KERNELS OF EPIMORPHISMS OF FINITELY GENERATED FREE LATTICES

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1. Introduction

1.1. **Bounded Homomorphisms.** The notation, definitions, ideas presented below are based on those we learned from the book by Freese, Jezek, Nation [FJN95], although the authors of that book credit to Ralph McKenzie's work on nonmodular lattice varieties [McK72], and Bjarni Jónsson's work on sublattices of free lattices [JN77].

If x, y are elements of a lattice L, and if $x \leqslant y$, then we write $[\![x, y]\!]$ to denote the sublattice of elements between x and y. That is,

$$[\![x, y]\!] := \{z \in L \mid x \leqslant z \leqslant y\}.$$

Let K and L be lattices and suppose L has bottom and top elements, 0_L and 1_L, resp. If $h\colon K \to L$ is a lattice homomorphism, then for each $a \in L$ we consider the sets $h^{-1}[\![a, 1]\!] = \{x \in K \mid h(x) \geqslant a\}$ and $h^{-1}[\![0, a]\!] = \{x \in K \mid h(x) \leqslant a\}$. When $h^{-1}[\![a, 1]\!]$ is nonempty, it is a filter of K; dually a nonempty $h^{-1}[\![0, a]\!]$ is an ideal. If K is infinite, then $h^{-1}[\![a, 1]\!]$ need not have a least element, nor $h^{-1}[\![0, a]\!]$ a greatest element. However, considering when such extrema exist leads to the notion of bounded homomorphism, which in turn helps us understand the structure of free lattices.

A lattice homomorphism $h\colon K \to L$ is **lower bounded** if for every $a \in L$, the set $h^{-1}[\![a, 1]\!]$ is either empty or has a least element. The least element of a nonempty $h^{-1}[\![a, 1]\!]$ is denoted by $\beta_h(a)$, or by $\beta(a)$ when h is clear from context. Thus, if h is a lower bounded homomorphism, then $\beta_h\colon L \rightharpoonup K$ is a partial mapping whose domain is an ideal of L.

Dually, h is an **upper bounded** homomorphism if, whenever the set $h^{-1}[\![0, a]\!]$ is nonempty, it has a greatest element, denoted by $\alpha_h(a)$, or $\alpha(a)$. For an upper bounded homomorphism, the domain of $\alpha_h\colon L \rightharpoonup K$ is clearly a filter of L. A **bounded** homomorphism is one that is both upper and lower bounded.

These definitions simplify when h is an epimorphism. In that case h is lower bounded if and only if each preimage $h^{-1}\{a\}$ has a least element. Likewise, if L is finite, then $h\colon K \to L$ is lower bounded if and only if $h^{-1}\{a\}$ has a least element whenever it is nonempty. On the other hand, every homomorphism h from a finite lattice K is bounded.

Note that β is monotonic and a left adjoint for h, i.e., $a \leqslant h(x)$ iff $\beta(a)x$. It then follows from a standard argument that β is a join preserving map on

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its domain: if $h^{-1}[\![a,1]\!] \neq \emptyset$ and $h^{-1}[\![b,1]\!] \neq \emptyset$, then $\beta(a \sqcup b) = \beta(a) \sqcup \beta(b)$. Similarly, α is a right adjoint for h, so that $h(y) \leqslant a$ iff $y \leqslant \alpha(a)$, and for $a, b \in \operatorname{dom}\alpha$, $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$. In particular, if h is an epimorphism, then α and β are respectively meet and join homomorphisms of L into K. For future reference, we note that α and β behave correctly with respect to composition.

Theorem 1.1 ([FJN95, Theorem 2.1])**.** *Let $f\colon K \to L$ and $g\colon L \to M$ be lattice homomorphisms. If f and g are lower bounded, then $gf\colon K \to M$ is lower bounded, and $\beta_{gf} = \beta_f\beta_g$. Similarly, if f and g are upper bounded, then $\alpha_{gf} = \alpha_f\alpha_g$.*

Proof. For $x \in K$ and $a \in M$, we have

$$a \leqslant gf(x) \quad \text{iff} \quad \beta_g(a) \leqslant f(x) \quad \text{iff} \quad \beta_f\beta_g(a) \leqslant x.$$

The upper bounded case is dual. □

We need a way to determine whether a lattice homomorphism $h\colon K \to L$ is upper or lower bounded. The most natural setting for this is when the lattice K is finitely generated, so from now on we assume K is generated by a finite set X. There are no special assumptions about L, nor do we assume that h is upper or lower bounded. We want to analyze the sets $h^{-1}[\![a,1]\!]$ for $a \in L$, with the possibility of lower boundedness in mind. The corresponding results for $h^{-1}[\![0,a]\!]$ are of course obtained by duality.

Note that because K is finitely generated, it has a greatest element $1_K = \bigvee X$, and that $h^{-1}[\![a,1]\!]$ is nonempty if and only if $a \leqslant h(1_K)$. Such elements are naturally the ones in which we are most interested.

Define a pair of closure operators, denoted by superscripts $^\wedge$ and $^\vee$, on subsets of an arbitrary lattice L as follows: for each subset $A \subseteq L$,

$$A^\wedge := \{\bigwedge B \mid B \text{ is a finite subset of } A\}.$$

We adopt the following convention: if L has a greatest element 1_L, then $\bigwedge\emptyset = 1_L$, and we include this in $A^\wedge$ for every $A \subseteq L$. (For lattices without a greatest element, $\bigwedge\emptyset$ is undefined.) The set $A^\vee$ is defined dually.

If $\langle K, \vee, \wedge\rangle$ is a lattice generated by a finite set X, then K is the union of a chain of subsets $H_0 \subseteq H_1 \subseteq \cdots$ defined inductively by setting $H_0 := X^\wedge$ and $H_{k+1} := (H_k)^{\vee\wedge}$, for all $k \geqslant 0$. By induction, each $H_n = X^{\wedge(\vee\wedge)^n}$ is a finite meet-closed subset of K, and $\bigcup H_n = K$, since X generates K.

Let $h\colon K \to L$ be a lattice epimorphism and define, for each $y \in L$ and $k < \omega$,

$$\beta_k(y) = \bigwedge\{w \in H_k : h(w) \geqslant a\}.$$

On page 30 of [FJN95], immediately after Theorem 2.4, the authors make the following remark, which is a crucial ingredient of our proof:

"...[h is lower bounded] if and only if for each $a \leqslant h(1_{\mathbf{K}})$ there exists $N \in \omega$ such that $\beta_n(a) = \beta_N(a)$ for all $n \geqslant N$."

wjd 2018-09-05: To do: verify this!

