KERNELS OF EPIMORPHISMS OF FINITELY GENERATED FREE LATTICES

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1. Introduction

2. Main Theorem

Let X be a finite set and $\mathbf{F} := \mathbf{F}(X)$ the free lattice generated by X.

Theorem 2.1. *Suppose $\mathbf{L} = \langle L, \wedge, \vee\rangle$ is a finite lattice and $h\colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ a lattice epimorphism. If h is bounded then the kernel of h is a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$.*

Proof. Assume h is bounded. That is, the preimage of each $y \in L$ under h is bounded. For each $y \in L$, let $\alpha y = \bigvee h^{-1}\{y\}$ and $\beta y = \bigwedge h^{-1}\{y\}$ denote the greatest and least elements of $h^{-1}\{y\}$, respectively (both of which exist by the boundedness assumption). Observe that $h\alpha h = h$, and $h\beta h = h$. In fact, α and β are adjoint to h. Indeed, it is easy to see that

$$hx \leqslant y \quad \Leftrightarrow \quad x \leqslant \alpha y,$$

$$y \leqslant hx \quad \Leftrightarrow \quad \beta y \leqslant x.$$

For each $y \in L$, let $X_y := X \cap h^{-1}\{y\}$, the set of generators that lie in the inverse image of y under h. Let G be the (finite) set of pairs in $\mathbf{F} \times \mathbf{F}$ defined as follows:

$$G = \bigcup_{y\in L}\{(x, \alpha y), (\alpha y, x), (x, \beta y), (\beta y, x), (\alpha y, \beta y), (\beta y, \alpha y) : x \in X_y\}.$$

We claim that G generates $\ker h$. To prove this, we first show, by induction on term complexity, that for every $y \in L$, for every $r \in h^{-1}\{y\}$, the pairs $(r, \alpha y)$ and $(r, \beta y)$ belong to the sublattice $\langle G\rangle \leqslant \mathbf{F} \times \mathbf{F}$ generated by G.

- **Case 0.** Suppose $r \in X$. Then $(r, \alpha y)$ and $(r, \beta y)$ belong to G itself, so there's nothing to prove.
- **Case 1.** Suppose $r = s \vee t$. Assume (the induction hypothesis) that $(s, \alpha h(s))$, $(s, \beta h(s))$, $(t, \alpha h(t))$, and $(t, \beta h(t))$ belong to $\langle G\rangle$. Then $y = h(r) = h(s \vee t) = h(s) \vee h(t)$, so

  $$h(\alpha h(s) \vee \alpha h(t)) = h\alpha h(s) \vee h\alpha h(t) = h(s) \vee h(t) = y.$$

  Likewise, $h(\beta h(s) \vee \beta h(t)) = h(s) \vee h(t) = y$. Therefore,

  $$\beta y \leqslant \beta h(s) \vee \beta h(t) \leqslant \alpha h(s) \vee \alpha h(t) \leqslant \alpha y.$$

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Also, $r \leqslant \alpha y$, so $r = \alpha y \wedge (s \vee t)$. Taken together, these observations yield

$$\begin{pmatrix} r \\ \beta y \end{pmatrix} = \begin{pmatrix} \alpha y \wedge (s \vee t) \\ \beta y \end{pmatrix} = \begin{pmatrix} \alpha y \wedge (s \vee t) \\ \beta y \wedge (\beta h(s) \vee \beta h(t)) \end{pmatrix}$$
$$= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \left[\begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \vee \begin{pmatrix} t \\ \beta h(t) \end{pmatrix} \right],$$

and each term in the last expression belongs to $\langle G \rangle$, so $(r, \beta y) \in \langle G \rangle$, as desired.

Similarly, $(r, \alpha y) \in \langle G \rangle$. Indeed, $\beta y \leqslant r$ implies $r = \beta y \vee s \vee t$, and $\beta h(s) \vee \beta h(t) \leqslant \alpha y$ implies $\alpha y = \alpha y \vee \beta h(s) \vee \beta h(t)$. Therefore,

$$\begin{pmatrix} r \\ \alpha y \end{pmatrix} = \begin{pmatrix} \beta y \vee s \vee t \\ \alpha y \vee \beta h(s) \vee \beta h(t) \end{pmatrix}$$
$$= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \vee \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}.$$

- **Case 2.** Suppose $r = s \wedge t$. Assume $(s, \alpha h(s))$, $(s, \beta h(s))$, $(t, \alpha h(t))$, and $(t, \beta h(t))$ belong to $\langle G \rangle$. Then $h(s \wedge t) = h(r) = y$, so $h(\alpha h(s) \wedge \alpha h(t)) = y = h(\beta h(s) \wedge \beta h(t))$, so $\beta y \leqslant \beta h(s) \wedge \beta h(t) \leqslant \alpha h(s) \wedge \alpha h(t) \leqslant \alpha y$. Also, $\beta y \leqslant r \leqslant \alpha y$ so $r = \alpha y \wedge s \wedge t$ and $r = \beta y \vee (s \wedge t)$. Altogether, we have

$$\begin{pmatrix} r \\ \alpha a \end{pmatrix} = \begin{pmatrix} \beta y \vee (s \wedge t) \\ \alpha y \vee (\alpha h(s) \wedge \alpha h(t)) \end{pmatrix}$$
$$= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \left[\begin{pmatrix} s \\ \alpha h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \alpha h(t) \end{pmatrix} \right],$$

and each term in the last expression belongs to $\langle Y \rangle$, as desired. Similarly,

$$\begin{pmatrix} r \\ \beta y \end{pmatrix} = \begin{pmatrix} \alpha y \wedge s \wedge t \\ \beta y \wedge \alpha h(s) \wedge \alpha h(t) \end{pmatrix}$$
$$= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \begin{pmatrix} s \\ \alpha h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \alpha h(t) \end{pmatrix}.$$

Note that, in both of the derivations above, we could have used β's instead of α's; that is,

$$\begin{pmatrix} r \\ \alpha y \end{pmatrix} = \begin{pmatrix} \beta y \vee (s \wedge t) \\ \alpha y \vee (\beta h(s) \wedge \beta h(t)) \end{pmatrix}$$
$$= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \left[\begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \beta h(t) \end{pmatrix} \right],$$

and

$$\begin{pmatrix} r \\ \beta y \end{pmatrix} = \begin{pmatrix} \alpha y \wedge s \wedge t \\ \beta y \wedge \beta h(s) \wedge \beta h(t) \end{pmatrix}$$
$$= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}.$$

In each case, we end up with an expression involving terms from $\langle G \rangle$, and this proves that $(r, \alpha y)$ and $(r, \beta y)$ belong to $\langle G \rangle$, as desired.

□

We conjecture the converse of Theorem 2.1.

Conjecture 1. Suppose $\mathbf{L} = \langle L, \wedge, \vee \rangle$ is a finite lattice and $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ a lattice epimorphism. If the kernel of h is a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$, then h is bounded.

We know how to prove this conjecture if we assume that, whenever h is not upper (lower) bounded, then there is a class of $\ker h$ that contains an infinite ascending (descending) chain as well as a meet (join) prime element of $\mathbf{F}$. (See Appendix A for the proof).

J.B. Nation brought the next result to our attention. It gives an example in which the unbounded classes of $\ker h$ do not contain generators of $\mathbf{F}$.

Proposition 2.2. *Let $\mathbf{F} = \mathbf{F}(x, y, z)$, and let $\mathbf{L} = \mathbf{F}_{\mathbf{M}_3}(0, 1, 2)$ (see Figure 1). Let $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ be the epimorphism induced by $x \mapsto 0$, $y \mapsto 1$, $z \mapsto 2$. Then $\ker h$ is not finitely generated.*

To prove this we will need the a technical lemma about the behavior of certain sequences of elements of $\mathbf{F}$.

Lemma 2.3. *For each $s \in \{x, y, z, m\}$, define the sequence $\{s_i : i < \omega\}$ of elements of $\mathbf{F}$ as follows: $x_0 = x$, $y_0 = y$, $z_0 = z$, and for $i \geqslant 0$,*

$$x_{i+1} = x \vee (y_i \wedge z_i), \quad y_{i+1} = y \vee (x_i \wedge z_i), \quad z_{i+1} = z \vee (x_i \wedge y_i),$$
$$m_i = (x_i \wedge y_i) \vee (x_i \wedge z_i) \vee (y_i \wedge z_i).$$

If $\{s_i : i < \omega\}$ is any one of the four sequences just defined, then for every $i \geqslant 1$, we have $s_{i+1} > s_i$ and $h(s_i) = h(s_1)$.

