Representing Finite Lattices as Congruence Lattices of Finite Algebras

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Abstract. This article describes various methods for representing a finite lattice as the congruence lattice of a finite algebra or for proving that such a representation exists. Using these methods, we show that with one possible exception every lattice with at most seven elements is isomorphic to the congruence lattice of a finite algebra.

1. Introduction

TODO: Expand the introduction.

1.1. Notation. Throughout this paper, we use $\mathscr{L}$ to denote the class of finite lattices that are isomorphic to congruence lattices of finite algebras. We call the lattices that belong to $\mathscr{L}$ *representable* lattices.

2. Closure properties of the class of representable lattices

This section describes some *closure properties* of the class $\mathscr{L}$. By closure properties, we mean the following: if O is an operation that can be applied to a lattice or collection of lattices, we say that $\mathscr{L}$ is *closed under* O provided $\mathsf{O}(\mathscr{K}) \subseteq \mathscr{L}$ for all $\mathscr{K} \subseteq \mathscr{L}$. For example, if $\mathsf{S}(\mathscr{K}) = \{\text{all sublattices of lattices in } \mathscr{K}\}$, it is unknown whether $\mathscr{L}$ is closed under S. If this were known to be true, then the *finite lattice representation problem* would be solved. The congruence lattice of the algebra consisting of a set X with no operations is the lattice of all equivalence relations on X, which we denote by $\operatorname{Eq} X$. By a celebrated theorem of Pudlák and Tůma [9], for every finite lattice L there is a finite set X such that $L \leqslant \operatorname{Eq} X$. Therefore, $\mathscr{L}$ would contain all finite lattices if it were closed under S.

The following is a list of operations under which $\mathscr{L}$ is known to be closed, along with the names of those who first (or independently) proved them. We discuss some of these results in greater detail later in this section. The class $\mathscr{L}$ of lattices isomorphic to congruence lattices of finite algebras is closed under

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(1) lattice duals[1] (Hans Kurzweil [4] and Raimund Netter [7], 1986),
(2) interval sublattices (follows from Kurzweil-Netter),
(3) direct products (Jiří Tůma [12], 1986),
(4) ordinal sums (Ralph McKenzie [5], 1984; John Snow [11], 2000),
(5) parallel sums (John Snow [11], 2000),
(6) certain sublattices of lattices in $\mathscr{L}$ – namely, those which are obtained as a union of a filter and an ideal of a lattice in $\mathscr{L}$ (John Snow [11], 2000).

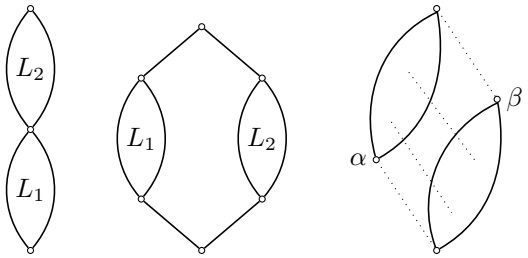


FIGURE 1. The adjoined ordinal (left) and parallel (middle) sum of the lattices L_1 and L_2; a sublattice obtained as a union of a filter $\alpha^{\uparrow}$ and an ideal $\beta^{\downarrow}$ (right).

Remarks. The first item in the list above says that if L is representable then so is the dual of L. It follows from this that any interval sublattice of a representable lattice is representable. For, let $[\alpha, \beta] := \{\theta \in L \mid \alpha \leqslant \theta \leqslant \beta\}$ be an interval in the representable lattice $L = \mathrm{Con}\ \mathbf{A}$. Then $[\alpha, 1_A] \cong \mathrm{Con}\ \mathbf{A}/\alpha$. By 1, the dual of $\ell := [\alpha, 1_A]$ is representable. Now take the filter above β' in ℓ' (where β' is the image of β under dualization) and we obtain a representation of a lattice isomorphic to the dual of $[\alpha, \beta]$. Apply 1 again and we have the desired representation of $[\alpha, \beta]$. Thus 2 follows from 1. By the ordinal sum of two lattices L_1, L_2, we mean the lattice on the left of Figure 2. By the parallel sum of two lattices L_1, L_2, we mean the lattice in the middle of Figure 2. Item 6 above is a very useful result which we will discuss it further in Section 2.2 below, where we present a short proof of this result. Whether the class $\mathscr{L}$ is closed under homomorphic images seems to be an open question.

2.1. Lattice duals: the theorem of Kurzweil and Netter. As mentioned above, the class $\mathscr{L}$ – the lattices isomorphic to congruence lattices of finite algebras – is closed under dualization. That is, if L is representable, then so is the dual of L. This was proved in 1986 by Raimund Netter [7], generalizing the idea of his advisor, Hans Kurzweil [4]. Though Kurzweil's article did appear

[1]Recall, the *dual of a lattice* is simply the lattice turned on its head, that is, the lattice obtained by reversing the partial order of the original lattice.

(in German), it is unclear whether Netter's article was ever published. In this section we present a proof of their result. The argument requires a fair bit of machinery, but it is a nice idea and well worth the effort.[2]

If G is a group and X a set, then the set $\{f \mid X \to G\}$ of functions from X into G is denoted by G^X. This is a group with binary operation $(f, g) \mapsto f \cdot g$, where, for each $x \in X$, $(f \cdot g)(x) = f(x)g(x)$ is simply multiplication in the group G. The identity of the group G^X is of course the constant map $f(x) = 1_G$ for all $x \in X$.

Let X be a finite totally ordered set, with order relation $\leqslant$, and consider the set X^X of functions mapping X into itself. The subset of X^X consisting of functions that are both idempotent and decreasing[3] will be denoted by $\mathcal{ID}(X)$. That is,

$$\mathcal{ID}(X) = \{f \in X^X \mid f^2 = f \text{ and } \forall x\ f(x) \leqslant x\}.$$

Define a partial order $\sqsubseteq$ on the set $\mathcal{ID}(X)$ by

$$f \sqsubseteq g \quad \Leftrightarrow \quad \ker f \leqslant \ker g, \tag{2.1}$$

where $\ker f = \{(x, y) \mid f(x) = f(y)\}$. It is easy to see that $f \sqsubseteq g$ holds if and only if $gf = g$. Moreover, under this partial ordering $\mathcal{ID}(X)$ is a lattice which is isomorphic to $\mathbf{Eq}(X)$ (viz. the map $\Theta : \operatorname{Eq} X \to \mathcal{ID}(X)$ given by $\Theta(\alpha) = f_\alpha$, where $f_\alpha(x) = \min\{y \in X \mid (x, y) \in \alpha\}$.)

Suppose S is a finite nonabelian simple group, and consider S^n, the direct power of n copies of S. An element of S^n may be viewed as a map from the set $n = \{0, 1, \dots, n-1\}$ into S. Thus, if $x = (x_0, x_1, \dots, x_{n-1}) \in S^n$, then by $\ker x$ we mean the relation $(i, j) \in \ker x$ if and only if $x_i = x_j$. The set of constant maps is a subgroup $D < S^n$, sometimes called the *diagonal subgroup*; that is, $D = \{(s, s, \dots, s) \mid s \in S\} \leqslant S^n$.

