

A CHARACTERIZATION OF BOUNDEDNESS FOR FINITE LATTICES

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ABSTRACT. Let X be a finite set, $\mathbf{F}(X)$ the free lattice generated by X , and $h: \mathbf{F}(X) \rightarrow \mathbf{L}$ a lattice epimorphism. Then \mathbf{L} is bounded if and only if the kernel of every such h is a finitely generated sublattice of $\mathbf{F}(X) \times \mathbf{F}(X)$.

1. INTRODUCTION

2. NOTATION AND DEFINITIONS

We present some standard notation and definitions needed for the statement and proof of our main result (Theorem 3.1). Many of these are found in the “Free Lattices” book by Freese, Jezek, Nation [FJN95], although the authors of that book point out that these concepts were first introduced in Ralph McKenzie’s work on nonmodular lattice varieties [McK72], and Bjarni Jónsson’s work on sublattices of free lattices [JN77].

2.1. Bounded Homomorphisms. Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be a lattice and suppose $x, y \in L$ and $x \leq y$. Then we write $\llbracket x, y \rrbracket$ to denote the sublattice of elements between x and y ; that is,

$$\llbracket x, y \rrbracket := \{z \in L \mid x \leq z \leq y\}.$$

Let \mathbf{K} and \mathbf{L} be lattices and suppose \mathbf{L} has bottom and top elements, 0 and 1, resp. If $h: \mathbf{K} \rightarrow \mathbf{L}$ is a lattice homomorphism, then for each $a \in L$ consider the sets $h^{-1}\llbracket a, 1 \rrbracket = \{x \in K \mid h(x) \geq a\}$ and $h^{-1}\llbracket 0, a \rrbracket = \{x \in K \mid h(x) \leq a\}$. If the domain \mathbf{K} has a top element, $1_{\mathbf{K}}$, then $h^{-1}\llbracket a, 1 \rrbracket$ is nonempty if and only if $a \leq h(1_{\mathbf{K}})$. If \mathbf{K} has a bottom element, $0_{\mathbf{K}}$, then $h^{-1}\llbracket 0, a \rrbracket$ is nonempty if and only if $a \geq h(0_{\mathbf{K}})$. When $h^{-1}\llbracket a, 1 \rrbracket$ is nonempty, it is a filter of \mathbf{K} ; dually a nonempty $h^{-1}\llbracket 0, a \rrbracket$ is an ideal. If \mathbf{K} is infinite, then $h^{-1}\llbracket a, 1 \rrbracket$ need not have a least element, nor $h^{-1}\llbracket 0, a \rrbracket$ a greatest element. However, considering when such extrema exist leads to the notion of bounded homomorphism, which is useful for studying the structure of free lattices.

A lattice homomorphism $h: \mathbf{K} \rightarrow \mathbf{L}$ is called **lower bounded** if for every $a \in L$, the set $h^{-1}\llbracket a, 1 \rrbracket$ is either empty or has a least element. The least element of a nonempty $h^{-1}\llbracket a, 1 \rrbracket$ is denoted by $\beta(a)$. Thus, if h is a lower bounded homomorphism, then $\beta: \mathbf{L} \rightarrow \mathbf{K}$ is a partial function whose domain is an ideal of \mathbf{L} .

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Dually, a homomorphism h is called **upper bounded** if every nonempty $h^{-1}[[0, a]]$ has a greatest element, called $\alpha(a)$. If h is an upper bounded homomorphism, then the domain of $\alpha: \mathbf{L} \rightarrow \mathbf{K}$ is a filter of \mathbf{L} . A **bounded** homomorphism is one that is both upper and lower bounded.

These definitions simplify when h is an epimorphism. In that case h is lower (upper) bounded if and only if each preimage $h^{-1}\{a\}$ has a least (greatest) element. Likewise, if L is finite, then $h: \mathbf{K} \rightarrow \mathbf{L}$ is lower (upper) bounded if and only if each $h^{-1}\{a\}$ has a least (greatest) element whenever it is nonempty. Of course, every lattice homomorphism on a finite domain is bounded.

