fg-sub-var

Research on finitely generated subvarieties of finitely generated varieties

Problem

Characterize the finitely generated varieties that have (or don't have) finitely generated proper subvarieties.

In [A-M], Aichinger and Mayr solve the problem in case $\mathbb{V}(\mathbf{A})$ is either congruence distributive or has an *edge term*. (TODO: check this and say more about cases handed in [A-M])

Here we consider case in which $\mathbf{A}$ is finite and idempotent, and $\mathbb{V}(\mathbf{A})$ is congruence modular, *not* congruence distributive, and does not have an edge term. (The latter is equivalent to having a so called cube term blocker.)

One such algebra is the groupoid $\mathbf{A} = \langle\{0, 1, 2, 3\}, *\rangle$ with the following operation table:

*	0	1	2	3
0	0	2	1	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

The variety $\mathbb{V}(\mathbf{A})$ generated by $\mathbf{A}$ is CM, not CD, and has no edge term—the latter, since $(\{1, 2, 3\}, \{0, 1, 2, 3\})$ is a cube term blocker.

Proper subvarieties

Every algebra is a subdirect product of subdirectly irreducible (SI) algebras, so varieties are generated by their SI members. To find a proper subvariety of $\mathbb{V}(\mathbf{A})$, we will look for an algebra $\mathbf{D} \in \mathbb{V}(\mathbf{A})$ and an SI in $\mathbb{V}(\mathbf{A})$ that does not belong to $\mathbb{V}(\mathbf{D})$.

Subdirectly Irreducibles

To begin we will try to identify the SI's of $\mathbb{V}(\mathbf{A})$.

First, recall the general form of an algebra in $\mathbb{V}(\mathbf{A})$. If $\mathbf{B} \in \mathbb{V}(\mathbf{A}) = \mathbf{HSP}(\mathbf{A})$, then there exists $n < \omega$, $\mathbf{C}$, and $\theta \in \operatorname{Con}(\mathbf{C})$ such that $\mathbf{C} \leq \mathbf{A}^n$ and $\mathbf{B} \cong \mathbf{C}/\theta$.

In our example, the only nontrivial proper subalgebra of $\mathbf{A}$ is the 3-element squag, which we denote by $\mathbf{S}$.

Let $\pi_i : C \to A$ denote the projection of C onto its i-th coordinate, with kernel $\eta_i = \ker \pi_i$. Let's assume for now that

$\pi_i C = A$, for $1 \leq i \leq m$, and

$\pi_i C = S$, for $m < i \leq n$.

Let $\eta_A = \bigcap_{1\leq i \leq m} \eta_i$ and $\eta_S = \bigcap_{m< i \leq n} \eta_i$.

Then $\mathbf{C} \leq_{sd} \mathbf{A} \times \cdots \times \mathbf{A} \times \mathbf{S} \times \cdots \times \mathbf{S}$ a project with m factors of $\mathbf{A}$, and $m-n$ factors of $\mathbf{S}$.

(Why don't we simply write $\mathbf{C} \leq_{sd} \mathbf{A}^m \times \mathbf{S}^{n-m}$? Because that would be a bit misleading, since $\mathbf{C}$ does not necessary project onto either of these two factors.)

Let $P := \mathrm{Proj}_{m+1,...,n} C$ be the (simultaneous) projection of C onto the $(m+1, \dots, n)$-factors of the product. (As indicated above, P may be a proper subset of S^{n-m}.) Let $\mathbf{P}$ be the algebra in $\mathbb{V}(\mathbf{A})$ with universe P. Then

$$\mathbf{C} \leq_{sd} \mathbf{A} \times \cdots \times \mathbf{A} \times \mathbf{P}.$$

Now, $\mathbf{A}$ is nonabelian. If we can show that $\mathbf{A}$ is also absorption-free, then it would follow by the Rectangularity Theorem that $\mathbf{C} = \mathbf{A}^m \times \mathbf{P}$.

Is $\mathbf{A}$ absorption-free?

TODO: check!

Finite powers of $\mathbf{A}$ are skew-free

Recall, [Theorem 2.3, K] says that if $\mathbf{A}$ is an idempotent simple algebra, then exactly one of the following is true:

1. $\mathbf{A}$ has a unique absorbing element.
2. $\mathbf{A}$ is abelian.
3. Every finite power of $\mathbf{A}$ is skew-free.

In our example, neither 1 nor 2 holds, so every finite power of $\mathbf{A}$ is skew-free.

TODO: What does this tell us about $\mathbf{B} \cong \mathbf{C}/\theta \leq \mathbf{A}^n$?

Subdirectly Irreducibles: results from the commutator book

Let $\mathbf{B} \cong \mathbf{C}/\theta$ for some $\mathbf{C} \leq \mathbf{A}^n$. Suppose $\mathbf{B}$ is SI and let μ be its unique minimal nontrivial congruence (*monolith*). Then there exists a corresponding congruence $\psi \in \mathrm{Con}(\mathbf{C})$ that is a join irreducible covering of θ, and satisfies $\mathbf{B}/\mu \cong \mathbf{C}/\psi$.

Let $\alpha = (0 : \mu)$ be the centralizer of μ in $\mathrm{Con}(\mathbf{B})$. That is, α is the largest congruence $\gamma \in \mathrm{Con}(\mathbf{B})$ such that $[\mu, \gamma] = 0$.

