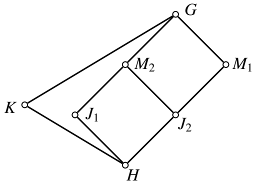
There exist finite lattices that cannot occur as upper intervals in the subgroup lattices of finite *solvable* groups.

Here are a few



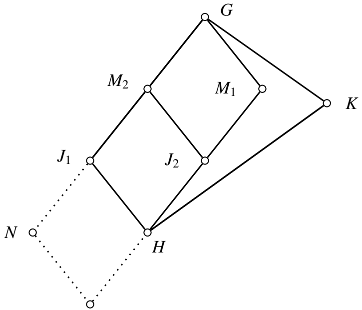


PROPOSITION

Suppose $H < G$, $\operatorname{core}_G(H) = 1$, and $L_7 \cong [\![H, G]\!]$. Then

(I) *G is a primitive permutation group.*
(II) *If $N \lhd G$, then $C_G(N) = 1$.*
(III) *G contains no nontrivial abelian normal subgroup.*
(IV) *G is not solvable.*
(V) *G is subdirectly irreducible.*
(VI) *With the possible exception of at most one maximal subgroup, M_1 or M_2, all proper subgroups in the interval $[\![H, G]\!]$ are core-free.*

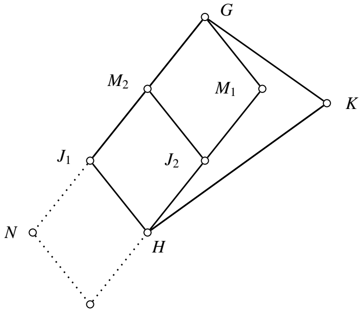
Example



Claim: J_1 and J_2 are core-free subgroups of G.

Proof:

Example



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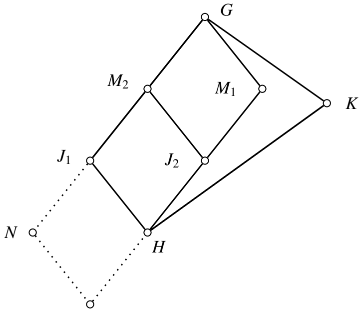

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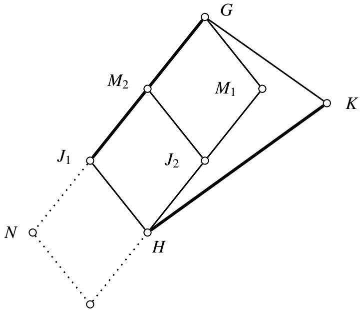

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- If $N \lhd G$ then NH permutes with each subgroup containing H.
- If $1 \neq N \leqslant J_1$, then $NH = J_1$, so J_1 and K permute.
- Since $J_1K = G$ and $J_1 \cap K = H$,

$$[\![J_1, G]\!] \cong [\![H, K]\!]^{J_1} = \{X \in [\![H, K]\!] \mid J_1X = XJ_1\}.$$

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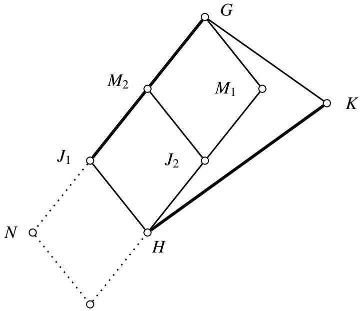

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Impossible!

The following are at least core-free interval enforceable:

- $\mathscr{G}_0 = \mathfrak{S}^c =$ the nonsolvable groups
- $\mathscr{G}_1 = \{G \in \mathfrak{G} \mid (\forall n < \omega)\ (G \neq A_n \text{ and } G \neq S_n)\}$
- $\mathscr{G}_2 =$ the subdirectly irreducible groups
- $\mathscr{G}_3 =$ groups with no nontrivial abelian normal subgroups
- $\mathscr{G}_4 = \{G \in \mathfrak{G} \mid C_G(M) = 1 \text{ for all } 1 \neq M \trianglelefteqslant G\}$.

If a lattice L is isomorphic to an interval in the subgroup lattice of a finite group, then we call L ***group representable***.

By the Pálfy-Pudlák Theorem, the FLRP has a negative answer if we can find a finite lattice that is not group representable.

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Suppose there exists property $\mathcal{P}$ such that both $\mathcal{P}$ and its negation $\neg\mathcal{P}$ are interval enforceable by the lattices L and L_c, respectively:

$$L \cong [\![H, G]\!] \implies G \text{ has property } \mathcal{P}$$

$$L_c \cong [\![H_c, G_c]\!] \implies G_c \text{ does not have property } \mathcal{P}$$

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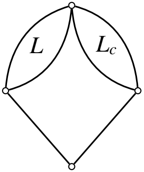
Then the lattice



could not be group representable.

But if a group property and its negation are interval enforceable by L and L_c, then already at least one of these lattices is not group representable.

LEMMA

If $\mathcal{P}$ is a group property that is interval enforceable by a group representable lattice, then $\neg\mathcal{P}$ is not interval enforceable by a group representable lattice.

Nonsolvability is interval enforceable, but solvability is not.

