Maltsev Conditions on the Feder-Vardi Reduction to Bipartite Graphs with Constants

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Plan

- Willard's Lemma about bipartite graphs and two sorted structures
- The Feder-Vardi construction.
- Identities that aren't preserved
- Identities that are preserved

Definition

Let $\mathbb{G} = (G, E)$ be a bipartite graph with bipartition $G = A \cup B$. That is, $E \subseteq A \times B \cup B \times A$ and $A \cap B = \varnothing$. Define $\overrightarrow{\mathbb{G}}$ to be the two sorted structure $(A, B; \overrightarrow{E})$ where $\overrightarrow{E} = E \cap A \times B$.

Definition

A polymorphism of $\overrightarrow{\mathbb{G}}$ is a pair $f = (f^A, f^B)$ of n-ary operations on A and B, respectively, that together preserve $E \cap A \times B$.

$$
\begin{array}{c}
\begin{array}{cc} f^A & f^B \\ \downarrow & \downarrow \end{array} \\
(a_1, b_1) \in \overrightarrow{E} \\
(a_2, b_2) \in \overrightarrow{E} \\
\vdots \\
(a_n, b_n) \in \overrightarrow{E} \\
\hline
(f^A(\overline{a}), f^B(\overline{b})) \in \overrightarrow{E}
\end{array}
$$

Lemma (Willard)

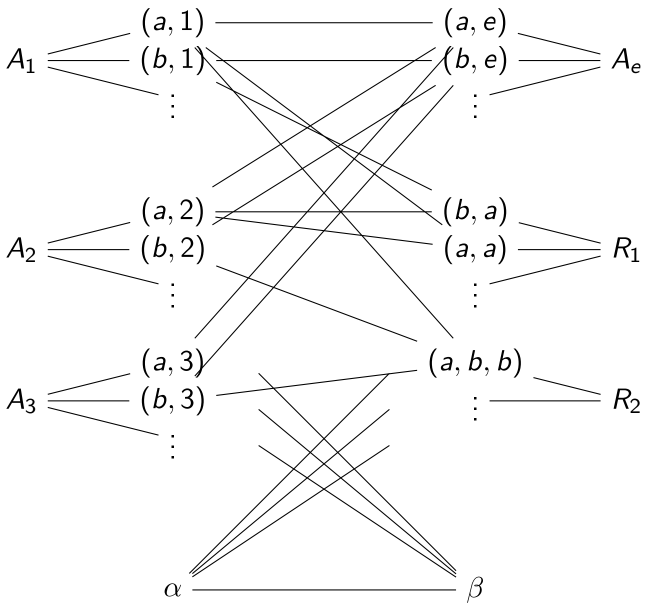
Let Σ be a Maltsev condition satisfied by the two element connected graph mentioning only two variables, and $\mathbb{G}$ be a bipartite graph. Then $\mathbb{G}$ satisfies Σ if and only if $\overrightarrow{\mathbb{G}}$ does.

- For a relational structure to satisfy an identity means that it has polymorphisms that do.
- For a two sorted structure to satisfy an identity, such as $s \approx t$ means that it has polymorphisms (s^A, s^B) and (t^A, t^B) satisfying $s^A \approx t^A$ and $s^B \approx t^B$.

The Feder-Vardi Construction

Let $\mathbb{A} = (A, \mathcal{R})$ be a relational structure. Let $\mathcal{R} = \{R_1, \ldots, R_m\}$, and k be the maximum arity of the R_i. The Feder-Vardi graph is constructed as follows:

- For $i = 1, \ldots, k$, there are vertices corresponding to each element of $A \times \{i\}$
- For $i = 1, \ldots, k$, there is a vertex A_i such that its neighbourhood is exactly $A \times \{i\}$.
- There is a vertex for each element of A, and each $a \in A$ is adjacent to (a, i) for each i.
- There is a vertex, A_e, adjacent to each $a \in A$.
- There is a vertex for each tuple in each relation, and $(x_1, x_2, \ldots, x_\ell)$ is adjacent to (x_i, i) for $i = 1, \ldots, \ell$.
- There is a vertex, R_i for each $R_i \in \mathcal{R}$ whose neighbourhood is precisely the set of tuples it contains.
- There is a vertex, α, adjacent to each tuple in each relation, and the elements of A.
- There is a vertex, β, adjacent to (a, i) for each $a \in A$ and $i = 1, \ldots, k$. It is also adjacent to α.



