(1) Not done: We only know $h\beta_0(a) \geq a$. But $\forall a \in L\ \exists \ell \in \mathbb{N}\ \forall k \geq \ell : h\beta_k(a) = a$

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$p_i \geqslant \beta_{\ell_i} h(p_i)$ and, by the induction hypothesis, $q_i \geqslant \beta_{\ell_i+n} h(p_i)$. Our goal once again is to prove $q \geqslant \beta_{\ell+n} h(p)$.

For all k, H_k is closed under meet, so the length of p is $\ell = \max\{\ell_1, \ell_2\}$ (in contrast to Case 1). Still, $\beta_{\ell_i+n} h(p_i) \geqslant \beta_{\ell+n} h(p_i)$, so

$$q = q_1 \wedge q_2 \geqslant \beta_{\ell_1+n} h(p_1) \wedge \beta_{\ell_1+n} h(p_2) \geqslant \beta_{\ell+n} h(p_1) \wedge \beta_{\ell+n} h(p_2).$$

Therefore, to complete the proof it this case, it suffices to show

$$\beta_{\ell+n} h(p_1) \wedge \beta_{\ell+n} h(p_2) \geqslant \beta_{\ell+n} h(p).$$

Recall that $\beta_{\ell+n} h(p_i)$ denotes the least term t_i of length at most $\ell + n$ such that $h(t_i) = h(p_i)$. Now, $t_1 \wedge t_2$ has length at most $\ell + n$, since $H_{\ell+n}$ is closed under meet. Also,

$$h(t_1 \wedge t_2) = h(t_1) \wedge h(t_2) = h(p_1) \wedge h(p_2) = h(p_1 \wedge p_2) = h(p),$$

so the image of $t_1 \wedge t_2$ under h is $h(p)$. Since $\beta_{\ell+n} h(p)$ is the least term of length at most $\ell + n$ that is mapped to $h(p)$, we have $t_1 \wedge t_2 \geqslant \beta_{\ell+n} h(p)$, as desired.

4. Proof of the Main Theorem

With Lemma 3.2 established, we are now in a position to prove Theorem 3.1. We begin with the converse direction. In fact, we will prove the contrapositive of the converse. Specifically, if $h : \mathbf{F} \to \mathbf{L}$ is not lower bounded, then $\ker h$ is not a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$. After proving the converse, we will assume h is (both upper- and lower-) bounded, and then prove that $\ker h$ is a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$.

Proof of Theorem 3.1.
($\Leftarrow$) Suppose $h : \mathbf{F} \to \mathbf{L}$ is not lower bounded, and let K be an arbitrary finite subset of $\ker h$. We prove that the subalgebra $\langle K \rangle \leqslant \mathbf{F} \times \mathbf{F}$ generated by K is not all of $\ker h$. Since K is an arbitrary finite subset of $\ker h$, this will prove that $\ker h$ is not finitely generated. $K \to C$

Let $h^{-1}(a)$ be a class of $\ker h$ that is not lower bounded. The ~~the~~ sequence $\beta_k(a)$ defined prior to Lemma 3.2 is an infinite decending chain. Of course, for every $k \in \mathbb{N}$, we have $h\beta_0(a) = h\beta_k(a)$, so $(\beta_0(a), \beta_k(a)) \in \ker h$. On the other hand, since K is a finite subset of $\ker h$, Lemma 3.2 asserts the existence of an $n \in \mathbb{N}$ such that, if $(p, q) \in \langle K \rangle$, then $p \geqslant \beta_\ell h(p)$ implies $p \geqslant \beta_{\ell+n} h(p)$. Therefore, $(\beta_0(a), \beta_{n+1}(a)) \in \ker h \setminus \langle K \rangle$. Thus $\langle K \rangle \neq \ker h$, as we set out to prove. (1)

($\Rightarrow$) Assume h is a bounded epimorphism so that the preimage of each $y \in L$ under h is bounded. For each $y \in L$, let $\alpha(y) = \bigvee h^{-1}\{y\}$ and $\beta(y) = \bigwedge h^{-1}\{y\}$ denote the greatest and least elements of $h^{-1}\{y\}$, respectively (both of which exist by the assumed boundedness and surjectivity of h). Observe that $h\alpha h = h$, and $h\beta h = h$. In fact, α and β are adjoint to h. Indeed, it is easy to see that $y \to a$

$$hx \leqslant y \quad \Leftrightarrow \quad x \leqslant \alpha y,$$

$$y \leqslant hx \quad \Leftrightarrow \quad \beta y \leqslant x.$$

For each $y \in L$, let $X_y := X \cap h^{-1}\{y\}$, the set of generators that lie in the inverse image of y under h. Let G be the (finite) set of pairs in $\mathbf{F} \times \mathbf{F}$ defined as follows:

$$G = \bigcup_{y \in L} \{(x, \alpha y), (\alpha y, x), (x, \beta y), (\beta y, x), (\alpha y, \beta y), (\beta y, \alpha y) : x \in X_y\}.$$

We claim that G generates $\ker h$. To prove this, we first show, by induction on term complexity, that for every $y \in L$, for every $r \in h^{-1}\{y\}$, the pairs $(r, \alpha y)$ and $(r, \beta y)$ belong to the sublattice $\langle G \rangle \leqslant \mathbf{F} \times \mathbf{F}$ generated by G.

fix $r \in F$, $y := h(r)$.

