# FINITELY GENERATED EQUATIONAL CLASSES

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Abstract. Classes of algebraic structures that are defined by equational laws are called *varieties* or *equational classes*. A variety is finitely generated if it is defined by the laws that hold in some fixed finite algebra. We show that every subvariety of a finitely generated congruence permutable variety is finitely generated; in fact, we prove the more general result that if a finitely generated variety has an edge term, then all its subvarieties are finitely generated as well. This applies in particular to all varieties of groups, loops, quasigroups and their expansions (e.g., modules, rings, Lie algebras, ...).

## 1. Introduction

Algebraic structures (or simply *algebras*) are often classified according to the equational laws they fulfill. For example, groups can be defined as those algebras with operation symbols $\cdot, ^{-1}, 1$ that satisfy the identities $(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$, $1 \cdot x \approx x$, and $x^{-1} \cdot x \approx 1$. A class of algebraic structures that is defined by equational laws is called an *equational class*, or simply a *variety*. We will consider equational classes that are generated by a single finite algebra. As an example, the variety generated by the symmetric group $S_3$ on 3 letters consists of those groups that satisfy all equational laws that are valid in $S_3$, such as $x^6 \approx 1$ or $x^2 \cdot y^2 \approx y^2 \cdot x^2$. Since the groups in a finitely generated variety have finite exponent, clearly not every variety is finitely generated. However, from the work of Oates and Powell [OP64], we obtain the following surprising fact for every finitely generated variety $V$ of groups:

(1.1) All subvarieties of $V$ are finitely generated.

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Kruse and L'vov proved that the same property is also satisfied for finitely generated varieties of rings [Kru73, L'v74]. For finitely generated varieties of lattices, (1.1) is a consequence of Jónsson's Lemma [BS81, Corollary IV.6.10].

In the main result of this paper, Theorem 1.2, we establish that property (1.1) holds for a much larger class of algebraic structures that includes groups, loops, rings and lattices as well as their expansions.

We continue with a discussion of those concepts from equational logic that we will use. First we fix one *type* of algebraic structures, this means, one set of operation symbols. An *identity* (of this type) is a formula of the form $s(x_1, \ldots, x_n) \approx t(x_1, \ldots, x_n)$, where $s, t$ are terms involving these operation symbols. The fundamental relation between *algebraic structures* and *identities* is whether the algebraic structure $\mathbf{A}$ *satisfies* the identity $s(x_1, \ldots, x_n) \approx t(x_1, \ldots, x_n)$, which is the case if $s^{\mathbf{A}}(a_1, \ldots, a_n) = t^{\mathbf{A}}(a_1, \ldots, a_n)$ for the induced term functions $s^{\mathbf{A}}, t^{\mathbf{A}}$ on $\mathbf{A}$ and for all $a_1, \ldots, a_n \in A$. In this case, we write $\mathbf{A} \models s \approx t$. Now for a set of identities $\Phi$, the class $\mathrm{Mod}(\Phi)$ is defined to be the class of all algebras such that $\mathbf{A} \models \varphi$ for all $\varphi \in \Phi$. It is not hard to see that $\mathrm{Mod}(\Phi)$ is closed under taking subalgebras, direct products, and homomorphic images. The fundamental HSP-Theorem of Birkhoff [Bir35, Theorem 10] tells that every class of algebras closed under taking subalgebras, direct products, and homomorphic images is of the form $\mathrm{Mod}(\Phi)$ for some (possibly infinite) set of identities. A class of algebraic structures that is of the form $\mathrm{Mod}(\Phi)$ is called an *equational class* or a *variety.* For an algebra $\mathbf{A}$, we consider the smallest variety that contains $\mathbf{A}$. This variety, $\mathbb{V}(\mathbf{A})$, is given by

$$\mathbb{V}(\mathbf{A}) = \{\mathbf{B} \mid \text{for all identities } \varphi \text{ with } \mathbf{A} \models \varphi, \text{ we have } \mathbf{B} \models \varphi\}.$$

By Birkhoff's Theorem, $\mathbb{V}(\mathbf{A}) = \mathbb{HSP}(\mathbf{A})$, where $\mathbb{HSP}(\mathbf{A})$ stands for the class of homomorphic images of subalgebras of direct powers of $\mathbf{A}$. We call an arbitrary variety $V$ of algebras *finitely generated* if there is a finite algebra $\mathbf{A}$ with $V = \mathbb{V}(\mathbf{A})$. For a finitely generated variety, we can effectively check whether an identity holds in all of its algebras, since it suffices to verify that it holds in the finite generating algebra. More details on varieties and identities can be found, e. g., in [BS81, McN92].

In this paper we consider finitely generated varieties $V$ that satisfy property (1.1), that is, all subvarieties of $V$ are finitely generated as well. That this condition is indeed nontrivial can be seen from the following examples: Oates MacDonald and Vaughan-Lee showed that the three-element Murskii groupoid $\mathbf{M}$ with zero generates a variety with subvarieties that are not finitely generated [OMVL78, Corollary 4.2]. Lee proved that property (1.1) does not hold for the variety generated by the five-element Brandt semigroup [Lee06, Proposition 6.7].

However, (1.1) above is true if $V$ is the variety generated by a finite lattice. In fact, for algebras generating congruence distributive varieties, a stronger result holds. By Jónsson's Lemma [BS81, Corollary IV.6.10], every finitely generated congruence distributive variety $V$ contains only finitely many subdirectly irreducible algebras. Therefore $V$ has only finitely many subvarieties and all of them are finitely generated. For varieties of groups, a similar result holds. Here, only certain subdirectly irreducible groups, namely *critical* groups, are used. A finite group $\mathbf{G}$ is *critical* if it does not lie in the variety generated by the groups in $\mathbb{HS}(\mathbf{G})$ of size smaller than $\mathbf{G}$. When Oates and Powell proved that for every finite group $\mathbf{G}$ the variety $\mathbb{V}(\mathbf{G})$ is finitely based (i.e., determined by a finite set of identities, [OP64]), they showed that $\mathbb{V}(\mathbf{G})$ contains only finitely many critical groups. Thus $\mathbb{V}(\mathbf{G})$ has only finitely many subvarieties, and all of them are finitely generated.

In this note, we consider a vast generalization of groups and lattices, namely those algebras that have an *edge operation* among their term functions. These edge operations were introduced by Berman, Idziak, Marković, McKenzie, Valeriote, and Willard in [BIM+10] as a common generalization of Mal'cev operations and near-unanimity operations. A ternary operation $d$ on a set $A$ is called a *Mal'cev operation* if $d(x,x,y) = d(y,x,x) = y$ for all $x, y \in A$. For a group, $d(x,y,z) := xy^{-1}z$ is a Mal'cev operation. On a quasigroup $(A, \cdot, \backslash, /)$ we have the Mal'cev operation $d(x,y,z) := (x/(y\backslash y)) \cdot (y\backslash z)$. For $k \geq 3$, a $k$-ary operation $m$ is a *near-unanimity operation* if $m(y,x,x,\ldots,x,x) = m(x,y,x,\ldots,x,x) = \cdots = m(x,x,x,\ldots,x,y) = x$ for all $x, y \in A$. On a lattice, $m(x_1, x_2, \ldots, x_k) := (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3)$ is an example of a $k$-ary near-unanimity operation. For $k \geq 2$, a function $t : A^{k+1} \to A$ is a *$k$-edge operation* if for all $x, y \in A$ we have

