KERNELS OF EPIMORPHISMS OF FINITELY GENERATED FREE LATTICES

WILLIAM DEMEO, PETER MAYR, AND NIK RUSKUC

1. Main Theorem

Let X be a finite set and $\mathbf{F} := \mathbf{F}(X)$ the free lattice generated by X.

Theorem 1. *Suppose $\mathbf{L} = \langle L, \wedge, \vee\rangle$ is a finite lattice and $h\colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ a lattice epimorphism. Then h is bounded if and only if $\ker h$ is finitely generated.*

Proof. ($\Rightarrow$) Assume h is bounded. That is, the preimage of each $y \in L$ under h is bounded. For each $y \in L$, let $\alpha y = \bigvee h^{-1}\{y\}$ and $\beta y = \bigwedge h^{-1}\{y\}$ denote the greatest and least elements of $h^{-1}\{y\}$, respectively (both of which exist by the boundedness assumption). Observe that $h\alpha h = h$, and $h\beta h = h$. In fact, α and β are adjoint to h. Indeed, it is easy to see that

$$hx \leqslant y \quad\Leftrightarrow\quad x \leqslant \alpha y,$$

$$y \leqslant hx \quad\Leftrightarrow\quad \beta y \leqslant x.$$

For each $y \in L$, let $X_y := X \cap h^{-1}\{y\}$, the set of generators that lie in the inverse image of y under h. Let G be the (finite) set of pairs in $\mathbf{F} \times \mathbf{F}$ defined as follows:

$$G = \bigcup_{y \in L} \{(x, \alpha y), (\alpha y, x), (x, \beta y), (\beta y, x), (\alpha y, \beta y), (\beta y, \alpha y) : x \in X_y\}.$$

We claim that G generates $\ker h$. To prove this, we first show, by induction on term complexity, that for every $y \in L$, for every $r \in h^{-1}\{y\}$, the pairs $(r, \alpha y)$ and $(r, \beta y)$ belong to the sublattice $\langle G\rangle \leqslant \mathbf{F} \times \mathbf{F}$ generated by G.
Case 0. If $r \in X$, then $(r, \alpha y)$ and $(r, \beta y)$ belong to G itself, so there's nothing to prove.
Case 1. Suppose $r = s \vee t$ and assume (the induction hypothesis) that $(s, \alpha h(s))$, $(s, \beta h(s))$, $(t, \alpha h(t))$, and $(t, \beta h(t))$ belong to $\langle G\rangle$. Then $y = h(r) = h(s \vee t) = h(s) \vee h(t)$, so

$$h(\alpha h(s) \vee \alpha h(t)) = h\alpha h(s) \vee h\alpha h(t) = h(s) \vee h(t) = y.$$

Similarly, $h(\beta h(s) \vee \beta h(t)) = h(s) \vee h(t) = y$. Therefore,

$$\beta y \leqslant \beta h(s) \vee \beta h(t) \leqslant \alpha h(s) \vee \alpha h(t) \leqslant \alpha y.$$

Also, $r \leqslant \alpha y$, so $r = \alpha y \wedge (s \vee t)$. Taken together, these observations yield

$$\begin{aligned}\begin{pmatrix} r \\ \beta y \end{pmatrix} &= \begin{pmatrix} \alpha y \wedge (s \vee t) \\ \beta y \end{pmatrix} = \begin{pmatrix} \alpha y \wedge (s \vee t) \\ \beta y \wedge (\beta h(s) \vee \beta h(t)) \end{pmatrix} \\ &= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \left[\begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \vee \begin{pmatrix} t \\ \beta h(t) \end{pmatrix} \right],\end{aligned}$$

and each term in the last expression belongs to $\langle G\rangle$, so $(r, \beta y) \in \langle G\rangle$, as desired.

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Similarly, $(r, \alpha y) \in \langle G\rangle$. Indeed, $\beta y \leqslant r$ implies $r = \beta y \vee s \vee t$, and $\beta h(s) \vee \beta h(t) \leqslant \alpha y$ implies $\alpha y = \alpha y \vee \beta h(s) \vee \beta h(t)$. Therefore,

$$\begin{pmatrix} r \\ \alpha y \end{pmatrix} = \begin{pmatrix} \beta y \vee s \vee t \\ \alpha y \vee \beta h(s) \vee \beta h(t) \end{pmatrix} = \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \vee \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}.$$

Case 2. Suppose $r = s \wedge t$ and assume $(s, \alpha h(s))$, $(s, \beta h(s))$, $(t, \alpha h(t))$, and $(t, \beta h(t))$ belong to $\langle G\rangle$. Then $h(s \wedge t) = h(r) = y$, so $h(\alpha h(s) \wedge \alpha h(t)) = y = h(\beta h(s) \wedge \beta h(t))$, so

$$\beta y \leqslant \beta h(s) \wedge \beta h(t) \leqslant \alpha h(s) \wedge \alpha h(t) \leqslant \alpha y.$$

Also, $\beta y \leqslant r \leqslant \alpha y$ so $r = \alpha y \wedge s \wedge t$ and $r = \beta y \vee (s \wedge t)$. Taken together, these observations yield

$$\begin{pmatrix} r \\ \alpha a \end{pmatrix} = \begin{pmatrix} \beta y \vee (s \wedge t) \\ \alpha y \vee (\alpha h(s) \wedge \alpha h(t)) \end{pmatrix} = \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \left[\begin{pmatrix} s \\ \alpha h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \alpha h(t) \end{pmatrix}\right],$$

and each term in the last expression belongs to $\langle Y\rangle$.

