Representing Finite Lattices as Congruence Lattices

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`http://math.hawaii.edu/~ralph/`

`http://uacalc.org/`

`https://github.com/UACalc/`

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The Problem

Theorem (Grätzer-Schmidt)

Every algebraic (so every finite) lattice is isomorphic to **Con** (**A**) *for some (unary) algebra* **A**.

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Since $\mathbf{Con}\,(\mathbf{A}) = \mathbf{Con}\,\langle A, \mathrm{Pol}_1(\mathbf{A})\rangle$, **we assume all algebras are unary.**

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Then **Con** $A \cong [H, G]$, the interval in the subgroup lattice.

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Example. The minimum size for $\mathbf{L}_6$ is 6:

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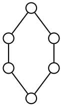

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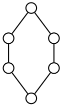
$\mathbf{L}_6$ 

$\mathbf{B}_6$	0	1	2	3	4	5
$f(x)$	2	2	1	5	5	4
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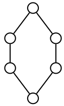

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Pálfy and Aschbacher have found groups $\mathbf{H} \leq \mathbf{G}$ representing this lattice. But Pálfy's example has $\mathbf{G} = \mathbf{A}_{11}$ and $|H| = 55$, so the size is $9! = 362880$.

Moral

Moral: Finding a representation with groups, (P4), may be much harder (and much bigger) than finding a (P1) representation.

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- overalgebras (DeMeo, 2013)

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- *If* (A) *and* (B′) *hold and* $\mathbf{L} \cong \langle A, F \rangle$ *is minimal, then* F *consists of permutations and constants.*

Representations by instransitive groups

Suppose $\mathbf{A} = \langle A, G \rangle$ is a G-set and let $\mathbf{A}_i = \langle A_i, G \rangle$, $i < k$, be the minimal subalgebras of $\mathbf{A}$; i.e. each set A_i is an orbit, or one-generated subuniverse, of $\mathbf{A}$.

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Define congruences on A by the partitions

$$\begin{aligned}
\tau &= |A_0|A_1| \cdots |A_{k-1}| && \text{(the blocks are the orbits)} \\
\tau_i &= |A_i| && \text{(at most one nontrivial block)} \\
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*Let $\theta \in$ **Con** $(\mathbf{A})$, where $\mathbf{A} = \langle A, G\rangle$ and G is a group. Then*

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   For $\theta_i \in \mathbf{Con}\,(\mathbf{A}_i)$, map $(\theta_0, \ldots, \theta_{k-1}) \mapsto \theta_0 \cup \cdots \cup \theta_{k-1}$.

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   If $\theta \nleq \gamma_i$, there are $a \in A_i$ and $b \notin A_i$ with $(a, b) \in \theta$. Since G acts transitively on each orbit, $\tau_i \leq \theta$. So $\tau \leq \theta$.

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   Follows from (3) and (4).

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   From (2) and (6).

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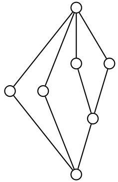
   From (8).

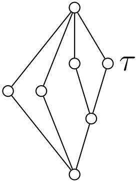
Examples: $\mathbf{L}_{14}$



Example

- $\mathbf{L}_{14}$ satisfies (A) and (B″) so a minimal representation is permutational.

Examples: $\mathbf{L}_{14}$



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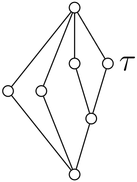
Examples: $\mathbf{L}_{14}$



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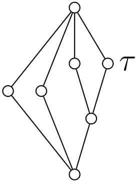

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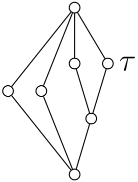

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- Is $\mathbf{L}_{14}$ representable? (Yes: as $[H, A_6]$ with $[A_6 : H] = 90$)

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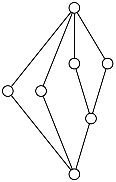

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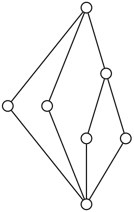
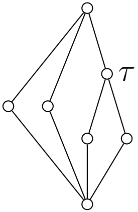
- Is $\mathbf{L}_{14}$ representable? (Yes: as $[H, A_6]$ with $[A_6 : H] = 90$)
- Is this a minimum representation? (Don't know)

Examples: $\mathbf{L}_{15}$, (the dual of $\mathbf{L}_{14}$)



Examples: $\mathbf{L}_{15}$, (the dual of $\mathbf{L}_{14}$)



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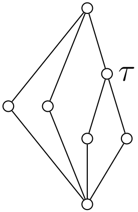

Example

- $\mathbf{L}_{15} \cong \mathbf{Con}\ \langle\{0, 1, 2, 3\}, G\rangle$, G the group generated by the double transposition $0 \leftrightarrow 1$, $2 \leftrightarrow 3$.

