Representing Finite Lattices as Congruence Lattices

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http://math.hawaii.edu/∼ralph/

http://uacalc.org/

https://github.com/UACalc/

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The Problem

Theorem (Grätzer-Schmidt)

Every algebraic (so every finite) lattice is isomorphic to **Con** (**A**) *for some (unary) algebra* **A**.

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Since $\mathbf{Con}\,(\mathbf{A}) = \mathbf{Con}\,\langle A, \mathrm{Pol}_1(\mathbf{A})\rangle$, **we assume all algebras are unary.**

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$$(\mathrm{P4}) \Leftrightarrow (\mathrm{P3}) \Rightarrow (\mathrm{P2}) \Rightarrow (\mathrm{P1})$$

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Then **Con** $A \cong [H, G]$, the interval in the subgroup lattice.

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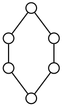
Example. The minimum size for $\mathbf{L}_6$ is 6:

$\mathbf{L}_6$ 

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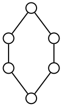

$\mathbf{B}_6$	0	1	2	3	4	5
$f(x)$	2	2	1	5	5	4
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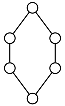

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Pálfy and Aschbacher have found groups $\mathbf{H} \leq \mathbf{G}$ representing this lattice. But Pálfy's example has $\mathbf{G} = \mathbf{A}_{11}$ and $|H| = 55$, so the size is $9! = 362880$.

Moral

Moral: Finding a representation with groups, (P4), may be much harder (and much bigger) than finding a (P1) representation.

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- overalgebras (DeMeo, 2013)

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$$(\mathrm{B}) \implies (\mathrm{B}'') \implies (\mathrm{B}').$$

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- *If* (A) *and* (B′) *hold and* $\mathbf{L} \cong \langle A, F \rangle$ *is minimal, then* F *consists of permutations and constants.*

Representations by instransitive groups

Suppose $\mathbf{A} = \langle A, G \rangle$ is a G-set and let $\mathbf{A}_i = \langle A_i, G \rangle$, $i < k$, be the minimal subalgebras of $\mathbf{A}$; i.e. each set A_i is an orbit, or one-generated subuniverse, of $\mathbf{A}$.

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Define congruences on A by the partitions

$$\tau = |A_0|A_1|\cdots|A_{k-1}| \qquad \text{(the blocks are the orbits)}$$
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*Let $\theta \in$ **Con** $(\mathbf{A})$, where $\mathbf{A} = \langle A, G \rangle$ and G is a group. Then*

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   For $\theta_i \in \mathbf{Con}\,(\mathbf{A}_i)$, map $(\theta_0, \ldots, \theta_{k-1}) \mapsto \theta_0 \cup \cdots \cup \theta_{k-1}$.

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   If $\theta \not\leq \gamma_i$, there are $a \in A_i$ and $b \notin A_i$ with $(a, b) \in \theta$. Since G acts transitively on each orbit, $\tau_i \leq \theta$. So $\tau \leq \theta$.

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   Follows from (3) and (4).

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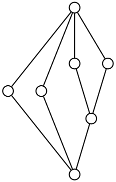
   From (8).

Examples: $\mathbf{L}_{14}$



Example

- $\mathbf{L}_{14}$ satisfies (A) and (B$''$) so a minimal representation is permutational.

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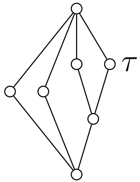

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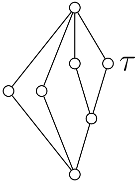

Example

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- $\mathbf{L}_{14} \cong \mathbf{Con}\,\langle A, G\rangle$ is not possible if G acts intransitively,

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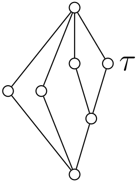

Example

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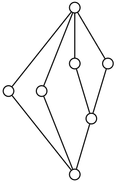
Examples: $\mathbf{L}_{14}$



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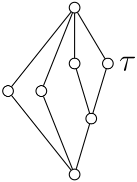

Example

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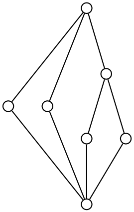
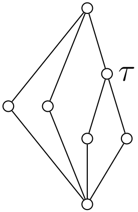
- if $\mathbf{Con}\, \langle A, F\rangle$ is a minimal representation, then F generates a transitive group.
- Is $\mathbf{L}_{14}$ representable? (Yes: as $[H, A_6]$ with $[A_6 : H] = 90$)
- Is this a minimum representation? (Don't know)

Examples: $\mathbf{L}_{15}$, (the dual of $\mathbf{L}_{14}$)



Examples: $\mathbf{L}_{15}$, (the dual of $\mathbf{L}_{14}$)



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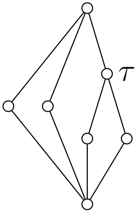

Example

- $\mathbf{L}_{15} \cong \mathbf{Con}\ \langle\{0, 1, 2, 3\}, G\rangle$, G the group generated by the double transposition $0 \leftrightarrow 1$, $2 \leftrightarrow 3$.