For each $f \in \mathcal{ID}(n)$, define

$$K_f = \{(x_{f(0)}, x_{f(1)}, \dots, x_{f(n-1)}) \mid x_{f(i)} \in S,\ i = 0, 1, \dots, n-1\}.$$

Then $D \leqslant K_f \leqslant S^n$, and K_f is the set of maps $K_f = \{xf \in S^n \mid x \in S^n\}$; i.e., compositions of the given map $f \in n^n$, followed by any $x \in S^n$. Thus, $K_f = \{y \in S^n \mid \ker f \leqslant \ker y\}$. For example, if $f = (0, 0, 2, 3, 2) \in \mathcal{ID}(5)$, then $\ker f = |0, 1|2, 4|3|$ and K_f is the subgroup of all $(y_0, y_1, \dots, y_4) \in S^5$ having $y_0 = y_1$ and $y_2 = y_4$. That is, $K_f = \{(x_0, x_0, x_2, x_3, x_2) \mid x \in S^5\}$.

Lemma 2.1. *The map $f \mapsto K_f$ is a dual lattice isomorphism from $\mathbf{Eq}(n)$ onto the interval sublattice $[D, S^n] \leqslant \operatorname{Sub}(S^n)$.*

Proof. This is clear since $\mathcal{ID}(n)$ is ordered by (2.1), and we have $f \sqsubseteq h$ if and only if $K_h = \{y \in S^n \mid \ker h \leqslant \ker y\} \leqslant \{y \in S^n \mid \ker f \leqslant \ker y\} = K_f$. □

[2]We learned of the main argument used in the proof from slides of a series of three lectures given by Péter Pálfy in 2009 [8]. Pálfy gives credit for the argument to Kurzweil and Netter.

[3]When we say that the map f is "decreasing" we mean $f(x) \leqslant x$ for all x. (We do not mean $x \leqslant y$ implies $f(y) \leqslant f(x)$.)

Theorem 2.2 (Kurzweil [4], Netter [7])**.** *If the finite lattice L is representable (as the congruence lattice of a finite algebra), then so is the dual lattice L'.*

Proof. Without loss of generality, we assume that L is represented as $L = \mathrm{Con}\,\langle n, F\rangle$. Also, by [6, Theorem 4.18], we can assume that F consists of unary operations: $F \subseteq n^n$. As above, let S be a nonabelian simple group and let D be the diagonal subgroup of S^n. Then the unary algebra $\langle S^n/D, S^n\rangle$ is a transitive S^n-set which (by Theorem 5.1 below) has congruence lattice isomorphic to the interval $[D, S^n]$. By Lemma 2.1, this is the dual of the lattice $\mathbf{Eq}(n)$. That is, $\mathrm{Con}\,\langle S^n/D, S^n\rangle \cong (\mathbf{Eq}(n))'$.

TODO: Replace the reference to Theorem 5.1 with a reference to the appropriate theorem in ALVIN, since the section containing Theorem 5.1 below will be deleted.

Now, each operation $\varphi \in F$ gives rise to an operation on S^n by composition:

$$\hat{\varphi}(\mathbf{s}) = \hat{\varphi}(s_0, s_1 \ldots, s_{n-1}) = (s_{\varphi(0)}, s_{\varphi(1)} \ldots, s_{\varphi(n-1)}).$$

Thus, φ induces an operation on S^n/D since, for $\mathbf{d} = (d, d, \ldots, d) \in D$ and $\mathbf{s} \in S^n$ we have $\mathbf{sd} = (s_0 d, s_1 d, \ldots, s_{n-1} d)$ and $\hat{\varphi}(\mathbf{sd}) = (s_{\varphi(0)} d, s_{\varphi(1)} d, \ldots, s_{\varphi(n-1)} d) = \hat{\varphi}(\mathbf{s})\mathbf{d}$, so $\hat{\varphi}(\mathbf{s}D) = \hat{\varphi}(\mathbf{s})D$. Finally, add the set of operations $\hat{F} = \{\hat{\varphi} \mid \varphi \in F\}$ to $\langle S^n/D, S^n\rangle$, yielding the new algebra $\langle S^n/D, S^n \cup \hat{F}\rangle$, and observe that a congruence $\theta \in \mathrm{Con}\,\langle S^n/D, S^n\rangle$ remains a congruence of $\langle S^n/D, S^n \cup \hat{F}\rangle$ if and only if it correponds to a partition on n that is invariant under F. □

TODO: Perhaps we should give more details in the last sentence of the proof.

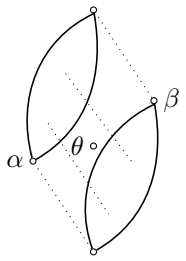
2.2. Union of a filter and ideal. The lemma in this section was originally proved by John Snow using primitive positive formulas. Since it provides such a useful tool for proving that certain finite lattices are representable as congruence lattices, we give our own direct proof of the result below.

Before stating the lemma, we need a couple of definitions. (These will be discussed in greater detail in Section 3.2.) Given a relation $\theta \subseteq X \times X$, we say that the map $f : X^n \to X$ *respects* θ and we write $f(\theta) \subseteq \theta$ provided $(x_i, y_i) \in \theta$ implies $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \theta$. For a set $L \subseteq \mathrm{Eq}\,X$ of equivalence relations we define

$$\lambda(L) = \{f \in X^X : (\forall \theta \in L)\ f(\theta) \subseteq \theta\},$$

which is the set of all unary maps on X which respect all relations in L.

Lemma 2.3. *Let X be a finite set. If $\mathbf{L} \leqslant \mathbf{Eq}(X)$ is representable and $\mathbf{L}_0 \leqslant \mathbf{L}$ is a sublattice with universe $\alpha^\uparrow \cup \beta^\downarrow$ where $\alpha^\uparrow = \{x \in L \mid \alpha \leqslant x\}$ and $\beta^\downarrow = \{x \in L \mid x \leqslant \beta\}$ for some $\alpha, \beta \in L$, then $\mathbf{L}_0$ is representable.*



$L_0 \leqslant L$

Proof. Assume $\mathbf{L}_0 \ncong \mathbf{2}$, otherwise the result holds trivially. Since $\mathbf{L} \leqslant \mathbf{Eq}(X)$ is representable, we have $\mathbf{L} = \mathbf{Con}\,\langle X, \lambda(L)\rangle$ (cf. Section 3.2). Take an arbitrary $\theta \in L \setminus L_0$. Since $\theta \notin \alpha^{\uparrow}$, there is a pair $(a, b) \in \alpha \setminus \theta$. Since $\theta \notin \beta^{\downarrow}$, there is a pair $(u, v) \in \theta \setminus \beta$. Define $h \in X^X$ as follows:

$$h(x) = \begin{cases} a, & x \in u/\beta, \\ b, & \text{otherwise.} \end{cases}$$

Then, $\beta \leqslant \ker h = (u/\beta)^2 \cup ((u/\beta)^c)^2$, where $(u/\beta)^c$ denotes the complement of the β class containing u. Therefore, h respects every $\gamma \leqslant \beta$. Furthermore, $(a, b) \in \gamma$ for all $\gamma \geqslant \alpha$, so h respects every γ above α. This proves that $h \in \lambda(L_0)$. Now, θ was arbitrary, so we have proved that for every $\theta \in L \setminus L_0$ there exists a function in $\lambda(L_0)$ which respects every $\gamma \in \alpha^{\uparrow} \cup \beta^{\downarrow} = L_0$, but violates θ. Finally, since $\mathbf{L}_0 \leqslant \mathbf{L}$, we have $\lambda(L) \subseteq \lambda(L_0)$. Combining these observations, we see that every $\theta \in \operatorname{Eq} X \setminus L_0$ is violated by some function in $\lambda(L_0)$. Therefore, $\mathbf{L}_0 = \mathbf{Con}\,\langle X, \lambda(L_0)\rangle$. □

2.3. Ordinal Sums I. TODO: Revise this section. Peter has written up some nice lemmas and theorems on this topic which now appear below in a subsection called "Ordinal Sums II," but that material should be integrated with and/or replace the material in this subsection.