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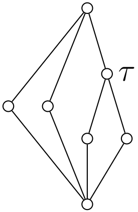

Example

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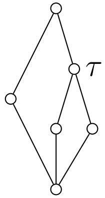
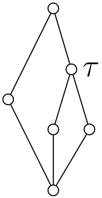
- $\mathbf{L}_{14} \cong \mathbf{L}_{15}^{d}$, which again proves $\mathbf{L}_{14}$ is representable.

Examples: $\mathbf{L}_4$.



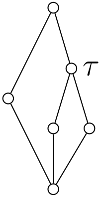
Examples: $\mathbf{L}_4$.



Example

- $\mathbf{L}_4$ satisfies (B″) but not (A) so minimal representations need not be permutational.

Examples: $\mathbf{L}_4$.

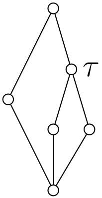


Example

- $\mathbf{L}_4$ satisfies (B$''$) but not (A) so minimal representations need not be permutational. In fact
- $\mathbf{L}_4 \cong \langle \{0, 1, 2, 3\}, f, g \rangle$, where

$\mathbf{B}_4$	0	1	2	3
$f(x)$	1	0	3	2
$g(x)$	0	0	2	2

Examples: $\mathbf{L}_4$.

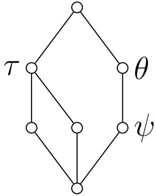


Example

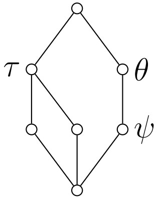
- $\mathbf{L}_4$ satisfies (B″) but not (A) so minimal representations need not be permutational.
- But $\mathbf{L}_4$ does have an intransitive representation on 6:

$\mathbf{B}_4'$	0	1	2	3	4	5
$f(x)$	1	2	0	4	5	3
$g(x)$	0	2	1	3	5	4

Examples: $\mathbf{L}_{19}$, a harder example:



Examples: $\mathbf{L}_{19}$, a harder example:



Lemma

Let $\mathbf{A} = \langle A, G \rangle$ be a finite algebra, where G is an intransitive group of permutations on A. Suppose the intransitivity congruence τ is a coatom. Then there do not exist congruences $0_{\mathbf{A}} < \psi < \theta$ in $\mathbf{Con}\,(\mathbf{A})$ with $\theta \wedge \tau = 0_{\mathbf{A}}$.

Proof

Lemma

Let $\mathbf{A} = \langle A, G \rangle$ *be a finite algebra, where* G *is an intransitive group of permutations on* A*. Suppose the intransitivity congruence* τ *is a coatom. Then there do not exist congruences* $0_{\mathbf{A}} < \psi < \theta$ *in* $\mathbf{Con}\,(\mathbf{A})$ *with* $\theta \wedge \tau = 0_{\mathbf{A}}$*.*

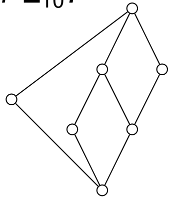
Proof.

Since τ is a coatom, there are exactly two orbits; call them B and C. Since $\theta \wedge \tau = 0_{\mathbf{A}}$, if $(x, y) \in \theta$ then $x = y$ or one is in B and the other is in C. So θ defines a bipartite graph between B and C. Since G acts transitively on both B and C, this graph corresponds to a bijection between B and C. The same applies to ψ. But equivalence relations corresponding to such graphs cannot be comparable. □

Small Lattices

Theorem

All lattices with at most 7 elements can be represented, with the one possible exception of $\mathbf{L}_{10}$*:*



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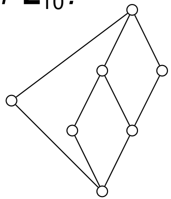

If $\mathbf{L}_{10} \cong \langle A, F \rangle$*, then* F *generates a transitive group on* A*.*

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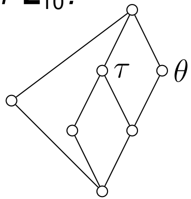

If $\mathbf{L}_{10} \cong \langle A, F \rangle$*, then* F *generates a transitive group on* A*.*

Proof.

