INDEPENDENCE OF ALGEBRAS WITH EDGE TERM

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Dedicated to our teacher Günter Pilz on the occasion of his 70th birthday

Abstract. In [4], two varieties V, W of the same type are defined to be *independent* if there is a binary term $t(x,y)$ such that $V \models t(x,y) \approx x$ and $W \models t(x,y) \approx y$. In this note, we give necessary and sufficient conditions for two finite algebras with a Mal'cev term (or, more generally, with an edge term) to generate independent varieties. In particular we show that the independence of finitely generated varieties with edge term can be decided by a polynomial time algorithm.

1. Independent varieties and algebras

In this note, we search for conditions on two varieties of the same type to be independent. This notion of independence was introduced in [4]. Foster calls a finite sequence $(V_i)_{i\in\{1,\dots,n\}}$ of subvarieties of a variety W *independent* if there exists a term $t(x_1,\dots,x_n)$ such that for each $i \in \{1,\dots,n\}$, $V_i \models t(x_1,\dots,x_n) \approx x_i$ [4, Lemma 2.1]. Grätzer, Lakser, and Płonka proved that for two independent varieties V_1 and V_2, every algebra in the join $V_1 \vee V_2$ is isomorphic to a direct product $\mathbf{A}_1 \times \mathbf{A}_2$ with $\mathbf{A}_1 \in V_1$ and $\mathbf{A}_2 \in V_2$. It is easy to see that two independent varieties V_1 and V_2 are *disjoint*, meaning that $V_1 \cap V_2$ only contains one element algebras. If V_1 and V_2 are subvarieties of a congruence permutable variety, then the converse holds: in fact, Hu and Kelenson proved that a sequence $(V_1,\dots,V_n)$ of subvarieties of a congruence permutable variety is independent if and only if V_i and V_j are disjoint for all distinct i,j [7, Corollary 2.9]. Freese and McKenzie showed that if V_1 and V_2 are disjoint subvarieties of a congruence modular such that at least one of the varieties is solvable, then V_1 and V_2 are independent [5, Theorem 11.3]. Jónsson and Tsinakis proved that the join of two independent finitely based varieties of finite type is again finitely based [8, Theorem 3.3]; a different finite axiomatization of the join is given in [11, Theorem 3.9]. In this

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paper, Kowalski, Paoli, and Ledda also gave a characterization of independence for disjoint varieties by a Mal'cev-type condition [11, Theorem 3.2].

Two algebras $\mathbf{A}$ and $\mathbf{B}$ from the same variety are called *independent* if they generate independent varieties; this is equivalent to the existence of a binary term $t(x, y)$ such that $\mathbf{A} \models t(x, y) \approx x$ and $\mathbf{B} \models t(x, y) \approx y$. Let $V = V(\mathbf{A})$, the variety generated by $\mathbf{A}$, let $W = V(\mathbf{B})$, and let $\mathbf{F}_V(2)$ and $\mathbf{F}_W(2)$ be the free algebras in V and W over 2 generators. It is not too hard to see (and will be proved in Lemma 5.1) that the following condition is equivalent to the independence of V and W:

$$\mathbf{F}_V(2) \times \mathbf{F}_W(2) \text{ is the only subdirect product of } \mathbf{F}_V(2) \times \mathbf{F}_W(2).$$

Hence the independence of V and W can be determined from the subuniverses of $\mathbf{A}^{A^2} \times \mathbf{B}^{B^2}$. In this note, we will see that for finite algebras $\mathbf{A}$ and $\mathbf{B}$, the independence of $V(\mathbf{A})$ and $V(\mathbf{B})$ can be determined from the subuniverses of $\mathbf{A}^2 \times \mathbf{B}^2$ if $\mathbf{A}$ and $\mathbf{B}$ have a common Mal'cev term, and from the subuniverses of $\mathbf{A}^{k-1} \times \mathbf{B}^{k-1}$ if $\mathbf{A}$ and $\mathbf{B}$ have a common k-edge term with $k \geq 3$. From this we obtain a polynomial time algorithm for deciding the independence of two finite algebras of finite type with edge term. As another application, we obtain a new proof of the description of polynomial functions on direct products without skew-congruences from [9].

2. Product subalgebras

In this section, we will describe the shape of subuniverses of direct products of powers of two algebras. For a direct product $\mathbf{E} \times \mathbf{F}$, we define $\pi_{\mathbf{E}}(e, f) = e$ and $\pi_{\mathbf{F}}(e, f) = f$ for all $e \in E$, $f \in F$.

Definition 2.1. Let $\mathbf{E}$ and $\mathbf{F}$ be two similar algebras. We call a subalgebra $\mathbf{C}$ of $\mathbf{E} \times \mathbf{F}$ a *product subalgebra* if $\mathbf{C} = \pi_{\mathbf{E}}(\mathbf{C}) \times \pi_{\mathbf{F}}(\mathbf{C})$.

Hence $\mathbf{C}$ is a product subalgebra of $\mathbf{E} \times \mathbf{F}$ if and only if for all $(e_1, f_1) \in C$ and $(e_2, f_2) \in C$, we have $(e_1, f_2) \in C$. We note that in this paper, the concept of product subalgebras only refers to subalgebras of direct products of *two* algebras. If we say that for similar algebras $\mathbf{A}$ and $\mathbf{B}$ and for $m, n \in \mathbb{N}$, a subalgebra $\mathbf{C}$ of $\mathbf{A}^m \times \mathbf{B}^n$ is a product subalgebra, we mean that $(a, b) \in C$ and $(c, d) \in C$ implies $(a, d) \in C$ for all $a, c \in A^m$ and $b, d \in B^n$. We recall that a *tolerance relation* of an algebra $\mathbf{A}$ is a subalgebra of $\mathbf{A} \times \mathbf{A}$ that is a reflexive and symmetric relation on A.