Note, we could have used β's instead:

$$\begin{pmatrix} r \\ \alpha y \end{pmatrix} = \begin{pmatrix} \beta y \vee (s \wedge t) \\ \alpha y \vee (\beta h(s) \wedge \beta h(t)) \end{pmatrix} = \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \left[\begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}\right].$$

Similarly,

$$\begin{pmatrix} r \\ \beta y \end{pmatrix} = \begin{pmatrix} \alpha y \wedge s \wedge t \\ \beta y \wedge \alpha h(s) \wedge \alpha h(t) \end{pmatrix} = \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \begin{pmatrix} s \\ \alpha h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \alpha h(t) \end{pmatrix}.$$

Again, we could have used β's instead:

$$\begin{pmatrix} r \\ \beta y \end{pmatrix} = \begin{pmatrix} \alpha y \wedge s \wedge t \\ \beta y \wedge \beta h(s) \wedge \beta h(t) \end{pmatrix} = \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}.$$

In each case, we end up with an expression involving terms from $\langle G\rangle$, and this proves that $(r, \alpha y)$ and $(r, \beta y)$ belong to $\langle G\rangle$.

($\Leftarrow$) Suppose h is not lower bounded. Then there exists an element $y_0 \in L$ such that $\beta_0(y_0) > \beta_1(y_0) > \cdots$ is an infinite descending chain. This is a consequence of the definitions and remarks excerpted from Freese, Jezek, Nation [FJN95] and paraphrased in the appendix below. (Specifically, see Section A.1 on *bounded homomorphisms*.)

Let K be a finite subset of $\ker h$, say, $K = \{(p_1, q_1), \dots, (p_m, q_m)\} \subseteq \ker h$. We prove $\langle K\rangle \neq \ker h$. (Since K is an arbitrary finite subset of $\ker h$, this will prove $\ker h$ is not finitely generated.)

Let $x_0 \in X$ be a generator of $\mathbf{F}$ that belongs to the class $h^{-1}\{y_0\}$ (so, $h(x_0) = y_0$).

Claim 1.1. There exists $N < \omega$ such that for all (p_i, q_i) in K, if $p_i \geqslant x_0$, then $q_i \geqslant \beta_N(y_0)$.

Proof. Fix i and $(p_i, q_i) \in K$ (so, $h(p_i) = h(q_i)$). Define N_i as follows:

Case 0. If $p_i \ngeqslant x_0$, let $N_i = 0$.

Case 1. If $p_i \geqslant x_0$, then $x_0 = x_0 \wedge p_i$, so $y_0 = h(x_0) = h(x_0) \wedge h(p_i) \leqslant h(p_i)$, so $y_0 \leqslant h(q_i)$. Also, $h(x_0 \wedge q_i) = h(x_0) \wedge h(q_i) = y_0$, so $x_0 \wedge q_i \in h^{-1}\{y_0\}$. Therefore (since $\{\beta_i(y_0)\}$ is an infinite descending chain in $h^{-1}\{y_0\}$) there exists $n_i > 0$ such that $x_0 \wedge q_i \geqslant \beta_n(y_0)$. Let $N_i = n_i$ in this case (so $q_i \geqslant \beta_{N_i}(y_0)$).

Since K is finite, we can find such N_i for each $(p_i, q_i) \in K$. Let $N = \max\{N_i : 1 \leqslant i \leqslant m\}$. Then for all $1 \leqslant i \leqslant m$ the following implication holds:

$$p_i \geqslant x_0 \quad \Longrightarrow \quad q_i \geqslant \beta_N(y_0). \tag{1}$$

Claim 1.2. There exists $N < \omega$ such that, for all $(p, q) \in \langle K\rangle$,

$$p \geqslant x_0 \quad\Longrightarrow\quad q \geqslant \beta_N(y_0). \tag{2}$$

Proof. Choose N as described in the proof of Claim 1.1 above so that for all $(p_i, q_i) \in K$ the implication (1) holds. Fix $(p, q) \in \langle K\rangle$. We prove (2) by induction on the complexity of (p, q). If $(p, q) \in K$, then there's nothing to prove.

