# Small Congruence Lattices

William DeMeo
williamdemeo@gmail.com
University of Colorado, Boulder

joint work with

Ralph Freese (U Hawaii) and Peter Jipsen (Chapman U)

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# The Problem

## Characterize congruence lattices of finite algebras.

For an arbitrary algebra, there is essentially no restriction on the shape of its congruence lattice.

**Theorem (Grätzer-Schmidt, 1963)**

*Every algebraic lattice is isomorphic to the congruence lattice of an algebra.*

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### Definition

We call a finite lattice representable if it is (isomorphic to) the congruence lattice of a finite algebra.

# A few important theorems

**Theorem (Pudlák and Tůma, 1980)**

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**Theorem (Berman, Quackenbush & Wolk, 1970)**

*Every finite distributive lattice is representable.*

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Given a finite lattice $\mathbf{L}$, prove there exists a finite algebra $\mathbf{A}$ with $\operatorname{Con}\mathbf{A} \cong \mathbf{L}$.

How to construct a finite algebra with a given congruence lattice?

# PART 1: METHODS

How to construct a finite algebra with a given congruence lattice?

How to prove the existence of a finite algebra with a given congruence lattice?

# How to prove a finite lattice is representable

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## 1. Use closure properties

Relate the given lattice to other lattices known to be representable.

- If $L$ is representable, so is
  - A. the dual of $L$ (Kurzweil 1985, Netter)
  - B. any interval sublattice of $L$ (follows from A.)
  - C. any sublattice that is the union of a principal filter and principal idea of $L$ (Snow, 2000)
- If $L_1$ and $L_2$ are representable, so is
  1. the direct product of $L_1$ and $L_2$ (Tůma1989)
  2. the ordinal sum of $L_1$ and $L_2$ (McKenzie 1984, Snow 2000)
  3. the parallel sum of $L_1$ and $L_2$ (Snow 2000)

# How to prove a finite lattice is representable

## 2. The closure method

Find a "closed" representation of $L$ in $\mathsf{Eq}(X)$.

- For $L \leqslant \mathsf{Eq}(X)$ define

$$\lambda(L) = \{f \in X^X : (\forall \theta \in L)\, f(\theta) \subseteq \theta\}$$

- For $F \subseteq X^X$ define

$$\rho(F) = \{\theta \in \mathsf{Eq}(X) : (\forall f \in F)\, f(\theta) \subseteq \theta\}$$

- The map $\rho\lambda$ is a *closure operator* on $\mathsf{Sub}[\mathsf{Eq}(X)]$.
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### Theorem

*A lattice $L \leqslant \mathsf{Eq}(X)$ is a congruence lattice if and only if it is closed, i.e. $\rho\lambda(L) = L$, in which case $L = \mathrm{Con}\,\langle X, \lambda(L)\rangle$.*

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Example: $M_3 \cong L \leqslant \mathrm{Eq}(5)$

### 3. THE G-SET METHOD

Find $L$ as an interval in a subgroup lattice of a finite group.

If $H \leqslant G$ are finite groups, then the filter above $H$ in Sub$(G)$,

$$[\![H, G]\!] := \{K : H \leqslant K \leqslant G\},$$

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## 4. The rabbit ears method (aka overalgebras, aka expansion-extension)

Build the required algebra by gluing together isomorphic copies of an algebra and adding new operations.

## The G-set method: details

For groups $H \leqslant G$, let $\mathbf{A} = \langle H \backslash G, \bar{G} \rangle$ denote the algebra with

- universe: the right cosets $H \backslash G = \{Hx : x \in G\}$
- operations: $\bar{G} = \{g^{\mathbf{A}} : g \in G\}$, where $g^{\mathbf{A}}(Hx) = Hxg$.

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*The isomorphism* $[\![H, G]\!] \ni K \mapsto \theta_K \in \operatorname{Con} \mathbf{A}$ *is given by*

$$\theta_K = \{(Hx, Hy) : xy^{-1} \in K\}.$$

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Aside: properties of such congruence lattices correspond to properties of subgroup lattices. For example,

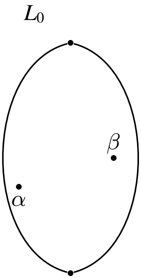
### LEMMA

*In* $\operatorname{Con}\langle H\backslash G, \bar{G}\rangle$*, two congruences,* $\theta_{K_1}$ *and* $\theta_{K_2}$*,* $n$*-permute if and only if the corresponding subgroups,* $K_1$ *and* $K_2$*,* $n$*-permute.*

## FILTER+IDEAL METHOD: DETAILS

### Lemma

*Suppose* $L_0 \cong \mathrm{Con}\,\langle A, F\rangle$, *and* $\alpha, \beta \in L_0 \setminus \{0, 1\}$.



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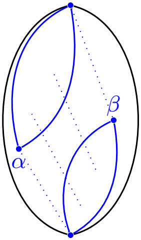
*Suppose* $L_0 \cong \operatorname{Con}\langle A, F\rangle$, *and* $\alpha, \beta \in L_0 \setminus \{0, 1\}$. *Consider* $L = \alpha^{\uparrow} \cup \beta^{\downarrow}$.

$L \leqslant L_0$



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*There exists a set* $F' \subset A^A$ *such that* $L \cong \operatorname{Con}\langle A, F \cup F'\rangle$.

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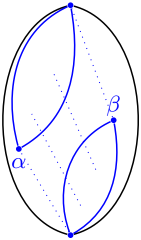

# FILTER+IDEAL METHOD: DETAILS

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*There exists a set* $F' \subset A^A$ *such that* $L \cong \mathrm{Con}\,\langle A, F \cup F'\rangle$.

*Proof:*

Fix $\theta \in L_0 \setminus L$. Then $\alpha \nleq \theta \nleq \beta$, so

- $\exists (a,b) \in \alpha \setminus \theta$,
- $\exists (u,v) \in \theta \setminus \beta$.