For if $L \cong [\![H, G]\!]$, then for any nonsolvable group K we have $L \cong [\![H \times K, G \times K]\!]$, and $G \times K$ is nonsolvable.

Note that the group $H \times K$ at the bottom of the interval is not core-free. So a more interesting question is whether a property and its negation could both be *core-free* IE.

Conjecture

If $\mathcal{P}$ is core-free interval enforceable by a group representable lattice, then $\neg\mathcal{P}$ is not core-free interval enforceable by a group representable lattice.

Any class of groups that omits certain wreath products cannot be core-free interval enforceable by a group representable lattice.

THEOREM

Suppose $\mathcal{P}$ is core-free interval enforceable by a group representable lattice. Then, for any finite nonabelian simple group S, there exists a wreath product group of the form $W = S \wr U$ that has property $\mathcal{P}$.

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COROLLARY

Solvable is not a core-free interval enforceable property.

Any class of groups that omits certain wreath products cannot be core-free interval enforceable by a group representable lattice.

Theorem

Suppose $\mathcal{P}$ is core-free interval enforceable by a group representable lattice. Then, for any finite nonabelian simple group S, there exists a wreath product group of the form $W = S \wr U$ that has property $\mathcal{P}$.

Corollary

Solvable is not a core-free interval enforceable property.

Corollary

Almost simple is not a core-free interval enforceable property.

Proof Sketch

Let L be a group representable lattice such that if $L \cong [\![H, G]\!]$ and $\mathrm{core}_G(H) = 1$ then G has property $\mathcal{P}$.

Since L is group representable, $\exists\, G$ with $L \cong [\![H, G]\!]$.

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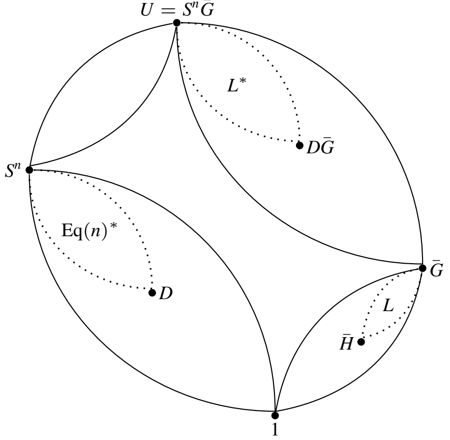
Since L is group representable, $\exists\, G$ with $L \cong [\![H, G]\!]$.

Use Kurzweil's method for representing duals of lattices (twice):

- Fix a finite nonabelian simple group S.
- Suppose the index of H in G is $|G : H| = n$.
- Then the action of G on the cosets of H induces an automorphism of the group S^n by permutation of coordinates.
- Denote this by $\varphi : G \to \mathrm{Aut}(S^n)$, and let $\varphi(G) = \bar{G} \leqslant \mathrm{Aut}(S^n)$.



The interval $[\![D, S^n]\!]$ is isomorphic to $\mathrm{Eq}(n)^*$, the dual of the lattice of partitions of an n-element set.

The dual lattice L^* is an upper interval of $\mathrm{Sub}(U)$, namely, $L^* \cong [\![D\bar{G}, U]\!]$.

The only technical part is proving the

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So a class of groups that omits wreath products of the form $S \wr G$, where S is an arbitrary finite nonabelian simple group, is not a core-free interval enforceable class.

Examples: the class of solvable groups, the class of almost simple groups.

Theorem

The following statements are equivalent:

(B) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

(C) *For every finite lattice* L *and every finite collection* $\mathscr{G}_1, \dots, \mathscr{G}_n$ *of cf-IE classes of groups,*

$$\exists\, G \in \bigcap_{i=1}^{n} \mathscr{G}_i \text{ such that } L \cong [\![H, G]\!] \text{ and } \operatorname{core}_G(H) = 1.$$

(D) *For every finite collection* $\mathscr{L}$ *of finite lattices, there exists a finite group* G *such that each* $L_i \in \mathscr{L}$ *is isomorphic to* $[\![H_i, G]\!]$ *for some core-free subgroup* $H_i \leqslant G$.

By (C), the FLRP would have a negative answer if we could find a collection $\mathscr{G}_1, \ldots, \mathscr{G}_n$ of cf-IE classes such that $\bigcap_{i=1}^{n} \mathscr{G}_i$ is empty.

By (D), it makes sense to consider finite collections of finite lattices and ask what can be proved about a group G if one assumes that all of these lattices are isomorphic to upper intervals of $\mathrm{Sub}(G)$.

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   What about property enforceable intervals?

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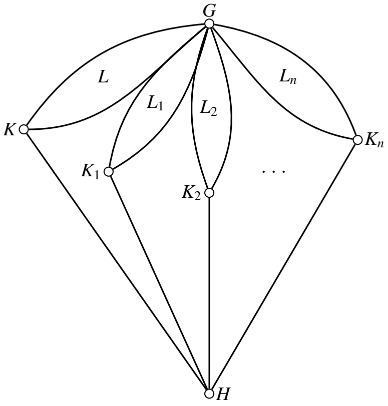
   What about property enforceable intervals?

2. Could PE intervals be used to prove Nice Boring Theorems? (cf. Peter Mayr's talk)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