# A CHARACTERIZATION OF BOUNDEDNESS FOR FINITE LATTICES

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ABSTRACT. Let $X$ be a finite set, $\mathbf{F}(X)$ the free lattice generated by $X$, and $h\colon \mathbf{F}(X) \to \mathbf{L}$ a lattice epimorphism. Then $\mathbf{L}$ is bounded if and only if the kernel of every such $h$ is a finitely generated sublattice of $\mathbf{F}(X) \times \mathbf{F}(X)$.

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## 1. Introduction

**wjd** 2018-10-07: TODO. Insert background on the problem, why it's interesting, applications, etc.

## 2. Notation and Definitions

We present some standard notation and definitions needed for the statement and proof of our main result (Theorem 3.1). Many of these are found in the "Free Lattices" book by Freese, Jezek, Nation [**?**], although the authors of that book point out that these concepts were first introduced in Ralph McKenzie's work on nonmodular lattice varieties [**?**], and Bjarni Jónsson's work on sublattices of free lattices [**?**].

2.1. **Bounded Homomorphisms.** Let $\mathbf{L} = \langle L, \wedge, \vee\rangle$ be a lattice and suppose $x, y \in L$ and $x \leqslant y$. Then we write $[\![x, y]\!]$ to denote the sublattice of elements between $x$ and $y$; that is,

$$[\![x, y]\!] := \{z \in L \mid x \leqslant z \leqslant y\}.$$

Let $\mathbf{K}$ and $\mathbf{L}$ be lattices and suppose $\mathbf{L}$ has bottom and top elements, 0 and 1, resp. If $h\colon \mathbf{K} \to \mathbf{L}$ is a lattice homomorphism, then for each $a \in L$ consider the sets $h^{-1}[\![a, 1]\!] = \{x \in K \mid h(x) \geqslant a\}$ and $h^{-1}[\![0, a]\!] = \{x \in K \mid h(x) \leqslant a\}$. If the domain $\mathbf{K}$ has a top element, $1_{\mathbf{K}}$, then $h^{-1}[\![a, 1]\!]$ is nonempty if and only if $a \leqslant h(1_{\mathbf{K}})$. If $\mathbf{K}$ has a bottom element, $0_{\mathbf{K}}$, then $h^{-1}[\![0, a]\!]$ is nonempty if and only if $a \geqslant h(0_{\mathbf{K}})$. When $h^{-1}[\![a, 1]\!]$ is nonempty, it is a filter of $\mathbf{K}$; dually a nonempty $h^{-1}[\![0, a]\!]$ is an ideal. If $\mathbf{K}$ is infinite, then $h^{-1}[\![a, 1]\!]$ need not have a least element, nor $h^{-1}[\![0, a]\!]$ a greatest element. However, considering when such extrema exist leads to the notion of bounded homomorphism, which is useful for studying the structure of free lattices.

A lattice homomorphism $h\colon \mathbf{K} \to \mathbf{L}$ is called **lower bounded** if for every $a \in L$, the set $h^{-1}[\![a, 1]\!]$ is either empty or has a least element. The least element of a nonempty $h^{-1}[\![a, 1]\!]$ is denoted by $\beta(a)$. Thus, if $h$ is a lower bounded homomorphism, then $\beta\colon \mathbf{L} \rightharpoonup \mathbf{K}$ is a partial function whose domain is an ideal of $\mathbf{L}$.

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Dually, a homomorphism $h$ is called **upper bounded** if every nonempty $h^{-1}[\![0, a]\!]$ has a greatest element, called $\alpha(a)$. If $h$ is an upper bounded homomorphism, then the domain of $\alpha \colon \mathbf{L} \rightharpoonup \mathbf{K}$ is a filter of $\mathbf{L}$. A **bounded** homomorphism is one that is both upper and lower bounded.