Proof. First observe that $h(x_1) = h(x \vee (y \wedge z)) = 0 \vee (1 \wedge 2)$. We begin by proving $h(x_2) = h(x_1)$. By definition, we have $h(x_2) = h(x \vee (y_1 \wedge z_1)) = h(x) \vee \big[h(y_1) \wedge h(z_1)\big]$, and $h(y_1) = h(y \vee (x \wedge z)) = h(y) \vee \big[h(x) \wedge h(z)\big] = 1 \vee (0 \wedge 2)$. Similarly, $h(z_1) = 2 \vee (0 \wedge 1)$. Therefore,

$$h(x_2) = 0 \vee \big\{[1 \vee (0 \wedge 2)] \wedge [2 \vee (0 \wedge 1)]\big\}. \tag{2.1}$$

Recall, the modular law: $x \leqslant b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$. Applying this law with $a = 1$ and $x = 0 \wedge 2 \leqslant 2 \vee (0 \wedge 1) = b$, we have

$$(0 \wedge 2) \vee \big\{1 \wedge [2 \vee (0 \wedge 1)]\big\} = [1 \vee (0 \wedge 2)] \wedge [2 \vee (0 \wedge 1)],$$

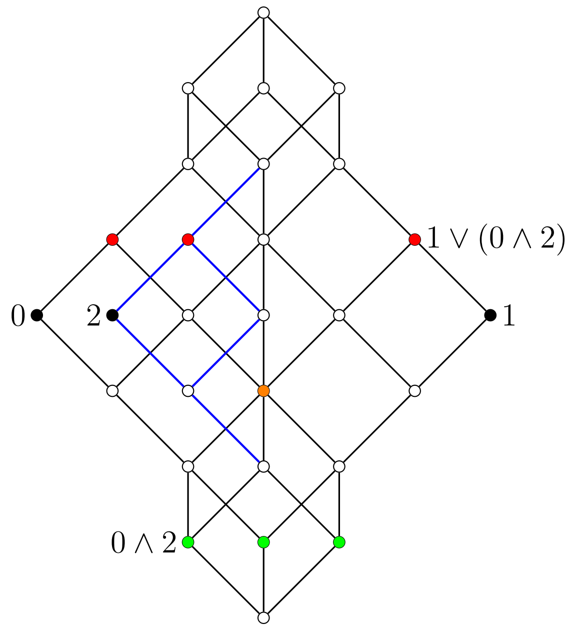

FIGURE 1. The free lattice over M_3 generated by $\{0, 1, 2\}$. Green dots identify elements $0 \wedge 1$, $0 \wedge 2$, and $1 \wedge 2$; red dots identify $0 \vee (1 \wedge 2)$, $1 \vee (0 \wedge 2)$, and $2 \vee (0 \wedge 1)$.

which is the right joinand in 2.1. Therefore,

$$h(x_2) = 0 \vee (0 \wedge 2) \vee \big\{1 \wedge [2 \vee (0 \wedge 1)]\big\} = 0 \vee \big\{1 \wedge [2 \vee (0 \wedge 1)]\big\}.$$

Applying the modular law once more to $1 \wedge [2 \vee (0 \wedge 1)]$, with $a = 2$ and $x = 0 \wedge 1 \leqslant 1 = b$, we have

$$1 \wedge [2 \vee (0 \wedge 1)] = (0 \wedge 1) \vee (1 \wedge 2).$$

Therefore,

$$h(x_2) = 0 \vee \big\{1 \wedge [2 \vee (0 \wedge 1)]\big\} = 0 \vee (0 \wedge 1) \vee (1 \wedge 2) = 0 \vee (1 \wedge 2),$$

as desired. Of course, $h(y_2) = h(y_1)$ and $h(z_2) = h(z_1)$ can be checked similarly.

With the base cases established, we proceed with the induction. Fix $n \geqslant 1$ and assume $h(x_n) = h(x_1)$, $h(y_n) = h(y_1)$, and $h(z_n) = h(z_1)$. Then,

$$h(x_{n+1}) = h(x \vee (y_n \wedge z_n)) = h(x) \vee (h(y_n) \wedge h(z_n)) = h(x) \vee (h(y_1) \wedge h(z_1)).$$

As observed above, this is equal to $h(x_2)$, which in turn is $h(x_1)$, as desired. By the same argument, $h(y_{n+1}) = h(y_1)$ and $h(z_{n+1}) = h(z_1)$. This proves that for all $n \geqslant 1$, we have $h(s_n) = h(s_1)$, when $\{s_n\}$ is $\{x_n\}$ or $\{y_n\}$ or $\{z_n\}$.

Finally, consider $\{m_n\}$. For all $n \geqslant 1$, we have

$$\begin{aligned} h(m_n) &= \big[h(x_n) \wedge h(y_n)\big] \vee \big[h(x_n) \wedge h(z_n)\big] \vee \big[h(y_n) \wedge h(z_n)\big] \\ &= \big[h(x_1) \wedge h(y_1)\big] \vee \big[h(x_1) \wedge h(z_1)\big] \vee \big[h(y_1) \wedge h(z_1)\big] \\ &= h(m_1). \end{aligned}$$

(In fact, in this case we can show that $h(m_1) = h(m_0)$, but this is unnecessary.)

It remains to show that $s_{i+1} > s_i$ for all $i < \omega$ (for each $s \in \{x, y, z, m\}$). For this we will need the following.

Claim 2.4. For all $i \geqslant 0$, $x \nleq y_i$ and $x \nleq z_i$.

Proof. When $i = 0$, the claim is that $x \nleq y$ and $x \nleq z$, which is clear. Fix $k \geqslant 0$ and suppose $x \nleq y_k$ and $x \nleq z_k$. We show $x \nleq y_{k+1}$ and $x \nleq z_{k+1}$. Assume the contrary, say, $x \leqslant y_{k+1} = y \vee (x_k \wedge z_k)$. Since x is a generator, it is join prime. Therefore, as $x \nleq y$, we must have $x \leqslant x_k \wedge z_k$. But this is impossible since $x \nleq z_k$, by the induction hypothesis. Thus, Claim 2.4 is proved. □

Observe that the claim just proved also yields the following: $\forall m, n$, $x_m \nleq y_n$. We will now prove $\forall i \geqslant 0$ that all of the following strict inequalities hold:

$$x_i < x_{i+1}, \quad y_i < y_{i+1}, \quad z_i < z_{i+1}$$
$$x_i \wedge y_i < x_{i+1} \wedge y_{i+1}, \quad x_i \wedge z_i < x_{i+1} \wedge z_{i+1}, \quad y_i \wedge z_i < y_{i+1} \wedge z_{i+1}. \tag{2.2}$$

Clearly $x_1 = x \vee (y \wedge z) > x = x_0$, by Theorem C.8. Similarly, $y_1 > y_0$ and $z_1 > z_0$. Also,

$$x_1 \wedge y_1 = \big[x \vee (y \wedge z)\big] \wedge \big[y \vee (x \wedge z)\big] > x \wedge y.$$

For, suppose on the contrary that $\big[x \vee (y \wedge z)\big] \wedge \big[y \vee (x \wedge z)\big] \leqslant x \wedge y$. Then, in particular, $\big[x \vee (y \wedge z)\big] \wedge \big[y \vee (x \wedge z)\big] \leqslant x$, but x is meet prime, so either $\big[x \vee (y \wedge z)\big] \leqslant x$ or $\big[y \vee (x \wedge z)\big] \leqslant x$. Both of these possibilities are easily ruled out. The arguments establishing the remaining inequalities in (2.2) in case $i = 0$ are similar.

Now, fix $n < \omega$ and suppose the strict inequalities in (2.2) hold when $i = n$. We show they also hold when $i = n + 1$. It suffices to prove just two of these, namely, $x_{n+1} > x_n$ and $x_{n+1} \wedge y_{n+1} > x_n \wedge y_n$.

Recall $x_{n+1} = x \vee (y_n \wedge z_n)$ and $x_n = x \vee (y_{n-1} \wedge z_{n-1})$, and the induction hypothesis implies that $y_n \wedge z_n > y_{n-1} \wedge z_{n-1}$. Therefore, $x_{n+1} \geqslant x_n$, so we must show $x_{n+1} \neq x_n$. Suppose on the contrary that

$$x_{n+1} = x \vee (y_n \wedge z_n) = x \vee (y_{n-1} \wedge z_{n-1}) = x_n. \tag{2.3}$$

Then $y_n \wedge z_n \leqslant x \vee (y_{n-1} \wedge z_{n-1})$. Recall Whitman's Theorem (C.7) asserting that $\mathbf{F}(X)$ satisfies condition (W+) repeated here for easy reference.

> If $v = v_1 \wedge \cdots \wedge v_r \wedge x_1 \wedge \cdots \wedge x_n \leqslant u_1 \vee \cdots \vee u_s \vee y_1 \vee \cdots \vee y_m = u$, where $x_i, y_j \in X$, then either $x_i = y_j$ for some i and j, or $v_i \leqslant u$ for some i, or $v \leqslant u_j$ for some j.

This condition and (2.3) together imply that one of the following must hold:

(1) $y_n \leqslant x \vee (y_{n-1} \wedge z_{n-1})$;
(2) $z_n \leqslant x \vee (y_{n-1} \wedge z_{n-1})$;
(3) $y_n \wedge z_n \leqslant y_{n-1} \wedge z_{n-1}$.

The first two inequalities are simply $y_n \leqslant x_n$ and $z_n \leqslant x_n$, which are both ruled out by the remark immediately following the proof of Claim 2.4, and

$y_n \wedge z_n \leqslant y_{n-1} \wedge z_{n-1}$ is ruled out by the induction hypothesis. By reaching this contradiction we have established that $x_{n+1} > x_n$.

