EVERY FINITE LATTICE IS THE CONGRUENCE LATTICE OF A FINITE PARTIAL ALGEBRA

1. Introduction

This note begins with a proof in Section 2 of the result stated in the title. Bill Lampe [1] pointed out that this result has been known for a long time and subsequently explained to DeMeo a more general viewpoint. In Section 3 we give some background about closure operators and then reiterate what Lampe explained. The presentation is imperfect, but hopefully it will elicit comments and criticisms that we can use to improve it.

2. A well known result

In this section we give a straight-forward proof of the fact that every finite lattice is the congruence lattice of a finite partial algebra.

Lemma 2.1. *Let X be a finite set, and let $\mathrm{Eq}(X)$ denote the lattice of equivalence relations on X. If $L \leq \mathrm{Eq}(X)$ is a 0-1-sublattice, and $\rho \in \mathrm{Eq}(X)$ and $\rho \notin L$, then for some $k < \omega$ there exists a partial operation $f\colon X^k \rightharpoonup X$ that is compatible with L and incompatible with ρ.*

Proof. First we focus on the relations in L that are above ρ. Let $\rho^{\uparrow} \cap L = \{\gamma \in L \mid \gamma \geq \rho\}$. Since $\rho \notin L$, we have $\gamma > \rho$ for all $\gamma \in \rho^{\uparrow} \cap L$. Now, $\rho^{\uparrow} \cap L$ has a least element $\rho^* = \bigwedge(\rho^{\uparrow} \cap L)$. Clearly $\rho^* \geq \rho$ and since $\rho^* \in L$ we have $\rho^* \neq \rho$, so $\rho^* > \rho$. Therefore, there exists $(u, v) \in \rho^* - \rho$.

Next consider the elements of L that are not above ρ. For each such $\alpha_i \in L - \rho^{\uparrow}$ there exists $(x_i, y_i) \in \rho - \alpha_i$. Let $(x_1, y_1), \dots, (x_k, y_k)$ be the list of all unique such pairs (i.e., each pair appears in the list exactly once). Define the partial function $f\colon X^k \rightharpoonup X$ at only two points of X^k; specifically, let

$$f(x_1, \dots, x_k) = u \quad \text{and} \quad f(y_1, \dots, y_k) = v.$$

Then, since $(\forall i)(x_i, y_i) \in \rho$ and $(u, v) \notin \rho$, f is incompatible with ρ. On the other hand, $(u, v) \in \rho^* = \bigwedge(\rho^{\uparrow} \cap L)$, so $(u, v) \in \gamma$ for every $\gamma \in \rho^{\uparrow} \cap L$, so f is compatible with every $\gamma \in \rho^{\uparrow} \cap L$.

Finally, for each $\alpha_i \in L$ not above ρ there is at least one pair $(x_i, y_i) \notin \alpha_i$. Therefore, it is impossible for f to be incompatible with any such α_i. $\square$

Theorem 2.2. *Let X be a finite set and let $L \leq \mathrm{Eq}(X)$ be a 0-1-sublattice. Then there exists a finite partial algebra $\mathbb{X} = \langle X, F\rangle$ with $\mathrm{Con}(\mathbb{X}) = L$.*

Proof. By the lemma, for each $\rho \in \mathrm{Eq}(X) - L$, there exists $k < \omega$ and $f_\rho\colon X^k \rightharpoonup X$ such that f_ρ is compatible with every relation in L and incompatible with ρ. Let $\mathcal{R}$ be the set $\mathrm{Eq}(X) - L$ of all equivalence relations on X that do not belong to L. Define, $F = \{f_\rho \mid \rho \in \mathcal{R}\}$. Evidently, $\mathrm{Con}\langle X, F\rangle = L$. $\square$

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3. Generalities

3.1. **Closure systems, closure operators, and Moore families.** First we recall some standard definitions. A ***closure system*** on a set X is a collection $\mathcal{C}$ of subsets of X that is closed under arbitrary intersection (including the empty intersection, so $\bigcap \emptyset = X \in \mathcal{C}$). Thus a closure system is a complete meet semilattice with respect to subset inclusion ordering. Since every complete meet semilattice is automatically a complete lattice (see [2, Theorem 2.5]), the closed sets of a closure system form a complete lattice.

Examples of closure systems that are especially relevant for our work are the following:

- order ideals of an ordered set
- subalgebras of an algebra
- equivalence relations on a set
- congruence relations of an algebra

Let $\mathbf{P} = \langle P, \leq \rangle$ be a poset. An function $\mathsf{C} \colon P \to P$ is called a ***closure operator*** on $\mathbf{P}$ if it satisfies the following axioms for all $x, y \in P$.