Define $f_\theta : A \to A$ by

$$f_\theta(x) = \begin{cases} a & x \in u/\beta, \\ b & x \notin u/\beta. \end{cases}$$

Then

- $(f_\theta(u), f_\theta(v)) = (a,b) \notin \theta$, so $f_\theta(\theta) \nsubseteq \theta$,
- $\ker f_\theta \geqslant \beta$, so $f_\theta(\gamma) \subseteq \gamma$ for all $\gamma \leqslant \beta$,
- $f_\theta(A) \subseteq \{a,b\}$, so $f_\theta(\gamma) \subseteq \gamma$ for all $\gamma \geqslant \alpha$.

Let $F' = \{f_\theta : \theta \in L_0 \setminus L\}$.

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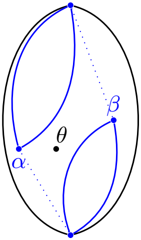
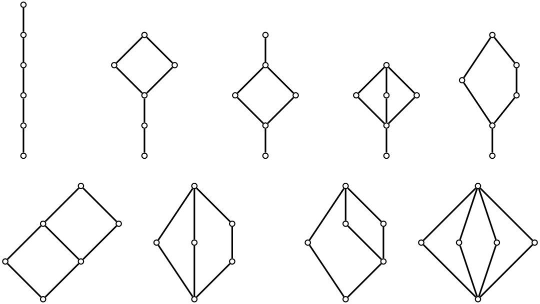
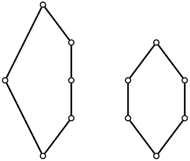


Which finite lattices are known to be representable?

# Lattices with at most 6 elements are representable.



Watatani (1996) *J. Funct. Anal.*



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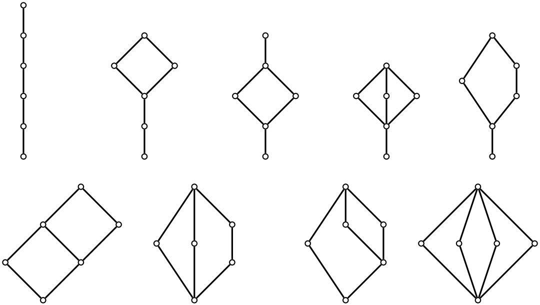

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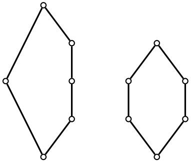

Aschbacher (2008) *JAMS*

**Theorem:** *Lattices with at most 6 elements are intervals in subgroup lattices of finite groups.*

# ARE ALL LATTICES WITH AT MOST 7 ELEMENTS REPRESENTABLE?

*As of Spring 2011...*

**LATTICES OF SIZE $\leq 7$ NOT YET KNOWN TO BE CONGRUENCE LATTICES OF FINITE ALGEBRAS**

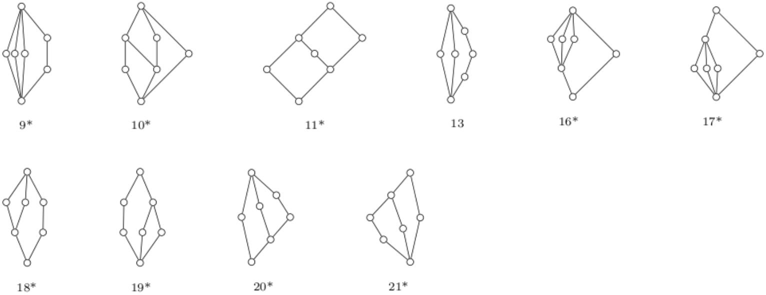


*Figure courtesy of Peter Jipsen.*

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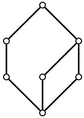
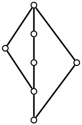
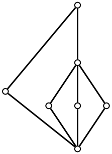


$L_{19}$



$L_{20}$



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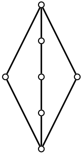

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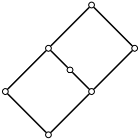

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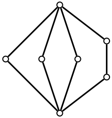
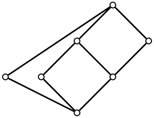


$L_{9}$



$L_{10}$

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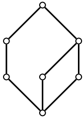

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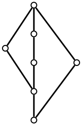

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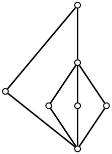
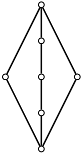


$L_{17}$



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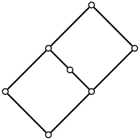

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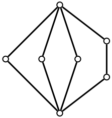

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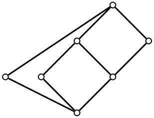

$L_{10}$

# Finding representations…

## …as intervals in subgroup lattices

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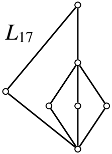

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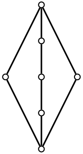
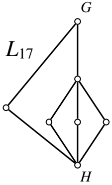


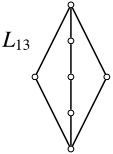
# FINDING REPRESENTATIONS...

## ...AS INTERVALS IN SUBGROUP LATTICES




SmallGroup(288,1025)

$|G : H| = 48$




- The group $G = (A_4 \times A_4) \rtimes C_2$ has a subgroup $H \cong S_3$ such that $[\![H, G]\!] \cong L_{17}$.
- ...so the dual $L_{16}$ is also representable.

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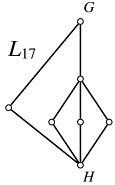


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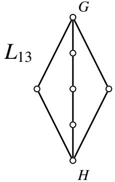
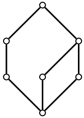



SmallGroup(960,11358)

$|G : H| = 80$

- The group $G = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_5$ has a subgroup $H \cong A_4$ such that $[\![H, G]\!] \cong L_{13}$.

# Are all lattices with at most 7 elements representable?



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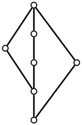

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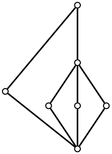

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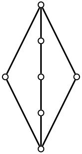

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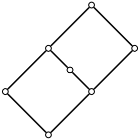

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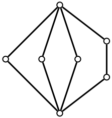
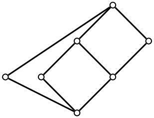


$L_9$

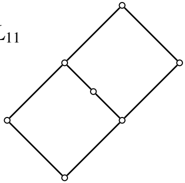


$L_{10}$

# Finding representations...

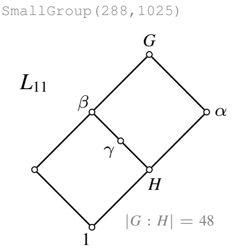
...using subgroup lattice intervals and the filter+ideal lemma.

