KERNELS OF EPIMORPHISMS OF FINITELY GENERATED FREE LATTICES

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1. Introduction

1.1. **Notation.** If A and B are two sets, then the notation $A \subseteq_\omega B$ means that A is a finite subset of B.

2. Main Theorem

Theorem 2.1. *Let X be a finite set and $\mathbf{F}(X)$ the free lattice generated by X. Suppose $\mathbf{L} = \langle L, \wedge, \vee\rangle$ is a finite lattice and $h\colon \mathbf{F}(X) \twoheadrightarrow \mathbf{L}$ a lattice epimorphism. If h is bounded then the kernel of h is a finitely generated sublattice of $\mathbf{F}(X) \times \mathbf{F}(X)$.*

Proof. Throughout the proof we let $\mathbf{F} := \mathbf{F}(X)$. Assume h is a bounded epimorphism so that the preimage of each $y \in L$ under h is bounded. For each $y \in L$, let $\alpha y = \bigvee h^{-1}\{y\}$ and $\beta y = \bigwedge h^{-1}\{y\}$ denote the greatest and least elements of $h^{-1}\{y\}$, respectively (both of which exist by the assumed boundedness and surjectivity of h). Observe that $h\alpha h = h$, and $h\beta h = h$. In fact, α and β are adjoint to h. Indeed, it is easy to see that

$$hx \leqslant y \quad \Leftrightarrow \quad x \leqslant \alpha y,$$
$$y \leqslant hx \quad \Leftrightarrow \quad \beta y \leqslant x.$$

For each $y \in L$, let $X_y := X \cap h^{-1}\{y\}$, the set of generators that lie in the inverse image of y under h. Let G be the (finite) set of pairs in $\mathbf{F} \times \mathbf{F}$ defined as follows:

$$G = \bigcup_{y\in L}\{(x, \alpha y), (\alpha y, x), (x, \beta y), (\beta y, x), (\alpha y, \beta y), (\beta y, \alpha y) : x \in X_y\}.$$

We claim that G generates $\ker h$. To prove this, we first show, by induction on term complexity, that for every $y \in L$, for every $r \in h^{-1}\{y\}$, the pairs $(r, \alpha y)$ and $(r, \beta y)$ belong to the sublattice $\langle G\rangle \leqslant \mathbf{F} \times \mathbf{F}$ generated by G.

- **Case 0.** Suppose $r \in X$. Then $(r, \alpha y)$ and $(r, \beta y)$ belong to G itself, so there's nothing to prove.
- **Case 1.** Suppose $r = s \vee t$. Assume (the induction hypothesis) that $(s, \alpha h(s))$, $(s, \beta h(s))$, $(t, \alpha h(t))$, and $(t, \beta h(t))$ belong to $\langle G\rangle$. Then $y = h(r) = h(s \vee t) = h(s) \vee h(t)$, so

$$h(\alpha h(s) \vee \alpha h(t)) = h\alpha h(s) \vee h\alpha h(t) = h(s) \vee h(t) = y.$$

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Likewise, $h(\beta h(s) \vee \beta h(t)) = h(s) \vee h(t) = y$. Therefore,

$$\beta y \leqslant \beta h(s) \vee \beta h(t) \leqslant \alpha h(s) \vee \alpha h(t) \leqslant \alpha y.$$

Also, $r \leqslant \alpha y$, so $r = \alpha y \wedge (s \vee t)$. Taken together, these observations yield

$$\begin{aligned}\begin{pmatrix} r \\ \beta y \end{pmatrix} &= \begin{pmatrix} \alpha y \wedge (s \vee t) \\ \beta y \end{pmatrix} = \begin{pmatrix} \alpha y \wedge (s \vee t) \\ \beta y \wedge (\beta h(s) \vee \beta h(t)) \end{pmatrix} \\ &= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \left[\begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \vee \begin{pmatrix} t \\ \beta h(t) \end{pmatrix} \right],\end{aligned}$$

and each term in the last expression belongs to $\langle G \rangle$, so $(r, \beta y) \in \langle G \rangle$, as desired.

Similarly, $(r, \alpha y) \in \langle G \rangle$. Indeed, $\beta y \leqslant r$ implies $r = \beta y \vee s \vee t$, and $\beta h(s) \vee \beta h(t) \leqslant \alpha y$ implies $\alpha y = \alpha y \vee \beta h(s) \vee \beta h(t)$. Therefore,

$$\begin{aligned}\begin{pmatrix} r \\ \alpha y \end{pmatrix} &= \begin{pmatrix} \beta y \vee s \vee t \\ \alpha y \vee \beta h(s) \vee \beta h(t) \end{pmatrix} \\ &= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \vee \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}.\end{aligned}$$

- **Case 2.** Suppose $r = s \wedge t$. Assume $(s, \alpha h(s))$, $(s, \beta h(s))$, $(t, \alpha h(t))$, and $(t, \beta h(t))$ belong to $\langle G \rangle$. Then $h(s \wedge t) = h(r) = y$, so $h(\alpha h(s) \wedge \alpha h(t)) = y = h(\beta h(s) \wedge \beta h(t))$, so $\beta y \leqslant \beta h(s) \wedge \beta h(t) \leqslant \alpha h(s) \wedge \alpha h(t) \leqslant \alpha y$. Also, $\beta y \leqslant r \leqslant \alpha y$ so $r = \alpha y \wedge s \wedge t$ and $r = \beta y \vee (s \wedge t)$. Altogether, we have

$$\begin{aligned}\begin{pmatrix} r \\ \alpha a \end{pmatrix} &= \begin{pmatrix} \beta y \vee (s \wedge t) \\ \alpha y \vee (\alpha h(s) \wedge \alpha h(t)) \end{pmatrix} \\ &= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \left[\begin{pmatrix} s \\ \alpha h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \alpha h(t) \end{pmatrix} \right],\end{aligned}$$

and each term in the last expression belongs to $\langle Y \rangle$, as desired. Similarly,

$$\begin{aligned}\begin{pmatrix} r \\ \beta y \end{pmatrix} &= \begin{pmatrix} \alpha y \wedge s \wedge t \\ \beta y \wedge \alpha h(s) \wedge \alpha h(t) \end{pmatrix} \\ &= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \begin{pmatrix} s \\ \alpha h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \alpha h(t) \end{pmatrix}.\end{aligned}$$

Note that, in both of the derivations above, we could have used β's instead of α's; that is,

$$\begin{aligned}\begin{pmatrix} r \\ \alpha y \end{pmatrix} &= \begin{pmatrix} \beta y \vee (s \wedge t) \\ \alpha y \vee (\beta h(s) \wedge \beta h(t)) \end{pmatrix} \\ &= \begin{pmatrix} \beta y \\ \alpha y \end{pmatrix} \vee \left[\begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \beta h(t) \end{pmatrix} \right],\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix} r \\ \beta y \end{pmatrix} &= \begin{pmatrix} \alpha y \wedge s \wedge t \\ \beta y \wedge \beta h(s) \wedge \beta h(t) \end{pmatrix}\\ &= \begin{pmatrix} \alpha y \\ \beta y \end{pmatrix} \wedge \begin{pmatrix} s \\ \beta h(s) \end{pmatrix} \wedge \begin{pmatrix} t \\ \beta h(t) \end{pmatrix}.\end{aligned}$$

In each case, we end up with an expression involving terms from $\langle G\rangle$, and this proves that $(r, \alpha y)$ and $(r, \beta y)$ belong to $\langle G\rangle$, as desired.

□

The converse of Theorem 2.1 is

Theorem 2.2. *Suppose $\mathbf{L} = \langle L, \wedge, \vee\rangle$ is a finite lattice and $h\colon \mathbf{F}(X) \twoheadrightarrow \mathbf{L}$ a lattice epimorphism. If the kernel of h is a finitely generated sublattice of $\mathbf{F}(X) \times \mathbf{F}(X)$, then h is bounded.*

Before proving the theorem, we present some standard notation and establish a lemma that will play a central role in the proof of Theorem 2.2.

Let $\mathbf{F} := \mathbf{F}(X)$. Define a pair of closure operators, denoted by ${}^\wedge$ and ${}^\vee$, on the powerset $\mathcal{P}(\mathbf{F})$ as follows: if $A \in \mathcal{P}(\mathbf{F})$, then

$$A^\wedge = \{\bigwedge B \mid B \subseteq_\omega A\} \quad \text{and} \quad A^\vee = \{\bigvee B \mid B \subseteq_\omega A\}.$$

Define also $H_0 = X^\wedge = \{\bigwedge B \mid B \subseteq_\omega X\}$, and for all $k \geqslant 0$, let

$$H_{k+1} = (H_k)^{\vee\wedge} = \{\bigwedge B \mid B \subseteq_\omega H_k^\vee\},$$

where $H_k^\vee = \{\bigvee A \mid A \subseteq_\omega H_k\}$. Evidentally, $\mathbf{F}$ is the union of the chain of subsets $H_0 \subseteq H_1 \subseteq \cdots$. Each $H_n = X^{\wedge(\vee\wedge)^n}$ is a finite meet-closed subset of $\mathbf{F}$ and, since X generates $\mathbf{F}$, it should be apparent that $\bigcup H_n = \mathbf{F}$.

To gain some intuition about H_k ($k \geqslant 0$), let's consider the complexity of the terms in $H_1 = (H_0)^{\vee\wedge}$.

$$\begin{aligned}H_1 &= \{\bigwedge B \mid B \subseteq_\omega (H_0)^\vee\} = \{\bigwedge B \mid B \subseteq_\omega X^{\wedge\vee}\}\\ &= \Big\{\bigwedge B \mid B \subseteq_\omega \{\bigvee A \mid A \subseteq_\omega X^\wedge\}\Big\}\\ &= \Big\{\bigwedge B \mid B \subseteq_\omega \{\bigvee A \mid A \subseteq_\omega \{\bigwedge B \mid B \subseteq_\omega X\}\}\Big\}.\end{aligned}$$

For a concrete example, if $x_i \in X$, then the following four terms belong to H_0,

$$t_1 = x_1 \wedge x_2 \wedge x_3, \quad t_2 = x_2 \wedge x_4, \quad t_3 = x_3 \wedge x_5, \quad t_4 = x_1,$$

and here's a term that belongs to $H_1 \setminus H_0$,

$$(t_1 \vee t_2) \wedge (t_3 \vee t_4) = \big[(x_1 \wedge x_2 \wedge x_3) \vee (x_2 \wedge x_4)\big] \wedge \big[(x_1 \wedge x_5) \vee x_2\big].$$

Let $h\colon \mathbf{F}(X) \to \mathbf{L}$ be an epimorphism and, for each $y \in L$ and $k < \omega$, define

$$\beta_k(y) = \bigwedge\{w \in H_k : h(w) \geqslant y\}.$$

Define the *length* of a term $t \in \mathbf{F}(X)$ as follows:

- t has *length* ℓ if $t \in H_\ell \setminus H_{\ell-1}$

- t has *length at most* ℓ if $t \in H_\ell$.

Observe that if t has length at most ℓ, then $t \geqslant \beta_\ell h(t)$.

Lemma 2.3. *Let h be an epimorphism from $\mathbf{F}(X)$ onto a finite lattice $\mathbf{L}$. Let K be a finite subset of the kernel of h. Then there exists $n > 0$ such that for all $(p, q) \in \langle K \rangle$, if p has length ℓ, then $q \geqslant \beta_{\ell+n} h(q)$.*

Remark. If $K \subseteq \ker h$, then $(p, q) \in \langle K \rangle$ implies $h(p) = h(q)$.