These definitions simplify when $h$ is an epimorphism. In that case $h$ is lower (upper) bounded if and only if each preimage $h^{-1}\{a\}$ has a least (greatest) element. Likewise, if $L$ is finite, then $h \colon \mathbf{K} \to \mathbf{L}$ is lower (upper) bounded if and only if each $h^{-1}\{a\}$ has a least (greatest) element whenever it is nonempty. Of course, every lattice homomorphism on a finite domain is bounded.

Note that $\beta$ is monotonic and a left adjoint for $h$; that is, $a \leqslant h(x)$ iff $\beta(a) \leqslant x$. Therefore, $\beta$ is a join preserving map on its domain: if $h^{-1}[\![a, 1]\!] \neq \emptyset \neq h^{-1}[\![b, 1]\!]$, then $\beta(a \vee b) = \beta(a) \vee \beta(b)$. Similarly, $\alpha$ is a right adjoint—$h(y) \leqslant a$ iff $y \leqslant \alpha(a)$)—so $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$, for all $a, b$ in the domain of $\alpha$. In particular, if $h$ is an epimorphism, then $\alpha$ is a meet homomorphism, and $\beta$ a join homomorphism, from $\mathbf{L}$ to $\mathbf{K}$.

2.2. **Unbounded Homomorphisms.** If $A$ and $B$ are sets, then by $A \subseteq_\omega B$ we mean that $A$ is a finite subset of $B$. Let $X$ be a set, and $\mathbf{F} := \mathbf{F}(X)$ the free lattice generated by $X$. Define a pair of closure operators, denoted by $^\wedge$ and $^\vee$, on the powerset $\mathcal{P}(\mathbf{F})$ as follows: for $A \in \mathcal{P}(\mathbf{F})$,

$$A^\wedge = \{\bigwedge B \mid B \subseteq_\omega A\} \quad \text{and} \quad A^\vee = \{\bigvee B \mid B \subseteq_\omega A\}.$$

Define also $H_0 = X^\wedge = \{\bigwedge B \mid B \subseteq_\omega X\}$, and for all $k \geqslant 0$, let

$$H_{k+1} = (H_k)^{\vee\wedge} = \{\bigwedge B \mid B \subseteq_\omega H_k^\vee\},$$

where $H_k^\vee = \{\bigvee A \mid A \subseteq_\omega H_k\}$. Evidentally, $\mathbf{F}$ is the union of the chain of subsets $H_0 \subseteq H_1 \subseteq \cdots$. Each $H_n = X^{\wedge(\vee\wedge)^n}$ is a finite meet-closed subset of $\mathbf{F}$ and, since $X$ generates $\mathbf{F}$, it should be apparent that $\bigcup H_n = \mathbf{F}$.

To gain some intuition about $H_k$ ($k \geqslant 0$), let's consider the complexity of the terms in $H_1 = (H_0)^{\vee\wedge}$.

$$\begin{aligned} H_1 &= \{\bigwedge B \mid B \subseteq_\omega (H_0)^\vee\} = \{\bigwedge B \mid B \subseteq_\omega X^{\wedge\vee}\} \\ &= \Big\{\bigwedge B \mid B \subseteq_\omega \{\bigvee A \mid A \subseteq_\omega X^\wedge\}\Big\} \\ &= \Big\{\bigwedge B \mid B \subseteq_\omega \{\bigvee A \mid A \subseteq_\omega \{\bigwedge B \mid B \subseteq_\omega X\}\}\Big\}. \end{aligned}$$

For a concrete example, if $x_i \in X$, then the following four terms belong to $H_0$,

$$t_1 = x_1 \wedge x_2 \wedge x_3, \quad t_2 = x_2 \wedge x_4, \quad t_3 = x_3 \wedge x_5, \quad t_4 = x_1,$$

and here's a term that belongs to $H_1 \setminus H_0$,

$$(t_1 \vee t_2) \wedge (t_3 \vee t_4) = \big[(x_1 \wedge x_2 \wedge x_3) \vee (x_2 \wedge x_4)\big] \wedge \big[(x_1 \wedge x_5) \vee x_2\big].$$

