

# Vanderbilt Open Problems List 2015

July 3, 2015

## 1 Maltsev conditions

**Question 1** (Matt Valeriote). Is there a strong Maltsev condition for a difference term for finite algebras? If so, what is it?

*Context.* It is known [5] that having a difference term corresponds to a Maltsev condition.

**Question 2** (Ralph McKenzie). Let  $V$  be a variety. Is it true that  $V$  has a difference term if and only if for every  $\mathbf{A} \in V$  and every  $\alpha, \beta \in \text{Con}(\mathbf{A})$  with  $\alpha$  Abelian we have  $\alpha \circ \beta = \beta \circ \alpha$ ?

**Question 3** (Ralph McKenzie). Is there a  $M$  strong Maltsev condition such that a variety  $V$  has a Taylor term iff  $V$  satisfies  $M$ ?

**Question 4** (Ralph McKenzie). Is there a  $M$  strong Maltsev condition such that a variety  $V$  is congruence meet semidistributive ( $\text{SD}(\wedge)$ ) iff  $V$  satisfies  $M$ ?

*Context.* Both above questions have positive answers for locally finite varieties.

**Question 5** (Libor Barto). Is congruence singularity (CS) characterized by a linear Maltsev condition for finitely generated varieties?

*Context.* CS means that for all (finite)  $\mathbf{A} \in V$  and all  $\alpha, \beta \in \text{Con}(\mathbf{A})$ , we have

$$|a/\alpha \wedge \beta| \cdot |a/\alpha \vee \beta| = |a/\alpha| \cdot |a/\beta|.$$

CS is known to be a Maltsev condition for finitely generated varieties.

**Question 6** (Libor Barto). Fill in the blanks: “... + finitely related = ...”.

*Context.* Two famous ways to fill in the blanks are “CM + finitely related = few subpowers” and “CD + finitely related = NU”. Are there more?

**Question 7** (Libor Barto). Let  $\mathbf{A}$  be an algebra that is

- a) idempotent and topologically closed (i.e. if  $f: A^n \rightarrow A$  is an operation such that for any finite  $B \subseteq A$  we have a  $g$  operation of  $\mathbf{A}$  such that  $g|_B = f|_B$  then  $f$  is also an operation of  $\mathbf{A}$ ), or
- b) oligomorphic (i.e. ) and topologically closed.

If the 2-element naked set (no nontrivial idempotent operations) can be found in  $HSP(\mathbf{A})$ , can we find it in  $HSP_{fin}(A)$ ?

*Context.* This double question is interesting for infinite algebras (it is known to be true for finite algebras). If having a Taylor term is not a strong Maltsev condition (which we don’t know) then the answer to a) above is “no”.

Naked set is an algebra with no nontrivial (non-projection) idempotent operations.

An algebra is oligomorphic if the following holds: Let  $G$  be the group of all unary terms of  $\mathbf{A}$  for which there is an inverse term in  $\mathbf{A}$ . Then for all  $n \in \mathbb{N}$  the number of orbits of  $G$  acting componentwise on  $A^n$  is finite. One can’t have an infinite algebra that would be oligomorphic and idempotent at the same time.

In general, there are algebras that have the 2-element naked set in  $HSP(\mathbf{A})$ , but not in  $HSP_{fin}(\mathbf{A})$ .

**Question 8** (Alex Kazda). Let  $\mathbf{A}$  be a finite 4-permutable algebra. Find an algorithm that solves  $CSP(\mathbf{A})$  in polynomial time.

**Question 9** (Jakub Opršal). Give a nice examples of 4-permutable finite algebras that are not congruence meet semidistributive, nor congruence modular.

**Question 10** (Alex Wires). Let  $\mathbf{A}$  be a 3-permutable algebra. Find an algorithm for  $CSP(\mathbf{A})$  that does not need to use the edge term.

**Question 11** (Ross Willard). Does there exist a polynomial time algorithm that, given a  $G$  finite bipartite graph would decide if  $G$  has a set of Hagemann-Mitschke polymorphisms?

