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ON INTERVALS IN SUBGROUP LATTICES OF FINITE GROUPS

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Recall that a *lattice* is a partially ordered set Λ such that for each $x, y \in \Lambda$, the least upper bound $x \vee y$ and greatest lower bound $x \wedge y$ of x and y in Λ exist. In this paper we concentrate on finite lattices. For $u, v \in \Lambda$, write $[u, v]$ for the sublattice of elements $x \in \Lambda$ such that $u \leq x \leq v$; thus $[u, v]$ is an *interval sublattice* of Λ.

Example 1. Let G be a finite group. Then the set of all subgroups of G, partially ordered by inclusion, is a lattice. For $H \leq G$ the sublattice $\mathcal{O}_G(H)$ of overgroups of H in G is the interval sublattice $[H, G]$. Call such a lattice a *finite group interval lattice*. There is a well-known open question as to whether every nonempty finite lattice is isomorphic to a finite group interval lattice. See [PP] for motivation for this question, and see [BL] for one possible approach to proving that the question has a negative answer.

Example 2. Let B be the algebra of bounded linear operators on a Hilbert space, M a type Π_1-factor of B, and N a subfactor of M such that $N' \cap M = \mathbf{C}$. Then the set $\mathcal{L}(N, M)$ of intermediate von Neumann subalgebras forms a lattice of great interest in the study of operator algebras. See for example [MN] and [W]. If the Jones index of M over N is finite, then (cf. Theorem 2.2 in [W]) $\mathcal{L}(N, M)$ is finite. Further, by a "crossed product" construction (cf. Example 2.1 in [W] or Chapter V, section 7 of [T]), given a finite group G and subgroup H of G, one can construct a pair of factors N, M with $\mathcal{L}(N, M) \cong \mathcal{O}_G(H)$. Using such constructions, in [W] Watatani shows that, with two possible exceptions, each finite lattice of order 6 is a lattice of intermediate subfactors. The two exceptions are denoted by L_{19} and L_{20} in Example 6.1 of [W]. The lattice L_{20} is the *hexagon*. According to [GJ], the question as to whether L_{19} and L_{20} are lattices of intermediate subfactors remained open at the time that paper was written.

In this paper we seek to determine whether certain finite lattices are isomorphic to interval sublattices in the subgroup lattice of some finite group and show that strong constraints are imposed on the structure of a group by the existence of such an interval. In particular given a finite lattice Λ, define $\mathcal{G}(\Lambda)$ to be the set of pairs (H, G) such that G is a finite group, $H \leq G$, and $\mathcal{O}_G(H)$ is isomorphic to Λ or its dual. Write $\mathcal{G}^*(\Lambda)$ for the set of pairs (H, G) such that $|G|$ is minimal subject to $(H, G) \in \mathcal{G}(\Lambda)$. One can attempt to show that for suitable choices of Λ and $(H, G) \in \mathcal{G}^*(\Lambda)$, the group G is *almost simple*: That is, G has a unique minimal normal subgroup D, and D is a nonabelian simple group. Then using the

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classification of the finite simple groups and knowledge of the maximal subgroups of almost simple groups, one can hope to decide whether Λ is really an interval lattice in some G. Most particularly, one can hope to find a counterexample to the question mentioned in Example 1, or in the other direction, show that certain lattices (such as L_{19} and L_{20} in Example 2) are indeed lattices of intermediate subfactors.

In this paper we concentrate on what we call "A-lattices", "D-lattices" and "CD-lattices". Let Λ be a nonempty finite lattice. Then Λ has a greatest element ∞ and least element 0. Define $m \in \Lambda$ to be *modular* if for all $a, b \in \Lambda$ with $a \leq b$, $(a \vee m) \wedge b = a \vee (m \wedge b)$. We say Λ is an *A-lattice* if $|\Lambda| > 2$ and 0 and ∞ are the only modular elements of Λ.

Regard Λ as an undirected graph with adjacency relation the comparability relation on Λ. We say that Λ is *disconnected* if the subgraph $\Lambda' = \Lambda - \{0, \infty\}$ is disconnected as a graph. We say Λ is a *D-lattice* if there exists a partition $\Lambda' = \Lambda'_1 \cup \Lambda'_2$ of Λ' such that for $i = 1$ and 2:

(D1) Λ'_i is a union of connected components of Λ', and

(D2) there exists a nontrivial chain $k_i < m_i$ in Λ'_i.

For example the hexagon is the smallest D-lattice. The lattice L_{19} of Watatani in Example 2 is not a D-lattice, as it fails the nondegeneracy condition (D2), but it is disconnected and closely related to the class of D lattices: Namely, L_{19} is the lattice such that L'_{19} has two connected components $U = \{x < y < z\}$ and $W = \{w\}$.

Observe (cf. (1.2)) that D-lattices are A-lattices.

Define Λ to be a *C^*-lattice* if:

(C) For all $x \in \Lambda'$ there exist maximal elements $m_1, \ldots, m_n$ of Λ' such that $x = m_1 \wedge \cdots \wedge m_n$.

A *C_*-lattice* is a lattice dual to a C^*-lattice, and a *C-lattice* is a lattice which is both a C^*-lattice and a C_*-lattice. Finally Λ is a *CD-lattice* if Λ is both a C-lattice and a D-lattice.

One consequence of our work is:

Proposition 1 *The hexagon and the Watatani lattice L_{19} are each isomorphic to an interval lattice $\mathcal{O}_G(H)$ for a suitable finite group G and subgroup H.*

More generally we study pairs G, H such that $\mathcal{O}_G(H)$ is an A-lattice, D-lattice or CD-lattice. If X is a normal subgroup of G contained in H and $G^* = G/X$, then $\mathcal{O}_G(H) \cong \mathcal{O}_{G^*}(H^*)$, so we may assume that the largest normal subgroup $\ker_H(G)$ of G contained in H is trivial. It is then easy to make the following reduction familiar to permutation theorists:

Proposition 2 *Assume G is a finite group and H is a subgroup of G such that $\ker_H(G) = 1$ and $\mathcal{O}_G(H)$ is an A-lattice. Then*

(1) *G has a unique minimal normal subgroup D.*

(2) *$G = HD$.*

(3) *D is the direct product of the set $\mathcal{L}$ of components of G, H is transitive on $\mathcal{L}$ via conjugation, and the components of G are nonabelian simple groups.*

(4) *The map $\varphi : \mathcal{O}_G(H) \to \mathcal{V}_D(H)$ is an isomorphism of posets, where $\varphi(U) = U \cap D$.*

Here $\mathcal{I}_D(H)$ is the set of H-invariant subgroups of D and $\mathcal{V}_D(H) = \mathcal{I}_D(H) \cap \mathcal{O}_D(H \cap D)$.

Let L be a nonabelian finite simple group. Define $\mathcal{T}(L)$ to be the set of triples $\tau = (H, N_H, I_H)$ such that:

(T1) H is a finite group and $N_H \leq H$.

(T2) $I_H \trianglelefteq N_H$ and $F^*(N_H/I_H) \cong L$.

The tuple $\tau \in \mathcal{T}(L)$ is said to be *faithful* if $\ker_{N_H}(H) = 1$.

Assume $\tau \in \mathcal{T}(L)$ and write N_0 for the preimage in N_H of $Inn(L)$ under the map of N_H into $Aut(L)$ supplied by (T2). Define

$$\mathcal{W} = \mathcal{W}_H(N_H, I_H) = \{W \in \mathcal{I}_H(N_H) : W \cap N_H = I_H\}$$

and

$$\mathcal{P} = \mathcal{P}(\tau) = \{(V, K) : V \in \mathcal{W},\ K \in \mathcal{O}_{N_H(V)}(VN_H),\ \text{and } N_0V/V = F^*(K/V)\}.$$

Partially order $\mathcal{P}$ by $(V_1, K_1) \leq (V_2, K_2)$ if $V_2 \leq V_1$ and $K_2 \leq K_1$. Let $\Lambda(\tau)$ be the poset obtained by adjoining an element 0 to $\mathcal{P}$ such that $0 < p$ for all $p \in \mathcal{P}$. We see in (7.1) that Λ is isomorphic to $\mathcal{O}_G(H)$ for a suitable overgroup G of H. Thus $\Lambda(\tau)$ is a lattice. We call such lattices *signalizer lattices.*

Theorem 3 *Assume Λ is a CD-lattice and $(H, G) \in \mathcal{G}^*(\Lambda)$. Assume G is* not *almost simple. Then*

(1) *$F^*(G)$ is the direct product of the set $\mathcal{L}$ of components of G, H is transitive on $\mathcal{L}$ via conjugation, and the components of G are nonabelian simple groups.*

(2) *H is a complement to $F^*(G)$ in G.*

(3) *Let $L \in \mathcal{L}$, $N_H = N_H(L)$, and $I_H = C_H(L)$. Then $Inn(L) \leq Aut_H(L)$, $\tau = (H, N_H, I_H) \in \mathcal{T}(L)$, and $\mathcal{O}_G(H)$ is isomorphic as a lattice to the signalizer lattice $\Lambda(\tau)$.*

The construction in (7.1) shows that given a simple group L and $\tau = (H, N_H, I_H) \in \mathcal{T}(L)$, there exists an overgroup G of H such that $\mathcal{O}_G(H) \cong \Lambda(\tau)$. This observation is used to construct examples (such as those establishing Proposition 1) in section 8. In the other direction, using the isomorphism in part (3) of Theorem 3, one can hope to show for suitable Λ that either the members of $\mathcal{G}^*(\Lambda)$ are almost simple or that Λ is a signalizer lattice in some almost simple group H. In a later paper, we achieve this reduction for a certain class of CD-lattices. The reduction can be compared to that in [BL] for a different class of lattices, where the notion of signalizer lattice does not appear, more cases arise, and it is unclear if the information in some of the cases can be effectively exploited.

See [FGT] for notation and terminology involving finite groups. While most of the group theory used here is fairly elementary, the classification of the finite simple groups and knowledge of the outer automorphism groups of the simple groups are used in [AS] and in the proof of (5.3).

Proposition 2 is proved in section 2, Theorem 3 is proved in section 6, and Proposition 1 is proved in section 8.

Section 1. Lattices

In this section Λ is a finite lattice. Define Λ to be a *B-lattice* if $\Lambda' \neq \varnothing$ and for all $x \in \Lambda'$ there exists $a, b \in \Lambda'$ with $a < b$, $x \wedge a = x \wedge b$, and $x \vee a = x \vee b$. Set $\Lambda^\# = \Lambda - \{0\}$.

(1.1) *Let G be a finite group, $H \le G$, and $K \trianglelefteq G$. Then HK is modular in the lattice $\mathcal{O}_G(H)$.*

Proof. For $H \le A \le B \le G$,

$$AHK \cap B = AK \cap B = A(K \cap B) = AH(K \cap B) = A(HK \cap B)$$

(cf. 1.14 in [FGT]). □

(1.2) (1) *B-lattices and D-lattices are A-lattices.*
(2) *D-lattices are B-lattices.*

Proof. Let $m \in \Lambda'$. Assume first that Λ is a B-lattice. Then there is $a < b$ in Λ' with $m \wedge a = m \wedge b$ and $m \vee a = m \vee b$. But if m is modular, then

$$b = (m \vee b) \wedge b = (m \vee a) \wedge b = a \vee (m \wedge b) = a \vee (m \wedge a) = a,$$

a contradiction. Thus B-lattices are A-lattices, so (1) follows from (2).

Next assume Λ is a D-lattice. Then there is a partition $\Lambda' = \Lambda'_1 \cup \Lambda'_2$ satisfying (D1) and (D2), and we may assume $m \in \Lambda'_1$. By (D2) there is $a < b$ in Λ'_2, and by (D1), $m \vee a = m \vee b = \infty$ and $m \wedge a = m \wedge b = 0$, so Λ is a B-lattice. This establishes (2), and hence also the lemma. □

Section 2. The proof of Proposition 2

In this section G is a finite group.

