

# KERNELS OF EPIMORPHISMS OF FINITELY GENERATED FREE LATTICES

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## 1. INTRODUCTION

## 2. MAIN THEOREM

Let  $X$  be a finite set and  $\mathbf{F} := \mathbf{F}(X)$  the free lattice generated by  $X$ .

**Theorem 2.1.** *Suppose  $\mathbf{L} = \langle L, \wedge, \vee \rangle$  is a finite lattice and  $h: \mathbf{F} \rightarrow \mathbf{L}$  a lattice epimorphism. If  $h$  is bounded then the kernel of  $h$  is a finitely generated sublattice of  $\mathbf{F} \times \mathbf{F}$ .*

*Proof.* Assume  $h$  is bounded. That is, the preimage of each  $y \in L$  under  $h$  is bounded. For each  $y \in L$ , let  $\alpha y = \bigvee h^{-1}\{y\}$  and  $\beta y = \bigwedge h^{-1}\{y\}$  denote the greatest and least elements of  $h^{-1}\{y\}$ , respectively (both of which exist by the boundedness assumption). Observe that  $h\alpha h = h$ , and  $h\beta h = h$ . In fact,  $\alpha$  and  $\beta$  are adjoint to  $h$ . Indeed, it is easy to see that

$$hx \leq y \iff x \leq \alpha y,$$

$$y \leq hx \iff \beta y \leq x.$$

For each  $y \in L$ , let  $X_y := X \cap h^{-1}\{y\}$ , the set of generators that lie in the inverse image of  $y$  under  $h$ . Let  $G$  be the (finite) set of pairs in  $\mathbf{F} \times \mathbf{F}$  defined as follows:

$$G = \bigcup_{y \in L} \{(x, \alpha y), (\alpha y, x), (x, \beta y), (\beta y, x), (\alpha y, \beta y), (\beta y, \alpha y) : x \in X_y\}.$$

We claim that  $G$  generates  $\ker h$ . To prove this, we first show, by induction on term complexity, that for every  $y \in L$ , for every  $r \in h^{-1}\{y\}$ , the pairs  $(r, \alpha y)$  and  $(r, \beta y)$  belong to the sublattice  $\langle G \rangle \leq \mathbf{F} \times \mathbf{F}$  generated by  $G$ .

- **Case 0.** Suppose  $r \in X$ . Then  $(r, \alpha y)$  and  $(r, \beta y)$  belong to  $G$  itself, so there's nothing to prove.
- **Case 1.** Suppose  $r = s \vee t$ . Assume (the induction hypothesis) that  $(s, \alpha h(s))$ ,  $(s, \beta h(s))$ ,  $(t, \alpha h(t))$ , and  $(t, \beta h(t))$  belong to  $\langle G \rangle$ . Then  $y = h(r) = h(s \vee t) = h(s) \vee h(t)$ , so

$$h(\alpha h(s) \vee \alpha h(t)) = h\alpha h(s) \vee h\alpha h(t) = h(s) \vee h(t) = y.$$

Likewise,  $h(\beta h(s) \vee \beta h(t)) = h(s) \vee h(t) = y$ . Therefore,

$$\beta y \leq \beta h(s) \vee \beta h(t) \leq \alpha h(s) \vee \alpha h(t) \leq \alpha y.$$

Also,  $r \leq \alpha y$ , so  $r = \alpha y \wedge (s \vee t)$ . Taken together, these observations yield

$$\begin{aligned} \begin{pmatrix} r \\ \beta y \end{pmatrix} &= \begin{pmatrix} \alpha y \wedge (s \vee t) \\ \beta y \end{pmatrix} = \begin{pmatrix} \alpha y \wedge (s \vee t) \\ \beta y \wedge (\beta h(s) \vee \beta h(t)) \end{pmatrix} \\ &= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \left[ \begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \vee \begin{pmatrix} t \\ \beta h(t) \end{pmatrix} \right], \end{aligned}$$

and each term in the last expression belongs to  $\langle G \rangle$ , so  $(r, \beta y) \in \langle G \rangle$ , as desired.

Similarly,  $(r, \alpha y) \in \langle G \rangle$ . Indeed,  $\beta y \leq r$  implies  $r = \beta y \vee s \vee t$ , and  $\beta h(s) \vee \beta h(t) \leq \alpha y$  implies  $\alpha y = \alpha y \vee \beta h(s) \vee \beta h(t)$ . Therefore,

$$\begin{aligned} \begin{pmatrix} r \\ \alpha y \end{pmatrix} &= \begin{pmatrix} \beta y \vee s \vee t \\ \alpha y \vee \beta h(s) \vee \beta h(t) \end{pmatrix} \\ &= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \vee \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}. \end{aligned}$$

- **Case 2.** Suppose  $r = s \wedge t$ . Assume  $(s, \alpha h(s))$ ,  $(s, \beta h(s))$ ,  $(t, \alpha h(t))$ , and  $(t, \beta h(t))$  belong to  $\langle G \rangle$ . Then  $h(s \wedge t) = h(r) = y$ , so  $h(\alpha h(s) \wedge \alpha h(t)) = y = h(\beta h(s) \wedge \beta h(t))$ , so  $\beta y \leq \beta h(s) \wedge \beta h(t) \leq \alpha h(s) \wedge \alpha h(t) \leq \alpha y$ . Also,  $\beta y \leq r \leq \alpha y$  so  $r = \alpha y \wedge s \wedge t$  and  $r = \beta y \vee (s \wedge t)$ . Altogether, we have

$$\begin{aligned} \begin{pmatrix} r \\ \alpha y \end{pmatrix} &= \begin{pmatrix} \beta y \vee (s \wedge t) \\ \alpha y \vee (\alpha h(s) \wedge \alpha h(t)) \end{pmatrix} \\ &= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \left[ \begin{pmatrix} s \\ \alpha h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \alpha h(t) \end{pmatrix} \right], \end{aligned}$$

and each term in the last expression belongs to  $\langle Y \rangle$ , as desired. Similarly,

$$\begin{aligned} \begin{pmatrix} r \\ \beta y \end{pmatrix} &= \begin{pmatrix} \alpha y \wedge s \wedge t \\ \beta y \wedge \alpha h(s) \wedge \alpha h(t) \end{pmatrix} \\ &= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \begin{pmatrix} s \\ \alpha h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \alpha h(t) \end{pmatrix}. \end{aligned}$$

Note that, in both of the derivations above, we could have used  $\beta$ 's instead of  $\alpha$ 's; that is,

$$\begin{aligned} \begin{pmatrix} r \\ \alpha y \end{pmatrix} &= \begin{pmatrix} \beta y \vee (s \wedge t) \\ \alpha y \vee (\beta h(s) \wedge \beta h(t)) \end{pmatrix} \\ &= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \left[ \begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \beta h(t) \end{pmatrix} \right], \end{aligned}$$

and

$$\begin{aligned} \begin{pmatrix} r \\ \beta y \end{pmatrix} &= \begin{pmatrix} \alpha y \wedge s \wedge t \\ \beta y \wedge \beta h(s) \wedge \beta h(t) \end{pmatrix} \\ &= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}. \end{aligned}$$

In each case, we end up with an expression involving terms from  $\langle G \rangle$ , and this proves that  $(r, \alpha y)$  and  $(r, \beta y)$  belong to  $\langle G \rangle$ , as desired.  $\square$

We conjecture the converse of Theorem 2.1.

**Conjecture 1.** Suppose  $\mathbf{L} = \langle L, \wedge, \vee \rangle$  is a finite lattice and  $h: \mathbf{F} \rightarrow \mathbf{L}$  a lattice epimorphism. If the kernel of  $h$  is a finitely generated sublattice of  $\mathbf{F} \times \mathbf{F}$ , then  $h$  is bounded.

We know how to prove this conjecture if we assume that, whenever  $h$  is not upper (lower) bounded, then there is a class of  $\ker h$  that contains an infinite ascending (descending) chain as well as a meet (join) prime element of  $\mathbf{F}$ . (See Appendix A for the proof).

J.B. Nation brought the next result to our attention. It gives an example in which the unbounded classes of  $\ker h$  do not contain generators of  $\mathbf{F}$ .

**Proposition 2.2.** Let  $\mathbf{F} = \mathbf{F}(x, y, z)$ , and let  $\mathbf{L} = \mathbf{F}_{\mathbf{M}_3}(0, 1, 2)$  (see Figure 1). Let  $h: \mathbf{F} \rightarrow \mathbf{L}$  be the epimorphism induced by  $x \mapsto 0$ ,  $y \mapsto 1$ ,  $z \mapsto 2$ . Then  $\ker h$  is not finitely generated.

To prove this we will need a technical lemma about the behavior of certain sequences of elements of  $\mathbf{F}$ .

**Lemma 2.3.** For each  $s \in \{x, y, z, m\}$ , define the sequence  $\{s_i : i < \omega\}$  of elements of  $\mathbf{F}$  as follows:  $x_0 = x$ ,  $y_0 = y$ ,  $z_0 = z$ , and for  $i \geq 0$ ,

$$\begin{aligned} x_{i+1} &= x \vee (y_i \wedge z_i), & y_{i+1} &= y \vee (x_i \wedge z_i), & z_{i+1} &= z \vee (x_i \wedge y_i), \\ m_i &= (x_i \wedge y_i) \vee (x_i \wedge z_i) \vee (y_i \wedge z_i). \end{aligned}$$

If  $\{s_i : i < \omega\}$  is any one of the four sequences just defined, then for every  $i \geq 1$ , we have  $s_{i+1} > s_i$  and  $h(s_i) = h(s_1)$ .