Case 1. Assume $(p, q) = (p_1, q_1) \wedge (p_2, q_2)$, where p_i, q_i $(i = 1, 2)$ satisfy (2). Assume $p \geqslant x_0$. Then $p = p_1 \wedge p_2 \geqslant x_0$, so $p_1 \geqslant x_0$ and $p_2 \geqslant x_0$, so (by the induction hypothesis) $q_1 \geqslant \beta_N(y_0)$ and $q_2 \geqslant \beta_N(y_0)$. Therefore, $q = q_1 \wedge q_2 \geqslant \beta_N(y_0)$, as desired.

Case 2. Assume $(p, q) = (p_1, q_1) \vee (p_2, q_2)$, where p_i, q_i $(i = 1, 2)$ satisfy (2). Assume $p \geqslant x_0$. Then $p = p_1 \vee p_2 \geqslant x$. Since x_0 is a generator, it is join prime in $\mathbf{F}(X)$, so either $p_1 \geqslant x_0$ or $p_2 \geqslant x_0$. Assume (wlog) $p_1 \geqslant x_0$. Then, (by induction hypothesis) $q_1 \geqslant \beta_N(y_0)$. Therefore, $q = q_1 \vee q_2 \geqslant q_1 \geqslant \beta_N(y_0)$, as desired.

Claim 1.3. K does not generate $\ker h$.

Proof. Let N be chosen as in the proof of Claim 1.2 above. Since $\beta_0(y_0) > \beta_1(y_0) > \cdots$ is an infinite descending chain, $\beta_N(y_0) > \beta_{N+1}(y_0)$. The pair $(p, q) = (x_0, \beta_{N+1}(y_0))$ does not belong to $\langle K\rangle$, however it does belong to the kernel of h. This proves that the finite subset K does not generate $\ker h$. Since K was an arbitrary finite subset of $\ker h$, we have proved that $\ker h$ is not finitely generated.

□

2. Examples

Let $\mathbf{M_3} = \langle\{0, a, b, c, 1\}, \wedge, \vee\rangle$, where $a \wedge b = a \wedge c = b \wedge c = 0$ and $a \vee b = a \vee c = b \vee c = 1$. Let $\mathbf{F} := \mathbf{F}(x, y, z)$ denote the free lattice generated by $\{x, y, z\}$.

Theorem 2. *Let $h\colon \mathbf{F} \twoheadrightarrow \mathbf{M_3}$ be the epimorphism that acts on the generators as follows: $x \mapsto a$, $y \mapsto b$, $z \mapsto c$. Then $\ker h$ is not finitely generated.*

Proof. Let $K := \ker h$, and for $u \in \{x, y, z\}$ let $C_u := u/K := \{v \in F : h(v) = h(u)\}$. Define sequences of elements in these classes by the following mutual recursions:

- for $i \in \mathbb{N}$,
$$m_{0,i} = (m_{x,i} \wedge m_{y,i}) \vee (m_{x,i} \wedge m_{z,i}) \vee (m_{y,i} \wedge m_{z,i});$$
- for $u \in \{x, y, z\}$,
$$m_{u,0} = u,$$
$$m_{u,i+1} = m_{u,i} \vee m_{0,i}.$$

Notice that $m_{0,0} = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ and $m_{x,i+1} = m_{x,i} \vee (m_{y,i} \wedge m_{z,i})$.

Let X be a finite subset of K. We will prove there exists $(p, q) \in K \setminus \langle X\rangle$. Fix $u \in \{0, x, y, z\}$. Since X is finite, Lemma 4 implies that there exists $M \in \mathbb{N}$ such that for every $(p, q) \in X$ with $p, q \in C_u$, we have $p, q \leqslant m_{u,M}$.

Claim 2.1. For $(p, q) \in \langle X\rangle$ and $u \in \{x, y, z\}$, the following implication holds:

$$q \leqslant u \quad\Longrightarrow\quad p \leqslant m_{u,M}. \tag{3}$$

We prove the claim by induction on the complexity of terms. Fix $(p, q) \in \langle X\rangle$. Then $p, q \in C_u$ for some $u \in \{x, y, z\}$.

Case 0. Suppose $(p, q) \in X$. Then by definition of M we have $p, q \leqslant m_{u,M}$.

Case 1. Suppose $(p,q) = (p_1,q_1) \wedge (p_2,q_2)$, where (p_i,q_i) satisfies (3) for $i = 1,2$. If $q = q_1 \wedge q_2 \leqslant u$, then, since generators in the free lattice are meet-prime (see Theorem 1.11 below), we have $q_1 \leqslant u$ or $q_2 \leqslant u$. Assume $q_1 \leqslant u$. Then, by the induction hypothesis, $p_1 \leqslant m_{u,M}$. Therefore, $p = p_1 \wedge p_2 \leqslant m_{u,M}$, as desired.

Case 2. Suppose $(p,q) = (p_1,q_1) \vee (p_2,q_2)$, where (p_i,q_i) satisfies (3) for $i = 1,2$. If $q = q_1 \vee q_2 \leqslant u$, then $q_i \leqslant u$ for $i = 1,2$. It now follows from the induction hypothesis that $p_i \leqslant m_{u,M}$ for $i = 1,2$, so $p = p_1 \vee p_2 \leqslant m_{u,M}$, as desired.