Definition 2.2. Let $\mathbf{A}$ and $\mathbf{B}$ be similar algebras, let α be a subset of $A \times A$, and let β be a subset of $B \times B$. Then the *product* $\alpha \times_c \beta$ is defined by

$$\alpha \times_c \beta = \{((a_1, b_1), (a_2, b_2)) \in (A \times B) \times (A \times B) \mid (a_1, a_2) \in \alpha, (b_1, b_2) \in \beta\}.$$

A *product tolerance* of the direct product $\mathbf{A} \times \mathbf{B}$ is a tolerance γ on $\mathbf{A}$ such that $\gamma = \alpha \times_c \beta$ for some tolerances α of $\mathbf{A}$ and β of $\mathbf{B}$; and γ is *product congruence* of $\mathbf{A} \times \mathbf{B}$ if there are $\alpha \in \operatorname{Con}(\mathbf{A})$ and $\beta \in \operatorname{Con}(\mathbf{B})$ such that $\gamma = \alpha \times_c \beta$.

Our main results are the following two theorems.

Theorem 2.3. *Let $\mathbf{A}, \mathbf{B}$ be algebras in a congruence permutable variety. We assume that*

(1) *all subalgebras of $\mathbf{A} \times \mathbf{B}$ are product subalgebras, and*
(2) *for all subalgebras $\mathbf{E}$ of $\mathbf{A}$ and $\mathbf{F}$ of $\mathbf{B}$, all congruences of $\mathbf{E} \times \mathbf{F}$ are product congruences.*

Then for all $m, n \in \mathbb{N}_0$, all subalgebras of $\mathbf{A}^m \times \mathbf{B}^n$ are product subalgebras.

The proof is given in Section 3. We will generalize this result from congruence permutable varieties to varieties with an edge term [2]. Let $k \in \mathbb{N}, k \geq 2$. A $(k+1)$-ary term t in the language of a variety V is a *k-edge term* if

$$t\begin{pmatrix} x & x & y & \dots & \dots & y \\ x & y & x & \ddots & & \vdots \\ y & y & y & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & y \\ y & y & y & \dots & y & x \end{pmatrix} \approx \begin{pmatrix} y \\ \vdots \\ \vdots \\ \vdots \\ y \end{pmatrix}.$$

We note that a variety has an 2-edge term if and only if it has a Mal'cev term. Every variety with a Mal'cev term or with a near-unanimity term has an edge term. The class of algebras with an edge term therefore contains all groups and their expansions (such as rings, vector spaces, Lie algebras ...), all quasigroups, loops, as well as all lattices and their expansions.

Theorem 2.4. *Let $k \geq 2$, let $\mathbf{A}, \mathbf{B}$ be algebras in a variety with k-edge term. We assume that*

(1) *for all $r, s \in \mathbb{N}$ with $r + s \leq \max(2, k-1)$, every subalgebra of $\mathbf{A}^r \times \mathbf{B}^s$ is a product subalgebra, and*
(2) *for all subalgebras $\mathbf{E}$ of $\mathbf{A}$ and $\mathbf{F}$ of $\mathbf{B}$, every tolerance of $\mathbf{E} \times \mathbf{F}$ is a product tolerance.*

Then for all $m, n \in \mathbb{N}_0$, every subalgebra of $\mathbf{A}^m \times \mathbf{B}^n$ is a product subalgebra.

The proof is given in Section 4. In an algebra with a Mal'cev term, all tolerances are congruences. Hence Theorem 2.3 is really a special case of Theorem 2.4.

In Theorem 5.2, 5.3, respectively, we show that finite algebras $\mathbf{A}, \mathbf{B}$ that satisfy the assumptions of Theorem 2.3, 2.4, respectively, are actually independent. In

Example 6.2 we provide examples that show that in general independence does not follow for infinite $\mathbf{A}$ and $\mathbf{B}$.

3. Mal'cev algebras

In this section, we give a proof of Theorem 2.3. From a logical point of view, this section could be omitted because Theorem 2.3 is a corollary of Theorem 2.4. However, we think it is instructive to see the ideas of the proof first in this case.

Proof of Theorem 2.3: Let $\mathbf{A}, \mathbf{B}$ satisfy the assumptions of Theorem 2.3. We will prove the claim by showing that for all $m, n \in \mathbb{N}_0$ and for every subalgebra $\mathbf{C}$ of $\mathbf{A}^m \times \mathbf{B}^n$, we have

$$\mathbf{C} = \pi_{A^m}(\mathbf{C}) \times \pi_{B^n}(\mathbf{C}). \tag{3.1}$$

We will proceed by induction on $n + m$.

For the induction base, we set $m := 0, n := 0$. The only subalgebra of $\mathbf{A}^0 \times \mathbf{B}^0$ is clearly a product subalgebra.

For the induction step, we let $n, m \in \mathbb{N}_0$ be such that $n + m \geq 1$. In the case that $m = 0$ or $n = 0$, the equality (3.1) clearly holds. Now we assume $n \geq 1$ and $m \geq 1$, and we let $\mathbf{C}$ be a subalgebra of $\mathbf{A}^m \times \mathbf{B}^n$. We define $\sigma : A^m \times B^n \to A^{m-1} \times B^n$ by

$$\sigma((a_1, \ldots, a_m), (b_1, \ldots, b_n)) = ((a_1, \ldots, a_{m-1}), (b_1, \ldots, b_n)),$$

and $\tau : A^m \times B^n \to A^m \times B^{n-1}$ by

$$\tau((a_1, \ldots, a_m), (b_1, \ldots, b_n)) = ((a_1, \ldots, a_m), (b_1, \ldots, b_{n-1}))$$

for all $a_1, \ldots, a_m \in A, b_1, \ldots, b_n \in B$.