It remains to prove $x_{n+1} \wedge y_{n+1} > x_n \wedge y_n$ under the assumption that all of the strict inequalities in (2.2) hold when $i = n$. Since we have already established that $x_{n+1} > x_n$ and (by an identical argument) that $y_{n+1} > y_n$, it's clear that $x_{n+1} \wedge y_{n+1} \geqslant x_n \wedge y_n$, so we just have to rule out equality. Suppose on the contrary that

$$\begin{aligned} x_n \wedge y_n &:= \big[x \vee (y_{n-1} \wedge z_{n-1})\big] \wedge \big[y \vee (x_{n-1} \wedge z_{n-1})\big] \\ &= \big[x \vee (y_n \wedge z_n)\big] \wedge \big[y \vee (x_n \wedge z_n)\big] =: x_{n+1} \wedge y_{n+1}. \end{aligned}$$

Then we have both $x_{n+1} \wedge y_{n+1} \leqslant x_n$ and $x_{n+1} \wedge y_{n+1} \leqslant y_n$. Equivalently,

$$\begin{aligned} \big[x \vee (y_n \wedge z_n)\big] \wedge \big[y \vee (x_n \wedge z_n)\big] &\leqslant x \vee (y_{n-1} \wedge z_{n-1}) \\ \big[x \vee (y_n \wedge z_n)\big] \wedge \big[y \vee (x_n \wedge z_n)\big] &\leqslant y \vee (x_{n-1} \wedge z_{n-1}). \end{aligned} \tag{2.4}$$

Again, by (W+), for the first of these inequalities to hold, we must have one of the following:

(1) $x \vee (y_n \wedge z_n) \leqslant x \vee (y_{n-1} \wedge z_{n-1})$;
(2) $y \vee (x_n \wedge z_n) \leqslant x \vee (y_{n-1} \wedge z_{n-1})$;
(3) $\big[x \vee (y_n \wedge z_n)\big] \wedge \big[y \vee (x_n \wedge z_n)\big] \leqslant y_{n-1} \wedge z_{n-1}$.

The first of these is equivalent to $x_{n+1} \leqslant x_n$, and the second to $y_{n+1} \leqslant x_n$; we have already ruled out both of these cases. Thus the only remaining possibility is the third, which can be restated as $x_{n+1} \wedge y_{n+1} \leqslant y_{n-1} \wedge z_{n-1}$. Let's assume this holds.

In order for the second of the inequalities in (2.4) to hold, according to (W+), one of the following must also be true:

(1) $x \vee (y_n \wedge z_n) \leqslant y \vee (x_{n-1} \wedge z_{n-1})$;
(2) $y \vee (x_n \wedge z_n) \leqslant y \vee (x_{n-1} \wedge z_{n-1})$;
(3) $\big[x \vee (y_n \wedge z_n)\big] \wedge \big[y \vee (x_n \wedge z_n)\big] \leqslant x_{n-1} \wedge z_{n-1}$.

Again, the first two—$x_{n+1} \leqslant y_n$ and $y_{n+1} \leqslant y_n$—have already been ruled out. So the only possibility is $x_{n+1} \wedge y_{n+1} \leqslant x_{n-1} \wedge z_{n-1}$. Let us assume we are in this case.

Taken all together, if $x_n \wedge y_n$ were equal to $x_{n+1} \wedge y_{n+1}$, we have would have

$$x_n \wedge y_n = x_{n+1} \wedge y_{n+1} \leqslant x_{n-1} \wedge y_{n-1} \wedge z_{n-1}.$$

In particular $x_n \wedge y_n \leqslant x_{n-1} \wedge y_{n-1}$. But this contradicts the inductive hypothesis. Thus the proof of Lemma 2.3 is complete. ☐

wjd 2018-09-17: TODO Prove Prop 2.2

Notice that the example above involves an infinite ascending chain $\{x_n\} \subseteq \mathbf{F}$ such that $0 = h(x_0) \neq h(x_1) = h(x_2) = \cdots$; thus, for $n \geqslant 1$, all x_n belong to the same (unbounded) class of $\ker h$, and this class contains $x_1 = x \vee (y \wedge z)$. However, x_1 is not meet prime, since $y \nleqslant x \vee (y \wedge z)$ and $z \nleqslant x \vee (y \wedge z)$, yet $y \wedge z \leqslant x \vee (y \wedge z)$, so we cannot use our previous proof technique.

As above, let $x_1 = x \vee (y \wedge z)$ and define $b = 0 \vee (1 \wedge 2)$. In particular, $h(x_1) = b$. To use the same inductive step that we used in our previous proofs, we would need to know that $p_1 \wedge p_2 \leqslant x_1$ implies either $p_1 \leqslant x_1$ or $p_2 \leqslant x_1$. But this is not the case here since we could have, for example, $p_1 = y$ and $p_2 = z$. In fact, by Lemma C.3, the only meet prime elements of $\mathbf{F}(X)$ have the form $\bigvee S$ for $S \subseteq X$.

Suppose $K \subseteq \mathbf{F} \times \mathbf{F}$ is a finite set. We wish to prove $\langle K \rangle \neq \ker h$. First, let's try to show there exists $N < \omega$ such that for all $(p, q) \in \mathbf{F} \times \mathbf{F}$ with $h(p) = b = h(q)$, the following implication holds:

$$\text{if } (p, q) \in \langle K \rangle \text{ and } p \leqslant x_1, \text{ then } q \leqslant \alpha_N(b). \tag{2.5}$$

Since $\alpha_n(b)$ is an infinite ascending chain, we can certainly find $N < \omega$ such that (2.5) holds for all of the (finitely many) pairs in K.

Now suppose $(p, q) = (p_1, q_1) \wedge (p_2, q_2)$ and $h(p) = b = h(q)$. Then p_1, p_2, q_1, q_2 must all belong to $h^{-1}\{b\}$ as well. Indeed, $b = h(p) = h(p_1 \wedge p_2) = h(p_1) \wedge h(p_2)$, and $b = 0 \vee (1 \wedge 2)$ is meet irreducible, so $h(p_1) = b = h(p_2)$. Similarly for q_1 and q_2.

Assume $(p, q) \in \langle K \rangle$ and $p = p_1 \wedge p_2 \leqslant x_1$. We must show $q_1 \wedge q_2 \leqslant \alpha_N(b)$. Assume the induction hypothesis that (p_i, q_i) satisfies (2.5) for both $i = 1, 2$.

If we knew that either $p_1 \leqslant x_1$ or $p_2 \leqslant x_1$, the we would have that either $q_1 \leqslant \alpha_N(b)$ or $q_2 \leqslant \alpha_N(b)$, by the induction hypothesis; in either case, we could conclude that $q_1 \wedge q_2 \leqslant \alpha_N(b)$, as desired. Unfortunately, we do not know that either $p_1 \leqslant x_1$ or $p_2 \leqslant x_1$, since x_1 is not meet prime.

To say that x_1 is meet prime means that *for all* p_1 and p_2 with $p_1 \wedge p_2 \leqslant x_1$, we have either $p_1 \leqslant x_1$ or $p_2 \leqslant x_1$. But this is actually more that we need. For our purposes, we only require that the condition

$$p_1 \wedge p_2 \leqslant x_1 \quad \Longrightarrow \quad p_1 \leqslant x_1 \text{ or } p_2 \leqslant x_1, \tag{2.6}$$

is satisfied for some special elements p_1 and p_2, namely...?

Maybe we should try to exploit a D^d-cycle here, involving

$$p_1 \wedge (p_1 \to x_1) \leqslant x_1$$

We know that the maximum element (denoted here by $p_1 \to x_1$), among those that meet with p_1 below x_1, exists by the existence of a D^d-cycle.

Appendix A. Proof of Conjecture under special assumptions

As usual, let X be a finite set and let $\mathbf{F} := \mathbf{F}(X)$ be the free lattice generated by X.

Proposition A.1. *Suppose $\mathbf{L} = \langle L, \wedge, \vee \rangle$ is a finite lattice and $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ a lattice epimorphism. Suppose also that there is a class of* $\ker h$ *containing an infinite descending chain as well as a join prime element of $\mathbf{F}$. Then the kernel of h is not a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$.*

Proof. Let $y_0 \in L$ and suppose $x_0 \in h^{-1}\{y_0\}$ is a join prime element of $\mathbf{F}$. Suppose also that the class $h^{-1}\{y_0\}$ contains an infinite descending chain, $\beta_0(y_0) > \beta_1(y_0) > \cdots$. Let K be a finite subset of $\ker h$, say, $K = \{(p_1, q_1), \dots, (p_m, q_m)\} \subseteq \ker h$. We prove $\langle K \rangle \neq \ker h$. Since K is an arbitrary finite subset of $\ker h$, this will prove that $\ker h$ is not finitely generated.

Claim A.2. There exists $N < \omega$ such that for all (p_i, q_i) in K, if $p_i \geqslant x_0$, then $q_i \geqslant \beta_N(y_0)$.

Proof. Fix i and $(p_i, q_i) \in K$ (so, $h(p_i) = h(q_i)$). Define N_i according to which of the following cases holds:

(1) If $p_i \ngeqslant x_0$, let $N_i = 0$.
(2) If $p_i \geqslant x_0$, then $x_0 = x_0 \wedge p_i$, so $y_0 = h(x_0) = h(x_0) \wedge h(p_i) \leqslant h(p_i)$, so $y_0 \leqslant h(q_i)$. Also, $h(x_0 \wedge q_i) = h(x_0) \wedge h(q_i) = y_0$, so $x_0 \wedge q_i \in h^{-1}\{y_0\}$. Therefore (since $\{\beta_i(y_0)\}$ is an infinite descending chain), there exists $n_i > 0$ such that $x_0 \wedge q_i \geqslant \beta_{n_i}(y_0)$. Let $N_i = n_i$ in this case (so $q_i \geqslant \beta_{N_i}(y_0)$).

After defining N_i this way for each pair $(p_i, q_i) \in K$, let $N := \max_i N_i$. Then the desired implication holds for all $1 \leqslant i \leqslant m$, that is,

$$p_i \geqslant x_0 \quad \Longrightarrow \quad q_i \geqslant \beta_N(y_0), \tag{A.1}$$

so Claim A.2 is proved. □

Claim A.3. There exists $N < \omega$ such that, for all $(p, q) \in \langle K \rangle$,

$$p \geqslant x_0 \quad \Longrightarrow \quad q \geqslant \beta_N(y_0). \tag{A.2}$$

Proof. Define N as in the proof of Claim A.2, so that for all $(p_i, q_i) \in K$ the implication (A.1) holds. Fix $(p, q) \in \langle K \rangle$. We prove (A.2) by induction on the complexity of (p, q). If $(p, q) \in K$, then there's nothing to prove. We split the induction step into two cases.