Note that β is monotonic and a left adjoint for h ; that is, $a \leq h(x)$ iff $\beta(a) \leq x$. Therefore, β is a join preserving map on its domain: if $h^{-1}[[a, 1]] \neq \emptyset \neq h^{-1}[[b, 1]]$, then $\beta(a \vee b) = \beta(a) \vee \beta(b)$. Similarly, α is a right adjoint— $h(y) \leq a$ iff $y \leq \alpha(a)$ —so $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$, for all a, b in the domain of α . In particular, if h is an epimorphism, then α is a meet homomorphism, and β a join homomorphism, from \mathbf{L} to \mathbf{K} .

2.2. Unbounded Homomorphisms. If A and B are sets, then by $A \subseteq_\omega B$ we mean that A is a finite subset of B . Let X be a set, and $\mathbf{F} := \mathbf{F}(X)$ the free lattice generated by X . Define a pair of closure operators, denoted by $^\wedge$ and $^\vee$, on the powerset $\mathcal{P}(\mathbf{F})$ as follows: for $A \in \mathcal{P}(\mathbf{F})$,

$$A^\wedge = \{\bigwedge B \mid B \subseteq_\omega A\} \quad \text{and} \quad A^\vee = \{\bigvee B \mid B \subseteq_\omega A\}.$$

Define also $H_0 = X^\wedge = \{\bigwedge B \mid B \subseteq_\omega X\}$, and for all $k \geq 0$, let

$$H_{k+1} = (H_k)^{\vee\wedge} = \{\bigwedge B \mid B \subseteq_\omega H_k^\vee\},$$

where $H_k^\vee = \{\bigvee A \mid A \subseteq_\omega H_k\}$. Evidently, \mathbf{F} is the union of the chain of subsets $H_0 \subseteq H_1 \subseteq \dots$. Each $H_n = X^{\wedge(\vee\wedge)^n}$ is a finite meet-closed subset of \mathbf{F} and, since X generates \mathbf{F} , it should be apparent that $\bigcup H_n = \mathbf{F}$.

To gain some intuition about H_k ($k \geq 0$), let's consider the complexity of the terms in $H_1 = (H_0)^{\vee\wedge}$.

$$\begin{aligned} H_1 &= \{\bigwedge B \mid B \subseteq_\omega (H_0)^\vee\} = \{\bigwedge B \mid B \subseteq_\omega X^{\wedge\vee}\} \\ &= \left\{ \bigwedge B \mid B \subseteq_\omega \left\{ \bigvee A \mid A \subseteq_\omega X^\wedge \right\} \right\} \\ &= \left\{ \bigwedge B \mid B \subseteq_\omega \left\{ \bigvee A \mid A \subseteq_\omega \left\{ \bigwedge B \mid B \subseteq_\omega X \right\} \right\} \right\}. \end{aligned}$$

For a concrete example, if $x_i \in X$, then the following four terms belong to H_0 ,

$$t_1 = x_1 \wedge x_2 \wedge x_3, \quad t_2 = x_2 \wedge x_4, \quad t_3 = x_3 \wedge x_5, \quad t_4 = x_1,$$

and here's a term that belongs to $H_1 \setminus H_0$,

$$(t_1 \vee t_2) \wedge (t_3 \vee t_4) = [(x_1 \wedge x_2 \wedge x_3) \vee (x_2 \wedge x_4)] \wedge [(x_3 \wedge x_5) \vee x_1].$$

Let $h: \mathbf{F}(X) \rightarrow \mathbf{L}$ be an epimorphism and, for each $y \in L$ and $k < \omega$, define

$$\beta_k(y) = \bigwedge \{w \in H_k : h(w) \geq y\}.$$

Theorem 2.1 ([FJN95, Thm. 2.2]). *Let \mathbf{K} be finitely generated, and let $h: \mathbf{K} \rightarrow \mathbf{L}$ be a lattice homomorphism. If $a \leq h(1_{\mathbf{K}})$, then*

- (1) $j \leq k$ implies $\beta_j(a) \geq \beta_k(a)$,
- (2) $\beta_k(a)$ is the least element of $H_k \cap h^{-1}[[a, 1]]$,
- (3) $h^{-1}[[a, 1]] = \bigcup_{k \in \omega} [[\beta_k(a), 1]]$.