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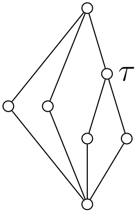
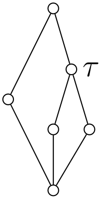


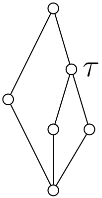
Example

- $\mathbf{L}_{15} \cong \mathbf{Con}\,\langle\{0,1,2,3\}, G\rangle$, G the group generated by the double transposition $0 \leftrightarrow 1$, $2 \leftrightarrow 3$.
- $\mathbf{L}_{14} \cong \mathbf{L}_{15}^{d}$, which again proves $\mathbf{L}_{14}$ is representable.

Examples: $\mathbf{L}_4$.



Examples: $\mathbf{L}_4$.



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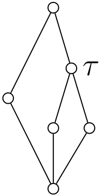
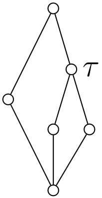


Example

- $\mathbf{L}_4$ satisfies (B″) but not (A) so minimal representations need not be permutational. In fact
- $\mathbf{L}_4 \cong \langle\{0, 1, 2, 3\}, f, g\rangle$, where

$\mathbf{B}_4$	0	1	2	3
$f(x)$	1	0	3	2
$g(x)$	0	0	2	2

Examples: $\mathbf{L}_4$.

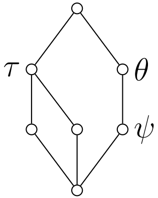


Example

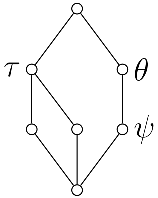
- $\mathbf{L}_4$ satisfies (B″) but not (A) so minimal representations need not be permutational.
- But $\mathbf{L}_4$ does have an intransitive representation on 6:

$\mathbf{B}_4'$	0	1	2	3	4	5
$f(x)$	1	2	0	4	5	3
$g(x)$	0	2	1	3	5	4

Examples: $\mathbf{L}_{19}$, a harder example:



Examples: $\mathbf{L}_{19}$, a harder example:



Lemma

Let $\mathbf{A} = \langle A, G \rangle$ be a finite algebra, where G is an intransitive group of permutations on A. Suppose the intransitivity congruence τ is a coatom. Then there do not exist congruences $0_{\mathbf{A}} < \psi < \theta$ in $\mathbf{Con}\,(\mathbf{A})$ with $\theta \wedge \tau = 0_{\mathbf{A}}$.

Proof

Lemma

Let $\mathbf{A} = \langle A, G\rangle$ *be a finite algebra, where* G *is an intransitive group of permutations on* A. *Suppose the intransitivity congruence* τ *is a coatom. Then there do not exist congruences* $0_{\mathbf{A}} < \psi < \theta$ *in* $\mathbf{Con}\,(\mathbf{A})$ *with* $\theta \wedge \tau = 0_{\mathbf{A}}$.

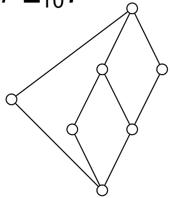
Proof.

Since τ is a coatom, there are exactly two orbits; call them B and C. Since $\theta \wedge \tau = 0_{\mathbf{A}}$, if $(x, y) \in \theta$ then $x = y$ or one is in B and the other is in C. So θ defines a bipartite graph between B and C. Since G acts transitively on both B and C, this graph corresponds to a bijection between B and C. The same applies to ψ. But equivalence relations corresponding to such graphs cannot be comparable. □

Small Lattices

Theorem

All lattices with at most 7 elements can be represented, with the one possible exception of $\mathbf{L}_{10}$*:*



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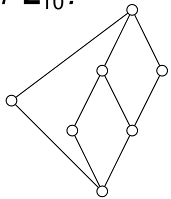

If $\mathbf{L}_{10} \cong \langle A, F \rangle$*, then* F *generates a transitive group on* A*.*

Small Lattices

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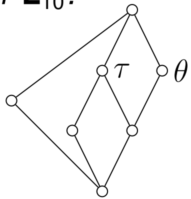

If $\mathbf{L}_{10} \cong \langle A, F \rangle$*, then* F *generates a transitive group on* A*.*

Proof.

