The purpose of this document is to outline a possible example of a finitely generated lattice N and an unbounded homomorphism ψ onto a finite lattice L such that the kernel of ψ is not finitely generated as a sublattice of $N \times N$.

The example is based on Example 7.5 from [1]. There, the construction begins with a 16-element lattice $L = \{0,1\} \cup \{a_i, b_i \ : \ i = 1, \dots, 7\}$. This lattice is then 'expanded' to and infinite lattice M 'by inflating each a_i, b_i to an infinite chain isomorphic to ω.' Thus $M = \{0,1\} \cup \bigcup_{i=1}^{7} A_i \cup \bigcup_{i=1}^{7} B_i$, where each $A_i = \{a_{ij} \ : \ j = 0, 1, \dots\}$ and $B_i = \{b_{ij} \ : \ j = 1, 2, \dots\}$ is an increasing chain. The comparisons between individual elements belonging to different 'blocks' is governed by equation (5) and illustrated in Figure 3.

To construct the example we need here, we extend each A_i, B_i by one further point, which is a new smallest element in its block; in other words:

$$A_i = \{a_{i,-\infty}\} \cup \{a_{ij} \ : \ j = 0, 1, \dots\} \quad \text{and} \quad B_i = \{b_{i,-\infty}\} \cup \{b_{ij} \ : \ j = 1, 2, \dots\}.$$

The new elements together with 0 and 1 form a sublattice isomorphic to L, and $b_{i,-\infty}$ is not above any other elements of the form a_{ij}. This means that (5) can remain unchanged:

$$a_{ij} \leq b_{kl} \Leftrightarrow \begin{cases} k = i, \ j \leq l \text{ or} \\ k = i+1, \ j+1 \leq l \text{ or} \\ k = i+3, \ j \leq l, \end{cases}$$

where $i, k \in \{1, \dots, 7\}$, $j \in \{-\infty\} \cup \{0, 1, \dots\}$ and $l \in \{-\infty\} \cup \{1, 2, \dots\}$. However, contrary to the patterns highlighted in Figure 3, note that $a_{i0} \not\leq b_{i,-\infty}$ and $a_{i0} \not\leq b_{i+3,-\infty}$. Denote the resulting lattice by N.

This lattice is finitely generated. Indeed, it contains the finitely generated lattice M as a sublattice, and the complement $N \setminus M$ is finite.

Now let

$$\psi : N \to L, \ 0 \mapsto 0, \ 1 \mapsto 1, \ a_{ij} \mapsto a_i, \ b_{ij} \mapsto b_i.$$

Clearly ψ is not bounded, since none of its kernel classes A_i, B_i have largest elements.

Let us now consider $D = \ker \psi$ as the subdirect product in $N \times N$. We claim that D is finitely generated. To prove this, consider the set

$$D_1 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \cup \bigcup_{i=1}^{7} \big(\{a_{i,-\infty}\} \times A_i\big) \cup \bigcup_{i=1}^{7} \big(\{b_{i,-\infty}\} \times B_i\big) \subseteq D.$$

Since $\{0,1\} \cup \{a_{i,-\infty}, b_{i,-\infty} \ : \ i = 1, \dots, 7\} \cong L$, it follows that $D_1 \cong N$. In particular, D_1 is finitely generated. By symmetry,

$$D_2 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \cup \bigcup_{i=1}^{7} (A_i \times \{a_{i,-\infty}\}) \cup \bigcup_{i=1}^{7} (B_i \times \{b_{i,-\infty}\})$$

is a lattice isomorphic to N and is finitely generated. A general element from D different from $\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}1\\1\end{pmatrix}$ has the form $\begin{pmatrix}a_{ij}\\a_{ik}\end{pmatrix}$ or $\begin{pmatrix}b_{ij}\\b_{ik}\end{pmatrix}$. Furthermore

$$\begin{pmatrix}a_{ij}\\a_{ik}\end{pmatrix} = \begin{pmatrix}a_{i,-\infty}\\a_{ik}\end{pmatrix} \vee \begin{pmatrix}a_{ij}\\a_{i,-\infty}\end{pmatrix} \in D_1 \vee D_2,$$

and a dual statement holds for $\begin{pmatrix}b_{ij}\\b_{ik}\end{pmatrix}$. It follows that D is generated by $D_1 \cup D_2$, and is hence finitely generated, as claimed.

References

[1] P. Mayr, N. Ruškuc, Generating subdirect products, submitted.