of order p^r and suppose that the composite $G/P \to \mathrm{Aut}(P) \to \mathrm{Aut}(\mathbb{Z}/p\mathbb{Z})$ has p-torsion kernel. Then the conjecture holds for G if it holds for G/P.

Remark 11. Finally, the idea in Remark 7 is fairly general. If G_0 and G_1 are two groups for which the conjecture holds, and if the orders of these two groups are relatively prime, then the conjecture holds for $G_0 \oplus G_1$. In particular, the conjecture holds for all finite nilpotent groups.

Given any irreducible rational representation ϕ of $G_0 \oplus G_1$ there exist unique irreducible rational representations ϕ_i of G_i such that ϕ is a constituent of $\phi_0 \otimes \phi_1$. The hypothesis that the orders of G_0 and G_1 are relatively prime ensure that there exists an integer r such that

$$r\phi = \phi_0 \otimes \phi_1.$$

This hypothesis further guarantees that Δ_ϕ and $\Delta_{\phi_0} \otimes_{\mathbb{Z}} \Delta_{\phi_1}$ are Morita equivalent since they are each $\mathbb{Z}[1/w_\phi]$-maximal orders in Morita equivalent simple algebras.

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On Feit's Examples of Intervals in Subgroup Lattices

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Introduction

This note is a small contribution concerning the following problem:

> Is every finite lattice isomorphic to an interval in the subgroup lattice of a finite group?

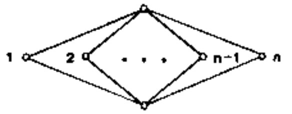
(By an interval $[H; G]$ we mean the lattice of all subgroups K such that $H \leqslant K \leqslant G$.) A similar problem with sublattices in place of intervals, due to P. M. Whitman [9, p. 509], was solved by P. Pudlák and J. Tůma [7, 8] only in 1977. The present problem seems to be much harder, although a negative answer is expected.

Our question arouse from a famous open problem in universal algebra:

> Is every finite lattice isomorphic to the congruence lattice of a finite algebra?

In a joint paper with P. Pudlák [6] we proved that these two problems are equivalent in the sense that the answer is positive to both or to neither of them. (However, in case of negative answers, it may happen that some lattice which is not representable as an interval in a subgroup lattice can be represented as a congruence lattice.)

Most work has been devoted to the lattices of length 2. The lattice $\mathscr{M}_n$ $(n \geqslant 1)$ consists of a least, a largest, and n pairwise incomparable elements:

$\mathscr{M}_n$ 

$\mathscr{M}_1$ is the subgroup lattice of a cyclic group of order p^2 (p prime), $\mathscr{M}_2$ is of the cyclic group of order $p_1 p_2$ (p_1, p_2 distinct primes). If $n-1$ is a prime

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power, then let F be a field of $n-1$ elements, and V be a 2-dimensional vector space over F and G the group of permutations $x \to \alpha x + v$ over the space V, where $\alpha \in F$, $\alpha \neq 0$, $v \in V$. Then the subgroups of G containing $H = \{\alpha x : \alpha \in F, \alpha \neq 0\}$ form an interval $\mathcal{M}_n$. Therefore the most investigated particular case of the problem is the following:

> Is $\mathcal{M}_n$ for $n \geqslant 3$, $n-1$ not a prime power (i.e., $n = 7, 11, 13, \ldots$) isomorphic to an interval in the subgroup lattice of a finite group?

If $N \lhd G$ and $N \leqslant H$ then $[H; G] \simeq [H/N; G/N]$; hence we may always assume without loss of generality that H contains no nontrivial normal subgroups of G. In [6] we proved that if $[H; G] \simeq \mathcal{M}_n$ with $n \geqslant 3$, $n-1$ not a prime power, then G has no nontrivial abelian normal subgroups; in particular, G is not solvable. Under the same assumptions P. Köhler [4] showed that G has a unique minimal normal subgroup.

It was quite a surprise when W. Feit [1] exhibited intervals $\mathcal{M}_7$ and $\mathcal{M}_{11}$ in the subgroup lattice of the alternating group of degree 31.

We tried to generalize these examples; however, as it has turned out, if an interval $[H; A_p]$ in the subgroup lattice of an alternating group of prime degree p is isomorphic to $\mathcal{M}_n$ then $n = 1, 2, 3, 5, 7$, or 11. We have considered groups of prime degree only, because in general no complete list of maximal subgroups of the alternating groups is available, whereas owing to the classification of finite simple groups all transitive permutation groups of prime degree are known.