(1) $x \leq \mathsf{C}\, x$ (extensivity)
(2) $x \leq y$ implies $\mathsf{C}\, x \leq \mathsf{C}\, y$ (monotonicity)
(3) $\mathsf{C}\,\mathsf{C}\, x = \mathsf{C}\, x$ (idempotence)

Thus, a closure operator is an extensive idempotent poset endomorphism, and the definition above is equivalent to the single axiom $(\forall\, x, y \in P)(x \leq \mathsf{C}\, y \longleftrightarrow \mathsf{C}\, x \leq \mathsf{C}\, y)$.

Example 3.1. Let X be a set and let $Y \subseteq X$. Define $\mathsf{C}_Y \colon \mathcal{P}(X) \to \mathcal{P}(X)$ by $\mathsf{C}_Y(W) = W \cup Y$. Then C_Y is a closure operator on $\langle \mathcal{P}(X), \subseteq \rangle$.

A ***fixpoint*** of a function $\mathsf{C} : P \to P$ is an $x \in P$ satisfying $\mathsf{C}\, x = x$. A fixpoint of a closure operator is called ***closed***. If the poset $\mathbf{P} = \langle P, \leq \rangle$ happens to be a complete lattice, then by extensivity the largest element $\top = \bigvee P$ is a fixpoint of every closure operator on $\mathbf{P}$. Also, the collection of fixpoints of a closure operator is closed under arbitrary meets. (Proof: If $\mathcal{A}$ is a set of fixpoints of C and $a \in \mathcal{A}$, then $\bigwedge \mathcal{A} \leq a$, so by monotonicity $\mathsf{C}\big(\bigwedge \mathcal{A}\big) \leq \mathsf{C}\, a = a$. Since a was arbitrary, $\mathsf{C}\big(\bigwedge \mathcal{A}\big) \leq \bigwedge \mathcal{A}$. By extensivity, $\bigwedge \mathcal{A} \leq \mathsf{C}\big(\bigwedge \mathcal{A}\big)$. Therefore, $\mathsf{C}\big(\bigwedge \mathcal{A}\big) = \bigwedge \mathcal{A}$.) The set of closure operators on $\mathbf{P}$ themselves form a complete lattice under the pointwise order: $\mathsf{C}_1 \leq \mathsf{C}_2$ iff $\mathsf{C}_1\, x \leq \mathsf{C}_2\, x$ for all $x \in P$.

Some of these observations can be restated as follows: if $\mathbf{P} = \langle P, \leq \rangle$ is a complete lattice, then the set $\mathcal{C} \subseteq P$ of fixedpoints of a closure operator is a ***Moore family*** on $\mathbf{P}$—that is, $\bigvee P \in \mathcal{C}$ and every nonempty subset of $\mathcal{C}$ is closed under arbitrary meets.

Conversely, if we are given a Moore family $\mathcal{C}$ on $\mathbf{P}$, and if we define $\mathsf{C} \colon P \to P$ by

$$\mathsf{C}\, a = \bigwedge \{ b \in \mathcal{C} \mid a \leq b \},$$

then C satisfies conditions (1)–(3) above, making it a closure operator. To summarize, $\mathcal{C} \subseteq P$ is the set of fixpoints (i.e., closed elements) of a closure operator on $\mathbf{P}$ if and only if $\mathcal{C}$ is a Moore family on $\mathbf{P}$.

Every Moore family on $\mathbf{P}$ is itself the universe of a complete lattice with the order inherited from $\mathbf{P}$, though the join may differ from the join of $\mathbf{P}$.

3.1.1. *More Moore families.* The name "closure system" is typically reserved for the special case in which $\mathbf{P}$ happens to be the powerset Boolean algebra of a set X—that is, $\mathbf{P} = \langle \mathcal{P}(X), \subseteq \rangle$; in that case, a Moore family on $\mathbf{P}$ is called a closure system on X.

$$\begin{tikzcd}
L \arrow[rr, "\phi"] \arrow[dr, "\sigma_{u,v}"'] & & K \\
& L_{u,v} \arrow[ur, dashed, "\exists\,\psi"'] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

FIGURE 1. Every join-homomorphism collapsing u and v is divisible by $\sigma_{u,v}$.