Given a subgroup H of G, write $\mathcal{O}(H) = \mathcal{O}_G(H)$ for the set of overgroups of H in G. Recall $\ker_H(G)$ denotes the largest normal subgroup of G contained in H. Write $\mathcal{M} = \mathcal{M}_G$ for the set of maximal subgroups of G, and set

$$\mathcal{M}(H) = \mathcal{M}_G(H) = \{M \in \mathcal{M} : H \le M\} = \mathcal{O}(H) \cap \mathcal{M}$$

and

$$\mathcal{M}^f(H) = \{M \in \mathcal{M}(H) : \ker_M(G) = 1\}.$$

Let $\mathcal{J}_G(H)$ be the set of nontrivial normal subgroups D of G such that $G = HD$. Let

$$\mathcal{J}^*_G(H) = \{D \in \mathcal{J}_G(H) : D \text{ is a minimal normal subgroup of } G\}.$$

(2.1) *Let $X \in \mathcal{J}_G(H)$. Then*
(1) *The map $\varphi : U \mapsto U \cap X$ is a bijection of $\mathcal{O}(H)$ with $\mathcal{V}_X(H)$.*
(2) *For each $U \in \mathcal{O}(H)$, $U = H\varphi(U)$.*

Proof. Let $U \in \mathcal{O}(H)$. Visibly $\varphi(U) \in \mathcal{V}_X(H)$. Also

$$U = U \cap G = U \cap XH = (U \cap X)H = \varphi(U)H,$$

so (2) holds and φ is injective. If $V \in \mathcal{V}_X(H)$, then $H \le HV \le G$, so $HV \in \mathcal{O}(H)$ and $\varphi(HV) = HV \cap X = (H \cap X)V = V$, as $V \in \mathcal{O}_X(H \cap X)$. Therefore φ is surjective, and hence a bijection. Thus (1) holds. □

We recall (cf. the introduction to [AS]) that if a group D is the direct product of a set $\mathcal{L}$ of subgroups and $\pi_L : D \to L$ is the projection of D onto $L \in \mathcal{L}$ with respect to this direct sum decomposition, then a subgroup F of D is a *full diagonal subgroup* of D if $\pi_L : F \to L$ is an isomorphism for each $L \in \mathcal{L}$.

(2.2) *Assume*

(a) $\ker_H(G) = 1$,
(b) *there exists a subgroup K of G such that $H < K < G$, and*
(c) $D \in \mathcal{J}^*_G(H)$.

Then

(1) $\mathcal{J}^*_G(H) = \{D\}$.
(2) *D is the direct product of the set $\mathcal{L}$ of components of D, H acts transitively on $\mathcal{L}$, and each component of D is simple.*

Proof. Suppose $E \in \mathcal{J}^*_G(H) - \{D\}$ and set $Y = H \cap ED$. Then $ED = E \times D$, and as $E, D \in \mathcal{J}_G(H)$, $ED = DY = EY$. Thus the projection maps $\pi_F : ED \to F$ for $F \in \{E, D\}$ are surjections on Y. Next $H \cap D$ is invariant under $EH = G$, so $H \cap D = 1$ by (a). Similarly $H \cap E = 1$, so the projections π_F are isomorphisms, and hence Y is a full diagonal subgroup of ED. Choose K as in (b). Then $Y < K \cap ED$ by (2.1)(2), so $1 \neq K \cap D$ is invariant under $HE = G$. Thus $D \leq K$ by minimality of D, so $G = HD \leq K$, contradicting $K \neq G$. This completes the proof of (1).

Next as $G = HD$ and D is a minimal normal subgroup of G, (2) holds unless D is an abelian p-group for some prime p (cf. 8.2 and 8.3 in [FGT]). But in that event as $H < K$, $H \cap D < K \cap D$ by (2.1)(1), so in particular, $K \cap D \neq 1$. However as D is abelian and $H \leq K$, $K \cap D$ is invariant under $HD = G$, so $K \cap D = D$ by minimality of D. Thus $G = HD \leq K$, contrary to the choice of K. Thus (2) holds. □

(2.3) *Suppose U is a normal subgroup of G contained in H and set $G^* = G/U$. Then the map $X \mapsto X^*$ is an isomorphism of the lattice $\mathcal{O}_G(H)$ with the lattice $\mathcal{O}_{G^*}(H^*)$.*

Proof. The map $X \mapsto X^*$ is a bijection between $\mathcal{O}_G(H)$ and $\mathcal{O}_{G^*}(H^*)$ preserving inclusion, intersection, and generation. □

We close this section with a proof of Proposition 2. Assume the hypotheses of Proposition 2 and let $\Lambda = \mathcal{O}_G(H)$. As the hypotheses of Proposition 2 are satisfied, hypothesis (a) of (2.2) holds and Λ is an A-lattice. As Λ is an A-lattice, hypothesis (b) of (2.2) holds.

Suppose $1 \neq K \trianglelefteq G$. Then as $\ker_H(G) = 1$, $K \not\leq H$, so $H < I = KH$. Further I is modular in Λ by (1.1), so as Λ is an A-lattice, $I = G$. That is, $\mathcal{J}_G(H)$ is the set of all nontrivial normal subgroups of G, so $\mathcal{J}^*_G(H)$ is the set of all minimal normal subgroups of G. In particular hypothesis (c) of (2.2) is satisfied by each minimal normal subgroup D of G. As $D \in \mathcal{J}_G(H)$, part (2) of proposition 2 holds. Parts (1) and (3) of the proposition follow from parts (1) and (2) of (2.2), respectively, and part (4) of the proposition follows from (2.1)(1).

Section 3. Full diagonal subgroups

Recall the definition of a "full diagonal subgroup" from section 2. In this section we assume the following hypothesis:

Hypothesis 3.1. Assume G is a finite group, $D \trianglelefteq G$, and H is a complement in G to D. Assume D is the direct product of the set $\mathcal{L}$ of components of D, H acts transitively on $\mathcal{L}$, and each component of D is simple. Assume for L a component of D that $Inn(L) \leq Aut_H(L)$.

In this section we adopt the following notation:

Notation 3.2. For $X \in \mathcal{L}$, let $\pi_X : D \to X$ be the projection of D on X with respect to the direct sum decomposition in Hypothesis 3.1. For $\gamma \subseteq \mathcal{L}$, set $D_\gamma = \langle \gamma \rangle$. Fix $L \in \mathcal{L}$, let $N_G = N_G(L)$, $\bar{G} = Aut_G(L)$, and $c : N_G \to \bar{G}$ be the conjugation map $c : g \mapsto c_g$, where $c_g : l \mapsto l^g$ for $l \in L$.

Set $N_H = N_H(L)$, $\bar{H} = Aut_H(L)$, $\bar{L} = Inn(L)$, and $L_H = N_H \cap LC_G(L) = H \cap c^{-1}(\bar{L})$.

For $U \in \mathcal{O}_H(N_H)$, set $\gamma(U) = L^U$ and $\Gamma(U) = \gamma(U)^H$. Write $\mathcal{P}(G)$ for the poset of all G-invariant partitions of $\mathcal{L}$ partially ordered by $\Sigma \leq \Gamma$ if Γ is a refinement of Σ. For $U \in \mathcal{O}_H(N_H)$, let $\mathcal{F}(U)$ be the set of U-invariant full diagonal subgroups of $D_{\gamma(U)}$. Set

$$\mathcal{F} = \mathcal{F}(G, D, H) = \bigcup_{U \in \mathcal{O}_H(N_H)} \mathcal{F}(U),$$

and for $F \in \mathcal{F}$ set

$$\gamma(F) = \{X \in \mathcal{L} : F\pi_X \neq 1\},\ \Gamma(F) = \gamma(F)^H,\ D(F) = \prod_{h \in H} F^h,\ \text{and } \xi(F) = HD(F).$$

(3.3) (1) $\bar{L}_H = \bar{L}$ *but* $D \cap H = 1$.

(2) *The map* $\Gamma : U \mapsto \Gamma(U)$ *is an isomorphism of the dual of the poset* $\mathcal{O}_H(N_H)$ *with the poset* $\mathcal{P}(G)$.

Proof. By Hypothesis 3.1, $Inn(L) \leq Aut_H(L)$, so from Notation 3.2, $\bar{L}_H = \bar{L}$. Similarly $H \cap D = 1$ by Hypothesis 3.1. Thus (1) holds. By Hypothesis 3.1, H is transitive on $\mathcal{L}$, and by definition N_H is the stabilizer in H of L in this representation, so (2) follows; cf. 5.18 in [FGT]. □

(3.4) *Let* $U \in \mathcal{O}_H(N_H)$ *and* $\gamma = \gamma(U)$. *Then*

(1) *A subgroup* F *of* D_γ *is a full diagonal subgroup of* D_γ *iff for each* $Y \in \gamma$, *the projection map* $\pi_{F,Y} : F \to Y$ *is an isomorphism.*

(2) *If* F *is a full diagonal subgroup of* D_γ, *then* $F = \{l\alpha_{L,F} : l \in L\}$, *where* $\alpha_{L,F} : L \to F$ *is the isomorphism defined by*

$$l\alpha_{L,F} = \prod_{Y \in \gamma} l\pi_{F,L,Y},$$

and for $X, Y \in \gamma$, $\pi_{F,X,Y} = \pi_{F,X}^{-1}\pi_{F,Y} : X \to Y$.

Proof. Part (1) is just the definition (cf. page 50 in [AS]) of a "full diagonal subgroup" of D_γ. Then (2) follows from 1.2 in [AS]. □

(3.5) *Let* $F \in \mathcal{F}$. *Then*

(1) *There exists a unique* $U(F) \in \mathcal{O}_H(N_H)$ *with* $F \in \mathcal{F}(U)$. *Moreover* $\gamma(F) = \gamma(U(F))$.

(2) *Define a relation* $\lesssim$ *on* $\mathcal{F}$ *by* $F \lesssim E$ *iff* $\gamma(E) \subseteq \gamma(F)$ *and* $E = F\pi_{D_{\gamma(E)}}$. *Then* $\lesssim$ *is a partial order on* $\mathcal{F}$.

(3) $D : \mathcal{F} \to \mathcal{V}_D(H)^\# = \mathcal{I}_D(H)^\#$ *is an isomorphism of posets.*

(4) $\xi : \mathcal{F} \to \mathcal{O}_G(H) - \{H\}$ *is an isomorphism of posets.*

(5) $\Gamma : \mathcal{F} \to \mathcal{P}(G)$ *is a map of posets, such that* $\Gamma(F) = \Gamma(E)$ *iff* $U(F) = U(E)$.

(6) $U : \mathcal{F} \to \mathcal{O}_H(N_H)$ *is a map of posets from* $\mathcal{F}$ *to the dual of* $\mathcal{O}_H(N_H)$.

Proof. Let $F \in \mathcal{F}$ and $\delta = \gamma(F)$. Then $F \in \mathcal{F}(U)$ for some $U \in \mathcal{O}_H(N_H)$ and by (3.4)(1), $\gamma(U) = \delta$. By (3.3)(2), γ is injective on $\mathcal{O}_H(U)$, so (1) holds. The proof of (2) is straightforward. If $F \lesssim E$, then $\gamma(U(E)) = \gamma(E) \subseteq \gamma(F) = \gamma(U(F))$, so $U(E) \leq U(F)$, and hence (6) holds. If for the moment we write $\hat{\gamma}$ for the map γ from $\mathcal{F}$ to the power set of $\mathcal{L}$, reserving the symbol γ for the map from $\mathcal{O}_H(N_H)$ to that power set, then by (1), $\hat{\gamma} = \gamma \circ U$, so (5) follows from (3.3)(2) and (6).