$L_{11}$



# FINDING REPRESENTATIONS...

## ...USING SUBGROUP LATTICE INTERVALS AND THE FILTER+IDEAL LEMMA.




- Let $G = (A_4 \times A_4) \rtimes C_2$.
- $G$ has a subgroup $H \cong C_6$ with $[\![H, G]\!] \cong N_5$.
- Let $[\![H, G]\!] = \{H, \alpha, \beta, \gamma, G\} \cong N_5$.

# FINDING REPRESENTATIONS…

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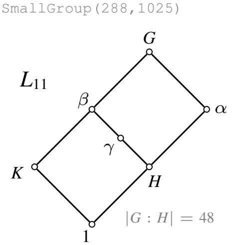


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- Let $[\![H, G]\!] = \{H, \alpha, \beta, \gamma, G\} \cong N_5$.

- Sub$(G)$ is a congruence lattice, so if there exists a subgroup $K \succ 1$, below $\beta$ and not below $\gamma$, then

$$L_{11} \cong K^{\downarrow} \cup H^{\uparrow}.$$

# Finding representations...

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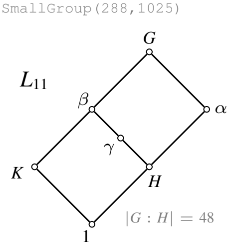


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- Let $[\![H, G]\!] = \{H, \alpha, \beta, \gamma, G\} \cong N_5$.
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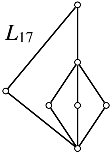
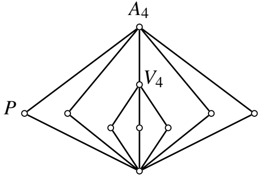


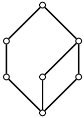




- Sub$(A_4)$ is a congruence lattice
  (of $A_4$ acting regularly on itself).
- Therefore,

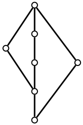
$$L_{17} \cong V_4^{\downarrow} \cup P^{\uparrow}$$

  is a congruence lattice.

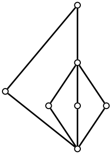
# Are all lattices with at most 7 elements representable?



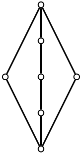
$L_{19}$ ✓



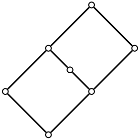
$L_{20}$ ✓



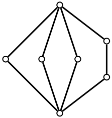
$L_{17}$ ✓



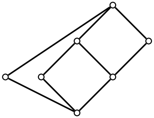
$L_{13}$ ✓



$L_{11}$ ✓

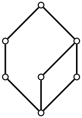


$L_9$

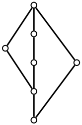


$L_{10}$

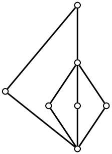
# Are all lattices with at most 7 elements representable?



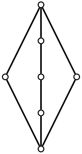
$L_{19}$ ✓



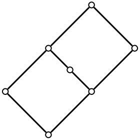
$L_{20}$ ✓



$L_{17}$ ✓



$L_{13}$ ✓



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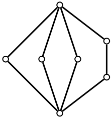

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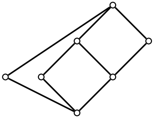
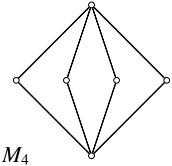
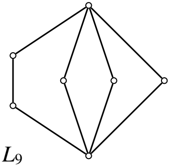


$L_{10}$

# CONSTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\operatorname{Con}\mathbf{A} \cong L_9$.



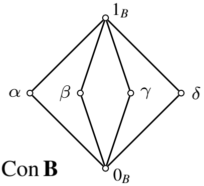
$M_4$



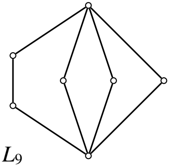
$L_9$

# CONSTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\operatorname{Con}\mathbf{A} \cong L_9$.

STEP 1 Take a permutational algebra $\mathbf{B} = \langle B, F\rangle$ with congruence lattice $\operatorname{Con}\mathbf{B} \cong M_4$.

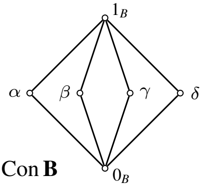



$\operatorname{Con}\mathbf{B}$



$L_9$

# Construction of an algebra $\mathbf{A}$ with $\operatorname{Con}\mathbf{A} \cong L_9$.




$\operatorname{Con}\mathbf{B}$

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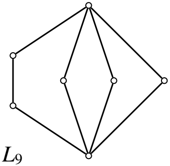
Example:

- Let $B = \{0, 1, \ldots, 5\}$ index the elements of $S_3$ and consider the right regular action of $S_3$ on itself.
- $g_0 = (0,4)(1,3)(2,5)$ and $g_1 = (0,1,2)(3,4,5)$ generate this action group, the image of $S_3 \hookrightarrow S_6$.
- $\operatorname{Con}\langle B, \{g_0, g_1\}\rangle \cong M_4$ with congruences

$$\alpha = |012|345|,\ \beta = |03|14|25|, \gamma = |04|15|23|,\ \delta = |05|13|24|.$$



$L_9$

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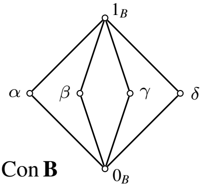


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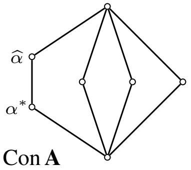
**Goal:** *expand* $\mathbf{B}$ *to an algebra* $\mathbf{A}$ *that has* $\alpha$ *"doubled" in* $\mathrm{Con}\,\mathbf{A}$.