$\mathbf{L}_{10}$ satisfies (A) and (B$''$). By part (5) of the intransitivity theorem, it cannot be represented with an intransitive group. □

Finding Reps: Methods and Algorithms

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- Closure Method

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(2) For each sublattice $\mathbf{L} \cong \mathbf{L}' \leq \mathrm{Eq}(X_k)$ found, find the unary polymorphs of the members of L'; that is, calculate the set F of all unary operations on X_k which respect all $\theta \in L'$.

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(3) For F found in the previous step, test if $\mathbf{Con}\,(\langle X_k, F\rangle) = L'$. If so then $\mathbf{A} = \langle X_k, F\rangle$ is a minimal representation. Otherwise continue the search.

Remarks

(a) **Find a small presentation of L:**
The procedure can be sped up by first finding a presentation of **L** with the minimal number of generators. Besides speeding up the search in $\mathrm{Eq}(k)$, it is enough in calculating the unary polymorphs to respect the generators.

Remarks

(b) **Subdirect Decompositions:**
Subdirect decompositions can be used to speed up finding unary polymorphs. For example, if $\theta_0, \theta_1 \in L' \leq \mathrm{Eq}(X_k)$ with $\theta_0 \wedge \theta_1 = 0$, then X_k is naturally embedded into $X_k/\theta_0 \times X_k/\theta_1$. Since the operations in a direct product are component-wise, this cuts the search space of possible unary polymorphs from k^k down to $r^r s^s$, where r and s are the number of blocks in θ_0 and θ_1.

Remarks

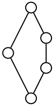
(c) **Uniform Equivalence Relations:**
If it can be shown that the algebra of a minimal representation of **L** has a transitive permutation group for its nonconstatant unary polynomials, then we can restrict our search in $\mathrm{Eq}(k)$ to uniform equivalence relations. Moreover the search for unary polymorphs can be restricted to permutations.

Remarks

(d) **Small generating set for the operations:**
Of course if $F' \subseteq F$ is a set of generators for the moniod F, we can take $\mathbf{A} = \langle X_k, F' \rangle$.

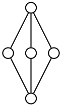
Nondist., linearly indec., small lattices

$\mathbf{L}_1$



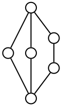
$\mathbf{B}_1$	0	1	2	3
$f(x)$	1	0	3	2
$g(x)$	1	0	1	0

$\mathbf{L}_2$



$\mathbf{B}_2$	0	1	2
$f(x)$	0	1	2

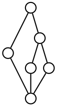
$\mathbf{L}_3$



$\mathbf{B}_3$	0	1	2	3	4	5	6
$f(x)$	0	1	2	1	2	1	0
$g(x)$	0	3	4	3	4	3	0
$h(x)$	6	5	2	5	2	5	6
$k(x)$	0	1	2	0	0	2	2

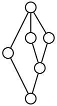
Method: overalgebras

$\mathbf{L}_4$



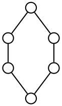
$\mathbf{B}_4$	0	1	2	3
$f(x)$	1	0	3	2
$g(x)$	0	0	2	2

$\mathbf{L}_5$



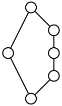
$\mathbf{B}_5$	0	1	2	3	4	5	6	7	8	9	10	11
$f(x)$	1	2	3	4	5	0	7	8	9	10	11	6
$g(x)$	6	11	10	9	8	7	0	5	4	3	2	1
$h(x)$	0	0	0	6	0	0	0	0	6	0	0	0

$\mathbf{L}_6$



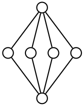
$\mathbf{B}_6$	0	1	2	3	4	5
$f(x)$	2	2	1	5	5	4
$g(x)$	3	4	4	0	1	1
$h(x)$	4	5	3	4	5	3

L_7



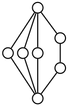
$\mathbf{B}_7$	0	1	2	3	4	5
$f(x)$	1	0	0	4	3	3
$g(x)$	4	5	5	1	2	2
$h(x)$	3	3	4	3	3	4

L_8



$\mathbf{B}_8$	0	1	2	3	4	5
$f(x)$	1	2	0	4	5	3
$g(x)$	3	5	4	0	2	1

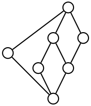
L_9



$\mathbf{B}_9$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(x)$	0	0	0	0	0	0	2	1	2	1	3	4	5	3	4	5
$g(x)$	0	0	0	0	0	0	6	7	6	7	10	11	12	10	11	12
$h(x)$	13	14	15	1	9	8	15	14	13	15	1	9	8	8	1	9

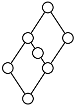
Method: overalgebras

$\mathbf{L}_{10}$



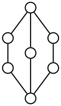
No finite algebra known with this as its congruence lattice.