Fact. The following are equivalent:

(1) h is not lower bounded;
(2) $(\exists y_0 \in L)(\forall N)(\exists n > N)\, \beta_n(a) \neq \beta_N(a)$;
(3) $(\exists y_0 \in L)(\exists N)(\forall n > N)\beta_n(a) \neq \beta_N(a)$.

2. Main Theorem

Let X be a finite set and $\mathbf{F} := \mathbf{F}(X)$ the free lattice generated by X.

Theorem 2.1. *Suppose $\mathbf{L} = \langle L, \wedge, \vee\rangle$ is a finite lattice and $h\colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ a lattice epimorphism. If h is bounded then the kernel of h is a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$.*

Proof. Assume h is bounded. That is, the preimage of each $y \in L$ under h is bounded. For each $y \in L$, let $\alpha y = \bigvee h^{-1}\{y\}$ and $\beta y = \bigwedge h^{-1}\{y\}$ denote the greatest and least elements of $h^{-1}\{y\}$, respectively (both of which exist by the boundedness assumption). Observe that $h\alpha h = h$, and $h\beta h = h$. In fact, α and β are adjoint to h. Indeed, it is easy to see that

$$\begin{aligned} hx \leqslant y \quad &\Leftrightarrow \quad x \leqslant \alpha y,\\ y \leqslant hx \quad &\Leftrightarrow \quad \beta y \leqslant x. \end{aligned}$$

For each $y \in L$, let $X_y := X \cap h^{-1}\{y\}$, the set of generators that lie in the inverse image of y under h. Let G be the (finite) set of pairs in $\mathbf{F} \times \mathbf{F}$ defined as follows:

$$G = \bigcup_{y\in L} \{(x, \alpha y), (\alpha y, x), (x, \beta y), (\beta y, x), (\alpha y, \beta y), (\beta y, \alpha y) : x \in X_y\}.$$

We claim that G generates $\ker h$. To prove this, we first show, by induction on term complexity, that for every $y \in L$, for every $r \in h^{-1}\{y\}$, the pairs $(r, \alpha y)$ and $(r, \beta y)$ belong to the sublattice $\langle G\rangle \leqslant \mathbf{F} \times \mathbf{F}$ generated by G.

- **Case 0.** Suppose $r \in X$. Then $(r, \alpha y)$ and $(r, \beta y)$ belong to G itself, so there's nothing to prove.
- **Case 1.** Suppose $r = s \vee t$. Assume (the induction hypothesis) that $(s, \alpha h(s))$, $(s, \beta h(s))$, $(t, \alpha h(t))$, and $(t, \beta h(t))$ belong to $\langle G\rangle$. Then $y = h(r) = h(s \vee t) = h(s) \vee h(t)$, so
$$h(\alpha h(s) \vee \alpha h(t)) = h\alpha h(s) \vee h\alpha h(t) = h(s) \vee h(t) = y.$$
Likewise, $h(\beta h(s) \vee \beta h(t)) = h(s) \vee h(t) = y$. Therefore,
$$\beta y \leqslant \beta h(s) \vee \beta h(t) \leqslant \alpha h(s) \vee \alpha h(t) \leqslant \alpha y.$$
Also, $r \leqslant \alpha y$, so $r = \alpha y \wedge (s \vee t)$. Taken together, these observations yield
$$\begin{aligned} \begin{pmatrix} r \\ \beta y \end{pmatrix} &= \begin{pmatrix} \alpha y \wedge (s \vee t) \\ \beta y \end{pmatrix} = \begin{pmatrix} \alpha y \wedge (s \vee t) \\ \beta y \wedge (\beta h(s) \vee \beta h(t)) \end{pmatrix}\\ &= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \left[\begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \vee \begin{pmatrix} t \\ \beta h(t) \end{pmatrix} \right], \end{aligned}$$
and each term in the last expression belongs to $\langle G\rangle$, so $(r, \beta y) \in \langle G\rangle$, as desired.

Similarly, $(r, \alpha y) \in \langle G\rangle$. Indeed, $\beta y \leqslant r$ implies $r = \beta y \vee s \vee t$, and $\beta h(s) \vee \beta h(t) \leqslant \alpha y$ implies $\alpha y = \alpha y \vee \beta h(s) \vee \beta h(t)$. Therefore,

$$\begin{aligned}\begin{pmatrix} r \\ \alpha y \end{pmatrix} &= \begin{pmatrix} \beta y \vee s \vee t \\ \alpha y \vee \beta h(s) \vee \beta h(t) \end{pmatrix}\\ &= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \vee \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}.\end{aligned}$$

- **Case 2.** Suppose $r = s \wedge t$. Assume $(s, \alpha h(s))$, $(s, \beta h(s))$, $(t, \alpha h(t))$, and $(t, \beta h(t))$ belong to $\langle G\rangle$. Then $h(s \wedge t) = h(r) = y$, so $h(\alpha h(s) \wedge \alpha h(t)) = y = h(\beta h(s) \wedge \beta h(t))$, so $\beta y \leqslant \beta h(s) \wedge \beta h(t) \leqslant \alpha h(s) \wedge \alpha h(t) \leqslant \alpha y$. Also, $\beta y \leqslant r \leqslant \alpha y$ so $r = \alpha y \wedge s \wedge t$ and $r = \beta y \vee (s \wedge t)$. Altogether, we have

$$\begin{aligned}\begin{pmatrix} r \\ \alpha a \end{pmatrix} &= \begin{pmatrix} \beta y \vee (s \wedge t) \\ \alpha y \vee (\alpha h(s) \wedge \alpha h(t)) \end{pmatrix}\\ &= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \left[\begin{pmatrix} s \\ \alpha h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \alpha h(t) \end{pmatrix}\right],\end{aligned}$$

and each term in the last expression belongs to $\langle Y\rangle$, as desired. Similarly,

$$\begin{aligned}\begin{pmatrix} r \\ \beta y \end{pmatrix} &= \begin{pmatrix} \alpha y \wedge s \wedge t \\ \beta y \wedge \alpha h(s) \wedge \alpha h(t) \end{pmatrix}\\ &= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \begin{pmatrix} s \\ \alpha h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \alpha h(t) \end{pmatrix}.\end{aligned}$$

Note that, in both of the derivations above, we could have used β's instead of α's; that is,

$$\begin{aligned}\begin{pmatrix} r \\ \alpha y \end{pmatrix} &= \begin{pmatrix} \beta y \vee (s \wedge t) \\ \alpha y \vee (\beta h(s) \wedge \beta h(t)) \end{pmatrix}\\ &= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \left[\begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}\right],\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix} r \\ \beta y \end{pmatrix} &= \begin{pmatrix} \alpha y \wedge s \wedge t \\ \beta y \wedge \beta h(s) \wedge \beta h(t) \end{pmatrix}\\ &= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}.\end{aligned}$$

In each case, we end up with an expression involving terms from $\langle G\rangle$, and this proves that $(r, \alpha y)$ and $(r, \beta y)$ belong to $\langle G\rangle$, as desired.