3. MAIN THEOREM

Theorem 3.1. *Let X be a finite set, $\mathbf{F}(X)$ the free lattice generated by X , $\mathbf{L} = \langle L, \wedge, \vee \rangle$ a finite lattice, and $h: \mathbf{F}(X) \rightarrow \mathbf{L}$ a lattice epimorphism that is one-to-one on X . Then h is bounded if and only if the kernel of h is a finitely generated sublattice of $\mathbf{F}(X) \times \mathbf{F}(X)$.*

A key ingredient in our proof of Theorem 3.1 is Lemma 3.2.

Lemma 3.2. *Let h be an epimorphism from $\mathbf{F}(X)$ onto a finite lattice \mathbf{L} . Let K be a finite subset of the kernel of h . Then there exists $N > 0$ such that for all $(p, q) \in \langle K \rangle$, if $p \geq \beta_k h(p)$, then $q \geq \beta_{k+N} h(q)$.*

Remark. If $K \subseteq \ker h$, then $(p, q) \in \langle K \rangle$ implies $h(p) = h(q)$. We will use the symbol a to denote this common element; that is, $h(p) = a = h(q)$. The context will make clear which p and q we mean.

Proof. Since K is finite, it is obvious that we can find an $N > 0$ such that the desired conclusion holds for all $(p, q) \in K$; that is, there exists $N > 0$ satisfying the following: $\forall (p, q) \in K$,

$$p \geq \beta_k(a) \implies q \geq \beta_{k+N}(a). \quad (3.1)$$

Fix $(p, q) \in \langle K \rangle$ and assume $p \geq \beta_k(a)$.

Case 1. Suppose $(p, q) = (p_1, q_1) \wedge (p_2, q_2)$, where for each i the pair $(p_i, q_i) \in \langle K \rangle$ satisfies the induction hypothesis (3.1). Since $p_1 \wedge p_2 = p \geq \beta_k(a)$, we have $p_1 \geq \beta_k(a)$ and $p_2 \geq \beta_k(a)$, so the induction hypothesis yields $q_1 \geq \beta_{k+N}(a)$ and $q_2 \geq \beta_{k+N}(a)$, so $q = q_1 \wedge q_2 \geq \beta_{k+N}(a)$, as desired.

Case 2. Suppose $(p, q) = (p_1, q_1) \vee (p_2, q_2)$, where for each i the pair $(p_i, q_i) \in \langle K \rangle$ satisfies the induction hypothesis (3.1). Since $p \geq \beta_k(a)$, there exists a smallest k' such that $p \geq \beta_{k'}(a)$, and since k' is the smallest, $p \geq \beta_{k'}(a) \geq \beta_k(a)$, and $p \not\geq \beta_{k'-1}(a)$. Thus,

$$p = p_1 \vee p_2 \geq \beta_{k'}(a) = \bigwedge \{w \in H_{k'} \mid h(w) \geq a\},$$

and

$$p = p_1 \vee p_2 \not\geq \beta_{k'-1}(a) = \bigwedge \{w \in H_{k'-1} \mid h(w) \geq a\}. \quad (3.2)$$

Recall,

$$H_{k'} = (H_{k'-1})^{\vee \wedge} = \{\bigwedge B \mid B \subseteq_{\omega} H_{k'-1}^{\vee}\},$$

so $w \in H_{k'}$ means $w = \bigwedge B$ for some finite set B containing joins of elements of $H_{k'-1}$. This puts $p = p_1 \vee p_2$ above a meet of the elements of B . Therefore