From Claim 2.1, and Lemma 3, it follows that $(m_{x,M+1}, x) \in K \setminus \langle X\rangle$, so the proof of the theorem is complete. □

Lemma 3. *For each $u \in \{0,x,y,z\}$, the sequence $\{m_{u,n} : n \in \mathbb{N}\}$ is a strictly ascending chain; that is, $m_{u,0} < m_{u,1} < m_{u,2} < \cdots$.*

Proof.
Case. $u \in \{x,y,z\}$.

For simplicity, assume $u = x$ for the remainder of the proof of this case. (Of course, the same argument goes through when u is y or z.) Fix $n \in \mathbb{N}$. We prove $m_{x,n} < m_{x,n+1}$.
Subclaim. For all $n \in \mathbb{N}$,

(1) $m_{x,n} \in C_x$,
(2) $m_{x,n} \ngeqslant y$, and $m_{x,n} \ngeqslant z$.

The first subclaim is obvious. For the second, if $m_{x,n} \geqslant y$, then $m_{x,n} \wedge y = y$, and then $0 = h(m_{x,n} \wedge y) = h(y) = b$. A similar contradiction is reach if we assume $m_{x,n} \geqslant z$, so the subclaim is proved.

Recall, $m_{x,n} = m_{x,n} \vee (m_{y,n} \wedge m_{z,n})$, so our desired conclusion, $m_{x,n} < m_{x,n+1}$, holds unless $m_{x,n} \geqslant m_{y,n} \wedge m_{z,n}$. So, by way of contradiction, suppose

$$m_{x,n} \geqslant m_{y,n} \wedge m_{z,n}. \tag{4}$$

Now, $m_{y,n} = y \vee (x \wedge z) \vee \cdots$, so clearly $m_{y,n} \geqslant y$. Similarly, $m_{z,n} \geqslant z$. This, together with (4), implies $m_{x,n} \geqslant m_{y,n} \wedge m_{z,n} \geqslant y \wedge z$. But then Theorem 1.11 below implies that either $m_{x,n} \geqslant y$ or $m_{x,n} \geqslant z$, which contradicts Subclaim 2 above.

Case. $u = 0$.

We first prove that $m_{0,0} = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ is strictly below $m_{0,1} = (m_{x,1} \wedge m_{y,1}) \vee (m_{x,1} \wedge m_{z,1}) \vee (m_{y,1} \wedge m_{z,1})$.

By symmetry, it suffices to show $x \wedge y < m_{x,1} \wedge m_{y,1}$; that is, $x \wedge y < (x \vee (y \wedge z)) \wedge (y \vee (x \wedge z))$.

Clearly $x \wedge y \leqslant (x \vee (y \wedge z)) \wedge (y \vee (x \wedge z))$. Suppose $x \wedge y = (x \vee (y \wedge z)) \wedge (y \vee (x \wedge z))$. Then $(x \vee (y \wedge z)) \wedge (y \vee (x \wedge z)) \leqslant x$. By Theorem 1.11, the latter holds iff $x \vee (y \wedge z) \leqslant x$ or $y \vee (x \wedge z) \leqslant x$ The first of these inequalities is clearly false, so it must be the case that $y \vee (x \wedge z) \leqslant x$. But then $y \leqslant x$, which is obviously false. We conclude that $x \wedge y < (x \vee (y \wedge z)) \wedge (y \vee (x \wedge z))$. This proves $m_{0,0} < m_{0,1}$.

Now fix $n \in \mathbb{N}$ and assume $m_{0,n} < m_{0,n+1}$. We show $m_{0,n+1} < m_{0,n+2}$.
(**To do:** complete the proof in this case; i.e., for $u = 0$.)
–scratch work–
$m_{0,n} := (m_{x,n} \wedge m_{y,n}) \vee (m_{x,n} \wedge m_{z,n}) \vee (m_{y,n} \wedge m_{z,n})$,
$m_{0,n+1} := (m_{x,n+1} \wedge m_{y,n+1}) \vee (m_{x,n+1} \wedge m_{z,n+1}) \vee (m_{y,n+1} \wedge m_{z,n+1})$,
By the first Case above, $m_{u,n} < m_{u,n+1}$.

□

Lemma 4. *For all $u \in \{x,y,z\}$ and $p \in C_u \cup C_0$ there exists $n \in \mathbb{N}$ such that $p \leqslant m_{u,n}$.*

Proof. (By induction on the complexity of p.)
Case 0. $p \in \{x, y, z\}$. Then $u = p = m_{p,0}$.