Let $h \colon \mathbf{F}(X) \to \mathbf{L}$ be an epimorphism and, for each $y \in L$ and $k < \omega$, define

$$\beta_k(y) = \bigwedge\{w \in H_k : h(w) \geqslant y\}.$$

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**Theorem 2.1** ([**?**, Thm. 2.2])**.** *Let* $\mathbf{K}$ *be finitely generated, and let* $h\colon \mathbf{K} \to \mathbf{L}$ *be a lattice homomorphism. If* $a \leqslant h(1_{\mathbf{K}})$*, then*

*(1)* $j \leqslant k$ *implies* $\beta_j(a) \geqslant \beta_k(a)$*,*
*(2)* $\beta_k(a)$ *is the least element of* $H_k \cap h^{-1}[\![a, 1]\!]$*,*
*(3)* $h^{-1}[\![a, 1]\!] = \bigcup_{k\in\omega} [\![\beta_k(a), 1]\!]$*.*

In the following lemma we distinguish between lattice terms $t$ over $X$ in the language $\wedge, \vee$ and the elements $\bar{t} \in \mathbf{F}(X)$ which they induce.

**Lemma.** Let $p = \bigvee P$ be a lattice term of minimal size such that $\bar{p} \in \bar{H}_1$ and all elements in $P$ are meets of subterms or variables in $X$. If $|P| > 1$, then $\bar{P} \subseteq \bar{H}_0$.

*Proof.* Assume $|P| > 1$. Let $r := \bigwedge_i(\bigvee R_i) \in H_1$ for sets $R_i \subseteq H_0$ such that $\bar{r} = \bar{p}$. First we will show that $r$ can actually be chosen as join of elements in $H_0$.

By Whitman's condition $\bigwedge_i(\bigvee R_i) \leqslant \bigvee P$ yields $r \leqslant t$ for some $t \in P$ or $\bigvee R_i \leqslant p$ for some $i$. In the former case $r \leqslant t \leqslant p$ yields $\bar{r} = \bar{t} = \bar{p}$ which contradicts the minimality of $p$. Hence we have the latter and $r \leqslant \bigvee R_i \leqslant p$ yields $\bigvee \bar{R}_i = \bar{p}$ for some $i$. We may choose $r = \bigvee R_i$ with $R_i \subseteq H_0$. In fact redefining the sets $R_i$ we may assume

$$r = \bigvee_i (\bigwedge R_i) \text{ for } R_i \subseteq X.$$

From $r \leqslant p$ it follows that $\bigwedge R_i \leqslant \bigvee P$ for all $i$. Fix $i$. By Whitman's condition either $\bigwedge R_i \leqslant t$ for some $t \in P$ or $x \leqslant \bigvee P$ for some $x \in R_i$. The latter case yields $x \leqslant t$ for some $t \in P$. Hence either way it follows that

$$\forall i \exists t \in P\colon \bigwedge R_i \leqslant t. \tag{2.1}$$

Conversely we claim

$$\forall t \in P \exists i\colon t \leqslant \bigwedge R_i. \tag{2.2}$$

For the proof let $t \in P$ and consider $p \leqslant r$. Then $t \leqslant \bigvee_i(\bigwedge R_i)$. By the assumption of the lemma either $t$ is a variable or a proper meet. If $t \in X$, then $t \leqslant \bigwedge R_i$ for some $i$ follows since variables are join prime. So assume $t = \bigwedge T$ for $|T| > 1$. By Whitman's condition either $t \leqslant \bigwedge R_i$ for some $i$ or $s \leqslant r$ for some $s \in T$. Suppose the latter. Note that $p = (\bigvee P \setminus \{t\}) \vee \bigwedge T$ and $\bigwedge T \leqslant s \leqslant r$. Hence $\overline{(\bigvee P \setminus \{t\}) \vee s} = \bar{r}$. Since $|T| > 1$, this contradicts the minimality of $p$. Thus (2.2) is proved.