$$t(y,y,x,\ldots,x) = t(y,x,y,x,\ldots,x) = x$$

and for all $i \in \{4, \ldots, k+1\}$ and for all $x, y \in A$, we have

$$t(x,\ldots,x,y,x,\ldots,x) = x, \text{ with } y \text{ in position } i.$$

Hence a ternary operation $d$ is a Mal'cev operation if and only if $t(x,y,z) := d(y,x,z)$ is a 2-edge operation, and for $k \geq 3$, a $k$-ary operation $m$ is a near-unanimity operation if and only if $t(x_1, \ldots, x_{k+1}) := m(x_2, \ldots, x_{k+1})$ is a $k$-edge operation. We say that an algebra has a Mal'cev term (near-unanimity term, edge term) if it has a Mal'cev operation (near-unanimity operation, edge operation) among its term functions. Hence every algebra with a Mal'cev term and every algebra with near-unanimity term has an edge term. The class of algebras with an edge term therefore contains all groups and their expansions (such as rings, vector spaces, Lie algebras ...), all quasigroups, loops, as well as all lattices and their expansions. In [BIM+10], we find the combinatorial characterization that a finite algebra $\mathbf{A}$ has an edge term if and only if there is a polynomial $p$ with real coefficients such that for all $n \in \mathbb{N}$, $\mathbf{A}^n$ has at most $2^{p(n)}$ subalgebras.

In the present paper, we obtain the following result on subvarieties of a finitely generated variety with edge term.

**Theorem 1.1.** *Let* $\mathbf{A}$ *be a finite algebra with an edge term. Then there is no infinite ascending chain* $V_1 \subset V_2 \subset \cdots$ *of subvarieties of* $\mathbb{V}(\mathbf{A})$.

From this result, one can infer that property (1.1) is true for finitely generated varieties with an edge term:

**Theorem 1.2.** *Let* $\mathbf{A}$ *be a finite algebra with an edge term. Then for every variety* $W \subseteq \mathbb{V}(\mathbf{A})$*, there is a finite algebra* $\mathbf{B} \in W$ *such that* $W = \mathbb{V}(\mathbf{B})$.

We will prove these theorems in Section 6. Unlike for groups and lattices, there are algebras with an edge term that generate varieties with infinitely many subvarieties. Thus a proof requires an approach that is different from the one for the classical cases. However, since a finitely generated variety contains, up to isomorphism, at most countably many finite algebras, we obtain, as a consequence of Theorem 1.2:

**Corollary 1.3.** *Every finitely generated variety with an edge term has at most countably many subvarieties.*

To illustrate that an arbitrary variety with an edge term may have uncountably many subvarieties, we recall two fundamental classical results: Evans and Neumann showed that the number of varieties of loops is continuum [EN53], and Ol'šanskiĭ proved that there are even continuum many distinct varieties of groups [Ol′70].

As an application of our results to the theory of loops we obtain the following.

**Corollary 1.4.** *Let* $\mathbf{A}$ *be a finite loop. Then* $\mathbb{V}(\mathbf{A})$ *has at most countably many subvarieties, and all of them are finitely generated.*

As mentioned above, for groups Theorem 1.2 was obtained in [OP64] in the course of proving that for every finite group $\mathbf{G}$, the variety $\mathbb{V}(\mathbf{G})$ is finitely based. In general, a variety $V$ of algebras of some type $\mathcal{F}$ is called *finitely based* if there is a finite set $\Phi$ of identities such that an algebra $\mathbf{A}$ of type $\mathcal{F}$ lies in $V$ if and only if $\mathbf{A} \models \Phi$. An algebra is called finitely based if the variety it generates is finitely based. Theorem 1.1 asserts that for a finite algebra $\mathbf{A}$ with an edge term, the class of subvarieties of $\mathbb{V}(\mathbf{A})$ has no infinite ascending chains. However, infinite descending chains may exist. As a corollary of the main result, we obtain several conditions that are equivalent to the fact that there are no infinite descending chains of subvarieties of $\mathbb{V}(\mathbf{A})$. We say that a variety $W$ contained in a variety $V$ is *finitely based relative to* $V$ if there is a finite set of identities $\Phi$ such that $W = \{\mathbf{A} \in V \mid \mathbf{A} \models \Phi\}$. An algebra $\mathbf{B} \in V$ is finitely based relative to $V$ if $\mathbb{V}(\mathbf{B})$ is finitely based relative to $V$. A finite algebra $\mathbf{A}$ is called *cardinality critical* if it does not lie in the variety generated by the algebras $\mathbf{B} \in \mathbb{V}(\mathbf{A})$ of size smaller than $\mathbf{A}$. Clearly, every cardinality critical finite group is critical. Now combining Theorem 1.2 with known facts, we obtain the following corollary.

**Corollary 1.5.** *Let $\mathbf{A}$ be a finite algebra with an edge term. Then the following are equivalent:*

1. *Each $\mathbf{B} \in \mathbb{V}(\mathbf{A})$ is finitely based relative to $\mathbb{V}(\mathbf{A})$.*
2. *$\mathbb{V}(\mathbf{A})$ has only finitely many subvarieties.*
3. *$\mathbb{V}(\mathbf{A})$ contains, up to isomorphism, only finitely many cardinality critical members.*
4. *There is no infinite descending chain of subvarieties of $\mathbb{V}(\mathbf{A})$.*

This corollary will be derived in Section 7.

There are examples of finite algebras with an edge term that satisfy none of the conditions (1) to (4) given in Corollary 1.5; one such example has been constructed in [Bry82]. This example is a pointed group. A *pointed group* is an algebra $\mathbf{A}$ in the language $\mathcal{P} = \{\cdot, {}^{-1}, 1, c\}$ such that its reduct $\mathbf{G} := \langle A, \cdot^{\mathbf{A}}, {}^{-1^{\mathbf{A}}}, 1^{\mathbf{A}} \rangle$ is a group, and $c$ is a nullary operation; we also write $(\mathbf{G}, c^{\mathbf{A}})$ for $\mathbf{A}$. Now [Bry82] provides a finite group $\mathbf{P}$ and an element $p \in P$ such that the pointed group $(\mathbf{P}, p)$ is not finitely based (in the language $\mathcal{P}$). The variety generated by $\mathbf{Q} := \prod_{g \in P} (\mathbf{P}, g)$ is finitely based. In fact, let $\Phi$ be the finite set of identities defining the variety generated by the group $\mathbf{P}$; such a set exists by the Oates-Powell-Theorem. Then $\Phi$ also axiomatizes the variety of pointed groups generated by $\mathbf{Q}$. Hence $(\mathbf{P}, p)$ is not finitely based relative to $\mathbb{V}(\mathbf{Q})$, and therefore $\mathbf{Q}$ does not satisfy any of the conditions (1) to (4) given in Corollary 1.5. Altogether, we have that the class of subvarieties of $\mathbb{V}(\mathbf{Q})$ satisfies the ascending chain condition, but not the descending chain condition.

The main technique for proving Theorem 1.1 will be an application of the arguments that were used in [Aic10, AMM11] to establish that every clone on a finite set that contains an edge operation is finitely related. In our setting, the role of clones is taken by a suitable encoding of the equational theory of a variety into a structure that we will call a *clonoid*. While a clonoid now represents the equational theory of a variety, the role of the invariant relations appearing in clone theory is taken by the algebras that lie in a variety.

## 2. Preliminaries from universal algebra

In this section, we review the relation between "finitely generated" and "no ascending chains of subvarieties", and between "finitely based" and "no descending chains of subvarieties".