For the remaining cases assume $u = x$, without loss of generality.
Case 1. $p = p_1 \vee p_2$. If $p \in C_x \cup C_0$, then $p_i \in C_x \cup C_0$ for $i = 1, 2$, and the induction hypothesis yields i and j for which $p_1 \leqslant m_{x,i}$ and $p_2 \leqslant m_{x,j}$. Letting $n = \max\{i, j\}$, we have $p_1, p_2 \leqslant m_{x,n}$, from which $p = p_1 \vee p_2 \leqslant m_{x,n}$, as desired.
Case 2. $p = p_1 \wedge p_2$. If $p \in C_x$, then we may assume $p_1 \in C_x$ and $p_2 \in C_x \cup C_0$. By the induction hypothesis, there exists $n \in \mathbb{N}$ such that $p_1 \leqslant m_{x,n}$, whence $p \leqslant p_1 \leqslant m_{x,n}$. If $p \in C_0$, then each p_i belongs to $C_u \cup C_0$ for some $u \in \{x, y, z\}$. If $p_1 \in C_x \cup C_0$, then $p_1 \leqslant m_{x,n}$, as above and we're done. Similarly, if $p_2 \in C_x \cup C_0$. So assume $p_1 \in C_y \cup C_0$ and $p_2 \in C_z \cup C_0$. Then the induction hypothesis implies that there exist i and j such that $p_1 \leqslant m_{y,i}$ and $p_2 \leqslant m_{z,j}$. If $n = \max\{i, j\}$, then $p_1 \leqslant m_{y,n}$ and $p_2 \leqslant m_{z,n}$. Then, by the above definition of the sequences, we have $p_1 \wedge p_2 \leqslant m_{y,n} \wedge m_{z,n} \leqslant m_{0,n} \leqslant m_{x,n+1}$. □

2.1. **Other Examples.** In each of the examples below, X is a finite set and $\mathbf{F} = \mathbf{F}(X)$ is the free lattice generated by X. The symbol F denotes the universe of $\mathbf{F}$.

Ex 1. Let $X = \{x, y, z\}$, and let $\mathbf{L} = \mathbf{2}$ be the 2-element chain. Then the kernel of an epimorphism $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ is a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$.

Ex 2. Let $X = \{x, y, z\}$ and let $\mathbf{L} = \mathbf{3}$ be the 3-element chain. Then the kernel of an epimorphism $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ is finitely generated.

Ex 4. Let $n > 2$, $X = \{x_0, x_1, \dots, x_{n-1}\}$, and $\mathbf{L} = \mathbf{2} \times \mathbf{2}$. Let $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ be an epimorphism. Then $K = \ker h$ is finitely generated.

Proof of Ex 4. Let the universe of $\mathbf{L} = \mathbf{2} \times \mathbf{2}$ be $\{0, a, b, 1\}$, where $a \vee b = 1$ and $a \wedge b = 0$. For each $y \in L$, denote by $X_y = X \cap h^{-1}\{y\}$ the set of generators mapped by h to y. Denote the least and greatest elements of $h^{-1}\{y\}$ (if they exist) by ℓ_y and g_y, respectively. For example, $\ell_a = \bigwedge h^{-1}\{a\} = \bigwedge\{x \in F : h(x) = a\}$, $\quad g_b = \bigvee\{x \in F : h(x) = b\}$, $\quad$ etc. In the present example, the least and greatest elements exist is each case, as we now show.

Claim 4.1. $h^{-1}\{a\}$ has least and greatest elements, namely $\ell_a = \bigwedge(X_a \cup X_1)$ and $g_a = \bigvee(X_a \cup X_0)$. (Similarly, $h^{-1}\{b\}$ has least and greatest elements, ℓ_b and g_b.)

Claim 4.2. $h^{-1}\{0\}$ has least and greatest elements, namely, $\ell_0 = \bigwedge X$ and $g_0 = g_a \wedge g_b$.

Claim 4.3. $h^{-1}\{1\}$ has least and greatest elements, namely $\ell_1 = \ell_a \vee \ell_b$ and $g_1 = \bigvee X$.

Proof of Claim 4.1. Let $M(a) := \bigwedge(X_a \cup X_1)$ and $J(a) := \bigvee(X_a \cup X_0)$ and note that these values exist in F, since the sets involved are finite. Also, Then $h(M(a)) = a = h(J(a))$. Fix $r \in h^{-1}\{a\}$.

Case 0. $r \in X_a$. Then $r \geqslant \bigwedge X_a \geqslant \bigwedge(X_a \cup X_1) = M(a)$.
Case 1. $r = s \vee t$, where $h(s) = a$ and $h(t) \in \{a, 0\}$. Assume (the induction hypothesis) that $s \geqslant M(a)$. Then $r = s \vee t \geqslant M(a)$.
Case 2. $r = s \wedge t$, where $h(s) = a$ and $h(t) \in \{a, 1\}$. Assume (the induction hypothesis) that $s, t \geqslant M(a)$. Then $s \wedge t \geqslant M(a)$. This proves that for each $r \in h^{-1}\{a\}$ we have $r \geqslant M(a)$, and as we noted at the outset, $M(a) \in h^{-1}\{a\}$. Therefore, $\ell_a = M(a)$ is the least element of $h^{-1}\{a\}$. Similarly, every $r \in h^{-1}\{a\}$ is below $J(a)$, so $g_a = J(a)$. The proofs of $\ell_b = M(b)$ and $g_b = J(b)$ are similar.