Let $t \in P$. By (2.2) and (2.1) we have $i$ and $s \in P$ such that $t \leqslant \bigwedge R_i \leqslant s$. If $s \neq t$, then $\bigvee \bar{P} = \bigvee \overline{P \setminus \{t\}}$ contradicts the minimality of $p$. Hence $s = t$ which yields $\bar{t} = \bigwedge \bar{R}_i$. Thus $\bar{P} \subseteq \bar{H}_0$. ☐

## 3. Main Theorem

**Theorem 3.1.** *Let* $X$ *be a finite set,* $\mathbf{F}(X)$ *the free lattice generated by* $X$*,* $\mathbf{L} = \langle L, \wedge, \vee\rangle$ *a finite lattice, and* $h\colon \mathbf{F}(X) \twoheadrightarrow \mathbf{L}$ *a lattice epimorphism. Then*

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*$h$ is bounded if and only if the kernel of $h$ is a finitely generated sublattice of $\mathbf{F}(X) \times \mathbf{F}(X)$.*

A key ingredient in our proof of Theorem 3.1 is Lemma 3.2, for which we require a practical definition of the length of a term. If $t \in \mathbf{F}(X)$, we define the **length** of $t$ as follows:

- $t$ has *length* $\ell$ if $t \in H_\ell \setminus H_{\ell-1}$
- $t$ has *length at most* $\ell$ if $t \in H_\ell$.

Observe that if $t$ has length at most $\ell$, then $t \geqslant \beta_\ell h(t)$.

**Lemma 3.2.** *Let $h$ be an epimorphism from $\mathbf{F}(X)$ onto a finite lattice $\mathbf{L}$. Let $K$ be a finite subset of the kernel of $h$. Then there exists $n > 0$ such that for all $(p, q) \in \langle K\rangle$, if $p$ has length $\ell$, then $q \geqslant \beta_{\ell+n} h(q)$.*

**Remark.** If $K \subseteq \ker h$, then $(p, q) \in \langle K\rangle$ implies $h(p) = h(q)$.

*Proof.* Let $\mathbf{F} := \mathbf{F}(X)$, let $K$ be a finite subset of $\ker h$, and define the following, where $S$ is an arbitrary subset of $\mathbf{F}^2$:

- $S_1 = \{s_1 \in \mathbf{F} \mid \exists s_2,\ (s_1, s_2) \in S\}$,
- $m = \min\{\ell \mid K_1 \cap H_\ell \neq \emptyset\}$,
- $M = \min\{\ell \mid K_1 \subseteq H_\ell\}$.

In other terms, $m$ and $M$ are the minimum and maximum lengths of terms $p$ such that $(p, q) \in K$. Clearly $m \leqslant M$, so for all $a \in L$ we have $\beta_m(a) \geqslant \beta_M(a)$. Also, $p \geqslant \beta_M h(p)$ for all pairs $(p, q) \in K$, since $K_1 \subseteq H_M$. Moreover, $\exists p \in K_1 \cap H_m$ and, by minimality of $m$, the length of this $p$ is exactly $m$ and $p \geqslant \beta_m h(p)$. Finally, since $K$ is a finite set, we can find $n > 0$ such that $q \geqslant \beta_{m+n} h(p)$ for all $(p, q) \in K$.

Let $(p, q) \in \langle K\rangle$ and let $\ell$ be the length of $p$, so $p \geqslant \beta_\ell h(p)$. We want to show $q \geqslant \beta_{\ell+n} h(p)$.

