EVERY FINITE LATTICE IS THE CONGRUENCE LATTICE OF A FINITE PARTIAL ALGEBRA

1. Introduction

This notes starts with a proof in Section 2 of the result stated in the title. Bill Lampe informed us that this result has been known for a long time and subsequently explained to DeMeo a more general view of the result. That view is developed in Section 3 below.

2. A simple proof of a well known result

Lemma 2.1. *Let X be a finite set, and let $\operatorname{Eq}(X)$ denote the lattice of equivalence relations on X. If $L \leq \operatorname{Eq}(X)$ is a 0-1-sublattice, and $\rho \in \operatorname{Eq}(X)$ and $\rho \notin L$, then for some $k < \omega$ there exists a partial operation $f \colon X^k \rightharpoonup X$ that is compatible with L and incompatible with ρ.*

Proof. First we focus on the relations in L that are above ρ. Let $\rho^{\uparrow} \cap L = \{\gamma \in L \mid \gamma \geq \rho\}$. Since $\rho \notin L$, we have $\gamma > \rho$ for all $\gamma \in \rho^{\uparrow} \cap L$. Now, $\rho^{\uparrow} \cap L$ has a least element $\rho^* = \bigwedge(\rho^{\uparrow} \cap L)$. Clearly $\rho^* \geq \rho$ and since $\rho^* \in L$ we have $\rho^* \neq \rho$, so $\rho^* > \rho$. Therefore, there exists $(u, v) \in \rho^* - \rho$.

Next consider the elements of L that are not above ρ. For each such $\alpha_i \in L - \rho^{\uparrow}$ there exists $(x_i, y_i) \in \rho - \alpha_i$. Let $(x_1, y_1), \dots, (x_k, y_k)$ be the list of all unique such pairs (i.e., each pair appears in the list exactly once). Define the partial function $f \colon X^k \rightharpoonup X$ at only two points of X^k; specifically, let

$$f(x_1, \dots, x_k) = u \quad \text{and} \quad f(y_1, \dots, y_k) = v.$$

Then, since $(\forall i)(x_i, y_i) \in \rho$ and $(u, v) \notin \rho$, f is incompatible with ρ. On the other hand, $(u, v) \in \rho^* = \bigwedge(\rho^{\uparrow} \cap L)$, so $(u, v) \in \gamma$ for every $\gamma \in \rho^{\uparrow} \cap L$, so f is compatible with every $\gamma \in \rho^{\uparrow} \cap L$.

Finally, for each $\alpha_i \in L$ not above ρ there is at least one pair $(x_i, y_i) \notin \alpha_i$. Therefore, it is impossible for f to be incompatible with any such α_i. □

Theorem 2.2. *Let X be a finite set and let $L \leq \operatorname{Eq}(X)$ be a 0-1-sublattice. Then there exists a finite partial algebra $\mathbb{X} = \langle X, F \rangle$ with $\operatorname{Con}(\mathbb{X}) = L$.*

Proof. By the lemma, for each $\rho \in \operatorname{Eq}(X) - L$, there exists $k < \omega$ and $f_\rho \colon X^k \rightharpoonup X$ such that f_ρ is compatible with every relation in L and incompatible with ρ. Let $\mathcal{R}$ be the set $\operatorname{Eq}(X) - L$ of all equivalence relations on X that do not belong to L. Define, $F = \{f_\rho \mid \rho \in \mathcal{R}\}$. Evidently, $\operatorname{Con}\langle X, F \rangle = L$. □

3. More general view

First we recall some standard definitions.[1] A *closure system* on a set X is a collection $\mathcal{C}$ of subsets of X that is closed under arbitrary intersection (including the empty intersection, so $\bigcap \emptyset = X \in \mathcal{C}$). Thus a closure system is a complete meet semilattice with respect to subset inclusion ordering. Since every complete meet semilattice is automatically a complete lattice (see [1, Theorem 2.5]), the closed sets of a closure system form a complete lattice.

For us, some key examples of closure systems are the

- the order ideals of an ordered set
- the subalgebras of an algebra
- the equivalence relations on a set

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[1]See J. B. Nation's notes [1], from which most of this section was directly lifted.

- the congruence relations of an algebra

A *closure operator* on a set X is a map $\mathsf{C} : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying, for all $A, B \in \mathcal{P}(X)$,

(a) $A \subseteq \mathsf{C}A$ (extensivity)
(b) $A \subseteq B$ implies $\mathsf{C}A \subseteq \mathsf{C}B$ (monotonicity)
(c) $\mathsf{C}\mathsf{C}A = \mathsf{C}A$ (idempotence)

A fixed point of a closure operator is called a *closed set*. The collection of closed sets of a closure operator is closed under arbitrary intersection, hence forms a closure system. Indeed, let $\mathcal{A}$ be a collection of fixed points of C. For each $A \in \mathcal{A}$ we have $\bigcap \mathcal{A} \subseteq A$, so $\mathsf{C}(\bigcap \mathcal{A}) \subseteq A$ by monotonicity. Therefore, $\bigcap \mathcal{A} \subseteq \mathsf{C}(\bigcap \mathcal{A}) \subseteq \bigcap \mathcal{A}$, which proves that $\bigcap \mathcal{A}$ is a fixed point of C.

A closure operator C is called *finitary* or *algebraic* iff

$$\mathsf{C}(A) = \bigcup \{\mathsf{C}(F) \mid F \subseteq A, F \text{ finite } \}.$$

References

[1] J. B. Nation. "Notes on Lattice Theory". Unpublished notes. 2007, 2016. URL: `http://www.math.hawaii.edu/~jb/`.

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