As F is U-invariant and $\Gamma(F) \in \mathcal{P}(G)$ by (5), $D(F)$ is a direct product of the groups F^h, as h varies over any choice of coset representatives for U in H. Thus $D(F) \in \mathcal{I}_D(H)$. Further, $\mathcal{I}_D(H) = \mathcal{V}_D(H)$ as $D \cap H = 1$. Visibly $D : \mathcal{F} \to \mathcal{I}_D(H)^{\#}$ is injective. On the other hand suppose $P \in \mathcal{I}_D(H)^{\#}$. Then P is nontrivial, so $P\pi_X \neq 1$ for some $X \in \mathcal{L}$, so $P\pi_X \neq 1$ for all $X \in \mathcal{L}$ by transitivity of H on $\mathcal{L}$. Thus as $P\pi_L$ is N_H-invariant and $\bar{L}_H = \bar{L}$, $P\pi_L = L$. Thus by 1.5 in [AS], $P = D(F)$ for some $F \in \mathcal{F}$, so D is a bijection. If $F \lesssim E$, then $E = F\pi_{D_{\hat{\gamma}(E)}}$, so $F \leq \langle E^{U(F)} \rangle \leq \langle E^H \rangle = D(E)$, and hence $D(F) = \langle F^H \rangle \leq D(E)$. This completes the proof of (3). Then (3) and (2.1) imply (4). □

(3.6) *Assume $J \in \mathcal{I}_H(N_H)$ with $N_J(L) = C_J(L)$, and set $U = JN_H$. Then*

(1) *$U \in \mathcal{O}_H(N_H)$ with $\gamma(U) = L^J$.*
(2) *There exists a unique member F of $\mathcal{F}(U)$ such that $J \leq C_U(F)$. Indeed $F = C_{D_{\gamma(U)}}(J)$.*

Proof. Part (1) is trivial. Let $\gamma = \gamma(U)$ and for $Y \in \gamma$ define

$$\mathcal{J}(Y) = \{j \in J : L^j = Y\}$$

and $\beta_Y : L \to Y$ by $\beta_Y = c_j$ for $j \in \mathcal{J}(Y)$. As $N_J(L) = C_J(L)$, β_Y is well defined. Define $\alpha : L \to D_\gamma$ by

$$l\alpha = \prod_{Y \in \gamma} l\beta_Y.$$

By construction, $F = L\alpha$ is a full diagonal subgroup of D_γ. Let I be a set of coset representatives for $N_J(L)$ in J, and let $j \in J$. Then Ij is also a set of coset representatives for $N_J(L)$ in J, so for $l \in L$,

$$(l\alpha)^j = (\prod_{i \in I}(l^i))^j = \prod_{i \in I} l^{ij} = \prod_{k \in Ij} l^k = l\alpha,$$

so J centralizes F. Let $E = C_{D_\gamma}(J)$. As $F \leq E$, $E\pi_X = X$ for all $X \in \gamma$, so by 1.4 in [AS], E is the direct product of full diagonal subgroups of the groups D_δ, $\delta \in \Delta$, for some U-invariant partition Δ of γ. As J is transitive on γ and centralizes E, $\Delta = \{\gamma\}$, so $E = F$. This completes the proof of (2). □

(3.7) *Assume $C_G(D) = 1$. Then*

(1) *$D = F^*(G)$.*
(2) *H acts faithfully on $\mathcal{L}$.*

Proof. As D is the product of components of G, $F^*(G) = DC_{F^*(G)}(D)$, so (1) follows as $C_G(D) = 1$. In particular, $G \leq Aut(D)$.

Let K be the kernel of the action of H on $\mathcal{L}$. Then K is contained in the kernel J of the action of $Aut(D)$ on $\mathcal{L}$, which is (cf. 1.1 in [AS]) the direct product of the groups $Aut(X)$, $X \in \mathcal{L}$. In particular $J^\infty = D$, so $K^\infty \leq H \cap D = 1$, so K is solvable.

Suppose $K \neq 1$. Then as H is transitive on $\mathcal{L}$, $1 \neq Aut_K(L)$. But then $\bar{L} \leq Aut_H(L) \leq N_{Aut(L)}(Aut_K(L))$, contradicting K solvable. Thus (2) holds. $\square$

Notation 3.8. Set $I_H = C_H(L)$ and write N_0 for the preimage in N_H of $Inn(L)$ under the conjugation map $c : N_H \to Aut(L)$. Observe that $\tau = (H, N_H, I_H)$ is in the set $\mathcal{T}(L)$ of the introduction, and define $\mathcal{W} = \mathcal{W}_H(N_H, I_H)$ and the poset $\mathcal{P} = \mathcal{P}(\tau)$ as in the introduction. For $(V, K) \in \mathcal{P}$ define $\zeta(V, K)$ to be the image of N_0 in $C_{D_{\gamma(K)}}(V)$ under the conjugation map $c_K : K \to Aut(C_{D_{\gamma(K)}}(V))$.

(3.9) (1) *For $(V, K) \in \mathcal{P}$, $\zeta(V, K) \in \mathcal{F}(K)$.*

(2) *The map $\zeta : \mathcal{P} \to \mathcal{F}$ is an isomorphism of posets.*

(3) *$\nu : \mathcal{F} \to \mathcal{P}$ is the inverse of ζ, where for $F \in \mathcal{F}$, $\nu(F) = (C_H(F), U(F))$.*

(4) *$U = N_H(\gamma(F)) = N_H(F)$ and $C_H(F) = C_{U(F)}(F)$.*

Proof. Let $(V, K) \in \mathcal{P}$. As $K \in \mathcal{O}_{N_H(V)}(VN_H)$, VN_H and K are in $\mathcal{O}_H(N_H)$. Set $\alpha = \gamma(VN_H)$ and $\gamma = \gamma(K)$.

Next as $V \in \mathcal{W}$, $N_V(L) = V \cap N_H = I_H$ centralizes L, so $C_{D_\alpha}(V) = F \in \mathcal{F}(VN_H)$ by (3.6)(2). Then as $VN_H \leq K$, γ is partitioned by α^K. Set $D' = \langle F^K \rangle$. Then D' is the direct product of the groups F^k, $k \in K$, and as $VN_0 \trianglelefteq K$, VN_0 fixes each F^k. Then for each F^k, $F^k = Aut_{N_0}(F^k)$, so $(VN_0)c_K = E$ is a subgroup of D' satisfying $E\pi_X = X$ for each $X \in F^K$. Thus by 1.4 in [AS], E is a product of full diagonal subgroups E_δ, $\delta \in \Delta$, for some K-invariant partition Δ of F^K. But $F \cong N_0/I_H$ and V centralizes D', so E is a full diagonal subgroup of D'. Thus E is also a full diagonal subgroup of D_γ, and as $VN_0 \trianglelefteq K$, $E \in \mathcal{F}(K)$. Indeed $E = \zeta(V, K)$, so (1) is established.

Next let $F \in \mathcal{F}$, $U = U(F)$, $V_F = C_H(F)$, and $\gamma = \gamma(F)$. By (3.5)(1), $U \in \mathcal{O}_H(N_H)$ and $\gamma = \gamma(U)$, so U is transitive on γ and $N_H \leq U$. Thus $U = N_H(\gamma)$ and by definition of $\gamma(F)$, $N_H(F) \leq N_H(\gamma)$, so (4) holds.

Now $V_F \cap N_H = C_{N_H}(F) = I_H$, so $V_F \in \mathcal{W}$. Let $\alpha = \gamma(V_F N_H)$. As in the proof of (1), $F_\alpha = C_{D_\alpha}(V_F) \in \mathcal{F}(V_F N_H)$ and F is the direct product of the groups $F_\delta = C_{D_\delta}(V_F)$, for $\delta \in \alpha^U$. As $Aut_L(N_0) = Inn(L)$, also $Aut_F(N_0) = Inn(F)$, so $N_0 V_F$ is the preimage in U of $Inn(F)$ under the conjugation map $U \to Aut(F)$. Therefore $N_0 V_F / V_F = F^*(U/V_F)$ as $F^*(Aut_U(F)) = Inn(F)$. Thus $\nu(F) \in \mathcal{P}$.

Next $(\zeta \circ \nu)(F) = \zeta(V_F, U)$ is the image of $N_0 V_F$ in $C_{D_\gamma}(V_F)$, so $F \leq \zeta(V_F, U)$ as $F \leq C_{D_\gamma}(V_F)$ and $F = Aut_{N_0 V_F}(F)$. Then as $F \cong L \cong \zeta(V_F, U)$, $F = \zeta(V_F, F) = (\zeta \circ \nu)(F)$. That is, $\zeta \circ \nu = 1$.

Similarly $(\nu \circ \zeta)(V, K) = \nu(E) = (C_H(E), U(E))$. We showed above that $E \in \mathcal{F}(K)$, so $K = U(E)$ by (3.5)(1). Similarly by construction of E, E is the image of $N_0 V$ in $C_{D_\gamma}(V)$, so $V \leq C_K(E)$. Then as $N_0 V/V = F^*(K/V)$, while $[C_K(E), N_0 V] \leq C_{N_0 V}(E) = V$, we have $V = C_K(E)$. Hence $(\nu \circ \zeta)(V, K) = (V, K)$, so $\nu \circ \zeta = 1$, establishing (3).

Suppose $F_i \in \mathcal{F}$ for $i = 1, 2$ with $F_1 \lesssim F_2$, and set $\gamma_i = \gamma(F_i)$ and $U_i = U(F_i)$. Then $\gamma_2 \subseteq \gamma_1$ and $F_2 = F_1 \pi_{D_{\gamma_2}}$. Then by (4), $U_2 = N_H(\gamma_2) \leq N_H(\gamma_1) = U_1$ and $C_H(F_2) \leq C_H(F_1)$ as $\pi_{D_{\gamma_2}}$ is $N_H(\gamma_2)$-equivariant. Thus ν is a map of posets.

Finally suppose $(V_i, K_i) \in \mathcal{P}$ for $i = 1, 2$ with $(V_1, K_1) \leq (V_2, K_2)$, and let $\gamma_i = \gamma(K_i)$ and $F_i = \zeta(V_i, K_i)$. Then $V_2 \leq V_1$ and $K_2 \leq K_1$. As $K_2 \leq K_1$, we have $\gamma_2 \subseteq \gamma_1$, while as $V_2 \leq V_1$, V_2 centralizes F_1. Therefore $P = F_1 \pi_{D_{\gamma_2}} \leq C_{D_{\gamma_2}}(V_2) = F_2$, and hence $P = F_2$ as $P \cong F_1 \cong F_2$. Hence $F_1 \lesssim F_2$, so ζ is a map of posets, completing the proof of (2) and the lemma. $\square$

(3.10) *Define the poset* $\mathcal{F} = \mathcal{F}(G, D, H)$ *and the map* $\xi : \mathcal{F} \to \mathcal{O}_G(H) - \{H\}$ *as in Notation 3.2 and* (3.5). *Define the poset* $\mathcal{P} = \mathcal{P}(H, N_H, I_H)$ *and the map* $\zeta : \mathcal{P} \to \mathcal{F}$ *as in Notation 3.8 and* (3.9). *Then* $\phi = \xi \circ \zeta : \mathcal{P} \to \mathcal{O}_G(H) - \{H\}$ *is an isomorphism of posets.*

Proof. This follows from (3.5)(4) and (3.9)(2). □

Section 4. Reduction to the almost simple case

In this section we assume the following hypothesis:

Hypothesis 4.1. Assume H is a proper subgroup of a finite group G, $\ker_H(G) = 1$, $D \in \mathcal{J}^*_G(H)$, and there exists a subgroup K of G with $H < K < G$.

(4.2) (1) $\mathcal{J}^*_G(H) = \{D\}$.
(2) $G = HD$.
(3) *Let* $\mathcal{L}$ *be the set of components of* D. *Then* D *is the direct product of its components,* H *acts transitively on* $\mathcal{L}$, *and each component of* D *is simple.*
(4) *The map* $\varphi : \mathcal{O}_G(H) \to \mathcal{D} = \mathcal{V}_D(H)$ *is a bijection, where* $\varphi(U) = U \cap D$.

Proof. By Hypothesis 4.1, $D \in \mathcal{J}^*_G(H)$, so (2) holds. Indeed by Hypothesis 4.1, conditions (a)–(c) of (2.2) are satisfied, so (1) and (3) follow from (2.2). Part (4) follows from (2.1). □

In the remainder of the section we adopt the following notation:

Notation 4.3. Adopt the notation of (4.2). For $X \in \mathcal{L}$, let $\pi_X : D \to X$ be the projection of D on X with respect to the direct sum decomposition in (4.2)(3). Fix $L \in \mathcal{L}$, let $N_G = N_G(L)$, $\bar{G} = Aut_G(L)$, and $c : N_G \to \bar{G}$ be the conjugation map $c : g \mapsto c_g$.

Set $N_H = N_H(L)$, $\bar{H} = Aut_H(L)$, $D_H = H \cap D$, $\bar{D}_H = D_H c$, $\bar{L} = Inn(L)$, and $L_H = N_H \cap LC_G(L) = H \cap c^{-1}(\bar{L})$.

Define $\mathcal{V} = \mathcal{I}_L(N_H) \cap \mathcal{O}_L(D_H \pi_L)$ and $\bar{\mathcal{V}} = \mathcal{I}_{\bar{L}}(\bar{H}) \cap \mathcal{O}_{\bar{L}}(\bar{D}_H)$. For $V \in \mathcal{V}$, set

$$V\theta = \prod_{h \in H} V^h.$$

(4.4) (1) *If* $H \leq Y < G$, *then* $\ker_Y(G) \leq C_G(D)$.
(2) $C_G(D) \cap D = Z(D) = 1$.
(3) $D = F^*(G)$ *iff* $C_G(D) = 1$.
(4) *If* $D = F^*(G)$, *then* $\mathcal{M}(H) = \mathcal{M}^f(H)$.

Proof. Let $H \leq Y < G$ and set $J = \ker_Y(G)$. By minimality of D, either $D \leq J$ or $D \cap J = 1$. In the former case $G = HD \leq Y$, contrary to the choice of Y. In the latter $[D, Y] \leq D \cap Y = 1$. Thus (1) holds.