$\mathbf{L}_{10}$ satisfies (A) and (B″). By part (5) of the intransitivity theorem, it cannot be represented with an intransitive group. □

Finding Reps: Methods and Algorithms

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- Closure Method

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(2) For each sublattice $\mathbf{L} \cong \mathbf{L}' \leq \mathrm{Eq}(X_k)$ found, find the unary polymorphs of the members of L'; that is, calculate the set F of all unary operations on X_k which respect all $\theta \in L'$.

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(3) For F found in the previous step, test if $\mathbf{Con}\,(\langle X_k, F\rangle) = L'$. If so then $\mathbf{A} = \langle X_k, F\rangle$ is a minimal representation. Otherwise continue the search.

Remarks

(a) **Find a small presentation of L:**
The procedure can be sped up by first finding a presentation of **L** with the minimal number of generators. Besides speeding up the search in $\mathrm{Eq}(k)$, it is enough in calculating the unary polymorphs to respect the generators.

Remarks

(b) **Subdirect Decompositions:**
Subdirect decompositions can be used to speed up finding unary polymorphs. For example, if $\theta_0, \theta_1 \in L' \leq \mathrm{Eq}(X_k)$ with $\theta_0 \wedge \theta_1 = 0$, then X_k is naturally embedded into $X_k/\theta_0 \times X_k/\theta_1$. Since the operations in a direct product are component-wise, this cuts the search space of possible unary polymorphs from k^k down to $r^r s^s$, where r and s are the number of blocks in θ_0 and θ_1.

Remarks

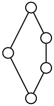
(c) **Uniform Equivalence Relations:**
If it can be shown that the algebra of a minimal representation of **L** has a transitive permutation group for its nonconstatant unary polynomials, then we can restrict our search in Eq(k) to uniform equivalence relations. Moreover the search for unary polymorphs can be restricted to permutations.

Remarks

(d) **Small generating set for the operations:**
Of course if $F' \subseteq F$ is a set of generators for the moniod F, we can take $\mathbf{A} = \langle X_k, F' \rangle$.

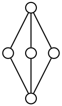
Nondist., linearly indec., small lattices

$\mathbf{L}_1$



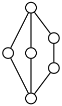
$\mathbf{B}_1$	0	1	2	3
$f(x)$	1	0	3	2
$g(x)$	1	0	1	0

$\mathbf{L}_2$



$\mathbf{B}_2$	0	1	2
$f(x)$	0	1	2

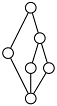
$\mathbf{L}_3$



$\mathbf{B}_3$	0	1	2	3	4	5	6
$f(x)$	0	1	2	1	2	1	0
$g(x)$	0	3	4	3	4	3	0
$h(x)$	6	5	2	5	2	5	6
$k(x)$	0	1	2	0	0	2	2

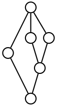
Method: overalgebras

$\mathbf{L}_4$



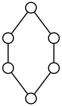
$\mathbf{B}_4$	0	1	2	3
$f(x)$	1	0	3	2
$g(x)$	0	0	2	2

$\mathbf{L}_5$



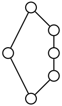
$\mathbf{B}_5$	0	1	2	3	4	5	6	7	8	9	10	11
$f(x)$	1	2	3	4	5	0	7	8	9	10	11	6
$g(x)$	6	11	10	9	8	7	0	5	4	3	2	1
$h(x)$	0	0	0	6	0	0	0	0	6	0	0	0

$\mathbf{L}_6$



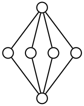
$\mathbf{B}_6$	0	1	2	3	4	5
$f(x)$	2	2	1	5	5	4
$g(x)$	3	4	4	0	1	1
$h(x)$	4	5	3	4	5	3

$\mathbf{L}_7$



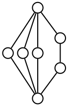
$\mathbf{B}_7$	0	1	2	3	4	5
$f(x)$	1	0	0	4	3	3
$g(x)$	4	5	5	1	2	2
$h(x)$	3	3	4	3	3	4