Proof of Claim 4.2. $\ell_0 = \bigwedge X$ is obvious, so we need only verify that $g_0 = g_a \wedge g_b$. Observe that $h(g_a \wedge g_b) = h(g_a) \wedge h(g_b) = a \wedge b = 0$, so $g_a \wedge g_b \in h^{-1}\{0\}$. It remains to prove that $r \leqslant g_a \wedge g_b$ holds for all $r \in h^{-1}\{0\}$. Fix $r \in h^{-1}\{0\}$. Then $h(r \vee g_a) = h(r) \vee h(g_a) = 0 \vee a = a$, which places $r \vee g_a$ in $h^{-1}\{a\}$. Therefore, by maximality of g_a, we have $r \vee g_a \leqslant g_a$, whence $r \leqslant g_a$. Similarly, $r \leqslant g_b$.

Proof of Claim 4.3. $g_1 = \bigvee X$ is obvious, so we need only verify that $\ell_1 = \ell_a \vee \ell_b$. Observe that $h(\ell_a \vee \ell_b) = h(\ell_a) \vee h(\ell_b) = a \vee b = 1$, so $\ell_a \vee \ell_b \in h^{-1}\{1\}$. It remains to prove that $r \geqslant \ell_a \vee \ell_b$ holds for all $r \in h^{-1}\{1\}$. Fix $r \in h^{-1}\{1\}$. Then $h(r \wedge \ell_a) = h(r) \wedge h(\ell_a) = 1 \wedge a = a$, which places $r \wedge \ell_a$ in $h^{-1}\{a\}$. Therefore, by minimality of ℓ_a, we have $r \wedge \ell_a \geqslant \ell_a$, whence $r \geqslant \ell_a$. Similarly, $r \geqslant \ell_b$. Now let $Y = \{(x, g_p), (g_p, x), (x, \ell_p), (\ell_p, x) : p \in \{0, a, b, 1\}, x \in X_p\}$.

Claim 4. If $r \in F$ and $h(r) = p$, then $(r, \ell_p), (r, g_p) \in \langle Y\rangle$.
Proof.
Case 0. If $r \in X_p$, then the pair belongs to Y and the claim is trivial.

Case 1. Suppose $r = s \wedge t$.

Subcase 1.1. $h(r) = 1$ implies $h(s) = h(t) = 1$. If we assume (the induction hypothesis) that $(s, \ell_1), (s, g_1), (t, \ell_1), (t, g_1)$ belong to $\langle Y\rangle$, then $(r, \ell_1) = (s \wedge t, \ell_1) = (s, \ell_1) \wedge (t, \ell_1) \in \langle Y\rangle$.

Subcase 1.2. $h(r) = a$ implies (wlog) $h(s) = a$ and $h(t) \in \{a, 1\}$. Assume (the induction hypothesis) that $(s, \ell_a), (s, g_a), (t, \ell_p), (t, g_p)$ belong to $\langle Y\rangle$. By Claim 1, $\ell_a \leqslant \ell_1$, so $\ell_a = \ell_a \wedge \ell_1$.
- If $h(t) = 1$, then $(r, \ell_a) = (s \wedge t, \ell_a \wedge \ell_1) = (s, \ell_a) \wedge (t, \ell_1) \in \langle Y\rangle$.
- If $h(t) = a$, then $(r, \ell_a) = (s \wedge t, \ell_a \wedge \ell_a) = (s, \ell_a) \wedge (t, \ell_a) \in \langle Y\rangle$.

Subcase 1.3. $h(r) = 0$ implies (wlog) that either (i) $h(s) = 0$, or (ii) $h(s) = a$, $h(t) = b$. If $h(s) = 0$, then $(s, \ell_0) \in \langle Y\rangle$ implies $(r, \ell_0) = (s \wedge t, \ell_0) = (s, \ell_0) \wedge (t, \ell_p) \in \langle Y\rangle$. If If $h(s) = a$, $h(t) = b$, and $(s, \ell_a), (t, \ell_b) \in \langle Y\rangle$, then $(r, \ell_0) = (s \wedge t, \ell_0) = (s, \ell_a) \wedge (t, \ell_b) \in \langle Y\rangle$.

Similarly, in each of the Subcases 1.1–1.3, we have $(r, g_p) \in \langle Y\rangle$.

Case 2. Suppose $r = s \vee t$.

Subcase 2.1. $h(r) = 0$ implies $h(s) = h(t) = 0$. If we assume (the induction hypothesis) that $(s, \ell_p), (s, g_p), (t, \ell_p), (t, g_p)$ belong to $\langle Y\rangle$, then $(r, \ell_p) = (s \vee t, \ell_p) = (s, \ell_p) \vee (t, \ell_p) \in \langle Y\rangle$.

