MR3034474 (Review) 20D06 20D30

Aschbacher, Michael (1-CAIT-NDM)

Overgroup lattices in finite groups of Lie type containing a parabolic. (English. English summary)

J. Algebra **382** (2013), 71–99.

The author addresses J. Shareshian's conjecture raised in [J. Algebra **268** (2003), no. 2, 677–686; MR2009327 (2004h:20032)], which asserts that $D\Lambda$-lattices are not overgroup lattices in any finite group. A $D\Lambda$-lattice is a disconnected lattice Λ, in the sense that Λ is a disconnected graph when its least and greatest elements are removed, and the components of the remaining graph are simplices $\Delta(m)$ of sets of cardinality $m \geq 3$ each. Shareshian's conjecture stems from P. P. Pálfy and P. Pudlák's earlier work [Algebra Universalis **11** (1980), no. 1, 22–27; MR0593011 (82g:08003)], which leads to asking whether or not each finite lattice is isomorphic to a lattice of overgroups of some subgroup in a finite group. Aschbacher conjectured that the answer is negative, though it is known that any group interval lattice can be realised from some von Neumann algebra. In previous articles [J. Amer. Math. Soc. **21** (2008), no. 3, 809–830; MR2393428 (2008m:20036); Michigan Math. J. **58** (2009), no. 1, 79–103; MR2526079 (2010d:20027)], Aschbacher reduced the question to almost simple groups. Then, on his own [J. Group Theory **15** (2012), no. 2, 151–225;MR2900223] and also in collaboration with Shareshian [J. Algebra **322** (2009), no. 7, 2449–2463; MR2553689 (2010h:20048)], he found the answer for alternating and symmetric groups and further reduced the problem to handling the conjecture for groups in which the generalised Fitting subgroup is of Lie type.

The main theorem in the article states that for G an almost simple group with unique minimal normal subgroup, its generalised Fitting subgroup $F^*(G)$ is of Lie type. Suppose further that there exists a selfnormalising subgroup H in G such that $G = HF^*(G)$, and such that the lattice Λ of overgroups of H in G satisfies the following conditions:

- Λ is a $D\Lambda$-lattice or $\Lambda \cong \Delta(m)$ with $m \geq 3$;
- H acts on a proper parabolic subgroup of $F^*(G)$;

then $\Lambda \cong \Delta(m)$ and $H \cap F^*(G)$ is a parabolic subgroup of $F^*(G)$.

The proof of the result involves a minute analysis of the maximal overgroups of H in G, and thus of the maximal parabolic subgroups of G. This forms the bulk of the article (Sections 4 to 8) and allows Aschbacher to obtain the key tool for the proof of the main result, which is the other notable result in the paper, namely the following. Consider a finite simple group of Lie type G and maximal parabolic subgroup M with unipotent radical R. Denote by $\mathfrak{Q}(M)$ the set of proper subgroups of M contained in R and by $\mathfrak{Q}^*(M)$ its maximal elements under inclusion. Let R_0 be the preimage in R of the Jacobson radical of the M/R-module $R/[R,R]$. Then one of the following holds:

- M/R is indecomposable on $R/[R,R]$ and faithful and irreducible on R/R_0. So $\mathfrak{Q}^*(M) = \{R_0\}$.
- $G \cong \mathrm{Sp}_{2n}(2)$ with $n \geq 3$ and M is the stabiliser of an $(n-1)$-dimensional subspace of the symplectic space for G.
- $G \cong G_2(2)' \cong U_3(3)$ and M is the centraliser of an involution.

More precisely, it is an extension of the above result which is needed in the proof of the main theorem.

As a by-product of the study summarised above, Aschbacher's article provides results of independent interest regarding embedding of subgroups in finite groups when the overgroup lattice is a simplex. *Nadia P. Mazza*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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Aschbacher, Michael (1-CAIT)

Lower signalizer lattices in alternating and symmetric groups. (English. English summary)

J. Group Theory **15** (2012), *no.* 2, 151–225.

The following question was suggested by a theorem of Pálfy and Pudlak: Is each nonempty finite lattice isomorphic to an overgroup lattice $O_G(H)$ for some finite group G and subgroup H of G? The answer is almost certainly negative. However, the question has remained open for almost 30 years.

