GENERATING SUBDIRECT PRODUCTS OF LATTICES

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1. Fiber products

Theorem 1.1. *Let C be a fiber product of finitely generated lattices A and B with respect to homomorphisms $g\colon A \to D, h\colon B \to D$ onto finite D.*

Then C is finitely generated if D is upper and lower bounded.

Proof. By definition

$$C = \{(a,b) \in A \times B \ : \ g(a) = h(b)\}.$$

Adapt the proof that $\ker h \leqslant F(X)^2$ is finitely generated if $h\colon F(X) \to D$ is bounded. □

Remark. The converse implication in Theorem 1.1 holds for $A = B = F(X)$ and $g = h$ by Lemma 2.2. More generally, the proof can be adapted immediately for A, B satisfying Whitman's condition and being generated by join prime elements and by meet prime elements.

The converse does not hold in general: Nik's example describes a lower but not upper bounded epimorphism $h\colon A \to D$ for A with 14 generators, D the subgroup lattice of $\mathbb{Z}_2^3$ (cf. projective plane) such that $\ker h$ is finitely generated.

2. Bounded Homomorphisms

Let $F(X)$ denote the free lattice over a finite set X. For $k \in \mathbb{N}$ define

$$H_0 := X^\wedge,\ \ H_{k+1} := H_k^{\vee\wedge}$$

the set of meets of elements of X, meets of joins of elements in H_k, respectively. Then

$$H_0 \subseteq H_1 \subseteq \dots \text{ and } F(X) = \bigcup_{k\in\mathbb{N}} H_k.$$

Fix an epimorphism $h\colon F(X) \to L$ onto a finite lattice L for the remainder of the section. For $k \in \mathbb{N}$ and $a \in L$ define

$$\beta_k(a) := \bigwedge\{w \in H_k \ : \ h(w) \geqslant a\}.$$

Note that for $k > 0$

$$\beta_k(a) := \bigwedge\{w \in H_{k-1}^\vee \ : \ h(w) \geqslant a\}.$$

Lemma 2.1. *Let $k \in \mathbb{N}$.*

(1) If $a \leqslant b$ in L, then $\beta_k(a) \leqslant \beta_k(b)$.

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(2) $\bigvee\{\beta_k(a) \;:\; a \in A\} \geqslant \beta_{k+1}(\bigvee A)$ *for all* $A \subseteq L$.

Proof. (1) is immediate. For (2) note that by definition $\beta_k(a) \in H_k$ and $h(\beta_k(a)) \geqslant a$ for all $a \in L$. Hence $w := \bigvee\{\beta_k(a) \;:\; a \in A\}$ is in $H_k^\vee$ and

$$h(w) = \bigvee\{h(\beta_k(a)) \;:\; a \in A\} \geqslant \bigvee A$$

yields the claim. □

Lemma 2.2. *For each finite subset* G *of* $\ker h$ *there exists* $N \in \mathbb{N}$ *such that*

$$\forall (p,q) \in \langle G\rangle \;\forall k \in \mathbb{N} \;\forall a \in L \colon p \geqslant \beta_k(a) \Rightarrow q \geqslant \beta_{k+N}(a). \tag{2.1}$$

Proof. First note that since G and L are finite, we have $N \in \mathbb{N}$ such that for all $(p,q) \in G, a \in L$ and $k \in \mathbb{N}$, the implication $p \geqslant \beta_k(a) \Rightarrow q \geqslant \beta_{k+N}(a)$ holds. We show (2.1) for this specific N by induction on the complexity of (p,q) over the generating set G. In the following we assume that (p,q) is not a generator.

Case $(p,q) = (p_1,q_1) \wedge (p_2,q_2)$ for $(p_1,q_1),(p_2,q_2) \in \langle G\rangle$: Immediate from the induction assumption.

Case $(p,q) = (p_1,q_1) \vee (p_2,q_2)$ for $(p_1,q_1),(p_2,q_2) \in \langle G\rangle$: Let $a \in L$. We use induction on $k \in \mathbb{N}$. For the base case $k = 0$, assume $p \geqslant \beta_0(a)$. By definition $\beta_0(a) = \bigwedge W$ for $W := \{x \in X \;:\; h(x) \geqslant a\}$. By Whitman's condition $p_1 \vee p_2 \geqslant \bigwedge W$ yields $p_1 \geqslant \beta_0(a)$ or $p_2 \geqslant \beta_0(a)$ or $p \geqslant x$ for some $x \in W$. Since generators in $F(X)$ are join prime, the latter case yields $p_1 \geqslant x$ or $p_2 \geqslant x$ which implies $p_1 \geqslant \beta_0(a)$ or $p_2 \geqslant \beta_0(a)$ again. Applying the induction assumption on term complexity, we find $q_1 \geqslant \beta_N(a)$ or $q_2 \geqslant \beta_N(a)$. Either way $q \geqslant \beta_N(a)$ and the base case is proved.

Assume $k > 0$ and $p \geqslant \beta_k(a)$ in the following. By definition $\beta_k(a) = \bigwedge W$ for $W := \{w \in H_{k-1}^\vee \;:\; h(w) \geqslant a\}$. By Whitman's condition $p_1 \vee p_2 \geqslant \bigwedge W$ yields $p_1 \geqslant \beta_k(a)$ or $p_2 \geqslant \beta_k(a)$ or $p \geqslant w$ for some $w \in W$. The first two cases are again straightforward using the induction assumption on term complexity. For the third case note that $w = \bigvee U$ for some $U \subseteq H_{k-1}$. For each $u \in U$, we have $p \geqslant u$ and $u \geqslant \beta_{k-1}(h(u))$, which yield $q \geqslant \beta_{k-1+N}(h(u))$ by the induction assumption on k. Hence

$$\begin{aligned}
q &\geqslant \bigvee\{\beta_{k-1+N}(h(u)) \;:\; u \in U\} \\
&\geqslant \beta_{k+N}(\bigvee\{h(u) \;:\; u \in U\}) && \text{by Lemma 2.1 (2)} \\
&= \beta_{k+N}(h(w)) \\
&\geqslant \beta_{k+N}(a) && \text{by } h(w) \geqslant a \text{ and Lemma 2.1 (1).}
\end{aligned}$$

□

Corollary 2.3. Let X be a finite set, L a finite lattice, and $h \colon F(X) \to L$ an epimorphism.

Then $\ker h$ is finitely generated as a sublattice of $F(X)^2$ iff h (equivalently L) is lower and upper bounded.