- **Case 0.** Suppose $r \in X$. Then $(r, \alpha y)$ and $(r, \beta y)$ belong to G itself, so there's nothing to prove.
- **Case 1.** Suppose $r = s \vee t$. ~~Assume~~ By (the induction hypothesis) ~~that~~ $(s, \alpha h(s))$, $(s, \beta h(s))$, $(t, \alpha h(t))$, and $(t, \beta h(t))$ belong to $\langle G \rangle$. Then $\cancel{y = h}(r) = h(s \vee t) = h(s) \vee h(t)$, so

$$h(\alpha h(s) \vee \alpha h(t)) = h\alpha h(s) \vee h\alpha h(t) = h(s) \vee h(t) = y.$$

Not needed

Likewise, $h(\beta h(s) \vee \beta h(t)) = h(s) \vee h(t) = y$. Therefore,

$$\beta y \leqslant \beta h(s) \vee \beta h(t) \leqslant \alpha h(s) \vee \alpha h(t) \leqslant \alpha y.$$

Clearly ~~Also~~, $r \leqslant \alpha y$, so $r = \alpha y \wedge (s \vee t)$. Taken together, these observations yield

$$\begin{aligned}
\begin{pmatrix} r \\ \beta y \end{pmatrix} = \begin{pmatrix} \alpha y \wedge (s \vee t) \\ \beta y \end{pmatrix} &= \begin{pmatrix} \alpha y \wedge (s \vee t) \\ \beta y \wedge (\beta h(s) \vee \beta h(t)) \end{pmatrix} \\
&= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \left[\begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \vee \begin{pmatrix} t \\ \beta h(t) \end{pmatrix} \right],
\end{aligned}$$

and each term in the last expression belongs to $\langle G \rangle$, so $(r, \beta y) \in \langle G \rangle$, as desired.

Similarly, $(r, \alpha y) \in \langle G \rangle$. Indeed, $\beta y \leqslant r$ implies $r = \beta y \vee s \vee t$, and $\beta h(s) \vee \beta h(t) \leqslant \alpha y$ implies $\alpha y = \alpha y \vee \beta h(s) \vee \beta h(t)$. Therefore,

$$\begin{aligned}
\begin{pmatrix} r \\ \alpha y \end{pmatrix} &= \begin{pmatrix} \beta y \vee s \vee t \\ \alpha y \vee \beta h(s) \vee \beta h(t) \end{pmatrix} \\
&= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \vee \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}.
\end{aligned}$$

- **Case 2.** Suppose $r = s \wedge t$. Assume $(s, \alpha h(s))$, $(s, \beta h(s))$, $(t, \alpha h(t))$, and $(t, \beta h(t))$ belong to $\langle G \rangle$. Then $h(s \wedge t) = h(r) = y$, so $h(\alpha h(s) \wedge \alpha h(t)) = y = h(\beta h(s) \wedge \beta h(t))$, so $\beta y \leqslant \beta h(s) \wedge \beta h(t) \leqslant \alpha h(s) \wedge \alpha h(t) \leqslant \alpha y$. Also, $\beta y \leqslant r \leqslant \alpha y$ so $r = \alpha y \wedge s \wedge t$ and $r = \beta y \vee (s \wedge t)$. Altogether,

symmetric to case 1. Probably omit

we have

$$\begin{pmatrix} r \\ \alpha a \end{pmatrix} = \begin{pmatrix} \beta y \vee (s \wedge t) \\ \alpha y \vee (\alpha h(s) \wedge \alpha h(t)) \end{pmatrix}$$
$$= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \left[\begin{pmatrix} s \\ \alpha h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \alpha h(t) \end{pmatrix} \right],$$

and each term in the last expression belongs to $\langle Y \rangle$, as desired. Similarly,

$$\begin{pmatrix} r \\ \beta y \end{pmatrix} = \begin{pmatrix} \alpha y \wedge s \wedge t \\ \beta y \wedge \alpha h(s) \wedge \alpha h(t) \end{pmatrix}$$
$$= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \begin{pmatrix} s \\ \alpha h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \alpha h(t) \end{pmatrix}.$$

Indicentally, in each of the last two derivations, we could have used β's instead of α's; in both cases the last meet could be replaced with

$$\begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}.$$

In each case, we end up with an expression involving terms from $\langle G \rangle$, and this proves that $(r, \alpha y)$ and $(r, \beta y)$ belong to $\langle G \rangle$, as desired. □

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Finally let $r, s \in F$ with $h(r) = h(s)$. Then

$$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} r \\ \beta h(r) \end{pmatrix} \vee \begin{pmatrix} \beta h(s) \\ s \end{pmatrix} \in \langle G \rangle.$$