We will first state a well-known lemma. For an algebra $\mathbf{A}$, we say that $\mathbf{A}$ is $k$-generated if it is generated by a subset $S \subseteq A$ with $|S| \leq k$.

**Lemma 2.1.** *Let $\mathcal{F}$ be a type of algebras, let $V$ be a variety of algebras of type $\mathcal{F}$, let $k \in \mathbb{N}$, let $\varphi := (s \approx t)$ be an identity over $\mathcal{F}$ that uses at most $k$ variables, and let $\mathbf{A}$ be a $k$-generated algebra of type $\mathcal{F}$. Then we have:*

(1) *$V \models \varphi$ if and only if every $k$-generated algebra in $V$ satisfies $\varphi$.*
(2) *$\mathbf{A} \in V$ if and only if $\mathbf{A}$ satisfies every identity with at most $k$ variables that holds in $V$.*

For groups, item (2) is explicitly given in [Neu67, Lemma 16.1]. A variety is called *locally finite* if all of its finitely generated members are finite. Every finitely generated variety is locally finite. From the lattice of subvarieties of a locally finite variety, finitely generated varieties can be recognized in the following way.

**Lemma 2.2** (cf. [OMVL78, p. 370])**.** *Let $V$ be a locally finite variety of arbitrary type $\mathcal{F}$, and let $W$ be a subvariety of $V$. Then the following are equivalent:*

(1) *There exists no infinite strictly ascending chain of varieties $V_1 \subset V_2 \subset V_3 \subset \cdots$ with $W := \mathbb{V}(\bigcup_{i \in \mathbb{N}} V_i)$.*
(2) *$W$ is finitely generated.*

Similarly, finitely based varieties will be singled out in the following lemma. We recall that for a set of identities $\Phi$ of type $\mathcal{F}$, $\mathrm{Mod}(\Phi)$ denotes the class of all algebras of type $\mathcal{F}$ satisfying $\Phi$.

**Lemma 2.3** (cf. [OMVL78, p. 370])**.** *Let $V$ be a locally finite variety of arbitrary type $\mathcal{F}$, and let $W$ be a subvariety of $V$. Then the following are equivalent:*

(1) *There exists no infinite strictly descending chain of varieties $V \supseteq V_1 \supset V_2 \supset V_3 \supset \cdots$ with $W := \bigcap_{i \in \mathbb{N}} V_i$.*
(2) *There is a $k \in \mathbb{N}$ and a set $\Sigma$ of identities such that each $\sigma \in \Sigma$ contains at most $k$ variables and $W = V \cap \mathrm{Mod}(\Sigma)$.*
(3) *There is a finite set $\Phi$ of identities such that $W = V \cap \mathrm{Mod}(\Phi)$.*

Finally, we need a lemma on cardinality critical algebras. We recall that a finite algebra $\mathbf{B}$ is called *cardinality critical* if $\mathbf{B} \notin \mathbb{V}(\{\mathbf{C} \mid \mathbf{C} \in \mathbb{V}(\mathbf{B}), |\mathbf{C}| < |\mathbf{B}|\})$. We have:

**Lemma 2.4.** *Let $V$ be a locally finite variety. Then each subvariety $W$ of $V$ is generated by the cardinality critical algebras it contains.*

*Proof:* Let $W_1 := \mathbb{V}(\{\mathbf{B} \in W \mid \mathbf{B} \text{ is finite and cardinality critical}\})$. Since $W$ and $W_1$ are both locally finite, it suffices to show that they have the same finite members. Seeking a contradiction, we let $\mathbf{C} \in W$ be finite and of minimal size with $\mathbf{C} \notin W_1$. Then $\mathbf{C}$ is not cardinality critical, hence it lies in the variety generated by $X := \{\mathbf{D} \mid |\mathbf{D}| < |\mathbf{C}|, \mathbf{D} \in \mathbb{V}(\mathbf{C})\}$. By the minimality of $\mathbf{C}$, we have $X \subseteq W_1$, and thus $\mathbf{C} \in W_1$, a contradiction. $\square$

## 3. Preliminaries from order theory

The techniques that we use from order theory have been used in a similar way in [Aic10], and in an almost identical way as they are used in the present note in [AMM11]. To keep our presentation self-contained, we report these results with only a slight modification in notation.

A partially ordered set $\langle X, \leq \rangle$ satisfies the *descending chain condition* (DCC) if there are no infinite descending chains $x_1 > x_2 > \cdots$ in $X$, and it satisfies *the ascending chain condition* (ACC) if there is no infinite ascending chain $x_1 < x_2 < \cdots$ in $X$. A subset $Y$ of $X$ is an *antichain* if for all $y_1, y_2 \in Y$ with $y_1 \leq y_2$, we have $y_1 = y_2$. The partially ordered set $\langle X, \leq \rangle$ is *well partially ordered* if it satisfies the (DCC) and has no infinite antichains. For a partially ordered set $\langle X, \leq \rangle$, a subset $Y$ of $X$ is *upward closed* if for all $y \in Y$ and $x \in X$ with $y \leq x$, we have $x \in Y$. The set of upward closed subsets of $X$ is denoted by $\mathcal{U}(X, \leq)$. If $\langle X, \leq \rangle$ is well partially ordered, then $\langle \mathcal{U}(X, \leq), \subseteq \rangle$ satisfies the (ACC) (cf. [Mil85, Theorem 1.2]).

For $A = \{1, 2, \ldots, t\}$, we will use the lexicographic ordering on $A^n$. For $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, we say $\mathbf{a} \leq_{\text{lex}} \mathbf{b}$ if

$$(\exists i \in \{1, \ldots, n\} : a_1 = b_1 \wedge \ldots \wedge a_{i-1} = b_{i-1} \wedge a_i < b_i) \text{ or}$$
$$(a_1, \ldots, a_n) = (b_1, \ldots, b_n).$$

For every finite set $A$, we let $A^+$ be the set $\bigcup\{A^n \mid n \in \mathbb{N}\}$. (We use $\mathbb{N}$ for the set of natural numbers $\{1, 2, 3, \ldots\}$.) We will now introduce an order relation on $A^+$. For $\mathbf{a} = (a_1, \ldots, a_n) \in A^+$ and $b \in A$, we define the *index of the first occurrence of $b$ in* $\mathbf{a}$, $\mathsf{firstOcc}\,(\mathbf{a}, b)$, by $\mathsf{firstOcc}\,(\mathbf{a}, b) := 0$ if $b \notin \{a_1, \ldots, a_n\}$, and $\mathsf{firstOcc}\,(\mathbf{a}, b) := \min\{i \in \{1, \ldots, n\} \mid a_i = b\}$ otherwise.

**Definition 3.1** ([AMM11, Definition 3.1])**.** Let $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ be in $A^+$. We say $\mathbf{a} \leq_{\mathrm{E}} \mathbf{b}$ if there is an injective and increasing function $h : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that

(1) for all $i \in \{1, \ldots, m\} : \; a_i = b_{h(i)}$,
(2) $\{a_1, \ldots, a_m\} = \{b_1, \ldots, b_n\}$,
(3) for all $c \in \{a_1, \ldots, a_m\}$: $h(\mathsf{firstOcc}\,(\mathbf{a}, c)) = \mathsf{firstOcc}\,(\mathbf{b}, c)$.

We will call such an $h$ a function *witnessing* $\mathbf{a} \leq_{\mathrm{E}} \mathbf{b}$.

Informally, $\mathbf{a} \leq_{\mathrm{E}} \mathbf{b}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ contain the same set of letters, and $\mathbf{b}$ can be obtained from $\mathbf{a}$ by certain insertions, where the insertion $\mathbf{xy} \to \mathbf{x}l\mathbf{y}$ is allowed only if the letter $l$ appears in the word $\mathbf{x}$.