As D is the direct product of nonabelian simple groups, $Z(D) = 1$, so (2) holds. Part (3) is straightforward. Then (1) and (3) imply (4). □

(4.5) (1) *For each* $d \in D$, $g \in G$, *and* $X \in \mathcal{L}$, *we have* $(d\pi_X)^g = (d^g)\pi_{X^g}$.
(2) $\pi_L \cdot c = c$ *on* D.
(3) *The map* $c : \mathcal{V} \to \bar{\mathcal{V}}$ *is a bijection.*
(4) *The map* θ *is an injection of* $\mathcal{V}$ *into* $\mathcal{V}_D(H)$. *Further, the image of* $\mathcal{V}^\# = \mathcal{V} - \{1\}$ *under* θ *is* $\mathcal{V}^I_D(H) = \{U \in \mathcal{V}_D(H) : 1 \neq U \cap L = U\pi_L\}$.

(5) $\mathcal{V}^{\#} = \{L\}$ *iff* $\bar{\mathcal{V}}^{\#} = \{\bar{L}\}$ *iff* $\mathcal{V}_{\bar{L}}(\bar{H}) = \{1, \bar{L}\}$ *iff* $\bar{L} = \bar{L}_H$.
(6) *If* $\bar{L} = \bar{L}_H$, *then* $\bar{D}_H = 1$ *or* $\bar{L}$.

Proof. Let $d \in D$ and $g \in G$. Then

$$d = \prod_{X \in \mathcal{L}} d\pi_X,$$

so

$$\prod_{X \in \mathcal{L}} d^g \pi_X = d^g = \prod_{X \in \mathcal{L}} (d\pi_X)^g,$$

with $(d\pi_X)^g \in X^g$, so (1) holds.

For $d \in D$, $d = ab$, where $a = d\pi_L$ and b is the projection of d on $C_D(L)$. Thus for $l \in L$, $l(dc) = l^d = l^a = l((d\pi_L)c) = l(d(\pi \cdot c))$, so (2) holds.

As $c : L \to \bar{L}$ is an N_H-equivariant isomorphism with $N_H c = \bar{H}$, $c : \mathcal{I}_L(N_H) \to \mathcal{I}_{\bar{L}}(\bar{H})$ is a bijection. By (2), $D_H \pi_L c = D_H c = \bar{D}_H$, so

$$\mathcal{O}_L(D_H \pi_L)c = \mathcal{O}_{\bar{L}}(D_H \pi_L c) = \mathcal{O}_{\bar{L}}(\bar{D}_H).$$

Thus (3) holds.

Let $V \in \mathcal{V}$. As V is N_H-invariant, for each $X \in \mathcal{L}$, $V_X = V^h$ is independent of $h \in \mathcal{H}_X = \{i \in H : L^i = X\}$. Thus H permutates $\{V_X : X \in \mathcal{L}\}$, and

$$V\theta = \prod_{X \in \mathcal{L}} V_X$$

is in $\mathcal{I}_D(H)$. Let $X \in \mathcal{L}$ and $h \in \mathcal{H}_X$. As $D_H \pi_L \leq V$, (1) says that

$$D_H \pi_X = D_H^h \pi_X = D_H^h \pi_{L^h} = (D_H \pi_L)^h \leq V^h = V_X,$$

so

$$D_H \leq \prod_{X \in \mathcal{L}} D_H \pi_X \leq V\theta,$$

and hence $V\theta \in \mathcal{V}_D(H)$. Thus $\theta : \mathcal{V} \to \mathcal{V}_D(H)$, and visibly θ is injective with $\mathcal{V}^{\#}\theta \subseteq \mathcal{V}_D^I(H)$. Conversely suppose $U \in \mathcal{V}_D^I(H)$. Then U is H-invariant, so $V = U \cap L$ is N_H-invariant. As $D_H \leq U$ and $U \cap L = U\pi_L$, $D_H \pi_L \leq U\pi_L = U \cap L = V$, so $V \in \mathcal{V}$. As $U \cap L \neq 1$, $V \neq 1$, so $V \in \mathcal{V}^{\#}$. Thus $\mathcal{V}^{\#}\theta = \mathcal{V}_D^I(H)$, completing the proof of (4).

We next prove (5). If $\bar{L} = \bar{L}_H$, then as L is simple, $\bar{\mathcal{V}}^{\#} = \{\bar{L}\}$ iff $\mathcal{V}^{\#} = \{L\}$ by (3). Similarly $\mathcal{V}_{\bar{L}}(\bar{H}) = \{\bar{L}\}$. Conversely suppose $\bar{L} \neq \bar{L}_H$. By (3), $V = \bar{L}_H c^{-1} \in \mathcal{V}$, so we may assume $\bar{L}_H = 1$. Then $\bar{D}_H = 1$ and an argument in the proof of 6.3 in [AS] shows $\mathcal{I}_{\bar{L}}(\bar{H}) \neq \{1, \bar{L}\}$, so $\bar{\mathcal{V}}^{\#} \neq \{\bar{L}\}$ and $\mathcal{V}_{\bar{L}}(\bar{H}) \nsubseteq \{1, \bar{L}\}$, completing the proof of (5).

Finally suppose $\bar{L} = \bar{L}_H$. Then as L is simple, $\mathcal{I}_{\bar{L}}(\bar{L}_H) = \{1, \bar{L}\}$, so as $\bar{D}_H \in \mathcal{I}_{\bar{L}}(\bar{L}_H)$, (6) holds. □

In the remainder of the section we assume:

Hypothesis 4.6. Hypothesis 4.1 holds and $F^*(G) = D$.

(4.7) *Let $M \in \mathcal{O}_G(H)$ and for $X \in \mathcal{L}$ let $M_X = M \cap X$ and $\bar{M}_L = M_L c$. Then $M \in \mathcal{M}(H)$ iff either*

(1) *$M \cap D$ is the direct product*

$$M \cap D = \prod_{X \in \mathcal{L}} M_X$$

and $\bar{M}_L \in \mathcal{V}^*_{\bar{L}}(\bar{H})$, where $\mathcal{V}^*_{\bar{L}}(\bar{H})$ is the set of maximal members of $\mathcal{V}_{\bar{L}}(\bar{H}) - \{\bar{L}\}$ under inclusion or

(2) *there exists $\Gamma(M) \in \mathcal{P}^*(G)$, the set of maximal G-invariant partitions of $\mathcal{L}$, such that for $\gamma \in \Gamma(M)$, $M_\gamma = M \cap D_\gamma$ is an H-invariant full diagonal subgroup of $D_\gamma = \langle \gamma \rangle$, and $M \cap D$ is the direct product*

$$M \cap D = \prod_{\gamma \in \Gamma(M)} M_\gamma.$$

Proof. As $F^*(G) = D$, $\mathcal{M}(H) = \mathcal{M}^f(H)$ by (4.4)(4). Thus the possibilities for members of $\mathcal{M}(H)$ are described in case (C) of Theorem 1 of [AS].

Assume $M \in \mathcal{M}$. By Hypothesis 4.1, there exists a subgroup K of G with $H < K < G$. Thus $H \notin \mathcal{M}$, so $H < M$, and therefore $H \cap D < M \cap D$ by (4.2)(4). In particular $M \cap D \neq 1$, so M is not in the set $\mathcal{N}^*_1$ of case (C).

Finally note that a member M of $\mathcal{O}_G(H)$ is in $\mathcal{N}^*_2$ iff M appears in (4.7)(2), while $M \in \mathcal{N}^*_3$ iff M appears in (4.7)(1). □

Notation 4.8. Write $\mathcal{M}^I(H)$ for the set of members of $\mathcal{M}(H)$ appearing in part (1) of (4.7), and write $\mathcal{M}^{II}(H)$ for the members appearing in part (2) of (4.7).

For $M \in \mathcal{M}^{II}(H)$ define $\Gamma(M)$ and M_γ and D_γ for $\gamma \in \Gamma(M)$ as in (4.7)(2). Define $\gamma(M)$ to be the member of $\Gamma(M)$ containing L. Let

$$\mathcal{M}^{III}(H) = \{M \in \mathcal{M}^{II} : Inn(M_{\gamma(M)}) \leq Aut_H(M_{\gamma(M)})\},$$

and set $\mathcal{M}^{IV}(H) = \mathcal{M}^{II}(H) - \mathcal{M}^{III}(H)$.

(4.9) (1) *$\mathcal{V}_{\bar{L}}(\bar{H})^\# = \mathcal{V}_{\bar{L}}(\bar{H}) - \{1\} \subseteq \bar{\mathcal{V}}$.*

(2) *The map $\Theta = c^{-1} \cdot \theta$ is an injection of $\mathcal{V}_{\bar{L}}(\bar{H})^\#$ into $\mathcal{V}^I_D(H)$ which induces a bijection between $\mathcal{V}^*_{\bar{L}}(\bar{H})$ and $\{M \cap D : M \in \mathcal{M}^I(H)\}$.*

(3) *For $\bar{V} \in \bar{\mathcal{V}}$ define $\bar{V}\tau = \bar{V}\Theta H$. Then $\tau : \mathcal{V}_{\bar{L}}(\bar{H})^\# \to \mathcal{O}_G(H)$ is an injection which induces a bijection $\tau : \mathcal{V}^*_{\bar{L}}(\bar{H}) \to \mathcal{M}^I(H)$.*

(4) *For $i = 1, 2$, let $\bar{V}_i \in \mathcal{V}_{\bar{L}}(\bar{H})^\#$, and set $M_i = \bar{V}_i\tau$. Then $M_1 \cap M_2 \cap D = (\bar{V}_1 \cap \bar{V}_2)\theta$, and $\langle M_1, M_2 \rangle \cap D = \langle \bar{V}_1, \bar{V}_2 \rangle\theta$.*

Proof. Suppose $U \in \mathcal{V}_{\bar{L}}(\bar{H})^\#$. Then $\bar{D}_H \leq \bar{L}_H \leq U$, so $U \in \bar{\mathcal{V}}$, establishing (1). Let $\bar{\mathcal{V}}^\# = \bar{\mathcal{V}} - \{1\}$. By parts (3) and (4) of (4.5), $\Theta : \bar{\mathcal{V}}^\# \to \mathcal{V}^I_D(H)$ is a bijection. By (4.7), $M \in \mathcal{M}^I(H)$ iff $M = U\Theta$ for some $U \in \mathcal{V}^*_{\bar{L}}(\bar{H})$. Together with (1), these observations imply (2). Then (2) and (4.2)(4) imply (3).

Assume the hypothesis of (4), and set $V_i = \bar{V}_i c^{-1}$. By (3) and the definition of Θ, $d \in D$ is in $M_1 \cap M_2$ iff $d \in \bar{V}_1\theta \cap \bar{V}_2\theta$ iff for each $X \in \mathcal{L}$, $d\pi_X \in V_1^h \cap V_2^h$, for $h \in H$ with $L^h = X$. Then as $V_1^h \cap V_2^h = (V_1 \cap V_2)^h$, the first statement in (4) follows. Similarly the second statement holds. □

(4.10) (1) *$\mathcal{M}^I(H) = \varnothing$ iff $\bar{L}_H = \bar{L}$.*

(2) *If $H \cap L \neq 1$, then $\mathcal{M}(H) = \mathcal{M}^I(H)$.*

(3) *Assume* $V \in \mathcal{V}$ *and* $M_2 \in \mathcal{M}^{II}(H)$. *Set* $M_1 = V\theta$, $E = M_1 \cap M_2 \cap D$ *and for* $\gamma \in \Gamma = \Gamma(M_2)$ *set* $E_\gamma = M_1 \cap M_{2,\gamma}$. *Then*

(a) E *is the direct product of the groups* E_γ, $\gamma \in \Gamma$.

(b) *Let* $\gamma_2 = \gamma(M_2)$ *and let* U *be the preimage in* M_{2,γ_2} *of* V *under* π_L. *Then*

$$E_{\gamma_2} = \bigcap_{h \in N_H(\gamma_2)} U^h.$$

Proof. By (4.9)(3), $\mathcal{M}^I(H) = \varnothing$ iff $\mathcal{V}^*_{\bar{L}}(\bar{H}) = \varnothing$; equivalently $\mathcal{V}_{\bar{L}}(\bar{H})^\# = \{\bar{L}\}$. By (4.5)(5), this is in turn equivalent to $\bar{L} = \bar{L}_H$.

If $M \in \mathcal{M}^{II}(H)$, then $M \cap D$ is the product of full diagonal subgroups M_γ, $\gamma \in \Gamma(M)$. Thus $M \cap L = 1$, so as $H \leq M$, also $H \cap L = 1$. This establishes (2).