*Proof.* First observe that  $h(x_1) = h(x \vee (y \wedge z)) = 0 \vee (1 \wedge 2)$ . We begin by proving  $h(x_2) = h(x_1)$ . By definition, we have  $h(x_2) = h(x \vee (y_1 \wedge z_1)) = h(x) \vee [h(y_1) \wedge h(z_1)]$ , and  $h(y_1) = h(y \vee (x \wedge z)) = h(y) \vee [h(x) \wedge h(z)] = 1 \vee (0 \wedge 2)$ . Similarly,  $h(z_1) = 2 \vee (0 \wedge 1)$ . Therefore,

$$h(x_2) = 0 \vee \{[1 \vee (0 \wedge 2)] \wedge [2 \vee (0 \wedge 1)]\}. \quad (2.1)$$

Recall, the modular law:  $x \leq b$  implies  $x \vee (a \wedge b) = (x \vee a) \wedge b$ . Applying this law with  $a = 1$  and  $x = 0 \wedge 2 \leq 2 \vee (0 \wedge 1) = b$ , we have

$$(0 \wedge 2) \vee \{1 \wedge [2 \vee (0 \wedge 1)]\} = [1 \vee (0 \wedge 2)] \wedge [2 \vee (0 \wedge 1)],$$

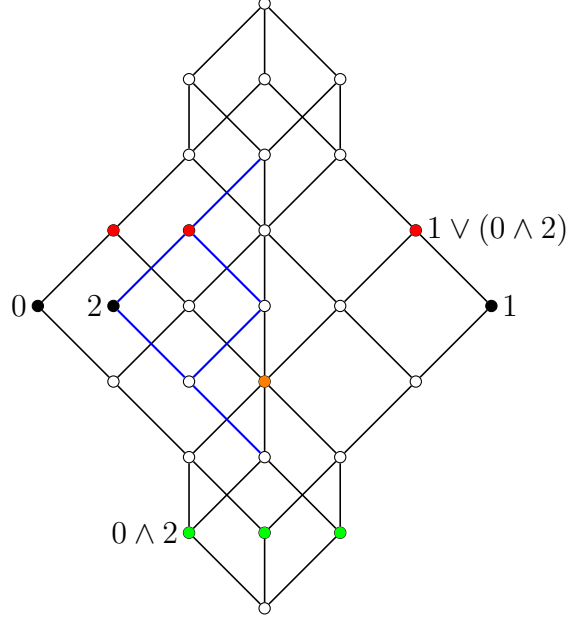


FIGURE 1. The free lattice over  $M_3$  generated by  $\{0, 1, 2\}$ . Green dots identify elements  $0 \wedge 1$ ,  $0 \wedge 2$ , and  $1 \wedge 2$ ; red dots identify  $0 \vee (1 \wedge 2)$ ,  $1 \vee (0 \wedge 2)$ , and  $2 \vee (0 \wedge 1)$ .

which is the right joinand in 2.1. Therefore,

$$h(x_2) = 0 \vee (0 \wedge 2) \vee \{1 \wedge [2 \vee (0 \wedge 1)]\} = 0 \vee \{1 \wedge [2 \vee (0 \wedge 1)]\}.$$

Applying the modular law once more to  $1 \wedge [2 \vee (0 \wedge 1)]$ , with  $a = 2$  and  $x = 0 \wedge 1 \leq 1 = b$ , we have

$$1 \wedge [2 \vee (0 \wedge 1)] = (0 \wedge 1) \vee (1 \wedge 2).$$

Therefore,

$$h(x_2) = 0 \vee \{1 \wedge [2 \vee (0 \wedge 1)]\} = 0 \vee (0 \wedge 1) \vee (1 \wedge 2) = 0 \vee (1 \wedge 2),$$

as desired. Of course,  $h(y_2) = h(y_1)$  and  $h(z_2) = h(z_1)$  can be checked similarly.

With the base cases established, we proceed with the induction. Fix  $n \geq 1$  and assume  $h(x_n) = h(x_1)$ ,  $h(y_n) = h(y_1)$ , and  $h(z_n) = h(z_1)$ . Then,

$$h(x_{n+1}) = h(x \vee (y_n \wedge z_n)) = h(x) \vee (h(y_n) \wedge h(z_n)) = h(x) \vee (h(y_1) \wedge h(z_1)).$$

As observed above, this is equal to  $h(x_2)$ , which in turn is  $h(x_1)$ , as desired. By the same argument,  $h(y_{n+1}) = h(y_1)$  and  $h(z_{n+1}) = h(z_1)$ . This proves that for all  $n \geq 1$ , we have  $h(s_n) = h(s_1)$ , when  $\{s_n\}$  is  $\{x_n\}$  or  $\{y_n\}$  or  $\{z_n\}$ .

Finally, consider  $\{m_n\}$ . For all  $n \geq 1$ , we have

$$\begin{aligned} h(m_n) &= [h(x_n) \wedge h(y_n)] \vee [h(x_n) \wedge h(z_n)] \vee [h(y_n) \wedge h(z_n)] \\ &= [h(x_1) \wedge h(y_1)] \vee [h(x_1) \wedge h(z_1)] \vee [h(y_1) \wedge h(z_1)] \\ &= h(m_1). \end{aligned}$$

(In fact, in this case we can show that  $h(m_1) = h(m_0)$ , but this is unnecessary.)

It remains to show that  $s_{i+1} > s_i$  for all  $i < \omega$  (for each  $s \in \{x, y, z, m\}$ ). For this we will need the following.

**Claim 2.4.** For all  $i \geq 0$ ,  $x \not\leq y_i$  and  $x \not\leq z_i$ .

*Proof.* When  $i = 0$ , the claim is that  $x \not\leq y$  and  $x \not\leq z$ , which is clear. Fix  $k \geq 0$  and suppose  $x \not\leq y_k$  and  $x \not\leq z_k$ . We show  $x \not\leq y_{k+1}$  and  $x \not\leq z_{k+1}$ . Assume the contrary, say,  $x \leq y_{k+1} = y \vee (x_k \wedge z_k)$ . Since  $x$  is a generator, it is join prime. Therefore, as  $x \not\leq y$ , we must have  $x \leq x_k \wedge z_k$ . But this is impossible since  $x \not\leq z_k$ , by the induction hypothesis. Thus, Claim 2.4 is proved.  $\square$

Observe that the claim just proved also yields the following:  $\forall m, n, x_m \not\leq y_n$ . We will now prove  $\forall i \geq 0$  that all of the following strict inequalities hold:

$$\begin{aligned} x_i < x_{i+1}, \quad y_i < y_{i+1}, \quad z_i < z_{i+1} \\ x_i \wedge y_i < x_{i+1} \wedge y_{i+1}, \quad x_i \wedge z_i < x_{i+1} \wedge z_{i+1}, \quad y_i \wedge z_i < y_{i+1} \wedge z_{i+1}. \end{aligned} \quad (2.2)$$

Clearly  $x_1 = x \vee (y \wedge z) > x = x_0$ , by Theorem C.8. Similarly,  $y_1 > y_0$  and  $z_1 > z_0$ . Also,

$$x_1 \wedge y_1 = [x \vee (y \wedge z)] \wedge [y \vee (x \wedge z)] > x \wedge y.$$

For, suppose on the contrary that  $[x \vee (y \wedge z)] \wedge [y \vee (x \wedge z)] \leq x \wedge y$ . Then, in particular,  $[x \vee (y \wedge z)] \wedge [y \vee (x \wedge z)] \leq x$ , but  $x$  is meet prime, so either  $[x \vee (y \wedge z)] \leq x$  or  $[y \vee (x \wedge z)] \leq x$ . Both of these possibilities are easily ruled out. The arguments establishing the remaining inequalities in (2.2) in case  $i = 0$  are similar.

Now, fix  $n \geq 0$  and suppose the strict inequalities in (2.2) hold when  $i = n$ . We show they also hold when  $i = n + 1$ . It suffices to prove just two of these, namely,  $x_{n+1} > x_n$  and  $x_{n+1} \wedge y_{n+1} > x_n \wedge y_n$ .

Recall  $x_{n+1} = x \vee (y_n \wedge z_n)$  and  $x_n = x \vee (y_{n-1} \wedge z_{n-1})$ , and the induction hypothesis implies that  $y_n \wedge z_n > y_{n-1} \wedge z_{n-1}$ . Therefore,  $x_{n+1} \geq x_n$ , so we must show  $x_{n+1} \neq x_n$ . Suppose on the contrary that

$$x_{n+1} = x \vee (y_n \wedge z_n) = x \vee (y_{n-1} \wedge z_{n-1}) = x_n. \quad (2.3)$$

Then  $y_n \wedge z_n \leq x \vee (y_{n-1} \wedge z_{n-1})$ . Recall Whitman's Theorem (C.7) asserting that  $\mathbf{F}(X)$  satisfies condition (W+) repeated here for easy reference.

If  $v = v_1 \wedge \cdots \wedge v_r \wedge x_1 \wedge \cdots \wedge x_n \leq u_1 \vee \cdots \vee u_s \vee y_1 \vee \cdots \vee y_m = u$ , where  $x_i, y_j \in X$ , then either  $x_i = y_j$  for some  $i$  and  $j$ , or  $v_i \leq u$  for some  $i$ , or  $v \leq u_j$  for some  $j$ .

This condition and (2.3) together imply that one of the following must hold:

- (1)  $y_n \leq x \vee (y_{n-1} \wedge z_{n-1})$ ;
- (2)  $z_n \leq x \vee (y_{n-1} \wedge z_{n-1})$ ;
- (3)  $y_n \wedge z_n \leq y_{n-1} \wedge z_{n-1}$ .

The first two inequalities are simply  $y_n \leq x_n$  and  $z_n \leq x_n$ , which are both ruled out by the remark immediately following the proof of Claim 2.4, and

$y_n \wedge z_n \leq y_{n-1} \wedge z_{n-1}$  is ruled out by the induction hypothesis. By reaching this contradiction we have established that  $x_{n+1} > x_n$ .

It remains to prove  $x_{n+1} \wedge y_{n+1} > x_n \wedge y_n$  under the assumption that all of the strict inequalities in (2.2) hold when  $i = n$ . Since we have already established that  $x_{n+1} > x_n$  and (by an identical argument) that  $y_{n+1} > y_n$ , it's clear that  $x_{n+1} \wedge y_{n+1} \geq x_n \wedge y_n$ , so we just have to rule out equality. Suppose on the contrary that

$$\begin{aligned} x_n \wedge y_n &:= [x \vee (y_{n-1} \wedge z_{n-1})] \wedge [y \vee (x_{n-1} \wedge z_{n-1})] \\ &= [x \vee (y_n \wedge z_n)] \wedge [y \vee (x_n \wedge z_n)] =: x_{n+1} \wedge y_{n+1}. \end{aligned}$$

Then we have both  $x_{n+1} \wedge y_{n+1} \leq x_n$  and  $x_{n+1} \wedge y_{n+1} \leq y_n$ . Equivalently,

$$\begin{aligned} [x \vee (y_n \wedge z_n)] \wedge [y \vee (x_n \wedge z_n)] &\leq x \vee (y_{n-1} \wedge z_{n-1}) \\ [x \vee (y_n \wedge z_n)] \wedge [y \vee (x_n \wedge z_n)] &\leq y \vee (x_{n-1} \wedge z_{n-1}). \end{aligned} \quad (2.4)$$

Again, by (W+), for the first of these inequalities to hold, we must have one of the following:

- (1)  $x \vee (y_n \wedge z_n) \leq x \vee (y_{n-1} \wedge z_{n-1})$ ;
- (2)  $y \vee (x_n \wedge z_n) \leq y \vee (x_{n-1} \wedge z_{n-1})$ ;
- (3)  $[x \vee (y_n \wedge z_n)] \wedge [y \vee (x_n \wedge z_n)] \leq y_{n-1} \wedge z_{n-1}$ .