**Case 1.** Suppose $(p, q) = (p_1, q_1) \vee (p_2, q_2)$, and assume the conclusion of Lemma 3.2 holds for $(p_1, q_1)$ and $(p_2, q_2)$. Let $\ell_i$ be the length of $p_i$. Then $p$ has length $\ell = \max\{\ell_1, \ell_2\} + 1$. For each $i \in \{1, 2\}$, we have $p_i \geqslant \beta_{\ell_i} h(p_i)$ and, by the induction hypothesis, $q_i \geqslant \beta_{\ell_i+n} h(p_i)$. Observe also that $\ell - 1 \geqslant \ell_i$, so $\beta_{\ell_i+n} h(p_i) \geqslant \beta_{\ell+n-1} h(p_i)$ for each $i \in \{1, 2\}$. Therefore,

$$q = q_1 \vee q_2 \geqslant \beta_{\ell_1+n} h(p_1) \vee \beta_{\ell_1+n} h(p_2) \geqslant \beta_{\ell+n-1} h(p_1) \vee \beta_{\ell+n-1} h(p_2).$$

To complete the proof in this case, then, it suffices to show,

$$\beta_{\ell+n-1} h(p_1) \vee \beta_{\ell+n-1} h(p_2) \geqslant \beta_{\ell+n} h(p). \tag{3.1}$$

Let $t_i = \beta_{\ell+n-1} h(p_i)$; that is, $t_i$ denotes the least term of length at most $\ell + n - 1$ that is mapped by $h$ to $h(p_i)$. Now $t_1 \vee t_2$ is mapped by $h$ to $h(p)$. Indeed, $h(t_1 \vee t_2) = h(t_1) \vee h(t_2) = h(p_1) \vee h(p_2) = h(p_1 \vee p_2) = h(p)$. Also, the length of $t_1 \vee t_2$ is at most $\ell + n$. Therefore, as $\beta_{\ell+n} h(p)$ denotes the least term of length at most $\ell + n$ that is mapped to $h(p)$, (3.1) holds.

**Case 2.** Suppose $(p, q) = (p_1, q_1) \wedge (p_2, q_2)$, and assume the conclusion of the Lemma 3.2 holds for $(p_1, q_1)$ and $(p_2, q_2)$. Let $\ell_i$ be the length of $p_i$. Then

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$p_i \geqslant \beta_{\ell_i} h(p_i)$ and, by the induction hypothesis, $q_i \geqslant \beta_{\ell_i+n} h(p_i)$. Our goal once again is to prove $q \geqslant \beta_{\ell+n} h(p)$.

For all $k$, $H_k$ is closed under meet, so the length of $p$ is $\ell = \max\{\ell_1, \ell_2\}$ (in contrast to Case 1). Still, $\beta_{\ell_i+n} h(p_i) \geqslant \beta_{\ell+n} h(p_i)$, so

$$q = q_1 \wedge q_2 \geqslant \beta_{\ell_1+n} h(p_1) \wedge \beta_{\ell_1+n} h(p_2) \geqslant \beta_{\ell+n} h(p_1) \wedge \beta_{\ell+n} h(p_2).$$

Therefore, to complete the proof it this case, it suffices to show

$$\beta_{\ell+n} h(p_1) \wedge \beta_{\ell+n} h(p_2) \geqslant \beta_{\ell+n} h(p).$$

Recall that $\beta_{\ell+n} h(p_i)$ denotes the least term $t_i$ of length at most $\ell + n$ such that $h(t_i) = h(p_i)$. Now, $t_1 \wedge t_2$ has length at most $\ell + n$, since $H_{\ell+n}$ is closed under meet. Also,

$$h(t_1 \wedge t_2) = h(t_1) \wedge h(t_2) = h(p_1) \wedge h(p_2) = h(p_1 \wedge p_2) = h(p),$$

so the image of $t_1 \wedge t_2$ under $h$ is $h(p)$. Since $\beta_{\ell+n} h(p)$ is the least term of length at most $\ell + n$ that is mapped to $h(p)$, we have $t_1 \wedge t_2 \geqslant \beta_{\ell+n} h(p)$, as desired.