□

We conjecture the converse of Theorem 2.1.

Conjecture 1. Suppose $\mathbf{L} = \langle L, \wedge, \vee\rangle$ is a finite lattice and $h\colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ a lattice epimorphism. If the kernel of h is a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$, then h is bounded.

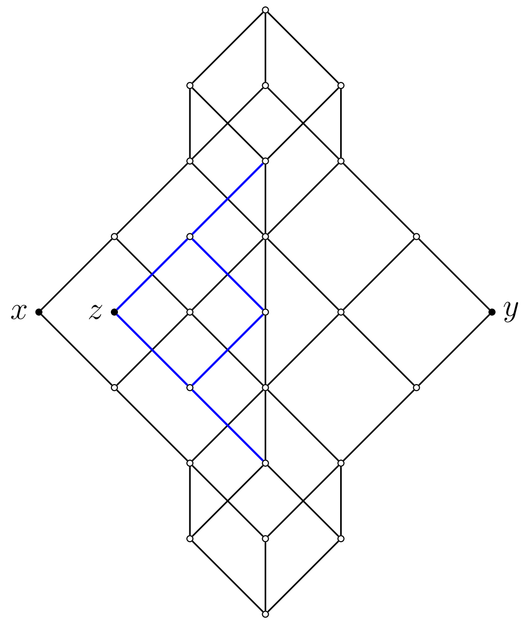

FIGURE 1. The free lattice over M_3.

If we could assume that whenever h is unbounded there is a class of $\ker h$ containing both an infinite chain and a generator of $\mathbf{F}$, then there is a straightforward proof of the conjecture. (See the Appendix for details.) Unfortunately, as the next result shows, this assumption is not always valid.

Proposition 2.2. *Let* $\mathbf{F} = \mathbf{F}(x, y, z)$*, and let* $\mathbf{L} = \mathbf{F}_{\mathbf{M}_3}(3)$ *(see Figure 1). Let* $h\colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ *be an epimorphism. Then* $K = \ker h$ *is not finitely generated.*

Proof. Define the sequences $\{m_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ $(n < \omega)$ of elements of $\mathbf{F}(X)$ as follows: let $x_0 = x$, $y_0 = y$, $z_0 = z$, and for $n \geqslant 0$,

$$m_n = (x_n \wedge y_n) \vee (x_n \wedge z_n) \vee (y_n \wedge z_n);$$
$$x_{n+1} = x_n \vee m_n = x_n \vee (y_n \wedge z_n).$$

Define y_{n+1} and z_{n+1} similarly.

Claim 2.3. If $\{s_n\}$ is any one of the four sequences just defined, then for $n > 0$, we have $s_{n+1} > s_n$ and $h(s_{n+1}) = h(s_n)$.

□

wjd 2018-09-05: Fill in proof of Claim 2.3

3. EXAMPLES

Let $\mathbf{M_3} = \langle\{0, a, b, c, 1\}, \wedge, \vee\rangle$, where $a \wedge b = a \wedge c = b \wedge c = 0$ and $a \vee b = a \vee c = b \vee c = 1$. Let $\mathbf{F} := \mathbf{F}(x, y, z)$ denote the free lattice generated by $\{x, y, z\}$.

Proposition 3.1. *Let* $h\colon \mathbf{F} \twoheadrightarrow \mathbf{M_3}$ *be the epimorphism that acts on the generators as follows:* $x \mapsto a$, $y \mapsto b$, $z \mapsto c$. *Then* $\ker h$ *is not finitely generated.*

Proof. Let $K := \ker h$, and for $u \in \{x, y, z\}$ let $C_u := u/K := \{v \in F : h(v) = h(u)\}$. Define sequences of elements in these classes by the following mutual recursions:

- for $i \in \mathbb{N}$,
$$m_{0,i} = (m_{x,i} \wedge m_{y,i}) \vee (m_{x,i} \wedge m_{z,i}) \vee (m_{y,i} \wedge m_{z,i});$$
- for $u \in \{x, y, z\}$,
$$m_{u,0} = u,$$
$$m_{u,i+1} = m_{u,i} \vee m_{0,i}.$$

Notice that $m_{0,0} = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ and $m_{x,i+1} = m_{x,i} \vee (m_{y,i} \wedge m_{z,i})$.

Let X be a finite subset of K. We will prove there exists $(p, q) \in K \setminus \langle X\rangle$. Fix $u \in \{0, x, y, z\}$. Since X is finite, Lemma 3.3 implies that there exists $M \in \mathbb{N}$ such that for every $(p, q) \in X$ with $p, q \in C_u$, we have $p, q \leqslant m_{u,M}$.

Subclaim 1. For $(p, q) \in \langle X\rangle$ and $u \in \{x, y, z\}$, the following implication holds:
$$q \leqslant u \quad\Longrightarrow\quad p \leqslant m_{u,M}. \tag{3.1}$$

We prove the subclaim by induction on the complexity of terms. Fix $(p, q) \in \langle X\rangle$. Then $p, q \in C_u$ for some $u \in \{x, y, z\}$.

- **Case 0.** Suppose $(p, q) \in X$. Then by definition of M we have $p, q \leqslant m_{u,M}$.
- **Case 1.** Suppose $(p, q) = (p_1, q_1) \wedge (p_2, q_2)$, where (p_i, q_i) satisfies (3.1) for $i = 1, 2$. If $q = q_1 \wedge q_2 \leqslant u$, then, since generators in the free lattice are meet-prime (see Theorem A.8 below), we have $q_1 \leqslant u$ or $q_2 \leqslant u$. Assume $q_1 \leqslant u$. Then, by the induction hypothesis, $p_1 \leqslant m_{u,M}$. Therefore, $p = p_1 \wedge p_2 \leqslant m_{u,M}$, as desired.
- **Case 2.** Suppose $(p, q) = (p_1, q_1) \vee (p_2, q_2)$, where (p_i, q_i) satisfies (3.1) for $i = 1, 2$. If $q = q_1 \vee q_2 \leqslant u$, then $q_i \leqslant u$ for $i = 1, 2$. It now follows from the induction hypothesis that $p_i \leqslant m_{u,M}$ for $i = 1, 2$, so $p = p_1 \vee p_2 \leqslant m_{u,M}$, as desired.