The first of these is equivalent to  $x_{n+1} \leq x_n$ , and the second to  $y_{n+1} \leq y_n$ ; we have already ruled out both of these cases. Thus the only remaining possibility is the third, which can be restated as  $x_{n+1} \wedge y_{n+1} \leq y_{n-1} \wedge z_{n-1}$ . Let's assume this holds.

In order for the second of the inequalities in (2.4) to hold, according to (W+), one of the following must also be true:

- (1)  $x \vee (y_n \wedge z_n) \leq y \vee (x_{n-1} \wedge z_{n-1})$ ;
- (2)  $y \vee (x_n \wedge z_n) \leq y \vee (x_{n-1} \wedge z_{n-1})$ ;
- (3)  $[x \vee (y_n \wedge z_n)] \wedge [y \vee (x_n \wedge z_n)] \leq x_{n-1} \wedge z_{n-1}$ .

Again, the first two— $x_{n+1} \leq y_n$  and  $y_{n+1} \leq y_n$ —have already been ruled out. So the only possibility is  $x_{n+1} \wedge y_{n+1} \leq x_{n-1} \wedge z_{n-1}$ . Let us assume we are in this case.

Taken all together, if  $x_n \wedge y_n$  were equal to  $x_{n+1} \wedge y_{n+1}$ , then we would have,

$$x_n \wedge y_n = x_{n+1} \wedge y_{n+1} \leq x_{n-1} \wedge y_{n-1} \wedge z_{n-1}.$$

In particular  $x_n \wedge y_n \leq x_{n-1} \wedge y_{n-1}$ . But this contradicts the inductive hypothesis.

To complete the proof, we must show that  $m_{k+1} > m_k$  for all  $k \geq 0$ .

$$m_0 = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z).$$

$$m_1 = (x_1 \wedge y_1) \vee (x_1 \wedge z_1) \vee (y_1 \wedge z_1).$$

Suppose  $m_1 \leq m_0$ . Then, in particular, we would have

$$x_1 \wedge y_1 \leq (x \wedge y) \vee (x \wedge z) \vee (y \wedge z).$$

Then, by Whitman's Theorem, one of the following must hold:

- (1)  $x_1 \leq (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ ,
- (2)  $y_1 \leq (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ ,
- (3)  $x_1 \wedge y_1 \leq x \wedge y$ ,
- (4)  $x_1 \wedge y_1 \leq x \wedge z$ ,
- (5)  $x_1 \wedge y_1 \leq y \wedge z$ .

Expanding the first of these, we would have

$$x_1 = x \vee (y \wedge z) \leq (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = m_0$$

in this case. This is easily ruled out since  $x$  is join prime, so it cannot lie below  $m_0$ . Similarly, the second case is ruled out because  $y$  does not lie below  $m_0$ . The third case is ruled out by what we showed earlier in the proof. The fourth case is  $x_1 \wedge y_1 \leq x \wedge z$ , which implies  $x_1 \wedge y_1 \leq x$ . Since  $x$  is meet prime, this puts either  $x_1$  or  $y_1$  below  $x$ , which, as we noted above, is impossible. The fifth case is  $x_1 \wedge y_1 \leq y \wedge z$ , which implies  $x_1 \wedge y_1 \leq y$  and, again, this is impossible since  $y$  is meet prime. Thus we have proved  $m_1 > m_0$ .

Finally, fix  $k > 0$  and assume  $m_k > m_{k-1}$ . We show  $m_{k+1} > m_k$ . It's clear that  $m_{k+1} \geq m_k$ , so as usual we must rule out equality. Suppose  $m_{k+1} \leq m_k$ ; that is,

$$(x_{k+1} \wedge y_{k+1}) \vee (x_{k+1} \wedge z_{k+1}) \vee (y_{k+1} \wedge z_{k+1}) \leq (x_k \wedge y_k) \vee (x_k \wedge z_k) \vee (y_k \wedge z_k).$$

Then, in particular,  $x_{k+1} \wedge y_{k+1} \leq (x_k \wedge y_k) \vee (x_k \wedge z_k) \vee (y_k \wedge z_k)$ . Whitman's Theorem then implies that one of the following must hold:

- (1)  $x_{k+1} \leq (x_k \wedge y_k) \vee (x_k \wedge z_k) \vee (y_k \wedge z_k)$ ;
- (2)  $y_{k+1} \leq (x_k \wedge y_k) \vee (x_k \wedge z_k) \vee (y_k \wedge z_k)$ ;
- (3)  $x_{k+1} \wedge y_{k+1} \leq x_k \wedge y_k$ ;
- (4)  $x_{k+1} \wedge y_{k+1} \leq x_k \wedge z_k$ ;
- (5)  $x_{k+1} \wedge y_{k+1} \leq y_k \wedge z_k$ .

We rule out the first three cases by the same argument used above. Suppose (4) holds. Then, in particular  $x_{k+1} \wedge y_{k+1} \leq z \vee (x_{k-1} \wedge y_{k-1})$ . By (W+) again, one of the following must hold:

- (1)  $x_{k+1} \wedge y_{k+1} \leq x_{k-1} \wedge y_{k-1}$ ;
- (2)  $x_{k+1} \leq z \vee (x_{k-1} \wedge y_{k-1})$ ;
- (3)  $y_{k+1} \leq z \vee (x_{k-1} \wedge y_{k-1})$ ;

The first two items are impossible since we already proved that for all  $n \geq 0$ ,  $x_{n+1} \wedge y_{n+1} > x_n \wedge y_n$  and  $x_{n+1} \not\leq z_n$  (cf. the observation immediately after Claim 2.4). Clearly the same argument can be used to rule out Case (5). This completes the proof of Lemma 2.3.  $\square$

Notice that the example above involves an infinite ascending chain  $\{x_n\} \subseteq \mathbf{F}$  such that  $0 = h(x_0) \neq h(x_1) = h(x_2) = \dots$ ; thus, for  $n \geq 1$ , all  $x_n$  belong to the same (unbounded) class of  $\ker h$ , and this class contains  $x_1 = x \vee (y \wedge z)$ . However,  $x_1$  is not meet prime, since  $y \not\leq x \vee (y \wedge z)$  and  $z \not\leq x \vee (y \wedge z)$ , yet  $y \wedge z \leq x \vee (y \wedge z)$ , so we cannot use our previous proof technique.

As above, let  $x_1 = x \vee (y \wedge z)$  and define  $b = 0 \vee (1 \wedge 2)$ . In particular,  $h(x_1) = b$ . To use the same inductive step that we used in our previous proofs, we would need to know that  $p_1 \wedge p_2 \leq x_1$  implies either  $p_1 \leq x_1$  or  $p_2 \leq x_1$ . But this is not the case here since we could have, for example,  $p_1 = y$  and  $p_2 = z$ . In fact, by Lemma C.3, the only meet prime elements of  $\mathbf{F}(X)$  have the form  $\bigvee S$  for  $S \subseteq X$ .

Suppose  $K \subseteq \mathbf{F} \times \mathbf{F}$  is a finite set. We wish to prove  $\langle K \rangle \neq \ker h$ . First, let's try to show there exists  $N < \omega$  such that for all  $(p, q) \in \mathbf{F} \times \mathbf{F}$  with  $h(p) = b = h(q)$ , the following implication holds:

$$\text{if } (p, q) \in \langle K \rangle \text{ and } p \leq x_1, \text{ then } q \leq \alpha_N(b). \quad (2.5)$$

Since  $\alpha_n(b)$  is an infinite ascending chain, we can certainly find  $N < \omega$  such that (2.5) holds for all of the (finitely many) pairs in  $K$ .

Now suppose  $(p, q) = (p_1, q_1) \wedge (p_2, q_2)$  and  $h(p) = b = h(q)$ . Then  $p_1, p_2, q_1, q_2$  must all belong to  $h^{-1}\{b\}$  as well. Indeed,  $b = h(p) = h(p_1 \wedge p_2) = h(p_1) \wedge h(p_2)$ , and  $b = 0 \vee (1 \wedge 2)$  is meet irreducible, so  $h(p_1) = b = h(p_2)$ . Similarly for  $q_1$  and  $q_2$ .

Assume  $(p, q) \in \langle K \rangle$  and  $p = p_1 \wedge p_2 \leq x_1$ . We must show  $q_1 \wedge q_2 \leq \alpha_N(b)$ . Assume the induction hypothesis that  $(p_i, q_i)$  satisfies (2.5) for both  $i = 1, 2$ .

If we knew that either  $p_1 \leq x_1$  or  $p_2 \leq x_1$ , then we would have that either  $q_1 \leq \alpha_N(b)$  or  $q_2 \leq \alpha_N(b)$ , by the induction hypothesis; in either case, we could conclude that  $q_1 \wedge q_2 \leq \alpha_N(b)$ , as desired. Unfortunately, we do not know that either  $p_1 \leq x_1$  or  $p_2 \leq x_1$ , since  $x_1$  is not meet prime.

To say that  $x_1$  is meet prime means that *for all*  $p_1$  and  $p_2$  with  $p_1 \wedge p_2 \leq x_1$ , we have either  $p_1 \leq x_1$  or  $p_2 \leq x_1$ . But this is actually more than we need. For our purposes, we only require that the condition

$$p_1 \wedge p_2 \leq x_1 \implies p_1 \leq x_1 \text{ or } p_2 \leq x_1, \quad (2.6)$$

is satisfied for some special elements  $p_1$  and  $p_2$ , namely...?

Maybe we should try to exploit a  $D^d$ -cycle here, involving

$$p_1 \wedge (p_1 \rightarrow x_1) \leq x_1$$

We know that the maximum element (denoted here by  $p_1 \rightarrow x_1$ ), among those that meet with  $p_1$  below  $x_1$ , exists by the existence of a  $D^d$ -cycle.



## APPENDIX A. PROOF OF CONJECTURE UNDER SPECIAL ASSUMPTIONS

As usual, let  $X$  be a finite set and let  $\mathbf{F} := \mathbf{F}(X)$  be the free lattice generated by  $X$ .