For a finite lattice L, let L' be the poset obtained by removing the minimum and maximum elements of L and, for $n \in \mathbb{N}$, let $\Delta(n)$ be the lattice of all subsets of $\{1,\dots,n\}$, ordered by inclusion. For positive integers $m_1,\dots,m_t$, the lattice $D\Delta(m_1,\dots,m_t)$ is defined as follows: $D\Delta(m_1,\dots,m_t)'$ is a disjoint union of posets $C_1,\dots,C_t$ with $C_i \cong \Delta(m_i)'$ for all i and if $i \neq j$ then no element of C_i is comparable to any element of C_j.

Shareshian advanced the conjecture that if $t > 1$ and $m_1 \geq m_2 \geq \cdots \geq m_t \geq 3$, then there does not exist a finite group G and subgroup H of G such that the overgroup lattice $O_G(H)$ is isomorphic to $D\Delta(m_1,\dots,m_t)$. The present paper is one step in a program to prove this conjecture. In a previous paper the author defined the a "signalizer lattice" and showed that if a lattice Λ in the class $D\Delta$ is a finite subgroup interval lattice, then there exists an almost simple group G such that either $\Lambda \cong O_G(H)$ for some $H \leq G$ or there exists a finite nonabelian simple group L and a pair $\gamma = (H,J)$ with $J \trianglelefteq H \leq G$ such that $F^*(H/J) \cong L$, $G = \langle W_0(\gamma), H\rangle$ and Λ is isomorphic to the lower signalizer lattice $\Xi(\gamma)$ (here $W_0(\gamma)$ is the set of the H-invariant subgroups W of G such that $H \cap W = J$ and $W \leq F^*(G)J$ while $\Xi(\gamma)$ is the lattice obtained by adjoining a greatest element to the poset $W_0(\gamma)$).

This reduces the study of Shareshian's conjecture to two questions about sublattices of the lattice of subgroups of almost simple groups. The obvious first test cases for the two questions are the alternating and symmetric groups. In a joint paper with Shareshian, the author proved that if G is an alternating or symmetric group and $H \leq G$, then $O_G(H)$ is not a $D\Delta$ lattice. The present paper treats the lower signalizer lattice case, proving that if G is an alternating or symmetric group, $t > 1$ and $m_1 \geq m_2 \geq \cdots \geq m_t \geq 3$, then there do not exist a simple group L and $\gamma = (H,J)$ such that $F^*(H/J) \cong L$, $G = \langle W_0(\gamma), H\rangle$ and $\Xi(\gamma) \cong D\Delta(m_1,\dots,m_t)$. *Andrea Lucchini*

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Aschbacher, Michael (1-CAIT)

Overgroups of primitive groups. (English. English summary)

J. Aust. Math. Soc. **87** (2009), *no.* 1, 37–82.

The author studies the structure of the lattice of primitive subgroups of the symmetric group $\mathrm{Sym}(\Omega)$ on a finite set Ω. In particular a qualitative description is given for the lattice of overgroups of a given primitive permutation group H on Ω. While such a description was given by the reviewer in [Proc. London Math. Soc. (3) **60** (1990), no. 1, 68–88; MR1023805 (90j:20009)], the author's description is "in terms of the embedding of generalized Fitting subgroups and relations among the structures preserved by the subgroups, particularly the product structures on Ω".

In particular, a partial order on product structures (studied for the larger class of innately transitive and quasiprimitive permutation groups in [R. W. Baddeley, C. E. Praeger and C. Schneider, J. Aust. Math. Soc. **77** (2004), no. 1, 55–72; MR2069025 (2005f:20002); Trans. Amer. Math. Soc. **358** (2006), no. 4, 1619–1641; MR2186989 (2006k:20003); J. Algebra **311** (2007), no. 1, 337–351; MR2309892 (2008c:20002); Trans. Amer. Math. Soc. **360** (2008), no. 2, 743–764; MR2346470 (2008g:20003)] allows a large portion of the poset of primitive subgroups of $\mathrm{Sym}(\Omega)$ to be described.

