Residually Finite Varieties

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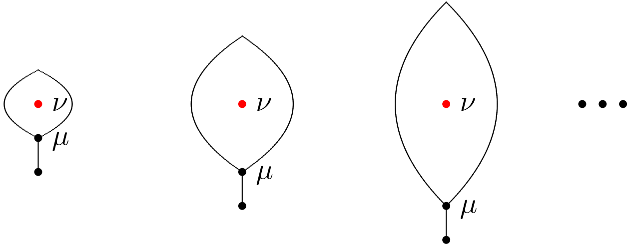
Conjecture

If $\mathcal{V}$ has finite type and satisfies $\mathrm{typ}\{\mathcal{V}\} \subseteq \{\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$, *then $\chi_{\mathcal{V}} \neq \omega$.*

How to prove $\chi_{\mathcal{V}} \neq \omega$

SI's: A_1, A_2, A_3, $|A| \nearrow \omega$

Con's:



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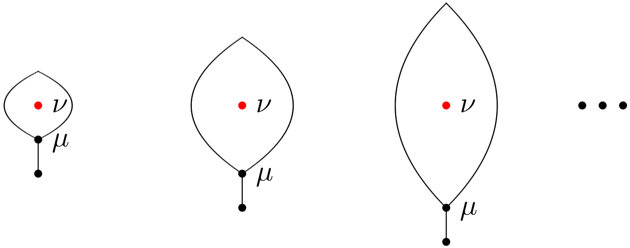

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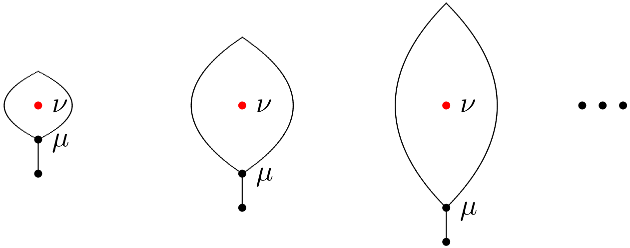

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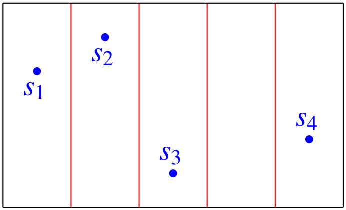

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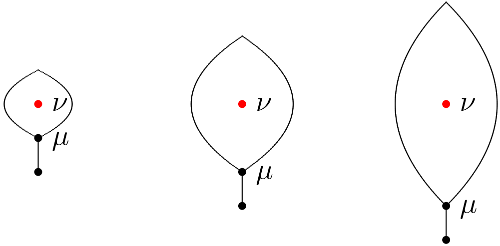

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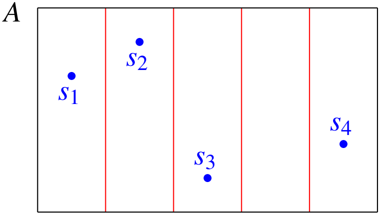

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$S = \{s_1, s_2, \ldots\}$
$= \text{a noncentralizing set}$

S 'noncentralizing' means: if $\{s,t\} \in S^{(2)}$ and $\{a,b\} \in A^{(2)}$, then $[\mathrm{Cg}(s,t), \mathrm{Cg}(a,b)] \geq [\mathrm{Cg}(s,t), \mu] > 0$.

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Key fact: If $\chi_{\mathcal{V}} \leq \omega$, $\mathcal{V}$ has finite type, and $\mathcal{V}$ omits nontrivial rectangular congruences, then "$[\mathrm{Cg}(w,x), \mathrm{Cg}(y,z)] = 0$" is definable.

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Hence $\exists$ formula $\Gamma(p,q,u,v)$ such that
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Example: (Rings, $\mathrm{Cg}(a,b) = \mathrm{Cg}(a-b,0)$)

$$[\mathrm{Cg}(a,0), \mathrm{Cg}(c,0)] = 0 \quad \Leftrightarrow (a)(c) = (0) = (c)(a)$$

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$$\Gamma(e, 0, a, 0) \quad = \exists s(e = sa \text{ or } e = as)$$

Bound the size of noncentralizing sets in SI's

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\{a, b\} \arrow[r, "\Gamma"] & \{u_1, v_1\} \arrow[r, "\Gamma"] & \{u_2, v_2\} \arrow[r, "\Gamma"] & \bullet\bullet\bullet \arrow[r, "\Gamma"] & \{u_m, v_m\}
\end{tikzcd}$$

Bound the size of noncentralizing sets in SI's

For pcf Γ', write $S \stackrel{\Gamma'}{\to} \{u,v\}$ to mean $A \models \Gamma'(u,v,s,t)$ for all $\{s,t\} \in S^{(2)}$.