**Proposition A.1.** *Suppose  $\mathbf{L} = \langle L, \wedge, \vee \rangle$  is a finite lattice and  $h: \mathbf{F} \rightarrow \mathbf{L}$  a lattice epimorphism. Suppose also that there is a class of  $\ker h$  containing an infinite descending chain as well as a join prime element of  $\mathbf{F}$ . Then the kernel of  $h$  is not a finitely generated sublattice of  $\mathbf{F} \times \mathbf{F}$ .*

*Proof.* Let  $y_0 \in L$  and suppose  $x_0 \in h^{-1}\{y_0\}$  is a join prime element of  $\mathbf{F}$ . Suppose also that the class  $h^{-1}\{y_0\}$  contains an infinite descending chain,  $\beta_0(y_0) > \beta_1(y_0) > \dots$ . Let  $K$  be a finite subset of  $\ker h$ , say,  $K = \{(p_1, q_1), \dots, (p_m, q_m)\} \subseteq \ker h$ . We prove  $\langle K \rangle \neq \ker h$ . Since  $K$  is an arbitrary finite subset of  $\ker h$ , this will prove that  $\ker h$  is not finitely generated.

**Claim A.2.** There exists  $N < \omega$  such that for all  $(p_i, q_i)$  in  $K$ , if  $p_i \geq x_0$ , then  $q_i \geq \beta_N(y_0)$ .

*Proof.* Fix  $i$  and  $(p_i, q_i) \in K$  (so,  $h(p_i) = h(q_i)$ ). Define  $N_i$  according to which of the following cases holds:

- (1) If  $p_i \not\geq x_0$ , let  $N_i = 0$ .
- (2) If  $p_i \geq x_0$ , then  $x_0 = x_0 \wedge p_i$ , so  $y_0 = h(x_0) = h(x_0) \wedge h(p_i) \leq h(p_i)$ , so  $y_0 \leq h(q_i)$ . Also,  $h(x_0 \wedge q_i) = h(x_0) \wedge h(q_i) = y_0$ , so  $x_0 \wedge q_i \in h^{-1}\{y_0\}$ . Therefore (since  $\{\beta_i(y_0)\}$  is an infinite descending chain), there exists  $n_i > 0$  such that  $x_0 \wedge q_i \geq \beta_{n_i}(y_0)$ . Let  $N_i = n_i$  in this case (so  $q_i \geq \beta_{N_i}(y_0)$ ).

After defining  $N_i$  this way for each pair  $(p_i, q_i) \in K$ , let  $N := \max_i N_i$ . Then the desired implication holds for all  $1 \leq i \leq m$ , that is,

$$p_i \geq x_0 \implies q_i \geq \beta_N(y_0), \quad (\text{A.1})$$

so Claim A.2 is proved.  $\square$

**Claim A.3.** There exists  $N < \omega$  such that, for all  $(p, q) \in \langle K \rangle$ ,

$$p \geq x_0 \implies q \geq \beta_N(y_0). \quad (\text{A.2})$$

*Proof.* Define  $N$  as in the proof of Claim A.2, so that for all  $(p_i, q_i) \in K$  the implication (A.1) holds. Fix  $(p, q) \in \langle K \rangle$ . We prove (A.2) by induction on the complexity of  $(p, q)$ . If  $(p, q) \in K$ , then there's nothing to prove. We split the induction step into two cases.

- (1) Assume  $(p, q) = (p_1, q_1) \wedge (p_2, q_2)$ , where  $p_i, q_i$  ( $i = 1, 2$ ) satisfy (A.2). Assume  $p \geq x_0$ . Then  $p = p_1 \wedge p_2 \geq x_0$ , so  $p_1 \geq x_0$  and  $p_2 \geq x_0$ , so (by the induction hypothesis)  $q_1 \geq \beta_N(y_0)$  and  $q_2 \geq \beta_N(y_0)$ . Therefore,  $q = q_1 \wedge q_2 \geq \beta_N(y_0)$ , as desired.
- (2) Assume  $(p, q) = (p_1, q_1) \vee (p_2, q_2)$ , where  $p_i, q_i$  ( $i = 1, 2$ ) satisfy (A.2). Assume  $p \geq x_0$ . Then  $p = p_1 \vee p_2 \geq x_0$ . Since  $x_0$  is join prime, either  $p_1 \geq x_0$  or  $p_2 \geq x_0$ . Assume (wlog)  $p_1 \geq x_0$ . Then, by the induction hypothesis,  $q_1 \geq \beta_N(y_0)$ . Therefore,  $q = q_1 \vee q_2 \geq q_1 \geq \beta_N(y_0)$ , as desired.

This completes the proof of Claim A.3.  $\square$

We can now see that  $K$  does not generate  $\ker h$ . Indeed, if  $N$  is as in the proof of Claim A.3, then  $\beta_N(y_0) > \beta_{N+1}(y_0)$ , so the pair  $(p, q) = (x_0, \beta_{N+1}(y_0))$  does not satisfy condition A.2, so does not belong to  $\langle K \rangle$ . Yet,  $(x_0, \beta_{N+1}(y_0)) \in \ker h$ . Thus,  $K$  does not generate  $\ker h$ .  $\square$

## APPENDIX B. EXAMPLES

Let  $\mathbf{M}_3 = \langle \{0, a, b, c, 1\}, \wedge, \vee \rangle$ , where  $a \wedge b = a \wedge c = b \wedge c = 0$  and  $a \vee b = a \vee c = b \vee c = 1$ . Let  $\mathbf{F} := \mathbf{F}(x, y, z)$  denote the free lattice generated by  $\{x, y, z\}$ .

**Proposition B.1.** *Let  $h: \mathbf{F} \twoheadrightarrow \mathbf{M}_3$  be the epimorphism that acts on the generators as follows:  $x \mapsto a$ ,  $y \mapsto b$ ,  $z \mapsto c$ . Then  $\ker h$  is not finitely generated.*

*Proof.* Let  $K := \ker h$ , and for  $u \in \{x, y, z\}$  let  $C_u := u/K := \{v \in F : h(v) = h(u)\}$ . Define sequences of elements in these classes as follows: let

$$\begin{aligned} x_0 &:= x, & y_0 &:= y, & z_0 &:= z, & \text{and for } i < \omega, \\ x_{i+1} &:= x \vee (y_i \wedge z_i), & y_{i+1} &:= y \vee (x_i \wedge z_i), & z_{i+1} &:= z \vee (x_i \wedge y_i), \\ m_i &:= (x_i \wedge y_i) \vee (x_i \wedge z_i) \vee (y_i \wedge z_i). \end{aligned}$$

Summarizing these observations,

$$\begin{aligned} m_0 &= (x \wedge y) \vee (x \wedge z) \vee (y \wedge z), \\ x_1 &= x \vee (y \wedge z), & y_1 &= y \vee (x \wedge z), & z_1 &= z \vee (x \wedge y), \\ m_1 &= (x_1 \wedge y_1) \vee (x_1 \wedge z_1) \vee (y_1 \wedge z_1), \\ x_2 &= x \vee \{[y \vee (x \wedge z)] \wedge [z \vee (x \wedge y)]\}, \\ y_2 &= y \vee \{[x \vee (y \wedge z)] \wedge [z \vee (x \wedge y)]\}, \\ z_3 &= z \vee (x_2 \wedge y_2) \\ &\vdots \end{aligned}$$

Let  $X$  be a finite subset of  $K$ . We will prove there exists  $(p, q) \in K \setminus \langle X \rangle$ . Fix  $u \in \{x, y, z, m\}$  and let  $\{u_i\}$  be the corresponding sequence defined above. Since  $X$  is finite, Lemma B.3 implies that there exists  $M \in \mathbb{N}$  such that for every  $(p, q) \in X$  with  $p, q \in C_u$ , we have  $p, q \leq u_M$ .

**Subclaim 1.** For  $(p, q) \in \langle X \rangle$  and  $u \in \{x, y, z\}$ , the following implication holds:

$$q \leq u \implies p \leq u_M. \quad (\text{B.1})$$

We prove the subclaim by induction on the complexity of terms. Fix  $(p, q) \in \langle X \rangle$ . Then  $p, q \in C_u$  for some  $u \in \{x, y, z\}$ .

- **Case 0.** Suppose  $(p, q) \in X$ . Then by definition of  $M$  we have  $p, q \leq u_M$ .
- **Case 1.** Suppose  $(p, q) = (p_1, q_1) \wedge (p_2, q_2)$ , where  $(p_i, q_i)$  satisfies (B.1) for  $i = 1, 2$ . If  $q = q_1 \wedge q_2 \leq u$ , then, since generators in the free lattice are meet-prime (see Theorem C.8 below), we have  $q_1 \leq u$  or  $q_2 \leq u$ . Assume  $q_1 \leq u$ . Then, by the induction hypothesis,  $p_1 \leq u_M$ . Therefore,  $p = p_1 \wedge p_2 \leq u_M$ , as desired.

- **Case 2.** Suppose  $(p, q) = (p_1, q_1) \vee (p_2, q_2)$ , where  $(p_i, q_i)$  satisfies (B.1) for  $i = 1, 2$ . If  $q = q_1 \vee q_2 \leq u$ , then  $q_i \leq u$  for  $i = 1, 2$ . It now follows from the induction hypothesis that  $p_i \leq u_M$  for  $i = 1, 2$ , so  $p = p_1 \vee p_2 \leq u_M$ , as desired.

This completes the proof of Subclaim 1. It now follows from Lemma B.2 that  $(x, x_{M+1}) \in K \setminus \langle X \rangle$ , so proposition is proved.  $\square$

**Lemma B.2.** *For each  $u \in \{x, y, z, m\}$ , the sequence  $\{u_i\}$  is a strictly ascending chain; that is,  $u_0 < u_1 < u_2 < \dots$ .*

*Proof.* We split the proof up into cases: either  $u \in \{x, y, z\}$ , or  $u = m$ .

- **Case 1.**  $u \in \{x, y, z\}$ .

For simplicity, assume  $u = x$  for the remainder of the proof of this case. (Of course, the same argument applies to the case when  $u$  is  $y$  or  $z$ .) Fix  $n < \omega$ . We prove  $x_n < x_{n+1}$ .

**Subclaim 2.** For all  $n < \omega$ ,

- (1)  $x_n \in C_x$ ,
- (2)  $x_n \not\geq y$ , and  $x_n \not\geq z$ .

*Proof of Subclaim 2.* The first item is obvious; for the second, if  $x_n \geq y$ , then  $x_n \wedge y = y$ , and then  $0 = h(x_n \wedge y) = h(y) = b$ . A similar contradiction is reached if we assume  $x_n \geq z$ , so the subclaim is proved.

Recall,  $x_{n+1} = x_n \vee (y_n \wedge z_n)$ , so  $x_{n+1} > x_n$  holds as long as  $x_n \not\geq y_n \wedge z_n$ . So, by way of contradiction, suppose

$$x_n \geq y_n \wedge z_n. \quad (\text{B.2})$$

Now,  $y_n = y \vee (x \wedge z) \vee \dots$ , so clearly  $y_n \geq y$ . Similarly,  $z_n \geq z$ . This, together with (B.2), implies  $x_n \geq y_n \wedge z_n \geq y \wedge z$ . But then Theorem C.8 below implies that either  $x_n \geq y$  or  $x_n \geq z$ , which contradicts Subclaim 2.