**Proposition 3.2** (cf. [AMM11, Lemma 3.2])**.** *Let $A$ be a finite set. Then $\langle A^+, \leq_{\mathrm{E}}\rangle$ is well partially ordered, and $\langle \mathcal{U}(A^+, \leq_{\mathrm{E}}), \subseteq\rangle$ satisfies the* (ACC)*.*

*Proof:* In [AMM11, Lemma 3.2], it is proved that $\langle A^+, \leq_{\mathrm{E}}\rangle$ is well partially ordered. Since $\langle A^+, \leq_{\mathrm{E}}\rangle$ is well partially ordered, [Mil85, Theorem 1.2] yields that $\langle \mathcal{U}(A^+, \leq_{\mathrm{E}}), \subseteq\rangle$ satisfies the (ACC). $\square$

We will now give a slight modification of [AMM11, Definition 3.3] and [AMM11, Lemma 3.4].

**Definition 3.3** (cf. [AMM11, Definition 3.3])**.** Let $A$ be a finite set, let $\mathbf{a} = (a_1, \ldots, a_m) \in A^m$, $\mathbf{b} = (b_1, \ldots, b_n) \in A^n$ be such that $\mathbf{a} \leq_{\mathrm{E}} \mathbf{b}$, and let $h$ be a function from $\{1, \ldots, m\} \to \{1, \ldots, n\}$ witnessing $\mathbf{a} \leq_{\mathrm{E}} \mathbf{b}$. We will now define a function $\tau_{\mathbf{a},\mathbf{b},h} : \{1, \ldots, n\} \to \{1, \ldots, m\}$. If $j \in \text{range}(h)$, then $\tau_{\mathbf{a},\mathbf{b},h}(j)$ is defined by

$$\tau_{\mathbf{a},\mathbf{b},h}(j) := h^{-1}(j).$$

If $j \notin \mathrm{range}(h)$, then

$$\tau_{\mathbf{a},\mathbf{b},h}(j) := i,$$

where $i$ is minimal in $\{1, \ldots, m\}$ with $a_i = b_j$.

**Lemma 3.4** ([AMM11, Lemma 3.4])**.** *Let $t \in \mathbb{N}$, let $A = \{1, 2, \ldots, t\}$, let $\mathbf{a} \in A^m$, $\mathbf{b} \in A^n$, and let $h : \{1, \ldots, m\} \to \{1, \ldots, n\}$ be a function witnessing $\mathbf{a} \leq_{\mathrm{E}} \mathbf{b}$. Let $\mathbf{c} \in A^m$ be such that $\mathbf{c} <_{\mathrm{lex}} \mathbf{a}$. Then we have*

(1) $\langle a_{\tau_{\mathbf{a},\mathbf{b},h}(j)} \mid j \in \{1, \ldots, n\}\rangle = \mathbf{b}$,
(2) $\langle c_{\tau_{\mathbf{a},\mathbf{b},h}(j)} \mid j \in \{1, \ldots, n\}\rangle <_{\mathrm{lex}} \mathbf{b}$.

The following proof is similar to the proof of [AMM11, Lemma 3.4] with some notational changes.

*Proof:* For proving (1), let $j \in \{1, \ldots, n\}$. We have to prove that $a_{\tau_{\mathbf{a},\mathbf{b},h}(j)} = b_j$. If $j \in \mathrm{range}(h)$, then $a_{\tau_{\mathbf{a},\mathbf{b},h}(j)} = a_{h^{-1}(j)} = b_{h(h^{-1}(j))} = b_j$. If $j \notin \mathrm{range}(h)$, then $\tau_{\mathbf{a},\mathbf{b},h}(j)$ has the property $a_{\tau_{\mathbf{a},\mathbf{b},h}(j)} = b_j$.

For proving (2), let $k$ be the index of the first place in which $\mathbf{c}$ differs from $\mathbf{a}$. Hence $\mathbf{c} = (a_1, \ldots, a_{k-1}, c_k, c_{k+1}, \ldots)$, $\mathbf{a} = (a_1, \ldots, a_{k-1}, a_k, a_{k+1}, \ldots)$, and $c_k < a_k$.

We first show that for all $j < h(k)$, we have $c_{\tau_{\mathbf{a},\mathbf{b},h}(j)} = a_{\tau_{\mathbf{a},\mathbf{b},h}(j)}$. If $j$ is in the range of $h$, we have $c_{\tau_{\mathbf{a},\mathbf{b},h}(j)} = c_{h^{-1}(j)}$ and $a_{\tau_{\mathbf{a},\mathbf{b},h}(j)} = a_{h^{-1}(j)}$. Since $h(h^{-1}(j)) < h(k)$, we have, by the monotonicity of $h$, $h^{-1}(j) < k$. Thus $c_{h^{-1}(j)} = a_{h^{-1}(j)}$, since $k$ is the first index at which $\mathbf{c}$ and $\mathbf{a}$ differ. We now consider the case that $j$ is not in the range of $h$. Since $\{b_1, \ldots, b_n\} = \{a_1, \ldots, a_m\}$, there is an $i_1 \in \{1, \ldots, m\}$ such that $a_{i_1} = b_j$. Let $i_2 := \mathsf{firstOcc}(\mathbf{a}, a_{i_1})$. By the definition of $\tau_{\mathbf{a},\mathbf{b},h}$, we have $\tau_{\mathbf{a},\mathbf{b},h}(j) := i_2$. Now let $j_2 := h(i_2)$. Then we have $j_2 = \mathsf{firstOcc}(\mathbf{b}, b_j)$, and therefore $j_2 \leq j$. Hence $j_2 < h(k)$, and thus $i_2 < k$. Therefore $c_{i_2} = a_{i_2}$, and thus $c_{\tau_{\mathbf{a},\mathbf{b},h}(j)} = a_{\tau_{\mathbf{a},\mathbf{b},h}(j)}$.

Since $\tau_{\mathbf{a},\mathbf{b},h}(h(k)) = k$, we have $a_{\tau_{\mathbf{a},\mathbf{b},h}(h(k))} = a_k$ and $c_{\tau_{\mathbf{a},\mathbf{b},h}(h(k))} = c_k$, and therefore $\langle c_{\tau_{\mathbf{a},\mathbf{b},h}(j)} \mid j \in \{1, \ldots, n\}\rangle <_{\mathrm{lex}} \langle a_{\tau_{\mathbf{a},\mathbf{b},h}(j)} \mid j \in \{1, \ldots, n\}\rangle = \mathbf{b}$. $\square$

## 4. Clonoids

In this section, we will introduce *clonoids*. They are sets of finitary functions from a set $A$ into the universe of an algebra $\mathbf{B}$ that are closed under the operations of $\mathbf{B}$ and under manipulation of arguments.

**Definition 4.1.** Let $\mathbf{B}$ be an algebra, and let $A$ be a nonempty set. For a subset $C$ of $\bigcup_{n\in\mathbb{N}} B^{A^n}$ and $k \in \mathbb{N}$, we let $C^{[k]} := C \cap B^{A^k}$. We call $C$ a *clonoid with source set* $A$ *and target algebra* $\mathbf{B}$ if

(1) for all $k \in \mathbb{N}$: $C^{[k]}$ is a subuniverse of $\mathbf{B}^{A^k}$, and
(2) for all $k, n \in \mathbb{N}$, for all $(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$, and for all $c \in C^{[k]}$, the function $c' : A^n \to B$ with $c'(a_1, \ldots, a_n) := c(a_{i_1}, \ldots, a_{i_k})$ lies in $C^{[n]}$.