*Context.* Hagemann-Mitschke terms witness  $n$ -permutability [4].

The goal here is to generalize from bipartite graphs to general graphs and to do the same for congruence modularity.

**Question 12** (Jakub Opršal). Is congruence modularity a linear meet of Maltsev and congruence distributivity?

*Context.* The meet here is taken in the lattice of interpretability of linear Maltsev conditions. Meet of two Maltsev conditions is the strongest linear Maltsev condition implied by each of the two original conditions.

**Question 13** (Ross Willard). Why are all interesting Maltsev conditions expressible in two variable equations?

**Question 14** (Libor Barto). Which implications among Maltsev conditions can be extended from the locally finite to oligomorphic closed clones by way of oligomorphic adjustment by automorphisms (i.e.  $t(x, y) = t(y, x)$  becomes  $t(x, y) = \sigma t(\theta(y), \lambda(x))$ )? Here  $\sigma, \theta, \lambda$  belong to the group of all unary terms of  $\mathbf{A}$  for which there is an inverse term in  $\mathbf{A}$ .

*Context.* We prefer closed clones, because they precisely correspond to polymorphisms of relational structures.

**Question 15** (Matt Moore). Which finite  $\text{SD}(\wedge)$  algebras are finitely related?

*Context.* We know that having a cube term is a sufficient condition here, but it is not necessary (consider the two element semilattice).

**Question 16** (Ralph Freese). Let  $V, W$  be idempotent congruence permutable varieties. Find Hagemann-Mitschke terms and the cube term for the variety  $V \circ W$  (where  $\circ$  denotes Maltsev product of varieties).

*Context.* It is known ([2]) that if  $V, W$  are idempotent congruence permutable varieties then  $V \circ W$  (join of varieties) is 3-permutable (it need not be permutable), but nobody knows how to write the Hagemann-Mitschke term for  $V \circ W$  as a composition of Maltsev operations in  $V, W$ .

We also know that  $V \circ W$  has a cube term since having a cube term (we don't need 3-permutability to prove this here since cube terms are robust with respect to  $\circ$ ).

**Question 17** (Cliff Bergman). Assume that  $V, W$  are idempotent varieties and that  $V \vee W$  (join of varieties of the same type with respect to inclusion; *not* with respect to interpretability) is 3-permutable. Is it then true that  $V \circ W$  is 3-permutable?

*Context.* It is known that for  $V, W$  idempotent we have “ $V \vee W$  permutable  $\Rightarrow V \circ W$  permutable.”

**Question 18** (folklore). Let  $P$  be a finite bounded poset, let  $\mathcal{C}$  be the clone of polymorphisms of  $P$ . Is it true that  $\mathcal{C}$  is a finitely generated clone iff  $\mathcal{C}$  contains an NU?

*Context.* “ $\Leftarrow$ ” is known by the Baker-Pixley theorem.

Smallest poset without NU is the Tardős’ poset [9]. The clone of polymorphisms of that poset does not have NU and is not finitely generated.

Also known is Larose and Zádori’s characterization of posets admitting an NU [6].

**Question 19** (Matt Valeriote). We say that a strong Maltsev condition  $M$  is local for idempotent algebras if there exists an  $n$  such that for every idempotent  $\mathbf{A}$  if for every  $B \subseteq A$  of size at most  $n$  there exist terms of  $\mathbf{A}$  that satisfy  $M$  for elements of  $B$  then  $\mathbf{A}$  as a whole satisfies  $M$ .

Which strong Maltsev conditions are local for idempotent algebras?

*Context.* The only provably non-local strong Maltsev condition we have so far is minority operation (see Question 20). On the other hand, we know that Maltsev operation,  $n$ -permutability (Hagemann-Mitschke terms),  $NU(k)$ , cube term,  $CD(k)$  and  $CM(k)$  are local for idempotent algebras. (Weak near unanimity of fixed arity is local if we restrict ourselves to Taylor idempotent algebras.)

**Question 20** (Alex Kazda). What is the computational complexity of deciding if  $\mathbf{A}$  finite idempotent algebra given by tables of its basic operations has a minority operation?