- **Case 2.**  $u = m$ .

We first prove that  $m_0 = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$  is strictly below  $m_1 = (x_1 \wedge y_1) \vee (x_1 \wedge z_1) \vee (y_1 \wedge z_1)$ . By symmetry, it suffices to show  $x \wedge y < x_1 \wedge y_1$ ; that is,  $x \wedge y < [x \vee (y \wedge z)] \wedge [y \vee (x \wedge z)]$ .

Clearly  $x \wedge y \leq [x \vee (y \wedge z)] \wedge [y \vee (x \wedge z)]$ . Suppose  $x \wedge y = [x \vee (y \wedge z)] \wedge [y \vee (x \wedge z)]$ . Then  $[x \vee (y \wedge z)] \wedge [y \vee (x \wedge z)] \leq x$ . By Theorem C.8, the latter holds iff  $x \vee (y \wedge z) \leq x$  or  $y \vee (x \wedge z) \leq x$ . The first of these inequalities is clearly false, so it must be the case that  $y \vee (x \wedge z) \leq x$ . But then  $y \leq x$ , which is obviously false. We conclude that  $x \wedge y < [x \vee (y \wedge z)] \wedge [y \vee (x \wedge z)]$ . This proves  $m_0 < m_1$ .

Now fix  $n < \omega$  and assume  $m_n < m_{n+1}$ . We show  $m_{n+1} < m_{n+2}$ .

$\downarrow$  begin scratch work  $\downarrow$

$$m_n := (x_n \wedge y_n) \vee (x_n \wedge z_n) \vee (y_n \wedge z_n),$$

$$m_{n+1} := (x_{n+1} \wedge y_{n+1}) \vee (x_{n+1} \wedge z_{n+1}) \vee (y_{n+1} \wedge z_{n+1}),$$

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Complete the proof  
of this case.

By the first Case above,  $u_n < u_{n+1}$ .  
 $\uparrow$  end scratch work  $\uparrow$

□

**Lemma B.3.** *For all  $u \in \{x, y, z\}$  and  $p \in C_u \cup C_0$  there exists  $n \in \mathbb{N}$  such that  $p \leq m_{u,n}$ .*

*Proof.* We prove this by induction on the complexity of  $p$ .

- **Case 0.**  $p \in \{x, y, z\}$ . Then  $u = p = m_{p,0}$ .

For the remaining cases assume  $u = x$ , without loss of generality.

- **Case 1.**  $p = p_1 \vee p_2$ .

If  $p \in C_x \cup C_0$ , then  $p_i \in C_x \cup C_0$  for  $i = 1, 2$ , and the induction hypothesis yields  $i$  and  $j$  for which  $p_1 \leq x_i$  and  $p_2 \leq x_j$ . Letting  $n = \max\{i, j\}$ , we have  $p_1, p_2 \leq x_n$ , from which  $p = p_1 \vee p_2 \leq x_n$ , as desired.

- **Case 2.**  $p = p_1 \wedge p_2$ .

If  $p \in C_x$ , then we may assume  $p_1 \in C_x$  and  $p_2 \in C_x \cup C_0$ . By the induction hypothesis, there exists  $n \in \mathbb{N}$  such that  $p_1 \leq x_n$ , whence  $p \leq p_1 \leq x_n$ . If  $p \in C_0$ , then each  $p_i$  belongs to  $C_u \cup C_0$  for some  $u \in \{x, y, z\}$ . If  $p_1 \in C_x \cup C_0$ , then  $p_1 \leq x_n$ , as above and we're done. Similarly, if  $p_2 \in C_x \cup C_0$ . So assume  $p_1 \in C_y \cup C_0$  and  $p_2 \in C_z \cup C_0$ . Then the induction hypothesis implies that there exist  $i$  and  $j$  such that  $p_1 \leq y_i$  and  $p_2 \leq z_j$ . If  $n = \max\{i, j\}$ , then  $p_1 \leq y_n$  and  $p_2 \leq z_n$ . Then, by the above definition of the sequences, we have  $p_1 \wedge p_2 \leq y_n \wedge z_n \leq m_n \leq x_{n+1}$ .

□

**B.1. Other Examples.** In each of the propositions in this section,  $X$  is a finite set and  $\mathbf{F} = \mathbf{F}(X)$  is the free lattice generated by  $X$ . The symbol  $F$  denotes the universe of  $\mathbf{F}$ . The proof in each case is straightforward, but tedious; we omit proofs of the first two, and give a detailed proof of the third.

**Prop. B.4.** *Let  $X = \{x, y, z\}$ , and let  $\mathbf{L} = \mathbf{2}$  be the 2-element chain. Then the kernel of an epimorphism  $h: \mathbf{F} \twoheadrightarrow \mathbf{L}$  is a finitely generated sublattice of  $\mathbf{F} \times \mathbf{F}$ .*

**Prop. B.5.** *Let  $X = \{x, y, z\}$  and let  $\mathbf{L} = \mathbf{3}$  be the 3-element chain. Then the kernel of an epimorphism  $h: \mathbf{F} \twoheadrightarrow \mathbf{L}$  is finitely generated.*

**Prop. B.6.** *Let  $n > 2$ ,  $X = \{x_0, x_1, \dots, x_{n-1}\}$ , and  $\mathbf{L} = \mathbf{2} \times \mathbf{2}$ . Let  $h: \mathbf{F} \twoheadrightarrow \mathbf{L}$  be an epimorphism. Then  $K = \ker h$  is finitely generated.*

*Proof.* Let the universe of  $\mathbf{L} = \mathbf{2} \times \mathbf{2}$  be  $\{0, a, b, 1\}$ , where  $a \vee b = 1$  and  $a \wedge b = 0$ . For each  $y \in L$ , denote by  $X_y = X \cap h^{-1}\{y\}$  the set of generators mapped by  $h$  to  $y$ . Denote the least and greatest elements of  $h^{-1}\{y\}$  (if they exist) by  $\ell_y$  and  $g_y$ , respectively. For example,  $\ell_a = \bigwedge h^{-1}\{a\} = \bigwedge \{x \in F : h(x) = a\}$ ,  $g_b = \bigvee \{x \in F : h(x) = b\}$ , etc. In the present example, the least and greatest elements exist in each case, as we now show.

**Subclaim 3.**  $h^{-1}\{a\}$  has least and greatest elements, namely  $\ell_a = \bigwedge(X_a \cup X_1)$  and  $g_a = \bigvee(X_a \cup X_0)$ . (Similarly,  $h^{-1}\{b\}$  has least and greatest elements,  $\ell_b$  and  $g_b$ .)

*Proof of Subclaim 3.* Let  $M(a) := \bigwedge(X_a \cup X_1)$  and  $J(a) := \bigvee(X_a \cup X_0)$  and note that these values exist in  $F$ , since the sets involved are finite. Also, Then  $h(M(a)) = a = h(J(a))$ . Fix  $r \in h^{-1}\{a\}$ .

- If  $r \in X_a$ , then  $r \geq \bigwedge X_a \geq \bigwedge(X_a \cup X_1) = M(a)$ .
- If  $r = s \vee t$ , where  $h(s) = a$  and  $h(t) \in \{a, 0\}$ , then assume (the induction hypothesis) that  $s \geq M(a)$ , and we have  $r = s \vee t \geq M(a)$ .
- If  $r = s \wedge t$ , where  $h(s) = a$  and  $h(t) \in \{a, 1\}$ , then assume (the induction hypothesis) that  $s, t \geq M(a)$ , and we have  $s \wedge t \geq M(a)$ . This proves that for each  $r \in h^{-1}\{a\}$  we have  $r \geq M(a)$ , and as we noted at the outset,  $M(a) \in h^{-1}\{a\}$ . Therefore,  $\ell_a = M(a)$  is the least element of  $h^{-1}\{a\}$ . Similarly, every  $r \in h^{-1}\{a\}$  is below  $J(a)$ , so  $g_a = J(a)$ . The proofs of  $\ell_b = M(b)$  and  $g_b = J(b)$  are similar.

This proves Subclaim 3.

**Subclaim 4.**  $h^{-1}\{0\}$  has least and greatest elements, namely,  $\ell_0 = \bigwedge X$  and  $g_0 = g_a \wedge g_b$ .

*Proof of Subclaim 4.*  $\ell_0 = \bigwedge X$  is obvious, so we need only verify that  $g_0 = g_a \wedge g_b$ . Observe that  $h(g_a \wedge g_b) = h(g_a) \wedge h(g_b) = a \wedge b = 0$ , so  $g_a \wedge g_b \in h^{-1}\{0\}$ . It remains to prove that  $r \leq g_a \wedge g_b$  holds for all  $r \in h^{-1}\{0\}$ . Fix  $r \in h^{-1}\{0\}$ . Then  $h(r \vee g_a) = h(r) \vee h(g_a) = 0 \vee a = a$ , which places  $r \vee g_a$  in  $h^{-1}\{a\}$ . Therefore, by maximality of  $g_a$ , we have  $r \vee g_a \leq g_a$ , whence  $r \leq g_a$ . Similarly,  $r \leq g_b$ . This proves Subclaim 4.

**Subclaim 5.**  $h^{-1}\{1\}$  has least and greatest elements, namely  $\ell_1 = \ell_a \vee \ell_b$  and  $g_1 = \bigvee X$ .

*Proof of Subclaim 5.*  $g_1 = \bigvee X$  is obvious, so we need only verify that  $\ell_1 = \ell_a \vee \ell_b$ . Observe that  $h(\ell_a \vee \ell_b) = h(\ell_a) \vee h(\ell_b) = a \vee b = 1$ , so  $\ell_a \vee \ell_b \in h^{-1}\{1\}$ . It remains to prove that  $r \geq \ell_a \vee \ell_b$  holds for all  $r \in h^{-1}\{1\}$ . Fix  $r \in h^{-1}\{1\}$ . Then  $h(r \wedge \ell_a) = h(r) \wedge h(\ell_a) = 1 \wedge a = a$ , which places  $r \wedge \ell_a$  in  $h^{-1}\{a\}$ . Therefore, by minimality of  $\ell_a$ , we have  $r \wedge \ell_a \geq \ell_a$ , whence  $r \geq \ell_a$ . Similarly,  $r \geq \ell_b$ . Now let  $Y = \{(x, g_p), (g_p, x), (x, \ell_p), (\ell_p, x) : p \in \{0, a, b, 1\}, x \in X_p\}$ . This proves Subclaim 5.