## 4. Proof of the Main Theorem

With Lemma 3.2 established, we are now in a position to prove Theorem 3.1. We begin with the converse direction. In fact, we will prove the contrapositive of the converse. Specifically, if $h : \mathbf{F} \to \mathbf{L}$ is not lower bounded, then $\ker h$ is not a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$. After proving the converse, we will assume $h$ is (both upper- and lower-) bounded, and then prove that $\ker h$ is a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$.

*Proof of Theorem 3.1.*
($\Leftarrow$) Suppose $h : \mathbf{F} \to \mathbf{L}$ is not lower bounded, and let $K$ be an arbitrary finite subset of $\ker h$. We prove that the subalgebra $\langle K \rangle \leqslant \mathbf{F} \times \mathbf{F}$ generated by $K$ is not all of $\ker h$. Since $K$ is an arbitrary finite subset of $\ker h$, this will prove that $\ker h$ is not finitely generated.

Let $h^{-1}(a)$ be a class of $\ker h$ that is not lower bounded. The the sequence $\beta_k(a)$ defined prior to Lemma 3.2 is an infinite decending chain. Of course, for every $k \in \mathbb{N}$, we have $h\beta_0(a) = h\beta_k(a)$, so $(\beta_0(a), \beta_k(a)) \in \ker h$. On the other hand, since $K$ is a finite subset of $\ker h$, Lemma 3.2 asserts the existence of an $n \in \mathbb{N}$ such that, if $(p, q) \in \langle K \rangle$, then $p \geqslant \beta_\ell h(p)$ implies $p \geqslant \beta_{\ell+n} h(p)$. Therefore, $(\beta_0(a), \beta_{n+1}(a)) \in \ker h \setminus \langle K \rangle$. Thus $\langle K \rangle \neq \ker h$, as we set out to prove.

($\Rightarrow$) Assume $h$ is a bounded epimorphism so that the preimage of each $y \in L$ under $h$ is bounded. For each $y \in L$, let $\alpha y = \bigvee h^{-1}\{y\}$ and $\beta y = \bigwedge h^{-1}\{y\}$ denote the greatest and least elements of $h^{-1}\{y\}$, respectively (both of which exist by the assumed boundedness and surjectivity of $h$). Observe that $h\alpha h = h$, and $h\beta h = h$. In fact, $\alpha$ and $\beta$ are adjoint to $h$. Indeed, it is easy to see that

$$hx \leqslant y \quad \Leftrightarrow \quad x \leqslant \alpha y,$$

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$$y \leqslant hx \quad \Leftrightarrow \quad \beta y \leqslant x.$$

For each $y \in L$, let $X_y := X \cap h^{-1}\{y\}$, the set of generators that lie in the inverse image of $y$ under $h$. Let $G$ be the (finite) set of pairs in $\mathbf{F} \times \mathbf{F}$ defined as follows:

$$G = \bigcup_{y \in L} \{(x, \alpha y), (\alpha y, x), (x, \beta y), (\beta y, x), (\alpha y, \beta y), (\beta y, \alpha y) : x \in X_y\}.$$

We claim that $G$ generates $\ker h$. To prove this, we first show, by induction on term complexity, that for every $y \in L$, for every $r \in h^{-1}\{y\}$, the pairs $(r, \alpha y)$ and $(r, \beta y)$ belong to the sublattice $\langle G \rangle \leqslant \mathbf{F} \times \mathbf{F}$ generated by $G$.