We are now ready to prove the non-trivial inclusion $\supseteq$ of (3.1). To this end, let $((a_1, \ldots, a_m), (b_1, \ldots, b_n)) \in \pi_{A^m}(\mathbf{C}) \times \pi_{B^n}(\mathbf{C})$. Then $((a_1, \ldots, a_{m-1}), (b_1, \ldots, b_n))$ is an element of $\pi_{A^{m-1}}(\sigma(\mathbf{C})) \times \pi_{B^n}(\sigma(\mathbf{C}))$. Therefore, by the induction hypothesis, we have $((a_1, \ldots, a_{m-1}), (b_1, \ldots, b_n)) \in \sigma(\mathbf{C})$. Thus, there is $c \in A$ such that

$$((a_1, \ldots, a_{m-1}, c), (b_1, \ldots b_n)) \in C. \tag{3.2}$$

Furthermore, we have

$$((a_1, \ldots, a_m), (b_1, \ldots, b_{n-1})) \in \pi_{A^m}(\tau(\mathbf{C})) \times \pi_{B^{n-1}}(\tau(\mathbf{C})).$$

Therefore, by the induction hypothesis, we have $((a_1, \ldots, a_m), (b_1, \ldots, b_{n-1})) \in \tau(\mathbf{C})$. Hence there is $d \in B$ such that

$$((a_1, \ldots, a_m), (b_1, \ldots, b_{n-1}, d)) \in C. \tag{3.3}$$

Next, we define a subset α of $(A \times B)^2$ by

(3.4)
$$\alpha := \{((x_m, y_n), (x'_m, y'_n)) \mid \exists (x_1, \dots, x_{m-1}) \in A^{m-1}, (y_1, \dots, y_{n-1}) \in B^{n-1} : \\ ((x_1, \dots, x_{m-1}, x_m), (y_1, \dots, y_{n-1}, y_n)) \in C \text{ and} \\ ((x_1, \dots, x_{m-1}, x'_m), (y_1, \dots, y_{n-1}, y'_n)) \in C\}.$$

It is easy to see that α is a reflexive relation on

$$S := \{(x_m, y_n) \mid ((x_1, \dots, x_m), (y_1, \dots, y_n)) \in C\}.$$

Furthermore, S is a subuniverse of $\mathbf{A} \times \mathbf{B}$, α is a subuniverse of $(\mathbf{A} \times \mathbf{B})^2$, and $\{(s,s) \mid s \in S\} \subseteq \alpha \subseteq S^2$. Since $\mathbf{S}$ has a Mal'cev term, this implies that $\alpha \in \operatorname{Con}(\mathbf{S})$.

From (3.2) and (3.3), we obtain $((c, b_n), (a_m, d)) \in \alpha$. Since $\mathbf{S}$ is a product subalgebra of $\mathbf{A} \times \mathbf{B}$, we obtain $(c, d) \in S$. We will prove next that

$$((c, b_n), (c, d)) \in \alpha. \tag{3.5}$$

All congruences of $\mathbf{S}$ are product congruences, and therefore, there are congruences $\alpha_1 \in \operatorname{Con}(\pi_{\mathbf{A}}(\mathbf{S}))$ and $\alpha_2 \in \operatorname{Con}(\pi_{\mathbf{B}}(\mathbf{S}))$ such that $\alpha = \alpha_1 \times_c \alpha_2$. Then $((c, b_n), (a_m, d)) \in \alpha$ yields $(b_n, d) \in \alpha_2$. Together with $(c, c) \in \alpha_1$, this implies (3.5).

Hence we have $u_1, \dots, u_{m-1} \in A$, $v_1, \dots, v_{n-1} \in B$ such that

$$\begin{array}{rcl} ((u_1, \dots, u_{m-1}, c), (v_1, \dots, v_{n-1}, b_n)) & \in & C, \\ ((u_1, \dots, u_{m-1}, c), (v_1, \dots, v_{n-1}, d)) & \in & C, \\ ((a_1, \dots, a_{m-1}, a_m), (b_1, \dots, b_{n-1}, d)) & \in & C. \end{array}$$

The last line above is equation (3.3). Applying the Mal'cev term to these 3 lines, we obtain

$$((a_1, \dots, a_m), (b_1, \dots, b_n)) \in C.$$

This completes the proof of (3.1), and hence the induction step. $\square$

4. Algebras with edge term

In this section, we will prove Theorem 2.4. To this end, we need some preparation about algebras with edge term. Let $k \in \mathbb{N}, k \geq 2$. A $(k+3)$-ary term p in the language of a variety V is a $(1, k-1)$-parallelogram term if

$$p \begin{pmatrix} x & x & y & z & y & \dots & \dots & y \\ y & x & x & y & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & y \\ y & x & x & y & \dots & \dots & y & z \end{pmatrix} \approx \begin{pmatrix} y \\ \vdots \\ \vdots \\ \vdots \\ y \end{pmatrix}.$$

A variety has a k-edge term iff it has a $(1, k-1)$-parallelogram term [10, Theorem 3.5].