We note that every clone $C$ on a set $A$ is a clonoid with source set $A$ and target algebra $\langle A, C\rangle$ (see [MMT87, Definition 4.2]).

Let $\mathbf{A}$ be an algebra, let $m \in \mathbb{N}$, and let $F$ be a subuniverse of $\mathbf{A}^m$. For $i \in \{1, \ldots, m\}$, we define the relation $\varphi_i(F)$ on $A$ by

$$\varphi_i(F) := \{(a_i, b_i) \mid (a_1, \ldots, a_m) \in F, (b_1, \ldots, b_m) \in F, (a_1, \ldots, a_{i-1}) = (b_1, \ldots, b_{i-1})\}. \tag{4.1}$$

An element of $\varphi_i(F)$ is also called a *fork* of $F$ at place $i$. We will now encode algebras by their forks. As a matter of fact, the sequence of forks does not contain the complete information about a subuniverse of $\mathbf{A}^m$, but if $\mathbf{A}$ has an edge term, then the information is helpful in distinguishing between two subalgebras of $\mathbf{A}^m$ such that one is contained in the other. For a subset $F$ of $A^m$ and $T \subseteq \{1, \ldots, m\}$, we will use $\pi_T(F)$ for the projection to the components with indices in $T$. Formally, seeing an element $\mathbf{a}$ as a function from $\{1, \ldots, m\}$ to $A$, we have $\pi_T(F) := \{\mathbf{a}|_T \mid \mathbf{a} \in F\}$.

**Lemma 4.2** ([AMM11, Lemma 4.1])**.** *Let $k, m \in \mathbb{N}, k \geq 2$, and let $\mathbf{A}$ be an algebra with a $k$-edge term. Let $F, G$ be subuniverses of $\mathbf{A}^m$ with $F \subseteq G$. Assume that $\varphi_i(G) = \varphi_i(F)$ for all $i \in \{1, \ldots, m\}$ and $\pi_T(F) = \pi_T(G)$ for all $T \subseteq \{1, \ldots, m\}$ with $|T| < k$. Then $F = G$.*

For each $k \in \mathbb{N}$, we will now encode the forks of the algebra of $k$-ary functions in a clonoid in a way that runs completely parallel to the encoding used for clones in [Aic10] and [AMM11].

Let $A$ be the set $\{1, \ldots, t\}$, let $\mathbf{B}$ be an algebra, and let $C$ be a clonoid with source set $A$ and target algebra $\mathbf{B}$. For $n \in \mathbb{N}$ and $\mathbf{a} \in A^n$, we define a binary relation $\boldsymbol{\varphi}(C, \mathbf{a})$ on $B$ by

$$\boldsymbol{\varphi}(C, \mathbf{a}) := \{(f(\mathbf{a}), g(\mathbf{a})) \mid f, g \in C^{[n]}, \forall \mathbf{c} \in A^n : \mathbf{c} <_{\text{lex}} \mathbf{a} \Rightarrow f(\mathbf{c}) = g(\mathbf{c})\}.$$

Hence the elements of $\boldsymbol{\varphi}(C, \mathbf{a})$ are the forks of $C^{[n]}$ at $\mathbf{a}$. We have the following lemma.

**Lemma 4.3** (cf. [AMM11, Lemma 5.2] and [Aic10, Lemma 4.1]). *Let $m, n, t \in \mathbb{N}$, let $\mathbf{B}$ be an algebra, and let $C$ be a clonoid with source set $A = \{1, 2, \ldots, t\}$ and target algebra $\mathbf{B}$. Let $\mathbf{a} \in A^m$, $\mathbf{b} \in A^n$ such that $\mathbf{a} \leq_{\mathrm{E}} \mathbf{b}$. Then $\boldsymbol{\varphi}(C, \mathbf{b}) \subseteq \boldsymbol{\varphi}(C, \mathbf{a})$.*

*Proof:* Let $(x, y) \in \boldsymbol{\varphi}(C, \mathbf{b})$. Then there are $f, g \in C^{[n]}$ such that $x = f(\mathbf{b})$, $y = g(\mathbf{b})$, and $f(\mathbf{c}) = g(\mathbf{c})$ for all $\mathbf{c} \in A^n$ with $\mathbf{c} <_{\mathrm{lex}} \mathbf{b}$. Let $h$ be a function from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$ witnessing $\mathbf{a} \leq_{\mathrm{E}} \mathbf{b}$. Now we define functions $f_1$ and $g_1$ from $A^m$ to $A$ by

$$\begin{aligned} f_1(\mathbf{x}) &:= f(\langle x_{\tau_{\mathbf{a},\mathbf{b},h}(j)} \mid j \in \{1, \ldots, n\}\rangle) \\ g_1(\mathbf{x}) &:= g(\langle x_{\tau_{\mathbf{a},\mathbf{b},h}(j)} \mid j \in \{1, \ldots, n\}\rangle) \end{aligned}$$

for $\mathbf{x} \in A^m$. By the closure properties of a clonoid, $f_1$ and $g_1$ lie in the clonoid $C$.

We will now show that $(f_1(\mathbf{a}), g_1(\mathbf{a}))$ is an element of $\boldsymbol{\varphi}(C, \mathbf{a})$. To this end, let $\mathbf{c} \in A^m$ be such that $\mathbf{c} <_{\mathrm{lex}} \mathbf{a}$. Then we have $f_1(\mathbf{c}) = f(\langle c_{\tau_{\mathbf{a},\mathbf{b},h}(j)} \mid j \in \{1, \ldots, n\}\rangle)$. Since $\mathbf{c} <_{\mathrm{lex}} \mathbf{a}$, Lemma 3.4 yields $\langle c_{\tau_{\mathbf{a},\mathbf{b},h}(j)} \mid j \in \{1, \ldots, n\}\rangle <_{\mathrm{lex}} \mathbf{b}$. Hence we have $f_1(\mathbf{c}) = f(\langle c_{\tau_{\mathbf{a},\mathbf{b},h}(j)} \mid j \in \{1, \ldots, n\}\rangle) = g(\langle c_{\tau_{\mathbf{a},\mathbf{b},h}(j)} \mid j \in \{1, \ldots, n\}\rangle = g_1(\mathbf{c})$. From this, we obtain $(f_1(\mathbf{a}), g_1(\mathbf{a})) \in \boldsymbol{\varphi}(C, \mathbf{a})$. Since by Lemma 3.4 (1), $(f_1(\mathbf{a}), g_1(\mathbf{a})) = (f(\mathbf{b}), g(\mathbf{b})) = (x, y)$, we obtain $(x, y) \in \boldsymbol{\varphi}(C, \mathbf{a})$. □

**Definition 4.4** (cf. [Aic10, Definition 4.2]). Let $t \in \mathbb{N}$, let $C$ be a clonoid with source set $A = \{1, 2, \ldots, t\}$ and target algebra $\mathbf{B}$, and let and let $\alpha \subseteq B \times B$. We define a subset $\Psi(C, \alpha)$ of $A^+$ by $\Psi(C, \alpha) := \{\mathbf{a} \in A^+ \mid \boldsymbol{\varphi}(C, \mathbf{a}) \subseteq \alpha\}$.

Hence $\Psi(C, \alpha)$ is the set of those places at which all forks from $C$ lie in $\alpha$.