Proof. Let $\mathbf{F} := \mathbf{F}(X)$, let K be a finite subset of $\ker h$, and define the following, where S is an arbitrary subset of $\mathbf{F}^2$:

- $S_1 = \{s_1 \in \mathbf{F} \mid \exists s_2,\ (s_1, s_2) \in S\}$,
- $m = \min\{\ell \mid K_1 \cap H_\ell \neq \emptyset\}$,
- $M = \min\{\ell \mid K_1 \subseteq H_\ell\}$.

In other terms, m and M are the minimum and maximum lengths of terms p such that $(p, q) \in K$. Clearly $m \leqslant M$, so for all $a \in L$ we have $\beta_m(a) \geqslant \beta_M(a)$. Also, $p \geqslant \beta_M h(p)$ for all pairs $(p, q) \in K$, since $K_1 \subseteq H_M$. Moreover, $\exists p \in K_1 \cap H_m$ and, by minimality of m, the length of this p is exactly m and $p \geqslant \beta_m h(p)$. Finally, since K is a finite set, we can find $n > 0$ such that $q \geqslant \beta_{m+n} h(p)$ for all $(p, q) \in K$.

Let $(p, q) \in \langle K \rangle$ and let ℓ be the length of p, so $p \geqslant \beta_\ell h(p)$. We want to show $q \geqslant \beta_{\ell+n} h(p)$.

Case 1: Suppose $(p, q) = (p_1, q_1) \vee (p_2, q_2)$, and assume the conclusion of the lemma holds for (p_1, q_1) and (p_2, q_2). Let ℓ_i be the length of p_i. Then p has length $\ell = \max\{\ell_1, \ell_2\} + 1$. For each $i \in \{1, 2\}$, we have $p_i \geqslant \beta_{\ell_i} h(p_i)$ and (by the induction hypothesis) $q_i \geqslant \beta_{\ell_i+n} h(p_i)$. Observe also that $\ell - 1 \geqslant \ell_i$, so $\beta_{\ell_i+n} h(p_i) \geqslant \beta_{\ell+n-1} h(p_i)$ for each $i \in \{1, 2\}$. Therefore,

$$q = q_1 \vee q_2 \geqslant \beta_{\ell_1+n} h(p_1) \vee \beta_{\ell_1+n} h(p_2) \geqslant \beta_{\ell+n-1} h(p_1) \vee \beta_{\ell+n-1} h(p_2).$$

To complete the proof in this case, then, it suffices to show

$$\beta_{\ell+n-1} h(p_1) \vee \beta_{\ell+n-1} h(p_2) \geqslant \beta_{\ell+n} h(p). \tag{2.1}$$

Let $t_i = \beta_{\ell+n-1} h(p_i)$ and recall that this denotes least term of length at most $\ell + n - 1$ that is mapped by h to $h(p_i)$. Now $t_1 \vee t_2$ is mapped by h to $h(p)$. Indeed, $h(t_1 \vee t_2) = h(t_1) \vee h(t_2) = h(p_1) \vee h(p_2) = h(p_1 \vee p_2) = h(p)$. Also, the length of $t_1 \vee t_2$ is at most $\ell + n$. Therefore, as $\beta_{\ell+n} h(p)$ denotes the least term of length at most $\ell + n$ that is mapped to $h(p)$, (2.1) holds.

Case 2: Suppose $(p, q) = (p_1, q_1) \wedge (p_2, q_2)$, and assume the conclusion of the lemma holds for (p_1, q_1) and (p_2, q_2). Let ℓ_i be the length of p_i. Then $p_i \geqslant \beta_{\ell_i} h(p_i)$ and (by the induction hypothesis) $q_i \geqslant \beta_{\ell_i+n} h(p_i)$. We must show $q \geqslant \beta_{\ell+n} h(p)$.

For all k, H_k is closed under meet, so the length of p is $\ell = \max\{\ell_1, \ell_2\}$ (in contrast to Case 1). Still, $\beta_{\ell_i+n} h(p_i) \geqslant \beta_{\ell+n} h(p_i)$, so

$$q = q_1 \wedge q_2 \geqslant \beta_{\ell_1+n} h(p_1) \wedge \beta_{\ell_1+n} h(p_2) \geqslant \beta_{\ell+n} h(p_1) \wedge \beta_{\ell+n} h(p_2).$$

To complete the proof it this case, it suffices to show

$$\beta_{\ell+n} h(p_1) \wedge \beta_{\ell+n} h(p_2) \geqslant \beta_{\ell+n} h(p).$$

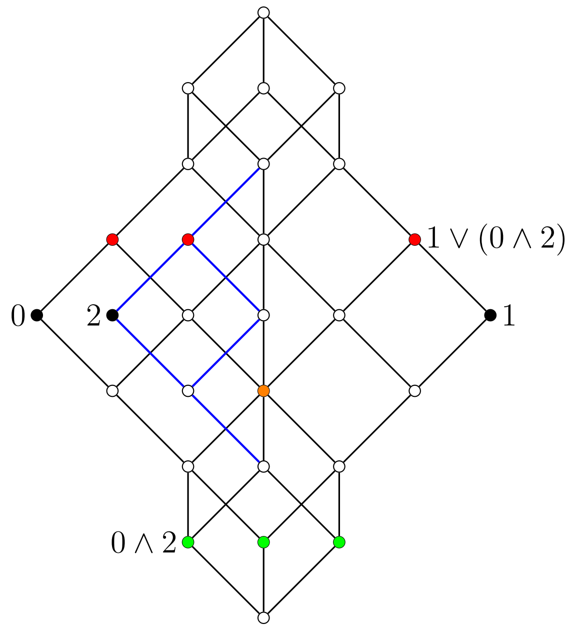

FIGURE 1. The free lattice over M_3 generated by $\{0, 1, 2\}$. Green dots identify elements $0 \wedge 1$, $0 \wedge 2$, and $1 \wedge 2$; red dots identify $0 \vee (1 \wedge 2)$, $1 \vee (0 \wedge 2)$, and $2 \vee (0 \wedge 1)$.

Recall that $\beta_{\ell+n}h(p_i)$ denotes the least term t_i of length at most $\ell + n$ such that $h(t_i) = h(p_i)$. Now, $t_1 \wedge t_2$ has length at most $\ell + n$, since $H_{\ell+n}$ is closed under meet. Also,

$$h(t_1 \wedge t_2) = h(t_1) \wedge h(t_2) = h(p_1) \wedge h(p_2) = h(p_1 \wedge p_2) = h(p),$$

so the image of $t_1 \wedge t_2$ under h is $h(p)$. Since $\beta_{\ell+n}h(p)$ is the least term of length at most $\ell + n$ that is mapped to $h(p)$, we have $t_1 \wedge t_2 \geqslant \beta_{\ell+n}h(p)$, as desired.

3. EXAMPLE

An example in which the unbounded classes of $\ker h$ do not contain generators of $\mathbf{F}$ we brought to our attention by J.B. Nation. Despite the fact that the unbounded kernel classes do not contain generators, we are still able to prove that the kernel is not a finitely generated subalgebra of $\mathbf{F} \times \mathbf{F}$.

Proposition 3.1. *Let* $\mathbf{F} = \mathbf{F}(x, y, z)$*, and let* $\mathbf{L} = \mathbf{F}_{\mathbf{M}_3}(0, 1, 2)$ *(see Figure 1). Let* $h\colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ *be the epimorphism induced by* $x \mapsto 0$*,* $y \mapsto 1$*,* $z \mapsto 2$*. Then* $\ker h$ *is not finitely generated.*

To prove this we will need a technical lemma about the behavior of certain sequences of elements of $\mathbf{F}$.

Lemma 3.2. *For each $s \in \{x, y, z, m\}$, define the sequence $\{s_i : i < \omega\}$ of elements of* $\mathbf{F}$ *as follows: $x_0 = x$, $y_0 = y$, $z_0 = z$, and for $i \geqslant 0$,*

$$x_{i+1} = x \vee (y_i \wedge z_i), \quad y_{i+1} = y \vee (x_i \wedge z_i), \quad z_{i+1} = z \vee (x_i \wedge y_i),$$
$$m_i = (x_i \wedge y_i) \vee (x_i \wedge z_i) \vee (y_i \wedge z_i).$$

If $\{s_i : i < \omega\}$ is any one of the four sequences just defined, then for every $i \geqslant 1$, we have $s_{i+1} > s_i$ and $h(s_i) = h(s_1)$.

Proof. First observe that $h(x_1) = h(x \vee (y \wedge z)) = 0 \vee (1 \wedge 2)$. We begin by proving $h(x_2) = h(x_1)$. By definition, we have $h(x_2) = h(x \vee (y_1 \wedge z_1)) = h(x) \vee \big[h(y_1) \wedge h(z_1)\big]$, and $h(y_1) = h(y \vee (x \wedge z)) = h(y) \vee \big[h(x) \wedge h(z)\big] = 1 \vee (0 \wedge 2)$. Similarly, $h(z_1) = 2 \vee (0 \wedge 1)$. Therefore,

$$h(x_2) = 0 \vee \big\{[1 \vee (0 \wedge 2)] \wedge [2 \vee (0 \wedge 1)]\big\}. \tag{3.1}$$

Recall, the modular law: $x \leqslant b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$. Applying this law with $a = 1$ and $x = 0 \wedge 2 \leqslant 2 \vee (0 \wedge 1) = b$, we have

$$(0 \wedge 2) \vee \big\{1 \wedge [2 \vee (0 \wedge 1)]\big\} = [1 \vee (0 \wedge 2)] \wedge [2 \vee (0 \wedge 1)],$$

which is the right joinand in 3.1. Therefore,

$$h(x_2) = 0 \vee (0 \wedge 2) \vee \big\{1 \wedge [2 \vee (0 \wedge 1)]\big\} = 0 \vee \big\{1 \wedge [2 \vee (0 \wedge 1)]\big\}.$$

Applying the modular law once more to $1 \wedge [2 \vee (0 \wedge 1)]$, with $a = 2$ and $x = 0 \wedge 1 \leqslant 1 = b$, we have

$$1 \wedge [2 \vee (0 \wedge 1)] = (0 \wedge 1) \vee (1 \wedge 2).$$

Therefore,

$$h(x_2) = 0 \vee \big\{1 \wedge [2 \vee (0 \wedge 1)]\big\} = 0 \vee (0 \wedge 1) \vee (1 \wedge 2) = 0 \vee (1 \wedge 2),$$

as desired. Of course, $h(y_2) = h(y_1)$ and $h(z_2) = h(z_1)$ can be checked similarly.

With the base cases established, we proceed with the induction. Fix $n \geqslant 1$ and assume $h(x_n) = h(x_1)$, $h(y_n) = h(y_1)$, and $h(z_n) = h(z_1)$. Then,

$$h(x_{n+1}) = h(x \vee (y_n \wedge z_n)) = h(x) \vee (h(y_n) \wedge h(z_n)) = h(x) \vee (h(y_1) \wedge h(z_1)).$$

As observed above, this is equal to $h(x_2)$, which in turn is $h(x_1)$, as desired. By the same argument, $h(y_{n+1}) = h(y_1)$ and $h(z_{n+1}) = h(z_1)$. This proves that for all $n \geqslant 1$, we have $h(s_n) = h(s_1)$, when $\{s_n\}$ is $\{x_n\}$ or $\{y_n\}$ or $\{z_n\}$.

Finally, consider $\{m_n\}$. For all $n \geqslant 1$, we have

$$\begin{aligned} h(m_n) &= \big[h(x_n) \wedge h(y_n)\big] \vee \big[h(x_n) \wedge h(z_n)\big] \vee \big[h(y_n) \wedge h(z_n)\big] \\ &= \big[h(x_1) \wedge h(y_1)\big] \vee \big[h(x_1) \wedge h(z_1)\big] \vee \big[h(y_1) \wedge h(z_1)\big] \\ &= h(m_1). \end{aligned}$$

(In fact, in this case we can show that $h(m_1) = h(m_0)$, but this is unnecessary.)