We give a slight generalization of representations for subpowers of algebras with edge terms from [2] to subalgebras of direct products. For $n \in \mathbb{N}$ and sets $A_1, \dots, A_n$, let $R \subseteq A_1 \times \cdots \times A_n$. For $i \in \{1, \dots, n\}$, we define the relation $\varphi_i(R)$ on A_i by

$$\varphi_i(R) := \{(a_i, b_i) \in A_i \times A_i \mid (a_1, \dots, a_n) \in R, (b_1, \dots, b_n) \in R, (a_1, \dots, a_{i-1}) = (b_1, \dots, b_{i-1})\}.$$

An element of $\varphi_i(R)$ is also called a *fork* of R at index i. If tuples $a := (a_1, \dots, a_n)$ and $b := (b_1, \dots, b_n)$ from R satisfy $(a_1, \dots, a_{i-1}) = (b_1, \dots, b_{i-1})$, then we say that a, b *witness* the fork (a_i, b_i) at index i in R.

For a tuple $a := (a_1, \dots, a_n)$ and $T \subseteq \{1, \dots, n\}$, let $\pi_T(a) := (a_i)_{i \in T}$.

Definition 4.1. Let $k, n \in \mathbb{N}, k \geq 2$, let $\mathbf{A}_1, \dots, \mathbf{A}_n$ be algebras in a variety with k-edge term, and let $B \leq \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$. Then $R \subseteq B$ is a *representation* of $\mathbf{B}$ if

(1) $\pi_T(R) = \pi_T(B)$ for all $T \subseteq \{1, \dots, n\}$ with $|T| < k$, and
(2) $\varphi_i(R) = \varphi_i(B)$ for all $i \in \{1, \dots, n\}$.

The present definition of a representation R differs from the original notion [2, Definition 3.2] in that it applies to products of algebras not only to powers of a single algebra. More importantly, we require witnesses for all forks to be in R whereas a representation in the sense of [2] only needs to contain witnesses for forks associated with minority indices.

Lemma 4.2. *Let $n \in \mathbb{N}$, let $\mathbf{A}_1, \dots, \mathbf{A}_n$ be algebras in a variety with k-edge term. Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ with representation R. Then R generates $\mathbf{B}$.*

Proof. Let $b \in B$, and let $\langle R \rangle$ denote the subalgebra of $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ that is generated by R. We will show that

$$\forall m \in \{1, \dots, n\}\ \exists f \in \langle R \rangle\ \forall i \leq m \colon f_i = b_i \tag{4.1}$$

by induction on m. The result then follows for $m = n$. From the definition of a representation (4.1) holds for $m < k$. Assume $m \geq k$ in the following. By the induction hypothesis we have $g \in \langle R \rangle$ such that $g_i = b_i$ for all $i \leq m-1$. Then (g_m, b_m) is in $\varphi_m(B)$ and hence in $\varphi_m(R)$. Hence we have $g', f' \in R$ that witness the fork (g_m, b_m) at m.

We claim that

$$\forall T \subseteq \{1, \dots, m-1\}\ \exists f^T \in \langle R \rangle\ \forall i \in T \cup \{m\} \colon f_i^T = b_i. \tag{4.2}$$

We will prove this by induction on $|T|$. Again, for $|T| \leq k-2$, we have such an $f^T \in R$ by the definition of a representation. Assume $|T| \geq k-1$ and

$T = \{i_1, \dots, i_{|T|}\}$. For $j \in \{1, \dots, k-1\}$, we let $U_j := T \setminus \{i_j\}$. Now for every $j \in \{1, \dots, k-1\}$ the induction hypothesis yields $f^{U_j} \in \langle R \rangle$ such that for all $i \in U_j \cup \{m\}$ we have $f_i^{U_j} = b_i$. Let p be the $(1, k-1)$-parallelogram term that exists in the variety by [10]. We define

$$f^T := p(g, g', f', g, f^{U_1}, \dots, f^{U_{k-1}}).$$

For $j \in \{1, \dots, k-1\}$ we obtain

$$f_{i_j}^T = p(b_{i_j}, g'_{i_j}, g'_{i_j}, b_{i_j}, b_{i_j}, \dots, b_{i_j}, f_{i_j}^{U_j}, b_{i_j}, \dots, b_{i_j}) = b_{i_j}.$$

For $i \in T \setminus \{i_1, \dots, i_{k-1}\}$ we have

$$f_i^T = p(b_i, g'_i, g'_i, b_i, b_i, \dots, b_i) = b_i.$$

Further

$$f_m^T = p(g_m, g_m, b_m, g_m, b_m, \dots, b_m) = b_m.$$

Thus the induction step of (4.2) is proved. Now (4.1) follows from (4.2) for $T = \{1, \dots, m-1\}$. $\square$

Proof of Theorem 2.4: Let $\mathbf{A}, \mathbf{B}$ satisfy the assumptions of Theorem 2.4 Let $m, n \in \mathbb{N}$, let $\mathbf{C}$ be a subalgebra of $\mathbf{A}^m \times \mathbf{B}^n$, and let $D := \pi_{A^m}(C) \times \pi_{B^n}(C)$. We will show that

$$C = D \tag{4.3}$$

by induction on $m + n$. The assertion is true if $m = 0$ or $n = 0$ or $m + n \leq \max(2, k-1)$ by assumption (1). So we assume $m > 0, n > 0$ and $m + n > \max(2, k-1)$. We consider $\mathbf{D}$ as subalgebra of $\underbrace{\mathbf{A} \times \dots \times \mathbf{A}}_{m} \times \underbrace{\mathbf{B} \times \dots \times \mathbf{B}}_{n}$ and claim that

$$C \text{ is a representation of } D. \tag{4.4}$$

First let $T \subseteq \{1, \dots, m+n\}$ with $|T| < k$. Then $\pi_T(C)$ is a product subalgebra by assumption (1). It follows that $\pi_T(C) = \pi_T(D)$.