**Lemma 4.5** (cf. [Aic10, Lemma 4.3]). *Let $t \in \mathbb{N}$, let $C$ be a clonoid with source set $A = \{1, 2, \ldots, t\}$ and target algebra $\mathbf{B}$, and let $\alpha \subseteq B \times B$. Then $\Psi(C, \alpha)$ is an upward closed subset of $\langle A^+, \leq_{\mathrm{E}}\rangle$.*

*Proof:* Let $\mathbf{a} \in \Psi(C, \alpha)$, and let $\mathbf{b} \in A^+$ such that $\mathbf{a} \leq_{\mathrm{E}} \mathbf{b}$. Since $\mathbf{a} \in \Psi(C, \alpha)$, we have $\boldsymbol{\varphi}(C, \mathbf{a}) \subseteq \alpha$. By Lemma 4.3, we have $\boldsymbol{\varphi}(C, \mathbf{b}) \subseteq \boldsymbol{\varphi}(C, \mathbf{a})$. Therefore, $\boldsymbol{\varphi}(C, \mathbf{b}) \subseteq \alpha$, and thus $\mathbf{b} \in \Psi(C, \alpha)$. □

## 5. Chains of Clonoids

The next lemma allows us to decide when two comparable clonoids are equal.

**Lemma 5.1.** *Let $k, t \in \mathbb{N}$ with $k \geq 2$, let $A := \{1, 2, \ldots, t\}$, let $\mathbf{B}$ be an algebra with $k$-edge term, let $(\mathcal{C}, \subseteq)$ be a linearly ordered set of clonoids with source set $A$ and target algebra $\mathbf{B}$, and let $C, D \in \mathcal{C}$. Then the following are equivalent:*

(1) $C \subseteq D$.
(2) $C^{[t^{k-1}]} \subseteq D^{[t^{k-1}]}$ *and for all* $\alpha \subseteq B \times B$, *we have* $\Psi(D, \alpha) \subseteq \Psi(C, \alpha)$.

*Proof:* (1)⇒(2): Let $\alpha \subseteq B \times B$, and let $\mathbf{a} \in \Psi(D, \alpha)$. Then $\boldsymbol{\varphi}(D, \mathbf{a}) \subseteq \alpha$. Since $C \subseteq D$, we therefore have $\boldsymbol{\varphi}(C, \mathbf{a}) \subseteq \alpha$, and thus $\mathbf{a} \in \Psi(C, \alpha)$.

(2)⇒(1): Since $\mathcal{C}$ is a linearly ordered set of clonoids, we either have $C \subseteq D$ or $D \subseteq C$. In the first case, there is nothing to prove, so we assume $D \subseteq C$.

We will now prove that in this case, we have $C^{[n]} \subseteq D^{[n]}$ for all $n \in \mathbb{N}$. Let us first assume $n \leq t^{k-1}$. In this case, the result is a consequence of the closure property of clonoids that is given in item (2) of Definition 4.1: indeed, if $f \in C^{[n]}$, then $f' : (x_1, \ldots, x_{t^{k-1}}) \mapsto f(x_1, \ldots, x_n)$ is an element of $C^{[t^{k-1}]}$, and hence of $D^{[t^{k-1}]}$. Hence $g : (x_1, \ldots, x_n) \mapsto f'(x_1, \ldots, x_n, x_n, \ldots, x_n)$ lies in $D^{[n]}$, and since $f = g$, we obtain $f \in D^{[n]}$. Let us now assume $n > t^{k-1}$. We consider $F := C^{[n]}$ and $G := D^{[n]}$ as subuniverses of $\mathbf{B}^{A^n}$, and we will employ Lemma 4.2 to show $F = G$. To this end, we first show that for all $T \subseteq A^n$ with $|T| < k$ we have

$$\pi_T(F) \subseteq \pi_T(G). \tag{5.1}$$

Let $f \in F$, and let $T \subseteq A^n$ with $|T| < k$.

For $\mathbf{a} = (a_1, \ldots, a_n)$ in $T$, let $\mu_{\mathbf{a}} := \{(i, j) \in \{1, \ldots, n\}^2 \mid a_i = a_j\}$. Then $\mu_{\mathbf{a}}$ is a partition of $\{1, \ldots, n\}$ into at most $t$ blocks. Let $\mu := \bigcap_{\mathbf{a} \in T} \mu_{\mathbf{a}}$, and let $q$ be the number of equivalence classes modulo $\mu$. Clearly $q \leq t^{k-1}$. Put differently, let $M$ be the $(|T| \times n)$-matrix with entries from $A$ whose rows are the elements of $T$, and let $(i, j) \in \mu$ if the $i$-th column of $M$ is equal to the $j$-th column. Since there are at most $|A|^{|T|}$ different column vectors, $\mu$ has at most $|A|^{|T|} \leq t^{k-1}$ equivalence classes.

Let $S_1, \ldots, S_q$ be the equivalence classes modulo $\mu$, and let $r_1 \in S_1, \ldots, r_q \in S_q$ be representatives of these classes. For $j \in \{1, \ldots, n\}$, let $c(j) \in \{1, \ldots, q\}$ be the unique element such that $j \in S_{c(j)}$. By the definition of a clonoid, $f_\mu : A^q \to B, (x_1, \ldots, x_q) \mapsto f(x_{c(1)}, \ldots, x_{c(n)})$ is in $\mathbf{C}^{[q]}$. Since $q \leq t^{k-1}$, it follows that $f_\mu \in \mathbf{D}^{[q]}$ as well. Hence the function $g \colon A^n \to B, (x_1, \ldots, x_n) \mapsto f_\mu(x_{r_1}, \ldots, x_{r_q})$ is in $\mathbf{D}^{[n]}$. Let $\mathbf{a} = (a_1, \ldots, a_n) \in T$. Then $g(a_1, \ldots, a_n) = f_\mu(a_{r_1}, \ldots, a_{r_q}) = f(a_{r_{c(1)}}, \ldots, a_{r_{c(n)}})$. Now for each $j \in \{1, \ldots, n\}$, we have $r_{c(j)} \in S_{c(j)}$ and $j \in S_{c(j)}$, thus $(r_{c(j)}, j) \in \mu$ and therefore $a_{r_{c(j)}} = a_j$. Therefore, $f(a_{r_{c(1)}}, \ldots, a_{r_{c(n)}}) = f(a_1, \ldots, a_n)$. Hence $\pi_T(f) = \pi_T(g)$, and thus $\pi_T(f) \in \pi_T(G)$, which proves (5.1).

Next we show that for all $\mathbf{a} \in A^n$,

$$\varphi(C, \mathbf{a}) \subseteq \varphi(D, \mathbf{a}). \tag{5.2}$$

In order to prove (5.2), we fix $\mathbf{a} \in A^n$. We obviously have $\mathbf{a} \in \Psi(D, \varphi(D, \mathbf{a}))$. By the assumption (2), we therefore have $\mathbf{a} \in \Psi(C, \varphi(D, \mathbf{a}))$ and (5.2) follows from the definition of $\Psi$.

Now from (5.1), (5.2), and Lemma 4.2 (with $m := |A|^n$), we obtain $C^{[n]} = D^{[n]}$. $\square$

Let $2^{B\times B}$ denote the power set of $B \times B$. On the set $\mathbb{U} := \mathcal{U}(A^+, \leq_{\mathrm{E}})^{2^{B\times B}}$, we define an order as follows: for $\mathbf{S} = \langle S(\alpha) \mid \alpha \subseteq B \times B\rangle$ and $\mathbf{T} = \langle T(\alpha) \mid \alpha \subseteq B \times B\rangle \in \mathbb{U}$, we say $\mathbf{S} \leq \mathbf{T}$ if $S(\alpha) \subseteq T(\alpha)$ for all $\alpha \subseteq B \times B$. Hence $\langle \mathbb{U}, \leq\rangle$ is isomorphic to the $2^{|B|^2}$-fold direct product of $\langle \mathcal{U}(A^+, \leq_{\mathrm{E}}), \subseteq\rangle$. Therefore, it follows from Proposition 3.2 that $(\mathbb{U}, \leq)$ satisfies the (ACC).