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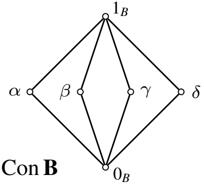


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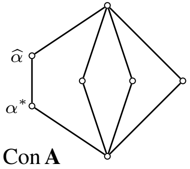
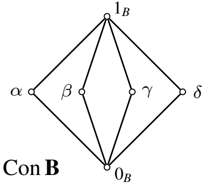



Step 2 Since $\alpha = \operatorname{Cg}^{\mathbf{B}}(0, 2)$, we let $A = B_0 \cup B_1 \cup B_2$ where

$$B_0 = \{0, 1, 2, 3, 4, 5\} = B$$
$$B_1 = \{0, 6, 7, 8, 9, 10\}$$
$$B_2 = \{11, 12, 2, 13, 14, 15\}.$$

# CONSTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\operatorname{Con}\mathbf{A} \cong L_9$.

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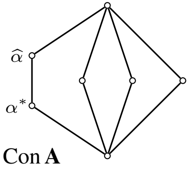


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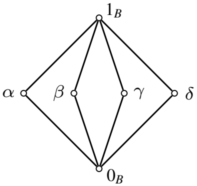
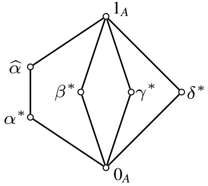
$$B_2 = \{11, 12, 2, 13, 14, 15\}.$$

STEP 3 Define unary operations $e_0, e_1, e_2, \ s, \ g_0e_0$, and $g_1e_0$.

# Construction of an algebra $\mathbf{A}$ with $\operatorname{Con}\mathbf{A} \cong L_9$.




$\operatorname{Con}\langle B, \{g_0, g_1\}\rangle$




$\operatorname{Con}\langle A, F_A\rangle$

$$\alpha = |0,1,2|3,4,5|$$

$$\beta = |0,3|1,4|2,5|$$

$$\gamma = |0,4|1,5|2,3|$$

$$\delta = |0,5|1,3|2,4|$$

$$\widehat{\alpha} = |0,1,2,6,7,11,12|3,4,5|8,9,10,13,14,15|$$

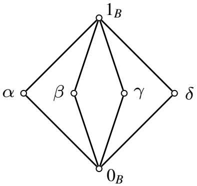
$$\alpha^* = |0,1,2,6,7,11,12|3,4,5|8,9,10|13,14,15|$$

$$\beta^* = |0,3,8|1,4|2,5,15|6,9|7,10|11,13|12,14|$$

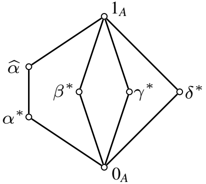
$$\gamma^* = |0,4,9|1,5|2,3,13|6,10|7,8|11,14|12,15|$$

$$\delta^* = |0,5,10|1,3|2,4,14|6,8|7,9,11,15|12,13|$$

## CONSTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\mathrm{Con}\,\mathbf{A} \cong L_9$.




$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$




$\mathrm{Con}\,\langle A, F_A\rangle$

$$\alpha = |0, 1, 2|3, 4, 5|$$

$$\beta = |0, 3|1, 4|2, 5|$$

$$\gamma = |0, 4|1, 5|2, 3|$$

$$\delta = |0, 5|1, 3|2, 4|$$

$$\widehat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$$

$$\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|$$

$$\beta^* = |0, 3, 8|1, 4|2, 5, 15|6, 9|7, 10|11, 13|12, 14|$$

$$\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$$

$$\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|$$

$$\alpha = \alpha^* \cap B^2 = \widehat{\alpha} \cap B^2, \quad \beta = \beta^* \cap B^2, \quad \ldots$$

# WHY DOES IT WORK?

$$\text{Con}\,\langle B, \{g_0, g_1\}\rangle$$

$$\alpha = |0, 1, 2|3, 4, 5|$$

$$\beta = |0, 3|1, 4|2, 5|$$

$$\gamma = |0, 4|1, 5|2, 3|$$

$$\delta = |0, 5|1, 3|2, 4|$$

$$\begin{array}{ccc} 0 & 1 & 2 \\ 3 & 4 & 5 \end{array}$$

$B_0$

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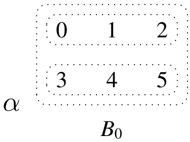

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$$\alpha = |0,1,2|3,4,5|$$

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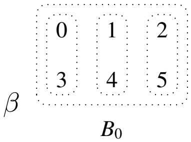
$$\gamma = |0,4|1,5|2,3|$$

$$\delta = |0,5|1,3|2,4|$$



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$$\alpha = |0, 1, 2|3, 4, 5|$$

$$\beta = |0, 3|1, 4|2, 5|$$

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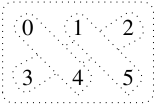
$$\delta = |0, 5|1, 3|2, 4|$$

$\gamma$




$B_0$

# WHY DOES IT WORK?

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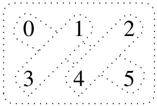
$$\beta = |0, 3|1, 4|2, 5|$$

$$\gamma = |0, 4|1, 5|2, 3|$$

$$\delta = |0, 5|1, 3|2, 4|$$

$\delta$




$B_0$

# Why does it work?

$\text{Con}\,\langle B, \{g_0, g_1\}\rangle$

$\alpha = |0, 1, 2|3, 4, 5|$

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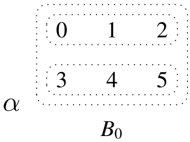
$\delta = |0, 5|1, 3|2, 4|$




$B_0 = \{\ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\ \}$

# WHY DOES IT WORK?

$$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$$

$$\alpha = |{\color{blue}0, 1, 2}|{\color{blue}3, 4, 5}|$$

$$\beta = |0, 3|1, 4|2, 5|$$

$$\gamma = |0, 4|1, 5|2, 3|$$

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$$\begin{array}{ccccccc} & B_1 & & & & B_2 & \\ 10 & 9 & 8 & & 15 & 14 & 13 \\ 7 & 6 & 0 & 1 & 2 & 12 & 11 \\ & & 3 & 4 & 5 & & \\ & & & B_0 & & & \end{array}$$