Assume the hypothesis of (3), let $d \in M_2 \cap D$, and for $X \in \mathcal{L}$ let $d_X = d\pi_X$. Then $d \in E$ iff $d_X \in M_1 \cap X$ for all $X \in \mathcal{L}$. Let $\sigma : M_2 \cap D \to M_{2,\gamma_2}$ be the projection and let $X \in \gamma_2$. Then $d_X \in M_1 \cap X$ iff $d\sigma\pi_X \in M_1 \cap X$ iff $d\sigma$ is in U_X, the preimage in M_{2,γ_2} of $M_1 \cap X$ under π_X. Further, for $h \in H$ with $L^h = X$,

$$M_1 \cap X = V^h = (U\pi_L)^h = U^h\pi_X$$

by (4.5)(1), so $U_X = U^h$. Therefore $d_X \in M_1 \cap X$ for all $X \in \gamma_2$ iff

$$d\sigma \in E_2 = \bigcap_{h \in N_H(\gamma_2)} U^h.$$

In particular it follows that $E_2 = E_{\gamma_2}$, establishing (3)(b). Further, $E\sigma \leq E_2$, so the projection of E on M_{2,γ_2} is contained in E. Then as H is transitive on Γ, (a) follows. □

(4.11) *Assume* $M_1 \in \mathcal{M}^I(H)$, $M_2 \in \mathcal{M}^{II}(H)$, *and* $H = M_1 \cap M_2$. *Set* $\gamma_2 = \gamma(M_2)$. *Then*

(1) *If* $1 \neq \bar{L}_H$, *then for each* $M \in \mathcal{M}^I(H)$, $M_1 \cap M \neq H$.

(2) *If* $M_2 \in \mathcal{M}^{IV}(H)$, *then either*

  (i) H *is maximal in* M_2 *or*

  (ii) *there exists* $M \in \mathcal{M}^I(H)$ *with* $M \cap M_2 \neq H$.

(3) *If* $D_H \neq 1$, *then* $1 \neq \bar{L}_H$ *and* $\mathcal{M}^{III}(H) = \varnothing$.

Proof. Set $P = M_{2,\gamma_2}$. As $M_1 \cap M_2 = H$, $E = M_1 \cap M_2 \cap D = D_H$.

Suppose first that $D_H \neq 1$. Then $1 \neq \bar{D}_H \leq \bar{L}_H$. Also for each $M \in \mathcal{M}^{II}(H)$, the projection of D_H on $Q = M_{\gamma(M)}$ is a nontrivial proper $N_H(\gamma(M))$-invariant subgroup of Q, so $Inn(Q) \not\leq Aut_H(Q)$, completing the proof of (3).

Next suppose $1 \neq \bar{L}_H$. As $\bar{L}_H \leq \bar{M}$ for each $M \in \mathcal{M}^I(H)$, the inverse image W of $\bar{L}_H$ in L is contained in each such M. Thus $W \leq M_1 \cap M$, while $H \cap L = 1$ by (4.10)(2), so $M_1 \cap M \neq H$, establishing (1).

Finally assume $M_2 \in \mathcal{M}^{IV}(H)$ but $H < J < M_2$. By (2.1)(1), $E < F = J \cap D < M_2 \cap D$. Let E_2, F_2 be the projections of E, F on P, respectively, and choose J with $F \cap P$ maximal. As E and F are H-invariant, E_2 and F_2 are $N_H(\gamma_2)$-invariant. Hence by (4.10)(3) and the maximality of $F \cap P$,

$$(*) \qquad E = \prod_{h \in H} E_2^h \text{ and if } F_2 \neq P, \text{ then } F = \prod_{h \in H} F_2^h.$$

As $E < F < M_2 \cap H$, we conclude from (*) that either $E_2 < F_2 < P$ or $F_2 = P$. In the latter case, maximality of $F \cap P$ says that $E_2 = F \cap P$ is a maximal proper $N_H(\gamma_2)$-invariant subgroup of P. But as $M_2 \in \mathcal{M}^{IV}(H)$, $Aut_H(P) \neq P$, so $E_2 \neq 1$. Therefore $P = [E_2, P] = [E_2, F_2] = [E_2, F] \leq F \cap P$, contradicting $J < M_2$. Thus $\bar{F}_2 \leq \bar{Y} \in \mathcal{V}^*_{\bar{L}}(\bar{H})$. Then by (4.9)(3), $M = \bar{Y}\tau \in \mathcal{M}^I(H)$ and as F_2 is $N_H(\gamma_2)$-invariant, $F_2 \leq M \cap M_2$ by (4.10)(3). Thus $M \cap M_2 \neq H$, and the proof is complete. □

(4.12) *Assume $\mathcal{M}(H) = \mathcal{M}^I(H)$ and there exist $M_1, M_2 \in \mathcal{M}(H)$ such that $M_1 \cap M_2 = H$. Then*

(1) *$M_1 \cap M_2 \cap L = H \cap L = L_H\pi_L$, $\bar{L}_H = \bar{D}_H$, and $H \cap L \in \mathcal{V}$ with $D_H = (H \cap L)\theta$ and $H = \bar{L}_H\tau$, where τ is the map of (4.9)(3).*

(2) *Define $\mu : \mathcal{O}_G(H) \to \mathcal{O}_{\bar{G}}(\bar{H})$ by $U\mu = Aut_U(L)$, and define $\eta : \mathcal{O}_{\bar{G}}(\bar{H}) \to \mathcal{O}_G(H)$ by $\bar{U}\eta = (\bar{U} \cap \bar{L})\tau$. Then μ and $\eta : \mathcal{O}_{\bar{G}}(\bar{H}) \to \mathcal{O}_G(H)$ are maps of posets.*

(3) *$\eta\mu = 1$.*

(4) *$\bar{G}\eta = G$ and $\bar{H}\eta = H$.*

(5) *$\mathcal{M}_{\bar{G}}(\bar{H})\eta = \mathcal{M}_G(H)$.*

(6) *For $\bar{U}_i \in \mathcal{O}_{\bar{G}}(\bar{H})$, $i = 1, 2$, $\langle \bar{U}_1, \bar{U}_2 \rangle \eta = \langle \bar{U}_1\eta, \bar{U}_2\eta \rangle$ and $\bar{U}_1\eta \cap \bar{U}_2\eta = (\bar{U}_1 \cap \bar{U}_2)\eta$.*

(7) *If $\mathcal{O}_G(H)$ is a C^*-lattice, then η is an isomorphism with inverse μ.*

Proof. Let $V_i = M_i \cap L$. As $H = M_1 \cap M_2$, $H \cap L = V_1 \cap V_2$. Further, $\bar{L}_H \leq \bar{V}_i$, so $P = L_H\pi_L \leq V_1 \cap V_2 = H \cap L$, and hence $P = H \cap L$. Now (1) follows.

Parts (2) and (3) are straightforward, while (4) follows as $L_H\pi_L = H \cap L$. Part (5) follows from (4.9)(3) and the hypothesis that $\mathcal{M}(H) = \mathcal{M}^I(H)$. Part (6) follows from (4.9)(4).

Finally suppose $\mathcal{O}_G(H)$ is a C^*-lattice. Then for $U \in \mathcal{O}_G(H)$, $U = K_1 \cap \cdots \cap K_n$ for some $K_i \in \mathcal{M}_G(H)$. By (5), $K_i = \bar{K}_i\eta$, where $\bar{K}_i = K_i\mu$. Then $\bar{U} = \bar{K}_1 \cap \cdots \cap \bar{K}_n \in \mathcal{O}_{\bar{G}}(\bar{H})$, and by (6), $\bar{U}\eta = \bar{K}_1\eta \cap \cdots \cap \bar{K}_n\eta = U$, so η is surjective. Thus η is a bijection with inverse μ by (3), and then (7) follows from (2). □

(4.13) *Assume $\bar{D}_H = \bar{L}$. Then*

(1) *$\mathcal{M}(H) = \mathcal{M}^{II}(H)$.*

(2) *There exists $\Gamma_0 \in \mathcal{P}(G)$ such that*

$$D_H = \prod_{\gamma \in \Gamma_0} D_{H,\gamma}$$

and $D_{H,\gamma}$ is a full diagonal subgroup of D_γ.

(3) *Let $\mathcal{G} = \{\Delta \in \mathcal{P}(G) : \Gamma_0 \leq \Delta\}$, and for $\delta \in \Delta \in \mathcal{G}$ let B_δ be the projection of D_H on D_δ, and let $B(\Delta) = \langle B_\delta : \delta \in \Delta \rangle$. Then the map $B : \Delta \to B(\Delta)$ is a bijection of $\mathcal{G}$ with $\mathcal{V}_D(H)$.*

(4) *Let H^* be the image of H in $Sym(\mathcal{L})$ under the representation of H via conjugation. Let H_0 be the stabilizer in H of the block in Γ_0 containing L, and for $U \in P = \mathcal{O}_{H_0^*}(N_H^*)$ set $\delta(U) = L^U$, $\Delta(U) = \delta(U)^H$, and $A(U) = B(\Delta(U))H$. Partially order P by the dual of inclusion. Then the maps $\Delta : P \to \mathcal{G}$ and $A : P \to \mathcal{O}_G(H)$ are isomorphisms of posets.*

Proof. Part (1) follows from (4.10)(1). Let $W \in \mathcal{V}_D(H)$. As $L = D_H\pi_L \leq W\pi_L$, as W is H-invariant, and as H is transitive on $\mathcal{L}$, $W\pi_X = X$ for each $X \in \mathcal{L}$.

Therefore (cf. 1.4 in [AS]) there exists an H-invariant partition $\Delta(W)$ of $\mathcal{L}$ such that for each $\delta \in \Delta(W)$, the projection W_δ of W on D_δ is a full diagonal subgroup of D_δ and

$$W = \prod_{\delta \in \Delta(W)} W_\delta.$$

As W is H-invariant, $\Delta(W) \in \mathcal{P}(G)$. In particular setting $\Gamma_0 = \Delta(D_H)$, (2) holds. Further, as $D_H \leq W$, $\Gamma_0 \leq \Delta(W)$, so $\Delta(W) \in \mathcal{G}$. Then as $B(\Delta(W)) = W$, the map B in (3) is surjective. By 1.6 in [AS], B is a well defined map from $\mathcal{G}$ into $\mathcal{V}_D(H)$. As B_δ is the projection of D_H on D_δ, the map $\delta \mapsto B_\delta$ is injective, so B is injective, completing the proof of (3).

For $\Sigma \in \mathcal{G}$, let δ_Σ be the block of Σ containing L, and let $H(\Sigma) = N_H(\delta_\Sigma)$. Then $H_0 = H(\Gamma_0)$ and $\Sigma \leq \Omega$ iff $H(\Sigma) \geq H(\Omega)$, so $H(\Sigma)^* \in \mathcal{O}_{H_0^*}(N_H^*) = P$. Conversely if $U \in P$, then $\delta(U)$ is the block in $\Delta(U) \in \mathcal{G}$ containing L, and the maps $\Sigma \mapsto H(\Sigma)^*$ and $U \mapsto \Delta(U)$ are inverses for each other and maps of posets. Similarly $A = \varphi^{-1} \circ B \circ \Delta$ is an isomorphism of posets as B and φ are isomorphisms by (3) and (4.2)(4). □

(4.14) *Assume $\mathcal{M}^I(H)$ and $\mathcal{M}^{III}(H)$ are nonempty. Then*

(1) *$\bar{H}$ is maximal in $\bar{G}$.*
(2) *$\bar{L}_H \neq 1$.*
(3) *$\mathcal{M}^I(H) = \{M_1\}$, where $M_1 = \bar{L}_H \tau$.*
(4) *Let $M \in \mathcal{M}^{III}(H)$, $\gamma = \gamma(M)$, $U = N_H(\gamma)$, and $P = M_\gamma$. Then N_H is maximal in U, $C_H(P) = C_{N_H}(P)$, and $U = N_H U_0$, where U_0 is the preimage of $Inn(P)$ in U under the conjugation map $c_U : U \to Aut(P)$.*
(5) *If $M, M' \in \mathcal{M}^{III}(H)$ are distinct then either*
   (i) *$M \cap M' = H$ or*
   (ii) *M, M', and M_1 are in the same connected component of $\mathcal{O}_G(H)'$.*

Proof. Let $M_1 \in \mathcal{M}^I(H)$, $M \in \mathcal{M}^{III}(H)$, $\Gamma = \Gamma(M)$, and adopt the notation in (4). By (4.7), $\Gamma \in \mathcal{P}^*(G)$, so U is primitive on γ. Thus N_H is maximal in U. As $M \in \mathcal{M}^{III}(H)$, $P^* = Inn(P) \leq Aut_U(P)$. The restriction σ of the projection map $\pi_L : D \to L$ to P is an isomorphism, which induces an isomorphism $\sigma^* : Aut(P) \to Aut(L)$ defined by $\sigma^* : \alpha \mapsto \sigma^{-1}\alpha\sigma$. Then $\bar{H}(\sigma^*)^{-1} = H^* < H^*P^*$ as $\bar{H} < \bar{G}$ by (4.10)(1). Also H^* is maximal in H^*P^* iff $\bar{H}$ is maximal in $\bar{G}$, and $H^* = N_H c_U$.