(1) Assume $(p, q) = (p_1, q_1) \wedge (p_2, q_2)$, where p_i, q_i $(i = 1, 2)$ satisfy (A.2). Assume $p \geqslant x_0$. Then $p = p_1 \wedge p_2 \geqslant x_0$, so $p_1 \geqslant x_0$ and $p_2 \geqslant x_0$, so (by the induction hypothesis) $q_1 \geqslant \beta_N(y_0)$ and $q_2 \geqslant \beta_N(y_0)$. Therefore, $q = q_1 \wedge q_2 \geqslant \beta_N(y_0)$, as desired.
(2) Assume $(p, q) = (p_1, q_1) \vee (p_2, q_2)$, where p_i, q_i $(i = 1, 2)$ satisfy (A.2). Assume $p \geqslant x_0$. Then $p = p_1 \vee p_2 \geqslant x_0$. Since x_0 is join prime, either $p_1 \geqslant x_0$ or $p_2 \geqslant x_0$. Assume (wlog) $p_1 \geqslant x_0$. Then, by the induction hypothesis, $q_1 \geqslant \beta_N(y_0)$. Therefore, $q = q_1 \vee q_2 \geqslant q_1 \geqslant \beta_N(y_0)$, as desired.

This completes the proof of Claim A.3. □

We can now see that K does not generate $\ker h$. Indeed, if N is as in the proof of Claim A.3, then $\beta_N(y_0) > \beta_{N+1}(y_0)$, so the pair $(p, q) = (x_0, \beta_{N+1}(y_0))$ does not satisfy condition A.2, so does not belong to $\langle K \rangle$. Yet, $(x_0, \beta_{N+1}(y_0)) \in \ker h$. Thus, K does not generate $\ker h$. □

Appendix B. Examples

Let $\mathbf{M_3} = \langle \{0, a, b, c, 1\}, \wedge, \vee \rangle$, where $a \wedge b = a \wedge c = b \wedge c = 0$ and $a \vee b = a \vee c = b \vee c = 1$. Let $\mathbf{F} := \mathbf{F}(x, y, z)$ denote the free lattice generated by $\{x, y, z\}$.

Proposition B.1. *Let $h \colon \mathbf{F} \twoheadrightarrow \mathbf{M_3}$ be the epimorphism that acts on the generators as follows: $x \mapsto a$, $y \mapsto b$, $z \mapsto c$. Then $\ker h$ is not finitely generated.*

Proof. Let $K := \ker h$, and for $u \in \{x, y, z\}$ let $C_u := u/K := \{v \in F : h(v) = h(u)\}$. Define sequences of elements in these classes as follows: let

$$x_0 := x, \quad y_0 := y, \quad z_0 := z, \qquad \text{and for } i < \omega,$$

$$x_{i+1} := x \vee (y_i \wedge z_i), \quad y_{i+1} := y \vee (x_i \wedge z_i), \quad z_{i+1} := z \vee (x_i \wedge y_i),$$

$$m_i := (x_i \wedge y_i) \vee (x_i \wedge z_i) \vee (y_i \wedge z_i).$$

Summarizing these observations,

$$\begin{aligned}
m_0 &= (x \wedge y) \vee (x \wedge z) \vee (y \wedge z),\\
x_1 &= x \vee (y \wedge z), \quad y_1 = y \vee (x \wedge z), \quad z_1 = z \vee (x \wedge y),\\
m_1 &= (x_1 \wedge y_1) \vee (x_1 \wedge z_1) \vee (y_1 \wedge z_1),\\
x_2 &= x \vee \big\{\big[y \vee (x \wedge z)\big] \wedge \big[z \vee (x \wedge y)\big]\big\},\\
y_2 &= y \vee \big\{\big[x \vee (y \wedge z)\big] \wedge \big[z \vee (x \wedge y)\big]\big\},\\
z_3 &= z \vee (x_2 \wedge y_2)\\
&\ \vdots
\end{aligned}$$

Let X be a finite subset of K. We will prove there exists $(p, q) \in K \setminus \langle X \rangle$. Fix $u \in \{x, y, z, m\}$ and let $\{u_i\}$ be the corresponding sequence defined above. Since X is finite, Lemma B.3 implies that there exists $M \in \mathbb{N}$ such that for every $(p, q) \in X$ with $p, q \in C_u$, we have $p, q \leqslant u_M$.

Subclaim 1. For $(p, q) \in \langle X \rangle$ and $u \in \{x, y, z\}$, the following implication holds:

$$q \leqslant u \quad \Longrightarrow \quad p \leqslant u_M. \tag{B.1}$$

We prove the subclaim by induction on the complexity of terms. Fix $(p, q) \in \langle X \rangle$. Then $p, q \in C_u$ for some $u \in \{x, y, z\}$.

- **Case 0.** Suppose $(p, q) \in X$. Then by definition of M we have $p, q \leqslant u_M$.
- **Case 1.** Suppose $(p, q) = (p_1, q_1) \wedge (p_2, q_2)$, where (p_i, q_i) satisfies (B.1) for $i = 1, 2$. If $q = q_1 \wedge q_2 \leqslant u$, then, since generators in the free lattice are meet-prime (see Theorem C.8 below), we have $q_1 \leqslant u$ or $q_2 \leqslant u$. Assume $q_1 \leqslant u$. Then, by the induction hypothesis, $p_1 \leqslant u_M$. Therefore, $p = p_1 \wedge p_2 \leqslant u_M$, as desired.

- **Case 2.** Suppose $(p, q) = (p_1, q_1) \vee (p_2, q_2)$, where (p_i, q_i) satisfies (B.1) for $i = 1, 2$. If $q = q_1 \vee q_2 \leqslant u$, then $q_i \leqslant u$ for $i = 1, 2$. It now follows from the induction hypothesis that $p_i \leqslant u_M$ for $i = 1, 2$, so $p = p_1 \vee p_2 \leqslant u_M$, as desired.

This completes the proof of Subclaim 1. It now follows from Lemma B.2 that $(x, x_{M+1}) \in K \setminus \langle X \rangle$, so proposition is proved. □

Lemma B.2. *For each $u \in \{x, y, z, m\}$, the sequence $\{u_i\}$ is a strictly ascending chain; that is, $u_0 < u_1 < u_2 < \cdots$.*

Proof. We split the proof up into cases: either $u \in \{x, y, z\}$, or $u = m$.

- **Case 1.** $u \in \{x, y, z\}$.
For simplicity, assume $u = x$ for the remainder of the proof of this case. (Of course, the same argument applies to the case when u is y or z.) Fix $n < \omega$. We prove $x_n < x_{n+1}$.

  Subclaim 2. For all $n < \omega$,
  (1) $x_n \in C_x$,
  (2) $x_n \ngeqslant y$, and $x_n \ngeqslant z$.

  Proof of Subclaim 2. The first item is obvious; for the second, if $x_n \geqslant y$, then $x_n \wedge y = y$, and then $0 = h(x_n \wedge y) = h(y) = b$. A similar contradiction is reached if we assume $x_n \geqslant z$, so the subclaim is proved.

  Recall, $x_{n+1} = x_n \vee (y_n \wedge z_n)$, so $x_{n+1} > x_n$ holds as long as $x_n \ngeqslant y_n \wedge z_n$. So, by way of contradiction, suppose

  $$x_n \geqslant y_n \wedge z_n. \tag{B.2}$$

  Now, $y_n = y \vee (x \wedge z) \vee \cdots$, so clearly $y_n \geqslant y$. Similarly, $z_n \geqslant z$. This, together with (B.2), implies $x_n \geqslant y_n \wedge z_n \geqslant y \wedge z$. But then Theorem C.8 below implies that either $x_n \geqslant y$ or $x_n \geqslant z$, which contradicts Subclaim 2.

- **Case 2.** $u = m$.
We first prove that $m_0 = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ is strictly below $m_1 = (x_1 \wedge y_1) \vee (x_1 \wedge z_1) \vee (y_1 \wedge z_1)$. By symmetry, it suffices to show $x \wedge y < x_1 \wedge y_1$; that is, $x \wedge y < \big[x \vee (y \wedge z)\big] \wedge \big[y \vee (x \wedge z)\big]$.

  Clearly $x \wedge y \leqslant \big[x \vee (y \wedge z)\big] \wedge \big[y \vee (x \wedge z)\big]$. Suppose $x \wedge y = \big[x \vee (y \wedge z)\big] \wedge \big[y \vee (x \wedge z)\big]$. Then $\big[x \vee (y \wedge z)\big] \wedge \big[y \vee (x \wedge z)\big] \leqslant x$. By Theorem C.8, the latter holds iff $x \vee (y \wedge z) \leqslant x$ or $y \vee (x \wedge z) \leqslant x$ The first of these inequalities is clearly false, so it must be the case that $y \vee (x \wedge z) \leqslant x$. But then $y \leqslant x$, which is obviously false. We conclude that $x \wedge y < \big[x \vee (y \wedge z)\big] \wedge \big[y \vee (x \wedge z)\big]$. This proves $m_0 < m_1$.

  Now fix $n < \omega$ and assume $m_n < m_{n+1}$. We show $m_{n+1} < m_{n+2}$.

  wjd 2018-09-17: Complete the proof of this case.