*Context.* We know that having local minorities (see Question 19) is not enough thanks to Dmitriy Zhuk’s example of idempotent algebras that have local minority operations (there is such an algebra for each value of  $n$ ) but no global minority.

## 2 Clones

**Question 21** (Ross Willard). Ralph McKenzie provided in [7] an algebra  $\mathbf{A}$  that has 4 elements, the variety generated by  $\mathbf{A}$  has residual character  $\omega$  and  $\text{Clo } \mathbf{A}$  is not finitely generated. Decide if  $\text{Clo } \mathbf{A}$  is finitely related. (Guess: No.)

**Question 22** (Dmitriy Zhuk). Fix a base set  $A$ . Let  $\mathcal{E}$  be the set of all clones on  $A$  generated by a single edge operation. Let  $C$  be an inclusion minimal member of  $\mathcal{E}$ . Is it true that  $C$  is always generated by a 4-ary edge operation?

**Question 23** (Dmitriy Zhuk). Fix a base set  $A$ . Let  $\mathcal{E}$  be the set of all clones on  $A$  generated by a single NU operation. Let  $C$  be an inclusion minimal member of  $\mathcal{E}$ . Is it true that  $C$  is always generated by a majority operation?

*Context.* An algebra  $\mathbf{A}$  has a  $k$ -ary NU if and only if  $\mathbf{A}$  is CD and has  $k+1$ -ary edge term, which makes the switch from 4-ary edge term to a majority operation plausible.

Both above questions have a positive answer when  $|A| = 2$ . In the Post's lattice, the only two clones that have an edge term and do not have a subclone with an edge term are:

- the clone generated by  $p(x, y, z) = x + y + z \pmod{2}$  and
- the clone generated by a majority operation.

The latter clone is also the only clone minimal among all clones with an NU operation.

**Question 24** (Dmitriy Zhuk). Give complexity classification or practical algorithms that decide if two definitions give the same clone. Cases:

- Given two finite sets of operations  $F_1, F_2$ , decide if  $\text{Clo}(F_1) = \text{Clo}(F_2)$ . (This is decidable.)
- Given two finite relations  $\rho_1, \rho_2$ , decide if  $\text{Pol}(\rho_1) = \text{Pol}(\rho_2)$ . (This is decidable.)
- Given a finite relation  $\rho$  and a finite set of operations  $F$ , decide if  $\text{Pol}(\rho) = \text{Clo}(F)$ . (We don't know if this is even decidable.)

- Given three finite sets of operations  $A, B, C$ , decide if  $\text{Clo}(A) \cap \text{Clo}(B) = \text{Clo}(C)$ . (We don't know if this is even decidable.)

**Question 25** (Dmitriy Zhuk). Given a clone, decide how many clones contain this clone. Variants:

- Given a finite set of operations  $F$ , decide how many clones contain  $\text{Clo}(F)$ . (Decidability unknown.)
- Given an operation, decide how many clones contain this operation. (Decidability unknown.)
- Given a weak near unanimity operation, decide how many clones contain this operation. (Decidability unknown.)
- Given a Maltsev operation, decide how many clones contain this operation. (Decidability unknown.)
- Given a relation  $\rho$ , decide how many clones contain  $\text{Pol}(\rho)$ . (Decidability unknown.)

**Question 26** (Dmitriy Zhuk). Find out how many clones contain a given minimal clone on 4 elements.

*Context.* All minimal clones on 4 elements were found (by combining results of Pöschel, Kaluzhnin, Szczepara, Waldhauser, Jezek, Quackenbush and Karsten Scholzer, the last from 2012)

**Question 27** (Dmitriy Zhuk). Find all minimal clones that are contained only in *finitely* many clones (i.e. correspond to a finite principal filter in the clone lattice).

**Question 28** (Dmitriy Zhuk). Find all minimal clones that are contained only in *countably* many clones (i.e. correspond to a finite principal filter in the clone lattice).