As $H^* < H^*P^*$ and H is maximal in U, H^* is maximal in H^*P^*, $Aut_U(P) = U^* = H^*P^*$, and $C_H(P) = C_U(P) = C_{N_H}(P)$. As $M \in \mathcal{M}^{III}(H)$, $P^* = U_0 c_U$. Then as $\ker(c_U) = C_U(P) \leq N_H$, $H^* = N_H c_H$, and $U^* = H^*P^*$, we have $U = N_H U_0$, completing the proof of (4). Further, H^* is maximal in H^*P^*, so $\bar{H}$ is maximal in $\bar{G}$ by the previous paragraph, establishing (1).

By (4.10)(1), $\bar{L} \neq \bar{L}_H$, while by (1), $\mathcal{V}_{\bar{L}}(\bar{H}) = \{\bar{L}, \bar{L}_H\}$, so (2) follows from (4.5)(5), and (3) follows from (4.9)(3).

Finally assume $M' \in \mathcal{M}^{III}(H) - \{M\}$ and $K = M \cap M' \neq H$. Let $\gamma' = \gamma(M')$, $U' = N_H(\gamma')$, $W = \langle U, U' \rangle$, and $\alpha = L^W$. As $K \neq H$, $D_K = K \cap D \neq 1$ by (4.2)(4). As H is transitive on $\mathcal{L}$, $D_K \pi_X \neq 1$ for each $X \in D_\gamma^H$. As $Inn(X) \leq Aut_M(X)$, $D_K \pi_X = X$. Then $D_K \pi_L = D_K \pi_{D_\gamma} \pi_L = D_\gamma \pi_L = L$. Hence by 1.4 in [AS], there exists $\Delta \in \mathcal{P}(G)$ such that D_K is the direct product of full diagonal subgroups K_δ, $\delta \in \Delta$, of D_δ. Let $L \in \delta \in \Delta$. Then $\gamma, \gamma' \subseteq \delta$, so $W \leq N_H(\delta)$ and hence $\alpha \subseteq \delta$. Then by 1.6 in [AS], $K_\delta \pi_{D_\alpha} = R$ is a full diagonal subgroup of D_α. Let $J = \langle R, H \rangle$.

Then $J \le K$, so J, M, M' are in the same connected component of $\mathcal{O}_G(H)'$. Hence to complete the proof of (5), it suffices to show that $J \cap M_1 \neq H$.

As $U = N_H U_0$ and $U' = N_H U_0'$, $W = N_H W_1$, where $W_1 = \langle U_0, U_0' \rangle$. As $Aut_{U_0}(P) = Inn(P)$ and $\pi_P : R \to P$ is U-equivariant, $Aut_{U_0}(R) = Inn(R)$. Similarly $Aut_{U_0'}(R) = Inn(R)$, so $W_1 \le W_0$, the preimage in W of $Inn(R)$ under the conjugation map $W \to Aut(R)$. Thus $W = N_H W_1 = N_H W_0 = UI$, where $I = C_H(R)$. But as $M \neq M'$, $L^U = \gamma$ is a proper subset of $\alpha = L^W = L^{UI}$, so $I \neq I_H$ and α is partitioned by γ^I. By (4.5)(3), $V = \bar{L}_H c^{-1} \in \mathcal{V}$. Let R_V be the inverse image in R of V under π_L. Then for $r \in R_V$ and $i \in I$, (4.5)(1) says that

$$r\pi_{L^i} = (r^i)\pi_{L^i} = (r\pi_L)^i \in V^i,$$

so

$$r = \prod_{i \in I} r\pi_{L^i} \in \prod_{i \in I} V^i \le M_1,$$

and hence $1 \neq R_V \le J \cap D \cap M_1$, completing the proof. $\square$

Section 5. $\mathcal{O}_G(H)$ for H a complement to $F^*(G)$ in an almost simple G

In this section we assume the following hypothesis:

Hypothesis 5.1. G is a finite group, $F^*(G) = L$ is a nonabelian simple group, and $1 \neq H$ is a complement to L in G.

(5.2) *Suppose $R \trianglelefteq H$ with $|R|$ prime. Then $\mathcal{I}_L(H)$ is a connected lattice.*

Proof. By 40.7 in [FGT], $K = C_L(R) \neq 1$. Let $\Delta = \mathcal{I}_L(H) - \{1, L\}$, $\mathcal{C} = \mathcal{C}(K)$ be the connected component of Δ containing K, and $\Gamma = \Delta - \mathcal{C}$.

Assume $J \in \Gamma$. Then $J \cap K \in \mathcal{I}_L(H)$, so $J \cap K = 1$; that is, $C_J(R) = 1$. Thus by a second application of 40.7 in [FGT], J is nilpotent. Let $p \in \pi(J)$ and $P = O_p(J)$. Then $X = N_L(P) \in \mathcal{C}(J)$, so $X \in \Gamma$, and hence by symmetry between J and X, X is nilpotent. In particular choosing J with $|J|_p$ maximal, it follows that $P \in Syl_p(L)$. Further, for $1 \neq U$ characteristic in P, $N_L(U) \in \mathcal{C}(J)$, so $N_L(U)$ is nilpotent. Thus if p is odd, then L has a normal p-complement by the Thompson Normal p-Complement Theorem (cf. 39.5 in [FGT]), a contradiction. Therefore $J \in Syl_2(L)$. Further, $J = N_L(Z(J))$, so $Z(J)$ is strongly closed in J with respect to L by 37.6 in [FGT]. But now Glauberman's Z^*-Theorem [Gl] supplies a contradiction and completes the proof. $\square$

(5.3) *$\mathcal{I}_L(H)$ is a connected lattice.*

Proof. Assume otherwise. Then by (5.2), H has no normal subgroup of prime order.

Suppose $X = Out(L)$ has a normal series $1 = X_0 \trianglelefteq \cdots \trianglelefteq X_n = X$ such that X_{i+1}/X_i is cyclic for each $0 \le i < n-1$ and X_n/X_{n-1} is abelian. Let j be the least i such that $H \cap X_i \neq 1$. If $j < n$, then $H \cap X_j$ is isomorphic to a subgroup of the cyclic group X_j/X_{j-1}, so H has a normal subgroup of prime order, a contradiction. On the other hand if $j = n$, then H is isomorphic to a subgroup of the abelian group X_n/X_{n-1}, and we obtain the same contradiction. Thus no such series exists.

By the previous paragraph, $Out(L)$ is nonabelian. Therefore (cf. 5.2.1 and Table 5.3 in [GLS3]) L is of Lie type. Adopt the notation of section 2.5 in [GLS3], and set $X_1 = Outdiag(L)$. By parts (a), (d), (e), and (f) of 2.5.12 in [GLS3], either X/X_1

is abelian, or $L \cong D_4(q)$ and $X/X_1 \cong \Phi_L \times \Gamma_L$, where $\Phi_L = Aut(\mathbf{F}_q)$ and $\Gamma_L \cong S_3$. Further, X_1 is described in part (c) of 2.5.12 of [GLS3], and in particular either X_1 is cyclic or q is odd, $L \cong D_{2m}(q)$ for some integer $m > 1$, and $X_1 \cong E_4$. We conclude from the second paragraph of this proof that $L \cong D_{2m}(q)$ with q odd and $m > 1$ an integer and that $X_1 \leq G/L$. Therefore by part (e) of 2.5.12 in [GLS3], $X/X_1 = \Phi_L \times \Gamma_L$, with Γ_L the group of symmetries of the Dynkin diagram of L. Hence $\Gamma_L \cong \mathbf{Z}_2$ if $m > 2$.

Let R be the preimage in H of X_1. As Φ_L centralizes X_1, either $Z(H) \cap R \neq 1$ or $m = 2$ and there exists a 3-element u in H acting nontrivially on R, and by (5.2) the latter holds. Then u^3 is a 3-element whose image in X/X_1 lies in Φ_L, so as H has no normal subgroup of order 3, $u^3 = 1$. Thus $U = \langle u \rangle R \cong A_4$.

Let $K = C_L(R)$. Thus $R \cong E_4$ and K is of even order so $K \neq 1$. As in the proof of (5.2), there is $1 \neq J \in \mathcal{I}_L(H)$ such that J is not in the connected component $\mathcal{C}(K)$ in Δ, so in particular $J \cap K = 1$. Thus $C_J(R) = 1$ so J is of odd order. Let P be a nontrivial elementary abelian p-subgroup of J for some prime p, such that $P \trianglelefteq HJ$; for example we could choose P to be a minimal normal subgroup of HJ contained in J. Choose J so that $P_J = O_p(J)$ is of maximal order.

Let $R^\# = \{r_1, r_2, r_2\}$. As $C_P(R) = 1$ and $A_4 \cong U \leq N_G(P)$, $P = P_1 \times P_2 \times P_3$, where $P_i = C_P(r_i)$ and U is transitive on $\{P_1, P_2, P_3\}$. In particular $m_p(P) \equiv 0$ mod 3 and $m_p(L) \geq 3$, so by 4.10.3.a in [GLS3], $m_p(L) = 4$ and p divides $q^2 - 1$.

Let $M = N_L(P)$. Then $M \in \mathcal{C}(P)$, so $M \notin \mathcal{C}(R)$, and hence $C_M(R) = 1$. Thus M is of odd order, and by 18.7 in [FGT], there is a unique R-invariant Sylow p-subgroup Q of M. Hence $Q \in \mathcal{I}_L(H)$, so $Q = P_J$ by maximality of P_J. Thus $Q \in Syl_p(L)$. Hence by 4.10.3.c in [GLS3], there is a unique E_{p^4}-subgroup S of Q. Thus H acts on S and as $m_p(S)$ is not a multiple of 3, the previous paragraph supplies our final contradiction. □

Section 6. Minimal representations of D-lattices as subgroup lattices

In this section we assume:

Hypothesis 6.1. Λ is a D-lattice, $\mathcal{Q} = \mathcal{G}^*(\Lambda)$, and $(H, G) \in \mathcal{Q}$. Further, G is *not* almost simple.

(6.2) *There exists a partition $\mathcal{O}_G(H)' = \mathcal{O}_1 \cup \mathcal{O}_2$ of $\mathcal{O}_G(H)' = \mathcal{O}_G(H) - \{H, G\}$ such that:*

(1) *For each $J_i \in \mathcal{O}_i$, $i = 1, 2$, $G = \langle J_1, J_2 \rangle$ and $J_1 \cap J_2 = H$.*
(2) *For $i = 1, 2$ there exists a nontrivial chain $K_i < M_i$ in $\mathcal{O}_i$.*

Proof. This is a restatement of the hypothesis that Λ is a D-lattice and $(H, G) \in \mathcal{Q}$, so that $\mathcal{O}_G(H)$ is isomorphic to Λ or its dual. □

(6.3) $\ker_H(G) = 1$.

Proof. Set $G^* = G/\ker_H(G)$. By (2.3), $(G^*, H^*) \in \mathcal{Q}$, so $\ker_H(G) = 1$ by minimality of $|G|$. □

(6.4) *$\mathcal{O}_G(H)$ is an A-lattice, so the conclusions of Proposition 2 hold, as does Hypothesis 4.6.*

Proof. As Λ is a D-lattice and $\mathcal{O}_G(H) \cong \Lambda$, $\mathcal{O}_G(H)$ is an A-lattice by (1.2)(1). Then by Proposition 2 and (6.3), the conclusions of Proposition 2 hold. Finally as the conclusions of Proposition 2 are satisfied and $\mathcal{O}_G(H)$ is a D-lattice, Hypothesis 4.6 is satisfied. □

Given (6.4), we may adopt the notation and terminology of section 4 and appeal to results in that section.