The following theorem is a consequence of McKenzie's shift product construction [5].

Theorem 2.4. *If $L_1, \ldots, L_n \in \mathscr{L}$ is a collection of representable lattices, then the ordinal sum and the adjoined ordinal sum, shown in Figure 2.3, are representable.*

A more direct proof of Theorem 2.4 follows the argument given by John Snow in [11]. As noted above, Jiří Tůma proved that the class of finite representable lattices is closed under direct products. Thus, if L_1 and L_2 are representable, then so is $L_1 \times L_2$. Now note that the adjoined ordinal sum of L_1 and L_2 is the union, $\alpha^{\uparrow} \cup \beta^{\downarrow}$, of a filter and ideal in the lattice $L_1 \times L_2$, where $\alpha = \beta = 1_{L_1} \times 0_{L_2}$. Therefore, by Lemma 2.3, the adjoined ordinal sum is representable. A trivial induction argument proves the result for adjoined

ordinal sums of n lattices. The same result for ordinal sums (Figure 2.3 left) follows since the two element lattice is obviously representable.

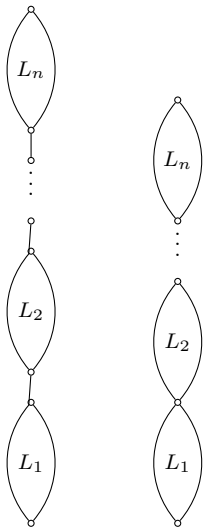


FIGURE 2. The ordinal sum (left) and the adjoined ordinal sum (right) of the lattices $L_1, \dots, L_n$.

2.4. Ordinal Sums II. TODO: Merge this subsection with the subsection "Ordinal Sums I."

The lattice of equivalence relations on an n-element set is denoted by $\mathrm{Equ}(n)$. For two lattices $\mathbf{L}, \mathbf{M}$ the *adjoined ordinal sum* is denoted by $\mathbf{L} \oplus_a \mathbf{M}$ and is defined on $L \uplus (M \setminus \{0\})$ by $x \leq y$ iff $x \in L, y \in M$ or $(x, y \in L$ and $x \leq^{\mathbf{L}} y)$ or $(x, y \in M$ and $x \leq^{\mathbf{M}} y)$.

Let $\mathbf{A}, \mathbf{B}$ be two finite algebras of cardinality m and n respectively, and let $\mathbf{L} = \mathrm{Con}(\mathbf{A})$ and $\mathbf{M} = \mathrm{Con}(\mathbf{B})$. In [11] it is proved that $\mathbf{L} \oplus_a \mathbf{M}$ is isomorphic to the congruence lattice of some finite algebra $\mathbf{C}$. Although the algebra $\mathbf{C}$ is not explicitly constructed, it is based on the direct product of $\mathbf{A}$ and $\mathbf{B}$, hence the ordinal sum is a sublattice of $\mathrm{Equ}(mn)$. Here we give a different construction of $\mathbf{C}$ that leads to a tighter representation of $\mathrm{Con}(\mathbf{C})$ as a sublattice of $\mathrm{Equ}(m + n - 1)$.

Define a unary algebra $\mathbf{A}_{m,n}$ with $m + n - 1$ elements as follows: the base set is $A \uplus B_1$ where $A = \{a_0, \dots, a_{m-1}\}$, $B_1 = \{b_1, \dots, b_{n-1}\}$ and for each function $h : B_1 \to A$ define a unary operation

$$\hat{h}(x) = \begin{cases} h(x) & \text{if } x \in B_1 \\ x & \text{otherwise.} \end{cases}$$

Lemma 2.5. *For $m, n \geq 1$ the lattice* $\mathrm{Con}(\mathbf{A}_{m,n})$ *is isomorphic to* $\mathrm{Equ}(m) \oplus_a \mathrm{Equ}(n)$.

Proof. Let α be the equivalence relation $A^2 \cup \{(b_1, b_1), \ldots, (b_{n-1}, b_{n-1})\}$, so as a partition it is $a_0, \ldots, a_{m-1}|b_1|b_2|\ldots|b_{n-1}$. Note that $\mathrm{Equ}(m) \oplus_a \mathrm{Equ}(n)$ is isomorphic to the sublattice of $\mathrm{Equ}(A \uplus B_1)$ of all equivalence relations comparable to α, since α has a unique non-singleton block of size m, and n blocks altogether. We claim that this sublattice is the congruence lattice of $\mathbf{A}_{m,n}$.

Suppose $\theta \leq \alpha$, and let $(x, y) \in \theta$. Then $x, y \in A$ or $x = y$, hence for any operation $\hat{h}$ we have $\hat{h}(x) = x$ and $\hat{h}(y) = y$ or $\hat{h}(x) = \hat{h}(y)$, so $(\hat{h}(x), \hat{h}(y)) \in \theta$.

Suppose $\alpha \leq \theta$, and let $(x, y) \in \theta$. Since $A^2 \subseteq \alpha$ and since the range of each $\hat{h}$ is A it follows that $(\hat{h}(x), \hat{h}(y)) \in \theta$.

Now suppose θ is incomparable with α. Then A^2 is not a subset of θ, hence there exist $(x, y) \in A^2 \setminus \theta$ and $(u, v) \in \theta \setminus \alpha$. If $u, v \in B_1$ then choose a function h (as in the definition of $\mathbf{A}_{m,n}$) such that $h(u) = x$ and $h(v) = y$, in which case $\hat{h}$ is an operation that shows θ is not a congruence. If $u \in B_1$, but $v \in A$, note that we cannot have both (x, v) and (y, v) in θ (else $(x, y) \in \theta$). Assume without loss of generality that $(x, v) \notin \theta$ and choose h such that $h(u) = x$, then again $\hat{h}$ shows that θ is not a congruence. The case $u \in A$, $v \in B_1$ is similar, and $u, v \in A$ is excluded since $(u, v) \notin \alpha$. □

Theorem 2.6. *Suppose* $\mathbf{A} = (A, F)$ *and* $\mathbf{B} = (B, G)$ *are unary algebras with* $A = \{a_0, \ldots, a_{m-1}\}$, $B = \{b_0, \ldots, b_{n-1}\}$ *and* $A \cap B = \{a_0\} = \{b_0\}$ *(so* a_0, b_0 *are identified). Let* $\mathbf{C}$ *be the algebra* $\mathbf{A}_{m,n}$ *expanded with the operations*

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in A \\ f(a_0) & \text{otherwise} \end{cases} \qquad \hat{g}(x) = \begin{cases} g(x) & \text{if } x \in B \\ g(b_0) & \text{otherwise} \end{cases}$$

for $f \in F$ and $g \in G$. Then $\mathrm{Con}(\mathbf{C})$ *is isomorphic to* $\mathrm{Con}(\mathbf{A}) \oplus_a \mathrm{Con}(\mathbf{B})$.