- $A = B_0 \cup B_1 \cup B_2$

$$B_1 = \{\ 0 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10\ \}$$

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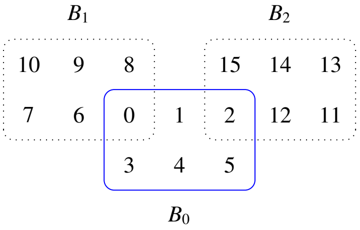


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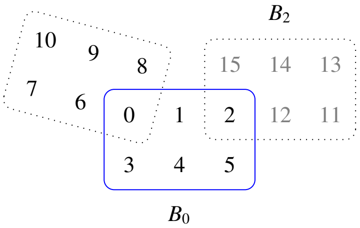


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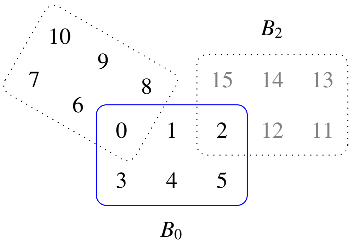


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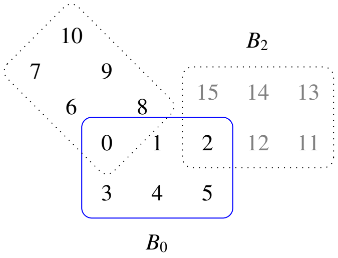


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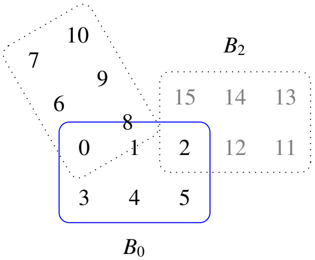


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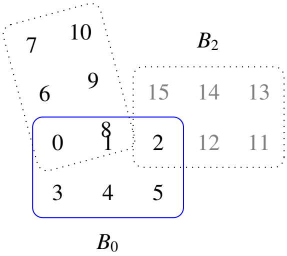


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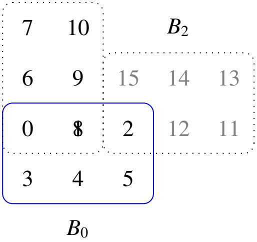


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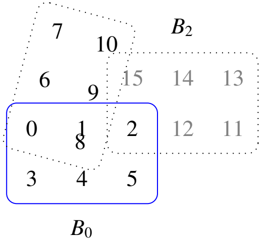


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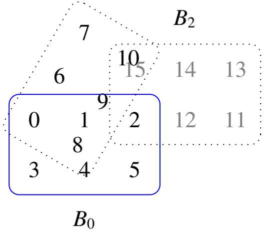


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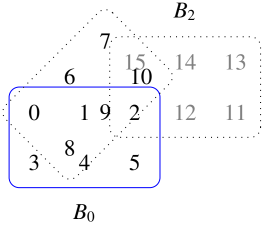


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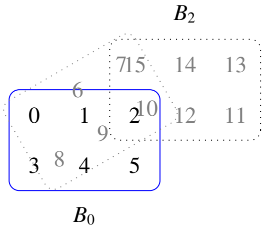


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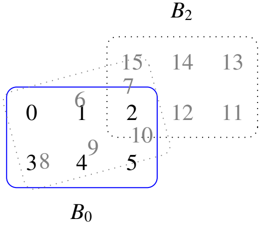


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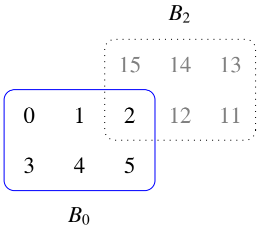


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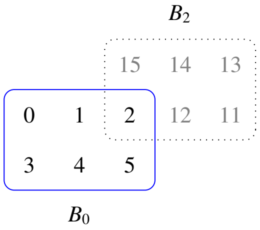


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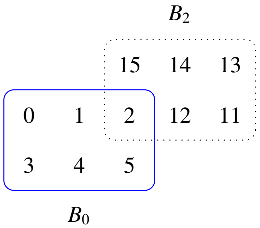


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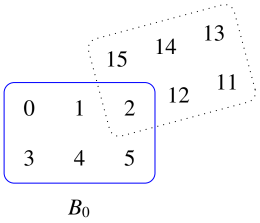


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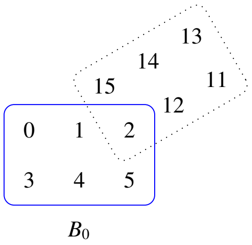


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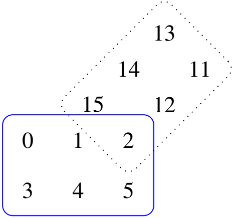


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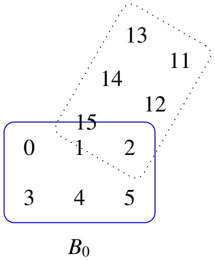


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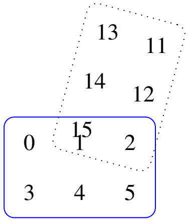


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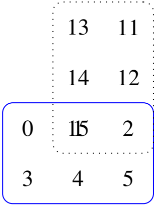


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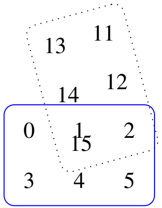


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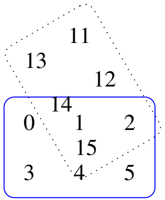


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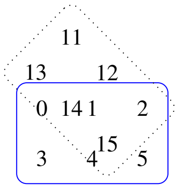


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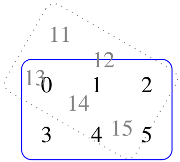


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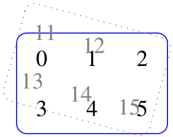


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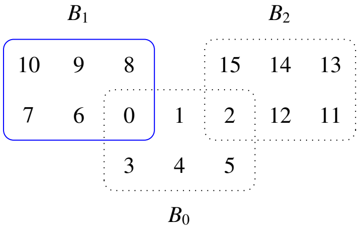


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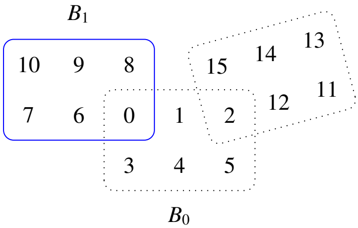