**Question 29** (Dmitriy Zhuk). Decide if a clone is finitely generated. Cases:

- Given a relation  $\rho$ , decide if  $\text{Pol}(\rho)$  is finitely generated. (Decidability unknown.)
- As above, but assume that  $\rho$  is a partial order relation with greatest and least element. (Decidability unknown.)

- Given two sets of operations  $B, C$ , decide if  $\text{Clo}(B) \cap \text{Clo}(C)$  is finitely generated. (Decidability unknown.)

*Context.* See [3] for finitely generated clones whose intersection is not finitely generated.

**Question 30** (Dmitriy Zhuk). Given a clone by a set of operations, decide whether the clone is finitely related. (Decidability unknown.)

**Question 31** (Dmitriy Zhuk). Given a clone  $C$  by a finite list of operations and knowing that  $C$  is finitely related, find a relation that defines this clone. (Complexity unknown.)

**Question 32** (Dmitriy Zhuk). Given a relation  $\rho$  such that  $\text{Pol}(\rho)$  is finitely generated, find a generating set for  $\text{Pol}(\rho)$ . (Complexity is unknown.)

**Question 33** (Dmitriy Zhuk). Does there exist a minimal clone that is not finitely related? Does there exist a minimal clone on 3 elements that is not finitely related?

**Question 34** (Dmitriy Zhuk). Given two finite sets of operations  $C$  and  $D$  such that  $C \subseteq D$  and the sublattice between two clones  $\text{Clo}(C)$  and  $\text{Clo}(D)$  is finite, find this sublattice. (Decidability unknown.)

**Question 35** (Dmitriy Zhuk). Does there exist a finite maximal chain of size greater than 52 on 3 elements?

**Question 36** (Dmitriy Zhuk). Is it true that a finite maximal chain of the maximal size always contains a minimal majority operation?

*Context.* The two above questions stem from the fact that there is a 52 element maximal chain of clones on three elements with a minimal majority operation.

**Question 37** (Dmitriy Zhuk). Is the size of a finite maximal chain of clones on  $A$  finite bounded?

*Context.* The bound is 7 when  $|A| = 2$ , we don't know for  $|A| = 3$ .

**Question 38** (Dmitriy Zhuk). Find a set of relations  $\Gamma$  of arity at most  $m$  such that  $\text{Pol}(\Gamma)$  is finitely generated, but any generating set of  $\text{Pol}(\Gamma)$  has to contain an operation of arity greater than  $|A|^m$  (or, alternatively,  $2^{2^m}$ ).

**Question 39** (Dmitriy Zhuk). Find a set of operations  $C$  of arity at most  $m$  such that  $\text{Clo}(C)$  is finitely related, but whenever  $\text{Clo}(C) = \text{Pol}(\Gamma)$  for some  $\Gamma$  set of relations, then  $\Gamma$  contains a relation of arity greater than  $|A|^m$  (or, alternatively,  $2^{2^m}$ ).

### 3 Commutators

**Question 40** (Alex Wires). Generalize results from commutator theory to the  $n$ -ary commutator by Bulatov [1] (see Question 41 for a concrete example of that).

**Question 41** (Alex Wires). Fill in the blank: An algebra  $\mathbf{A}$  is supernilpotent if and only if  $\mathbf{A}$  is polynomially equivalent to  $\dots$

*Context.* One guess for the “ $\dots$ ” above is “affine algebra”. See the results on 2-supernilpotent algebras by Nebojsa Mudrinski [8].

### 4 Algorithmic topics

**Question 42** (Ross Willard). Is the following decidable? Given  $\mathbf{A}$  finite algebra and  $n \in \mathbb{N}$ , are all subdirectly irreducible algebras in the variety generated by  $\mathbf{A}$  of cardinality at most  $n$ ?

*Context.* The problem is decidable in the  $\text{SD}(\wedge)$  and CM cases (because then we can decide if  $\mathbf{A}$  has a finite residual bound).

**Question 43** (Matt Valeriote). Find a strong idempotent nontrivial Maltsev condition  $M$  such that deciding if an algebra  $\mathbf{A}$  satisfies  $M$  is provably *not* EXPTIME-complete. (Modulo reasonable computational complexity conjectures.)