Proof. Since $\mathbf{C}$ is an expansion of $\mathbf{A}_{m,n}$ it follows from the preceding lemma that $\mathrm{Con}(\mathbf{C})$ is a sublattice of $\{\theta \in \mathrm{Equ}(A \cup B) : \theta \leq \alpha \text{ or } \alpha \leq \theta\}$ where, as before, $\alpha = A^2 \cup \mathrm{id}_B$. Note that $\alpha \in \mathrm{Con}(\mathbf{C})$, so it suffices to show that $\{\theta \in \mathrm{Con}(\mathbf{C}) : \theta \leq \alpha\}$ is isomorphic to $\mathrm{Con}(\mathbf{A})$ and $\{\theta \in \mathrm{Con}(\mathbf{C}) : \theta \leq \alpha\}$ is isomorphic to $\mathrm{Con}(\mathbf{B})$. The second isomorphism follows from the observation that $\mathbf{C}/\alpha$ is isomorphic to $\mathbf{B}$ via the map $A \mapsto b_0$ and $\{b_i\} \mapsto b_i$ for $i \geq 1$. For the first isomorphism, note that the operations $\hat{g}$, $\hat{h}$ preserve all equivalence relations below α. Similarly it is straight forward to check that $\hat{f}$ preserves $\theta \leq \alpha$ iff f preserves $\theta \cap A^2$. Hence the map $\theta \mapsto \theta \cap A^2$ is the required isomorphism. □

Now assume $\mathbf{A}$ and $\mathbf{B}$ are minimal-size algebras with congruence lattices isomorphic to $\mathbf{L}$ and $\mathbf{M}$ respectively. Then it seems likely that the algebra $\mathbf{C}$ above is minimal in size with respect to having a congruence lattice isomorphic to $\mathbf{L} \oplus_a \mathbf{M}$. Suppose $\mathbf{C}'$ is a smaller algebra with the same congruence lattice, and let α be the congruence in the "middle" of the ordinal sum. Then $\mathbf{C}'/\alpha$

has a congruence lattice isomorphic to $\mathbf{M}$, hence $|\mathbf{C}'/\alpha| = |\mathbf{B}|$ = number of blocks in α. However, α does not need to have a unique nontrivial block, so it is not clear at the moment how to obtain a smaller algebra $\mathbf{A}'$ such that $\mathrm{Con}(\mathbf{A}')$ is isomorphic to $\mathbf{L}$.

3. Concrete Representations

In this section, we introduce a strategy that has proven very useful for showing that a given lattice is representable as a congruence lattice of a finite algebra. We call it the *closure method*, and it has become especially useful with the advent of powerful computers which can search for such representations. Here, $\mathrm{Eq}\, X$ denotes the lattice of equivalence relations on X. Sometimes we abuse notation and take $\mathrm{Eq}\, X$ to mean the lattice of partitions of the set X. This never causes problems because these two lattices are isomorphic.

3.1. Concrete versus abstract representations. As Bjarni Jónsson explains in [3], there are two types of representation problems for congruence lattices, the concrete and the abstract. The *concrete representation problem* asks whether a specific family of equivalence relations on a set A is equal to $\mathrm{Con}\,\mathbf{A}$ for some algebra $\mathbf{A}$ with universe A. The *abstract representation problem* asks whether a given lattice is isomorphic to $\mathrm{Con}\,\mathbf{A}$ for some algebra $\mathbf{A}$.

These two problems are closely related, and have become even more so since the publication in 1980 of [9], in which Pavel Pudlák and Jiří Tůma prove that every finite lattice can be embedded as a spanning sublattice[4] of the lattice $\mathrm{Eq}\, X$ of equivalence relations on a finite set X. Given this result, we see that even if our goal is to solve the abstract representation problem for some (abstract) lattice L, then we can embed L into $\mathrm{Eq}\, X$ as $L \cong L_0 \leqslant \mathrm{Eq}\, X$, for some finite set X, and then try to solve the concrete representation problem for L_0.

A point of clarification is in order here. The term *representation* has become a bit overused in the literature about the finite lattice representation problem. On the one hand, given a finite lattice L, if there is a finite algebra $\mathbf{A}$ such that $L \cong \mathrm{Con}\,\mathbf{A}$, then L is called a *representable lattice*. On the other hand, given a sublattice $L_0 \leqslant \mathrm{Eq}\, X$, if $L_0 \cong L$, then L_0 is sometimes called a *concrete representation* of the lattice L (whether or not it is the congruence lattice of an algebra). Below we will define the notion of a *closed concrete representation*, and if we have this special kind of concrete representation of a give lattice, then that lattice is indeed representable in the first sense.

As we will see below, there are many examples in which a particular concrete representation $L_0 \leqslant \mathrm{Eq}\, X$ of L is not a congruence lattice of a finite algebra. (In fact, we will describe general situations in which we can guarantee that

[4]Recall, by a *spanning sublattice* of a bounded lattice L_0, we mean a sublattice $L \leqslant L_0$ that has the same top and bottom as L_0. That is $1_L = 1_{L_0}$ and $0_L = 0_{L_0}$.

there are no non-trivial[5] operations which respect the equivalence relations of L_0.) This does not imply that $L \notin \mathscr{L}$. It may simply mean that L_0 is not the "right" concrete representation of L, and perhaps we can find some other $L \cong L_1 \leqslant \operatorname{Eq} X$ such that $L_1 = \operatorname{Con}\langle X, \lambda(L_1)\rangle$.

3.2. The closure method. The idea described in this section first appeared in *Topics in Universal Algebra* [3], pages 174–175, where Jónsson states, "these or related results were discovered independently by at least three different parties during the summer and fall of 1970: by Stanley Burris, Henry Crapo, Alan Day, Dennis Higgs and Warren Nickols at the University of Waterloo, by R. Quackenbush and B. Wolk at the University of Manitoba, and by B. Jónsson at Vanderbilt University."

Let X^X denote the set of all (unary) maps from the set X to itself, and let $\operatorname{Eq} X$ denote the lattice of equivalence relations on the set X. If $\theta \in \operatorname{Eq} X$ and $h \in X^X$, we write $h(\theta) \subseteq \theta$ and say that "h respects θ" if and only if for all $(x, y) \in X^2$ $(x, y) \in \theta$ implies $(h(x), h(y)) \in \theta$. If $h(\theta) \nsubseteq \theta$, we sometimes say that "h violates θ."

For $L \subseteq \operatorname{Eq} X$ define

$$\lambda(L) = \{h \in X^X : (\forall \theta \in L)\ h(\theta) \subseteq \theta\}.$$

For $H \subseteq X^X$ define

$$\rho(H) = \{\theta \in \operatorname{Eq} X \mid (\forall h \in H)\ h(\theta) \subseteq \theta\}.$$

The map $\rho\lambda$ is a *closure operator* on $\operatorname{Sub}[\operatorname{Eq} X]$. That is, $\rho\lambda$ is

- *idempotent:*[6] $\rho\lambda\rho\lambda = \rho\lambda$;
- *extensive:* $L \subseteq \rho\lambda(L)$ for every $L \leqslant \operatorname{Eq} X$;
- *order preserving:* $\rho\lambda(L) \leqslant \rho\lambda(L_0)$ if $L \leqslant L_0$.