So Feit's examples proved to be exceptional, and the problem of representing $\mathcal{M}_n$, $n-1$ not a prime power, as an interval in the subgroup lattice of a finite group remains open for $n \geqslant 13$.

Preliminaries

Throughout the paper we choose $G = A_p$ for some prime number p. (However, several auxiliary results, e.g., Lemmas 1 and 5, hold for arbitrary degree.) We will determine all subgroups H for which $[H; A_p] \simeq \mathcal{M}_n$ with $n \geqslant 2$. Since A_3 has only trivial subgroups, $p \geqslant 5$.

If there are subgroups $H < L < K < G$ then we shall say that H is *deep*, in that case $[H; G]$ cannot be isomorphic to $\mathcal{M}_n$. Otherwise, either H is maximal in G or H is contained only in maximal subgroups of G and $[H; G] \simeq \mathcal{M}_n$ for some n.

Let A_p act on the p-element set X. If $X_1, \ldots, X_k$ is a partition of the set X then $G(X_1, \ldots, X_k)$ will denote the group of all even permutations which map each subset X_i $(1 \leqslant i \leqslant k)$ onto itself.

The following is obvious.

LEMMA 1. *The intransitive maximal subgroups of A_p $(p \geqslant 5)$ have the form $G(X_1, X_2)$ for some partition X_1, X_2.*

Owing to the classification of finite simple groups all transitive permutation groups of prime degree are known (see [5; 2, p. 56]):

THEOREM 2. *If T is a transitive permutation group of prime degree p, then there exists a simple normal subgroup S of T and T/S is cyclic. The possible choices for S are listed in Table I. Furthermore, any two isomorphic transitive subgroups of the symmetric group S_p are conjugate in S_p.*

From this theorem we can derive the list of transitive maximal subgroups of A_p.

COROLLARY 3. *The transitive maximal subgroups of A_p $(p \geqslant 5)$ are the following:*

(a) *solvable groups of order $p(p-1)/2$, for all p except* $p = 7, 11, 17, 23$;

(b) M_{11} *for* $p = 11$;

(c) M_{23} *for* $p = 23$;

(d) $\mathrm{P\Gamma L}(2, 2^{2^m})$ $(m \geqslant 2)$ *for* $p = 2^{2^m} + 1$;

(e) $\mathrm{P\Gamma L}(d, q)$ $(d \geqslant 3)$ *for* $p = (q^d - 1)/(q-1)$.

Proof. It is known that $\mathrm{PSL}(2, 11) < M_{11}$. For a Sylow p-subgroup P in a nonsolvable transitive group $T < A_p$, the normalizer $N_T(P)$ has order

TABLE I

p	S	$N(S)^a$	$\lvert N_T(P)\rvert^b$
p	Cyclic of order p	Solvable of order $p(p-1)$	$p \cdot \lvert T:S\rvert,\ \lvert T:S\rvert \mid p-1$
$p \geqslant 5$	A_p	S_p	$p \cdot \frac{p-1}{2} \cdot \lvert T:S\rvert,\ \lvert T:S\rvert \mid 2$
$\frac{q^d-1}{q-1} > 5^c$	$\mathrm{PSL}(d, q)$	$\mathrm{P\Gamma L}(d, q)$	$p \cdot d \cdot \lvert T:S\rvert,\ \lvert T:S\rvert \mid d^m$
11	$\mathrm{PSL}(2, 11)$	$= S$	$11 \cdot 5$
11	M_{11}	$= S$	$11 \cdot 5$
23	M_{23}	$= S$	$23 \cdot 11$

[a] The normalizers are taken in S_p.
[b] P is a Sylow p-subgroup of T.
[c] In order to $(q^d-1)/(q-1)$ be a prime it is necessary that d be a prime and $q = q_0^{d^m}$ for some prime number q_0 and $m \geqslant 0$. (If $m = 0$ then $\mathrm{P\Gamma L}(d, q) = \mathrm{PSL}(d, q)$.)

$p(p-1)/2$ if and only if T is one of the groups PSL(3, 2), PSL(2, 11), M_{11}, PΓL(2, 16) and M_{23} (cf. case 7 in the proof of our theorem). The rest is clear.