This completes the proof of Subclaim 1. It now follows from Lemma 3.2 that $(m_{x,M+1}, x) \in K \setminus \langle X\rangle$, so proposition is proved. ☐

Lemma 3.2. *For each $u \in \{0, x, y, z\}$, the sequence $\{m_{u,n} : n \in \mathbb{N}\}$ is a strictly ascending chain; that is, $m_{u,0} < m_{u,1} < m_{u,2} < \cdots$.*

Proof. We split the proof up into cases: either $u \in \{x, y, z\}$, or $u = 0$.

- **Case 1.** $u \in \{x, y, z\}$.
  For simplicity, assume $u = x$ for the remainder of the proof of this case. (Of course, the same argument goes through when u is y or z.) Fix $n \in \mathbb{N}$. We prove $m_{x,n} < m_{x,n+1}$.

  Subclaim 2. For all $n \in \mathbb{N}$,
  (1) $m_{x,n} \in C_x$,
  (2) $m_{x,n} \not\geqslant y$, and $m_{x,n} \not\geqslant z$.

Proof of Subclaim 2. The first item is obvious; for the second, if $m_{x,n} \geqslant y$, then $m_{x,n} \wedge y = y$, and then $0 = h(m_{x,n} \wedge y) = h(y) = b$. A similar contradiction is reached if we assume $m_{x,n} \geqslant z$, so the subclaim is proved.

Recall, $m_{x,n} = m_{x,n} \vee (m_{y,n} \wedge m_{z,n})$, so our desired conclusion, $m_{x,n} < m_{x,n+1}$, holds unless $m_{x,n} \geqslant m_{y,n} \wedge m_{z,n}$. So, by way of contradiction, suppose

$$m_{x,n} \geqslant m_{y,n} \wedge m_{z,n}. \tag{3.2}$$

Now, $m_{y,n} = y \vee (x \wedge z) \vee \cdots$, so clearly $m_{y,n} \geqslant y$. Similarly, $m_{z,n} \geqslant z$. This, together with (3.2), implies $m_{x,n} \geqslant m_{y,n} \wedge m_{z,n} \geqslant y \wedge z$. But then Theorem A.8 below implies that either $m_{x,n} \geqslant y$ or $m_{x,n} \geqslant z$, which contradicts Subclaim 2.

- **Case 2.** $u = 0$.
  We first prove that $m_{0,0} = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ is strictly below $m_{0,1} = (m_{x,1} \wedge m_{y,1}) \vee (m_{x,1} \wedge m_{z,1}) \vee (m_{y,1} \wedge m_{z,1})$.

  By symmetry, it suffices to show $x \wedge y < m_{x,1} \wedge m_{y,1}$; that is, $x \wedge y < (x \vee (y \wedge z)) \wedge (y \vee (x \wedge z))$.

  Clearly $x \wedge y \leqslant (x \vee (y \wedge z)) \wedge (y \vee (x \wedge z))$. Suppose $x \wedge y = (x \vee (y \wedge z)) \wedge (y \vee (x \wedge z))$. Then $(x \vee (y \wedge z)) \wedge (y \vee (x \wedge z)) \leqslant x$. By Theorem A.8, the latter holds iff $x \vee (y \wedge z) \leqslant x$ or $y \vee (x \wedge z) \leqslant x$ The first of these inequalities is clearly false, so it must be the case that $y \vee (x \wedge z) \leqslant x$. But then $y \leqslant x$, which is obviously false. We conclude that $x \wedge y < (x \vee (y \wedge z)) \wedge (y \vee (x \wedge z))$. This proves $m_{0,0} < m_{0,1}$.

  Now fix $n \in \mathbb{N}$ and assume $m_{0,n} < m_{0,n+1}$. We show $m_{0,n+1} < m_{0,n+2}$.

  wjd 2018-09-05: Complete the proof of this case.

  ↓ *begin scratch work* ↓
  $m_{0,n} := (m_{x,n} \wedge m_{y,n}) \vee (m_{x,n} \wedge m_{z,n}) \vee (m_{y,n} \wedge m_{z,n})$,
  $m_{0,n+1} := (m_{x,n+1} \wedge m_{y,n+1}) \vee (m_{x,n+1} \wedge m_{z,n+1}) \vee (m_{y,n+1} \wedge m_{z,n+1})$,
  By the first Case above, $m_{u,n} < m_{u,n+1}$.
  ↑ *end scratch work* ↑

□

Lemma 3.3. *For all $u \in \{x, y, z\}$ and $p \in C_u \cup C_0$ there exists $n \in \mathbb{N}$ such that $p \leqslant m_{u,n}$.*

Proof. We prove this by induction on the complexity of p.

- **Case 0.** $p \in \{x, y, z\}$. Then $u = p = m_{p,0}$.
  For the remaining cases assume $u = x$, without loss of generality.
- **Case 1.** $p = p_1 \vee p_2$.
  If $p \in C_x \cup C_0$, then $p_i \in C_x \cup C_0$ for $i = 1, 2$, and the induction hypothesis yields i and j for which $p_1 \leqslant m_{x,i}$ and $p_2 \leqslant m_{x,j}$. Letting $n = \max\{i, j\}$, we have $p_1, p_2 \leqslant m_{x,n}$, from which $p = p_1 \vee p_2 \leqslant m_{x,n}$, as desired.
- **Case 2.** $p = p_1 \wedge p_2$.
  If $p \in C_x$, then we may assume $p_1 \in C_x$ and $p_2 \in C_x \cup C_0$. By the

induction hypothesis, there exists $n \in \mathbb{N}$ such that $p_1 \leqslant m_{x,n}$, whence $p \leqslant p_1 \leqslant m_{x,n}$. If $p \in C_0$, then each p_i belongs to $C_u \cup C_0$ for some $u \in \{x, y, z\}$. If $p_1 \in C_x \cup C_0$, then $p_1 \leqslant m_{x,n}$, as above and we're done. Similarly, if $p_2 \in C_x \cup C_0$. So assume $p_1 \in C_y \cup C_0$ and $p_2 \in C_z \cup C_0$. Then the induction hypothesis implies that there exist i and j such that $p_1 \leqslant m_{y,i}$ and $p_2 \leqslant m_{z,j}$. If $n = \max\{i, j\}$, then $p_1 \leqslant m_{y,n}$ and $p_2 \leqslant m_{z,n}$. Then, by the above definition of the sequences, we have $p_1 \wedge p_2 \leqslant m_{y,n} \wedge m_{z,n} \leqslant m_{0,n} \leqslant m_{x,n+1}$.