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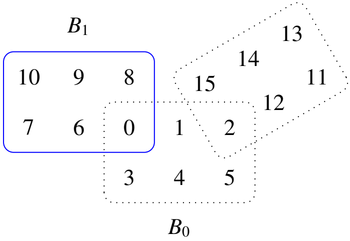


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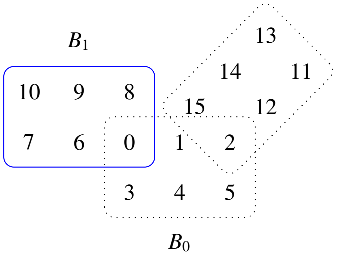


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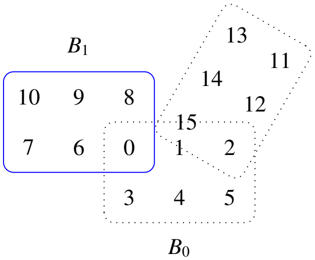


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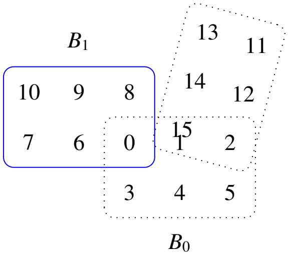


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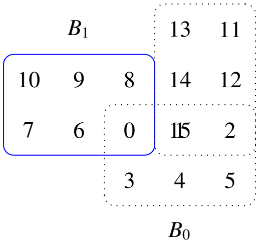


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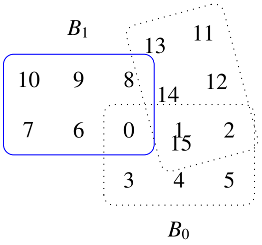


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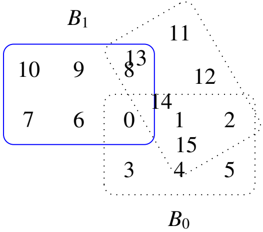


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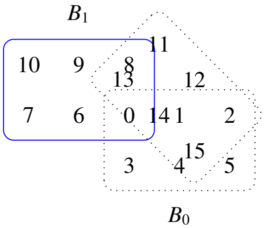


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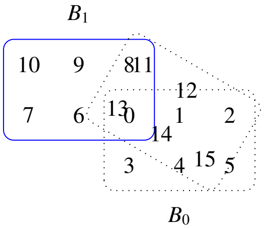


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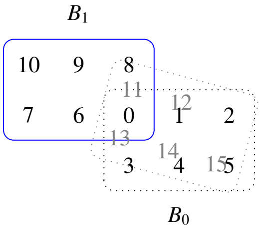


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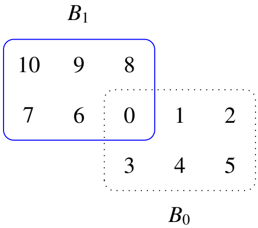


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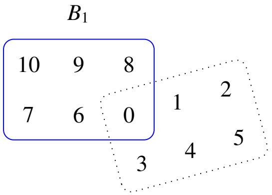


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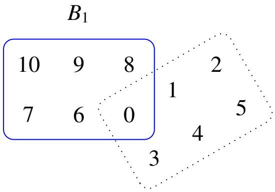


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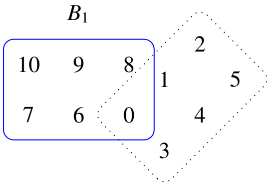


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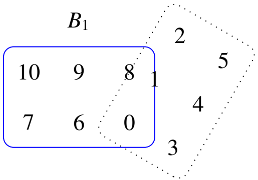


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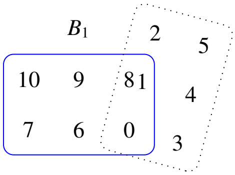


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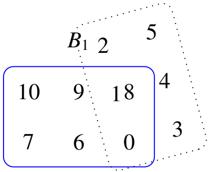


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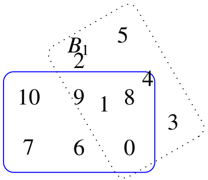


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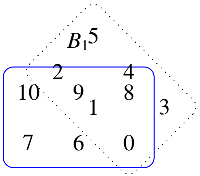


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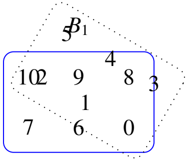


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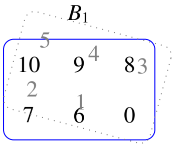


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| | | |
|---|---|---|
| 10 | 9 | 8 |
| 7 | 6 | 0 |

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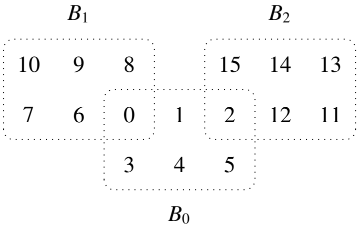


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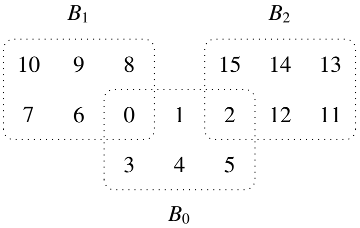


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$$ge_0 \colon A \overset{e_0}{\twoheadrightarrow} B_0 \overset{g}{\to} B_0$$

$$B_1 = \{\ 0 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10\ \}$$

$$B_0 = \{\ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\ \}$$

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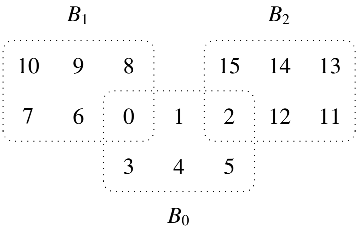
# Why does it work?