Given $L \leqslant \operatorname{Eq} X$, if $\rho\lambda(L) = L$, then we say L is a *closed* sublattice of $\operatorname{Eq} X$, in which case we clearly have

$$L = \operatorname{Con}\langle X, \lambda(L)\rangle.$$

This suggests the following strategy for solving the representation problem for a given abstract finite lattice L: search for a concrete representation $L \cong L_0 \leqslant \operatorname{Eq} X$, compute $\lambda(L_0)$, compute $\rho\lambda(L_0)$, and determine whether $\rho\lambda(L_0) = L_0$. If so, then we have solved the abstract representation problem for L, by finding a *closed concrete representation*, or simply *closed representation*, of L_0. We call this strategy the *closure method*.

We now state without proof a well known theorem which shows that the finite lattice representation problem can be formulated in terms of closed concrete representations (cf. [3]).

Theorem 3.1. *If* $\mathbf{L} \leqslant \mathbf{Eq}(X)$*, then* $\mathbf{L} = \mathbf{Con}\,\mathbf{A}$ *for some algebra* $\mathbf{A} = \langle X, F\rangle$ *if and only if* $\mathbf{L}$ *is closed.*

---

[5] By a *non-trivial function* we mean a function that is not constant and not the identity.

[6] In fact, $\rho\lambda\rho = \rho$ and $\lambda\rho\lambda = \lambda$.

Before proceeding, we introduce a slightly different set-up than the one introduced above that we have found particularly useful for implementing the closure method on a computer. Instead of considering the set of equivalence relations on a finite set, we work with the set of idempotent decreasing maps. These were introduced above in Section 2.1, but we briefly review the definitions here for convenience.

Given a totally ordered set X, let the set $\mathcal{ID}(X) = \{f \in X^X : f^2 = f \text{ and } f(x) \leqslant x\}$ be partially ordered by $\sqsubseteq$ as follows:

$$f \sqsubseteq g \quad \Leftrightarrow \quad \ker f \leqslant \ker g.$$

As noted above, this makes $\mathcal{ID}(X)$ into a lattice that is isomorphic to $\mathbf{Eq}(X)$. Define a relation R on $X^X \times \mathcal{ID}(X)$ as follows:

$$(h, f) \in R \quad \Leftrightarrow \quad (\forall (x, y) \in \ker f)\ (h(x), h(y)) \in \ker f.$$

If $h\, R\, f$, we say that h *respects* f.

Let $\mathscr{F} = \mathscr{P}(\mathcal{ID}(X))$ and $\mathscr{H} = \mathscr{P}(X^X)$ be partially ordered by set inclusion, and define the maps $\lambda : \mathscr{F} \to \mathscr{H}$ and $\rho : \mathscr{H} \to \mathscr{F}$ as follows:

$$\lambda(F) = \{h \in X^X : \forall f \in F,\ h\, R\, f\} \quad (F \in \mathscr{F})$$

$$\rho(H) = \{f \in \mathcal{ID}(X) : \forall h \in H,\ h\, R\, f\} \quad (H \in \mathscr{H})$$

The pair (λ, ρ) defines a *Galois correspondence* between $\mathcal{ID}(X)$ and X^X. That is, λ and ρ are antitone maps such that $\lambda\rho \geqslant \mathrm{id}_{\mathscr{H}}$ and $\rho\lambda \geqslant \mathrm{id}_{\mathscr{F}}$. In particular, for any set $F \in \mathscr{F}$ we have $F \subseteq \rho\lambda(F)$. These statements are all trivial verifications, and a couple of easy consequences are:

(1) $\rho\lambda\rho = \rho$ and $\lambda\rho\lambda = \lambda$,
(2) $\rho\lambda$ and $\lambda\rho$ are idempotent.

Since the map $\rho\lambda$ from $\mathscr{F}$ to itself is idempotent, extensive, and order preserving, it is a *closure operator* on $\mathscr{F}$, and we say a set $F \in \mathscr{F}$ is *closed* if and only if $\rho\lambda(F) = F$. Equivalently, F is closed if and only if $F = \rho(H)$ for some $H \in \mathscr{H}$.

4. Distributive lattices

TODO: Cite Birkhoff result about every finite distributive lattice being the congruence lattice of a finite lattice.

A lattice $\mathbf{L}$ is called *strongly representable* if whenever $\mathbf{L} \cong \mathbf{L}_0 \leqslant \mathbf{Eq}(X)$ for some X then there is an algebra based on X whose congruence lattice is $\mathbf{L}_0$. In other words, *every* distributive spanning sublattice of the lattice of equivalence relations on a finite set X is equal to the congruence lattice of an algebra $\langle X, F\rangle$, for some collection F of operations on X.

Theorem 4.1 (Berman [1], Quackenbush and Wolk [10])**.** *Every finite distributive lattice is strongly representable.*

Remarks. By Theorem 3.1 above, the result of Berman, Quackenbush and Wolk says, if $\mathbf{L}$ is a finite distributive lattice then every embedding $\mathbf{L} \cong \mathbf{L}_0 \leqslant \mathbf{Eq}(X)$ is closed. The following proof is only slightly shorter than to the original in [10], and the methods are similar.

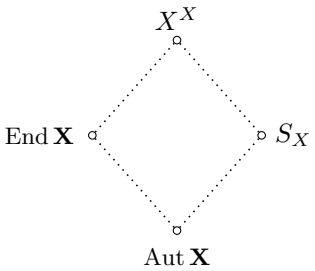
Proof. Without loss of generality, suppose $\mathbf{L} \leqslant \mathbf{Eq}(X)$. Fix $\theta \in \operatorname{Eq} X \setminus L$ and define $\theta^* = \bigwedge\{\gamma \in L \mid \gamma \geqslant \theta\}$ and $\theta_* = \bigvee\{\gamma \in L \mid \gamma \leqslant \theta\}$. Let α be a join irreducible in L below θ^* and not below θ_*. Note that α is not below θ. Let $\beta = \bigvee\{\gamma \in L \mid \gamma \not\geqslant \alpha\}$. If β were above θ, then β would be above θ^*, and so β would be above α. But α is join prime, so β is not above θ.

Choose $(u, v) \in \alpha \setminus \theta$ and note that $u \neq v$. Choose $(x, y) \in \theta \setminus \beta$ and note that $x \neq y$. Let B be the β block of y and define $h \in X^X$ as in (**??**). Then it is clear that h violates θ, h respects all elements in the sets $\alpha^{\uparrow} = \{\gamma \in L : \alpha \leqslant \gamma\}$ and $\beta^{\downarrow} = \{\gamma \in L : \gamma \leqslant \beta\}$, and $L = \alpha^{\uparrow} \cup \beta^{\downarrow}$. Since θ was an arbitrary element of $\operatorname{Eq} X \setminus L$, we can construct such an $h = h_\theta$ for each $\theta \in \operatorname{Eq} X \setminus L$. Let $\mathscr{H} = \{h_\theta : \theta \in \operatorname{Eq} X \setminus L\}$ and let $\mathbf{A}$ be the algebra $\langle X, \mathscr{H} \rangle$. Then, $\mathbf{L} = \mathbf{Con}(\mathbf{A})$. □

5. Congruence Lattices of Group Actions

Let X be a finite set and consider the set X^X of all maps from X to itself, which, when endowed with composition of maps and the identity mapping, forms a monoid, $\langle X^X, \circ, \mathrm{id}_X \rangle$. The submonoid S_X of all bijective maps in X^X is a group, the *symmetric group on* X. When the underlying set is more complicated, or for emphasis, we denote the symmetric group on X by $\operatorname{Sym}(X)$. When the underlying set isn't important, we usually write S_n to denote the symmetric group on an n-element set.