- **Case 0.** Suppose $r \in X$. Then $(r, \alpha y)$ and $(r, \beta y)$ belong to $G$ itself, so there's nothing to prove.
- **Case 1.** Suppose $r = s \vee t$. Assume (the induction hypothesis) that $(s, \alpha h(s))$, $(s, \beta h(s))$, $(t, \alpha h(t))$, and $(t, \beta h(t))$ belong to $\langle G \rangle$. Then $y = h(r) = h(s \vee t) = h(s) \vee h(t)$, so

  $$h(\alpha h(s) \vee \alpha h(t)) = h\alpha h(s) \vee h\alpha h(t) = h(s) \vee h(t) = y.$$

  Likewise, $h(\beta h(s) \vee \beta h(t)) = h(s) \vee h(t) = y$. Therefore,

  $$\beta y \leqslant \beta h(s) \vee \beta h(t) \leqslant \alpha h(s) \vee \alpha h(t) \leqslant \alpha y.$$

  Also, $r \leqslant \alpha y$, so $r = \alpha y \wedge (s \vee t)$. Taken together, these observations yield

  $$\begin{aligned} \begin{pmatrix} r \\ \beta y \end{pmatrix} = \begin{pmatrix} \alpha y \wedge (s \vee t) \\ \beta y \end{pmatrix} &= \begin{pmatrix} \alpha y \wedge (s \vee t) \\ \beta y \wedge (\beta h(s) \vee \beta h(t)) \end{pmatrix} \\ &= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \left[ \begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \vee \begin{pmatrix} t \\ \beta h(t) \end{pmatrix} \right], \end{aligned}$$

  and each term in the last expression belongs to $\langle G \rangle$, so $(r, \beta y) \in \langle G \rangle$, as desired.

  Similarly, $(r, \alpha y) \in \langle G \rangle$. Indeed, $\beta y \leqslant r$ implies $r = \beta y \vee s \vee t$, and $\beta h(s) \vee \beta h(t) \leqslant \alpha y$ implies $\alpha y = \alpha y \vee \beta h(s) \vee \beta h(t)$. Therefore,

  $$\begin{aligned} \begin{pmatrix} r \\ \alpha y \end{pmatrix} &= \begin{pmatrix} \beta y \vee s \vee t \\ \alpha y \vee \beta h(s) \vee \beta h(t) \end{pmatrix} \\ &= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \vee \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}. \end{aligned}$$

- **Case 2.** Suppose $r = s \wedge t$. Assume $(s, \alpha h(s))$, $(s, \beta h(s))$, $(t, \alpha h(t))$, and $(t, \beta h(t))$ belong to $\langle G \rangle$. Then $h(s \wedge t) = h(r) = y$, so $h(\alpha h(s) \wedge \alpha h(t)) = y = h(\beta h(s) \wedge \beta h(t))$, so $\beta y \leqslant \beta h(s) \wedge \beta h(t) \leqslant \alpha h(s) \wedge \alpha h(t) \leqslant \alpha y$. Also, $\beta y \leqslant r \leqslant \alpha y$ so $r = \alpha y \wedge s \wedge t$ and $r = \beta y \vee (s \wedge t)$. Altogether,

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we have

$$\begin{pmatrix} r \\ \alpha a \end{pmatrix} = \begin{pmatrix} \beta y \vee (s \wedge t) \\ \alpha y \vee (\alpha h(s) \wedge \alpha h(t)) \end{pmatrix}$$
$$= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \left[ \begin{pmatrix} s \\ \alpha h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \alpha h(t) \end{pmatrix} \right],$$

and each term in the last expression belongs to $\langle Y \rangle$, as desired. Similarly,

$$\begin{pmatrix} r \\ \beta y \end{pmatrix} = \begin{pmatrix} \alpha y \wedge s \wedge t \\ \beta y \wedge \alpha h(s) \wedge \alpha h(t) \end{pmatrix}$$
$$= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \begin{pmatrix} s \\ \alpha h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \alpha h(t) \end{pmatrix}.$$

Indicentally, in each of the last two derivations, we could have used $\beta$'s instead of $\alpha$'s; in both cases the last meet could be replaced with

$$\begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}.$$

In each case, we end up with an expression involving terms from $\langle G \rangle$, and this proves that $(r, \alpha y)$ and $(r, \beta y)$ belong to $\langle G \rangle$, as desired.

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