*Context.* Maltsev (not always strong) conditions known to be EXPTIME-complete [10]:

- omitting type 1
- omitting types 1 and 2
- omitting types 1 and 5
- omitting types 1, 2, and 5
- having a semilattice operation
- congruence modularity
- congruence distributivity



- having a chain of Jónsson terms of fixed length  $n$

Deciding a strong Maltsev condition is in EXPTIME, because we can always build up the  $n$ -generated free algebra (with  $n$  corresponding to the arities of operations in the Maltsev condition) in  $V(\mathbf{A})$  and examine all its members.

**Question 44** (Matt Valeriote). Given a general (not idempotent) algebra  $\mathbf{A}$ , how hard is it to decide if  $\mathbf{A}$ :

- a) has Maltsev operation,
- b) has a majority operation,
- c) has a Pixley operation,
- d) is primal.

**Question 45** (Matt Valeriote). Find a strong idempotent nontrivial Maltsev condition  $M$  such that deciding if an *idempotent* algebra  $\mathbf{A}$  satisfies  $M$  is provably *not* in P (modulo reasonable computational complexity conjectures.), or show that we can decide all such Maltsev conditions in polynomial time.

*Context.* Maltsev (not always strong) conditions known to be in P for  $\mathbf{A}$  idempotent:

- Jónsson terms (directed, undirected, with a fixed length of chain or with a chain of any length)
- Gumm terms (all variants)
- Maltsev operation
- $NU(k)$  for a fixed  $k$  or any  $k$
- cube term of fixed dimension or any dimension
- Pixley term
- Hagemann-Mitschke terms for fixed length of chain or any length of chain

- having some weak near unanimity operation (this is due to Bulatov, see [10, Theorem 6.3] for an algorithm) and having a weak near unanimity of a fixed arity
- having a flat semilattice operation (only two levels)

Complexity is open for (among others):

- semilattice operation (not linear, conjectured to be EXPTIME-complete even for  $\mathbf{A}$  idempotent; see [ ] for an algebra with local semilattice operations, but no global semilattice)
- minority (this is Question 20)
- Having totally symmetric idempotent operation of arity  $k$  for  $k$  fixed
- omitting types 1 and 5
- Day terms (the last two should be easy to decide).

**Question 46.** What is the complexity of deciding if a given relational structure has a Maltsev polymorphisms?

**Question 47.** How hard is it to tell if a relational structure has a Taylor polymorphism?

**Question 48.** How hard is it to tell if a relational structure has a semilattice polymorphism?

## 5 McKenzie-Wagner Circuits

In this section,  $\mathbf{A}^+$  is the algebra with the universe  $2^\omega$  and basic operations  $\cap$ ,  $\cup$  (binary set operations),  $+$ ,  $\cdot$  (binary elementwise arithmetics; for example  $\{1, 2\} + \{4, 5\} = \{5, 6, 7\}$ ),  $^c$  (complement) and constants  $\emptyset$ ,  $\omega$ ,  $\{0\}$ , and  $\{1\}$ .

The algebra  $\mathbf{M}$  is the minimum subalgebra of  $\mathbf{A}^+$ , i.e.  $\mathbf{M}$  contains all sets we can describe using  $\{\emptyset, \omega, \{0\}, \{1\}\}$  and basic operations.

$\mathbf{M}$  has many interesting members: finite sets, cofinite sets, all primes, ...

**Question 49** (Cliff Bergman). Does  $\mathbf{M}$  contain  $\{u^2 : u \in \omega\}$ ?

**Question 50** (Cliff Bergman). Is the word problem in  $\mathbf{M}$  decidable? Or, to put it differently, can we decide if a term of  $\mathbf{M}$  describes the empty set (problem EMPTY)?

*Context.* (Pierre) McKenzie and Wagner have shown that EMPTY is NEXPTIME-hard, and NEXPTIME-complete when we leave out the complement operation.

Among instances of EMPTY is the Goldbach conjecture.

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