If we have defined some set F of basic operations on X, so that $\mathbf{X} = \langle X, F \rangle$ is an algebra, then two other important submonoids of X^X are $\operatorname{End} \mathbf{X}$, the set of maps in X^X which respect all operations in F, and $\operatorname{Aut} \mathbf{X}$, the set of bijective maps in X^X which respect all operations in F. It is apparent from the definition that $\operatorname{Aut} \mathbf{X} = S_X \cap \operatorname{End} \mathbf{X}$, and $\operatorname{Aut} \mathbf{X}$ is a submonoid of $\operatorname{End} \mathbf{X}$ and a subgroup of S_X. These four fundamental monoids associated with the algebra $\mathbf{X}$, and their relative ordering under inclusion, are shown in the diagram below.



Given a finite group G, and an algebra $\mathbf{X} = \langle X, F\rangle$, a *representation* of G on $\mathbf{X}$ is a group homomorphism from G into $\operatorname{Aut}\mathbf{X}$. That is, a representation of G is a mapping $\varphi : G \to \operatorname{Aut}\mathbf{X}$ which satisfies $\varphi(g_1 g_2) = \varphi(g_1) \circ \varphi(g_2)$, where (as above) $\circ$ denotes composition of maps in $\operatorname{Aut}\mathbf{X}$.

5.1. Transitive G-sets. A representation $\varphi : G \to \operatorname{Aut}\mathbf{X}$ defines an action by G on the set X, as follows: $\varphi(g) : x \mapsto x^{\varphi(g)}$. If $\varphi(G) \leqslant \operatorname{Aut}\mathbf{X}$ denotes the image of G under φ, we call the algebra $\langle X, \varphi(G)\rangle$ a *G-set*. The action is called *transitive* if for each pair $x, y \in X$ there is some $g \in G$ such that $x^{\varphi(g)} = y$. A group that acts transitively on some set is called a *transitive group*. (Without specifying the set, however, this term is meaningless, since every group acts transitively on some sets and intransitively on others.) A representation φ is called *transitive* if the resulting action is transitive.

A representation $\varphi : G \to \operatorname{Aut}\mathbf{X}$ is called *faithful* if it is a monomorphism, in which case G is isomorphic to its image under φ, which is a subgroup of $\operatorname{Aut}\mathbf{X}$. We also say, in this case, that the group G acts faithfully, and call it a *permutation group*.

The *degree* of a group action on a set X is the cardinality of X. Finally, a *primitive group* is a group that contains a core-free maximal subgroup.

For our purposes the most important representation of a group G is its action on the set of cosets of a subgroup. That is, for any subgroup $H \leqslant G$, we define a transitive permutation representation of G, which we will denote by ρ_H. Specifically, ρ_H is a group homomorphism from G into the symmetric group $\operatorname{Sym}(G/H)$ of permutations on the set $G/H = \{H, Hx_1, Hx_2, \dots\}$ of *right* cosets of H in G.

When The action is simply right multiplication by elements of G. That is, $(Hx)^{\rho(g)} = Hxg$. Each Hx is a point in the set G/H, and the *point stabilizer* of Hx in G is defined by $G_{Hx} = \{g \in G \mid Hxg = Hx\}$. Notice that $G_H = \{g \in G \mid Hg = H\} = H$ is the point stabilizer of H in G, and

$$G_{Hx} = \{g \in G \mid Hxgx^{-1} = H\} = x^{-1}G_H x = x^{-1}Hx = H^x.$$

Thus, the kernel of the homomorphism ρ is

$$\ker\rho = \{g \in G \mid \forall x \in G,\ Hxg = Hx\} = \bigcap_{x\in G} G_{Hx} = \bigcap_{x\in G} x^{-1}Hx = \bigcap_{x\in G} H^x.$$

Note that $\ker\rho$ is the largest normal subgroup of G contained in H, also known as the *core* of H in G, which we denote by $\operatorname{core}_G(H)$.

If the subgroup H happens to be *core-free*, that is, $\operatorname{core}_G(H) = 1$, then $\rho : G \hookrightarrow \operatorname{Sym}(G/H)$, an embedding, so ρ is a faithful representation; hence G is a permutation group.

Theorem 5.1 (G-set Isomorphism Theorem)**.** *Let $\mathbf{A} = \langle A, G\rangle$ be a transitive G-set and fix $a \in A$. Then the lattice* Con $\mathbf{A}$ *is isomorphic to the interval $[\![G_a, G]\!]$ in the subgroup lattice of G.*

Since the foregoing theorem is so central to our work, we provide an alternative statement of it. This is the version typically found in group theory textbooks (e.g., [2]). Keeping these two alternative perspectives in mind can be useful.

Theorem 5.2 (G-set Isomorphism Theorem, version 2)**.** *Let $\mathbf{A} = \langle A, \varphi(G)\rangle$ be a transitive G-set and fix $a \in A$. Let $\mathscr{B}$ be the set of all blocks B that contain a. Then there is a bijection $\Psi : \mathscr{B} \to [\![G_a, G]\!]$ given by $\Psi(B) = G(B)$, with inverse mapping $\Phi : [\![G_a, G]\!] \to \mathscr{B}$ given by $\Phi(H) = \{a^{\varphi(h)} \mid h \in H\}$. The mapping Ψ is order-preserving in the sense that $B_1 \subseteq B_2 \Leftrightarrow \Psi(B_1) \leqslant \Psi(B_2)$.*

Briefly, the poset $\langle \mathscr{B}, \subseteq\rangle$ is order-isomorphic to the poset $\langle [G_a, G], \leqslant\rangle$.

Corollary 5.3. *Let G act transitively on a set with at least two points. Then G is primitive if and only if each stabilizer G_a is a maximal subgroup of G.*

Since the point stabilizers of a transitive group are all conjugate, one stabilizer is maximal only when all of the stabilizers are maximal. In particular, a regular permutation group is primitive if and only if it has prime degree.

Next we describe (up to equivalence) all transitive permutation representations of a given group G. We call two representations (or actions) *equivalent* provided the associated G-sets are isomorphic. The foregoing implies that every transitive permutation representation of G is equivalent to $\hat{\lambda}_H$ for some subgroup $H \leqslant G$. The following lemma[7] shows that we need only consider a single representative H from each of the conjugacy classes of subgroups.

Lemma 5.4. *Suppose G acts transitively on two sets, A and B. Fix $a \in A$ and let G_a be the stabilizer of a (under the first action). Then the two actions are equivalent if and only if the subgroup G_a is also a stabilizer under the second action of some point $b \in B$.*

The point stabilizers of the action $\hat{\lambda}_H$ described above are the conjugates of H in G. Therefore, the lemma implies that, for any two subgroups $H, K \leqslant G$, the representations $\hat{\lambda}_H$ and $\hat{\lambda}_K$ are equivalent precisely when $K = xHx^{-1}$ for some $x \in G$. Hence, the transitive permutation representations of G are given, up to equivalence, by $\hat{\lambda}_{K_i}$ as K_i runs over a set of representatives of conjugacy classes of subgroups of G.