Now, as a corollary of Lemma 5.1, we obtain:

**Lemma 5.2.** *Let $k, t \in \mathbb{N}$ with $k \geq 2$, let $A := \{1, 2, \ldots, t\}$, and let $\mathbf{B}$ be an algebra with a $k$-edge term. Let $(\mathcal{C}, \subseteq)$ be a linearly ordered set of clonoids with source set $A$ and target algebra $\mathbf{B}$, let $\mathbb{T}$ denote the power set of $B^{A^{t^{k-1}}}$, and let $\mathbb{U} := \mathcal{U}(A^+, \leq_{\mathrm{E}})^{2^{B\times B}}$. Let $R : \mathcal{C} \to \mathbb{T} \times \mathbb{U}$ be defined by*

$$R(C) := (C^{[t^{k-1}]}, \langle \Psi(C, \alpha) \mid \alpha \subseteq B \times B\rangle)$$

*for $C \in \mathcal{C}$. Then $R$ is injective, and for all $C, D \in \mathcal{C}$ with $C \subseteq D$, we have $R(C)_1 \subseteq R(D)_1$ and $R(D)_2 \leq R(C)_2$.*

**Theorem 5.3.** *Let $\mathbf{B}$ be a finite algebra with an edge term, and let $A$ be a nonempty finite set. Let $\mathcal{C} := \{C \mid C$ is clonoid with source set $A$ and target algebra $\mathbf{B}\}$. Then $(\mathcal{C}, \subseteq)$ satisfies the descending chain condition.*

*Proof:* Let $\mathcal{C}$ be an infinite descending chain of clonoids with source set $A$ and target algebra $\mathbf{B}$. From these clonoids, Lemma 5.2 produces an infinite ascending chain in $\langle \mathcal{U}(A^+, \leq_{\mathrm{E}}), \subseteq\rangle^{2^{B\times B}}$, which contradicts Proposition 3.2. $\square$

## 6. Varieties

For an algebra $\mathbf{A}$, we will introduce a clonoid $C$ with source set $A$ and target algebra $\mathbf{A} \times \mathbf{A}$ such that the equational theory of each subvariety of $\mathbb{V}(\mathbf{A})$ corresponds to a "subclonoid" of $C$.

**Definition 6.1.** Let $\mathbf{A}$ be an algebra, and let $W$ be a subvariety of $\mathbb{V}(\mathbf{A})$. By $\mathrm{Th}_{\mathbf{A}}(W)$ we denote the clonoid with source set $A$ and target algebra $\mathbf{A} \times \mathbf{A}$ defined by

$$\begin{aligned}(6.1)\quad \mathrm{Th}_{\mathbf{A}}(W) := \{(a_1, \dots, a_k) \mapsto \begin{pmatrix} s^{\mathbf{A}}(\mathbf{a}) \\ t^{\mathbf{A}}(\mathbf{a}) \end{pmatrix} \mid k \in \mathbb{N}, \\ s, t \text{ are } k\text{-variable terms in the language of } \mathbf{A} \text{ with } W \models s \approx t\}.\end{aligned}$$

We note that $\mathrm{Th}_{\mathbf{A}}(W)$ clearly satisfies the closure properties from the definition of a clonoid.

**Lemma 6.2.** *Let* $\mathbf{A}$ *be an algebra, and let* $W_1$ *and* $W_2$ *be subvarieties of* $\mathbb{V}(\mathbf{A})$. *Then we have:*

$$(6.2)\qquad W_1 \subseteq W_2 \textit{ if and only if } \mathrm{Th}_{\mathbf{A}}(W_2) \subseteq \mathrm{Th}_{\mathbf{A}}(W_1).$$

*Proof:* For the "only if"-direction, we observe that for all terms $s, t$ in the language of $\mathbf{A}$ with $W_2 \models s \approx t$, we have $W_1 \models s \approx t$. This implies that $\mathrm{Th}_{\mathbf{A}}(W_2) \subseteq \mathrm{Th}_{\mathbf{A}}(W_1)$. For the "if"-direction, we show that every identity that is satisfied in $W_2$ is also satisfied in $W_1$. To this end, we let $k \in \mathbb{N}$, and we let $s, t$ be two $k$-variable terms such that $W_2 \models s \approx t$. Then the $k$-ary function $\mathbf{a} \mapsto \begin{pmatrix} s^{\mathbf{A}}(\mathbf{a}) \\ t^{\mathbf{A}}(\mathbf{a}) \end{pmatrix}$ lies in $\mathrm{Th}_{\mathbf{A}}(W_2)$, and therefore, by the assumption, also in $\mathrm{Th}_{\mathbf{A}}(W_1)$. Hence there exist $k$-ary terms $s_1, t_1$ such that $s^{\mathbf{A}} = s_1^{\mathbf{A}}$, $t^{\mathbf{A}} = t_1^{\mathbf{A}}$, and $W_1 \models s_1 \approx t_1$. Since $W_1 \subseteq \mathbb{V}(\mathbf{A})$, we have $W_1 \models s \approx s_1$ and $W_1 \models t \approx t_1$. Hence $W_1 \models s \approx t$. Therefore, every identity satisfied in $W_2$ is satisfied in $W_1$, and thus $W_1 \subseteq W_2$. $\Box$

We can now give the proof of the main results:

*Proof of Theorem 1.1:* Let $W_1 \subset W_2 \subset W_3 \subset \dots$ be a strictly increasing sequence of subvarieties of $\mathbb{V}(\mathbf{A})$. Then by Lemma 6.2, $\mathrm{Th}_{\mathbf{A}}(W_1) \supset \mathrm{Th}_{\mathbf{A}}(W_2) \supset \mathrm{Th}_{\mathbf{A}}(W_3) \supset \dots$ is an infinite strictly decreasing subsequence of clonoids with source set $A$ and target algebra $\mathbf{A} \times \mathbf{A}$, contradicting Theorem 5.3. Hence the subvarieties of $\mathbb{V}(A)$ satisfy the ascending chain condition. $\Box$

*Proof of Theorem 1.2:* The result follows from Theorem 1.1 and the implication (1)⇒(2) of Lemma 2.2. $\Box$

## 7. On the descending chain condition for subvarieties

In this section, we will investigate the relation of Theorem 1.2 to the finite basis property. For varieties $V$ and $W$ of the same type, we write $V \prec W$ if $V \subset W$ and there is no variety $V'$ with $V \subset V' \subset W$, and we say that $V$ is a *subcover* of $W$. We will first prove a lemma that states that in the lattice of all varieties of a given type, every finitely generated variety has only finitely many subcovers. This property resembles the fact that every finitely related clone on a finite set has only finitely many covers [PK79, p. 93, 4.1.3].

**Lemma 7.1.** *Let* $\mathbf{A}$ *be a finite algebra. Then we have:*

(1) *There are only finitely many varieties* $W$ *with* $W \prec \mathbb{V}(\mathbf{A})$.
(2) *For each variety* $U \subset \mathbb{V}(\mathbf{A})$ *there is a variety* $W$ *with* $U \subseteq W \prec \mathbb{V}(\mathbf{A})$.