  ↓ *begin scratch work* ↓
  $m_n := (x_n \wedge y_n) \vee (x_n \wedge z_n) \vee (y_n \wedge z_n)$,
  $m_{n+1} := (x_{n+1} \wedge y_{n+1}) \vee (x_{n+1} \wedge z_{n+1}) \vee (y_{n+1} \wedge z_{n+1})$,

By the first Case above, $u_n < u_{n+1}$.
$\uparrow$ *end scratch work* $\uparrow$

$\square$

Lemma B.3. *For all $u \in \{x, y, z\}$ and $p \in C_u \cup C_0$ there exists $n \in \mathbb{N}$ such that $p \leqslant m_{u,n}$.*

Proof. We prove this by induction on the complexity of p.

- **Case 0.** $p \in \{x, y, z\}$. Then $u = p = m_{p,0}$.
  For the remaining cases assume $u = x$, without loss of generality.
- **Case 1.** $p = p_1 \vee p_2$.
  If $p \in C_x \cup C_0$, then $p_i \in C_x \cup C_0$ for $i = 1, 2$, and the induction hypothesis yields i and j for which $p_1 \leqslant x_i$ and $p_2 \leqslant x_j$. Letting $n = \max\{i, j\}$, we have $p_1, p_2 \leqslant x_n$, from which $p = p_1 \vee p_2 \leqslant x_n$, as desired.
- **Case 2.** $p = p_1 \wedge p_2$.
  If $p \in C_x$, then we may assume $p_1 \in C_x$ and $p_2 \in C_x \cup C_0$. By the induction hypothesis, there exists $n \in \mathbb{N}$ such that $p_1 \leqslant x_n$, whence $p \leqslant p_1 \leqslant x_n$. If $p \in C_0$, then each p_i belongs to $C_u \cup C_0$ for some $u \in \{x, y, z\}$. If $p_1 \in C_x \cup C_0$, then $p_1 \leqslant x_n$, as above and we're done. Similarly, if $p_2 \in C_x \cup C_0$. So assume $p_1 \in C_y \cup C_0$ and $p_2 \in C_z \cup C_0$. Then the induction hypothesis implies that there exist i and j such that $p_1 \leqslant y_i$ and $p_2 \leqslant z_j$. If $n = \max\{i, j\}$, then $p_1 \leqslant y_n$ and $p_2 \leqslant z_n$. Then, by the above definition of the sequences, we have $p_1 \wedge p_2 \leqslant y_n \wedge z_n \leqslant m_n \leqslant x_{n+1}$.

$\square$

B.1. **Other Examples.** In each of the propositions in this section, X is a finite set and $\mathbf{F} = \mathbf{F}(X)$ is the free lattice generated by X. The symbol F denotes the universe of $\mathbf{F}$. The proof in each case is straightforward, but tedious; we omit proofs of the first two, and give a detailed proof of the third.

Prop. B.4. *Let $X = \{x, y, z\}$, and let $\mathbf{L} = \mathbf{2}$ be the 2-element chain. Then the kernel of an epimorphism $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ is a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$.*

Prop. B.5. *Let $X = \{x, y, z\}$ and let $\mathbf{L} = \mathbf{3}$ be the 3-element chain. Then the kernel of an epimorphism $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ is finitely generated.*

Prop. B.6. *Let $n > 2$, $X = \{x_0, x_1, \dots, x_{n-1}\}$, and $\mathbf{L} = \mathbf{2} \times \mathbf{2}$. Let $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ be an epimorphism. Then $K = \ker h$ is finitely generated.*

Proof. Let the universe of $\mathbf{L} = \mathbf{2} \times \mathbf{2}$ be $\{0, a, b, 1\}$, where $a \vee b = 1$ and $a \wedge b = 0$. For each $y \in L$, denote by $X_y = X \cap h^{-1}\{y\}$ the set of generators mapped by h to y. Denote the least and greatest elements of $h^{-1}\{y\}$ (if they exist) by ℓ_y and g_y, respectively. For example, $\ell_a = \bigwedge h^{-1}\{a\} = \bigwedge\{x \in F : h(x) = a\}$, $g_b = \bigvee\{x \in F : h(x) = b\}$, etc. In the present example, the least and greatest elements exist is each case, as we now show.

Subclaim 3. $h^{-1}\{a\}$ has least and greatest elements, namely $\ell_a = \bigwedge(X_a \cup X_1)$ and $g_a = \bigvee(X_a \cup X_0)$. (Similarly, $h^{-1}\{b\}$ has least and greatest elements, ℓ_b and g_b.)

Proof of Subclaim 3. Let $M(a) := \bigwedge(X_a \cup X_1)$ and $J(a) := \bigvee(X_a \cup X_0)$ and note that these values exist in F, since the sets involved are finite. Also, Then $h(M(a)) = a = h(J(a))$. Fix $r \in h^{-1}\{a\}$.

- If $r \in X_a$, then $r \geqslant \bigwedge X_a \geqslant \bigwedge(X_a \cup X_1) = M(a)$.
- If $r = s \vee t$, where $h(s) = a$ and $h(t) \in \{a, 0\}$, then assume (the induction hypothesis) that $s \geqslant M(a)$, and we have $r = s \vee t \geqslant M(a)$.
- If $r = s \wedge t$, where $h(s) = a$ and $h(t) \in \{a, 1\}$, then assume (the induction hypothesis) that $s, t \geqslant M(a)$, and we have $s \wedge t \geqslant M(a)$. This proves that for each $r \in h^{-1}\{a\}$ we have $r \geqslant M(a)$, and as we noted at the outset, $M(a) \in h^{-1}\{a\}$. Therefore, $\ell_a = M(a)$ is the least element of $h^{-1}\{a\}$. Similarly, every $r \in h^{-1}\{a\}$ is below $J(a)$, so $g_a = J(a)$. The proofs of $\ell_b = M(b)$ and $g_b = J(b)$ are similar.

This proves Subclaim 3.

Subclaim 4. $h^{-1}\{0\}$ has least and greatest elements, namely, $\ell_0 = \bigwedge X$ and $g_0 = g_a \wedge g_b$.

Proof of Subclaim 4. $\ell_0 = \bigwedge X$ is obvious, so we need only verify that $g_0 = g_a \wedge g_b$. Observe that $h(g_a \wedge g_b) = h(g_a) \wedge h(g_b) = a \wedge b = 0$, so $g_a \wedge g_b \in h^{-1}\{0\}$. It remains to prove that $r \leqslant g_a \wedge g_b$ holds for all $r \in h^{-1}\{0\}$. Fix $r \in h^{-1}\{0\}$. Then $h(r \vee g_a) = h(r) \vee h(g_a) = 0 \vee a = a$, which places $r \vee g_a$ in $h^{-1}\{a\}$. Therefore, by maximality of g_a, we have $r \vee g_a \leqslant g_a$, whence $r \leqslant g_a$. Similarly, $r \leqslant g_b$. This proves Subclaim 4.

Subclaim 5. $h^{-1}\{1\}$ has least and greatest elements, namely $\ell_1 = \ell_a \vee \ell_b$ and $g_1 = \bigvee X$.

Proof of Subclaim 5. $g_1 = \bigvee X$ is obvious, so we need only verify that $\ell_1 = \ell_a \vee \ell_b$. Observe that $h(\ell_a \vee \ell_b) = h(\ell_a) \vee h(\ell_b) = a \vee b = 1$, so $\ell_a \vee \ell_b \in h^{-1}\{1\}$. It remains to prove that $r \geqslant \ell_a \vee \ell_b$ holds for all $r \in h^{-1}\{1\}$. Fix $r \in h^{-1}\{1\}$. Then $h(r \wedge \ell_a) = h(r) \wedge h(\ell_a) = 1 \wedge a = a$, which places $r \wedge \ell_a$ in $h^{-1}\{a\}$. Therefore, by minimality of ℓ_a, we have $r \wedge \ell_a \geqslant \ell_a$, whence $r \geqslant \ell_a$. Similarly, $r \geqslant \ell_b$. Now let $Y = \{(x, g_p), (g_p, x), (x, \ell_p), (\ell_p, x) : p \in \{0, a, b, 1\}, x \in X_p\}$. This proves Subclaim 5.

Subclaim 6. If $r \in F$ and $h(r) = p$, then $(r, \ell_p), (r, g_p) \in \langle Y \rangle$.

Proof of Subclaim 6. Either $r \in X_p$ or $r = s \wedge t$ or $r = s \vee t$. If $r \in X_p$, then the pair belongs to Y and the claim is trivial.

Suppose $r = s \wedge t$.

- If $h(r) = 1$, then $h(s) = h(t) = 1$. Assume (the induction hypothesis) that $\{(s, \ell_1), (s, g_1), (t, \ell_1), (t, g_1)\} \subseteq \langle Y \rangle$. Then $(r, \ell_1) = (s \wedge t, \ell_1) = (s, \ell_1) \wedge (t, \ell_1) \in \langle Y \rangle$.