$\mathbf{L}_8$



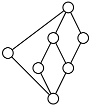
$\mathbf{B}_8$	0	1	2	3	4	5
$f(x)$	1	2	0	4	5	3
$g(x)$	3	5	4	0	2	1

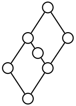
$\mathbf{L}_9$

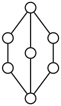


$\mathbf{B}_9$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(x)$	0	0	0	0	0	0	2	1	2	1	3	4	5	3	4	5
$g(x)$	0	0	0	0	0	0	6	7	6	7	10	11	12	10	11	12
$h(x)$	13	14	15	1	9	8	15	14	13	15	1	9	8	8	1	9

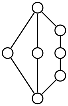
Method: overalgebras

$\mathbf{L}_{10}$  **No finite algebra known with this as its congruence lattice.**

$\mathbf{L}_{11}$  A finite algebra with 108 elements known.

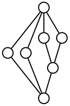
$\mathbf{L}_{12}$ 

$\mathbf{B}_{12}$	0	1	2	3	4	5	6	7	8
$f(x)$	0	0	3	3	3	6	6	6	0
$g(x)$	0	0	8	8	8	1	1	1	0
$h(x)$	0	5	5	4	0	0	5	4	4
$k(x)$	4	2	2	3	4	4	2	3	3
$l(x)$	5	5	7	7	7	6	6	6	5

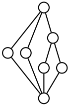
$\mathbf{L}_{13}$ 

$\mathbf{B}_{13}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$f(x)$	0	1	2	1	2	1	0	0	1	2	2	1	0	0	1	2	1	2	0
$g(x)$	0	1	2	0	0	2	2	0	3	4	0	4	4	6	5	2	6	6	2
$h(x)$	0	1	2	3	4	5	6	0	1	2	4	5	6	0	1	2	3	4	6
$k(x)$	7	8	9	3	10	11	12	3	3	3	3	3	3	11	11	11	11	11	11
$l(x)$	13	14	15	16	17	5	18	13	16	17	17	16	13	5	5	5	5	5	5

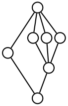
Method: overalgebras

$\mathbf{L}_{14}$ 

Upper interval in Sub(A_6),
algebra of size 90

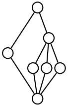
$\mathbf{L}_{15}$ 

$\mathbf{B}_{15}$	0	1	2	3
$f(x)$	1	0	3	2

$\mathbf{L}_{16}$ 

Upper interval in Sub($C_2.A_6$)
algebra of size 180

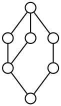
$\mathbf{L}_{17}$



$\mathbf{B}_{17}$	0	1	2	3	4	5	6	7	8	9	10	11
$f(x)$	1	0	3	2	5	4	7	6	9	8	11	10
$g(x)$	4	7	5	6	8	11	9	10	0	3	1	2
$h(x)$	0	0	0	0	5	5	5	5	10	10	10	10

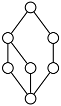
Method: filter-ideal in Sub(A_4)

$\mathbf{L}_{18}$



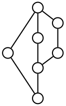
Dual of 19, no explicit
small representation known

$\mathbf{L}_{19}$



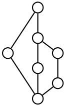
$\mathbf{B}_{19}$	0	1	2	3	4	5	6	7
$f(x)$	0	1	1	0	4	5	5	4
$g(x)$	0	2	3	1	0	2	3	1
$h(x)$	7	6	6	7	3	2	2	3

$\mathbf{L}_{20}$



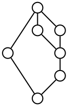
Method: filter-ideal in SmallGroup(216,153) in GAP

$\mathbf{L}_{21}$



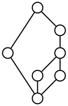
$\mathbf{B}_{21}$	0	1	2	3	4	5	6	7	8
$f(x)$	3	3	4	8	8	2	2	3	4
$g(x)$	0	0	6	1	1	0	0	5	6
$h(x)$	4	5	5	7	8	8	7	4	4

$\mathbf{L}_{22}$



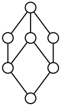
Dual of 23, no explicit small representation known

$\mathbf{L}_{23}$



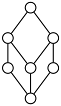
$\mathbf{B}_{23}$	0	1	2	3	4	5
$f(x)$	0	1	0	1	4	4
$g(x)$	1	1	3	3	4	5
$h(x)$	3	2	3	2	5	5
$k(x)$	4	1	5	3	4	5