Claim. $\exists$ pcf Γ_n such that for any noncentralizing $S \subseteq A \in \mathcal{V}_{SI}$, $|S| = n$, $\forall \{a,b\} \in A^{(2)}\ \exists \{u,v\} \in A^{(2)} (\{a,b\} \stackrel{\Gamma_n}{\to} \{u,v\} \stackrel{\Gamma_n}{\leftarrow} S)$:

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 & & & & \{u,v\}
\end{tikzcd}$$

Unbounded noncentralizing sets yield infinite SI's

If SI's have noncentralizing sets of unbounded size, then it is consistent with $\mathrm{Th}(\mathcal{V})$ that some SI $A \in \mathcal{V}$ contains elements related as follows.

$$\begin{tikzcd}
 & S_5 \arrow[d, "\Gamma_5"] & S_4 \arrow[d, "\Gamma_4"] & S_3 \arrow[d, "\Gamma_3"] & S_2 \arrow[d, "\Gamma_2"] \\
\bullet\ \bullet\ \bullet \arrow[r, "\Gamma_5"] & \{u_5, v_5\} \arrow[r, "\Gamma_4"] & \{u_4, v_4\} \arrow[r, "\Gamma_3"] & \{u_3, v_3\} \arrow[r, "\Gamma_2"] & \{u_2, v_2\}
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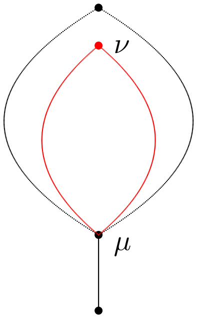
Hence $\mathcal{V}$ has an infinite SI.

Bound on $|S|$ = bound on $[1 : \nu]$

Hypotheses

Con(A), A SI



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- $\chi\nu \leq \omega$
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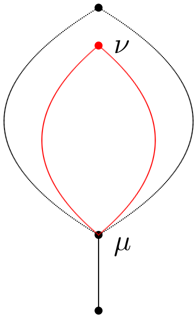

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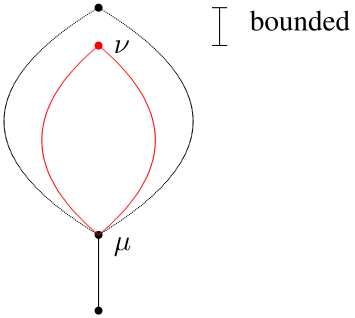

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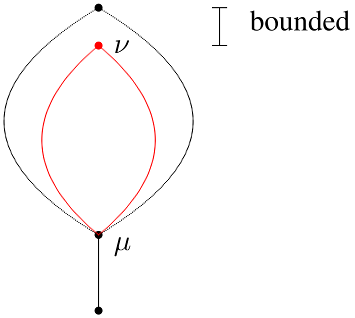

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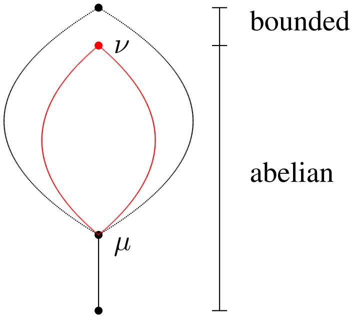

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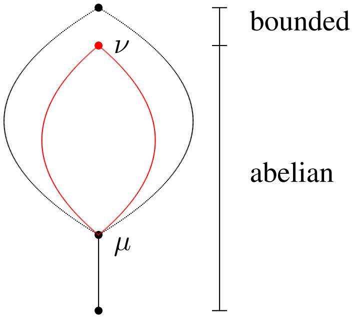

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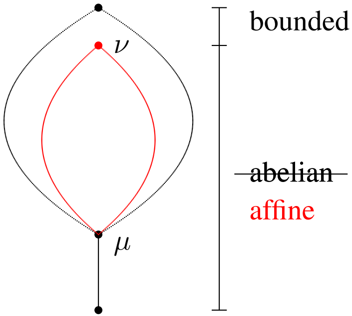

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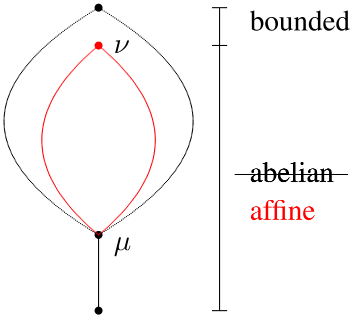

If $[\mathrm{Cg}(w, x), \mathrm{Cg}(y, z)] = 0$ is definable and $\mathcal{V}$ omits type **1**, then proving $\chi_{\mathcal{V}} \neq \omega$ can be reduced to the case of modules.

Varieties of modules

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Corollary

If $[\mathrm{Cg}(w, x), \mathrm{Cg}(y, z)] = 0$ *is definable, and* $\mathcal{V}$ *omits type* $\mathbf{1}$, *then* $\chi_{\mathcal{V}} \neq \omega$.