- If $h(r) = a$, then (wlog) $h(s) = a$ and $h(t) \in \{a, 1\}$. Assume (the induction hypothesis) that $\{(s, \ell_a), (s, g_a), (t, \ell_p), (t, g_p)\} \subseteq \langle Y \rangle$. By Claim 1, $\ell_a \leqslant \ell_1$, so $\ell_a = \ell_a \wedge \ell_1$.
  If $h(t) = 1$, then
  $$(r, \ell_a) = (s \wedge t, \ell_a \wedge \ell_1) = (s, \ell_a) \wedge (t, \ell_1) \in \langle Y \rangle,$$
  while If $h(t) = a$, then $(r, \ell_a) = (s \wedge t, \ell_a \wedge \ell_a) = (s, \ell_a) \wedge (t, \ell_a) \in \langle Y \rangle$.
- If $h(r) = 0$, then (wlog) that either (i) $h(s) = 0$, or (ii) $h(s) = a$, $h(t) = b$. If $h(s) = 0$, then $(s, \ell_0) \in \langle Y \rangle$ implies $(r, \ell_0) = (s \wedge t, \ell_0) = (s, \ell_0) \wedge (t, \ell_p) \in \langle Y \rangle$. If If $h(s) = a$, $h(t) = b$, and $(s, \ell_a), (t, \ell_b) \in \langle Y \rangle$, then $(r, \ell_0) = (s \wedge t, \ell_0) = (s, \ell_a) \wedge (t, \ell_b) \in \langle Y \rangle$.

This proves that $(r, \ell_1) \in \langle Y \rangle$ if $r = s \wedge t$. A similar argument shows that $(r, g_p) \in \langle Y \rangle$ in each of the three subcases. We have thus proved that $\{(r, \ell_1), (r, g_p)\} \subseteq \langle Y \rangle$, if $r = s \wedge t$.

Suppose $r = s \vee t$.

- If $h(r) = 0$, then $h(s) = h(t) = 0$. Assume (the induction hypothesis) that $\{(s, \ell_p), (s, g_p), (t, \ell_p), (t, g_p)\} \subseteq \langle Y \rangle$. Then $(r, \ell_p) = (s \vee t, \ell_p) = (s, \ell_p) \vee (t, \ell_p) \in \langle Y \rangle$.
- If $h(r) = a$, then (wlog) $h(s) = a$ and $h(t) \in \{a, 0\}$. If we assume (the induction hypothesis) that $(s, \ell_p), (s, g_p), (t, \ell_p), (t, g_p)$ belong to $\langle Y \rangle$, then $(r, \ell_p) = (s \vee t, \ell_p) = (s, \ell_p) \vee (t, \ell_p) \in \langle Y \rangle$.
- If $h(r) = 1$, then (wlog) that either (i) $h(s) = 1$, or (ii) $h(s) = a$, $h(t) = b$.
  In the first case, $(s, \ell_1) \in \langle Y \rangle$ implies $(r, \ell_1) = (s \vee t, \ell_1) = (s, \ell_1) \vee (t, \ell_p) \in \langle Y \rangle$. In the second case $h(s) = a$, $h(t) = b$, and $(s, \ell_a), (t, \ell_b) \in \langle Y \rangle$. Then $(r, \ell_1) = (s \vee t, \ell_1) = (s, \ell_a) \vee (t, \ell_b) \in \langle Y \rangle$.

Similarly, in each of these three subcases, we have $(r, g_p) \in \langle Y \rangle$. This proves Subclaim 6, and completes the proof of Prop B.6. □

Appendix C. Background

The notation, definitions, ideas presented below are based on those we learned from the book by Freese, Jezek, Nation [FJN95],

Definition C.1 (length of a term)**.** Let X be a set. Each element of X is a term of length 1, also known as a **variable**. If $t_1, \dots, t_n$ are terms of lengths $k_1, \dots, k_n$, then $t_1 \vee \cdots \vee t_n$ and $t_1 \wedge \cdots \wedge t_n$ are both terms of length $1 + k_1 + \cdots + k_n$.

Examples. By the above definition, the terms

$$x \vee y \vee z \qquad x \vee (y \vee z) \qquad (x \vee y) \vee z$$

have lengths 4, 5, and 5, respectively. Reason: variables have length 1, so $x \vee y \vee z$ has length $1+1+1+1$. On the other hand, $x \vee y$ is a term of length 3, so $(x \vee y) \vee z$ has length $1+3+1$. Similarly, $x \vee (y \vee z)$ has length $1+1+3$.

Lemma C.2 ([FJN95, Lem. 1.2])**.** *Let $\mathcal{V}$ be a nontrivial variety of lattices and let $\mathbf{F}_{\mathcal{V}}(X)$ be the relatively free lattice in $\mathcal{V}$ over X. Then,*

$$\bigwedge S \leqslant \bigvee T \text{ implies } S \cap T \neq \emptyset \text{ for each pair of finite subsets } S, T \subseteq X. \tag{C.1}$$

Lemma C.3 ([FJN95, Lem. 1.4])**.** *Let $\mathbf{L}$ be a lattice generated by a set X and let $a \in L$.*

(1) If a is join prime, then $a = \bigwedge S$ for some finite subset $S \subseteq X$.
(2) If a is meet prime, then $a = \bigvee S$ for some finite subset $S \subseteq X$.
If X satisfies condition (C.1) above, then
(3) for every finite, nonempty subset $S \subset X$, $\bigwedge S$ is join prime and $\bigvee S$ is meet prime.

Corollary C.4 ([FJN95, Cor. 1.5])**.** Let $\mathcal{V}$ be a nontrivial variety of lattices and let $\mathbf{F}_{\mathcal{V}}(X)$ be the relatively free lattice in $\mathcal{V}$ over X. For each finite nonempty subset $S \subseteq X$, $\bigwedge S$ is join prime and $\bigvee S$ is meet prime. In particular, every $x \in X$ is both join and meet prime. Moreover, if $x \leqslant y$ for $x, y \in X$, then $x = y$.

Theorem C.5 (Whitman's Condition, ver. 1)**.** *The free lattice $\mathbf{F}(X)$ satisfies the following condition:*

(W) If $v = v_1 \wedge \cdots \wedge v_r \leqslant u_1 \vee \cdots \vee u_s = u$, then either $v_i \leqslant u$ for some i, or $v \leqslant u_j$ for some j.

Corollary C.6 ([FJN95, Cor. 1.9])**.** Every sublattice of a free lattice satisfies (W). Every element of a lattice satisfying (W) is either join or meet irreducible.

Theorem C.7 (Whitman's Condition, ver. 2)**.** *The free lattice $\mathbf{F}(X)$ satisfies the following condition:*

(W+) If $v = v_1 \wedge \cdots \wedge v_r \wedge x_1 \wedge \cdots \wedge x_n \leqslant u_1 \vee \cdots \vee u_s \vee y_1 \vee \cdots \vee y_m = u$, where $x_i, y_j \in X$, then either $x_i = y_j$ for some i and j, or $v_i \leqslant u$ for some i, or $v \leqslant u_j$ for some j.

Theorem C.8 ([FJN95, Thm. 1.11])**.** *If $s = s(x_1, \dots, x_n)$ and $t = t(x_1, \dots, x_n)$ are terms and $x_1, \dots, x_n \in X$, then the truth of*

$$s^{\mathbf{F}(X)} \leqslant t^{\mathbf{F}(X)} \tag{C.2}$$

can be determined by applying the following rules.

(1) *If $s = x_i$ and $t = x_j$, then (C.2) holds iff $x_i = x_j$.*
(2) *If $s = s_1 \vee \cdots \vee s_k$ is a formal join, then (C.2) holds iff $s_i^{\mathbf{F}(X)} \leqslant t^{\mathbf{F}(X)}$ for all i.*
(3) *If $t = t_1 \wedge \cdots \wedge t_k$ is a formal meet, then (C.2) holds iff $s^{\mathbf{F}(X)} \leqslant t_i^{\mathbf{F}(X)}$ for all i.*
(4) *If $s = x_i$ and $t = t_1 \vee \cdots \vee t_k$ is a formal join, then (C.2) holds iff $x_i \leqslant t_j^{\mathbf{F}(X)}$ for some j.*
(5) *If $s = s_1 \wedge \cdots \wedge s_k$ is a formal meet and $t = x_i$, then (C.2) holds iff $s_j^{\mathbf{F}(X)} \leqslant x_i$ for some j.*
(6) *If $s = s_1 \wedge \cdots \wedge s_k$ is a formal meet and and $t = t_1 \vee \cdots \vee t_m$ is a formal join, then (C.2) holds iff $s_i^{\mathbf{F}(X)} \leqslant t^{\mathbf{F}(X)}$ for some i or $s^{\mathbf{F}(X)} \leqslant t_j^{\mathbf{F}(X)}$ for some j*

Theorem C.9 ([FJN95, Thm. 1.17])**.** *For each $w \in \mathbf{F}(X)$ there is a term of minimal length representing w, unique up to commutativity. This term is called the* ***canonical form*** *of w.*

Let $w \in \mathbf{F}(X)$ be join reducible and suppose $t = t_1 \vee \cdots \vee t_n$ (with $n > 1$) is the canonical form of w. Let $w_i = t_i^{\mathbf{F}(X)}$. Then $\{w_1, \dots, w_n\}$ are called the **canonical joinands** of w. We also say $w = w_1 \vee \cdots \vee w_n$ *canonically* and that $w = w_1 \vee \cdots \vee w_n$ is the **canonical join representation** of w. If w is join irreducible, we define the canonical joinands of w to be the set $\{w\}$. Of course the **canonical meet representation** and **canonical meetands** of an element in a free lattice are defined dually.

A join representation $a = a_1 \vee \cdots \vee a_n$ in an arbitrary lattice is said to be a **minimal join representation** if $a = b_1 \vee \cdots \vee b_m$ and $\{b_1, \dots, b_m\} \ll \{a_1, \dots, a_n\}$ imply $\{a_1, \dots, a_n\} \subseteq \{b_1, \dots, b_m\}$. Equivalently, a join representation is minimal if it is an antichain and nonrefinable.

