**Examples.** In each of the examples below, $X$ is a finite set and $\mathbf{F} = \mathbf{F}(X)$ is the free lattice generated by $X$. The symbol $F$ denotes the universe of $\mathbf{F}$.

- Ex 1. Let $X = \{x, y, z\}$, and let $\mathbf{L} = \mathbf{2}$ be the 2-element chain. Then the kernel of an epimorphism $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ is a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$.
- Ex 2. Let $X = \{x, y, z\}$ and let $\mathbf{L} = \mathbf{3}$ be the 3-element chain. Then the kernel of an epimorphism $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ is finitely generated.
- Ex 3. Let $n > 2$, $X = \{x_0, x_1, \dots, x_{n-1}\}$, and $\mathbf{L} = \mathbf{2} \times \mathbf{2}$. Then the kernel of an epimorphism $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ is finitely generated.

(Proof of Ex 3 appears below.)

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Here are some lemmas that may be useful.

**Lemma 1.2** (Freese, Jezek, Nation) Let $\mathcal{V}$ be a nontrivial variety of lattices and let $\mathbf{F}_{\mathcal{V}}(X)$ be the relatively free lattice in $\mathcal{V}$ over $X$. Then,

($\bigstar$) $\quad \bigwedge S \leq \bigvee T$ implies $S \cap T \neq \emptyset$ for each pair of finite subsets $S, T \subseteq X$.

**Lemma 1.4** (Freese, Jezek, Nation) Let $\mathbf{L}$ be a lattice generated by a set $X$ and let $a \in L$. Then

- (1) if $a$ is join prime, then $a = \bigwedge S$ for some finite subset $S \subseteq X$.
- (2) if $a$ is meet prime, then $a = \bigvee S$ for some finite subset $S \subseteq X$.

If $X$ satisfies condition ($\bigstar$) above, then

- (3) for every finite, nonempty subset $S \subset X$, $\bigwedge S$ is join prime and $\bigvee S$ is meet prime.

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*Ex 3 Proof.*
Let the universe of $\mathbf{L} = \mathbf{2} \times \mathbf{2}$ be $\{0, a, b, 1\}$, where $a \vee b = 1$ and $a \wedge b = 0$.

For each $y \in L$, denote by $X_y = X \cap h^{-1}\{y\}$ the set of generators mapped by $h$ to $y$.

Denote the least and greatest elements of $h^{-1}\{y\}$ (if they exist) by $y_*$ and $y^*$, respectively. For example,

$$a_* = \bigwedge h^{-1}\{a\} = \bigwedge\{x \in F : h(x) = a\}, \qquad b^* = \bigvee\{x \in F : h(x) = b\}, \qquad \text{etc.}$$

In the present example, the least and greatest elements exist is each case, as we now show.

**Claim 1.** $h^{-1}\{a\}$ has least and greatest elements, namely $a_* = \bigwedge(X_a \cup X_1)$ and $a^* = \bigvee(X_a \cup X_0)$, resp. (Similarly, for $b_*$ and $b^*$.)

**Claim 2.** $h^{-1}\{1\}$ has least and greatest elements, namely $1_* = a_* \vee b_*$ and $1^* = \bigvee X$, resp.

**Claim 3.** $h^{-1}\{0\}$ has least and greatest elements, namely, $0_* = \bigwedge X$ and $0^* = a^* \wedge b^*$, resp.

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**Proof of Claim 1.** Let $l_a = \bigwedge(X_a \cup X_1)$ and $g_a = \bigvee(X_a \cup X_0)$ (which exist in $F$, since the sets involved are finite). Then $h(l_a) = a = h(g_a)$.

Fix $r \in h^{-1}\{a\}$.

*Case 0.* If $r \in X_a$, then $r \geq \bigwedge X_a \geq \bigwedge(X_a \cup X_1) = l_a$.

*Case 1.* Let $r = s \vee t$, where $h(s) = a$ and $h(t) \in \{a, 0\}$. Assume (the induction hypothesis) that $s \geq l_a$. Then $r = s \vee t \geq l_a$.

*Case 2.* Let $r = s \wedge t$, where $h(s) = a$ and $h(t) \in \{a, 1\}$. Assume (the induction hypothesis) that $s, t \geq l_a$. Then $s \wedge t \geq l_a$.

This proves that for each $r \in h^{-1}\{a\}$ we have $r \geq l_a$. Thus $a_* = l_a$.

The proof that every $r \in h^{-1}\{a\}$ is above $g_a$ is similar, as are the analogous proofs for $b_*$ and $b^*$.

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**Proof of Claim 2.**
$1^* = \bigvee X$ is obvious.

We now verify that $1_* = a_* \vee b_*$ (**TO DO:** fill in details)

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**Proof of Claim 3.**
$0_* = \bigwedge X$ is obvious.

We verify that $0^* = a^* \wedge b^*$ (**TO DO:** fill in details)

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Now let $Y = \{(x, p^*), (p^*, x), (x, p_*), (p_*, x) : x \in X_p, p \in \{0, a, b, 1\}\}$.

**Claim 4.** If $r \in F$ and $h(r) = p$, then $(r, p_*), (r, p^*) \in \langle Y \rangle$.

**Proof.**
*Case 0.* If $r \in X_p$, then the pair belongs to $Y$ and the claim is trivial.

*Case 1.* Suppose $r = s \wedge t$.

- *Subcase 1.1.*
  $h(r) = 1$ implies $h(s) = h(t) = 1$. If we assume (the induction hypothesis) that $(s, p_*), (s, p^*), (t, p_*), (t, p^*)$ belong to $\langle Y \rangle$, then $(r, p_*) = (s \wedge t, p_*) = (s, p_*) \wedge (t, p_*) \in \langle Y \rangle$. Similarly, $(r, p^*) \in \langle Y \rangle$.

- *Subcase 1.2.*
  $h(r) = a$ implies (wlog) $h(s) = a$ and $h(t) \in \{a, 1\}$. If we assume (the induction hypothesis) that $(s, p_*), (s, p^*), (t, p_*), (t, p^*)$ belong to $\langle Y \rangle$, then
  $(r, p_*) = (s \wedge t, p_*) = (s, p_*) \wedge (t, p_*) \in \langle Y \rangle$. Similarly, $(r, p^*) \in \langle Y \rangle$.
- *Subcase 1.3.*
  $h(r) = 0$ implies (wlog) that either (i) $h(s) = 0$, or (ii) $h(s) = a$, $h(t) = b$.
  If $h(s) = 0$, then $(s, 0_*) \in \langle Y \rangle$ implies $(r, 0_*) = (s \wedge t, 0_*) = (s, 0_*) \wedge (t, p_*) \in \langle Y \rangle$.
  If If $h(s) = a$, $h(t) = b$, and $(s, a_*), (t, b_*) \in \langle Y \rangle$, then
  $(r, 0_*) = (s \wedge t, 0_*) = (s, a_*) \wedge (t, b_*) \in \langle Y \rangle$.