**Subclaim 6.** If  $r \in F$  and  $h(r) = p$ , then  $(r, \ell_p), (r, g_p) \in \langle Y \rangle$ .

*Proof of Subclaim 6.* Either  $r \in X_p$  or  $r = s \wedge t$  or  $r = s \vee t$ . If  $r \in X_p$ , then the pair belongs to  $Y$  and the claim is trivial.

Suppose  $r = s \wedge t$ .

- If  $h(r) = 1$ , then  $h(s) = h(t) = 1$ . Assume (the induction hypothesis) that  $\{(s, \ell_1), (s, g_1), (t, \ell_1), (t, g_1)\} \subseteq \langle Y \rangle$ . Then  $(r, \ell_1) = (s \wedge t, \ell_1) = (s, \ell_1) \wedge (t, \ell_1) \in \langle Y \rangle$ .

- If  $h(r) = a$ , then (wlog)  $h(s) = a$  and  $h(t) \in \{a, 1\}$ . Assume (the induction hypothesis) that  $\{(s, \ell_a), (s, g_a), (t, \ell_p), (t, g_p)\} \subseteq \langle Y \rangle$ . By Claim 1,  $\ell_a \leq \ell_1$ , so  $\ell_a = \ell_a \wedge \ell_1$ .

If  $h(t) = 1$ , then

$$(r, \ell_a) = (s \wedge t, \ell_a \wedge \ell_1) = (s, \ell_a) \wedge (t, \ell_1) \in \langle Y \rangle,$$

while If  $h(t) = a$ , then  $(r, \ell_a) = (s \wedge t, \ell_a \wedge \ell_a) = (s, \ell_a) \wedge (t, \ell_a) \in \langle Y \rangle$ .

- If  $h(r) = 0$ , then (wlog) that either (i)  $h(s) = 0$ , or (ii)  $h(s) = a$ ,  $h(t) = b$ . If  $h(s) = 0$ , then  $(s, \ell_0) \in \langle Y \rangle$  implies  $(r, \ell_0) = (s \wedge t, \ell_0) = (s, \ell_0) \wedge (t, \ell_p) \in \langle Y \rangle$ . If  $h(s) = a$ ,  $h(t) = b$ , and  $(s, \ell_a), (t, \ell_b) \in \langle Y \rangle$ , then  $(r, \ell_0) = (s \wedge t, \ell_0) = (s, \ell_a) \wedge (t, \ell_b) \in \langle Y \rangle$ .

This proves that  $(r, \ell_1) \in \langle Y \rangle$  if  $r = s \wedge t$ . A similar argument shows that  $(r, g_p) \in \langle Y \rangle$  in each of the three subcases. We have thus proved that  $\{(r, \ell_1), (r, g_p)\} \subseteq \langle Y \rangle$ , if  $r = s \wedge t$ .

Suppose  $r = s \vee t$ .

- If  $h(r) = 0$ , then  $h(s) = h(t) = 0$ . Assume (the induction hypothesis) that  $\{(s, \ell_p), (s, g_p), (t, \ell_p), (t, g_p)\} \subseteq \langle Y \rangle$ . Then  $(r, \ell_p) = (s \vee t, \ell_p) = (s, \ell_p) \vee (t, \ell_p) \in \langle Y \rangle$ .
- If  $h(r) = a$ , then (wlog)  $h(s) = a$  and  $h(t) \in \{a, 0\}$ . If we assume (the induction hypothesis) that  $(s, \ell_p), (s, g_p), (t, \ell_p), (t, g_p)$  belong to  $\langle Y \rangle$ , then  $(r, \ell_p) = (s \vee t, \ell_p) = (s, \ell_p) \vee (t, \ell_p) \in \langle Y \rangle$ .
- If  $h(r) = 1$ , then (wlog) that either (i)  $h(s) = 1$ , or (ii)  $h(s) = a$ ,  $h(t) = b$ .

In the first case,  $(s, \ell_1) \in \langle Y \rangle$  implies  $(r, \ell_1) = (s \vee t, \ell_1) = (s, \ell_1) \vee (t, \ell_p) \in \langle Y \rangle$ . In the second case  $h(s) = a$ ,  $h(t) = b$ , and  $(s, \ell_a), (t, \ell_b) \in \langle Y \rangle$ . Then  $(r, \ell_1) = (s \vee t, \ell_1) = (s, \ell_a) \vee (t, \ell_b) \in \langle Y \rangle$ .

Similarly, in each of these three subcases, we have  $(r, g_p) \in \langle Y \rangle$ . This proves Subclaim 6, and completes the proof of Prop B.6.  $\square$

## APPENDIX C. BACKGROUND

The notation, definitions, ideas presented below are based on those we learned from the book by Freese, Jezek, Nation [FJN95],

**Definition C.1** (length of a term). Let  $X$  be a set. Each element of  $X$  is a term of length 1, also known as a **variable**. If  $t_1, \dots, t_n$  are terms of lengths  $k_1, \dots, k_n$ , then  $t_1 \vee \dots \vee t_n$  and  $t_1 \wedge \dots \wedge t_n$  are both terms of length  $1 + k_1 + \dots + k_n$ .

**Examples.** By the above definition, the terms

$$x \vee y \vee z \quad x \vee (y \vee z) \quad (x \vee y) \vee z$$

have lengths 4, 5, and 5, respectively. Reason: variables have length 1, so  $x \vee y \vee z$  has length  $1 + 1 + 1 + 1$ . On the other hand,  $x \vee y$  is a term of length 3, so  $(x \vee y) \vee z$  has length  $1 + 3 + 1$ . Similarly,  $x \vee (y \vee z)$  has length  $1 + 1 + 3$ .

**Lemma C.2** ([FJN95, Lem. 1.2]). Let  $\mathcal{V}$  be a nontrivial variety of lattices and let  $\mathbf{F}_{\mathcal{V}}(X)$  be the relatively free lattice in  $\mathcal{V}$  over  $X$ . Then,

$$\bigwedge S \leq \bigvee T \text{ implies } S \cap T \neq \emptyset \text{ for each pair of finite subsets } S, T \subseteq X. \quad (\text{C.1})$$

**Lemma C.3** ([FJN95, Lem. 1.4]). Let  $\mathbf{L}$  be a lattice generated by a set  $X$  and let  $a \in L$ .

- (1) If  $a$  is join prime, then  $a = \bigwedge S$  for some finite subset  $S \subseteq X$ .
- (2) If  $a$  is meet prime, then  $a = \bigvee S$  for some finite subset  $S \subseteq X$ .
- If  $X$  satisfies condition (C.1) above, then
- (3) for every finite, nonempty subset  $S \subseteq X$ ,  $\bigwedge S$  is join prime and  $\bigvee S$  is meet prime.

**Corollary C.4** ([FJN95, Cor. 1.5]). Let  $\mathcal{V}$  be a nontrivial variety of lattices and let  $\mathbf{F}_{\mathcal{V}}(X)$  be the relatively free lattice in  $\mathcal{V}$  over  $X$ . For each finite nonempty subset  $S \subseteq X$ ,  $\bigwedge S$  is join prime and  $\bigvee S$  is meet prime. In particular, every  $x \in X$  is both join and meet prime. Moreover, if  $x \leq y$  for  $x, y \in X$ , then  $x = y$ .

**Theorem C.5** (Whitman's Condition, ver. 1). The free lattice  $\mathbf{F}(X)$  satisfies the following condition:

(W) If  $v = v_1 \wedge \dots \wedge v_r \leq u_1 \vee \dots \vee u_s = u$ , then either  $v_i \leq u$  for some  $i$ , or  $v \leq u_j$  for some  $j$ .

**Corollary C.6** ([FJN95, Cor. 1.9]). Every sublattice of a free lattice satisfies (W). Every element of a lattice satisfying (W) is either join or meet irreducible.

**Theorem C.7** (Whitman's Condition, ver. 2). The free lattice  $\mathbf{F}(X)$  satisfies the following condition:

(W+) If  $v = v_1 \wedge \dots \wedge v_r \wedge x_1 \wedge \dots \wedge x_n \leq u_1 \vee \dots \vee u_s \vee y_1 \vee \dots \vee y_m = u$ , where  $x_i, y_j \in X$ , then either  $x_i = y_j$  for some  $i$  and  $j$ , or  $v_i \leq u$  for some  $i$ , or  $v \leq u_j$  for some  $j$ .



**Theorem C.8** ([FJN95, Thm. 1.11]). *If  $s = s(x_1, \dots, x_n)$  and  $t = t(x_1, \dots, x_n)$  are terms and  $x_1, \dots, x_n \in X$ , then the truth of*

$$s^{\mathbf{F}(X)} \leq t^{\mathbf{F}(X)} \quad (\text{C.2})$$

*can be determined by applying the following rules.*

- (1) *If  $s = x_i$  and  $t = x_j$ , then (C.2) holds iff  $x_i = x_j$ .*
- (2) *If  $s = s_1 \vee \dots \vee s_k$  is a formal join, then (C.2) holds iff  $s_i^{\mathbf{F}(X)} \leq t^{\mathbf{F}(X)}$  for all  $i$ .*
- (3) *If  $t = t_1 \wedge \dots \wedge t_k$  is a formal meet, then (C.2) holds iff  $s^{\mathbf{F}(X)} \leq t_i^{\mathbf{F}(X)}$  for all  $i$ .*
- (4) *If  $s = x_i$  and  $t = t_1 \vee \dots \vee t_k$  is a formal join, then (C.2) holds iff  $x_i \leq t_j^{\mathbf{F}(X)}$  for some  $j$ .*
- (5) *If  $s = s_1 \wedge \dots \wedge s_k$  is a formal meet and  $t = x_i$ , then (C.2) holds iff  $s_j^{\mathbf{F}(X)} \leq x_i$  for some  $j$ .*
- (6) *If  $s = s_1 \wedge \dots \wedge s_k$  is a formal meet and  $t = t_1 \vee \dots \vee t_m$  is a formal join, then (C.2) holds iff  $s_i^{\mathbf{F}(X)} \leq t^{\mathbf{F}(X)}$  for some  $i$  or  $s^{\mathbf{F}(X)} \leq t_j^{\mathbf{F}(X)}$  for some  $j$ .*

**Theorem C.9** ([FJN95, Thm. 1.17]). *For each  $w \in \mathbf{F}(X)$  there is a term of minimal length representing  $w$ , unique up to commutativity. This term is called the **canonical form** of  $w$ .*

Let  $w \in \mathbf{F}(X)$  be join reducible and suppose  $t = t_1 \vee \dots \vee t_n$  (with  $n > 1$ ) is the canonical form of  $w$ . Let  $w_i = t_i^{\mathbf{F}(X)}$ . Then  $\{w_1, \dots, w_n\}$  are called the **canonical joinands** of  $w$ . We also say  $w = w_1 \vee \dots \vee w_n$  *canonically* and that  $w = w_1 \vee \dots \vee w_n$  is the **canonical join representation** of  $w$ . If  $w$  is join irreducible, we define the canonical joinands of  $w$  to be the set  $\{w\}$ . Of course the **canonical meet representation** and **canonical meetands** of an element in a free lattice are defined dually.