□

3.1. **Other Examples.** In each of the propositions in this section, X is a finite set and $\mathbf{F} = \mathbf{F}(X)$ is the free lattice generated by X. The symbol F denotes the universe of $\mathbf{F}$. The proof in each case is straightforward, but tedious; we omit proofs of the first two, and give a detailed proof of the third.

Prop. 3.4. *Let $X = \{x, y, z\}$, and let $\mathbf{L} = \mathbf{2}$ be the 2-element chain. Then the kernel of an epimorphism $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ is a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$.*

Prop. 3.5. *Let $X = \{x, y, z\}$ and let $\mathbf{L} = \mathbf{3}$ be the 3-element chain. Then the kernel of an epimorphism $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ is finitely generated.*

Prop. 3.6. *Let $n > 2$, $X = \{x_0, x_1, \dots, x_{n-1}\}$, and $\mathbf{L} = \mathbf{2} \times \mathbf{2}$. Let $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ be an epimorphism. Then $K = \ker h$ is finitely generated.*

Proof. Let the universe of $\mathbf{L} = \mathbf{2} \times \mathbf{2}$ be $\{0, a, b, 1\}$, where $a \vee b = 1$ and $a \wedge b = 0$. For each $y \in L$, denote by $X_y = X \cap h^{-1}\{y\}$ the set of generators mapped by h to y. Denote the least and greatest elements of $h^{-1}\{y\}$ (if they exist) by ℓ_y and g_y, respectively. For example, $\ell_a = \bigwedge h^{-1}\{a\} = \bigwedge\{x \in F : h(x) = a\}$, $g_b = \bigvee\{x \in F : h(x) = b\}$, etc. In the present example, the least and greatest elements exist is each case, as we now show.

Subclaim 3. $h^{-1}\{a\}$ has least and greatest elements, namely $\ell_a = \bigwedge(X_a \cup X_1)$ and $g_a = \bigvee(X_a \cup X_0)$. (Similarly, $h^{-1}\{b\}$ has least and greatest elements, ℓ_b and g_b.)

Proof of Subclaim 3. Let $M(a) := \bigwedge(X_a \cup X_1)$ and $J(a) := \bigvee(X_a \cup X_0)$ and note that these values exist in F, since the sets involved are finite. Also, Then $h(M(a)) = a = h(J(a))$. Fix $r \in h^{-1}\{a\}$.

- If $r \in X_a$, then $r \geqslant \bigwedge X_a \geqslant \bigwedge(X_a \cup X_1) = M(a)$.
- If $r = s \vee t$, where $h(s) = a$ and $h(t) \in \{a, 0\}$, then assume (the induction hypothesis) that $s \geqslant M(a)$, and we have $r = s \vee t \geqslant M(a)$.
- If $r = s \wedge t$, where $h(s) = a$ and $h(t) \in \{a, 1\}$, then assume (the induction hypothesis) that $s, t \geqslant M(a)$, and we have $s \wedge t \geqslant M(a)$. This proves that for each $r \in h^{-1}\{a\}$ we have $r \geqslant M(a)$, and as we noted at the outset, $M(a) \in h^{-1}\{a\}$. Therefore, $\ell_a = M(a)$ is the least element of $h^{-1}\{a\}$. Similarly, every $r \in h^{-1}\{a\}$ is below $J(a)$, so $g_a = J(a)$. The proofs of $\ell_b = M(b)$ and $g_b = J(b)$ are similar.

This proves Subclaim 3.

Subclaim 4. $h^{-1}\{0\}$ has least and greatest elements, namely, $\ell_0 = \bigwedge X$ and $g_0 = g_a \wedge g_b$.

Proof of Subclaim 4. $\ell_0 = \bigwedge X$ is obvious, so we need only verify that $g_0 = g_a \wedge g_b$. Observe that $h(g_a \wedge g_b) = h(g_a) \wedge h(g_b) = a \wedge b = 0$, so $g_a \wedge g_b \in h^{-1}\{0\}$. It remains to prove that $r \leqslant g_a \wedge g_b$ holds for all $r \in h^{-1}\{0\}$. Fix $r \in h^{-1}\{0\}$. Then $h(r \vee g_a) = h(r) \vee h(g_a) = 0 \vee a = a$, which places $r \vee g_a$ in $h^{-1}\{a\}$. Therefore, by maximality of g_a, we have $r \vee g_a \leqslant g_a$, whence $r \leqslant g_a$. Similarly, $r \leqslant g_b$. This proves Subclaim 4.

Subclaim 5. $h^{-1}\{1\}$ has least and greatest elements, namely $\ell_1 = \ell_a \vee \ell_b$ and $g_1 = \bigvee X$.

Proof of Subclaim 5. $g_1 = \bigvee X$ is obvious, so we need only verify that $\ell_1 = \ell_a \vee \ell_b$. Observe that $h(\ell_a \vee \ell_b) = h(\ell_a) \vee h(\ell_b) = a \vee b = 1$, so $\ell_a \vee \ell_b \in h^{-1}\{1\}$. It remains to prove that $r \geqslant \ell_a \vee \ell_b$ holds for all $r \in h^{-1}\{1\}$. Fix $r \in h^{-1}\{1\}$. Then $h(r \wedge \ell_a) = h(r) \wedge h(\ell_a) = 1 \wedge a = a$, which places $r \wedge \ell_a$ in $h^{-1}\{a\}$. Therefore, by minimality of ℓ_a, we have $r \wedge \ell_a \geqslant \ell_a$, whence $r \geqslant \ell_a$. Similarly, $r \geqslant \ell_b$. Now let $Y = \{(x, g_p), (g_p, x), (x, \ell_p), (\ell_p, x) : p \in \{0, a, b, 1\}, x \in X_p\}$. This proves Subclaim 5.

Subclaim 6. If $r \in F$ and $h(r) = p$, then $(r, \ell_p), (r, g_p) \in \langle Y \rangle$.

Proof of Subclaim 6. Either $r \in X_p$ or $r = s \wedge t$ or $r = s \vee t$. If $r \in X_p$, then the pair belongs to Y and the claim is trivial.

Suppose $r = s \wedge t$.