Two kinds of results are given in terms of four types of primitive permutation groups (called, by the author, affine, semisimple, strongly diagonal, and almost simple), and two further types of primitive groups called doubled (primitive groups with two minimal normal subgroups) and complemented (primitive groups with a unique minimal normal subgroup that is nonabelian and regular). The results are concerned with inclusions

$H < M < \mathrm{Sym}(\Omega)$ such that M does not contain the alternating group $\mathrm{Alt}(\Omega)$. The first kind of result specifies the possible types of primitive subgroups H of a primitive group M of a given type; while the second identifies the possible types of overgroups M of a primitive group H of a given type.

The author plans to apply these results to study the question of P. P. Pálfy and P. Pudlák [Algebra Universalis **11** (1980), no. 1, 22–27; MR0593011 (82g:08003)] of whether each nonempty finite lattice is isomorphic to an interval in the lattice of subgroups of a finite group. To illustrate how these results might be used two further results are proved.

(1) A qualitative description of pairs of distinct subgroups M_1, M_2 of $\mathrm{Sym}(\Omega)$ or $\mathrm{Alt}(\Omega)$, with each M_i maximal subject to $\mathrm{Alt}(\Omega) \not\leq M_i$, such that $M_1 \cap M_2$ is primitive.

(2) A description of the set $\mathcal{M}$ of maximal overgroups of a primitive group H such that, for all $M \in \mathcal{M}$, there exists $M' \in \mathcal{M} \setminus \{M\}$ such that $H = M \cap M'$.

This program of study is pursued further by the author and J. Shareshian [M. G. Aschbacher and J. Shareshian, J. Algebra **322** (2009), no. 7, 2449–2463; MR2553689 (2010h:20048); M. G. Aschbacher, J. Algebra **322** (2009), no. 5, 1586–1626; MR2543625 (2010h:20004); Michigan Math. J. **58** (2009), no. 1, 79–103; MR2526079 (2010d:20027); J. Amer. Math. Soc. **21** (2008), no. 3, 809–830; MR2393428 (2008m:20036)].

Cheryl E. Praeger

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Aschbacher, Michael (1-CAIT); **Shareshian, John** (1-WASN)

Restrictions on the structure of subgroup lattices of finite alternating and symmetric groups. (English. English summary)

J. Algebra **322** (2009), *no.* 7, 2449–2463.

The paper deals with an important and well-known open question: Which finite lattices can be represented as intervals in the subgroup lattice of a finite group? In particular the authors exhibit restrictions on the structure of the lattice $\mathcal{O}_G(H)$ of subgroups of G containing $H \leq G$ when G is a finite alternating or symmetric group. For a finite lattice L, let L' be the poset obtained by removing the minimum and maximum elements of L and, for $n \in \mathbb{N}$, let $\Delta(n)$ be the lattice of all subsets of $\{1,\dots,n\}$, ordered by inclusion. For positive integers $m_1,\dots,m_t$, the authors consider the lattice $D\Delta(m_1,\dots,m_t)$ defined as follows: $D\Delta(m_1,\dots,m_t)'$ is a disjoint union of posets $\mathcal{C}_1,\dots,\mathcal{C}_t$ with $C_i \cong \Delta(m_i)'$ for all i and if $i \neq j$ then no element of C_i is comparable to any element of C_j. The authors prove that if $t > 1$ and $m_1 \geq m_2 \geq \cdots \geq m_t \geq 3$, then $G \in \{\mathrm{Alt}(n), \mathrm{Sym}(n)\}$ contains no subgroup H with $\mathcal{O}_G(H) \cong D\Delta(m_1,\dots,m_t)$. *Andrea Lucchini*