(6.5) *If $\mathcal{M}(H) = \mathcal{M}^I(H)$, then Λ is not a CD-lattice.*

Proof. Suppose $\mathcal{M}(H) = \mathcal{M}^I(H)$ and Λ is a CD-lattice. Then by (4.12)(7), $\mathcal{O}_G(H) \cong \mathcal{O}_{\bar{G}}(\bar{H})$. But as G is not almost simple, $|\bar{G}| < |G|$, contrary to the minimality of G. □

(6.6) *Assume $\mathcal{M}^I(H)$ and $\mathcal{M}^{II}(H)$ are nonempty. Then*

(1) *Either $\bar{L}_H = 1$ or $\mathcal{M}^{III}(H) \neq \varnothing$.*
(2) *$D_H = 1$.*
(3) *Assume Λ is a CD-lattice. Then*
  (a) *$\bar{L}_H = 1$, and*
  (b) *$\mathcal{M}^{III}(H) = \varnothing$.*

Proof. We may choose notation so that $M_i \in \mathcal{O}_i$ for $i = 1, 2$, $M_1 \in \mathcal{M}^I(H)$, and $M_2 \in \mathcal{M}^{II}(H)$. Thus $M_1 \cap M_2 = H$ by (6.2).

Assume $\bar{L}_H \neq 1$. Then by (4.11)(1), for each $M \in \mathcal{M}^I(H)$, $M \cap M_1 \neq H$, so $M \in \mathcal{O}_1$ by (6.2). Therefore $\mathcal{O}_2 \cap \mathcal{M}(H) \subseteq \mathcal{M}^{II}(H)$, and for each $M' \in \mathcal{O}_2 \cap \mathcal{M}(H)$, $M \cap M' = H$ by (6.2). Hence if $M' \in \mathcal{M}^{IV}(H)$, then by (4.11)(2), H is maximal in M', so M' is an isolated point in the graph of $\mathcal{O}_G(H)$, But by (6.2), $\mathcal{O}_2$ contains an edge, so we may choose $M_2 \in \mathcal{M}^{III}(H)$. This establishes (1). Then (1) and (4.11)(3) imply (2).

It remains to prove (3), so assume Λ is a CD-lattice. As $\mathcal{O}_2$ contains an edge, we may assume $M' \in \mathcal{O}_2 \cap \mathcal{M}(H) - \{M_2\}$ with $M_2 \cap M' \neq H$. By the previous paragraph, $M_2, M' \in \mathcal{M}^{III}(H)$. But now by (4.14)(5), M_2 is in the same connected component of $\mathcal{O}_G(H)'$ as M_1, contrary to (6.2)(1) and the choice of $M_i \in \mathcal{O}_i$. This completes the proof of (3)(a). Then (3)(b) follows from (3)(a) and (4.14)(2). □

(6.7) *Assume $\mathcal{M}^I(H) = \varnothing$. Then*

(1) *$\mathcal{M}(H) = \mathcal{M}^{II}(H) \neq \varnothing$.*
(2) *$|\mathcal{L}| > 1$.*
(3) *$\bar{L}_H = \bar{L}$.*
(4) *H is a complement to D in G.*
(5) *Hypothesis 3.1 is satisfied.*
(6) *Define the poset $\mathcal{F}$ and $\xi : \mathcal{F} \to \mathcal{O}_G(H) - \{H\}$ as in Notation 3.2 and (3.5). Then ξ is an isomorphism of posets.*
(7) *Define the poset $\mathcal{P}$ and the map $\phi : \mathcal{P} \to \mathcal{O}_G(H) - \{H\}$ as in Notation 3.8 and (3.10). Then ϕ is an isomorphism.*
(8) *The representation of H on $\mathcal{L}$ via conjugation is faithful.*

Proof. As $\mathcal{M}^I(H) = \varnothing$, (1) is a consequence of (4.7), while (3) is a consequence of (4.10)(1). As G is not almost simple, (2) holds.

By (3) and (4.5)(6), $\bar{D}_H$ is 1 or $\bar{L}$. Suppose the latter case holds. Pick Γ_0 as in (4.13)(2), let H^* be the image of H in $Sym(\mathcal{L})$ under the representation of H on $\mathcal{L}$ via conjugation, and let H_0 be the stabilizer in H of the block of Γ_0 containing

L. Then by (4.13)(4), $\mathcal{O}_G(H)$ is isomorphic to the dual of $\mathcal{O}_{H_0^*}(N_H^*)$. But then as $\mathcal{G}(\Lambda) = \mathcal{G}(\Lambda^*)$, where Λ^* is the dual of Λ, the interval $[N_H^*, H_0^*]$ supplies a contradiction to the minimality of $|G|$.

Therefore $\bar{D}_H = 1$, so $D_H = 1$, establishing (4). Now (5) follows from (6.4), (3), and (4). By (6.4), $D = F^*(G)$ is the direct product of nonabelian simple groups, so $C_G(D) = 1$. Then (8) follows from (3.7)(2). Part (6) follows from (5) and (3.5)(4), while part (7) follows from (5) and (3.10). □

(6.8) *If Λ is a CD-lattice, then $\mathcal{M}^I(H) = \varnothing$.*

Proof. Assume otherwise. By (6.5), $\mathcal{M}^{II}(H) \neq \varnothing$. Then by (6.6)(3), $\bar{L}_H = 1$ and $\mathcal{M}^{II}(H) = \mathcal{M}^{IV}(H)$.

Suppose $\bar{H} = 1$. Then $N_H = I_H$. As in the proof of (3.6), for $Y \in \mathcal{L}$ define $\beta_Y : L \to Y$ by $\beta_Y = c_j$ for $j \in \mathcal{H}_Y = \{j \in H : L^j = Y\}$. Then define $\alpha : L \to D$ by

$$l\alpha = \prod_{Y \in \mathcal{L}} l\beta_Y.$$

As is the proof of (3.6), $F = L\alpha$ is a full diagonal subgroup of D, and $F = C_D(H)$. In particular $HF \in \mathcal{O}_G(H)$.

Next by (4.9)(3), $\tau : \mathcal{V}_{\bar{L}}^*(\bar{H}) \to \mathcal{M}^I(H)$ is a bijection. Let $\bar{V} \in \mathcal{V}_{\bar{L}}^*(\bar{H})$, $V = \bar{V}c^{-1}$, and $M_V = \bar{V}\tau$. Thus $M_V\pi_Y = V^j$ for $j \in \mathcal{H}_Y$. Hence $V\alpha \leq M_Y \cap HF$, so $M_Y \cap HF \neq H$. Therefore $\mathcal{M}^I(H)$ is contained in the connected component of HF in $\mathcal{O}_G(H)'$, so we may take $\mathcal{M}^I(H) \subseteq \mathcal{O}_1$.

Next suppose $\bar{H} \neq 1$. Then as $\bar{L}_H = 1$, $\bar{H}$ is a nontrivial complement to $\bar{L}$ in $\bar{G}$, so $\mathcal{I}_{\bar{L}}(\bar{H})$ is a connected lattice by (5.3). Hence by (4.9), $\mathcal{M}^I(H)$ is contained in a connected component of $\mathcal{O}_G(H)'$, so again we may take $\mathcal{M}^I(H) \subseteq \mathcal{O}_1$.

Finally let $M_1 \in \mathcal{M}^I(H)$ and $M_2 \in \mathcal{O}_2$. Then $M_1 \cap M_2 = H$ by (6.2)(1), and $M_2 \in \mathcal{M}^{II}(H)$ as $\mathcal{M}^I(H) \subseteq \mathcal{O}_1$. Now $M_2 \in \mathcal{M}^{IV}(H)$ by the first paragraph of this proof. By (6.2) we may assume $H < K_2 < M_2$. Thus H is not maximal in M_2, so by (4.11)(2), there exists $M \in \mathcal{M}^I(H)$ with $M_1 \cap M_2 \neq H$. This contradicts (6.2)(1) and completes the proof. □

We are now in a position to prove Theorem 3. By (6.4) and Proposition 2, $F^*(G) = D$ satisfies conclusion (1) of Theorem 3. By (6.8), $\mathcal{M}^I(H) = \varnothing$, so we may appeal to (6.7). In particular conclusion (2) of Theorem 3 is satisfied by (6.7)(4). Further, $Inn(L) \leq Aut_H(L)$ by (6.7)(3), while Hypothesis 3.1 is satisfied by (6.7)(5). Then by Notation 3.8, $\tau = (H, N_H, I_H) \in \mathcal{T}(L)$, and by (6.7)(7), $\mathcal{O}_G(H)$ is isomorphic to $\Lambda(\tau)$. This completes the proof of Theorem 3.

Section 7. Recovering G from H

In this section L is a nonabelian finite simple group, and $\tau = (H, N_H, I_H) \in \mathcal{T}(L)$. We show how to construct a triple (G, H, L) satisfying Hypothesis 3.1 from τ, with the property that $\mathcal{O}_G(H) \cong \Lambda(\tau)$.

Let $\eta : H \to H_0$ be an isomorphism and set $N_0 = N_H\eta$ and $I_0 = I_H\eta$. Form the coset space $\Omega = H_0/N_0$, and write ω_0 for the coset $N_0 \in \Omega$. Let $L_0 = N_H/I_H$ and let $p : N_H \to L_0$ be the natural map with kernel I_H. Form a direct product

$$D_0 = \prod_{\omega \in \Omega} L_\omega$$

of $|\Omega|$ copies L_ω, $\omega \in \Omega$, of L_0, and identify L_0 with L_{ω_0}. Let $S = Sym(\Omega)$ and embed S in $Aut(D_0)$, in such a way that for $s \in S$ and $\omega \in \Omega$, $L_\omega^s = L_{\omega s}$, and SD_0 is the wreath product of S with L_0; that is, S_{ω_0} centralizes L_0. Let $\rho : H_0 \to S$ be the representation of H_0 on Ω by right multiplication, and let $c_0 : S \to Aut(D_0)$ be the conjugation map. Write G_0 for the the semidirect product of D_0 by H_0 with respect to ρc_0.

Let $N = N_{G_0}(L_0)$, and observe $N = D_0 N_0$. Set

$$D' = \prod_{\omega \in \Omega - \{\omega_0\}} L_\omega,$$

and observe that as $I_0 \trianglelefteq N_0$ and N_0 centralizes L_0, we have $I_0 D' \trianglelefteq N$. Finally set $N^* = N/I_0 D'$. Then $N^* = L_0^* \times N_0^*$. Define $\beta : L_0^* \to N_0^*$ by $\beta : (I_H x)^* \mapsto (x\eta)^*$, and observe that β is an isomorphism. Let M^* be the full diagonal subgroup

$$M^* = \{l^* \cdot l^* \beta : l \in L_0\}$$

of N^*. By Theorem 2 in [AS], there is a complement $\hat{H}$ to D_0 in G_0 such that $N_{\hat{H}}(L_0)^* = M^*$.

As H_0 and $\hat{H}$ are complements to D_0 in G_0, there is an isomorphism $\mu : H_0 \to \hat{H}$ defined by $h_0\mu = \hat{H} \cap D_0 h_0$ for $h_0 \in H_0$. Then $\chi = \eta\mu : H \to \hat{H}$ is an isomorphism. Notice that $N_H\chi = N_0\mu = N_{\hat{H}}(L_0)$. Further, $N_0 D'$ centralizes L_0, so as $N_{\hat{H}}(L_0)^* = M^*$, it follows that $C_{\hat{H}}(L_0) = \hat{H} \cap I_0 D'$. Now for $x \in N_H$, $x\chi = x\eta D \cap \hat{H}$, so

$$(x\chi)^* = (x\eta)^* \cdot (x\eta)^*\beta = (x\eta)^* \cdot (xI_H)^*,$$

and hence $x\chi \in I_0 D'$ iff $(x\chi)^* = 1$ iff $(xI_H)^* = 1$ iff $xI_H = 1$ iff $x \in I_H$. That is, $I_H\chi = C_{\hat{H}}(L_0)$.

We identify H with $\hat{H}$ via the isomorphism χ. Subject to this convention, $H = \hat{H} \leq G_0$ with $N_H = N_H(L_0)$ and $I_H = C_H(L_0)$.

Define $D = D(\tau)$ to be $F^*(D_0)$. Thus D is the direct product of the groups $F^*(L_\omega)$, $\omega \in \Omega$, each of which is isomorphic to $F^*(L_0) \cong L$. Let $G(\tau) = G$ be the subgroup HD of G_0. Identify L with $F^*(L_0)$, so that subject to this convention, L^H is the set $\mathcal{L}$ of components of D.