Next, by the induction hypothesis, we have $\pi_{\{1,\dots,m+n-1\}}(C) = \pi_{\{1,\dots,m+n-1\}}(D)$. Consequently $\varphi_i(C) = \varphi_i(D)$ for all $i \in \{1, \dots, m+n-1\}$. It remains to show $\varphi_{m+n}(C) = \varphi_{m+n}(D)$. The inclusion $\subseteq$ is immediate. For the converse consider

$$\gamma := \{((f_1, f_{m+n}), (g_1, g_{m+n})) \in (A \times B)^2 \mid f, g \in C \text{ and } f_i = g_i \text{ for all } i \in \{2, \dots, m+n-1\}\}.$$

Clearly γ is a tolerance of $\pi_{\{1,m+n\}}(\mathbf{C})$, which is a subalgebra of $\mathbf{A} \times \mathbf{B}$. By assumption (1) we have $E \leq \mathbf{A}, F \leq \mathbf{B}$ such that $\pi_{1,m+n}(\mathbf{C}) = \mathbf{E} \times \mathbf{F}$. By assumption (2) we have tolerances α of $\mathbf{E}$ and β of $\mathbf{F}$ such that $\gamma = \alpha \times_c \beta$.

Let $(u,v) \in \varphi_{m+n}(D)$ be a fork that is witnessed by $f, g \in D$. By the induction hypothesis we have $f', g' \in C$ such that $f'_i = f_i$ and $g'_i = g_i$ for all $i > 1$. Then

$$((f'_1, \underbrace{f'_{m+n}}_{u}), (g'_1, \underbrace{g'_{m+n}}_{v})) \in \gamma.$$

In particular $(u,v) \in \beta$ and

$$((f'_1, u), (f'_1, v)) \in \alpha \times_c \beta = \gamma.$$

By the definition of γ we then have $f'', g'' \in C$ that witness the fork (u,v) at $m+n$. Hence $(u,v) \in \varphi_{m+n}(C)$ and (4.4) is proved. By Lemma 4.2 it follows that $C = D$. $\square$

5. Independent algebras

In this section we will relate our results on product subalgebras to independent varieties and algebras. The following lemma explains basic relations between these concepts. The implication (5)⇒(1) comes from [7, Corollary 2.9] (cf. [8, Theorem 3.2]).

Lemma 5.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be similar algebras. Then the following are equivalent:*

(1) $\mathbf{A}$ *and* $\mathbf{B}$ *are independent.*
(2) *For all (possibly infinite) sets* I, J*, every subalgebra* $\mathbf{E}$ *of* $\mathbf{A}^I \times \mathbf{B}^J$ *is a product subalgebra.*
(3) *For all sets* I, J *with* $|I| \leq |A|^2$ *and* $|J| \leq |B|^2$*, every subalgebra* $\mathbf{E}$ *of* $\mathbf{A}^I \times \mathbf{B}^J$ *is a product subalgebra.*
(4) *For all subdirect products* $\mathbf{E}$ *of* $\mathbf{F}_{V(\mathbf{A})}(2)$ *and* $\mathbf{F}_{V(\mathbf{B})}(2)$*, we have* $\mathbf{E} = \mathbf{F}_{V(\mathbf{A})}(2) \times \mathbf{F}_{V(\mathbf{B})}(2)$*.*

If $\mathbf{A}$ *and* $\mathbf{B}$ *lie in the same congruence permutable variety* V*, then these four items are furthermore equivalent to*

(5) $V(\mathbf{A})$ *and* $V(\mathbf{B})$ *are disjoint.*

Proof: (1)⇒(2): Let t be a binary term witnessing the independence of $\mathbf{A}$ and $\mathbf{B}$. Let (a,b) and (c,d) be elements of $\mathbf{E}$. Then $t^{\mathbf{A}^I \times \mathbf{B}^J}((a,b),(c,d)) = (a,d)$, and hence $(a,d) \in E$.

(2)⇒(3): Obvious.

(3)⇒(4): Let $\mathbf{E}$ be a subdirect product of $\mathbf{F}_{V(\mathbf{A})}(2) \times \mathbf{F}_{V(\mathbf{B})}(2)$. $\mathbf{F}_{V(\mathbf{A})}(2)$ is isomorphic to a subalgebra $\mathbf{A}'$ of $\mathbf{A}^{A^2}$, and similarly $\mathbf{F}_{V(\mathbf{B})}(2) \cong \mathbf{B}' \leq \mathbf{B}^{B^2}$. Via these isomorphisms, we obtain an isomorphic copy $\mathbf{E}'$ of $\mathbf{E}$ such that $\mathbf{E}'$ is a subalgebra of $\mathbf{A}^{A^2} \times \mathbf{B}^{B^2}$. Using item (3) and the fact that $\mathbf{E}'$ is a subdirect product of $\mathbf{A}'$ and $\mathbf{B}'$, we obtain $\mathbf{E}' = \mathbf{A}' \times \mathbf{B}'$, and thus $\mathbf{E} = \mathbf{F}_{V(\mathbf{A})}(2) \times \mathbf{F}_{V(\mathbf{B})}(2)$.

(4)⇒(1): We let $F_{V(\mathbf{A})}(2) = F_{V(\mathbf{A})}(x, y)$, and for two terms $s(x, y)$ and $t(x, y)$, we write $s{\sim_{\mathbf{A}}}t$ if $\mathbf{A} \models s \approx t$. Now $E := \{(s/{\sim_{\mathbf{A}}}, s/{\sim_{\mathbf{B}}}) \mid s \text{ is a term in } x, y\}$ is the universe of a subdirect product of $F_{V(\mathbf{A})}(x, y) \times F_{V(\mathbf{B})}(x, y)$. By (4), $(x/{\sim_{\mathbf{A}}}, y/{\sim_{\mathbf{B}}}) \in E$, and therefore there exists a binary term $t(x, y)$ such that $t{\sim_{\mathbf{A}}}x$ and $t{\sim_{\mathbf{B}}}y$. Thus $\mathbf{A}$ and $\mathbf{B}$ are independent.