In order to determine the number of those transitive maximal subgroups of a given isomorphism type which contain H we shall use the following simple lemma.

LEMMA 4. *Let $H \leqslant K \leqslant G_0$ be groups. Suppose that whenever $xHx^{-1} \leqslant K$ for some $x \in G_0$ then there exists a $y \in K$ such that $xHx^{-1} = yHy^{-1}$. Then the number of those conjugates of K which contain H is*

$$\frac{|N(H) : K \cap N(H)|}{|N(K) : K|},$$

where the normalizers are taken in G_0.

Proof. Each of the $|G_0 : N(H)|$ conjugates of H is contained in the same number, say, s conjugates of K. By assumption, each of the $|G_0 : N(K)|$ conjugates of K contains $|K : N_K(H)|$ conjugates of H. Hence for the total number of containments we have

$$|G_0 : N(H)| \cdot s = |G_0 : N(K)| \cdot |K : N_K(H)|,$$

therefore,

$$s = \frac{|N(H)|}{|N_K(H)|} \cdot \frac{|K|}{|N(K)|}.$$

When applying this lemma we will always take $G_0 = S_p$, and we will leave the computations to the reader.

The following classical result of C. Jordan (see [10, p. 39]) will be used several times. Note that any transitive group of prime degree is primitive.

LEMMA 5. *If a primitive group $K \leqslant A_p$ contains a 3-cycle then $K = A_p$.*

WHEN IS $[H; A_p] \simeq \mathcal{M}_n$?

THEOREM. *Let H be a subgroup of the alternating group A_p of prime degree $p \geqslant 5$ such that the interval $[H; A_p] \simeq \mathcal{M}_n$ for some $n \geqslant 2$. Then either $n = 3$ and $H = G(X_1, X_2, X_3)$ for some partition X_1, X_2, X_3 of the p-element set of permuted elements with $|X_1| < |X_2| < |X_3|$ or $|X_1| = |X_2| = 1$, $|X_3| = p-2$, or one of the following occurs:*

TABLE II

n	p	H
2	7	Stabilizer of a line in PSL(3, 2)
2	7	Normalizer of a Sylow p-subgroup in PSL(3, 2)
2	17	Normalizer of a Sylow p-subgroup in PΓL(2, 16)
2	23	Normalizer of a Sylow p-subgroup in M_{23}
3	7	Stabilizer of a point in PSL(3, 2)
3	11	Stabilizer of a point in M_{11}
3	23	Stabilizer of a point in M_{23}
5	13	Normalizer of a Sylow p-subgroup in PSL(3, 3)
7	31	Normalizer of a Sylow p-subgroup in PSL(5, 2)
11	31	Normalizer of a Sylow p-subgroup in PSL(3, 5)

Proof. Let the orbits of H be $X_1, X_2, ..., X_k$ $(k \geqslant 1)$. We shall distinguish seven cases as follows: (1) $k \geqslant 4$; (2) $k = 3$; (3) $k = 2$, $|X_1| = 1$; (4) $k = 2$, $|X_1| = 2$; (5) $k = 2$, $|X_1|$, $|X_2| \geqslant 3$; (6) $k = 1$, H is nonsolvable; (7) $k = 1$, H is solvable.

(1) *H has more than three orbits.* If $H = 1$, then as $p \geqslant 5$ we have $H = 1 < A_3 < A_4 < A_p$, so H is deep. If $H \neq 1$ then, say, $|X_1| > 1$. Then $H \leqslant G(X_1, X_2, X_3, X_4, ...) < G(X_1 \cup X_2, X_3, X_4, ...) < G(X_1 \cup X_2 \cup X_3, X_4, ...) < A_p$, so H is deep again.

(2) *H has three orbits.* If, say, $|X_1| = |X_2| > 1$, let $G(X_1 | X_2, X_3)$ denote the group of all even permutations which map X_3 onto itself, and either X_1 onto X_1 and X_2 onto X_2 or X_1 onto X_2 and X_2 onto X_1. Now $H \leqslant G(X_1, X_2, X_3) < G(X_1 | X_2, X_3) < G(X_1 \cup X_2, X_3) < A_p$, so H is deep. If we list the orbits in increasing order then either $|X_1| < |X_2| < |X_3|$ or $|X_1| = |X_2| = 1$ and $|X_3| = p-2$. In any case, $|X_3| \geqslant 3$. Now $H \leqslant G(X_1, X_2, X_3) < G(X_1, X_2 \cup X_3) < A_p$, hence $H = G(X_1, X_2, X_3)$. It follows that H contains 3-cycles, therefore by Lemma 5 H is not contained in any proper transitive subgroup of A_p. Now it is easy to see that the interval $[H; A_p] \simeq \mathcal{M}_3$ with the intermediate subgroups being $G(X_1 \cup X_2, X_3)$, $G(X_1 \cup X_3, X_2)$, and $G(X_2 \cup X_3, X_1)$.