(7.1) *Let L be a nonabelian finite simple group, $\tau = (H, N_H, I_H) \in \mathcal{T}(L)$, $G = G(\tau)$, and $D = D(\tau)$. Then*

1. *Hypothesis 3.1 is satisfied by G, H, D, L, and $\mathcal{L}$.*
2. *$N_H = N_H(L)$ and $I_H = C_H(L)$.*
3. *Define $\mathcal{F}$ and ξ as in Notataion 3.2 and (3.5). Then the map $\xi : F \mapsto HD(F)$ is an isomorphism of the poset $\mathcal{F}$ with $\mathcal{O}_G(H) - \{H\}$.*
4. *Define $\phi : \mathcal{P}(\tau) \to \mathcal{O}_G(H) - \{H\}$ as in (3.10). Then ϕ is an isomorphism of posets, which extends to an isomorphism $\mathcal{O}_G(H) \cong \Lambda(\tau)$ of lattices.*
5. *If τ is faithful, then H is faithful on $\mathcal{L}$ and $D = F^*(G)$.*

Proof. From the discussion above of the construction of G, D is the direct product of the set $\mathcal{L} = L^H$ of components of D, H is a complement to D in G, and (2) holds. By (2), $Aut_H(L) = Aut_{N_H}(L) \cong N_H/I_H$, so as $F^*(N_H/I_H) \cong L$, it follows that $Inn(L) \leq Aut_H(L)$. This completes the proof of (1).

Part (3) follows from (1) and (3.5)(4), while part (4) follows from (1), (2), and (3.10).

Finally assume τ is faithful. Then $\ker_{N_H}(H) = 1$, so H is faithful on Ω. Then as the representations of H on Ω and $\mathcal{L}$ are equivalent, H is faithful on $\mathcal{L}$.

Let $K = C_G(D)$. Then K is contained in the kernel of the action of G on $\mathcal{L}$, which is D as $G = HD$ and H is faithful on $\mathcal{L}$. Then as $Z(D) = 1$, $K = 1$, completing the proof of (5). $\square$

Section 8. Constructing examples

In this section we construct some examples of finite group interval lattices which are D-lattices and in particular prove Proposition 1.

In this section we assume:

Hypothesis 8.1. L is a nonabelian finite simple group, and $H = H_1 \times H_2$, with H_i a nonabelian finite simple group for $i = 1$ and 2, such that there exist subgroups K_i of H_i isomorphic to L. Pick an isomorphism $\alpha_0 : K_1 \to K_2$, and let

$$N_H = \{k \cdot k\alpha_0 : k \in K_1\}$$

be the full diagonal subgroup of $K_1 \times K_2$ determined by α_0. Let $I_H = 1$ and $\tau = (H, N_H, I_H)$. Observe that $\tau \in \mathcal{T}(L)$ and form the group $G = G(\tau)$ as in section 7. Let $\mathcal{L} = \{L_\omega : \omega \in \Omega\}$ be the set of components of G. Set $\mathcal{P} = \mathcal{P}(\tau)$.

Let $\pi_i : H \to H_i$ be the projection of H on H_i for $i = 1, 2$. Thus $K_i = N_H\pi_i$.

Usually we assume the following hypothesis:

Hypothesis 8.2. Hypothesis 8.1 holds, and for $i = 1$ and 2, $N_{H_i}(K_i) = K_i$ and $\mathcal{W}_i = \{1\}$, where $\mathcal{W}_i = \{W \in \mathcal{I}_{H_i}(K_i) : K_i \cap W = 1\}$.

(8.3) *$\mathcal{W}_i = \mathcal{W}_{H_i}(K_i, 1)$ and $\mathcal{W}_H(N_H, I_H) = \{W \in \mathcal{I}_H(N_H) : W \cap N_H = 1\}$.*

Proof. As $I_H = 1$, this is immediate from the definition of $\mathcal{W}_{H_i}(K_i, 1)$ and $\mathcal{W}_H(N_H, I_H)$ in the introduction. $\square$

(8.4) *Assume Hypothesis 8.2. Then*

(1) $\mathcal{W}^\# = \mathcal{W}_H(N_H, I_H) - \{1\} = \mathcal{O}_{H_1}(K_1) \cup \mathcal{O}_{H_2}(K_2)$.

(2) $\mathcal{P}' = \mathcal{P}(H, N_H, I_H) - \{(I_H, N_H)\}$ *has two connected components* $\mathcal{P}_1$ *and* $\mathcal{P}_2$, *where* $\mathcal{P}_i = \{(U_i, U_iN_H) : U_i \in \mathcal{O}_{H_i}(K_i)\}$.

(3) *The map* $\phi_i : U_i \mapsto (U_i, U_iN_H)$ *is an isomorphism of the dual of* $\mathcal{O}_{H_i}(K_i)$ *with* $\mathcal{P}_i$.

(4) $\mathcal{O}_G(H)' = \mathcal{O}_G(H) - \{H, G\}$ *has two connected components* $\mathcal{O}_1$ *and* $\mathcal{O}_2$, *and* $\mathcal{O}_i \cong \mathcal{P}_i$.

Proof. We first prove (1). Let $\mathcal{W} = \mathcal{W}_H(N_H, I_H)$, $W \in \mathcal{W}^\#$, and set $W_i = W\pi_i$ for $i = 1, 2$. Then $\mathcal{W}_i \in \mathcal{I}_{H_i}(K_i)$ and $W_j \neq 1$ for some $j \in \{1, 2\}$. Then by Hypothesis 8.2, $1 \neq W_j \cap K_j$. Hence as K_j is simple and $W_j \cap K_j \trianglelefteq K_j$, we conclude that $K_j = W_j \cap K_j \leq W_j$, so $W_j \in \mathcal{O}_{H_j}(K_j)$. Then if in addition $W_{3-j} = 1$, then $W = W_j \in \mathcal{O}_{H_j}(K_j)$. Conversely if $U \in \mathcal{O}_{H_j}(K_j)$, then $U \in \mathcal{I}_H(N_H)$ and $U \cap N_H = 1$, so that $U \in \mathcal{W}$.

So assume $W_i \neq 1$ for $i = 1, 2$. Then $W_i \in \mathcal{O}_{H_i}(K_i)$ by the previous paragraph. Let $U_i = W \cap H_i$. Then $U_i \in \mathcal{I}_{H_i}(K_i)$, so if $U_i \neq 1$, then from the previous paragraph, $U_i \in \mathcal{O}_{H_i}(K_i)$. In particular if $U_i \neq 1$ for $i = 1$ and 2, then $N_H \leq K_1K_2 \leq W$, contradicting $W \in \mathcal{W}$. Therefore we may assume $U_1 = 1$.

If $U_2 \neq 1$, let $V = U_2$, while if $U_2 = 1$, let $V = K_2$. Thus in any event $K_2 \leq V$. Claim that V is W-invariant. If $V = U_2$, this holds as $U_2 = \ker(\pi_1) \cap W$. On the otherhand if $U_2 = 1$, then $V = H_2 \cap WN_H \trianglelefteq WN_H$: Namely as $K_2 \leq W_2$ and $W \cap N_H = 1$, we conclude that $1 \neq P = H_2 \cap WN_H$. Then since $P \in \mathcal{I}_{H_2}(N_H)$, as usual $K_2 \leq P$. But as $U_2 = 1$, $|P| \leq |WN_H : W| = |K_2|$, so it follows that $P = K_2$. Thus the claim is established.

For $h \in H$, let $h_i = h\pi_i$. Let $x \in N_H$ and $w \in W$. Then $y = [x, w] = [x_1, w_1][x_2, w_2] \in W$, with $[x_i, w_i] = y_i$. Now $x_2 \in K_2 \leq V$, so as V is W-invariant, $y_2 \in V$. Pick $w \in J = K_1\pi_2^{-1} \cap W$. Then $y_1 \in K_1$, and as K_1 is simple, $K_1 = \langle [x_1, w_1] : x \in N_H, w \in J \rangle$. Thus $K_1 = Q\pi_1$, where $Q = W \cap K_1V$. But if $V = U_2$, then $K_2 \leq V \leq W$, so as $U_1 = 1$, we have $W \cap K_1V = (W \cap K_1)V = V$, and hence $Q\pi_1 = V\pi_1 = 1$, a contradiction. Thus $U_2 = 1$ and $V = K_2$. As $U_1 = U_2 = 1$ and $Q\pi_1 = K_1$, Q is a full diagonal subgroup of K_1K_2. Then as N_H is also a full diagonal subgroup of K_1K_2 and Q is N_H-invariant, it follows that $N_H = Q \leq W$, contradicting $W \in \mathcal{W}$. This completes the proof of (1).

Next let $(V, X) \in \mathcal{P}'$. By Hypothesis 8.2, $N_{H_i}(K_i) = K_i$, so $N_H(N_H) = N_H$. Therefore $V \in \mathcal{W}^{\#}$, so we may take $V \in \mathcal{O}_{H_1}(K_1)$ by (1). Thus $N_H(V) = N_{H_1}(V)H_2$, and $X \in \mathcal{O}_{N_H(V)}(VN_H)$ with $N_HV/V = F^*(X/V)$. In particular $X\pi_2$ acts on $K_2 = (VN_H)\pi_2$, so as $N_{H_2}(K_2) = K_2$ by Hypothesis 8.2, it follows that $X\pi_2 = K_2$. Thus $X = K_2X_1$, where $X_1 = X \cap H_1$. Now $[N_H, X_1] \leq N_HV \cap H_1 = V$, so $X_1/V \leq C_{X/V}(N_HV/V) = 1$ since $N_HV/V = F^*(X/V)$. Thus $X_1 = V$, so $X = K_2X_1 = K_2V = N_HV$. That is, $(V, X) \in \mathcal{P}_1$, so we have shown that $\mathcal{P}' = \mathcal{P}_1 \cup \mathcal{P}_2$. Further, (K_i, K_iN_H) is the greatest member of $\mathcal{P}_i$, so $\mathcal{P}_i$ is connected. On the other hand for $(V_i, V_iN_H) \in \mathcal{P}_i$, $V_1 \cap V_2 = 1$, so $\mathcal{P}_1$ and $\mathcal{P}_2$ are the connected components of $\mathcal{P}'$, completing the proof of (2).

Part (3) is immediate from the definition of $\mathcal{P}_i$ in (2) and the definition of the partial order $\leq$ on $\mathcal{P}$. Then (4) is a consequence of the isomorphism $\mathcal{P} \cong \mathcal{O}_G(H) - \{H\}$ in (7.1)(4). □

Example 8.5. Here is the smallest example leading to the hexagon. Let $L \cong A_5$ and $H_1 \cong H_2 \cong A_6$. Then $\mathcal{I}_{H_i}(K_i) = \{1, K_i, H_i\}$, so Hypothesis 8.2 is satisfied, and then $\mathcal{O}_G(H)$ is the hexagon by (8.4).

Example 8.6. Given a positive integer n, write $\Lambda(n)$ for the poset of all positive integer divisors of n, partially ordered by $d \leq e$ iff d divides e. Fix positive integers n_1 and n_2, and let $H_i = L_2(4^{n_i})$ for $i = 1, 2$. Let σ_i be a field automorphism of H_i of order n_i, and for $d \in \Lambda(n_i)$, set $U_{i,d} = C_{H_i}(\sigma_i^d)$. Then $U_{i,d} \cong L_2(4^{n_i/d})$, and $U_{i,e} \leq U_{i,d}$ iff $d \leq e$ in $\Lambda(n_i)$.

Take $L \cong L_2(4)$ and $K_i = U_{i,n_i}$. From the structure of H_i,

$$\mathcal{O}_{H_i}(K_i) = \{U_{i,d} : d \in \Lambda(n_i)\},$$

and from the previous paragraph, the dual of $\mathcal{O}_{H_i}(K_i)$ is isomorphic to $\Lambda(n_i)$. Therefore by (8.4), $\mathcal{O}_G(H)'$ has two connected components $\mathcal{O}_1$ and $\mathcal{O}_2$, with $\mathcal{O}_i \cong \Lambda(n_i)$. In particular if $n_1 > 1 < n_2$, then $\mathcal{O}_G(H)$ is a D-lattice.

Example 8.7. Let m_1 and m_2 be positive integers. Define a lattice Λ to be a (m_1, m_2)-*lattice* if Λ' has two connected components Λ_1 and Λ_2, with

$$\Lambda_1 = \{x_1 < \cdots < x_{m_1}\} \text{ and } \Lambda_2 = \{y_1 < \cdots < y_{m_2}\}.$$

Observe that the chain $\{x_1 < \cdots < x_m\}$ is isomorphic to $\Lambda(2^{m-1})$. Thus from Example 8.6, if we take $L \cong L_2(4)$ and $H_i \cong L_2(4^{2^{m_i-1}})$, then $\mathcal{O}_G(H)$ is an (m_1, m_2)-gon. In particular when $m_1 = m_2 = 2$, $\mathcal{O}_G(H)$ is the hexagon, while when $m_1 = 3$ and $m_2 = 1$, $\mathcal{O}_G(H)$ is the Watatani lattice L_{19}. Thus we have proved Proposition 1.

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