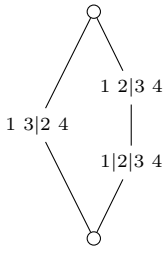
6. Small unary algebras for congruence lattices of size ≤ 7

Distributive lattices and lattices that are ordinal sums of smaller lattices are omitted. The base set of each algebra is $\{0, 1, \ldots, n-1\}$, and each unary operation is specified by a vector of values of these elements. Algebras of size less than 11 are known to be minimal-size algebras that produce the corresponding congruence lattice. The algebra for 33 (M_5) is also known to

[7] Lemma 1.6B of [2].

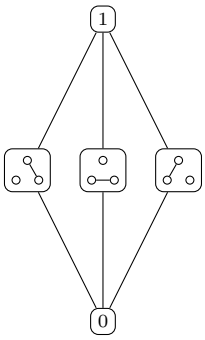
be minimal in size. Currently only one of the lattices (10) is not known to be the congruence lattice of a finite algebra.

x	0	1	2	3
$f(x)$	1	0	3	2
$g(x)$	1	0	1	0



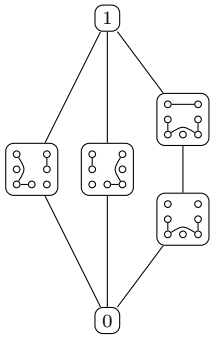
1 (N_5)

(1,0,3,2)
(2,3,0,1)



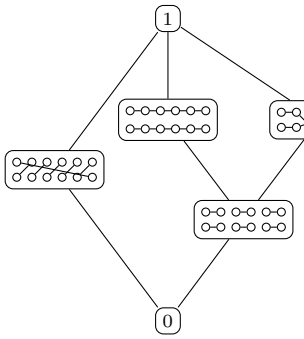
2 (M_3)

p_0 = (0,1,2,1,2,1,0)
p_1 = (0,3,4,3,4,3,0)
p_2 = (6,5,2,5,2,5,6)
p_3 = (0,1,2,0,0,2,2)



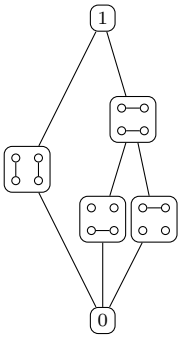
3 (V_1)

(1,2,3,4,5,0,7,8,9,10,11,6)
(6,11,10,9,8,7,0,5,4,3,2,1)
(0, 0, 0,6,0,0,0,0,6,0,0,0)



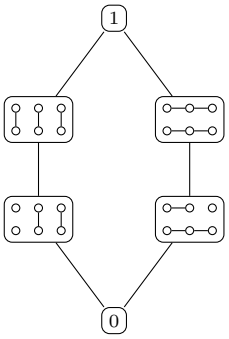
4 (L_5)

(1,0,3,2)
(0,0,2,2)



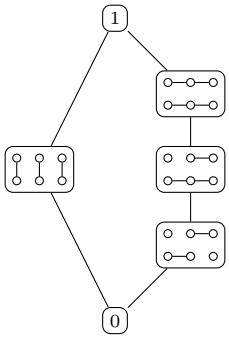
5 (L_4)

(2,2,1,5,5,4)
(3,4,4,0,1,1)
(4,5,3,4,5,3)



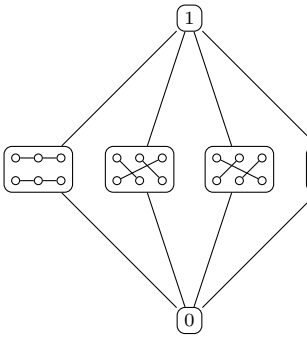
6

(1,0,0,4,3,3)
(4,5,5,1,2,2)
(3,3,4,3,3,4)



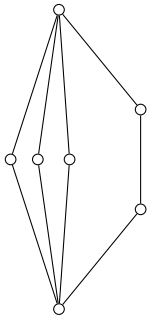
7

(1,2,0,4,5,3)
(3,5,4,0,2,1)



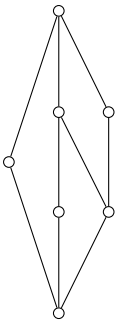
8 (M_4)

(0,0,0,0,0,0,2,1,2,1,3,4,5,3,4,5)
(0,0,0,0,0,0,6,7,6,7,10,11,12,10,11,12)
(13,14,15,1,9,8,15,14,13,15,1,9,8,8,1,9)
(R. Freese “rabbit ears”)



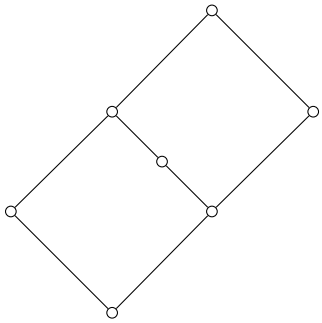
9

No finite algebra known with this congruence lattice



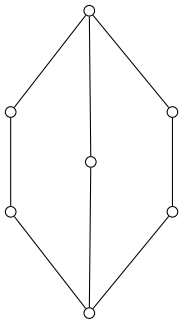
10

Finite algebra with 108 elements known (W. DeMeo)



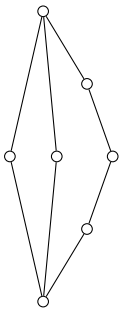
11

(0,0,3,3,3,6,6,6,0)
(0,0,8,8,8,1,1,1,0)
(0,5,5,4,0,0,5,4,4)
(4,2,2,3,4,4,2,3,3)
(5,5,7,7,7,6,6,6,5)
(found by W. DeMeo)



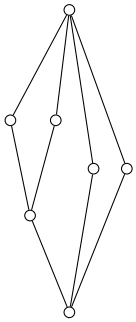
12

(0,1,2,1,2,1,0,0,1,2,2,1,0,0,1,2,1,2,0)
(0,1,2,0,0,2,2,0,3,4,0,4,4,6,5,2,6,6,2)
(0,1,2,3,4,5,6,0,1,2,4,5,6,0,1,2,3,4,6)
(7,8,9,3,10,11,12,3,3,3,3,3,3,11,11,11,
11,11,11),(13,14,15,16,17,5,18,13,16,17,
17,16,13,5,5,5,5,5,5)("rabbit ears")



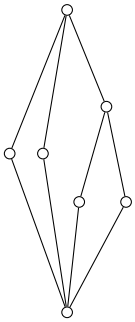
13

Upper interval in Sub(A_6)
algebra of size 90 (W. DeMeo)



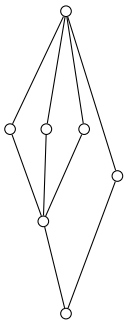
14

(1,0,3,2)



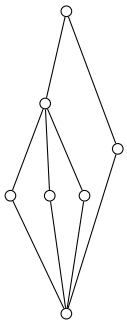
15

Upper interval in Sub($C_2.A_6$)
algebra of size 180 (W. DeMeo)



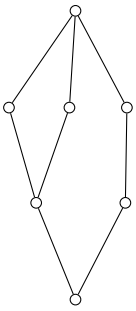
16

(1,0,3,2,5,4,7,6,9,8,11,10)
(4,7,5,6,8,11,9,10,0,3,1,2)
(0,0,0,0,5,5,5,5,10,10,10,10)
(W. Demeo, filter-ideal in Sub(A_4))



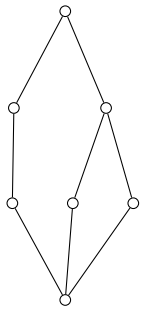
17

Dual of 19, no explicit
small representation known?