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Aschbacher, Michael (1-CAIT)

Overgroups of primitive groups. II. (English. English summary)

J. Algebra **322** (2009), *no.* 5, 1586–1626.

Assume that Ω is a set of finite order and G is the symmetric group or the alternating group on Ω, and let H be a subgroup of G primitive on Ω. The author continues his study (begun in Part I [J. Aust. Math. Soc. **87** (2009), no. 1, 37–82;MR 2011a:20002]) of the set $\mathcal{O}_G(H)$ of overgroups of H in G. In this paper, attention is concentrated on the case where H is the intersection of many pairs of maximal subgroups of G. The overgroup lattice $\mathcal{O}_G(H)$ is described in terms of generalized Fitting subgroups of overgroups and certain natural structures on Ω associated to the overgroups, most particularly regular product structures on Ω. One motivation is the theorem of P. P. Pálfy and P. Pudlák [Algebra Universalis **11** (1980), no. 1, 22–27; MR0593011 (82g:08003)], which focused attention on the question of whether each nonempty finite lattice is isomorphic to an interval in the lattice of subgroups of finite groups. That question is still open. The author and J. Shareshian have begun a program to show that certain finite lattices are not of the form $\mathcal{O}_G(H)$ for any finite group G and subgroup H of G. The lattices that they consider are particular examples of I-lattices, where Λ is called an I-lattice if the poset L' obtained by removing the minimum and maximum elements of L is non-empty and for each maximal member x of L' there exists a maximal member y of L' with $x \wedge y = 0$. In this paper all the possible I-lattices of the form $\mathcal{O}_G(H)$ with $G \in \{\mathrm{Alt}(\Omega), \mathrm{Sym}(\Omega)\}$ and H primitive are classified. This has been used by the author and Shareshian in a subsequent paper [J. Algebra **322** (2009), no. 7, 2449–2463;MR2553689] to prove that there exists an infinite class of finite lattices none of which is isomorphic to an interval $\mathcal{O}_G(H)$ in the subgroup lattice of a finite or alternating symmetric group G. However, the work is of larger and independent interest and can be viewed as a development of the study of the inclusions between primitive permutation groups, a problem to which several authors have devoted particular attention [see, for example, C. E. Praeger, Proc. London Math. Soc. (3) **60** (1990), no. 1, 68–88; MR1023805 (90j:20009)].

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Signalizer lattices in finite groups.

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The following question was suggested by a theorem of Pálfy and Pudlak: Is each nonempty finite lattice isomorphic to an overgroup lattice $\mathcal{O}_G(H)$ for some finite group G and subgroup H of G? The answer is almost certainly negative. However, the question has remained open for almost 30 years since the paper of Pálfy and Pudlak was published [P. P. Pálfy and P. Pudlák, Algebra Universalis **11** (1980), no. 1, 22–27; MR0593011 (82g:08003)].

In a previous paper [J. Amer. Math. Soc. **21** (2008), no. 3, 809–830; MR2393428 (2008m:20036)] the author started to deal with this question, showing that the structure of G is highly restricted if the lattice $\mathcal{O}_G(H)$ is "disconnected". He defined the notion of a "signalizer lattice" and showed that for suitable disconnected lattices Λ (the CD-lattices), if G is minimal subject to $\mathcal{O}_G(H)$ being isomorphic to Λ or its dual, then either G is almost simple or H admits a signalizer lattice isomorphic to Λ or its dual.

In this paper the author initiates the study of signalizer lattices, with the hope of finding a suitable CD-lattice which is not a finite group interval lattice. For a given finite nonabelian simple group L, let $\mathcal{T}(L)$ be the set of triples $\tau=(H,N_H,I_H)$ with H a finite group, $I_H \trianglelefteq N_H \leq H$ and $F^*(N_H/I_H)\cong L$. For $\tau\in\mathcal{T}(L)$, write N_0 for the preimage in N_H of $\mathrm{Inn}(L)$ and consider the set $\mathcal{P}$ of the pairs (V,K) where V is a N_H-invariant subgroup of H with $V\cap N_H=I_H$, $K\in\mathcal{O}_{N_H(V)}(VN_H)$ and $N_0V/V=F^*(K/V)$. Partially order $\mathcal{P}$ by $(V_1,K_1)\leq(V_2,K_2)$ if $V_2\leq V_1$ and $K_2\leq K_1$: the signalizer lattice $\Lambda(\tau)$ is the poset obtained by adjoining a least element to $\mathcal{P}$.