It remains to show that $s_{i+1} > s_i$ for all $i < \omega$ (for each $s \in \{x, y, z, m\}$). For this we will need the following.

Claim 3.3. For all $i \geqslant 0$, $x \nleq y_i$ and $x \nleq z_i$.

Proof. When $i = 0$, the claim is that $x \nleqslant y$ and $x \nleqslant z$, which is clear. Fix $k \geqslant 0$ and suppose $x \nleqslant y_k$ and $x \nleqslant z_k$. We show $x \nleqslant y_{k+1}$ and $x \nleqslant z_{k+1}$. Assume the contrary, say, $x \leqslant y_{k+1} = y \vee (x_k \wedge z_k)$. Since x is a generator, it is join prime. Therefore, as $x \nleqslant y$, we must have $x \leqslant x_k \wedge z_k$. But this is impossible since $x \nleqslant z_k$, by the induction hypothesis. Thus, Claim 3.3 is proved. ☐

Observe that the claim just proved also yields the following: $\forall m, n, x_m \nleqslant y_n$. We will now prove $\forall i \geqslant 0$ that all of the following strict inequalities hold:

$$x_i < x_{i+1}, \quad y_i < y_{i+1}, \quad z_i < z_{i+1}$$
$$x_i \wedge y_i < x_{i+1} \wedge y_{i+1}, \quad x_i \wedge z_i < x_{i+1} \wedge z_{i+1}, \quad y_i \wedge z_i < y_{i+1} \wedge z_{i+1}. \tag{3.2}$$

Clearly $x_1 = x \vee (y \wedge z) > x = x_0$, by Theorem C.8. Similarly, $y_1 > y_0$ and $z_1 > z_0$. Also,

$$x_1 \wedge y_1 = \big[x \vee (y \wedge z)\big] \wedge \big[y \vee (x \wedge z)\big] > x \wedge y.$$

For otherwise we would have $\big[x \vee (y \wedge z)\big] \wedge \big[y \vee (x \wedge z)\big] = x \wedge y$, which implies $\big[x \vee (y \wedge z)\big] \wedge \big[y \vee (x \wedge z)\big] \leqslant x$; but x is meet prime, so we would have either $x \vee (y \wedge z) \leqslant x$ or $y \vee (x \wedge z) \leqslant x$, both of which are clearly false. The arguments establishing the remaining inequalities in (3.2) in case $i = 0$ are similar.

Now, fix $n \geqslant 0$ and suppose the strict inequalities in (3.2) hold when $i = n$. We show they also hold when $i = n + 1$. It suffices to prove just two of these, namely, $x_{n+1} > x_n$ and $x_{n+1} \wedge y_{n+1} > x_n \wedge y_n$; the other cases are symmetric.

Recall $x_{n+1} = x \vee (y_n \wedge z_n)$ and $x_n = x \vee (y_{n-1} \wedge z_{n-1})$, and the induction hypothesis implies that $y_n \wedge z_n > y_{n-1} \wedge z_{n-1}$. Therefore, $x_{n+1} \geqslant x_n$, so we must show $x_{n+1} \neq x_n$. Suppose on the contrary that

$$x_{n+1} = x \vee (y_n \wedge z_n) = x \vee (y_{n-1} \wedge z_{n-1}) = x_n. \tag{3.3}$$

Then $y_n \wedge z_n \leqslant x \vee (y_{n-1} \wedge z_{n-1})$. Recall Whitman's Theorem (C.7) asserting that $\mathbf{F}(X)$ satisfies condition (W+), repeated here for easy reference.

> If $v = v_1 \wedge \cdots \wedge v_r \wedge x_1 \wedge \cdots \wedge x_n \leqslant u_1 \vee \cdots \vee u_s \vee y_1 \vee \cdots \vee y_m = u$, where $x_i, y_j \in X$, then either $x_i = y_j$ for some i and j, or $v_i \leqslant u$ for some i, or $v \leqslant u_j$ for some j.

This condition and (3.3) together imply that one of the following must hold:

(1) $y_n \leqslant x \vee (y_{n-1} \wedge z_{n-1})$;
(2) $z_n \leqslant x \vee (y_{n-1} \wedge z_{n-1})$;
(3) $y_n \wedge z_n \leqslant y_{n-1} \wedge z_{n-1}$.

The first two inequalities are simply $y_n \leqslant x_n$ and $z_n \leqslant x_n$, which are both ruled out by the remark immediately following the proof of Claim 3.3, and $y_n \wedge z_n \leqslant y_{n-1} \wedge z_{n-1}$ is ruled out by the induction hypothesis. By reaching this contradiction we have established that $x_{n+1} > x_n$.

It remains to prove $x_{n+1} \wedge y_{n+1} > x_n \wedge y_n$ under the assumption that all of the strict inequalities in (3.2) hold when $i = n$. Since we have already established that $x_{n+1} > x_n$ and (by an identical argument) that $y_{n+1} > y_n$, it's clear that $x_{n+1} \wedge y_{n+1} \geqslant x_n \wedge y_n$, so we just have to rule out equality. Suppose on the contrary that

$$\begin{aligned} x_n \wedge y_n &:= \big[x \vee (y_{n-1} \wedge z_{n-1})\big] \wedge \big[y \vee (x_{n-1} \wedge z_{n-1})\big] \\ &= \big[x \vee (y_n \wedge z_n)\big] \wedge \big[y \vee (x_n \wedge z_n)\big] =: x_{n+1} \wedge y_{n+1}. \end{aligned}$$

Then we have both $x_{n+1} \wedge y_{n+1} \leqslant x_n$ and $x_{n+1} \wedge y_{n+1} \leqslant y_n$. Equivalently,

$$\begin{aligned} \big[x \vee (y_n \wedge z_n)\big] \wedge \big[y \vee (x_n \wedge z_n)\big] &\leqslant x \vee (y_{n-1} \wedge z_{n-1}) \\ \big[x \vee (y_n \wedge z_n)\big] \wedge \big[y \vee (x_n \wedge z_n)\big] &\leqslant y \vee (x_{n-1} \wedge z_{n-1}). \end{aligned} \tag{3.4}$$

Again, by (W+), for the first of these inequalities to hold, we must have one of the following:

(1) $x \vee (y_n \wedge z_n) \leqslant x \vee (y_{n-1} \wedge z_{n-1})$;
(2) $y \vee (x_n \wedge z_n) \leqslant x \vee (y_{n-1} \wedge z_{n-1})$;
(3) $\big[x \vee (y_n \wedge z_n)\big] \wedge \big[y \vee (x_n \wedge z_n)\big] \leqslant y_{n-1} \wedge z_{n-1}$.

The first of these is equivalent to $x_{n+1} \leqslant x_n$, and the second to $y_{n+1} \leqslant x_n$; we have already ruled out both of these cases. Thus the only remaining possibility is the third, which can be restated as $x_{n+1} \wedge y_{n+1} \leqslant y_{n-1} \wedge z_{n-1}$. Let's assume this holds.

In order for the second of the inequalities in (3.4) to hold, according to (W+), one of the following must also be true:

(1) $x \vee (y_n \wedge z_n) \leqslant y \vee (x_{n-1} \wedge z_{n-1})$;
(2) $y \vee (x_n \wedge z_n) \leqslant y \vee (x_{n-1} \wedge z_{n-1})$;
(3) $\big[x \vee (y_n \wedge z_n)\big] \wedge \big[y \vee (x_n \wedge z_n)\big] \leqslant x_{n-1} \wedge z_{n-1}$.

Again, the first two—$x_{n+1} \leqslant y_n$ and $y_{n+1} \leqslant y_n$—have already been ruled out. So the only possibility is $x_{n+1} \wedge y_{n+1} \leqslant x_{n-1} \wedge z_{n-1}$. Let us assume we are in this case. Taken all together, if $x_n \wedge y_n$ were equal to $x_{n+1} \wedge y_{n+1}$, then we would have,

$$x_n \wedge y_n = x_{n+1} \wedge y_{n+1} \leqslant x_{n-1} \wedge y_{n-1} \wedge z_{n-1}.$$

In particular $x_n \wedge y_n \leqslant x_{n-1} \wedge y_{n-1}$. But this contradicts the inductive hypothesis.

To complete the proof of Lemma 3.2, we must show that $m_{k+1} > m_k$ for all $k \geqslant 0$. Recall,

$$\begin{aligned} m_0 &= (x \wedge y) \vee (x \wedge z) \vee (y \wedge z), \text{ and} \\ m_1 &= (x_1 \wedge y_1) \vee (x_1 \wedge z_1) \vee (y_1 \wedge z_1). \end{aligned}$$

Suppose $m_1 \leqslant m_0$. Then, in particular, we would have

$$x_1 \wedge y_1 \leqslant (x \wedge y) \vee (x \wedge z) \vee (y \wedge z).$$

Then, by Whitman's Theorem, one of the following must hold:

(1) $x_1 \leqslant (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$,
(2) $y_1 \leqslant (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$,
(3) $x_1 \wedge y_1 \leqslant x \wedge y$,
(4) $x_1 \wedge y_1 \leqslant x \wedge z$,
(5) $x_1 \wedge y_1 \leqslant y \wedge z$.

Expanding the first of these, we would have

$$x_1 = x \vee (y \wedge z) \leqslant (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = m_0$$

in this case. This is easily ruled out since x is join prime, so it cannot lie below m_0. Similarly, the second case is ruled out because y does not lie below m_0. The third case is ruled out by what we showed earlier in the proof. The fourth case is $x_1 \wedge y_1 \leqslant x \wedge z$, which implies $x_1 \wedge y_1 \leqslant x$. Since x is meet prime, this puts either x_1 or y_1 below x, which, as we noted above, is impossible. The fifth case is $x_1 \wedge y_1 \leqslant y \wedge z$, which implies $x_1 \wedge y_1 \leqslant y$ and, again, this is impossible since y is meet prime. Thus we have proved $m_1 > m_0$.

Finally, fix $k > 0$ and assume $m_k > m_{k-1}$. We show $m_{k+1} > m_k$. It's clear that $m_{k+1} \geqslant m_k$, so as usual we must rule out equality. Suppose $m_{k+1} \leqslant m_k$; that is,

$$(x_{k+1} \wedge y_{k+1}) \vee (x_{k+1} \wedge z_{k+1}) \vee (y_{k+1} \wedge z_{k+1}) \leqslant (x_k \wedge y_k) \vee (x_k \wedge z_k) \vee (y_k \wedge z_k).$$

Then, in particular, $x_{k+1} \wedge y_{k+1} \leqslant (x_k \wedge y_k) \vee (x_k \wedge z_k) \vee (y_k \wedge z_k)$. Whitman's Theorem then implies that one of the following must hold:

(1) $x_{k+1} \leqslant (x_k \wedge y_k) \vee (x_k \wedge z_k) \vee (y_k \wedge z_k)$;
(2) $y_{k+1} \leqslant (x_k \wedge y_k) \vee (x_k \wedge z_k) \vee (y_k \wedge z_k)$;
(3) $x_{k+1} \wedge y_{k+1} \leqslant x_k \wedge y_k$;
(4) $x_{k+1} \wedge y_{k+1} \leqslant x_k \wedge z_k$;
(5) $x_{k+1} \wedge y_{k+1} \leqslant y_k \wedge z_k$.