Whitman's condition implies that either $p_i \geq \beta_{k'}(a)$ for some $i \in \{1, 2\}$, or $p_1 \vee p_2 \geq w$ for some meetand w of $\beta_{k'}(a)$. In the former case, $q_i \geq \beta_{k'+N}(a) \geq \beta_{k+N}(a)$, so $q = q_1 \vee q_2 \geq \beta_{k+N}(a)$, as desired. In the latter case, $p = p_1 \vee p_2 \geq w$ for some $w \in H_{k'-1}^\vee$. But then p is above each of the joinands of w , so $p \geq u$ for some $u \in H_{k'-1}$, yielding $p \geq u \geq \beta_{k'-1}(a)$ contradicting (3.2). \square

4. PROOF OF THE MAIN THEOREM

With Lemma 3.2 established, we are now in a position to prove Theorem 3.1. We begin with the converse direction. In fact, we will prove the contrapositive of the converse. Specifically, if $h : \mathbf{F} \rightarrow \mathbf{L}$ is not lower bounded, then $\ker h$ is not a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$. Finally, we will assume h is (both upper- and lower-) bounded, and prove that $\ker h$ is a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$ in this case.

Proof of Theorem 3.1.

(\Leftarrow) Suppose $h : \mathbf{F} \rightarrow \mathbf{L}$ is not lower bounded, and let K be an arbitrary finite subset of $\ker h$. We prove that the subalgebra $\langle K \rangle \leq \mathbf{F} \times \mathbf{F}$ generated by K is not all of $\ker h$. Since K is an arbitrary finite subset of $\ker h$, this will prove that $\ker h$ is not finitely generated.

Let $h^{-1}(a)$ be a class of $\ker h$ that is not lower bounded. The sequence $\beta_k(a)$ defined prior to Lemma 3.2 is an infinite descending chain. Of course, for every $k \in \mathbb{N}$, we have $h\beta_0(a) = h\beta_k(a)$, so $(\beta_0(a), \beta_k(a)) \in \ker h$. On the other hand, since K is a finite subset of $\ker h$, Lemma 3.2 asserts the existence of an $N \in \mathbb{N}$ such that, if $(p, q) \in \langle K \rangle$, if $p \geq \beta_\ell(a)$, and if $p \not\geq \beta_{\ell-1}(a)$, then $p \geq \beta_{\ell+N}h(p)$. Therefore, $(\beta_0(a), \beta_{N+1}(a)) \in \ker h \setminus \langle K \rangle$. Thus $\langle K \rangle \neq \ker h$, as we set out to prove.

(\Rightarrow) Assume h is a bounded epimorphism so that the preimage of each $y \in L$ under h is bounded. For each $y \in L$, let $\alpha y = \bigvee h^{-1}\{y\}$ and $\beta y = \bigwedge h^{-1}\{y\}$ denote the greatest and least elements of $h^{-1}\{y\}$, respectively (both of which exist by the assumed boundedness and surjectivity of h). Observe that $h\alpha h = h$, and $h\beta h = h$. In fact, α and β are adjoint to h . Indeed, it is easy to see that

$$\begin{aligned} hx \leq y &\Leftrightarrow x \leq \alpha y, \\ y \leq hx &\Leftrightarrow \beta y \leq x. \end{aligned}$$

For each $y \in L$, let $X_y := X \cap h^{-1}\{y\}$, the set of generators that lie in the inverse image of y under h . Let G be the (finite) set of pairs in $\mathbf{F} \times \mathbf{F}$ defined as follows:

$$G = \bigcup_{y \in L} \{(x, \alpha y), (\alpha y, x), (x, \beta y), (\beta y, x), (\alpha y, \beta y), (\beta y, \alpha y) : x \in X_y\}.$$

We claim that G generates $\ker h$. To prove this, we first show, by induction on term complexity, that for every $y \in L$, for every $r \in h^{-1}\{y\}$, the pairs $(r, \alpha y)$ and $(r, \beta y)$ belong to the sublattice $\langle G \rangle \leq \mathbf{F} \times \mathbf{F}$ generated by G .

- **Case 0.** Suppose $r \in X$. Then $(r, \alpha y)$ and $(r, \beta y)$ belong to G itself, so there's nothing to prove.