*Proof:* Let $k \in \mathbb{N}$ be such that $\mathbf{A}$ is generated by $k$ elements. We first show that for all varieties $W \subseteq \mathbb{V}(\mathbf{A})$ we have:

$$\text{If } \mathrm{Th}^{[k]}_{\mathbf{A}}(W) = \mathrm{Th}^{[k]}_{\mathbf{A}}(\mathbb{V}(\mathbf{A})), \text{ then } W = \mathbb{V}(\mathbf{A}). \tag{7.1}$$

To this end, we show $\mathbf{A} \in W$. By Lemma 2.1, it is sufficient to check that every identity $s \approx t$ of $W$ with at most $k$ variables is satisfied by $\mathbf{A}$ . Letting $s^{\mathbf{A}}$ and $t^{\mathbf{A}}$ be the $k$-ary term functions induced by $s$ and $t$, we see that $(s^{\mathbf{A}}, t^{\mathbf{A}}) \in \mathrm{Th}^{[k]}_{\mathbf{A}}(W)$. Therefore $(s^{\mathbf{A}}, t^{\mathbf{A}}) \in \mathrm{Th}^{[k]}_{\mathbf{A}}(\mathbb{V}(\mathbf{A}))$. This implies that there are $k$-ary terms $s_1, t_1$ such that $(s_1^{\mathbf{A}}, t_1^{\mathbf{A}}) = (s^{\mathbf{A}}, t^{\mathbf{A}})$ and $\mathbb{V}(\mathbf{A}) \models s_1 \approx t_1$. From this, we get $s^{\mathbf{A}} = t^{\mathbf{A}}$, and hence $\mathbf{A} \models s \approx t$. This completes the proof of (7.1).

Let $R$ be a set of representatives of the $k$-ary terms in the language of $\mathbf{A}$ modulo the equivalence relation defined by $s \sim_{\mathbf{A}} t \Leftrightarrow s^{\mathbf{A}} = t^{\mathbf{A}}$. Let

$$S := \{\mathbb{V}(\mathbf{A}) \cap \mathrm{Mod}(s \approx t) \mid s, t \in R, s \not\sim_{\mathbf{A}} t\}.$$

Note that $R$ and hence $S$ is finite. Since no variety in $S$ contains $\mathbf{A}$, all varieties in $S$ are strictly smaller than $\mathbb{V}(\mathbf{A})$. We now show that for every variety $U$ with $U \subset \mathbb{V}(\mathbf{A})$ there is a variety $W \in S$ such that $U \subseteq W$. Since $U \subset \mathbb{V}(\mathbf{A})$, (7.1) yields $k$-ary terms $s, t$ such that $U \models s \approx t$ and $s^{\mathbf{A}} \neq t^{\mathbf{A}}$. Let $s_1, t_1 \in R$ with $s_1^{\mathbf{A}} = s^{\mathbf{A}}$ and $t_1^{\mathbf{A}} = t^{\mathbf{A}}$. Then, since $U \subseteq \mathbb{V}(\mathbf{A})$, we have $U \models s_1 \approx s$ and $U \models t_1 \approx t$. Therefore $U \models s_1 \approx t_1$, and thus $U \subseteq \mathbb{V}(\mathbf{A}) \cap \mathrm{Mod}(s_1 \approx t_1)$, and this last variety lies in $S$.

Let $U$ be a variety with $U \prec \mathbb{V}(\mathbf{A})$. Then there is $W \in S$ with $U \subseteq W \subset \mathbb{V}(\mathbf{A})$, and therefore $U = W$ and thus $U \in S$. Therefore all varieties $U$ with $U \prec \mathbb{V}(\mathbf{A})$ are among the finitely many elements of $S$; in fact, the cardinality of $S$ satisfies $|S| \leq |\mathrm{Clo}_k(\mathbf{A})|^2$, where $\mathrm{Clo}_k(\mathbf{A})$ denotes the set of $k$-ary term functions of $\mathbf{A}$. This completes the proof of item (1).

For proving (2), we let $U$ be a variety with $U \subset \mathbb{V}(\mathbf{A})$. Let $W$ be maximal in $\{W' \in S \mid U \subseteq W'\}$. We next show $W \prec \mathbb{V}(\mathbf{A})$. Suppose there is $W_1$ with $W \subset W_1 \subset \mathbb{V}(\mathbf{A})$. Then there is $W_2 \in S$ with $W_1 \subseteq W_2 \subset \mathbb{V}(\mathbf{A})$. Then

$W \subset W_2$, contradicting the maximality of $W$. Therefore $W \prec \mathbb{V}(\mathbf{A})$, and $U \subseteq W \prec \mathbb{V}(\mathbf{A})$. $\square$

*Proof of Corollary 1.5:* (1)⇒(2): Let $S := \{W \mid W \leq \mathbb{V}(\mathbf{A})\}$ be the class of subvarieties of $\mathbb{V}(\mathbf{A})$. By Theorem 1.2, every variety in $S$ is finitely generated, and so the assumptions yield that every variety in $S$ is finitely based relative to $\mathbb{V}(\mathbf{A})$. From Lemma 2.3, we obtain that $S$ has no infinite strictly descending chains. Thus, supposing that $S$ is infinite, we may pick $W$ minimal in $\{W' \mid W'$ has infinitely many subvarieties$\}$. Since $W$ is a subvariety of $\mathbb{V}(\mathbf{A})$, it is finitely generated by Theorem 1.2, and therefore, by Lemma 7.1, there must be a $W_1 \prec W$ such that $W_1$ has infinitely many subvarieties. This contradicts the minimality of $W$. Hence $\mathbb{V}(\mathbf{A})$ has only finitely many subvarieties.

(2)⇒(3): Let $(\mathbf{A}_i)_{i\in\mathbb{N}}$ be an infinite sequence of nonisomorphic cardinality critical algebras in $\mathbb{V}(\mathbf{A})$. By the assumption, there is an infinite subset $T$ of $\mathbb{N}$ such that $\mathbb{V}(\mathbf{A}_s) = \mathbb{V}(\mathbf{A}_t)$ for all $s, t \in T$. We will now show that $|\mathbf{A}_s| = |\mathbf{A}_t|$ for all $s, t \in T$. Suppose $|\mathbf{A}_s| < |\mathbf{A}_t|$. Since $\mathbf{A}_s \in \mathbb{V}(\mathbf{A}_t)$, we have $\mathbb{V}(\mathbf{A}_s) \subseteq \mathbb{V}(\{\mathbf{B} \mid \mathbf{B} \in \mathbb{V}(\mathbf{A}_t), |\mathbf{B}| < |\mathbf{A}_t|\})$. Since $\mathbf{A}_t \in \mathbb{V}(\mathbf{A}_s)$, this yields that $\mathbf{A}_t$ lies in the variety generated by the members of $\mathbb{V}(\mathbf{A}_t)$ of cardinality $< |\mathbf{A}_t|$, contradicting the fact that $\mathbf{A}_t$ is cardinality critical. Since $\mathbb{V}(\mathbf{A})$ is locally finite, it contains only finitely many nonisomorphic members of the same finite cardinality because each $k$-element algebra is a homomorphic image of the free algebra $\mathbf{F}_{\mathbb{V}(\mathbf{A})}(k)$. Hence $T$ must be finite, a contradiction.

(3)⇒(4): This follows from Lemma 2.4.

(4)⇒(1): This follows from Lemma 2.3.

$\square$

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