We rule out the first three cases by the same argument used above. Suppose (4) holds. Then, in particular $x_{k+1} \wedge y_{k+1} \leqslant z \vee (x_{k-1} \wedge y_{k-1})$. By (W+) again, one of the following must hold:

(1) $x_{k+1} \wedge y_{k+1} \leqslant x_{k-1} \wedge y_{k-1}$;
(2) $x_{k+1} \leqslant z \vee (x_{k-1} \wedge y_{k-1})$;
(3) $y_{k+1} \leqslant z \vee (x_{k-1} \wedge y_{k-1})$;

The first two items are impossible since we already proved that for all $n \geqslant 0$, $x_{n+1} \wedge y_{n+1} > x_n \wedge y_n$ and $x_{n+1} \nleqslant z_n$ (cf. the observation immediately after Claim 3.3). Clearly the same argument can be used to rule out Case (5). This completes the proof. □

Proof of Proposition 3.1. As above, define

$$\begin{aligned} x_1 &= x \vee (y \wedge z), & a &= 0 \vee (1 \wedge 2),\\ y_1 &= y \vee (x \wedge z), & b &= 1 \vee (0 \wedge 2),\\ z_1 &= z \vee (x \wedge y), & c &= 2 \vee (0 \wedge 1),\\ m_1 &= (x_1 \wedge y_1) \vee (x_1 \wedge z_1) \vee (y_1 \wedge z_1), \\ d &= (a \wedge b) \vee (a \wedge c) \vee (b \wedge c). \end{aligned}$$

so that $h(x_1) = a$, $h(y_1) = b$, $h(z_1) = c$, and $h(m_1) = d$.

Actually, by modularity, since $a \wedge b \leqslant a$, we have

$$(a \wedge b) \vee (a \wedge c) = a \wedge ((a \wedge b) \vee c).$$

In fact, we claim that $a \wedge b \leqslant c$, so the last equation becomes

$$(a \wedge b) \vee (a \wedge c) = a \wedge c.$$

Therefore, $d = (a \wedge c) \vee (b \wedge c)$. Similarly, since $c \geqslant b \wedge c$, modularity implies

$$d = (a \vee (b \wedge c)) \wedge c.$$

If we also have $b \wedge c \leqslant a$, then this reduces further to $d = a \wedge c$.

We now prove that, indeed, $a \wedge b \leqslant c$, and from this follows both $b \wedge c \leqslant a$ and $a \wedge c \leqslant b$, by symmetry. Recall that $a \wedge b = \big[0 \vee (1 \wedge 2)\big] \wedge \big[1 \vee (0 \wedge 2)\big]$. Since $1 \wedge 2 \leqslant 1 \vee (0 \wedge 2)$, modularity implies $a \wedge b = \big\{0 \wedge \big[1 \vee (0 \wedge 2)\big]\big\} \vee (1 \wedge 2)$, and since $0 \geqslant 0 \wedge 2$, modularity also implies

$$a \wedge b = (0 \wedge 1) \vee (0 \wedge 2) \vee (1 \wedge 2).$$

Each joinand in the last expression is below c, hence $a \wedge b \leqslant c$, as claimed. From what we just proved, not only do we have $d = a \wedge c$, but also $d = a \wedge b = a \wedge c = b \wedge c$.

Now, suppose $K \subseteq \mathbf{F} \times \mathbf{F}$ is a finite set. We wish to prove $\langle K \rangle \neq \ker h$. First, we show there exists $N < \omega$ such that for all $(p, q) \in \langle K \rangle$ the following implications hold:

$$p \leqslant x_1 \implies q \leqslant x_N; \tag{3.5}$$

$$p \leqslant y_1 \implies q \leqslant y_N; \tag{3.6}$$

$$p \leqslant z_1 \implies q \leqslant z_N; \tag{3.7}$$

$$p \leqslant m_1 \implies q \leqslant m_N. \tag{3.8}$$

As we proved above, for each $w \in \{x, y, z, m\}$, the sequence $\{w_n\}$ is an infinite ascending chain, so we can certainly find, for each of the four implications above, a number $N_w < \omega$ such that the given implication holds for $N = N_w$ and for all of the (finitely many) pairs in K. Therefore, all four implications will hold for $(p, q) \in K$ if we take $N = \max\{N_a, N_b, N_c, N_d\}$. In fact, we will now see that the same N works for all $(p, q) \in \langle K \rangle$.

Claim 3.4. The implications (3.5)–(3.8) hold for all $(p, q) \in \langle K \rangle$.

Proof. Fix $(u, v) \in \langle K \rangle$. Obviously, if we manage to prove that the implication (3.5) is satisfied when $(p, q) = (u, v)$, then the same argument will prove (u, v) also satisfies implications (3.6) and (3.7). So we need only establish the implications (3.5) and (3.8) when $(p, q) = (u, v)$.

In case (u, v) belongs to the finite set K, all four implications were already established in the paragraph immediately preceding the statement of Claim 3.4.

Suppose $(u, v) = (p_1, q_1) \wedge (p_2, q_2)$, and assume (the induction hypothesis) that implications (3.5)–(3.8) are satisfied for $(p, q) = (p_i, q_i)$ $i \in \{1, 2\}$. We now prove that, under these assumptions, (u, v) satisfies implications (3.5) and (3.8).

Assume $u = p_1 \wedge p_2 \leqslant x_1$; we must show $v = q_1 \wedge q_2 \leqslant x_N$. The relation $p_1 \wedge p_2 \leqslant x \vee (y \wedge z)$, along with Whitman's Condition (W+), implies that one of the following holds:

(1) $p_1 = x$;
(2) $p_2 = x$;
(3) $p_1 \leqslant x_1$;
(4) $p_2 \leqslant x_1$;
(5) $p_1 \wedge p_2 \leqslant y \wedge z$;

In the first case we have $p_1 = x \leqslant x \vee (y \wedge z) = x_1$, so the induction hypothesis implies $q_1 \leqslant x_N$, so $v = q_1 \wedge q_2 \leqslant x_N$, as desired. Notice that the second, third, and fourth items in the list submit to the same argument, so we are left with the fifth case, in which $p_1 \wedge p_2 \leqslant y \wedge z$. This implies $p_1 \wedge p_2 \leqslant y$, and y is meet prime, so $p_i \leqslant y$ for some $i \in \{1, 2\}$. But then, $p_i \leqslant y_1 = y \vee (x \wedge z)$, so by the induction hypothesis, $q_i \leqslant y_N$. Similarly, $p_1 \wedge p_2 \leqslant z$ implies $p_j \leqslant z$ for some $j \in \{1, 2\}$, so $q_j \leqslant z_N$. This yields the desired result,

$$v = q_1 \wedge q_2 \leqslant y_N \wedge z_N \leqslant x \vee (y_N \wedge z_N) = x_N.$$

Next, suppose $u = p_1 \vee p_2 \leqslant x_1$; we show $v = q_1 \vee q_2 \leqslant x_N$. The relation $p_1 \vee p_2 \leqslant x_1$ implies $p_1 \leqslant x_1$ and $p_2 \leqslant x_1$, so by the induction hypothesis we have $q_1 \leqslant x_N$ and $q_2 \leqslant x_N$. Therefore, $v = q_1 \vee q_2 \leqslant x_N$, as desired. We have thus shown that (u, v) satisfies the implication (3.5).

Now we turn our attention to implication (3.8). Supposing $u \leqslant m_1$, we must show $v \leqslant m_N$.

Assume $u = p_1 \wedge p_2$ and $v = q_1 \wedge q_2$, where the pairs (p_i, q_i) satisfy implications (3.5)–(3.8). Since $p_1 \wedge p_2 \leqslant m_1 = (x_1 \wedge y_1) \vee (x_1 \wedge z_1) \vee (y_1 \wedge z_1)$, condition (W+) implies that one of the following holds:

(1) $p_1 \leqslant m_1$;
(2) $p_2 \leqslant m_1$;
(3) $p_1 \wedge p_2 \leqslant x_1 \wedge y_1$;
(4) $p_1 \wedge p_2 \leqslant x_1 \wedge z_1$;
(5) $p_1 \wedge p_2 \leqslant y_1 \wedge z_1$.

Cases (1) and (2) are easily handled. Indeed, if there is an i in $\{1, 2\}$ for which $p_i \leqslant m_1$, then $q_i \leqslant m_N$ by the induction hypothesis; therefore, $v = q_1 \wedge q_2 \leqslant q_i \leqslant m_N$, which is the desired result.

It suffices to handle Case (3), since Cases (4) and (5) are analogous. In Case (3), we have $p_1 \wedge p_2 \leqslant x_1 \wedge y_1$. In particular, both of the following hold:

$$p_1 \wedge p_2 \leqslant x_1 \text{ and} \tag{3.9}$$

$$p_1 \wedge p_2 \leqslant y_1. \tag{3.10}$$

Let's first consider $p_1 \wedge p_2 \leqslant x_1 = x \vee (y \wedge z)$. This and Whitman's condition (W+) imply that one of the following must hold:

(a) $p_1 = x$;
(b) $p_2 = x$;
(c) $p_1 \leqslant x_1$;

(d) $p_2 \leqslant x_1$;
(e) $p_1 \wedge p_2 \leqslant y \wedge z$.

Case (e) can be handled without appealing to constraint (3.10), so we dispense with that case first. If $p_1 \wedge p_2 \leqslant y \wedge z$, then $p_1 \wedge p_2 \leqslant y$ and $p_1 \wedge p_2 \leqslant z$. Since y and z are both meet prime, we have, for some $i, j \in \{1, 2\}$, $p_i \leqslant y$ and $p_j \leqslant z$. But then $p_i \leqslant y \vee (x \wedge z) = y_1$, so the induction hypothesis implies that $q_i \leqslant y_N$. Similarly, $p_j \leqslant z \leqslant z \vee (x \wedge y) = z_1$ implies $q_j \leqslant z_N$. It follows that $v = q_1 \wedge q_2 \leqslant y_N \wedge z_N \leqslant m_N$, as desired.

Next, we turn to Case (a). In this case we have $p_1 = x \leqslant x \vee (y \wedge z) = x_1$, so once we prove the desired result in Case (c), Case (a) will follow. Of course, the same comments apply to Cases (b) and (d), by symmetry.

In Case (c), we have $p_1 \leqslant x_1$, so the induction hypothesis implies $q_1 \leqslant x_N$. To conclude that $v \leqslant m_N$ in this case, we need to exploit condition (3.10); that is, $p_1 \wedge p_2 \leqslant y_1 = y \vee (x \wedge z)$. From this and (W+), one of the following holds:

(i) $p_1 = y$;
(ii) $p_2 = y$;
(iii) $p_1 \leqslant y_1$;
(iv) $p_2 \leqslant y_1$;
(v) $p_1 \wedge p_2 \leqslant x \wedge z$.

Again, the first two of these imply the third and fourth, respectively. So we show how to handle the third and fifth cases in this list. Keep in mind, we have already shown $q_1 \leqslant x_N$ at this point.

In the third case, we have $p_1 \leqslant y_1 = y \vee (x \wedge z)$. So the induction hypothesis implies $q_1 \leqslant y_N$. This and $q_1 \leqslant x_N$ implies $q_1 \leqslant x_N \wedge y_N \leqslant m_N$. Therefore, $v = q_1 \wedge q_2 \leqslant m_N$, as desired.

Assume we are in the fifth case, that is $p_1 \wedge p_2 \leqslant x \wedge z$. Then $p_1 \wedge p_2 \leqslant x$ and $p_1 \wedge p_2 \leqslant z$. Since x and z are both meet prime, we have $p_i \leqslant x \leqslant x_1$ and $p_j \leqslant z \leqslant z_1$ for some $i, j \in \{1, 2\}$. By the induction hypothesis, then, $q_i \leqslant x_N$ and $q_j \leqslant z_N$, whence, $v = q_1 \wedge q_2 \leqslant x_N \wedge z_N \leqslant m_N$, as desired.