- If $h(r) = 1$, then $h(s) = h(t) = 1$. Assume (the induction hypothesis) that $\{(s, \ell_1), (s, g_1), (t, \ell_1), (t, g_1)\} \subseteq \langle Y \rangle$. Then $(r, \ell_1) = (s \wedge t, \ell_1) = (s, \ell_1) \wedge (t, \ell_1) \in \langle Y \rangle$.
- If $h(r) = a$, then (wlog) $h(s) = a$ and $h(t) \in \{a, 1\}$. Assume (the induction hypothesis) that $\{(s, \ell_a), (s, g_a), (t, \ell_p), (t, g_p)\} \subseteq \langle Y \rangle$. By Claim 1, $\ell_a \leqslant \ell_1$, so $\ell_a = \ell_a \wedge \ell_1$.
  If $h(t) = 1$, then
  $$(r, \ell_a) = (s \wedge t, \ell_a \wedge \ell_1) = (s, \ell_a) \wedge (t, \ell_1) \in \langle Y \rangle,$$
  while If $h(t) = a$, then $(r, \ell_a) = (s \wedge t, \ell_a \wedge \ell_a) = (s, \ell_a) \wedge (t, \ell_a) \in \langle Y \rangle$.
- If $h(r) = 0$, then (wlog) that either (i) $h(s) = 0$, or (ii) $h(s) = a$, $h(t) = b$. If $h(s) = 0$, then $(s, \ell_0) \in \langle Y \rangle$ implies $(r, \ell_0) = (s \wedge t, \ell_0) = (s, \ell_0) \wedge (t, \ell_p) \in \langle Y \rangle$. If If $h(s) = a$, $h(t) = b$, and $(s, \ell_a), (t, \ell_b) \in \langle Y \rangle$, then $(r, \ell_0) = (s \wedge t, \ell_0) = (s, \ell_a) \wedge (t, \ell_b) \in \langle Y \rangle$.

This proves that $(r, \ell_1) \in \langle Y \rangle$ if $r = s \wedge t$. A similar argument shows that $(r, g_p) \in \langle Y \rangle$ in each of the three subcases. We have thus proved that $\{(r, \ell_1), (r, g_p)\} \subseteq \langle Y \rangle$, if $r = s \wedge t$.

Suppose $r = s \vee t$.

- If $h(r) = 0$, then $h(s) = h(t) = 0$. Assume (the induction hypothesis) that $\{(s, \ell_p), (s, g_p), (t, \ell_p), (t, g_p)\} \subseteq \langle Y \rangle$. Then $(r, \ell_p) = (s \vee t, \ell_p) = (s, \ell_p) \vee (t, \ell_p) \in \langle Y \rangle$.
- If $h(r) = a$, then (wlog) $h(s) = a$ and $h(t) \in \{a, 0\}$. If we assume (the induction hypothesis) that $(s, \ell_p), (s, g_p), (t, \ell_p), (t, g_p)$ belong to $\langle Y \rangle$, then $(r, \ell_p) = (s \vee t, \ell_p) = (s, \ell_p) \vee (t, \ell_p) \in \langle Y \rangle$.
- If $h(r) = 1$, then (wlog) that either (i) $h(s) = 1$, or (ii) $h(s) = a$, $h(t) = b$.
  In the first case, $(s, \ell_1) \in \langle Y \rangle$ implies $(r, \ell_1) = (s \vee t, \ell_1) = (s, \ell_1) \vee (t, \ell_p) \in \langle Y \rangle$. In the second case $h(s) = a$, $h(t) = b$, and $(s, \ell_a), (t, \ell_b) \in \langle Y \rangle$. Then $(r, \ell_1) = (s \vee t, \ell_1) = (s, \ell_a) \vee (t, \ell_b) \in \langle Y \rangle$.

Similarly, in each of these three subcases, we have $(r, g_p) \in \langle Y \rangle$. This proves Subclaim 6, and completes the proof of Prop 3.6. □

Appendix A. Background

Here are some useful definitions and results from the Free Lattices book by Freese, Jezek, and Nation [FJN95].

Definition A.1 (length of a term)**.** Let X be a set. Each element of X is a term of length 1, also known as a *variable*. If $t_1, \dots, t_n$ are terms of lengths $k_1, \dots, k_n$, then $t_1 \vee \cdots \vee t_n$ and $t_1 \wedge \cdots \wedge t_n$ are both terms of length $1 + k_1 + \cdots + k_n$.

Examples. By the above definition, the terms
$x \vee y \vee z \qquad x \vee (y \vee z) \qquad (x \vee y) \vee z$
have lengths 4, 5, and 5, respectively. Reason: variables have length 1, so $x \vee y \vee z$ has length $1+1+1+1$. On the other hand, $x \vee y$ is a term of length 3, so $(x \vee y) \vee z$ has length $1+3+1$. Similarly, $x \vee (y \vee z)$ has length $1+1+3$.

Lemma A.2 ([FJN95, Lem. 1.2])**.** *Let $\mathcal{V}$ be a nontrivial variety of lattices and let $\mathbf{F}_{\mathcal{V}}(X)$ be the relatively free lattice in $\mathcal{V}$ over X. Then,*

$$\bigwedge S \leqslant \bigvee T \text{ implies } S \cap T \neq \emptyset \text{ for each pair of finite subsets } S, T \subseteq X. \tag{A.1}$$

Lemma A.3 ([FJN95, Lem. 1.4])**.** *Let $\mathbf{L}$ be a lattice generated by a set X and let $a \in L$. Then*

(1) if a is join prime, then $a = \bigwedge S$ for some finite subset $S \subseteq X$.
(2) if a is meet prime, then $a = \bigvee S$ for some finite subset $S \subseteq X$.
If X satisfies condition (A.1) above, then
(3) for every finite, nonempty subset $S \subset X$, $\bigwedge S$ is join prime and $\bigvee S$ is meet prime.

Corollary A.4 ([FJN95, Cor. 1.5])**.** Let $\mathcal{V}$ be a nontrivial variety of lattices and let $\mathbf{F}_{\mathcal{V}}(X)$ be the relatively free lattice in $\mathcal{V}$ over X. For each finite nonempty subset $S \subseteq X$, $\bigwedge S$ is join prime and $\bigvee S$ is meet prime. In particular, every $x \in X$ is both join and meet prime. Moreover, if $x \leqslant y$ for $x, y \in X$, then $x = y$.

Theorem A.5 (Whitman's Condition, ver. 1)**.** *The free lattice $\mathbf{F}(X)$ satisfies the following condition:*

(W) If $v = v_1 \wedge \cdots \wedge v_r \leqslant u_1 \vee \cdots \vee u_s = u$, then either $v_i \leqslant u$ for some i, or $v \leqslant u_j$ for some j.

Corollary A.6 ([FJN95, Cor. 1.9])**.** Every sublattice of a free lattice satisfies (W). Every element of a lattice satisfying (W) is either join or meet irreducible.

Theorem A.7 (Whitman's Condition, ver. 2)**.** *The free lattice $\mathbf{F}(X)$ satisfies the following condition:*

(W+) If $v = v_1 \wedge \cdots \wedge v_r \wedge x_1 \wedge \cdots \wedge x_n \leqslant u_1 \vee \cdots \vee u_s \vee y_1 \vee \cdots \vee y_m = u$, where $x_i, y_j \in X$, then either $x_i = y_j$ for some i and j, or $v_i \leqslant u$ for some i, or $v \leqslant u_j$ for some j.