- **Case 1.** Suppose $r = s \vee t$. Assume (the induction hypothesis) that $(s, \alpha h(s))$, $(s, \beta h(s))$, $(t, \alpha h(t))$, and $(t, \beta h(t))$ belong to $\langle G \rangle$. Then $y = h(r) = h(s \vee t) = h(s) \vee h(t)$, so

$$h(\alpha h(s) \vee \alpha h(t)) = h\alpha h(s) \vee h\alpha h(t) = h(s) \vee h(t) = y.$$

Likewise, $h(\beta h(s) \vee \beta h(t)) = h(s) \vee h(t) = y$. Therefore,

$$\beta y \leq \beta h(s) \vee \beta h(t) \leq \alpha h(s) \vee \alpha h(t) \leq \alpha y.$$

Also, $r \leq \alpha y$, so $r = \alpha y \wedge (s \vee t)$. Taken together, these observations yield

$$\begin{aligned} \begin{pmatrix} r \\ \beta y \end{pmatrix} &= \begin{pmatrix} \alpha y \wedge (s \vee t) \\ \beta y \end{pmatrix} = \begin{pmatrix} \alpha y \wedge (s \vee t) \\ \beta y \wedge (\beta h(s) \vee \beta h(t)) \end{pmatrix} \\ &= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \left[\begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \vee \begin{pmatrix} t \\ \beta h(t) \end{pmatrix} \right], \end{aligned}$$

and each term in the last expression belongs to $\langle G \rangle$, so $(r, \beta y) \in \langle G \rangle$, as desired.

Similarly, $(r, \alpha y) \in \langle G \rangle$. Indeed, $\beta y \leq r$ implies $r = \beta y \vee s \vee t$, and $\beta h(s) \vee \beta h(t) \leq \alpha y$ implies $\alpha y = \alpha y \vee \beta h(s) \vee \beta h(t)$. Therefore,

$$\begin{aligned} \begin{pmatrix} r \\ \alpha y \end{pmatrix} &= \begin{pmatrix} \beta y \vee s \vee t \\ \alpha y \vee \beta h(s) \vee \beta h(t) \end{pmatrix} \\ &= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \vee \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}. \end{aligned}$$

- **Case 2.** Suppose $r = s \wedge t$. Assume $(s, \alpha h(s))$, $(s, \beta h(s))$, $(t, \alpha h(t))$, and $(t, \beta h(t))$ belong to $\langle G \rangle$. Then $h(s \wedge t) = h(r) = y$, so $h(\alpha h(s) \wedge \alpha h(t)) = y = h(\beta h(s) \wedge \beta h(t))$, so $\beta y \leq \beta h(s) \wedge \beta h(t) \leq \alpha h(s) \wedge \alpha h(t) \leq \alpha y$. Also, $\beta y \leq r \leq \alpha y$ so $r = \alpha y \wedge s \wedge t$ and $r = \beta y \vee (s \wedge t)$. Altogether, we have

$$\begin{aligned} \begin{pmatrix} r \\ \alpha y \end{pmatrix} &= \begin{pmatrix} \beta y \vee (s \wedge t) \\ \alpha y \vee (\alpha h(s) \wedge \alpha h(t)) \end{pmatrix} \\ &= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \left[\begin{pmatrix} s \\ \alpha h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \alpha h(t) \end{pmatrix} \right], \end{aligned}$$

and each term in the last expression belongs to $\langle Y \rangle$, as desired. Similarly,

$$\begin{aligned} \begin{pmatrix} r \\ \beta y \end{pmatrix} &= \begin{pmatrix} \alpha y \wedge s \wedge t \\ \beta y \wedge \alpha h(s) \wedge \alpha h(t) \end{pmatrix} \\ &= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \begin{pmatrix} s \\ \alpha h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \alpha h(t) \end{pmatrix}. \end{aligned}$$

Indicentally, in each of the last two derivations, we could have used β 's instead of α 's; in both cases the last meet could be replaced with

$$\begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}.$$

In each case, we end up with an expression involving terms from $\langle G \rangle$, and this proves that $(r, \alpha y)$ and $(r, \beta y)$ belong to $\langle G \rangle$, as desired. \square

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