A_1
A_2
A_3
$(a,1)$
$(b,1)$
$(a,2)$
$(b,2)$
$(a,3)$
$(b,3)$
(a,e)
(b,e)
(b,a)
(a,a)
(a,b,b)
A_e
R_1
R_2
α
β

Properties

Denote by $\mathbb{G}_{\mathbb{A}}$ the graph obtained in this way from $\mathbb{A}$.

1. $\mathbb{G}_{\mathbb{A}}$ is always bipartite
2. $\mathrm{CSP}((\mathbb{G}_{\mathbb{A}})^c) \equiv_P \mathrm{CSP}(\mathbb{A})$. [^{c} means “with constants"]
3. $\mathbb{A}$ pp-interprets in $(\mathbb{G}_{\mathbb{A}})^c$, but not the other way around.
4. Item 3 implies that any idempotent Maltsev condition satisfied by $\mathbb{G}_{\mathbb{A}}$ is satisfied by $\mathbb{A}$. So the interesting questions about preservation of idempotent Maltsev conditions are of the form “if $\mathbb{A}$ satisfies Σ, does $\mathbb{G}_{\mathbb{A}}$?

Some polymorphisms were never meant to happen

Definition

Let $\mathbb{G} = (G, E)$ be a graph. For $a \in G$, define $N(a) = \{b \in G : (a, b) \in E\}$.

Lemma

Let $\mathbb{G}$ be a graph containing a 6 cycle, $u - a - c - v - d - b$ satisfying

1. $N(u) \cap N(c) = \{a\}$
2. $N(a) \cap N(v) = \{c\}$.

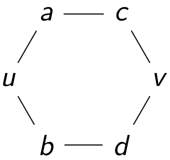
Then $\mathbb{G}$ has no NU-polymorphism, no edge polymorphism, and no Maltsev polymorphism.

Proof of Lemma

For Maltsev.

Let m be an idempotent polymorphism of $\mathbb{G}$ satisfying $m(x, y, y) \approx x$. We will show that $m(a, a, b) = a$.

- By assumption, $m(a, b, b) = a$ and $m(v, b, b) = v$.
- $m(c, u, d)$ is adjacent to both $m(a, b, b)$ and (v, b, b) since m is a polymorphism.
- This forces $m(c, u, d) = c$.
- $m(a, a, b)$ is adjacent to $m(c, u, d) = c$ and $m(u, u, u) = u$.
- Therefore, $m(a, a, b) = a$.



$\square$

What about the Feder-Vardi graphs?

Theorem

Let $\mathbb{A} = (A, \mathcal{R})$ be a relational structure with $|A| \geq 2$, and at least one nonempty relation with positive arity. Then the Feder-Vardi graph, $\mathbb{G}_{\mathbb{A}}$ of $\mathbb{A}$ has no NU-polymorphism, no edge polymorphism, and no Maltsev polymorphism.

Proof.

Let $a, b \in A$ with $a \neq b$. Then $A_1 - (a, 1) - a - A_e - b - (b, 1)$ satisfies the conditions of the lemma. □

Darn! No matter what polymorphism $\mathbb{A}$ satisfies, its Feder-Vardi graph can't possibly have any of these nice polymorphisms. Is anything preserved?

Some Maltsev conditions do survive

- Thankfully, $(\mathbb{G}_{\mathbb{A}})^c$ has a WNU polymorphism if and only if $\mathbb{A}$ does.
- In fact, any identity of the form $f(\text{variables}) \approx g(\text{variables})$ is satisfied by one iff it is satisfied by the other. This includes an unpublished Maltsev condition due to Kozik, Krokhin, Valeriote, and Willard for omitting types **1** and **2**.
- If $\mathbb{A}$ has n-permutable Hagemann-Mitchke polymorphisms for some $n \geq 2$, then $(\mathbb{G}_{\mathbb{A}})^c$ has $(n+4)$-permutable Hagemann-Mitchke polymorphisms.
- If $\mathbb{A}$ has Hobby-Mckenzie polymorphisms (for omitting types **1** and **5**), so does $(\mathbb{G}_{\mathbb{A}})^c$. (Need to add 8 extra operations).

Our proofs of these results rely on the result stated about two-sorted structures.

Thank you!

References and Acknowledgements

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