(1)⇒(5): Let $t(x, y)$ be a binary term witnessing the independence of $\mathbf{A}$ and $\mathbf{B}$. Then $V(\mathbf{A}) \cap V(\mathbf{B}) \models x \approx t(x, y) \approx y$, and thus this intersection contains only one element algebras.

For the implication (5)⇒(2), we assume that $\mathbf{A}$ and $\mathbf{B}$ lie in a congruence permutable variety. Let I and J be sets, and let $\mathbf{E}$ be a subalgebra of $\mathbf{A}^I \times \mathbf{B}^J$. Then $\mathbf{E}$ is a subdirect product of $\pi_{\mathbf{A}^I}(\mathbf{E}) \times \pi_{\mathbf{B}^J}(\mathbf{E})$. Now by Fleischer's Lemma [3, Lemma IV.10.1], there is an algebra $\mathbf{D}$ and there are surjective homomorphisms $\alpha' : \pi_{\mathbf{A}^I}(\mathbf{E}) \to \mathbf{D}$ and $\beta' : \pi_{\mathbf{B}^J}(\mathbf{E}) \to \mathbf{D}$ such that $E = \{(x, y) \in \pi_{\mathbf{A}^I}(\mathbf{E}) \times \pi_{\mathbf{B}^J}(\mathbf{E}) \mid \alpha'(x) = \beta'(y)\}$. Since $\mathbf{D}$ lies in $V(\mathbf{A}) \cap V(\mathbf{B})$, we have $|D| = 1$. Thus $\mathbf{E} = \pi_{\mathbf{A}^I}(\mathbf{E}) \times \pi_{\mathbf{B}^J}(\mathbf{E})$, and it is therefore a product subalgebra. □

Now our results from Section 2 give the following characterizations of independence:

Theorem 5.2. *Let* $\mathbf{A}$, $\mathbf{B}$ *be finite algebras in a congruence permutable variety. Then the following are equivalent:*

(1) $\mathbf{A}$ *and* $\mathbf{B}$ *are independent.*
(2) *All subalgebras of* $\mathbf{A} \times \mathbf{B}$ *are product subalgebras, and all congruences of all subalgebras of* $\mathbf{A} \times \mathbf{B}$ *are product congruences.*
(3) *All subalgebras of* $\mathbf{A}^2 \times \mathbf{B}^2$ *are product subalgebras.*
(4) $HS(\mathbf{A}^2) \cap HS(\mathbf{B}^2)$ *contains only one element algebras.*

Proof. (2)⇒(1): We assume that (2) holds. Then by Theorem 2.3, for all $m, n \in \mathbb{N}_0$ with $m \leq |A|^2$ and $n := |B|^2$, all subalgebras of $\mathbf{A}^m \times \mathbf{B}^n$ are product subalgebras. Now by Lemma 5.1, $\mathbf{A}$ and $\mathbf{B}$ are independent.

(3)⇒(2): Let γ be a congruence of $\mathbf{A} \times \mathbf{B}$. Let $\mathbf{E}(\gamma)$ be the subalgebra of $\mathbf{A}^2 \times \mathbf{B}^2$ given by

$$E(\gamma) = \{((a_1, a_2), (b_1, b_2)) \mid ((a_1, b_1), (a_2, b_2)) \in \gamma\}.$$

From the fact that $\mathbf{E}(\gamma)$ is a product subalgebra, we obtain that γ is a product congruence of $\mathbf{A} \times \mathbf{B}$.

(4)⇒(3): We let $\mathbf{C}$ be a subalgebra of $\mathbf{A}^2 \times \mathbf{B}^2$. By Fleischer's Lemma, there are a subalgebra $\mathbf{A}'$ of $\mathbf{A}^2$, a subalgebra $\mathbf{B}'$ of $\mathbf{B}^2$, an algebra $\mathbf{D}$, and epimorphisms $\alpha' : \mathbf{A}' \to \mathbf{D}$, $\beta' : \mathbf{B}' \to \mathbf{D}$ such that $C = \{(a', b') \in A' \times B' \mid \alpha'(a') = \beta'(b')\}$. Since $\mathbf{D} \in HS(\mathbf{A}^2) \cap HS(\mathbf{B}^2)$, $\mathbf{D}$ is a one element algebra, and therefore $C = A' \times B'$, and it is therefore a product subalgebra.

(1)⇒(4) is proved in the same way as (1)⇒(5) of Lemma 5.1. □

Theorem 5.3. *Let $k \geq 2$, and let* $\mathbf{A}$, $\mathbf{B}$ *be finite algebras in a variety with k-edge term. Then the following are equivalent:*

(1) $\mathbf{A}$ *and* $\mathbf{B}$ *are independent.*
(2) *For all $r, s \in \mathbb{N}$ with $r + s \leq \max(2, k-1)$, every subalgebra of $\mathbf{A}^r \times \mathbf{B}^s$ is a product subalgebra, and for all $E \leq \mathbf{A}, F \leq \mathbf{B}$, every tolerance of $\mathbf{E} \times \mathbf{F}$ is a product tolerance.*
(3) *For all $r, s \in \mathbb{N}$ with $r + s \leq \max(4, k-1)$, every subalgebra of $\mathbf{A}^r \times \mathbf{B}^s$ is a product subalgebra.*

Proof. (2)⇒(1) follows from Theorem 2.4 and the implication (3)⇒(1) of Lemma 5.1.

(3)⇒(2), We associate a subalgebra $\mathbf{E}(\tau)$ of $\mathbf{A}^2 \times \mathbf{B}^2$ with every tolerance τ of $\mathbf{A} \times \mathbf{B}$ and proceed as in the proof of (3)⇒(2) of Theorem 5.2.