The author proves that if H admits a CD-signalizer, then the structure of H is highly restricted. More precisely, let Λ be a CD-lattice and assume $\tau(H,N_H,I_H)\in\mathcal{T}(L)$, Λ is isomorphic to $\Lambda(\tau)$ or its dual, and H is minimal subject to this constraint: then $F^*(H)$ is the direct product of nonabelian simple subgroups permuted transitively by H. A stronger result is obtained if Λ is a $D\Delta(m_1,\dots,m_t)$-lattice (i.e., the lattice Λ' obtained by removing from Λ the greatest and the least element has t connected components $C_1,\dots,C_t$ and $C_i\cong\Delta(m_i)'$, where $\Delta(m_i)$ is the lattice of all subsets of a m_i-set): if $t>1$, $m_i>2$ and $\Lambda=D\Delta(m_1,\dots,m_t)$ is a finite group interval lattice, then there exists an almost simple group G such that either $\Lambda\cong\mathcal{O}_G(H)$ for some $H\leq G$ or there exists L and $\gamma=(G,N_G,I_G)\in\mathcal{T}(L)$ with the following properties: if $\mathcal{W}_1$ is the set of the N_H-invariant subgroups W of H with $W\cap N_H=I_H$ and $W\leq F^*(H)I_H$, then $G=F^*(G)N_G=\langle\mathcal{W}_1,N_H\rangle$ and Λ is isomorphic to the poset obtained by adjoining a greatest element to $\mathcal{W}_1$.

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Note: This list, extracted from the PDF form of the original paper, may contain data conversion errors, almost all limited to the mathematical expressions.

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On intervals in subgroup lattices of finite groups.

J. Amer. Math. Soc. **21** (2008), *no.* 3, 809–830.

The following question was suggested by a theorem of P. P. Pálfy and P. Pudlák [Algebra Universalis **11** (1980), no. 1, 22–27; MR0593011 (82g:08003)]: Is each nonempty finite lattice isomorphic to an overgroup lattice $\mathcal{O}_G(H)$ for some finite group G and subgroup H of G? The answer is almost certainly negative. However, the question has remained open for almost 30 years since the paper of Pálfy and Pudlák was published. In this paper the author deals with this question, showing that the structure of G is highly restricted if the lattice $\mathcal{O}_G(H)$ is disconnected. He defines the notion of a "signalizer lattice" in H and shows that for suitable disconnected lattices Λ, if G is minimal subject to $\mathcal{O}_G(H)$ being isomorphic to Λ or its dual, then either G is almost simple or H admits a signalizer lattice isomorphic to Λ or its dual.

In particular he uses this theory to answer a question in functional analysis raised by Y. Watatani [J. Funct. Anal. **140** (1996), no. 2, 312–334; MR1409040 (98c:46134)]. Watatani showed that, with two possible exceptions (one is the hexagon), each finite lattice of order 6 is a lattice of intermediate subfactors. The question as to whether the two remaining lattices are lattices of intermediate subfactors remained open for a long time and is now solved by the author, proving that the two lattices are each isomorphic to an overgroup lattice $\mathcal{O}_G(H)$.

In trying to prove that there exist finite lattices that are not isomorphic to an overgroup lattice, one can hope to obtain a reduction on questions about almost simple groups, showing for suitable lattices Λ that either Λ can be realized as an overgroup lattice in an almost simple group G or Λ is a signalizer lattice in some almost simple group H. The author announces that a result of this kind will be proved in a forthcoming paper.

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