$$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$$




$$\alpha = |0,1,2|3,4,5|$$

$$\beta = |0,3|1,4|2,5|$$

$$\gamma = |0,4|1,5|2,3|$$

$$\delta = |0,5|1,3|2,4|$$

- $A = B_0 \cup B_1 \cup B_2$
- Unary operations

$$e_0\colon A \twoheadrightarrow B_0$$

$$e_1\colon A \twoheadrightarrow B_1$$

$$e_2\colon A \twoheadrightarrow B_2$$

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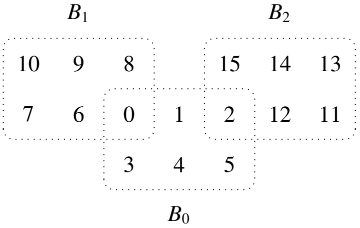


Con $\langle A, F_A\rangle$

$$\widehat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$$

$$\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|$$

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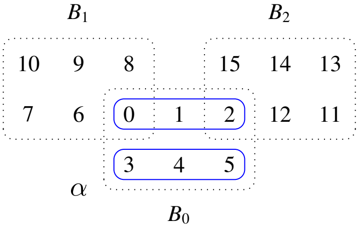


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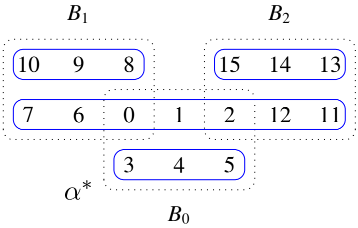


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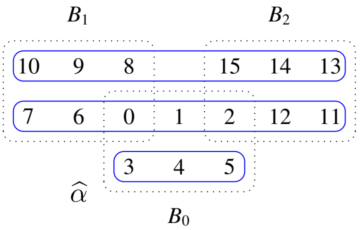


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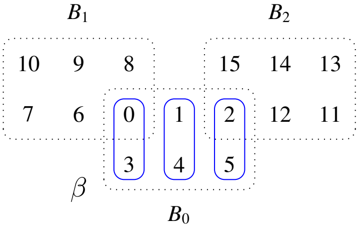


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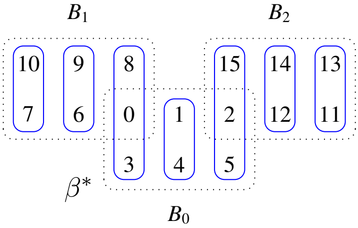


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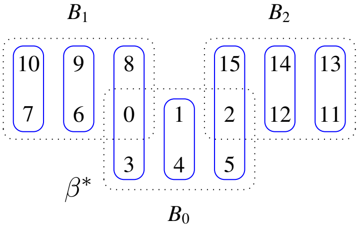


Con $\langle A, F_A\rangle$

*Why don't the $\beta$ classes of $B_1$ and $B_2$ mix?*

$$\widehat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$$

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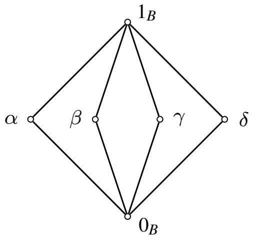
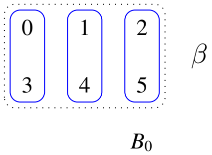
# Variations on the same example...

- Suppose we want $\beta = \mathrm{Cg}^{\mathbf{B}}(0,3) = |0,3|2,5|1,4|$ to have non-trivial inverse image $\beta|_{_B}^{-1} = [\beta^*, \widehat{\beta}]$.



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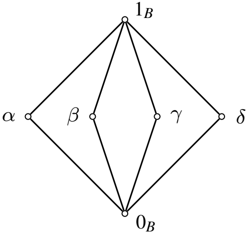


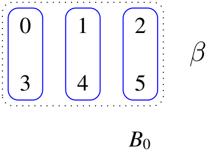

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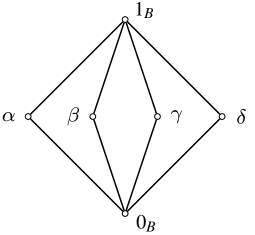


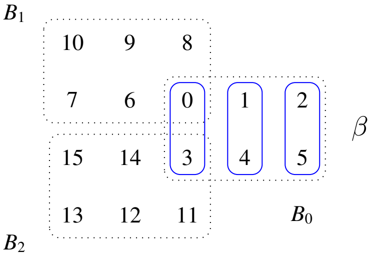

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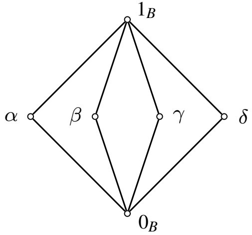


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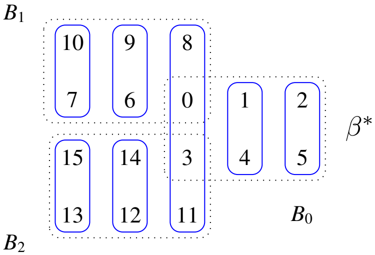


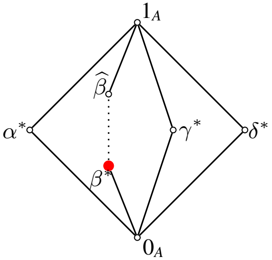

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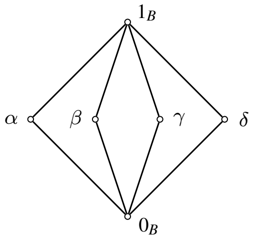


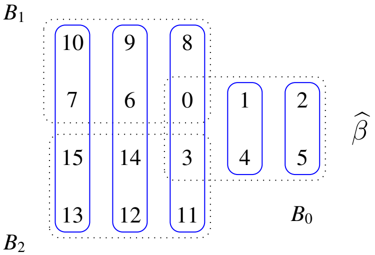


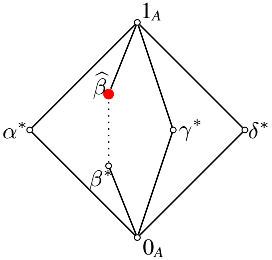

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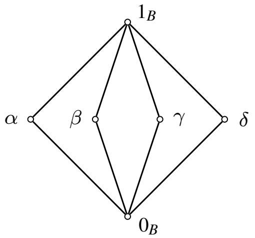