A join representation  $a = a_1 \vee \dots \vee a_n$  in an arbitrary lattice is said to be a **minimal join representation** if  $a = b_1 \vee \dots \vee b_m$  and  $\{b_1, \dots, b_m\} \ll \{a_1, \dots, a_n\}$  imply  $\{a_1, \dots, a_n\} \subseteq \{b_1, \dots, b_m\}$ . Equivalently, a join representation is minimal if it is an antichain and nonrefinable.

**Theorem C.10** ([FJN95, Thm. 1.19]). *Let  $w = w_1 \vee \dots \vee w_n$  canonically in  $\mathbf{F}(X)$ . If also  $w = u_1 \vee \dots \vee u_m$ , then  $\{w_1, \dots, w_n\} \ll \{u_1, \dots, u_m\}$ . Thus  $w = w_1 \vee \dots \vee w_n$  is the unique minimal join representation of  $w$ .*

**Theorem C.11** ([FJN95, Thm. 1.20]). *Let  $w \in \mathbf{F}(X)$  and let  $u$  be a join irreducible element in  $\mathbf{F}(X)$ . Then  $u$  is a canonical joinand of  $w$  if and only if there is an element  $a$  such that  $w = u \vee a$  and  $w > v \vee a$  for every  $v < u$ .*

**Definition C.12** (up directed, continuous). A subset  $A$  of a lattice  $L$  is said to be **up directed** if every finite subset of  $A$  has an upper bound in  $A$ . It suffices to check this for pairs.  $A$  is up directed iff for all  $a, b \in A$  there exists  $c \in A$  such that  $a \leq c$  and  $b \leq c$ . A lattice is **upper continuous** if whenever

$A \subseteq L$  is an up directed set having a least upper bound  $u = \bigvee A$ , then for every  $b$ ,

$$\bigvee_{a \in A} (a \wedge b) = \bigvee_{a \in A} a \wedge b = u \wedge b.$$

**Down directed** and **down continuous** are defined dually. A lattice that is both up and down continuous is called **continuous**.

**Theorem C.13** ([FJN95, Thm. 1.22]). *Free lattices are continuous.*

**C.1. Bounded Homomorphisms.** We continue to follow [FJN95] very closely, although the authors of that book indicate that the ideas in this subsection have their roots in Ralph McKenzie's work on nonmodular lattice varieties [McK72], and Bjarni Jónsson's work on sublattices of free lattices [JN77].

If  $x, y$  are elements of a lattice  $L$ , and if  $x \leq y$ , then we write  $\llbracket x, y \rrbracket$  to denote the sublattice of elements between  $x$  and  $y$ . That is,

$$\llbracket x, y \rrbracket := \{z \in L \mid x \leq z \leq y\}.$$

Let  $\mathbf{K}$  and  $\mathbf{L}$  be lattices and suppose  $\mathbf{L}$  has bottom and top elements,  $0_{\mathbf{L}}$  and  $1_{\mathbf{L}}$ , resp. If  $h: \mathbf{K} \rightarrow \mathbf{L}$  is a lattice homomorphism, then for each  $a \in L$  we consider the sets  $h^{-1}\llbracket a, 1 \rrbracket = \{x \in K \mid h(x) \geq a\}$  and  $h^{-1}\llbracket 0, a \rrbracket = \{x \in K \mid h(x) \leq a\}$ . When  $h^{-1}\llbracket a, 1 \rrbracket$  is nonempty, it is a filter of  $\mathbf{K}$ ; dually a nonempty  $h^{-1}\llbracket 0, a \rrbracket$  is an ideal. If  $K$  is infinite, then  $h^{-1}\llbracket a, 1 \rrbracket$  need not have a least element, nor  $h^{-1}\llbracket 0, a \rrbracket$  a greatest element. However, considering when such extrema exist leads to the notion of bounded homomorphism, which in turn helps us understand the structure of free lattices.

A lattice homomorphism  $h: \mathbf{K} \rightarrow \mathbf{L}$  is **lower bounded** if for every  $a \in L$ , the set  $h^{-1}\llbracket a, 1 \rrbracket$  is either empty or has a least element. The least element of a nonempty  $h^{-1}\llbracket a, 1 \rrbracket$  is denoted by  $\beta_h(a)$ , or by  $\beta(a)$  when  $h$  is clear from context. Thus, if  $h$  is a lower bounded homomorphism, then  $\beta_h: \mathbf{L} \rightarrow \mathbf{K}$  is a partial mapping whose domain is an ideal of  $\mathbf{L}$ .

Dually,  $h$  is an **upper bounded** homomorphism if, whenever the set  $h^{-1}\llbracket 0, a \rrbracket$  is nonempty, it has a greatest element, denoted by  $\alpha_h(a)$ , or  $\alpha(a)$ . For an upper bounded homomorphism, the domain of  $\alpha_h: \mathbf{L} \rightarrow \mathbf{K}$  is clearly a filter of  $\mathbf{L}$ . A **bounded** homomorphism is one that is both upper and lower bounded.

These definitions simplify when  $h$  is an epimorphism. In that case  $h$  is lower bounded if and only if each preimage  $h^{-1}\{a\}$  has a least element. Likewise, if  $L$  is finite, then  $h: \mathbf{K} \rightarrow \mathbf{L}$  is lower bounded if and only if  $h^{-1}\{a\}$  has a least element whenever it is nonempty. On the other hand, every homomorphism  $h$  from a finite lattice  $\mathbf{K}$  is bounded.

Note that  $\beta$  is monotonic and a left adjoint for  $h$ , i.e.,  $a \leq h(x)$  iff  $\beta(a)x$ . It then follows from a standard argument that  $\beta$  is a join preserving map on its domain: if  $h^{-1}\llbracket a, 1 \rrbracket \neq \emptyset$  and  $h^{-1}\llbracket b, 1 \rrbracket \neq \emptyset$ , then  $\beta(a \vee b) = \beta(a) \vee \beta(b)$ . Similarly,  $\alpha$  is a right adjoint for  $h$ , so that  $h(y) \leq a$  iff  $y \leq \alpha(a)$ , and for  $a, b \in \text{dom } \alpha$ ,  $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$ . In particular, if  $h$  is an epimorphism, then  $\alpha$  and  $\beta$  are respectively meet and join homomorphisms of  $\mathbf{L}$  into  $\mathbf{K}$ .

For future reference, we note that  $\alpha$  and  $\beta$  behave correctly with respect to composition.

**Theorem C.14** ([FJN95, Thm. 2.1]). *Let  $f: \mathbf{K} \rightarrow \mathbf{L}$  and  $g: \mathbf{L} \rightarrow \mathbf{M}$  be homomorphisms. If  $f, g$  are lower bounded, then  $gf: \mathbf{K} \rightarrow \mathbf{M}$  is lower bounded and  $\beta_{gf} = \beta_f \beta_g$ . Similarly, if  $f, g$  are upper bounded, then  $\alpha_{gf} = \alpha_f \alpha_g$ .*

*Proof.* For  $x \in K$  and  $a \in M$ , we have

$$a \leq gf(x) \quad \text{iff} \quad \beta_g(a) \leq f(x) \quad \text{iff} \quad \beta_f \beta_g(a) \leq x.$$

The upper bounded case is dual.  $\square$

We need a way to determine whether a lattice homomorphism  $h: \mathbf{K} \rightarrow \mathbf{L}$  is upper or lower bounded. The most natural setting for this is when the lattice  $\mathbf{K}$  is finitely generated, so from now on we assume  $\mathbf{K}$  is generated by a finite set  $X$ . We want to analyze the sets  $h^{-1}[[a, 1]]$  for  $a \in L$ , with the possibility of lower boundedness in mind. (The corresponding results for  $h^{-1}[[0, a]]$  are obtained by duality.) Note that  $\mathbf{K}$  has a greatest element  $1_{\mathbf{K}} = \bigvee X$ , and that  $h^{-1}[[a, 1]]$  is nonempty if and only if  $a \leq h(1_{\mathbf{K}})$ .

Define a pair of closure operators, denoted by  $\wedge$  and  $\vee$ , on subsets of an arbitrary lattice  $L$  as follows: for each  $A \subseteq L$ ,

$$A^\wedge := \{\bigwedge B \mid B \text{ is a finite subset of } A\}.$$

We adopt the following convention: if  $\mathbf{L}$  has a greatest element  $1_{\mathbf{L}}$ , then  $\bigwedge \emptyset = 1_{\mathbf{L}}$ , and we include this in  $A^\wedge$  for every  $A \subseteq L$ ; otherwise,  $\bigwedge \emptyset$  is undefined. The set  $A^\vee$  is defined dually.

We can write  $K$  as the union of a chain of subsets  $H_0 \subseteq H_1 \subseteq \dots$  defined inductively by setting  $H_0 := X^\wedge$  and  $H_{k+1} := (H_k)^\vee$ , for all  $k \geq 0$ . By induction, each  $H_n = X^{\wedge(\vee)^\wedge n}$  is a finite meet-closed subset of  $K$ , and  $\bigcup H_n = K$ , since  $X$  generates  $K$ .

Let  $h: \mathbf{K} \rightarrow \mathbf{L}$  be an epimorphism and, for each  $y \in L$  and  $k < \omega$ , define

$$\beta_k(y) = \bigwedge \{w \in H_k : h(w) \geq y\}.$$

**Theorem C.15** ([FJN95, Thm. 2.2]). *Let  $\mathbf{K}$  be finitely generated, and let  $h: \mathbf{K} \rightarrow \mathbf{L}$  be a lattice homomorphism. If  $a \leq h(1_{\mathbf{K}})$ , then*

- (1)  $j \leq k$  implies  $\beta_j(a) \geq \beta_k(a)$ ,
- (2)  $\beta_k(a)$  is the least element of  $H_k \cap h^{-1}[[a, 1]]$ ,
- (3)  $h^{-1}[[a, 1]] = \bigcup_{k \in \omega} [\beta_k(a), 1]$ .

**C.1.1. Minimal join covers and refinement.** A **join cover** of the element  $a \in L$  is a finite subset  $S \subseteq L$  such that  $a \leq \bigvee S$ . A join cover  $S$  of  $a$  is *nontrivial* if  $a \not\leq s$  for all  $s \in S$ . Let  $\mathcal{C}(a)$  be the set of all nontrivial join covers of  $a$  in  $\mathbf{L}$ .