(1)⇒(3) follows from (1)⇒(2) of Lemma 5.1. □

6. Polynomial functions

As observed in [4, 6], term functions of independent algebras $\mathbf{A}$ and $\mathbf{B}$ can be *paired* in the sense that for all k-ary terms r and s, the term $u := t(r, s)$ satisfies $\mathbf{A} \models u \approx r$ and $\mathbf{B} \models u \approx s$, where t is a binary term witnessing the independence of $\mathbf{A}$ and $\mathbf{B}$. Hence we have:

Lemma 6.1. *Let $\mathbf{A}$ and $\mathbf{B}$ be similar independent algebras, and let $k \in \mathbb{N}$. Then the mapping $\phi : \mathrm{Clo}_k(\mathbf{A}) \times \mathrm{Clo}_k(\mathbf{B}) \to \mathrm{Clo}_k(\mathbf{A} \times \mathbf{B})$ with*

$$\phi(f, g)((a_1, b_1), \ldots, (a_k, b_k)) = (f(a), g(b))$$

for $f \in \mathrm{Clo}_k(\mathbf{A}), g \in \mathrm{Clo}_k(\mathbf{B}), a \in A^k, b \in B^k$ is a bijection.

Proof: We first show that for $f \in \mathrm{Clo}_k(\mathbf{A})$ and $g \in \mathrm{Clo}_k(\mathbf{B})$, we have $\phi(f, g) \in \mathrm{Clo}_k(\mathbf{A} \times \mathbf{B})$. Let r and s be k-variable terms with $r^{\mathbf{A}} = f$ and $s^{\mathbf{B}} = g$. Let t be a term witnessing the independence of $\mathbf{A}$ and $\mathbf{B}$. Then for $u := t(r, s)$ we have $u^{\mathbf{A} \times \mathbf{B}}((a_1, b_1), \ldots, (a_k, b_k)) = (r^{\mathbf{A}}(a), s^{\mathbf{B}}(b))$ for all $a \in A^k$ and $b \in \mathbf{B}^k$. The mapping ϕ is clearly injective, and for proving that ϕ is surjective, let u be a term. Then $\phi(u^{\mathbf{A}}, u^{\mathbf{B}}) = u^{\mathbf{A} \times \mathbf{B}}$, and hence the range of ϕ contains $\mathrm{Clo}_k(\mathbf{A} \times \mathbf{B})$. □

We can now easily provide an example showing that Theorem 5.2 does not hold for infinite algebras.

Example 6.2. Let p and q be different primes, and let $\mathbf{A}$ be the Prüfer group $\mathbb{Z}_{p^\infty}$ and $\mathbf{B} := \mathbb{Z}_{q^\infty}$. It is easy to see that for all $m, n \in \mathbb{N}$, all subalgebras of $\mathbf{A}^m \times \mathbf{B}^n$ are product subalgebras. Since all binary term functions of $\mathbf{A} \times \mathbf{B}$ are of the form $((a_1, b_1), (a_2, b_2)) \mapsto (z_1 a_1 + z_2 a_2, z_1 b_1 + z_2 b_2)$, there is no term inducing the function $((a_1, b_1), (a_2, b_2)) \mapsto (a_1, b_2)$. Hence $\mathbf{A}$ and $\mathbf{B}$ are not independent.

We will now consider polynomial functions on direct products of two algebras. Pilz conjectured that for expanded groups $\mathbf{A}$ and $\mathbf{B}$ such that all congruences of $\mathbf{A} \times \mathbf{B}$ are product congruences the following holds: for all unary polynomial functions f on $\mathbf{A}$ and g on $\mathbf{B}$, the function $(a, b) \mapsto (f(a), g(b))$ is a polynomial function on $\mathbf{A}\times\mathbf{B}$ (cf. [12, Conjecture 2.10]). In [1], the conjecture was verified for finite $\mathbf{A}$ and $\mathbf{B}$, and [9] generalized this result to finite algebras with a Mal'cev or a majority term. The following theorem generalizes two of their results to algebras with a 3-edge term.

Theorem 6.3. *Let $\mathbf{A}$ and $\mathbf{B}$ be finite algebras in a variety with a 3-edge term, and let $k \in \mathbb{N}$. We assume that every tolerance of $\mathbf{A} \times \mathbf{B}$ is a product tolerance. Let $\psi : \mathrm{Pol}_k(\mathbf{A}) \times \mathrm{Pol}_k(\mathbf{B}) \to (A \times B)^{(A\times B)^k}$ be the mapping defined by*

$$\psi(f, g)\,((a_1, b_1), \ldots, (a_k, b_k)) := (f(a), g(b))$$

for $f \in \mathrm{Pol}_k(\mathbf{A}), g \in \mathrm{Pol}_k(\mathbf{B})$, $a \in A^k$, and $b \in B^k$. Then ψ is a bijection from $\mathrm{Pol}_k(\mathbf{A}) \times \mathrm{Pol}_k(\mathbf{B})$ to $\mathrm{Pol}_k(\mathbf{A} \times \mathbf{B})$.