By [Theorem 10.1, F-M], if $\varphi = (\theta : \psi) \in \mathrm{Con}(\mathbf{C})$ is the centralizer of ψ over θ—that is, $[\psi, \varphi] \leq \theta$ and φ is the largest such congruence—then $\mathbf{B}/\alpha \cong \mathbf{C}/\varphi$, and $\mathbf{B}/\alpha \in \mathbf{HS}(\mathbf{A})$.

Note that, in our example, $\mathbf{HS}(\mathbf{A})$ is merely $\{\mathbf{A}, \mathbf{S}, \{*\}\}$. (Here $\{*\}$ is any of the 1-element subalgebras of $\mathbf{A}$.)

So, we have two possibilities: Either

1. $\alpha \geq \mu$, in which case $[\mu, \mu] = 0$ (i.e., μ is abelian), or
2. $\alpha = 0$, in which case $\mathbf{B} \in \{\mathbf{A}, \mathbf{S}, \{*\}\}$.

Since we are interested in finding SI's other than the trivial ones appearing in case 2, we focus on case 1, and will assume from now on that $\mathbf{B}$ is a finite SI in $\mathbb{V}(\mathbf{A})$ with abelian monolith μ.

Other 4-Element Examples

Here are the operation tables of a few other examples of idempotent groupoids with the properties mentioned (i.e., $\mathbb{V}(\mathbf{A})$ is CM, not CD, CTB).

$\circ_1$	**0**	**1**	**2**	**3**
0	0	2	1	3
1	0	1	3	2
2	1	3	2	1
3	1	2	1	3

$\circ_2$	**0**	**1**	**2**	**3**
0	0	2	3	1
1	0	1	3	2
2	2	3	2	1
3	2	2	1	3

$\circ_3$	**0**	**1**	**2**	**3**
0	0	3	2	1
1	0	1	3	2
2	3	3	2	1
3	3	2	1	3

$\circ_4$	**0**	**1**	**2**	**3**
0	0	3	1	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

6-Element Examples

There are also 6-element algebras with these properties. Take, for example, the algebra $\mathbf{A}_6 = \langle \{0, 1, 2, 3, 4, 5\}, \circ \rangle$ with operation table shown below.

*	0	1	2	3	4	5
0	0	2	1	1	1	1
1	0	1	3	5	2	4
2	0	3	2	4	5	1
3	0	5	4	3	1	2
4	0	2	5	1	4	3
5	1	4	1	2	3	5

More generally, the following table yields a $2n$-element algebra with the same properties (i.e., $\mathbb{V}(\mathbf{A})$ is CM, not CD, CTB):

*	0	1	2	3	...	$2n-1$
0	0	2	1	1	$\cdots$	1
1	0					
2	0					

*	0	1	2	3	...	
$\vdots$	$\vdots$				Squag_{2n-1}	
$2n-2$	0					
$2n-1$	1					

Specializing to 3-permutability for now, a **local Hagemann-Mitchke sequence** for the set

$$\{(a_0, b_0, 0), (a_1, b_1, 1), (a_2, b_2, 2)\} \subseteq A \times A \times \{0, 1, 2\}$$

is a triple, $\mathbf{p} = (p_0, p_1, p_2)$, of 3-place terms satisfying

$$\begin{aligned} a_0 &= p_0(a_0, a_0, b_0) = p_1(a_0, b_0, b_0) \\ &= p_1(a_1, a_1, b_1) = p_2(a_1, b_1, b_1) \\ &= p_2(a_2, a_2, b_2) = p_3(a_2, b_2, b_2) = b_2 \end{aligned}$$

Where we take p_0 and p_3 to be the first and third projections, respectively.

More generally, for $0 \leq i < m$ and $(a_i, b_i, i) \in A \times A \times \{0, 1, \dots, m-1\}$, we call $p = (p_0, p_1, \dots, p_m)$ a **local Hagemann-Mitchke sequence** for (a_i, b_i, i) if $p_i(a_i, a_i, b_i) = p_{i+1}(a_i, b_i, b_i)$, where p_0, p_m are the first and third projections, respectively.

Valeriote and Willard prove in [V-W] that an idempotent algebra $\mathbf{A}$ generates a congruence $(n+1)$-permutable variety if and only if every subset of $A \times A \times \{0, 1, \dots, n\}$ has a local Hagemann-Mitchke sequence.

Question: Can the pair of terms below be used as the (non-projection) terms in a local Hagemann-Mitchke sequence $\mathbf{p} = (p_0, p_1, p_2, p_3)$ for the examples above.

$$p_1(x, y, z) = (x(yz))x$$

$$p_2(x, y, z) = (x(x(yz)))x$$

Answer: No, these don't work. (insert counterexample)

Here are some other tables that also yield algebras generating a CM, non-distributive variety with no cube term:

Checking the examples in UACalc

All of the examples mentioned above, and the three properties claimed (CM, not CD, no edge term), can be checked by pasting the following xml code into a file (say, `Groupoids.ua`) and then opening that file in the UACalc. To check the properties, from the "Idempotent Alg" menu, select the following:

- Is V(A) CD (answer "no")
- Is V(A) CM (answer "yes")
- Edge term for some k (answer "no")

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References

- **[A-M]** Aichinger and Mayr, “Finitely generated equational classes,” 2016.
- **[F-M]** Freese and McKenzie, “Commutator theory for congruence modular varieties,” 1987.
- **[K]** Kearnes, “Idempotent simple algebras,” 1994.
- **[V-W]** Valeriote and Willard, “Idempotent $\{n\}$-permutable varieties,” 2014.

BibTeX entries for references cited


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