Subcase 2.2. $h(r) = a$ implies (wlog) $h(s) = a$ and $h(t) \in \{a, 0\}$. If we assume (the induction hypothesis) that $(s, \ell_p), (s, g_p), (t, \ell_p), (t, g_p)$ belong to $\langle Y\rangle$, then $(r, \ell_p) = (s \vee t, \ell_p) = (s, \ell_p) \vee (t, \ell_p) \in \langle Y\rangle$.

Subcase 2.3. $h(r) = 1$ implies (wlog) that either (i) $h(s) = 1$, or (ii) $h(s) = a$, $h(t) = b$.
- If $h(s) = 1$, then $(s, \ell_1) \in \langle Y\rangle$ implies $(r, \ell_1) = (s \vee t, \ell_1) = (s, \ell_1) \vee (t, \ell_p) \in \langle Y\rangle$.
- If $h(s) = a$, $h(t) = b$, and $(s, \ell_a), (t, \ell_b) \in \langle Y\rangle$, then $(r, \ell_1) = (s \vee t, \ell_1) = (s, \ell_a) \vee (t, \ell_b) \in \langle Y\rangle$.

Similarly, in each of the Subcases 2.1–2.3, we have $(r, g_p) \in \langle Y\rangle$.

Appendix A. Background

Here are some useful definitions and results from the Free Lattices book by Freese, Jezek, and Nation [FJN95].

Definition 5 (length of a term)**.** Let X be a set. Each element of X is a term of length 1, also known as a *variable*. If $t_1, \dots, t_n$ are terms of lengths $k_1, \dots, k_n$, then $t_1 \vee \dots \vee t_n$ and $t_1 \wedge \dots \wedge t_n$ are both terms of length $1 + k_1 + \dots + k_n$.

Examples. By the above definition, the terms
$x \vee y \vee z \qquad x \vee (y \vee z) \qquad (x \vee y) \vee z$
have lengths 4, 5, and 5, respectively. Reason: variables have length 1, so $x \vee y \vee z$ has length $1+1+1+1$. On the other hand, $x \vee y$ is a term of length 3, so $(x \vee y) \vee z$ has length $1+3+1$. Similarly, $x \vee (y \vee z)$ has length $1+1+3$.

Lemma 6 ([FJN95, Lem. 1.2])**.** *Let $\mathcal{V}$ be a nontrivial variety of lattices and let $\mathbf{F}_{\mathcal{V}}(X)$ be the relatively free lattice in $\mathcal{V}$ over X. Then,*

$$\bigwedge S \leqslant \bigvee T \text{ implies } S \cap T \neq \emptyset \text{ for each pair of finite subsets } S, T \subseteq X. \tag{5}$$

Lemma 7 ([FJN95, Lem. 1.4])**.** *Let $\mathbf{L}$ be a lattice generated by a set X and let $a \in L$. Then*

(1) if a is join prime, then $a = \bigwedge S$ for some finite subset $S \subseteq X$.
(2) if a is meet prime, then $a = \bigvee S$ for some finite subset $S \subseteq X$.
If X satisfies condition (5) above, then
(3) for every finite, nonempty subset $S \subset X$, $\bigwedge S$ is join prime and $\bigvee S$ is meet prime.

Corollary 8 ([FJN95, Cor. 1.5])**.** *Let $\mathcal{V}$ be a nontrivial variety of lattices and let $\mathbf{F}_{\mathcal{V}}(X)$ be the relatively free lattice in $\mathcal{V}$ over X. For each finite nonempty subset $S \subseteq X$, $\bigwedge S$ is join prime and $\bigvee S$ is meet prime. In particular, every $x \in X$ is both join and meet prime. Moreover, if $x \leqslant y$ for $x, y \in X$, then $x = y$.*

Theorem 9 (Whitman's Condition, ver. 1)**.** *The free lattice $\mathbf{F}(X)$ satisfies the following condition:*

(W) If $v = v_1 \wedge \dots \wedge v_r \leqslant u_1 \vee \dots \vee u_s = u$, then either $v_i \leqslant u$ for some i, or $v \leqslant u_j$ for some j.

Corollary 10 ([FJN95, Cor. 1.9])**.** *Every sublattice of a free lattice satisfies (W). Every element of a lattice satisfying (W) is either join or meet irreducible.*

Theorem 11 (Whitman's Condition, ver. 2)**.** *The free lattice $\mathbf{F}(X)$ satisfies the following condition:*

(W+) If $v = v_1 \wedge \dots \wedge v_r \wedge x_1 \wedge \dots \wedge x_n \leqslant u_1 \vee \dots \vee u_s \vee y_1 \vee \dots \vee y_m = u$, where $x_i, y_j \in X$, then either $x_i = y_j$ for some i and j, or $v_i \leqslant u$ for some i, or $v \leqslant u_j$ for some j.