(3) *H has two orbits, having length 1 and $p-1$.* Since $n \geqslant 2$ and H is contained in a unique intransitive maximal subgroup of A_p, namely in $G(X_1, X_2)$, it follows that $H = G(X_1, X_2) \cap K$ for some transitive maximal subgroup K of A_p. As $X_1 = \{x\}$, we have $H = K_x$, the stabilizer of a point in K. Moreover, H is transitive on X_2 $(|X_2| = p-1)$ and maximal in $G(X_1, X_2) \simeq A_{p-1}$. Now we can go through the list of maximal transitive subgroups of A_p (Corollary 3) and check the stabilizers of them.

If K is solvable then $|K_x| = (p-1)/2$ and K_x has three orbits. If $K = M_{11}$ then K_x is really maximal in $G(X_1, X_2) \simeq A_{10}$. For any transitive proper subgroup $K' \geqslant H = K_x$ we have $K' \simeq M_{11}$, $K'_x = K_x$. By Lemma 4 the number of such subgroups is

$$\frac{|N(H) : K \cap N(H)|}{|N(K) : K|} = 2$$

(the normalizers are taken in S_{11}). Together with the intransitive maximal subgroup there are three subgroups containing H, so $[H; A_{11}] \simeq \mathcal{M}_3$. If $K = M_{23}$ then a similar argument yields that for $H = K_x$ we have $[H; A_{23}] \simeq \mathcal{M}_3$. Let now $K = \mathrm{P\Gamma L}(2, 2^{2^m})$ $(m \geqslant 2)$, then K_x contains an elementary abelian normal subgroup E. It is easy to see that $H = K_x < N(E) < G(X_1, X_2) < A_p$, since $N(E)$ is triply transitive on X_2 but K_x is not. Hence for this choice of K, H is deep. If $K = \mathrm{P\Gamma L}(d, q)$ for $d \geqslant 3$, then K_x is imprimitive with $(q^{d-1} - 1)/(q-1)$ blocks of size q. In order to be a maximal subgroup of $G(X_1, X_2)$ it should act as the symmetric group on the set of blocks. However, the action of K_x on the set of blocks is $\mathrm{P\Gamma L}(d-1, q)$, which is the symmetric group only if $d = 3$ and $q = 2, 3, 4$. For $q = 4$, $(q^d - 1)/(q-1)$ is not a prime number. If $q = 3$ then $H = K_x < H^* < G(X_1, X_2) < A_{13}$ with a maximal imprimitive subgroup H^* of order $3!^4 \cdot 4!/2 = 15{,}552 > |K_x| = 432$. So there remains to consider the case $K = \mathrm{PSL}(3, 2)$, $p = 7$. Now K_x is really maximal in $G(X_1, X_2)$; moreover, for any transitive proper subgroup $K' \geqslant K_x$, we have $K' \simeq \mathrm{PSL}(3, 2)$ and by Lemma 4 there are

$$\frac{|N(H) : K \cap N(H)|}{|N(K) : K|} = 2$$

such subgroups. Thus $[H; A_7] \simeq \mathcal{M}_3$.