To complete the proof, we must consider the case in which $u = p_1 \vee p_2$ and $v = q_1 \vee q_2$, where $u \leqslant m_N$ and where (p_i, q_i) satisfy the same induction hypotheses as above. We must prove that $v \leqslant m_N$ under these assumptions. If $p_1 \vee p_2 \leqslant m_1$, then $p_1 \leqslant m_1$, so $q_1 \leqslant m_N$, but the induction hypothesis. Similarly, $p_2 \leqslant m_1$, so $q_2 \leqslant m_N$. Therefore, $v = q_1 \vee q_2 \leqslant m_N$, and the proof of Claim 3.4 is finally complete. □

It follows Claim 3.4 that $\langle K \rangle \neq \ker h$. Indeed, since $\{m_n\}$ is a strictly increasing sequence, $m_N < m_{N+1}$, so the pair $(p, q) = (m_1, m_{N+1})$ belongs to $\ker h$, as $h(m_1) = d = h(m_{N+1})$, but does not satisfy condition (3.8). On the other hand, we just proved that all pairs in $\langle K \rangle$ satisfy (3.5)–(3.8) when K is a finite set. The proof of Proposition 3.1 is now complete. □

Appendix A. Proof of Conjecture under special assumptions

As usual, let X be a finite set and let $\mathbf{F} := \mathbf{F}(X)$ be the free lattice generated by X.

Proposition A.1. *Suppose $\mathbf{L} = \langle L, \wedge, \vee\rangle$ is a finite lattice and $h\colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ a lattice epimorphism. Suppose also that there is a class of* $\ker h$ *containing an infinite descending chain as well as a join prime element of* $\mathbf{F}$. *Then the kernel of h is not a finitely generated sublattice of* $\mathbf{F} \times \mathbf{F}$.

Proof. Let $y_0 \in L$ and suppose $x_0 \in h^{-1}\{y_0\}$ is a join prime element of $\mathbf{F}$. Suppose also that the class $h^{-1}\{y_0\}$ contains an infinite descending chain, $\beta_0(y_0) > \beta_1(y_0) > \cdots$. Let K be a finite subset of $\ker h$, say, $K = \{(p_1, q_1), \dots, (p_m, q_m)\} \subseteq \ker h$. We prove $\langle K\rangle \neq \ker h$. Since K is an arbitrary finite subset of $\ker h$, this will prove that $\ker h$ is not finitely generated.

Claim A.2. There exists $N < \omega$ such that for all (p_i, q_i) in K, if $p_i \geqslant x_0$, then $q_i \geqslant \beta_N(y_0)$.

Proof. Fix i and $(p_i, q_i) \in K$ (so, $h(p_i) = h(q_i)$). Define N_i according to which of the following cases holds:

(1) If $p_i \ngeqslant x_0$, let $N_i = 0$.
(2) If $p_i \geqslant x_0$, then $x_0 = x_0 \wedge p_i$, so $y_0 = h(x_0) = h(x_0) \wedge h(p_i) \leqslant h(p_i)$, so $y_0 \leqslant h(q_i)$. Also, $h(x_0 \wedge q_i) = h(x_0) \wedge h(q_i) = y_0$, so $x_0 \wedge q_i \in h^{-1}\{y_0\}$. Therefore (since $\{\beta_i(y_0)\}$ is an infinite descending chain), there exists $n_i > 0$ such that $x_0 \wedge q_i \geqslant \beta_{n_i}(y_0)$. Let $N_i = n_i$ in this case (so $q_i \geqslant \beta_{N_i}(y_0)$).

After defining N_i this way for each pair $(p_i, q_i) \in K$, let $N := \max_i N_i$. Then the desired implication holds for all $1 \leqslant i \leqslant m$, that is,

$$p_i \geqslant x_0 \quad \Longrightarrow \quad q_i \geqslant \beta_N(y_0), \tag{A.1}$$

so Claim A.2 is proved. ☐

Claim A.3. There exists $N < \omega$ such that, for all $(p, q) \in \langle K\rangle$,

$$p \geqslant x_0 \quad \Longrightarrow \quad q \geqslant \beta_N(y_0). \tag{A.2}$$

Proof. Define N as in the proof of Claim A.2, so that for all $(p_i, q_i) \in K$ the implication (A.1) holds. Fix $(p, q) \in \langle K\rangle$. We prove (A.2) by induction on the complexity of (p, q). If $(p, q) \in K$, then there's nothing to prove. We split the induction step into two cases.

(1) Assume $(p, q) = (p_1, q_1) \wedge (p_2, q_2)$, where p_i, q_i $(i = 1, 2)$ satisfy (A.2). Assume $p \geqslant x_0$. Then $p = p_1 \wedge p_2 \geqslant x_0$, so $p_1 \geqslant x_0$ and $p_2 \geqslant x_0$, so (by the induction hypothesis) $q_1 \geqslant \beta_N(y_0)$ and $q_2 \geqslant \beta_N(y_0)$. Therefore, $q = q_1 \wedge q_2 \geqslant \beta_N(y_0)$, as desired.
(2) Assume $(p, q) = (p_1, q_1) \vee (p_2, q_2)$, where p_i, q_i $(i = 1, 2)$ satisfy (A.2). Assume $p \geqslant x_0$. Then $p = p_1 \vee p_2 \geqslant x_0$. Since x_0 is join prime, either $p_1 \geqslant x_0$ or $p_2 \geqslant x_0$. Assume (wlog) $p_1 \geqslant x_0$. Then, by the induction hypothesis, $q_1 \geqslant \beta_N(y_0)$. Therefore, $q = q_1 \vee q_2 \geqslant q_1 \geqslant \beta_N(y_0)$, as desired.

This completes the proof of Claim A.3. □

We can now see that K does not generate $\ker h$. Indeed, if N is as in the proof of Claim A.3, then $\beta_N(y_0) > \beta_{N+1}(y_0)$, so the pair $(p, q) = (x_0, \beta_{N+1}(y_0))$ does not satisfy condition A.2, so does not belong to $\langle K \rangle$. Yet, $(x_0, \beta_{N+1}(y_0)) \in \ker h$. Thus, K does not generate $\ker h$. □

Appendix B. Examples

Let $\mathbf{M_3} = \langle \{0, a, b, c, 1\}, \wedge, \vee \rangle$, where $a \wedge b = a \wedge c = b \wedge c = 0$ and $a \vee b = a \vee c = b \vee c = 1$. Let $\mathbf{F} := \mathbf{F}(x, y, z)$ denote the free lattice generated by $\{x, y, z\}$.

Proposition B.1. *Let $h \colon \mathbf{F} \twoheadrightarrow \mathbf{M_3}$ be the epimorphism that acts on the generators as follows: $x \mapsto a$, $y \mapsto b$, $z \mapsto c$. Then $\ker h$ is not finitely generated.*

Proof. Let $K := \ker h$, and for $u \in \{x, y, z\}$ let $C_u := u/K := \{v \in F : h(v) = h(u)\}$. Define sequences of elements in these classes as follows: let

$$x_0 := x, \quad y_0 := y, \quad z_0 := z, \qquad \text{and for } i < \omega,$$
$$x_{i+1} := x \vee (y_i \wedge z_i), \quad y_{i+1} := y \vee (x_i \wedge z_i), \quad z_{i+1} := z \vee (x_i \wedge y_i),$$
$$m_i := (x_i \wedge y_i) \vee (x_i \wedge z_i) \vee (y_i \wedge z_i).$$

Summarizing these observations,

$$\begin{aligned}
m_0 &= (x \wedge y) \vee (x \wedge z) \vee (y \wedge z),\\
x_1 &= x \vee (y \wedge z), \quad y_1 = y \vee (x \wedge z), \quad z_1 = z \vee (x \wedge y),\\
m_1 &= (x_1 \wedge y_1) \vee (x_1 \wedge z_1) \vee (y_1 \wedge z_1),\\
x_2 &= x \vee \big\{ \big[y \vee (x \wedge z)\big] \wedge \big[z \vee (x \wedge y)\big] \big\},\\
y_2 &= y \vee \big\{ \big[x \vee (y \wedge z)\big] \wedge \big[z \vee (x \wedge y)\big] \big\},\\
z_3 &= z \vee (x_2 \wedge y_2)\\
&\ \ \vdots
\end{aligned}$$

Let X be a finite subset of K. We will prove there exists $(p, q) \in K \setminus \langle X \rangle$. Fix $u \in \{x, y, z, m\}$ and let $\{u_i\}$ be the corresponding sequence defined above. Since X is finite, Lemma B.3 implies that there exists $M \in \mathbb{N}$ such that for every $(p, q) \in X$ with $p, q \in C_u$, we have $p, q \leqslant u_M$.

Subclaim 1. For $(p, q) \in \langle X \rangle$ and $u \in \{x, y, z\}$, the following implication holds:

$$q \leqslant u \quad \Longrightarrow \quad p \leqslant u_M. \tag{B.1}$$

We prove the subclaim by induction on the complexity of terms. Fix $(p, q) \in \langle X \rangle$. Then $p, q \in C_u$ for some $u \in \{x, y, z\}$.

- **Case 0.** Suppose $(p, q) \in X$. Then by definition of M we have $p, q \leqslant u_M$.

- **Case 1.** Suppose $(p, q) = (p_1, q_1) \wedge (p_2, q_2)$, where (p_i, q_i) satisfies (B.1) for $i = 1, 2$. If $q = q_1 \wedge q_2 \leqslant u$, then, since generators in the free lattice are meet-prime (see Theorem C.8 below), we have $q_1 \leqslant u$ or $q_2 \leqslant u$. Assume $q_1 \leqslant u$. Then, by the induction hypothesis, $p_1 \leqslant u_M$. Therefore, $p = p_1 \wedge p_2 \leqslant u_M$, as desired.
- **Case 2.** Suppose $(p, q) = (p_1, q_1) \vee (p_2, q_2)$, where (p_i, q_i) satisfies (B.1) for $i = 1, 2$. If $q = q_1 \vee q_2 \leqslant u$, then $q_i \leqslant u$ for $i = 1, 2$. It now follows from the induction hypothesis that $p_i \leqslant u_M$ for $i = 1, 2$, so $p = p_1 \vee p_2 \leqslant u_M$, as desired.

This completes the proof of Subclaim 1. It now follows from Lemma B.2 that $(x, x_{M+1}) \in K \setminus \langle X \rangle$, so proposition is proved. ☐

Lemma B.2. *For each $u \in \{x, y, z, m\}$, the sequence $\{u_i\}$ is a strictly ascending chain; that is, $u_0 < u_1 < u_2 < \cdots$.*

Proof. We split the proof up into cases: either $u \in \{x, y, z\}$, or $u = m$.

- **Case 1.** $u \in \{x, y, z\}$.
  For simplicity, assume $u = x$ for the remainder of the proof of this case. (Of course, the same argument applies to the case when u is y or z.) Fix $n < \omega$. We prove $x_n < x_{n+1}$.

  Subclaim 2. For all $n < \omega$,
  (1) $x_n \in C_x$,
  (2) $x_n \ngeqslant y$, and $x_n \ngeqslant z$.

  Proof of Subclaim 2. The first item is obvious; for the second, if $x_n \geqslant y$, then $x_n \wedge y = y$, and then $0 = h(x_n \wedge y) = h(y) = b$. A similar contradiction is reached if we assume $x_n \geqslant z$, so the subclaim is proved.