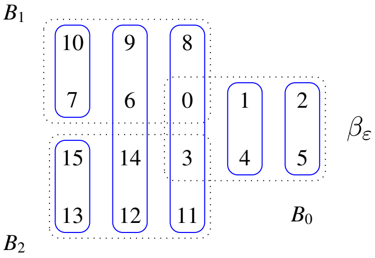


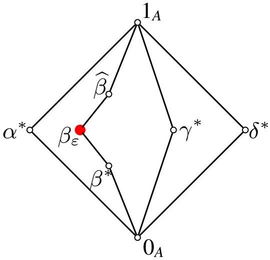

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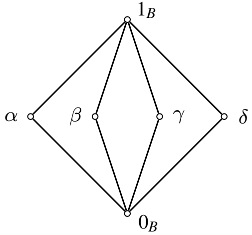


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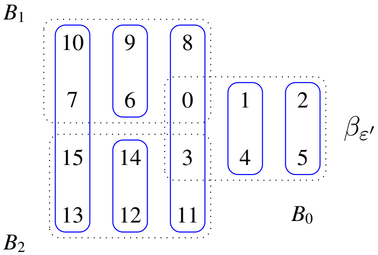


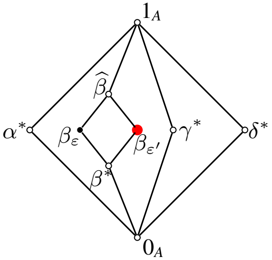

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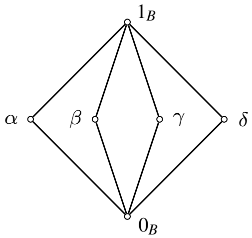
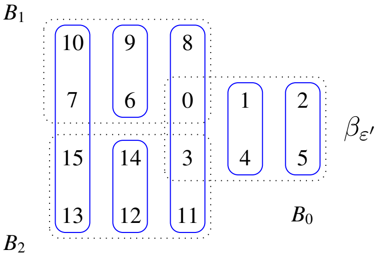
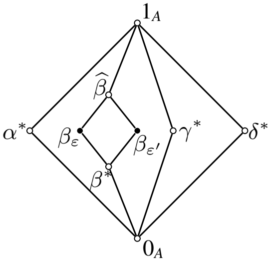
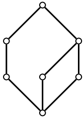


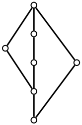



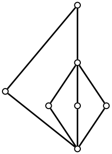
# Seven element lattices: summary



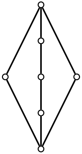
$L_{19}$



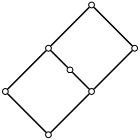
$L_{20}$



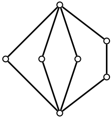
$L_{17}$



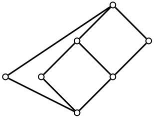
$L_{13}$



$L_{11}$



$L_9$



$L_7$

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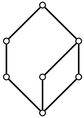
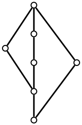
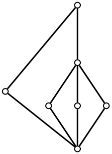


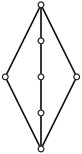
$L_{19}$



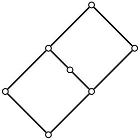
$L_{20}$



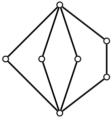
$L_{17}$



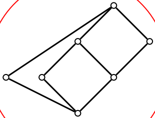
$L_{13}$



$L_{11}$



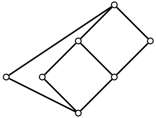
$L_{9}$



$L_{7}$

# Interval enforceable properties

- A minimal representation of $L_7$ must come from a transitive $G$-set.

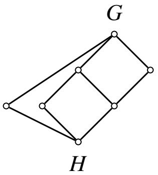


$L_7$

# INTERVAL ENFORCEABLE PROPERTIES

- A minimal representation of $L_7$ must come from a transitive $G$-set.
- Suppose $L_7 \cong [\![H, G]\!]$ for some finite groups $H < G$.

  What can we say about the group $G$?



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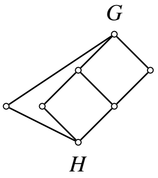

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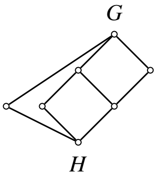


### PROPOSITION

Suppose $H < G, \quad \mathrm{core}_G(H) = 1, \quad L_7 \cong [\![H, G]\!]$.

(I) $G$ is a primitive permutation group.
(II) If $N \lhd G$, then $C_G(N) = 1$.
(III) $G$ contains no non-trivial abelian normal subgroup.
(IV) $G$ is not solvable.
(V) $G$ is subdirectly irreducible.
(VI) With the possible exception of at most one maximal subgroup, all proper subgroups in the interval $[\![H, G]\!]$ are core-free.

# OPEN PROBLEMS

1. Are homomorphic images of representable lattices representable?

2. Are subdirect products of representable lattices representable?

3. Does representable imply “group representable?”

   i.e., is the congruence lattice of a finite algebra isomorphic to an interval in the subgroup lattice of a finite group?

4. Is the class of representable lattices recursive?

# Remarks on the problems

- Are homomorphic images of representable lattices representable?

  If $L = \operatorname{Con}\langle A, F\rangle$ and $\tilde{L}$ is the homomorphic image $\{\theta \cap B^2 \mid \theta \in L\}$, where $B = e(A)$ for some operation $e^2 = e \in F$, then $\tilde{L} = \operatorname{Con}\langle B, F|_B\rangle$ .

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  This question asks whether it's possible to write a program that, when given a finite lattice $L$, halts with output True if $L$ is representable and False otherwise.

  A negative answer would solve the finite lattice representation problem.

One approach: try to find a computable function $f$ such that, if $L$ is a representable lattice of size $n$, then $L$ is representable as the congruence lattice of an algebra of cardinality $f(n)$ or smaller.

This would answer the decidability question, as follows:

```
IsRepresentable( L ) {

    n:= Size( L )
    N:= f( n )

    for each L′ in Sub[Eq(N)] {

        if ( L′ isomorphic to L ) and ( L′ is closed ):
            return True

    }

    return False

}
```