Theorem C.10 ([FJN95, Thm. 1.19])**.** *Let $w = w_1 \vee \cdots \vee w_n$ canonically in $\mathbf{F}(X)$. If also $w = u_1 \vee \cdots \vee u_m$, then $\{w_1, \dots, w_n\} \ll \{u_1, \dots, u_m\}$. Thus $w = w_1 \vee \cdots \vee w_n$ is the unique minimal join representation of w.*

Theorem C.11 ([FJN95, Thm. 1.20])**.** *Let $w \in \mathbf{F}(X)$ and let u be a join irreducible element in $\mathbf{F}(X)$. Then u is a canonical joinand of w if and only if there is an element a such that $w = u \vee a$ and $w > v \vee a$ for every $v < u$.*

Definition C.12 (up directed, continuous)**.** A subset A of a lattice L is said to be **up directed** if every finite subset of A has an upper bound in A. It suffices to check this for pairs. A is up directed iff for all $a, b \in A$ there exists $c \in A$ such that $a \leqslant c$ and $b \leqslant c$. A lattice is **upper continuous** if whenever

$A \subseteq L$ is an up directed set having a least upper bound $u = \bigvee A$, then for every b,

$$\bigvee_{a \in A}(a \wedge b) = \bigvee_{a \in A} a \wedge b = u \wedge b.$$

Down directed and **down continuous** are defined dually. A lattice that is both up and down continuous is called **continuous**.

Theorem C.13 ([FJN95, Thm. 1.22])**.** *Free lattices are continuous.*

C.1. **Bounded Homomorphisms.** We continue to follow [FJN95] very closely, although the authors of that book indicate that the ideas in this subsection have their roots in Ralph McKenzie's work on nonmodular lattice varieties [McK72], and Bjarni Jónsson's work on sublattices of free lattices [JN77].

If x, y are elements of a lattice L, and if $x \leqslant y$, then we write $[\![x, y]\!]$ to denote the sublattice of elements between x and y. That is,

$$[\![x, y]\!] := \{z \in L \mid x \leqslant z \leqslant y\}.$$

Let $\mathbf{K}$ and $\mathbf{L}$ be lattices and suppose $\mathbf{L}$ has bottom and top elements, $0_{\mathbf{L}}$ and $1_{\mathbf{L}}$, resp. If $h \colon \mathbf{K} \to \mathbf{L}$ is a lattice homomorphism, then for each $a \in L$ we consider the sets $h^{-1}[\![a, 1]\!] = \{x \in K \mid h(x) \geqslant a\}$ and $h^{-1}[\![0, a]\!] = \{x \in K \mid h(x) \leqslant a\}$. When $h^{-1}[\![a, 1]\!]$ is nonempty, it is a filter of $\mathbf{K}$; dually a nonempty $h^{-1}[\![0, a]\!]$ is an ideal. If K is infinite, then $h^{-1}[\![a, 1]\!]$ need not have a least element, nor $h^{-1}[\![0, a]\!]$ a greatest element. However, considering when such extrema exist leads to the notion of bounded homomorphism, which in turn helps us understand the structure of free lattices.

A lattice homomorphism $h \colon \mathbf{K} \to \mathbf{L}$ is **lower bounded** if for every $a \in L$, the set $h^{-1}[\![a, 1]\!]$ is either empty or has a least element. The least element of a nonempty $h^{-1}[\![a, 1]\!]$ is denoted by $\beta_h(a)$, or by $\beta(a)$ when h is clear from context. Thus, if h is a lower bounded homomorphism, then $\beta_h \colon \mathbf{L} \rightharpoonup \mathbf{K}$ is a partial mapping whose domain is an ideal of $\mathbf{L}$.

Dually, h is an **upper bounded** homomorphism if, whenever the set $h^{-1}[\![0, a]\!]$ is nonempty, it has a greatest element, denoted by $\alpha_h(a)$, or $\alpha(a)$. For an upper bounded homomorphism, the domain of $\alpha_h \colon \mathbf{L} \rightharpoonup \mathbf{K}$ is clearly a filter of $\mathbf{L}$. A **bounded** homomorphism is one that is both upper and lower bounded.

These definitions simplify when h is an epimorphism. In that case h is lower bounded if and only if each preimage $h^{-1}\{a\}$ has a least element. Likewise, if L is finite, then $h \colon \mathbf{K} \to \mathbf{L}$ is lower bounded if and only if $h^{-1}\{a\}$ has a least element whenever it is nonempty. On the other hand, every homomorphism h from a finite lattice $\mathbf{K}$ is bounded.

Note that β is monotonic and a left adjoint for h, i.e., $a \leqslant h(x)$ iff $\beta(a)x$. It then follows from a standard argument that β is a join preserving map on its domain: if $h^{-1}[\![a, 1]\!] \neq \emptyset$ and $h^{-1}[\![b, 1]\!] \neq \emptyset$, then $\beta(a \vee b) = \beta(a) \vee \beta(b)$. Similarly, α is a right adjoint for h, so that $h(y) \leqslant a$ iff $y \leqslant \alpha(a)$, and for $a, b \in \operatorname{dom} \alpha$, $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$. In particular, if h is an epimorphism, then α and β are respectively meet and join homomorphisms of $\mathbf{L}$ into $\mathbf{K}$.

For future reference, we note that α and β behave correctly with respect to composition.

Theorem C.14 ([FJN95, Thm. 2.1])**.** *Let $f\colon \mathbf{K} \to \mathbf{L}$ and $g\colon \mathbf{L} \to \mathbf{M}$ be homomorphisms. If f, g are lower bounded, then $gf\colon \mathbf{K} \to \mathbf{M}$ is lower bounded and $\beta_{gf} = \beta_f \beta_g$. Similarly, if f, g are upper bounded, then $\alpha_{gf} = \alpha_f \alpha_g$.*

Proof. For $x \in K$ and $a \in M$, we have

$$a \leqslant gf(x) \quad \text{iff} \quad \beta_g(a) \leqslant f(x) \quad \text{iff} \quad \beta_f\beta_g(a) \leqslant x.$$

The upper bounded case is dual. □

We need a way to determine whether a lattice homomorphism $h\colon \mathbf{K} \to \mathbf{L}$ is upper or lower bounded. The most natural setting for this is when the lattice $\mathbf{K}$ is finitely generated, so from now on we assume $\mathbf{K}$ is generated by a finite set X. We want to analyze the sets $h^{-1}[\![a, 1]\!]$ for $a \in L$, with the possibility of lower boundedness in mind. (The corresponding results for $h^{-1}[\![0, a]\!]$ are obtained by duality.) Note that $\mathbf{K}$ has a greatest element $1_{\mathbf{K}} = \bigvee X$, and that $h^{-1}[\![a, 1]\!]$ is nonempty if and only if $a \leqslant h(1_{\mathbf{K}})$.

Define a pair of closure operators, denoted by $^\wedge$ and $^\vee$, on subsets of an arbitrary lattice L as follows: for each $A \subseteq L$,

$$A^\wedge := \{\bigwedge B \mid B \text{ is a finite subset of } A\}.$$

We adopt the following convention: if $\mathbf{L}$ has a greatest element $1_{\mathbf{L}}$, then $\bigwedge \emptyset = 1_{\mathbf{L}}$, and we include this in $A^\wedge$ for every $A \subseteq L$; otherwise, $\bigwedge \emptyset$ is undefined. The set $A^\vee$ is defined dually.

We can write K as the union of a chain of subsets $H_0 \subseteq H_1 \subseteq \cdots$ defined inductively by setting $H_0 := X^\wedge$ and $H_{k+1} := (H_k)^{\vee\wedge}$, for all $k \geqslant 0$. By induction, each $H_n = X^{\wedge(\vee\wedge)^n}$ is a finite meet-closed subset of K, and $\bigcup H_n = K$, since X generates K.

Let $h\colon \mathbf{K} \to \mathbf{L}$ be an epimorphism and, for each $y \in L$ and $k < \omega$, define

$$\beta_k(y) = \bigwedge\{w \in H_k : h(w) \geqslant a\}.$$

Theorem C.15 ([FJN95, Thm. 2.2])**.** *Let $\mathbf{K}$ be finitely generated, and let $h\colon \mathbf{K} \to \mathbf{L}$ be a lattice homomorphism. If $a \leqslant h(1_{\mathbf{K}})$, then*

(1) $j \leqslant k$ implies $\beta_j(a) \geqslant \beta_k(a)$,
(2) $\beta_k(a)$ is the least element of $H_k \cap h^{-1}[\![a, 1]\!]$,
(3) $h^{-1}[\![a, 1]\!] = \bigcup_{k\in\omega} [\![\beta_k(a), 1]\!]$.

C.1.1. *Minimal join covers and refinement.* A **join cover** of the element $a \in L$ is a finite subset $S \subseteq L$ such that $a \leqslant \bigvee S$. A join cover S of a is *nontrivial* if $a \nleq s$ for all $s \in S$. Let $\mathcal{C}(a)$ be the set of all nontrivial join covers of a in $\mathbf{L}$.