Theorem A.8 ([FJN95, Thm. 1.11])**.** *If $s = s(x_1, \dots, x_n)$ and $t = t(x_1, \dots, x_n)$ are terms and $x_1, \dots, x_n \in X$, then the truth of*

$$s^{\mathbf{F}(X)} \leqslant t^{\mathbf{F}(X)} \tag{A.2}$$

can be determined by applying the following rules.

(1) *If $s = x_i$ and $t = x_j$, then (A.2) holds iff $x_i = x_j$.*
(2) *If $s = s_1 \vee \cdots \vee s_k$ is a formal join, then (A.2) holds iff $s_i^{\mathbf{F}(X)} \leqslant t^{\mathbf{F}(X)}$ for all i.*
(3) *If $t = t_1 \wedge \cdots \wedge t_k$ is a formal meet, then (A.2) holds iff $s^{\mathbf{F}(X)} \leqslant t_i^{\mathbf{F}(X)}$ for all i.*
(4) *If $s = x_i$ and $t = t_1 \vee \cdots \vee t_k$ is a formal join, then (A.2) holds iff $x_i \leqslant t_j^{\mathbf{F}(X)}$ for some j.*
(5) *If $s = s_1 \wedge \cdots \wedge s_k$ is a formal meet and $t = x_i$, then (A.2) holds iff $s_j^{\mathbf{F}(X)} \leqslant x_i$ for some j.*
(6) *If $s = s_1 \wedge \cdots \wedge s_k$ is a formal meet and and $t = t_1 \vee \cdots \vee t_m$ is a formal join, then (A.2) holds iff $s_i^{\mathbf{F}(X)} \leqslant t^{\mathbf{F}(X)}$ for some i or $s^{\mathbf{F}(X)} \leqslant t_j^{\mathbf{F}(X)}$ for some j*

Definition A.9 (up directed, continuous)**.** A subset A of a lattice L is said to be *up directed* if every finite subset of A has an upper bound in A. It suffices to check this for pairs. A is up directed iff for all $a, b \in A$ there exists $c \in A$ such that $a \leqslant c$ and $b \leqslant c$. A lattice is *upper continuous* if whenever $A \subseteq L$ is an up directed set having a least upper bound $u = \bigvee A$, then for every b,

$$\bigvee_{a \in A} (a \wedge b) = \bigvee_{a \in A} a \wedge b = u \wedge b.$$

Down directed and *down continuous* are defined dually. A lattice that is both up and down continuous is called *continuous*.

Theorem A.10 ([FJN95, Thm. 1.22])**.** *Free lattices are continuous.*

Appendix B. Proof of Conjecture under special assumptions

Prop. B.1. *Suppose $\mathbf{L} = \langle L, \wedge, \vee \rangle$ is a finite lattice and $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ a lattice epimorphism. Suppose also that whenever h is unbounded then there is a class of $\ker h$ containing both an infinite chain and a generator of $\mathbf{F}$. Then h is bounded whenever its kernel is a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$.*

Proof. Suppose h is not lower bounded. Then by Fact 1.1 there is an element $y_0 \in L$ such that $\beta_0(y_0) > \beta_1(y_0) > \cdots$ is an infinite descending chain.

Let K be a finite subset of $\ker h$, say, $K = \{(p_1, q_1), \dots, (p_m, q_m)\} \subseteq \ker h$. We prove $\langle K \rangle \neq \ker h$. (Since K is an arbitrary finite subset of $\ker h$, this will prove $\ker h$ is not finitely generated.)

Let $x_0 \in X$ be a generator of $\mathbf{F}$ that belongs to the class $h^{-1}\{y_0\}$ (so, $h(x_0) = y_0$).

Claim 1.1. There exists $N < \omega$ such that for all (p_i, q_i) in K, if $p_i \geqslant x_0$, then $q_i \geqslant \beta_N(y_0)$.

Proof. Fix i and $(p_i, q_i) \in K$ (so, $h(p_i) = h(q_i)$). Define N_i as follows:

Case 0. If $p_i \ngeqslant x_0$, let $N_i = 0$.

Case 1. If $p_i \geqslant x_0$, then $x_0 = x_0 \wedge p_i$, so $y_0 = h(x_0) = h(x_0) \wedge h(p_i) \leqslant h(p_i)$, so $y_0 \leqslant h(q_i)$. Also, $h(x_0 \wedge q_i) = h(x_0) \wedge h(q_i) = y_0$, so $x_0 \wedge q_i \in h^{-1}\{y_0\}$. Therefore (since $\{\beta_i(y_0)\}$ is an infinite descending chain in $h^{-1}\{y_0\}$) there exists $n_i > 0$ such that $x_0 \wedge q_i \geqslant \beta_n(y_0)$. Let $N_i = n_i$ in this case (so $q_i \geqslant \beta_{N_i}(y_0)$).

Since K is finite, we can find such N_i for each $(p_i, q_i) \in K$. Let $N = \max\{N_i : 1 \leqslant i \leqslant m\}$. Then for all $1 \leqslant i \leqslant m$ the following implication holds:

$$p_i \geqslant x_0 \quad \Longrightarrow \quad q_i \geqslant \beta_N(y_0). \tag{B.1}$$

Claim 1.2. There exists $N < \omega$ such that, for all $(p, q) \in \langle K \rangle$,

$$p \geqslant x_0 \quad \Longrightarrow \quad q \geqslant \beta_N(y_0). \tag{B.2}$$

Proof. Choose N as described in the proof of Claim 1.1 above so that for all $(p_i, q_i) \in K$ the implication (B.1) holds. Fix $(p, q) \in \langle K \rangle$. We prove (B.2) by induction on the complexity of (p, q). If $(p, q) \in K$, then there's nothing to prove.

Case 1. Assume $(p, q) = (p_1, q_1) \wedge (p_2, q_2)$, where p_i, q_i $(i = 1, 2)$ satisfy (B.2). Assume $p \geqslant x_0$. Then $p = p_1 \wedge p_2 \geqslant x_0$, so $p_1 \geqslant x_0$ and $p_2 \geqslant x_0$, so (by the induction hypothesis) $q_1 \geqslant \beta_N(y_0)$ and $q_2 \geqslant \beta_N(y_0)$. Therefore, $q = q_1 \wedge q_2 \geqslant \beta_N(y_0)$, as desired.