(4) *H has two orbits, having length 2 and $p-2$.* Again, $H = G(X_1, X_2) \cap K$, where K is a transitive maximal subgroup of A_p. We must have H maximal in $G(X_1, X_2) \simeq S_{p-2}$. If K is solvable then $|H| \leqslant 2$, hence H has more than two orbits. If $K = M_{11}$ then H of order $9 \cdot 8 \cdot 2$ contains an elementary abelian normal subgroup E of order 9, and $H < N(E) < G(X_1, X_2) < A_{11}$, hence H is deep in this case. If $K = M_{23}$ then $H \simeq \mathrm{P\Sigma L}(3, 4) < \mathrm{P\Gamma L}(3, 4) < S_{21}$, and H proves to be deep again. Let now $K = \mathrm{P\Gamma L}(2, 2^{2^m})$, $m \geqslant 2$. Since K is doubly transitive, we may assume without loss of generality that $X_1 = \{0, \infty\}$. Then $H = \{cx^{\varepsilon 2^j} : c \in GF(2^{2^m}),$ $c \neq 0, j = 0, 1, ..., 2^m - 1, \varepsilon = +1, -1\}$ is imprimitive on $X_2 = GF(2^{2^m}) - \{0\}$ with u and v belonging to the same block iff $u^3 = v^3$. However, it is obviously not a maximal imprimitive subgroup, hence H is deep in this case as well. Finally, for the choice $K = \mathrm{P\Gamma L}(d, q)$, $d \geqslant 3$, H has three orbits: $X_1 = \{x, y\}$, the remaining points of the line connecting x and y, and the complement of this line.

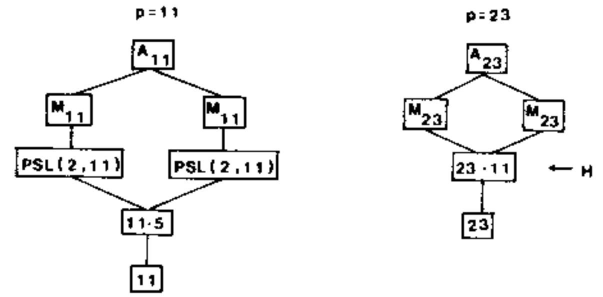
(5) *H has two orbits, both having length $\geqslant 3$.* Now $H = G(X_1, X_2) \cap K$, where K is transitive; and H must be maximal in $G(X_1, X_2)$. Let the action of H on X_1 be H_1 and denote by $\mathrm{Sym}(X_1)$ the full symmetric group over X_1. If $H_1 < \mathrm{Sym}(X_1)$ then $H \leqslant (H_1 \times \mathrm{Sym}(X_2)) \cap A_p < G(X_1, X_2)$, hence $H = (H_1 \times \mathrm{Sym}(X_2)) \cap A_p$. As $|X_2| \geqslant 3$, H contains 3-cycles, hence no transitive proper subgroup K contains H by Lemma 5. Therefore H induces $\mathrm{Sym}(X_1)$ on X_1 and, by symmetry, $\mathrm{Sym}(X_2)$ on X_2. Now for N_1, the restriction of the pointwise stabilizer of X_2 onto X_1, we have $N_1 \lhd \mathrm{Sym}(X_1)$, and similarly $N_2 \lhd \mathrm{Sym}(X_2)$; moreover, $\mathrm{Sym}(X_1)/N_1 \simeq \mathrm{Sym}(X_2)/N_2$, see [3, p. 63]. As $H < G(X_1, X_2)$, it follows that $N_1 < \mathrm{Alt}(X_1)$ and $N_2 < \mathrm{Alt}(X_2)$. Since $|X_1| \neq |X_2|$ such normal subgroups exist only for $|X_1| = 3$, $|X_2| = 4$ (or reversely), when $N_1 = 1$ and N_2 is an elementary abelian group of order four. Hence $p = 7$, so $K = \mathrm{PSL}(3, 2)$. If the three points in X_1 are not collinear then $H = K \cap G(X_1, X_2)$ has three orbits, which is not the case. If X_1 is a line then $H \simeq S_4$ is in fact maximal in both K and $G(X_1, X_2)$. If a transitive proper subgroup K' contains H then $K' \simeq \mathrm{PSL}(3, 2)$. The number of such subgroups is

$$\frac{|N(H) : K \cap N(H)|}{|N(K) : K|} = 1$$

(the normalizers are taken in S_7); hence $[H; A_7] \simeq \mathcal{M}_2$ in this case.

(6) *H is transitive and nonsolvable.* Let $H < K < A_p$ and P be a Sylow p-subgroup of H. Then $N_H(P) \leqslant N_K(P)$ and a look at the orders of these normalizers (see Table I) yields that either $H \lhd K$ or $p = 11$, $H \simeq \mathrm{PSL}(2, 11)$ and $K \simeq M_{11}$. In any case H is contained in a unique maximal subgroup of A_p.