**Theorem C.16** ([FJN95, Thm. 2.3]). *Suppose  $\mathbf{K}$  is generated by a finite set  $X$ ,  $h: \mathbf{K} \rightarrow \mathbf{L}$  is a homomorphism,  $a \leq h(1_{\mathbf{K}})$ , and  $k \in \omega$ . Then,*

$$\beta_0(a) = \bigwedge \{x \in X \mid h(x) \geq a\},$$

$$\beta_{k+1}(a) = \beta_0(a) \wedge \bigwedge_{\substack{S \in \mathcal{C}(a) \\ \bigvee S \leq h(1_{\mathbf{K}})}} \bigvee_{s \in S} \beta_k(s).$$

In general, the expression for  $\beta_{k+1}(a)$  has some redundant terms, which we can exclude if  $\mathbf{L}$  satisfies a weak finiteness condition that we now define. For finite subsets  $A, B \subseteq L$ , we say  $A$  *join refines*  $B$  and write  $A \ll B$  if for every  $a \in A$  there exists  $b \in B$  with  $a \leq b$ . Theorem C.10 states that if  $w \in \mathbf{F}(X)$  and  $w = \bigvee B$ , then the set of canonical joinands of  $w$  join refines  $B$ .

Define a **minimal nontrivial join cover** of  $a \in L$  to be a nontrivial join cover  $S$  with the property that whenever  $a \leq \bigvee T$  and  $T \ll S$ , then  $S \subseteq T$ . This formulation is equivalent to our more intuitive notion of what minimality ought to mean: *a nontrivial join cover  $S$  of  $a$  is minimal if and only if*

- (1)  *$S$  is an antichain of join irreducible elements of  $L$ , and*
- (2) *if an element of  $S$  is deleted or replaced by a (finite) set of strictly smaller elements, then the resulting set is no longer a join cover of  $a$ .*

Let  $\mathcal{M}(a)$  denote the set of minimal nontrivial join covers of  $a \in L$ . Let us say that  $\mathbf{L}$  has the **minimal join cover refinement property** if for each  $a \in L$ ,  $\mathcal{M}(a)$  is finite and every nontrivial join cover of  $a$  refines to a minimal one. Clearly every finite lattice has the minimal join cover refinement property, but so do free lattices. The following reformulation of Theorem C.16 simplifies the calculation of  $\beta_k$  whenever the minimal join cover refinement property holds.

**Theorem C.17** ([FJN95, Thm. 2.4]). *Let  $\mathbf{K}$  be generated by the finite set  $X$ , and let  $h: \mathbf{K} \rightarrow \mathbf{L}$  be a lattice homomorphism. If  $\mathbf{L}$  has the minimal join cover refinement property, then for each  $a \in L$  with  $a \leq h(1_{\mathbf{K}})$  and  $k \in \omega$ , we have*

$$\beta_0(a) = \bigwedge \{x \in X \mid h(x) \geq a\},$$

$$\beta_{k+1}(a) = \beta_0(a) \wedge \bigwedge_{\substack{S \in \mathcal{M}(a) \\ \bigvee S \leq h(1_{\mathbf{K}})}} \bigvee_{s \in S} \beta_k(s).$$

We now look for a condition on the  $\mathbf{L}$  that will insure the homomorphism  $h: \mathbf{K} \rightarrow \mathbf{L}$  is lower bounded. From Theorem C.15, this will happen iff for each  $a \leq h(1_{\mathbf{K}})$  there exists  $N \in \omega$  such that  $\beta_n(a) = \beta_N(a)$  for all  $n \geq N$ . In this case,  $\beta(a) = \beta_N(a)$  for all  $a \in \text{dom } \beta = \llbracket 0, h(1_{\mathbf{K}}) \rrbracket$ , where  $N$  depends upon  $a$ .

**Fact.** The following are equivalent:

- (1)  $h$  is not lower bounded;
- (2)  $(\exists y_0 \in L)(\forall N)(\exists n > N) \beta_n(a) \neq \beta_N(a)$ ;
- (3)  $(\exists y_0 \in L)(\exists N)(\forall n > N) \beta_n(a) \neq \beta_N(a)$ .

wjd 2018-09-17: To do: verify this!

Let  $D_0(\mathbf{L})$  be the set of all join prime elements of  $\mathbf{L}$ , i.e., the set of elements that have no nontrivial join cover. Given  $D_k(\mathbf{L})$ , define  $D_{k+1}(\mathbf{L})$  to be the set of  $p \in L$  such that every nontrivial join cover of  $p$  refines to a join cover contained in  $D_k(\mathbf{L})$ , i.e.,  $p \leq \bigvee S$  nontrivially implies there exists  $T \ll S$  with  $p \leq \bigvee T$  and  $T \subseteq D_k(\mathbf{L})$ . Note that if  $L$  has the minimal join cover refinement property, then  $p \in D_{k+1}(\mathbf{L})$  iff every minimal nontrivial join cover of  $p$  is contained in  $D_k(\mathbf{L})$ .

The definition clearly implies  $D_0(\mathbf{L}) \subseteq D_1(\mathbf{L}) \subseteq D_2(\mathbf{L}) \subseteq \dots$ . Let  $D(\mathbf{L}) = \bigcup D_i$ . For  $a \in D(\mathbf{L})$ , define the *D-rank*,  $\rho(a)$ , to be the least integer  $N$  such that  $a \in D_N(\mathbf{L})$ ; for  $a \notin D(\mathbf{L})$ ,  $\rho(a)$  is undefined. The duals of  $D_k(\mathbf{L})$ ,  $D(\mathbf{L})$ , and  $\rho(a)$  are denoted by  $D_k^d(\mathbf{L})$ ,  $D^d(\mathbf{L})$ , and  $\rho^d(a)$ , respectively.

## APPENDIX D. MISCELLANEOUS NOTES

Let  $K$  be a finite subset of  $\ker h$ . Since  $K$  is finite, we can find an  $N < \omega$  such that for all  $\binom{p}{q} \in K$ , the following implications are satisfied:

$$\begin{aligned} p \leq x &\implies q \leq x_N \\ p \leq y &\implies q \leq y_N \\ p \leq z &\implies q \leq z_N \end{aligned} \tag{D.1}$$

$$\begin{aligned} p \leq x \vee (y \wedge z) &\implies q \leq x_{N+1} \\ p \leq y \vee (x \wedge z) &\implies q \leq y_{N+1} \\ p \leq z \vee (x \wedge y) &\implies q \leq z_{N+1} \end{aligned} \tag{D.2}$$

**Claim 4.3** If  $N$  is chosen as just described, and if  $\binom{p}{q} \in \langle K \rangle$  then the implications D.1 and D.2 hold.

*Proof.* As usual, we proceed by induction on term complexity. If  $\binom{p}{q} \in K$ , then by choice of  $N$ , there is nothing to prove.

*Case 1.* Suppose  $\binom{p}{q} = \binom{p_1}{q_1} \vee \binom{p_2}{q_2}$ , where  $\binom{p_1}{q_1}$  and  $\binom{p_2}{q_2}$  satisfy (D.1) and (D.2). We show that  $\binom{p}{q}$  satisfies these two implications as well. Recall, in the notation above,  $x_1 := x \vee (y \wedge z)$ .

Assume  $p \leq x_1$ . We show  $q \leq x_{N+1}$ . Since  $p = p_1 \vee p_2 \leq x_1$ , we have  $p_1 \leq x_1$  and  $p_2 \leq x_1$ , so by the induction hypothesis,  $q_1 \leq x_{N+1}$  and  $q_2 \leq x_{N+1}$ . Therefore,  $q = q_1 \vee q_2 \leq x_{N+1}$ , as desired.

Now assume  $p \leq x$ . We show  $q \leq x_N$ . Since  $p = p_1 \vee p_2 \leq x$ , we have  $p_1 \leq x$  and  $p_2 \leq x$ , so by the induction hypothesis,  $q_1 \leq x_N$  and  $q_2 \leq x_N$ . Therefore,  $q = q_1 \vee q_2 \leq x_N$ , as desired.

*Case 2.* Suppose  $\binom{p}{q} = \binom{p_1}{q_1} \wedge \binom{p_2}{q_2}$ , where  $\binom{p_1}{q_1}$  and  $\binom{p_2}{q_2}$  satisfy (D.1) and (D.2).

Assume  $p \leq x_1 = x \vee (y \wedge z)$ . We must show  $q \leq x_{N+1}$ . Since  $p_1 \wedge p_2 \leq x_1$ , then according to Theorem C.8, at least one of the following inequalities must hold:

- (1)  $p_1 \leq x_1$ ;
- (2)  $p_2 \leq x_1$ ;
- (3)  $p_1 \wedge p_2 \leq x$ ;
- (4)  $p_1 \wedge p_2 \leq y \wedge z$ .

By the induction hypothesis, (1) implies  $q_1 \leq x_{N+1}$  and (2) implies  $q_2 \leq x_{N+1}$ . In either case,  $q = q_1 \wedge q_2 \leq x_{N+1}$ , as desired. In case (3), Theorem C.8 implies that either  $p_1 \leq x$  or  $p_2 \leq x$ , since  $x$  is a generator. Therefore,  $q_1 \leq x_N$  or  $q_2 \leq x_N$  and we conclude that  $q \leq x_N \leq x_{N+1}$ , as desired. It remains to prove  $q \leq x_{N+1}$  for the final case in which  $p_1 \wedge p_2 \leq y \wedge z$ .

If  $p_1 \wedge p_2 \leq y \wedge z$ , then  $p_1 \wedge p_2 \leq y$  and  $p_1 \wedge p_2 \leq z$ . Therefore, both of the following disjunctions hold:

- $p_1 \leq y$  or  $p_2 \leq y$ , and
- $p_1 \leq z$  or  $p_2 \leq z$ .

If  $p_1 \leq y$  and  $p_1 \leq z$ , then  $p_1 \leq x \vee (y \wedge z) = x_1$ , so  $q_1 \leq x_{N+1}$ , so  $q = q_1 \wedge q_2 \leq x_{N+1}$ , as desired. Similarly, if  $p_2 \leq y$  and  $p_2 \leq z$ , the desired conclusion holds. Finally, consider the case in which  $p_1 \leq y$  and  $p_2 \leq z$ . In this case  $q_1 \leq y_N$  and  $q_2 \leq z_N$ . Therefore,  $q = q_1 \wedge q_2 \leq y_N \wedge z_N \leq x_N \vee (y_N \wedge z_N) = x_{N+1}$ , as desired.  $\square$

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