  Recall, $x_{n+1} = x_n \vee (y_n \wedge z_n)$, so $x_{n+1} > x_n$ holds as long as $x_n \ngeqslant y_n \wedge z_n$. So, by way of contradiction, suppose
  $$x_n \geqslant y_n \wedge z_n. \tag{B.2}$$
  Now, $y_n = y \vee (x \wedge z) \vee \cdots$, so clearly $y_n \geqslant y$. Similarly, $z_n \geqslant z$. This, together with (B.2), implies $x_n \geqslant y_n \wedge z_n \geqslant y \wedge z$. But then Theorem C.8 below implies that either $x_n \geqslant y$ or $x_n \geqslant z$, which contradicts Subclaim 2.
- **Case 2.** $u = m$.
  We first prove that $m_0 = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ is strictly below $m_1 = (x_1 \wedge y_1) \vee (x_1 \wedge z_1) \vee (y_1 \wedge z_1)$. By symmetry, it suffices to show $x \wedge y < x_1 \wedge y_1$; that is, $x \wedge y < \big[x \vee (y \wedge z)\big] \wedge \big[y \vee (x \wedge z)\big]$.

  Clearly $x \wedge y \leqslant \big[x \vee (y \wedge z)\big] \wedge \big[y \vee (x \wedge z)\big]$. Suppose $x \wedge y = \big[x \vee (y \wedge z)\big] \wedge \big[y \vee (x \wedge z)\big]$. Then $\big[x \vee (y \wedge z)\big] \wedge \big[y \vee (x \wedge z)\big] \leqslant x$. By Theorem C.8, the latter holds iff $x \vee (y \wedge z) \leqslant x$ or $y \vee (x \wedge z) \leqslant x$ The first of these inequalities is clearly false, so it must be the case that $y \vee (x \wedge z) \leqslant x$. But then $y \leqslant x$, which is obviously false. We conclude that $x \wedge y < \big[x \vee (y \wedge z)\big] \wedge \big[y \vee (x \wedge z)\big]$. This proves $m_0 < m_1$.

wjd 2018-10-04: Complete the proof of this case.

Now fix $n < \omega$ and assume $m_n < m_{n+1}$. We show $m_{n+1} < m_{n+2}$.

↓ *begin scratch work* ↓
$m_n := (x_n \wedge y_n) \vee (x_n \wedge z_n) \vee (y_n \wedge z_n)$,
$m_{n+1} := (x_{n+1} \wedge y_{n+1}) \vee (x_{n+1} \wedge z_{n+1}) \vee (y_{n+1} \wedge z_{n+1})$,
By the first Case above, $u_n < u_{n+1}$.
↑ *end scratch work* ↑

□

Lemma B.3. *For all $u \in \{x, y, z\}$ and $p \in C_u \cup C_0$ there exists $n \in \mathbb{N}$ such that $p \leqslant m_{u,n}$.*

Proof. We prove this by induction on the complexity of p.

- **Case 0.** $p \in \{x, y, z\}$. Then $u = p = m_{p,0}$.
  For the remaining cases assume $u = x$, without loss of generality.
- **Case 1.** $p = p_1 \vee p_2$.
  If $p \in C_x \cup C_0$, then $p_i \in C_x \cup C_0$ for $i = 1, 2$, and the induction hypothesis yields i and j for which $p_1 \leqslant x_i$ and $p_2 \leqslant x_j$. Letting $n = \max\{i, j\}$, we have $p_1, p_2 \leqslant x_n$, from which $p = p_1 \vee p_2 \leqslant x_n$, as desired.
- **Case 2.** $p = p_1 \wedge p_2$.
  If $p \in C_x$, then we may assume $p_1 \in C_x$ and $p_2 \in C_x \cup C_0$. By the induction hypothesis, there exists $n \in \mathbb{N}$ such that $p_1 \leqslant x_n$, whence $p \leqslant p_1 \leqslant x_n$. If $p \in C_0$, then each p_i belongs to $C_u \cup C_0$ for some $u \in \{x, y, z\}$. If $p_1 \in C_x \cup C_0$, then $p_1 \leqslant x_n$, as above and we're done. Similarly, if $p_2 \in C_x \cup C_0$. So assume $p_1 \in C_y \cup C_0$ and $p_2 \in C_z \cup C_0$. Then the induction hypothesis implies that there exist i and j such that $p_1 \leqslant y_i$ and $p_2 \leqslant z_j$. If $n = \max\{i, j\}$, then $p_1 \leqslant y_n$ and $p_2 \leqslant z_n$. Then, by the above definition of the sequences, we have $p_1 \wedge p_2 \leqslant y_n \wedge z_n \leqslant m_n \leqslant x_{n+1}$.

□

B.1. **Other Examples.** In each of the propositions in this section, X is a finite set and $\mathbf{F} = \mathbf{F}(X)$ is the free lattice generated by X. The symbol F denotes the universe of $\mathbf{F}$. The proof in each case is straightforward, but tedious; we omit proofs of the first two, and give a detailed proof of the third.

Prop. B.4. *Let $X = \{x, y, z\}$, and let $\mathbf{L} = \mathbf{2}$ be the 2-element chain. Then the kernel of an epimorphism $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ is a finitely generated sublattice of $\mathbf{F} \times \mathbf{F}$.*

Prop. B.5. *Let $X = \{x, y, z\}$ and let $\mathbf{L} = \mathbf{3}$ be the 3-element chain. Then the kernel of an epimorphism $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ is finitely generated.*

Prop. B.6. *Let $n > 2$, $X = \{x_0, x_1, \dots, x_{n-1}\}$, and $\mathbf{L} = \mathbf{2} \times \mathbf{2}$. Let $h \colon \mathbf{F} \twoheadrightarrow \mathbf{L}$ be an epimorphism. Then $K = \ker h$ is finitely generated.*

Proof. Let the universe of $\mathbf{L} = \mathbf{2}\times\mathbf{2}$ be $\{0, a, b, 1\}$, where $a \vee b = 1$ and $a \wedge b = 0$. For each $y \in L$, denote by $X_y = X \cap h^{-1}\{y\}$ the set of generators mapped by

h to y. Denote the least and greatest elements of $h^{-1}\{y\}$ (if they exist) by ℓ_y and g_y, respectively. For example, $\ell_a = \bigwedge h^{-1}\{a\} = \bigwedge\{x \in F : h(x) = a\}$, $g_b = \bigvee\{x \in F : h(x) = b\}$, etc. In the present example, the least and greatest elements exist is each case, as we now show.

Subclaim 3. $h^{-1}\{a\}$ has least and greatest elements, namely $\ell_a = \bigwedge(X_a \cup X_1)$ and $g_a = \bigvee(X_a \cup X_0)$. (Similarly, $h^{-1}\{b\}$ has least and greatest elements, ℓ_b and g_b.)

Proof of Subclaim 3. Let $M(a) := \bigwedge(X_a \cup X_1)$ and $J(a) := \bigvee(X_a \cup X_0)$ and note that these values exist in F, since the sets involved are finite. Also, Then $h(M(a)) = a = h(J(a))$. Fix $r \in h^{-1}\{a\}$.

- If $r \in X_a$, then $r \geqslant \bigwedge X_a \geqslant \bigwedge(X_a \cup X_1) = M(a)$.
- If $r = s \vee t$, where $h(s) = a$ and $h(t) \in \{a, 0\}$, then assume (the induction hypothesis) that $s \geqslant M(a)$, and we have $r = s \vee t \geqslant M(a)$.
- If $r = s \wedge t$, where $h(s) = a$ and $h(t) \in \{a, 1\}$, then assume (the induction hypothesis) that $s, t \geqslant M(a)$, and we have $s \wedge t \geqslant M(a)$. This proves that for each $r \in h^{-1}\{a\}$ we have $r \geqslant M(a)$, and as we noted at the outset, $M(a) \in h^{-1}\{a\}$. Therefore, $\ell_a = M(a)$ is the least element of $h^{-1}\{a\}$. Similarly, every $r \in h^{-1}\{a\}$ is below $J(a)$, so $g_a = J(a)$. The proofs of $\ell_b = M(b)$ and $g_b = J(b)$ are similar.

This proves Subclaim 3.

Subclaim 4. $h^{-1}\{0\}$ has least and greatest elements, namely, $\ell_0 = \bigwedge X$ and $g_0 = g_a \wedge g_b$.

Proof of Subclaim 4. $\ell_0 = \bigwedge X$ is obvious, so we need only verify that $g_0 = g_a \wedge g_b$. Observe that $h(g_a \wedge g_b) = h(g_a) \wedge h(g_b) = a \wedge b = 0$, so $g_a \wedge g_b \in h^{-1}\{0\}$. It remains to prove that $r \leqslant g_a \wedge g_b$ holds for all $r \in h^{-1}\{0\}$. Fix $r \in h^{-1}\{0\}$. Then $h(r \vee g_a) = h(r) \vee h(g_a) = 0 \vee a = a$, which places $r \vee g_a$ in $h^{-1}\{a\}$. Therefore, by maximality of g_a, we have $r \vee g_a \leqslant g_a$, whence $r \leqslant g_a$. Similarly, $r \leqslant g_b$. This proves Subclaim 4.

Subclaim 5. $h^{-1}\{1\}$ has least and greatest elements, namely $\ell_1 = \ell_a \vee \ell_b$ and $g_1 = \bigvee X$.

Proof of Subclaim 5. $g_1 = \bigvee X$ is obvious, so we need only verify that $\ell_1 = \ell_a \vee \ell_b$. Observe that $h(\ell_a \vee \ell_b) = h(\ell_a) \vee h(\ell_b) = a \vee b = 1$, so $\ell_a \vee \ell_b \in h^{-1}\{1\}$. It remains to prove that $r \geqslant \ell_a \vee \ell_b$ holds for all $r \in h^{-1}\{1\}$. Fix $r \in h^{-1}\{1\}$. Then $h(r \wedge \ell_a) = h(r) \wedge h(\ell_a) = 1 \wedge a = a$, which places $r \wedge \ell_a$ in $h^{-1}\{a\}$. Therefore, by minimality of ℓ_a, we have $r \wedge \ell_a \geqslant \ell_a$, whence $r \geqslant \ell_a$. Similarly, $r \geqslant \ell_b$. Now let $Y = \{(x, g_p), (g_p, x), (x, \ell_p), (\ell_p, x) : p \in \{0, a, b, 1\}, x \in X_p\}$. This proves Subclaim 5.

Subclaim 6. If $r \in F$ and $h(r) = p$, then $(r, \ell_p), (r, g_p) \in \langle Y \rangle$.

Proof of Subclaim 6. Either $r \in X_p$ or $r = s \wedge t$ or $r = s \vee t$. If $r \in X_p$, then the pair belongs to Y and the claim is trivial.

Suppose $r = s \wedge t$.

- If $h(r) = 1$, then $h(s) = h(t) = 1$. Assume (the induction hypothesis) that $\{(s, \ell_1), (s, g_1), (t, \ell_1), (t, g_1)\} \subseteq \langle Y \rangle$. Then $(r, \ell_1) = (s \wedge t, \ell_1) = (s, \ell_1) \wedge (t, \ell_1) \in \langle Y \rangle$.
- If $h(r) = a$, then (wlog) $h(s) = a$ and $h(t) \in \{a, 1\}$. Assume (the induction hypothesis) that $\{(s, \ell_a), (s, g_a), (t, \ell_p), (t, g_p)\} \subseteq \langle Y \rangle$. By Claim 1, $\ell_a \leqslant \ell_1$, so $\ell_a = \ell_a \wedge \ell_1$.
  If $h(t) = 1$, then
  $$(r, \ell_a) = (s \wedge t, \ell_a \wedge \ell_1) = (s, \ell_a) \wedge (t, \ell_1) \in \langle Y \rangle,$$
  while If $h(t) = a$, then $(r, \ell_a) = (s \wedge t, \ell_a \wedge \ell_a) = (s, \ell_a) \wedge (t, \ell_a) \in \langle Y \rangle$.
- If $h(r) = 0$, then (wlog) that either (i) $h(s) = 0$, or (ii) $h(s) = a$, $h(t) = b$. If $h(s) = 0$, then $(s, \ell_0) \in \langle Y \rangle$ implies $(r, \ell_0) = (s \wedge t, \ell_0) = (s, \ell_0) \wedge (t, \ell_p) \in \langle Y \rangle$. If If $h(s) = a$, $h(t) = b$, and $(s, \ell_a), (t, \ell_b) \in \langle Y \rangle$, then $(r, \ell_0) = (s \wedge t, \ell_0) = (s, \ell_a) \wedge (t, \ell_b) \in \langle Y \rangle$.