$\mathbf{L}_{11}$



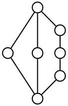
A finite algebra with 108 elements known.

$\mathbf{L}_{12}$



$\mathbf{B}_{12}$	0	1	2	3	4	5	6	7	8
$f(x)$	0	0	3	3	3	6	6	6	0
$g(x)$	0	0	8	8	8	1	1	1	0
$h(x)$	0	5	5	4	0	0	5	4	4
$k(x)$	4	2	2	3	4	4	2	3	3
$l(x)$	5	5	7	7	7	6	6	6	5

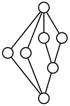
$\mathbf{L}_{13}$



$\mathbf{B}_{13}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$f(x)$	0	1	2	1	2	1	0	0	1	2	2	1	0	0	1	2	1	2	0
$g(x)$	0	1	2	0	0	2	2	0	3	4	0	4	4	6	5	2	6	6	2
$h(x)$	0	1	2	3	4	5	6	0	1	2	4	5	6	0	1	2	3	4	6
$k(x)$	7	8	9	3	10	11	12	3	3	3	3	3	3	11	11	11	11	11	11
$l(x)$	13	14	15	16	17	5	18	13	16	17	17	16	13	5	5	5	5	5	5

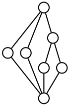
Method: overalgebras

$\mathbf{L}_{14}$



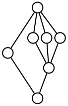
Upper interval in $\mathrm{Sub}(A_6)$,
algebra of size 90

$\mathbf{L}_{15}$



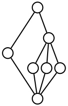
$\mathbf{B}_{15}$	0	1	2	3
$f(x)$	1	0	3	2

$\mathbf{L}_{16}$



Upper interval in $\mathrm{Sub}(C_2.A_6)$
algebra of size 180

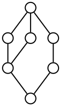
$\mathbf{L}_{17}$



$\mathbf{B}_{17}$	0	1	2	3	4	5	6	7	8	9	10	11
$f(x)$	1	0	3	2	5	4	7	6	9	8	11	10
$g(x)$	4	7	5	6	8	11	9	10	0	3	1	2
$h(x)$	0	0	0	0	5	5	5	5	10	10	10	10

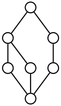
Method: filter-ideal in Sub(A_4)

$\mathbf{L}_{18}$



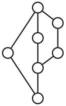
Dual of 19, no explicit
small representation known

$\mathbf{L}_{19}$



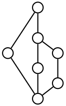
$\mathbf{B}_{19}$	0	1	2	3	4	5	6	7
$f(x)$	0	1	1	0	4	5	5	4
$g(x)$	0	2	3	1	0	2	3	1
$h(x)$	7	6	6	7	3	2	2	3

$\mathbf{L}_{20}$



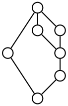
Method: filter-ideal in SmallGroup(216,153) in GAP

$\mathbf{L}_{21}$



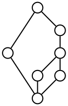
$\mathbf{B}_{21}$	0	1	2	3	4	5	6	7	8
$f(x)$	3	3	4	8	8	2	2	3	4
$g(x)$	0	0	6	1	1	0	0	5	6
$h(x)$	4	5	5	7	8	8	7	4	4

$\mathbf{L}_{22}$



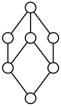
Dual of 23, no explicit small representation known

$\mathbf{L}_{23}$



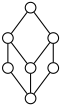
$\mathbf{B}_{23}$	0	1	2	3	4	5
$f(x)$	0	1	0	1	4	4
$g(x)$	1	1	3	3	4	5
$h(x)$	3	2	3	2	5	5
$k(x)$	4	1	5	3	4	5

$\mathbf{L}_{24}$



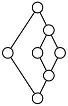
$\mathbf{B}_{24}$	0	1	2	3
$f(x)$	1	1	2	2
$g(x)$	2	3	3	2