Proof: For each $a \in A, b \in B$, we add a constant operation $c_{(a,b)}$ to our language. By $\mathbf{A}^*$, we denote the expansion of $\mathbf{A}$ satisfying $c^{\mathbf{A}}_{(a,b)}() = a$, and we write $\mathbf{B}^*$ for the expansion of $\mathbf{B}$ with $c^{\mathbf{B}}_{(a,b)}() = b$ for all $a \in A, b \in B$. It is easy to see that $\mathrm{Clo}_k(\mathbf{A}^*) = \mathrm{Pol}_k(\mathbf{A})$, $\mathrm{Clo}_k(\mathbf{B}^*) = \mathrm{Pol}_k(\mathbf{B})$, and $\mathrm{Clo}_k(\mathbf{A}^* \times \mathbf{B}^*) = \mathrm{Pol}_k(\mathbf{A} \times \mathbf{B})$. By its construction, $\mathbf{A}^* \times \mathbf{B}^*$ has no proper subuniverses. Since $\mathbf{A} \times \mathbf{B}$ and $\mathbf{A}^* \times \mathbf{B}^*$ have the same tolerances, every tolerance of $\mathbf{A}^* \times \mathbf{B}^*$ is a product tolerance. We apply Theorem 2.4 and obtain that $\mathbf{A}^*$ and $\mathbf{B}^*$ are independent. Now the result follows from Lemma 6.1. ☐

As a corollary, we obtain Corollary 2 and Theorem 3 of [9].

Corollary 6.4 ([9])**.** *Let $\mathbf{A}$ and $\mathbf{B}$ be algebras in the variety V, and let $k \in \mathbb{N}$. If either*

(1) *V has a majority term, or*
(2) *V is congruence permutable, and every congruence of $\mathbf{A} \times \mathbf{B}$ is a product congruence,*

then for all polynomial functions $f \in \mathrm{Pol}_k(\mathbf{A})$ and $g \in \mathrm{Pol}_k(\mathbf{B})$, there is a polynomial function $h \in \mathrm{Pol}_k(\mathbf{A} \times \mathbf{B})$ with $h((a_1, b_1), \ldots, (a_k, b_k)) = (f(a), g(b))$ for all $a \in A^k$ and $b \in B^k$.

Proof: Let us first assume that V has a majority term $m(x, y, z)$. Then the term $e(x_1, x_2, x_3, x_4) := m(x_2, x_3, x_4)$ is a 3-edge term. All tolerances of $\mathbf{A} \times \mathbf{B}$ are product tolerances. To see this, let ε be a tolerance of $\mathbf{A} \times \mathbf{B}$, and let $((a_1, b_1), (a_2, b_2)) \in \varepsilon$ and $((a_3, b_3), (a_4, b_4)) \in \varepsilon$. Writing m for the induced term operation $m^{(\mathbf{A}\times\mathbf{B})\times(\mathbf{A}\times\mathbf{B})}$ we obtain

$$\begin{aligned} m\bigl(&m(((a_1,b_1),(a_2,b_2)),((a_1,b_3),(a_1,b_3)),((a_2,b_3),(a_2,b_3))),\\ &m(((a_1,b_1),(a_2,b_2)),((a_1,b_4),(a_1,b_4)),((a_2,b_4),(a_2,b_4))),\\ &m(((a_2,b_3),(a_2,b_3)),((a_2,b_4),(a_2,b_4)),((a_3,b_3),(a_4,b_4)))\bigr)\\ &\qquad = ((a_1,b_3),(a_2,b_4)) \in \varepsilon. \end{aligned}$$

Thus ε is a product tolerance. Now Theorem 6.3 yields the required polynomial h. In the case that V has a Mal'cev term d, then $e(x_1,x_2,x_3,x_4) := d(x_2,x_1,x_3)$ is a 3-edge term, and all tolerances of algebras in V are congruences. Hence all tolerances of $\mathbf{A}\times\mathbf{B}$ are product tolerances. The result follows from Theorem 6.3. □

7. Algorithms

Let $\mathbf{A},\mathbf{B}$ be finite algebras of fixed finite type. For $i \in \{1,2\}$ let $e_i\colon A^2 \to A, (x_1,x_2)\mapsto x_i$, and $f_i\colon B^2 \to B, (x_1,x_2)\mapsto x_i$, denote the i-th projection on A^2 and B^2, respectively. Then $e_i \in A^{A^2}$ and $f_i \in B^{B^2}$. From the definition, $\mathbf{A}$ and $\mathbf{B}$ are independent iff there exists a binary term operation on $\mathbf{A}\times\mathbf{B}$ that is e_1 on the factor A and f_2 on the factor B. Equivalently, $\mathbf{A}$ and $\mathbf{B}$ are independent iff

(e_1,f_2) lies in the subalgebra of $\mathbf{A}^{A^2}\times\mathbf{B}^{B^2}$ that is generated by $(e_1,f_1),(e_2,f_2)$.

Using a straightforward closure algorithm that enumerates all elements of the generated algebra, the last condition can be checked in time exponential in $\max(|A|,|B|)$. Hence deciding independence of two arbitrary finite algebras is in EXPTIME. The results from Section 5 yield easy polynomial time algorithms for algebras with a Mal'cev term or more generally an edge term.

Theorem 7.1. *Let $k \geq 2$, and let $\mathbf{A}$, $\mathbf{B}$ be finite algebras in a variety of finite type with k-edge term. Then the independence of $\mathbf{A}$ and $\mathbf{B}$ can be decided in time polynomial in $\max(|A|,|B|)$.*

Proof. Let $m := \max(4,k-1)$ and $n := \max(|A|,|B|)$. By Theorem 5.3 (3) the algebras $\mathbf{A}$ and $\mathbf{B}$ are independent iff for all $r,s \in \mathbb{N}$ with $r+s \leq m$ and for all $a,c \in A^r, b,d \in B^s$

$$(a,d) \in \langle (a,b),(c,d)\rangle.$$

To verify the latter condition we need to solve at most $(m-1)n^{2m}$ membership problems for algebras of size at most n^m. Each of these membership problems can be decided using a closure algorithm in time polynomial in n with the actual degree of the polynomial depending on m and the arities of the basic operations of the algebras. □

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