This proves that $(r, \ell_1) \in \langle Y \rangle$ if $r = s \wedge t$. A similar argument shows that $(r, g_p) \in \langle Y \rangle$ in each of the three subcases. We have thus proved that $\{(r, \ell_1), (r, g_p)\} \subseteq \langle Y \rangle$, if $r = s \wedge t$.

Suppose $r = s \vee t$.

- If $h(r) = 0$, then $h(s) = h(t) = 0$. Assume (the induction hypothesis) that $\{(s, \ell_p), (s, g_p), (t, \ell_p), (t, g_p)\} \subseteq \langle Y \rangle$. Then $(r, \ell_p) = (s \vee t, \ell_p) = (s, \ell_p) \vee (t, \ell_p) \in \langle Y \rangle$.
- If $h(r) = a$, then (wlog) $h(s) = a$ and $h(t) \in \{a, 0\}$. If we assume (the induction hypothesis) that $(s, \ell_p), (s, g_p), (t, \ell_p), (t, g_p)$ belong to $\langle Y \rangle$, then $(r, \ell_p) = (s \vee t, \ell_p) = (s, \ell_p) \vee (t, \ell_p) \in \langle Y \rangle$.
- If $h(r) = 1$, then (wlog) that either (i) $h(s) = 1$, or (ii) $h(s) = a$, $h(t) = b$.
  In the first case, $(s, \ell_1) \in \langle Y \rangle$ implies $(r, \ell_1) = (s \vee t, \ell_1) = (s, \ell_1) \vee (t, \ell_p) \in \langle Y \rangle$. In the second case $h(s) = a$, $h(t) = b$, and $(s, \ell_a), (t, \ell_b) \in \langle Y \rangle$. Then $(r, \ell_1) = (s \vee t, \ell_1) = (s, \ell_a) \vee (t, \ell_b) \in \langle Y \rangle$.

Similarly, in each of these three subcases, we have $(r, g_p) \in \langle Y \rangle$. This proves Subclaim 6, and completes the proof of Prop B.6. ☐

Appendix C. Background

The notation, definitions, ideas presented below are based on those we learned from the book by Freese, Jezek, Nation [FJN95],

Definition C.1 (length of a term)**.** Let X be a set. Each element of X is a term of length 1, also known as a **variable**. If $t_1, \dots, t_n$ are terms of lengths $k_1, \dots, k_n$, then $t_1 \vee \cdots \vee t_n$ and $t_1 \wedge \cdots \wedge t_n$ are both terms of length $1 + k_1 + \cdots + k_n$.

Examples. By the above definition, the terms

$$x \vee y \vee z \qquad x \vee (y \vee z) \qquad (x \vee y) \vee z$$

have lengths 4, 5, and 5, respectively. Reason: variables have length 1, so $x \vee y \vee z$ has length $1+1+1+1$. On the other hand, $x \vee y$ is a term of length 3, so $(x \vee y) \vee z$ has length $1+3+1$. Similarly, $x \vee (y \vee z)$ has length $1+1+3$.

Lemma C.2 ([FJN95, Lem. 1.2]). *Let $\mathcal{V}$ be a nontrivial variety of lattices and let $\mathbf{F}_{\mathcal{V}}(X)$ be the relatively free lattice in $\mathcal{V}$ over X. Then,*

$$\bigwedge S \leqslant \bigvee T \text{ implies } S \cap T \neq \emptyset \text{ for each pair of finite subsets } S, T \subseteq X. \tag{C.1}$$

Lemma C.3 ([FJN95, Lem. 1.4]). *Let $\mathbf{L}$ be a lattice generated by a set X and let $a \in L$.*

(1) If a is join prime, then $a = \bigwedge S$ for some finite subset $S \subseteq X$.
(2) If a is meet prime, then $a = \bigvee S$ for some finite subset $S \subseteq X$.
If X satisfies condition (C.1) above, then
(3) for every finite, nonempty subset $S \subset X$, $\bigwedge S$ is join prime and $\bigvee S$ is meet prime.

Corollary C.4 ([FJN95, Cor. 1.5]). Let $\mathcal{V}$ be a nontrivial variety of lattices and let $\mathbf{F}_{\mathcal{V}}(X)$ be the relatively free lattice in $\mathcal{V}$ over X. For each finite nonempty subset $S \subseteq X$, $\bigwedge S$ is join prime and $\bigvee S$ is meet prime. In particular, every $x \in X$ is both join and meet prime. Moreover, if $x \leqslant y$ for $x, y \in X$, then $x = y$.

Theorem C.5 (Whitman's Condition, ver. 1). *The free lattice $\mathbf{F}(X)$ satisfies the following condition:*

(W) If $v = v_1 \wedge \cdots \wedge v_r \leqslant u_1 \vee \cdots \vee u_s = u$, then either $v_i \leqslant u$ for some i, or $v \leqslant u_j$ for some j.

Corollary C.6 ([FJN95, Cor. 1.9]). Every sublattice of a free lattice satisfies (W). Every element of a lattice satisfying (W) is either join or meet irreducible.

Theorem C.7 (Whitman's Condition, ver. 2). *The free lattice $\mathbf{F}(X)$ satisfies the following condition:*

(W+) If $v = v_1 \wedge \cdots \wedge v_r \wedge x_1 \wedge \cdots \wedge x_n \leqslant u_1 \vee \cdots \vee u_s \vee y_1 \vee \cdots \vee y_m = u$, where $x_i, y_j \in X$, then either $x_i = y_j$ for some i and j, or $v_i \leqslant u$ for some i, or $v \leqslant u_j$ for some j.

Theorem C.8 ([FJN95, Thm. 1.11]). *If $s = s(x_1, \ldots, x_n)$ and $t = t(x_1, \ldots, x_n)$ are terms and $x_1, \ldots, x_n \in X$, then the truth of*

$$s^{\mathbf{F}(X)} \leqslant t^{\mathbf{F}(X)} \tag{C.2}$$

can be determined by applying the following rules.

(1) If $s = x_i$ and $t = x_j$, then (C.2) holds iff $x_i = x_j$.
(2) If $s = s_1 \vee \cdots \vee s_k$ is a formal join, then (C.2) holds iff $s_i^{\mathbf{F}(X)} \leqslant t^{\mathbf{F}(X)}$ for all i.
(3) If $t = t_1 \wedge \cdots \wedge t_k$ is a formal meet, then (C.2) holds iff $s^{\mathbf{F}(X)} \leqslant t_i^{\mathbf{F}(X)}$ for all i.

(4) If $s = x_i$ and $t = t_1 \vee \cdots \vee t_k$ is a formal join, then (C.2) holds iff $x_i \leqslant t_j^{\mathbf{F}(X)}$ for some j.
(5) If $s = s_1 \wedge \cdots \wedge s_k$ is a formal meet and $t = x_i$, then (C.2) holds iff $s_j^{\mathbf{F}(X)} \leqslant x_i$ for some j.
(6) If $s = s_1 \wedge \cdots \wedge s_k$ is a formal meet and and $t = t_1 \vee \cdots \vee t_m$ is a formal join, then (C.2) holds iff $s_i^{\mathbf{F}(X)} \leqslant t^{\mathbf{F}(X)}$ for some i or $s^{\mathbf{F}(X)} \leqslant t_j^{\mathbf{F}(X)}$ for some j

Theorem C.9 ([FJN95, Thm. 1.17])**.** *For each $w \in \mathbf{F}(X)$ there is a term of minimal length representing w, unique up to commutativity. This term is called the* ***canonical form*** *of w.*

Let $w \in \mathbf{F}(X)$ be join reducible and suppose $t = t_1 \vee \cdots \vee t_n$ (with $n > 1$) is the canonical form of w. Let $w_i = t_i^{\mathbf{F}(X)}$. Then $\{w_1, \dots, w_n\}$ are called the **canonical joinands** of w. We also say $w = w_1 \vee \cdots \vee w_n$ *canonically* and that $w = w_1 \vee \cdots \vee w_n$ is the **canonical join representation** of w. If w is join irreducible, we define the canonical joinands of w to be the set $\{w\}$. Of course the **canonical meet representation** and **canonical meetands** of an element in a free lattice are defined dually.

A join representation $a = a_1 \vee \cdots \vee a_n$ in an arbitrary lattice is said to be a **minimal join representation** if $a = b_1 \vee \cdots \vee b_m$ and $\{b_1, \dots, b_m\} \ll \{a_1, \dots, a_n\}$ imply $\{a_1, \dots, a_n\} \subseteq \{b_1, \dots, b_m\}$. Equivalently, a join representation is minimal if it is an antichain and nonrefinable.

Theorem C.10 ([FJN95, Thm. 1.19])**.** *Let $w = w_1 \vee \cdots \vee w_n$ canonically in $\mathbf{F}(X)$. If also $w = u_1 \vee \cdots \vee u_m$, then $\{w_1, \dots, w_n\} \ll \{u_1, \dots, u_m\}$. Thus $w = w_1 \vee \cdots \vee w_n$ is the unique minimal join representation of w.*

Theorem C.11 ([FJN95, Thm. 1.20])**.** *Let $w \in \mathbf{F}(X)$ and let u be a join irreducible element in $\mathbf{F}(X)$. Then u is a canonical joinand of w if and only if there is an element a such that $w = u \vee a$ and $w > v \vee a$ for every $v < u$.*

Definition C.12 (up directed, continuous)**.** A subset A of a lattice L is said to be **up directed** if every finite subset of A has an upper bound in A. It suffices to check this for pairs. A is up directed iff for all $a, b \in A$ there exists $c \in A$ such that $a \leqslant c$ and $b \leqslant c$. A lattice is **upper continuous** if whenever $A \subseteq L$ is an up directed set having a least upper bound $u = \bigvee A$, then for every b,

$$\bigvee_{a \in A} (a \wedge b) = \bigvee_{a \in A} a \wedge b = u \wedge b.$$

Down directed and **down continuous** are defined dually. A lattice that is both up and down continuous is called **continuous**.

Theorem C.13 ([FJN95, Thm. 1.22])**.** *Free lattices are continuous.*

C.1. **Bounded Homomorphisms.** We continue to follow [FJN95] very closely, although the authors of that book indicate that the ideas in this subsection have their roots in Ralph McKenzie's work on nonmodular lattice varieties [McK72], and Bjarni Jónsson's work on sublattices of free lattices [JN77].

If x, y are elements of a lattice L, and if $x \leqslant y$, then we write $[\![x, y]\!]$ to denote the sublattice of elements between x and y. That is,

$$[\![x, y]\!] := \{z \in L \mid x \leqslant z \leqslant y\}.$$

Let $\mathbf{K}$ and $\mathbf{L}$ be lattices and suppose $\mathbf{L}$ has bottom and top elements, $0_{\mathbf{L}}$ and $1_{\mathbf{L}}$, resp. If $h \colon \mathbf{K} \to \mathbf{L}$ is a lattice homomorphism, then for each $a \in L$ we consider the sets $h^{-1}[\![a, 1]\!] = \{x \in K \mid h(x) \geqslant a\}$ and $h^{-1}[\![0, a]\!] = \{x \in K \mid h(x) \leqslant a\}$. When $h^{-1}[\![a, 1]\!]$ is nonempty, it is a filter of $\mathbf{K}$; dually a nonempty $h^{-1}[\![0, a]\!]$ is an ideal. If K is infinite, then $h^{-1}[\![a, 1]\!]$ need not have a least element, nor $h^{-1}[\![0, a]\!]$ a greatest element. However, considering when such extrema exist leads to the notion of bounded homomorphism, which in turn helps us understand the structure of free lattices.