(7) *H is transitive and solvable.* Let us first consider the exceptional cases $p = 11$ and $p = 23$. One can easily check that the interval formed by the subgroups containing a given subgroup of order p is the following:



There is one suitable choice for H, when we have $[H; A_{23}] \simeq \mathcal{M}_2$.

Now let us suppose $p \neq 11, 23$. H is contained in a unique maximal solvable subgroup, namely $N_G(H) \geq H$. Since $n \geq 2$, there is some nonsolvable maximal subgroup containing H, hence $H < \mathrm{P\Gamma L}(d, q) < A_p$, where $p = (q^d - 1)/(q-1)$. As H is maximal in $\mathrm{P\Gamma L}(d, q)$ it has to be the normalizer of a Sylow p-subgroup of $\mathrm{P\Gamma L}(d, q)$, so $|H| = p \cdot d^{m+1}$ (see Table I). Since either $N_G(H) = H$ or H is maximal in $N_G(H)$, we have

$$|N_G(H) : H| = \frac{p-1}{2d^{m+1}} = r, \tag{1}$$

where $r = 1$ or r is a prime number. If K is any nonsolvable proper subgroup containing H then the order of the normalizers of the Sylow p-subgroups in K is divisible by pd^{m+1}, hence $K \simeq \mathrm{P\Gamma L}(d, q)$. By Lemma 4 the number of subgroups isomorphic to K and containing H is

$$\frac{|N(H) : K \cap N(H)|}{|N(K) : K|} = 2r$$

(the normalizers are taken in S_p). If $r = 1$ then this is the total number of subgroups containing H, if $r > 1$ then there is also a solvable maximal subgroup containing H, hence $[H; A_p] \simeq \mathcal{M}_n$, where

$$n = \begin{cases} 2 & \text{if } \ r = 1, \\ 2r+1 & \text{if } \ r \text{ is prime.} \end{cases} \tag{2}$$

We have to determine when $(p-1)/2d^{m+1}$ is a prime number or equal to one. First let $d = 2$, then $q = 2^{2^m}$, $m \geq 2$, $p = q + 1$, hence

$$r = \frac{p-1}{2 \cdot 2^{m+1}} = 2^{2^m - m - 2}.$$

If $m = 2$ then $p = 17$, $r = 1$, and $n = 2$ by (2). If $m > 2$ then r is never a prime number.

Now let $d = 2t + 1$ be an odd prime, $q = q_0^{d^m}$, where q_0 is a prime and $m \geq 0$. Then (1) can be rewritten as

$$2rd^{m+1} = \frac{q^d - 1}{q - 1} - 1 = q(q^t + 1)\frac{q^t - 1}{q - 1}. \tag{3}$$

We distinguish three cases.

(a) $q_0 = 2$. Comparing the highest powers of 2 dividing the two sides of (3) we obtain $4 \geq q = 2^{d^m}$; hence $m = 0$, $q = 2$, and

$$2rd = 2(2^t + 1)(2^t - 1).$$

Now either $t = 1$, $d = 2t + 1 = 3$, $r = 1$, $n = 2$ by (2), and $p = (q^d - 1)/(q - 1) = 7$, or for $t > 1$, both $2^t + 1$ and $2^t - 1$ are primes; hence $t = 2$, $d = 5$, $r = 3$, $n = 7$, and $p = 31$.

(b) $q_0 = d$. Comparing the highest powers of d dividing the two sides of (3) we obtain $d^{m+2} \geq d^{d^m}$, so $m = 0$ or $m = 1$, $d = 3$. In the latter case $q = 27$ and $r = 42$ which is not prime. Thus $m = 0$, $q = d$, and

$$2rd = d(d^t + 1)\frac{d^t - 1}{d - 1}.$$

Since $d^t + 1$ is even and > 2, we have

$$2r = d^t + 1 \qquad \text{and} \qquad 1 = \frac{d^t - 1}{d - 1},$$

hence $t = 1$, $d = 3$, $r = 2$, $n = 5$, $p = 13$.

(c) $q_0 \neq 2, d$. Comparing the highest powers of q_0 dividing the two sides of (3) we get $m = 0$, $r = q = q_0$, and

$$2qd = q(q^t + 1)\frac{q^t - 1}{q - 1}.$$

Again, $q^t + 1$ is even and > 2, hence $t = 1$, $d = 3$, and $2d = q + 1$, so $q = 5$, $r = 5$, $n = 11$, and $p = 31$.

The proof is complete.

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