Case 2. Assume $(p, q) = (p_1, q_1) \vee (p_2, q_2)$, where p_i, q_i $(i = 1, 2)$ satisfy (B.2). Assume $p \geqslant x_0$. Then $p = p_1 \vee p_2 \geqslant x$. Since x_0 is a generator, it is join prime in $\mathbf{F}(X)$, so either $p_1 \geqslant x_0$ or $p_2 \geqslant x_0$. Assume (wlog) $p_1 \geqslant x_0$. Then, (by induction hypothesis) $q_1 \geqslant \beta_N(y_0)$. Therefore, $q = q_1 \vee q_2 \geqslant q_1 \geqslant \beta_N(y_0)$, as desired.

Claim 1.3. K does not generate $\ker h$.

Proof. Let N be chosen as in the proof of Claim 1.2 above. Since $\beta_0(y_0) > \beta_1(y_0) > \cdots$ is an infinite descending chain, $\beta_N(y_0) > \beta_{N+1}(y_0)$. The pair $(p, q) = (x_0, \beta_{N+1}(y_0))$ does not belong to $\langle K \rangle$, however it does belong to the kernel of h. This proves that the finite subset K does not generate $\ker h$. Since K was an arbitrary finite subset of $\ker h$, we have proved that $\ker h$ is not finitely generated. □

Appendix C. Miscellaneous Notes

Let K be a finite subset of $\ker h$. Since K is finite, we can find an $N < \omega$ such that for all $\binom{p}{q} \in K$, the following implications are satisfied:

$$\begin{aligned} p \leqslant x &\implies q \leqslant x_N \\ p \leqslant y &\implies q \leqslant y_N \\ p \leqslant z &\implies q \leqslant z_N \end{aligned} \tag{C.1}$$

$$\begin{aligned} p \leqslant x \vee (y \wedge z) &\implies q \leqslant x_{N+1} \\ p \leqslant y \vee (x \wedge z) &\implies q \leqslant y_{N+1} \\ p \leqslant z \vee (x \wedge y) &\implies q \leqslant z_{N+1} \end{aligned} \tag{C.2}$$

Claim 4.3 If N is chosen as just described, and if $\binom{p}{q} \in \langle K\rangle$ then the implications C.1 and C.2 hold.

Proof. As usual, we proceed by induction on term complexity. If $\binom{p}{q} \in K$, then by choice of N, there is nothing to prove.

Case 1. Suppose $\binom{p}{q} = \binom{p_1}{q_1} \vee \binom{p_2}{q_2}$, where $\binom{p_1}{q_1}$ and $\binom{p_2}{q_2}$ satisfy (C.1) and (C.2). We show that $\binom{p}{q}$ satisfies these two implications as well. Recall, in the notation above, $x_1 := x \vee (y \wedge z)$.

Assume $p \leqslant x_1$. We show $q \leqslant x_{N+1}$. Since $p = p_1 \vee p_2 \leqslant x_1$, we have $p_1 \leqslant x_1$ and $p_2 \leqslant x_1$, so by the induction hypothesis, $q_1 \leqslant x_{N+1}$ and $q_2 \leqslant x_{N+1}$. Therefore, $q = q_1 \vee q_2 \leqslant x_{N+1}$, as desired.

Now assume $p \leqslant x$. We show $q \leqslant x_N$. Since $p = p_1 \vee p_2 \leqslant x$, we have $p_1 \leqslant x$ and $p_2 \leqslant x$, so by the induction hypothesis, $q_1 \leqslant x_N$ and $q_2 \leqslant x_N$. Therefore, $q = q_1 \vee q_2 \leqslant x_N$, as desired.

Case 2. Suppose $\binom{p}{q} = \binom{p_1}{q_1} \wedge \binom{p_2}{q_2}$, where $\binom{p_1}{q_1}$ and $\binom{p_2}{q_2}$ satisfy (C.1) and (C.2).

Assume $p \leqslant x_1 = x \vee (y \wedge z)$. We must show $q \leqslant x_{N+1}$. Since $p_1 \wedge p_2 \leqslant x_1$, then according to Theorem A.8, at least one of the following inequalities must hold:

(1) $p_1 \leqslant x_1$;
(2) $p_2 \leqslant x_1$;
(3) $p_1 \wedge p_2 \leqslant x$;
(4) $p_1 \wedge p_2 \leqslant y \wedge z$.

By the induction hypothesis, (1) implies $q_1 \leqslant x_{N+1}$ and (2) implies $q_2 \leqslant x_{N+1}$. In either case, $q = q_1 \wedge q_2 \leqslant x_{N+1}$, as desired. In case (3), Theorem A.8 implies that either $p_1 \leqslant x$ or $p_2 \leqslant x$, since x is a generator. Therefore, $q_1 \leqslant x_N$ or $q_2 \leqslant x_N$ and we conclude that $q \leqslant x_N \leqslant x_{N+1}$, as desired. It remains to prove $q \leqslant x_{N+1}$ for the final case in which $p_1 \wedge p_2 \leqslant y \wedge z$.

If $p_1 \wedge p_2 \leqslant y \wedge z$, then $p_1 \wedge p_2 \leqslant y$ and $p_1 \wedge p_2 \leqslant z$. Therefore, both of the following disjunctions hold:

- $p_1 \leqslant y$ or $p_2 \leqslant y$, and
- $p_1 \leqslant z$ or $p_2 \leqslant z$.

If $p_1 \leqslant y$ and $p_1 \leqslant z$, then $p_1 \leqslant x \vee (y \wedge z) = x_1$, so $q_1 \leqslant x_{N+1}$, so $q = q_1 \wedge q_2 \leqslant x_{N+1}$, as desired. Similarly, if $p_2 \leqslant y$ and $p_2 \leqslant z$, the desired conclusion holds. Finally, consider the case in which $p_1 \leqslant y$ and $p_2 \leqslant z$. In this case $q_1 \leqslant y_N$ and $q_2 \leqslant z_N$. Therefore, $q = q_1 \wedge q_2 \leqslant y_N \wedge z_N \leqslant x_N \vee (y_N \wedge z_N) = x_{N+1}$, as desired. □

References

[FJN95] Ralph Freese, Jaroslav Ježek, and J. B. Nation. *Free lattices*, volume 42 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1995. URL: http://dx.doi.org/10.1090/surv/042, doi:10.1090/surv/042.

[JN77] B. Jónsson and J. B. Nation. A report on sublattices of a free lattice. In *Contributions to universal algebra (Colloq., József Attila Univ., Szeged, 1975)*, pages 223–257. Colloq. Math. Soc. János Bolyai, Vol. 17. North-Holland, Amsterdam, 1977.

[McK72] Ralph McKenzie. Equational bases and nonmodular lattice varieties. *Trans. Amer. Math. Soc.*, 174:1–43, 1972.

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