Theorem 12 ([FJN95, Thm. 1.11])**.** *If $s = s(x_1, \dots, x_n)$ and $t = t(x_1, \dots, x_n)$ are terms and $x_1, \dots, x_n \in X$, then the truth of*

$$s^{\mathbf{F}(X)} \leqslant t^{\mathbf{F}(X)} \tag{6}$$

can be determined by applying the following rules.

(1) If $s = x_i$ and $t = x_j$, then (6) holds iff $x_i = x_j$.
(2) If $s = s_1 \vee \dots \vee s_k$ is a formal join, then (6) holds iff $s_i^{\mathbf{F}(X)} \leqslant t^{\mathbf{F}(X)}$ for all i.
(3) If $t = t_1 \wedge \dots \wedge t_k$ is a formal meet, then (6) holds iff $s^{\mathbf{F}(X)} \leqslant t_i^{\mathbf{F}(X)}$ for all i.
(4) If $s = x_i$ and $t = t_1 \vee \dots \vee t_k$ is a formal join, then (6) holds iff $x_i \leqslant t_j^{\mathbf{F}(X)}$ for some j.

(5) *If* $s = s_1 \wedge \cdots \wedge s_k$ *is a formal meet and* $t = x_i$*, then (6) holds iff* $s_j^{\mathbf{F}(X)} \leqslant x_i$ *for some* j*.*
(6) *If* $s = s_1 \wedge \cdots \wedge s_k$ *is a formal meet and and* $t = t_1 \vee \cdots \vee t_m$ *is a formal join, then (6) holds iff* $s_i^{\mathbf{F}(X)} \leqslant t^{\mathbf{F}(X)}$ *for some* i *or* $s^{\mathbf{F}(X)} \leqslant t_j^{\mathbf{F}(X)}$ *for some* j

Definition 13 (up directed, continuous)**.** A subset A of a lattice L is said to be *up directed* if every finite subset of A has an upper bound in A. It suffices to check this for pairs. A is up directed iff for all $a, b \in A$ there exists $c \in A$ such that $a \leqslant c$ and $b \leqslant c$. A lattice is *upper continuous* if whenever $A \subseteq L$ is an up directed set having a least upper bound $u = \bigvee A$, then for every b,

$$\bigvee_{a\in A}(a \wedge b) = \bigvee_{a \in A} a \wedge b = u \wedge b.$$

Down directed and *down continuous* are defined dually. A lattice that is both up and down continuous is called *continuous*.

Theorem 14 ([FJN95, Thm. 1.22])**.** *Free lattices are continuous.*

A.1. **Bounded Homomorphisms.** Following Freese, Jezek, Nation [FJN95], we define a pair of closure operators, denoted by superscripts $^\wedge$ and $^\vee$, on subsets of an arbitrary lattice $\mathbf{L} = \langle L, \vee, \wedge\rangle$ as follows: For each $A \subseteq L$, let

$$A^\wedge = \{\bigwedge B : B \text{ is a finite subset of } A\}.$$

We adopt the convention that if $\mathbf{L}$ has a greatest element $1_{\mathbf{L}}$, then $\bigwedge \emptyset = 1_{\mathbf{L}}$, and we include this in $A^\wedge$ for every $A \subseteq L$. (For lattices without a greatest element, $\bigwedge\emptyset$ is undefined.) The set $A^\vee$ is defined dually.

If $\mathbf{K} = \langle K, \vee, \wedge\rangle$ is a lattice generated by a finite set X, then we can write K as a union of a chain of subsets $H_0 \subseteq H_1 \subseteq \cdots$ defined inductively by setting $H_0 = X^\wedge$ and $H_{k+1} = (H_k)^{\vee\wedge}$, for all $k \geqslant 0$. By induction, each $H_n = X^{\wedge(\vee\wedge)^n}$ is a finite meet-closed subset of K, and $\bigcup H_n = K$, since X generates $\mathbf{X}$.

Let $h\colon \mathbf{K} \to \mathbf{L}$ be a lattice epimorphism and, for each $y \in L$ and $k < \omega$, define

$$\beta_k(y) = \bigwedge\{w \in H_k : h(w) \geqslant a\}.$$

On page 30 of [FJN95], immediately after Theorem 2.4, the authors make the following remark, which is a crucial ingredient of our proof:

"...[h is lower bounded] if and only if for each $a \leqslant h(1_{\mathbf{K}})$ there exists $N \in \omega$ such that $\beta_n(a) = \beta_N(a)$ for all $n \geqslant N$."

Equivalently,

$$\begin{aligned} h \text{ is not lower bounded} \quad &\iff (\exists y_0 \in L)(\forall N)(\exists n > N)\,\beta_n(a) \neq \beta_N(a)\\ &\iff (\exists y_0 \in L)(\exists N)(\forall n > N)\beta_n(a) \neq \beta_N(a). \end{aligned}$$

References

[FJN95] Ralph Freese, Jaroslav Ježek, and J. B. Nation. *Free lattices*, volume 42 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1995. URL: `http://dx.doi.org/10.1090/surv/042`, `doi:10.1090/surv/042`.

E-mail address: `williamdemeo@gmail.com`