$\mathbf{L}_{25}$



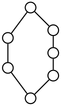
$\mathbf{B}_{25}$	0	1	2	3	4
$f(x)$	0	0	2	2	2
$g(x)$	0	1	0	1	1
$h(x)$	1	1	4	4	4
$k(x)$	2	3	2	3	3

$\mathbf{L}_{26}$



$\mathbf{B}_{26}$	0	1	2	3	4	5
$f(x)$	1	0	3	2	0	2
$g(x)$	4	4	5	5	1	3
$h(x)$	0	0	0	0	1	1
$k(x)$	3	5	3	5	3	3

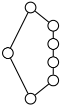
$\mathbf{L}_{27}$



$\mathbf{B}_{27}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(x)$	0	1	2	3	4	5	0	0	0	0	0	2	2	2	2	2
$g(x)$	4	5	3	4	5	3	5	3	4	5	3	4	5	4	5	3
$h(x)$	2	2	1	5	5	4	2	1	5	5	4	2	2	5	5	4
$k(x)$	3	4	4	0	1	1	4	4	0	1	1	3	4	0	1	1
$l(x)$	0	6	7	8	9	10	6	7	8	9	10	0	6	8	9	10
$m(x)$	11	12	2	13	14	15	12	2	13	14	15	11	12	13	14	15

Method: overalgebras

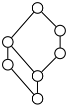
$\mathbf{L}_{28}$



$\mathbf{B}_{28}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(x)$	0	1	2	3	4	5	0	0	0	0	0	2	2	2	2	2
$g(x)$	3	3	4	3	3	4	3	4	3	3	4	3	3	3	3	4
$h(x)$	1	0	0	4	3	3	0	0	4	3	3	1	0	4	3	3
$k(x)$	4	5	5	1	2	2	5	5	1	2	2	4	5	1	2	2
$l(x)$	0	6	7	8	9	10	6	7	8	9	10	0	6	8	9	10
$m(x)$	11	12	2	13	14	15	12	2	13	14	15	11	12	13	14	15

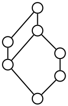
Method: overalgebras

$\mathbf{L}_{29}$



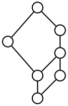
$\mathbf{B}_{29}$	0	1	2	3	4
$f(x)$	1	0	3	2	2
$g(x)$	2	4	2	4	3

$\mathbf{L}_{30}$



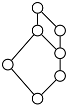
$\mathbf{B}_{30}$	0	1	2	3	4
$f(x)$	0	3	4	3	4
$g(x)$	2	2	1	4	3

$\mathbf{L}_{31}$



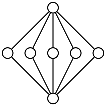
$\mathbf{B}_{31}$	0	1	2	3	4
$f(x)$	0	1	1	0	0
$g(x)$	1	1	2	2	2
$h(x)$	3	2	2	4	4

$\mathbf{L}_{32}$



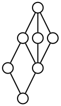
$\mathbf{B}_{32}$	0	1	2	3	4
$f(x)$	0	1	1	3	3
$g(x)$	1	2	2	4	4
$h(x)$	3	3	4	3	4

$\mathbf{L}_{33}$



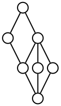
$\mathbf{B}_{33}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(x)$	1	3	2	0	9	11	10	8	13	15	14	12	5	7	6	4
$g(x)$	11	8	10	9	7	4	6	5	15	12	14	13	3	0	2	1
$h(x)$	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1

$\mathbf{L}_{34}$



$\mathbf{B}_{34}$	0	1	2	3
$f(x)$	0	1	3	2

$\mathbf{L}_{35}$



$\mathbf{B}_{35}$	0	1	2	3
$f(x)$	1	1	2	3
$g(x)$	2	3	3	3

Problem

- What about nonunary algebras?

Problem

- What about nonunary algebras?

Problem

Which (finite) lattices can be represented as **Con A**, *where* **A** *has a Taylor term?*

Resources

- SmallLatticeReps.ua, a UACalc file with most of the $\mathbf{B}_i$'s:

  `http://math.hawaii.edu/~ralph/`

- The slides of this talk are there too.
- SmallLatticeReps.ua and other algebra files:

  `https://github.com/UACalc/SmallAlgebras/`

- Get UACalc at

  `http://uacalc.org/`

- Source at

  `https://github.com/UACalc/uacalcsrc/`