Example 3.2. Let X be a set, let $\mathrm{Eq}(X)$ denote the lattice of equivalence relations on X, and let $\mathsf{O}(X) = \bigcup_{n<\omega} X^{X^n}$ be the set of all operations on X. Define $F : \mathcal{P}(\mathrm{Eq}(X)) \to \mathcal{P}(\mathsf{O}(X))$ and $G : \mathcal{P}(\mathsf{O}(X)) \to \mathcal{P}(\mathrm{Eq}(X))$ as follows: if $A \subseteq \mathrm{Eq}(X)$ and $B \subseteq \mathsf{O}(X)$, then

$$F(A) = \{f \in \mathsf{O}(X) \mid f \text{ is compatible with every relation in } A\}$$

and

$$G(B) = \{\alpha \in \mathrm{Eq}(X) \mid \alpha \text{ is compatible with every operation in } B\}$$

Then $G \circ F$ is a closure operator.

Example 3.3 (Pudlák-Tůma [3])**.** Let $\mathbf{L} = \langle L, \vee, \wedge\rangle$ be a lattice, $u, v \in L$, and $u \leq v$. Define a subset $L_{u,v}$ of L by $L_{u,v} = \{x \in L \mid v \leq x \text{ or } u \nleq x\}$. The partial order relation of the lattice $\mathbf{L}$ induces a lattice order on $L_{u,v}$. Denote the resulting lattice by $\mathbf{L}_{u,v}$. Then the meet of $\mathbf{L}_{u,v}$ is that of $\mathbf{L}$, whereas the join of $\mathbf{L}_{u,v}$ is

$$x \vee_{u,v} y = \begin{cases} x \vee y, & \text{if } u \nleq x \vee y, \\ x \vee y \vee v, & \text{if } u \leq x \vee y. \end{cases}$$

Define a mapping $\sigma_{u,v} \colon L \to L_{u,v}$ as follows:

$$\sigma_{u,v}(x) = \begin{cases} x, & \text{if } u \nleq x, \\ x \vee v, & \text{if } u \leq x. \end{cases}$$

Then $\sigma_{u,v}$ is a surjective join-homomorphism. In fact, every join-homomorphism $\phi \colon L \to K$ satisfying $\phi(u) = \phi(v)$ splits as $\phi = \psi \circ \sigma_{u,v}$ for some $\psi \colon L_{u,v} \to K$. (See the commutative diagram in Figure 1.)

As a mapping from $\mathbf{L}$ to itself, $\sigma_{u,v}$ does not preserve joins. However, $\sigma_{u,v} \colon L \to L$ is a closure operator and $L_{u,v}$ is the set of fixpoints of $\sigma_{u,v}$ (the closed sets).

3.2. Algebraicity.

A subset D of an ordered set P is called ***up-directed*** if for every $x, y \in D$ there exists $z \in D$ such that $x \leq z$ and $y \leq z$. A closure operator $\mathsf{C} : \mathcal{P}(X) \to \mathcal{P}(X)$ is called ***algebraic*** provided, for all $A \subseteq X$,

$$\mathsf{C}\, A = \bigcup\{\mathsf{C}\, F \mid F \subseteq A \text{ and } F \text{ finite }\}.$$

The collection $\mathcal{C}$ of closed sets of an algebraic closure operator is called an ***algebraic closure system***.

Theorem 3.4 (cf. [2] Thm 3.1)**.** *Let $\mathcal{C}$ be the closure system of fixed points of the closure operator $\mathsf{C} : \mathcal{P}(X) \to \mathcal{P}(X)$. The following are equivalent:*

(1) C *is an algebraic closure operator*
(2) $\mathcal{C}$ *is an algebraic closure system*
(3) If $D \subseteq \mathcal{C}$ *is up-directed and* $C \subseteq D$ *is a chain, then* $\bigcup C$ *is closed.*
(4) If $D \subseteq \mathcal{C}$ *is up-directed, then* $\bigcup D$ *is closed.*
(5) If $C \subseteq \mathcal{C}$ *is a chain, then* $\bigcup C$ *is closed.*

References

[1] Bill Lampe. personal communication. October 17. 2016.
[2] J. B. Nation. "Notes on Lattice Theory". Unpublished notes. 2007, 2016. URL: http://www.math.hawaii.edu/~jb/.
[3] Pavel Pudlák and Jiří Tůma. "Every finite lattice can be embedded in a finite partition lattice". In: *Algebra Universalis* 10.1 (1980), pp. 74–95. ISSN: 0002-5240. DOI: 10.1007/BF02482893. URL: http://dx.doi.org/10.1007/BF02482893.

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