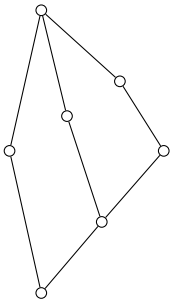
18

(0,1,1,0,4,5,5,4)
(0,2,3,1,0,2,3,1)
(7,6,6,7,3,2,2,3)
(P. Jipsen, search in Equ(8))



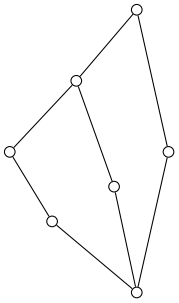
19

(W. DeMeo in GAP
SmallGroup(216,153))



20

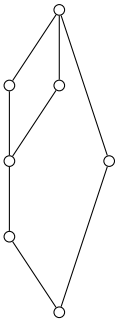
(3,3,4,8,8,2,2,3,4)
(0,0,6,1,1,0,0,5,6)
(4,5,5,7,8,8,7,4,4)
(R. Freese, in Equ(9))



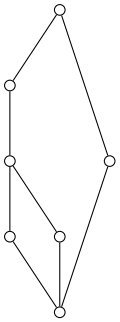
21

Dual of 23, no
explicit small
representation
known?

(0,1,0,1,4,4)
(1,1,3,3,4,5)
(3,2,3,2,5,5)
(4,1,5,3,4,5)

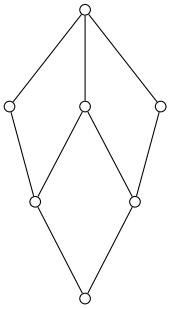


22



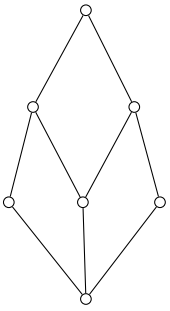
23

(1,1,2,2)
(2,3,3,2)



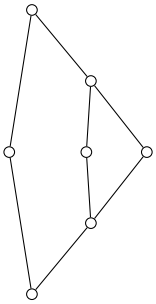
24 (L_2)

(0,0,2,2,2)
(0,1,0,1,1)
(1,1,4,4,4)
(2,3,2,3,3)



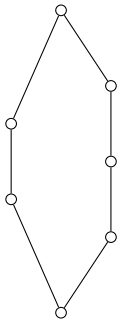
25 (L_1)

(1,0,3,2,0,2)
(4,4,5,5,1,3)
(0,0,0,0,1,1)
(3,5,3,5,3,3)



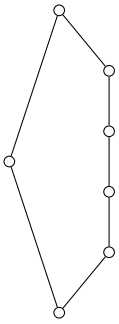
26

(0,1,2,3,4,5,0,0,0,0,0,2,2,2,2,2)
(4,5,3,4,5,3,5,3,4,5,3,4,5,4,5,3)
(2,2,1,5,5,4,2,1,5,5,4,2,2,5,5,4)
(3,4,4,0,1,1,4,4,0,1,1,3,4,0,1,1)
(0,6,7,8,9,10,6,7,8,9,10,0,6,8,9,10)
(11,12,2,13,14,15,12,2,13,14,15,
11,12,13,14,15) (“rabbit ears”)



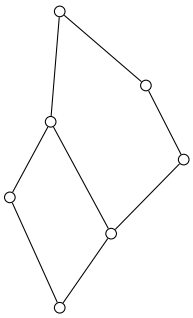
27

(0,1,2,3,4,5,0,0,0,0,0,2,2,2,2,2)
(3,3,4,3,3,4,3,4,3,3,4,3,3,3,3,4)
(1,0,0,4,3,3,0,0,4,3,3,1,0,4,3,3)
(4,5,5,1,2,2,5,5,1,2,2,4,5,1,2,2)
(0,6,7,8,9,10,6,7,8,9,10,0,6,8,9,10)
(11,12,2,13,14,15,12,2,13,14,15,
11,12,13,14,15) (“rabbit ears”)

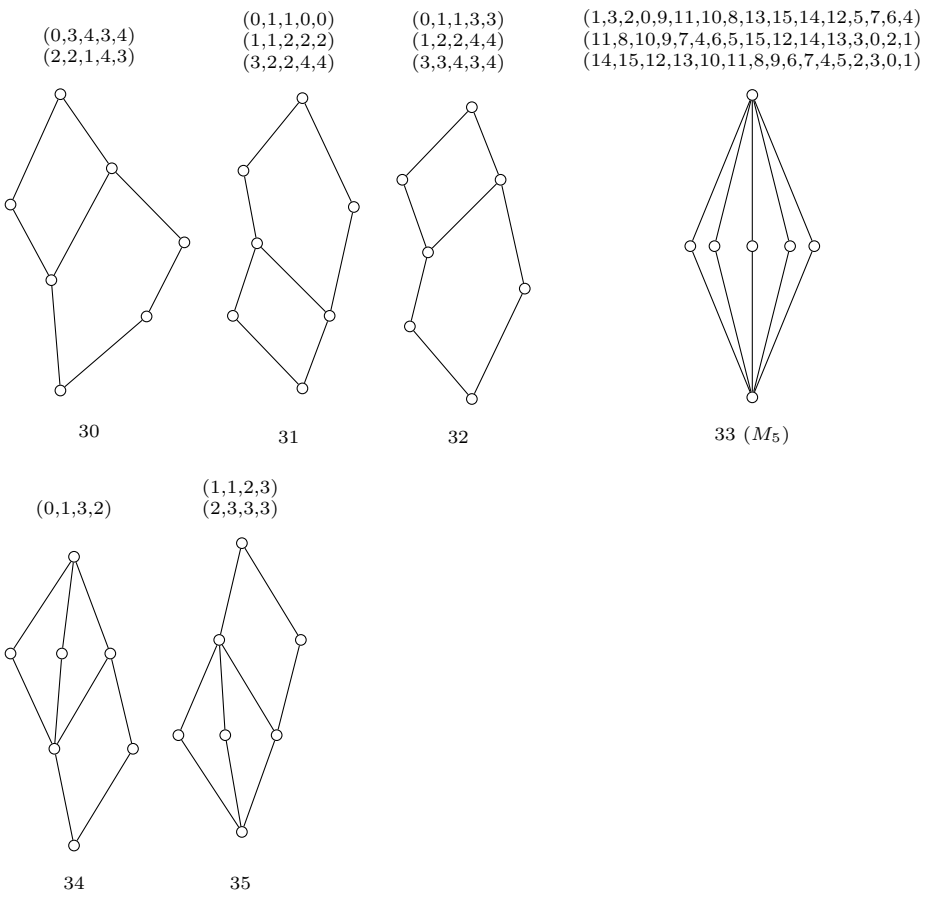


28

(1,0,3,2,2)
(2,4,2,4,3)



29



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