$\mathbf{L}_{24}$



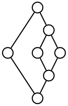
$\mathbf{B}_{24}$	0	1	2	3
$f(x)$	1	1	2	2
$g(x)$	2	3	3	2

$\mathbf{L}_{25}$



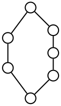
$\mathbf{B}_{25}$	0	1	2	3	4
$f(x)$	0	0	2	2	2
$g(x)$	0	1	0	1	1
$h(x)$	1	1	4	4	4
$k(x)$	2	3	2	3	3

$\mathbf{L}_{26}$



$\mathbf{B}_{26}$	0	1	2	3	4	5
$f(x)$	1	0	3	2	0	2
$g(x)$	4	4	5	5	1	3
$h(x)$	0	0	0	0	1	1
$k(x)$	3	5	3	5	3	3

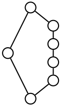
$\mathbf{L}_{27}$



$\mathbf{B}_{27}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(x)$	0	1	2	3	4	5	0	0	0	0	0	2	2	2	2	2
$g(x)$	4	5	3	4	5	3	5	3	4	5	3	4	5	4	5	3
$h(x)$	2	2	1	5	5	4	2	1	5	5	4	2	2	5	5	4
$k(x)$	3	4	4	0	1	1	4	4	0	1	1	3	4	0	1	1
$l(x)$	0	6	7	8	9	10	6	7	8	9	10	0	6	8	9	10
$m(x)$	11	12	2	13	14	15	12	2	13	14	15	11	12	13	14	15

Method: overalgebras

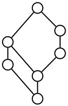
$\mathbf{L}_{28}$



$\mathbf{B}_{28}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(x)$	0	1	2	3	4	5	0	0	0	0	0	2	2	2	2	2
$g(x)$	3	3	4	3	3	4	3	4	3	3	4	3	3	3	3	4
$h(x)$	1	0	0	4	3	3	0	0	4	3	3	1	0	4	3	3
$k(x)$	4	5	5	1	2	2	5	5	1	2	2	4	5	1	2	2
$l(x)$	0	6	7	8	9	10	6	7	8	9	10	0	6	8	9	10
$m(x)$	11	12	2	13	14	15	12	2	13	14	15	11	12	13	14	15

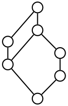
Method: overalgebras

$\mathbf{L}_{29}$



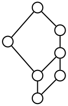
$\mathbf{B}_{29}$	0	1	2	3	4
$f(x)$	1	0	3	2	2
$g(x)$	2	4	2	4	3

$\mathbf{L}_{30}$



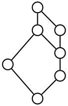
$\mathbf{B}_{30}$	0	1	2	3	4
$f(x)$	0	3	4	3	4
$g(x)$	2	2	1	4	3

$\mathbf{L}_{31}$



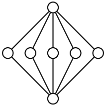
$\mathbf{B}_{31}$	0	1	2	3	4
$f(x)$	0	1	1	0	0
$g(x)$	1	1	2	2	2
$h(x)$	3	2	2	4	4

$\mathbf{L}_{32}$



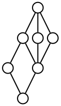
$\mathbf{B}_{32}$	0	1	2	3	4
$f(x)$	0	1	1	3	3
$g(x)$	1	2	2	4	4
$h(x)$	3	3	4	3	4

$\mathbf{L}_{33}$



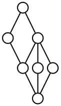
$\mathbf{B}_{33}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(x)$	1	3	2	0	9	11	10	8	13	15	14	12	5	7	6	4
$g(x)$	11	8	10	9	7	4	6	5	15	12	14	13	3	0	2	1
$h(x)$	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1

$\mathbf{L}_{34}$



$\mathbf{B}_{34}$	0	1	2	3
$f(x)$	0	1	3	2

$\mathbf{L}_{35}$



$\mathbf{B}_{35}$	0	1	2	3
$f(x)$	1	1	2	3
$g(x)$	2	3	3	3

Problem

- What about nonunary algebras?

Problem

- What about nonunary algebras?

Problem

Which (finite) lattices can be represented as **Con A**, *where* **A** *has a Taylor term?*

Resources

- SmallLatticeReps.ua, a UACalc file with most of the $\mathbf{B}_i$'s:

  `http://math.hawaii.edu/~ralph/`

- The slides of this talk are there too.
- SmallLatticeReps.ua and other algebra files:

  `https://github.com/UACalc/SmallAlgebras/`

- Get UACalc at

  `http://uacalc.org/`

- Source at

  `https://github.com/UACalc/uacalcsrc/`