Theorem C.16 ([FJN95, Thm. 2.3])**.** *Suppose* $\mathbf{K}$ *is generated by a finite set* X*,* $h\colon \mathbf{K} \to \mathbf{L}$ *is a homomorphism,* $a \leqslant h(1_{\mathbf{K}})$*, and* $k \in \omega$*. Then,*

$$\beta_0(a) = \bigwedge\{x \in X \mid h(x) \geqslant a\},$$
$$\beta_{k+1}(a) = \beta_0(a) \wedge \bigwedge_{\substack{S \in \mathcal{C}(a) \\ \bigvee S \leqslant h(1_{\mathbf{K}})}} \bigvee_{s \in S} \beta_k(s).$$

In general, the expression for $\beta_{k+1}(a)$ has some redundant terms, which we can exclude if $\mathbf{L}$ satisfies a weak finiteness condition that we now define. For finite subsets A, $B \subseteq L$, we say A *join refines* B and write $A \ll B$ if for every $a \in A$ there exists $b \in B$ with $a \leqslant b$. Theorem C.10 states that if $w \in \mathbf{F}(X)$ and $w = \bigvee B$, then the set of canonical joinands of w join refines B.

Define a **minimal nontrivial join cover** of $a \in L$ to be a nontrivial join cover S with the property that whenever $a \leqslant \bigvee T$ and $T \ll S$, then $S \subseteq T$. This formulation is equivalent to our more intuitive notion of what minimality ought to mean: *a nontrivial join cover* S *of* a *is minimal if and only if*

(1) *S is an antichain of join irreducible elements of L, and*
(2) *if an element of S is deleted or replaced by a (finite) set of strictly smaller elements, then the resulting set is no longer a join cover of a.*

Let $\mathcal{M}(a)$ denote the set of minimal nontrivial join covers of $a \in L$. Let us say that $\mathbf{L}$ has the **minimal join cover refinement property** if for each $a \in L$, $\mathcal{M}(a)$ is finite and every nontrivial join cover of a refines to a minimal one. Clearly every finite lattice has the minimal join cover refinement property, but so do free lattices. The following reformulation of Theorem C.16 simplifies the calculation of β_k whenever the minimal join cover refinement property holds.

Theorem C.17 ([FJN95, Thm. 2.4])**.** *Let* $\mathbf{K}$ *be generated by the finite set* X*, and let* $h\colon \mathbf{K} \to \mathbf{L}$ *be a lattice homomorphism. If* $\mathbf{L}$ *has the minimal join cover refinement property, then for each* $a \in L$ *with* $a \leqslant h(1_{\mathbf{K}})$ *and* $k \in \omega$*, we have*

$$\beta_0(a) = \bigwedge\{x \in X \mid h(x) \geqslant a\},$$
$$\beta_{k+1}(a) = \beta_0(a) \wedge \bigwedge_{\substack{S \in \mathcal{M}(a) \\ \bigvee S \leqslant h(1_{\mathbf{K}})}} \bigvee_{s \in S} \beta_k(s).$$

We now look for a condition on the $\mathbf{L}$ that will insure the homomorphism $h\colon \mathbf{K} \to \mathbf{L}$ is lower bounded. From Theorem C.15, this will happen iff for each $a \leqslant h(1_{\mathbf{K}})$ there exists $N \in \omega$ such that $\beta_n(a) = \beta_N(a)$ for all $n \geqslant N$. In this case, $\beta(a) = \beta_N(a)$ for all $a \in \operatorname{dom} \beta = [\![0, h(1_{\mathbf{K}})]\!]$, where N depends upon a.

Fact. The following are equivalent:

wjd 2018-09-17: To do: verify this!

(1) h is not lower bounded;
(2) $(\exists y_0 \in L)(\forall N)(\exists n > N)\, \beta_n(a) \neq \beta_N(a)$;
(3) $(\exists y_0 \in L)(\exists N)(\forall n > N)\beta_n(a) \neq \beta_N(a)$.

Let $D_0(\mathbf{L})$ be the set of all join prime elements of $\mathbf{L}$, i.e., the set of elements that have no nontrivial join cover. Given $D_k(\mathbf{L})$, define $D_{k+1}(\mathbf{L})$ to be the set of $p \in L$ such that every nontrivial join cover of p refines to a join cover contained in $D_k(\mathbf{L})$, i.e., $p \leqslant \bigvee S$ nontrivially implies there exists $T \ll S$ with $p \leqslant \bigvee T$ and $T \subseteq D_k(\mathbf{L})$. Note that if L has the minimal join cover refinement property, then $p \in D_{k+1}(\mathbf{L})$ iff every minimal nontrivial join cover of p is contained in $D_k(\mathbf{L})$.

The definition clearly implies $D_0(\mathbf{L}) \subseteq D_1(\mathbf{L}) \subseteq D_2(\mathbf{L}) \subseteq \cdots$. Let $D(\mathbf{L}) = \bigcup D_i$. For $a \in D(\mathbf{L})$, define the D-**rank**, $\rho(a)$, to be the least integer N such that $a \in D_N(\mathbf{L})$; for $a \notin D(L)$, $\rho(a)$ is undefined. The duals of $D_k(\mathbf{L})$, $D(\mathbf{L})$, and $\rho(a)$ are denoted by $D_k^d(\mathbf{L})$, $D^d(\mathbf{L})$, and $\rho^d(a)$, respectively.

Appendix D. Miscellaneous Notes

Let K be a finite subset of $\ker h$. Since K is finite, we can find an $N < \omega$ such that for all $\binom{p}{q} \in K$, the following implications are satisfied:

$$\begin{aligned} p \leqslant x &\implies q \leqslant x_N \\ p \leqslant y &\implies q \leqslant y_N \\ p \leqslant z &\implies q \leqslant z_N \end{aligned} \tag{D.1}$$

$$\begin{aligned} p \leqslant x \vee (y \wedge z) &\implies q \leqslant x_{N+1} \\ p \leqslant y \vee (x \wedge z) &\implies q \leqslant y_{N+1} \\ p \leqslant z \vee (x \wedge y) &\implies q \leqslant z_{N+1} \end{aligned} \tag{D.2}$$

Claim 4.3 If N is chosen as just described, and if $\binom{p}{q} \in \langle K \rangle$ then the implications D.1 and D.2 hold.

Proof. As usual, we proceed by induction on term complexity. If $\binom{p}{q} \in K$, then by choice of N, there is nothing to prove.

Case 1. Suppose $\binom{p}{q} = \binom{p_1}{q_1} \vee \binom{p_2}{q_2}$, where $\binom{p_1}{q_1}$ and $\binom{p_2}{q_2}$ satisfy (D.1) and (D.2). We show that $\binom{p}{q}$ satisfies these two implications as well. Recall, in the notation above, $x_1 := x \vee (y \wedge z)$.

Assume $p \leqslant x_1$. We show $q \leqslant x_{N+1}$. Since $p = p_1 \vee p_2 \leqslant x_1$, we have $p_1 \leqslant x_1$ and $p_2 \leqslant x_1$, so by the induction hypothesis, $q_1 \leqslant x_{N+1}$ and $q_2 \leqslant x_{N+1}$. Therefore, $q = q_1 \vee q_2 \leqslant x_{N+1}$, as desired.

Now assume $p \leqslant x$. We show $q \leqslant x_N$. Since $p = p_1 \vee p_2 \leqslant x$, we have $p_1 \leqslant x$ and $p_2 \leqslant x$, so by the induction hypothesis, $q_1 \leqslant x_N$ and $q_2 \leqslant x_N$. Therefore, $q = q_1 \vee q_2 \leqslant x_N$, as desired.

Case 2. Suppose $\binom{p}{q} = \binom{p_1}{q_1} \wedge \binom{p_2}{q_2}$, where $\binom{p_1}{q_1}$ and $\binom{p_2}{q_2}$ satisfy (D.1) and (D.2).

Assume $p \leqslant x_1 = x \vee (y \wedge z)$. We must show $q \leqslant x_{N+1}$. Since $p_1 \wedge p_2 \leqslant x_1$, then according to Theorem C.8, at least one of the following inequalities must hold:

(1) $p_1 \leqslant x_1$;
(2) $p_2 \leqslant x_1$;
(3) $p_1 \wedge p_2 \leqslant x$;
(4) $p_1 \wedge p_2 \leqslant y \wedge z$.

By the induction hypothesis, (1) implies $q_1 \leqslant x_{N+1}$ and (2) implies $q_2 \leqslant x_{N+1}$. In either case, $q = q_1 \wedge q_2 \leqslant x_{N+1}$, as desired. In case (3), Theorem C.8 implies that either $p_1 \leqslant x$ or $p_2 \leqslant x$, since x is a generator. Therefore, $q_1 \leqslant x_N$ or $q_2 \leqslant x_N$ and we conclude that $q \leqslant x_N \leqslant x_{N+1}$, as desired. It remains to prove $q \leqslant x_{N+1}$ for the final case in which $p_1 \wedge p_2 \leqslant y \wedge z$.

If $p_1 \wedge p_2 \leqslant y \wedge z$, then $p_1 \wedge p_2 \leqslant y$ and $p_1 \wedge p_2 \leqslant z$. Therefore, both of the following disjunctions hold:

- $p_1 \leqslant y$ or $p_2 \leqslant y$, and
- $p_1 \leqslant z$ or $p_2 \leqslant z$.

If $p_1 \leqslant y$ and $p_1 \leqslant z$, then $p_1 \leqslant x \vee (y \wedge z) = x_1$, so $q_1 \leqslant x_{N+1}$, so $q = q_1 \wedge q_2 \leqslant x_{N+1}$, as desired. Similarly, if $p_2 \leqslant y$ and $p_2 \leqslant z$, the desired conclusion holds. Finally, consider the case in which $p_1 \leqslant y$ and $p_2 \leqslant z$. In this case $q_1 \leqslant y_N$ and $q_2 \leqslant z_N$. Therefore, $q = q_1 \wedge q_2 \leqslant y_N \wedge z_N \leqslant x_N \vee (y_N \wedge z_N) = x_{N+1}$, as desired. $\square$

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