A lattice homomorphism $h \colon \mathbf{K} \to \mathbf{L}$ is **lower bounded** if for every $a \in L$, the set $h^{-1}[\![a, 1]\!]$ is either empty or has a least element. The least element of a nonempty $h^{-1}[\![a, 1]\!]$ is denoted by $\beta_h(a)$, or by $\beta(a)$ when h is clear from context. Thus, if h is a lower bounded homomorphism, then $\beta_h \colon \mathbf{L} \rightharpoonup \mathbf{K}$ is a partial mapping whose domain is an ideal of $\mathbf{L}$.

Dually, h is an **upper bounded** homomorphism if, whenever the set $h^{-1}[\![0, a]\!]$ is nonempty, it has a greatest element, denoted by $\alpha_h(a)$, or $\alpha(a)$. For an upper bounded homomorphism, the domain of $\alpha_h \colon \mathbf{L} \rightharpoonup \mathbf{K}$ is clearly a filter of $\mathbf{L}$. A **bounded** homomorphism is one that is both upper and lower bounded.

These definitions simplify when h is an epimorphism. In that case h is lower bounded if and only if each preimage $h^{-1}\{a\}$ has a least element. Likewise, if L is finite, then $h \colon \mathbf{K} \to \mathbf{L}$ is lower bounded if and only if $h^{-1}\{a\}$ has a least element whenever it is nonempty. On the other hand, every homomorphism h from a finite lattice $\mathbf{K}$ is bounded.

Note that β is monotonic and a left adjoint for h, i.e., $a \leqslant h(x)$ iff $\beta(a)x$. It then follows from a standard argument that β is a join preserving map on its domain: if $h^{-1}[\![a, 1]\!] \neq \emptyset$ and $h^{-1}[\![b, 1]\!] \neq \emptyset$, then $\beta(a \vee b) = \beta(a) \vee \beta(b)$. Similarly, α is a right adjoint for h, so that $h(y) \leqslant a$ iff $y \leqslant \alpha(a)$, and for $a, b \in \operatorname{dom} \alpha$, $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$. In particular, if h is an epimorphism, then α and β are respectively meet and join homomorphisms of $\mathbf{L}$ into $\mathbf{K}$. For future reference, we note that α and β behave correctly with respect to composition.

Theorem C.14 ([FJN95, Thm. 2.1])**.** *Let $f \colon \mathbf{K} \to \mathbf{L}$ and $g \colon \mathbf{L} \to \mathbf{M}$ be homomorphisms. If f, g are lower bounded, then $gf \colon \mathbf{K} \to \mathbf{M}$ is lower bounded and $\beta_{gf} = \beta_f \beta_g$. Similarly, if f, g are upper bounded, then $\alpha_{gf} = \alpha_f \alpha_g$.*

Proof. For $x \in K$ and $a \in M$, we have

$$a \leqslant gf(x) \quad \text{iff} \quad \beta_g(a) \leqslant f(x) \quad \text{iff} \quad \beta_f \beta_g(a) \leqslant x.$$

The upper bounded case is dual. □

We need a way to determine whether a lattice homomorphism $h \colon \mathbf{K} \to \mathbf{L}$ is upper or lower bounded. The most natural setting for this is when the lattice $\mathbf{K}$ is finitely generated, so from now on we assume $\mathbf{K}$ is generated by a finite set X. We want to analyze the sets $h^{-1}[\![a, 1]\!]$ for $a \in L$, with the possibility of lower boundedness in mind. (The corresponding results for $h^{-1}[\![0, a]\!]$ are obtained by duality.) Note that $\mathbf{K}$ has a greatest element $1_{\mathbf{K}} = \bigvee X$, and that $h^{-1}[\![a, 1]\!]$ is nonempty if and only if $a \leqslant h(1_{\mathbf{K}})$.

Define a pair of closure operators, denoted by $^\wedge$ and $^\vee$, on subsets of an arbitrary lattice L as follows: for each $A \subseteq L$,

$$A^\wedge := \{\bigwedge B \mid B \text{ is a finite subset of } A\}.$$

We adopt the following convention: if $\mathbf{L}$ has a greatest element $1_{\mathbf{L}}$, then $\bigwedge \emptyset = 1_{\mathbf{L}}$, and we include this in $A^\wedge$ for every $A \subseteq L$; otherwise, $\bigwedge \emptyset$ is undefined. The set $A^\vee$ is defined dually.

We can write K as the union of a chain of subsets $H_0 \subseteq H_1 \subseteq \cdots$ defined inductively by setting $H_0 := X^\wedge$ and $H_{k+1} := (H_k)^{\vee\wedge}$, for all $k \geqslant 0$. By induction, each $H_n = X^{\wedge(\vee\wedge)^n}$ is a finite meet-closed subset of K, and $\bigcup H_n = K$, since X generates K.

Let $h \colon \mathbf{K} \to \mathbf{L}$ be an epimorphism and, for each $y \in L$ and $k < \omega$, define

$$\beta_k(y) = \bigwedge\{w \in H_k : h(w) \geqslant y\}.$$

Theorem C.15 ([FJN95, Thm. 2.2])**.** *Let* $\mathbf{K}$ *be finitely generated, and let* $h \colon \mathbf{K} \to \mathbf{L}$ *be a lattice homomorphism. If* $a \leqslant h(1_{\mathbf{K}})$*, then*

(1) $j \leqslant k$ *implies* $\beta_j(a) \geqslant \beta_k(a)$*,*
(2) $\beta_k(a)$ *is the least element of* $H_k \cap h^{-1}[\![a, 1]\!]$*,*
(3) $h^{-1}[\![a, 1]\!] = \bigcup_{k \in \omega} [\![\beta_k(a), 1]\!]$*.*

C.1.1. *Minimal join covers and refinement.* A **join cover** of the element $a \in L$ is a finite subset $S \subseteq L$ such that $a \leqslant \bigvee S$. A join cover S of a is *nontrivial* if $a \nleqslant s$ for all $s \in S$. Let $\mathcal{C}(a)$ be the set of all nontrivial join covers of a in $\mathbf{L}$.

Theorem C.16 ([FJN95, Thm. 2.3])**.** *Suppose* $\mathbf{K}$ *is generated by a finite set* X*,* $h \colon \mathbf{K} \to \mathbf{L}$ *is a homomorphism,* $a \leqslant h(1_{\mathbf{K}})$*, and* $k \in \omega$*. Then,*

$$\beta_0(a) = \bigwedge\{x \in X \mid h(x) \geqslant a\},$$

$$\beta_{k+1}(a) = \beta_0(a) \wedge \bigwedge_{\substack{S \in \mathcal{C}(a) \\ \bigvee S \leqslant h(1_{\mathbf{K}})}} \bigvee_{s \in S} \beta_k(s).$$

In general, the expression for $\beta_{k+1}(a)$ has some redundant terms, which we can exclude if $\mathbf{L}$ satisfies a weak finiteness condition that we now define. For finite subsets $A, B \subseteq L$, we say A *join refines* B and write $A \ll B$ if for every $a \in A$ there exists $b \in B$ with $a \leqslant b$. Theorem C.10 states that if $w \in \mathbf{F}(X)$ and $w = \bigvee B$, then the set of canonical joinands of w join refines B.

Define a **minimal nontrivial join cover** of $a \in L$ to be a nontrivial join cover S with the property that whenever $a \leqslant \bigvee T$ and $T \ll S$, then $S \subseteq T$. This formulation is equivalent to our more intuitive notion of what minimality ought to mean: *a nontrivial join cover* S *of* a *is minimal if and only if*

(1) *S is an antichain of join irreducible elements of* L*, and*

(2) *if an element of S is deleted or replaced by a (finite) set of strictly smaller elements, then the resulting set is no longer a join cover of a.*

Let $\mathcal{M}(a)$ denote the set of minimal nontrivial join covers of $a \in L$. Let us say that $\mathbf{L}$ has the **minimal join cover refinement property** if for each $a \in L$, $\mathcal{M}(a)$ is finite and every nontrivial join cover of a refines to a minimal one. Clearly every finite lattice has the minimal join cover refinement property, but so do free lattices. The following reformulation of Theorem C.16 simplifies the calculation of β_k whenever the minimal join cover refinement property holds.

Theorem C.17 ([FJN95, Thm. 2.4])**.** *Let $\mathbf{K}$ be generated by the finite set X, and let $h\colon \mathbf{K} \to \mathbf{L}$ be a lattice homomorphism. If $\mathbf{L}$ has the minimal join cover refinement property, then for each $a \in L$ with $a \leqslant h(1_{\mathbf{K}})$ and $k \in \omega$, we have*

$$\beta_0(a) = \bigwedge\{x \in X \mid h(x) \geqslant a\},$$

$$\beta_{k+1}(a) = \beta_0(a) \wedge \bigwedge_{\substack{S\in\mathcal{M}(a)\\ \bigvee S \leqslant h(1_{\mathbf{K}})}} \bigvee_{s\in S} \beta_k(s).$$

We now look for a condition on the $\mathbf{L}$ that will insure the homomorphism $h\colon \mathbf{K} \to \mathbf{L}$ is lower bounded. From Theorem C.15, this will happen iff for each $a \leqslant h(1_{\mathbf{K}})$ there exists $N \in \omega$ such that $\beta_n(a) = \beta_N(a)$ for all $n \geqslant N$. In this case, $\beta(a) = \beta_N(a)$ for all $a \in \operatorname{dom}\beta = [\![0, h(1_{\mathbf{K}})]\!]$, where N depends upon a.

Fact. The following are equivalent:

wjd 2018-10-04: To do: verify this!

(1) h is not lower bounded;
(2) $(\exists y_0 \in L)(\forall N)(\exists n > N)\, \beta_n(a) \neq \beta_N(a)$;
(3) $(\exists y_0 \in L)(\exists N)(\forall n > N)\beta_n(a) \neq \beta_N(a)$.

Let $D_0(\mathbf{L})$ be the set of all join prime elements of $\mathbf{L}$, i.e., the set of elements that have no nontrivial join cover. Given $D_k(\mathbf{L})$, define $D_{k+1}(\mathbf{L})$ to be the set of $p \in L$ such that every nontrivial join cover of p refines to a join cover contained in $D_k(\mathbf{L})$, i.e., $p \leqslant \bigvee S$ nontrivially implies there exists $T \ll S$ with $p \leqslant \bigvee T$ and $T \subseteq D_k(\mathbf{L})$. Note that if L has the minimal join cover refinement property, then $p \in D_{k+1}(\mathbf{L})$ iff every minimal nontrivial join cover of p is contained in $D_k(\mathbf{L})$.

The definition clearly implies $D_0(\mathbf{L}) \subseteq D_1(\mathbf{L}) \subseteq D_2(\mathbf{L}) \subseteq \cdots$. Let $D(\mathbf{L}) = \bigcup D_i$. For $a \in D(\mathbf{L})$, define the D-**rank**, $\rho(a)$, to be the least integer N such that $a \in D_N(\mathbf{L})$; for $a \notin D(L)$, $\rho(a)$ is undefined. The duals of $D_k(\mathbf{L})$, $D(\mathbf{L})$, and $\rho(a)$ are denoted by $D^d_k(\mathbf{L})$, $D^d(\mathbf{L})$, and $\rho^d(a)$, respectively.

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