EVERY FINITE LATTICE IS THE CONGRUENCE LATTICE OF A FINITE PARTIAL ALGEBRA

1. Introduction

This note begins with a proof in Section 2 of the result stated in the title. Bill Lampe [3] pointed out that this result has been known for a long time and subsequently explained to DeMeo a more general viewpoint. In Section 3 we give some background about closure operators and then reiterate what Lampe explained. The presentation is imperfect, but hopefully it will elicit comments and criticisms that we can use to improve it.

2. A well known result

In this section we give a straight-forward proof of the fact that every finite lattice is the congruence lattice of a finite partial algebra.

Lemma 2.1. *Let X be a finite set, and let $\mathrm{Eq}(X)$ denote the lattice of equivalence relations on X. If $L \leq \mathrm{Eq}(X)$ is a 0-1-sublattice, and $\rho \in \mathrm{Eq}(X)$ and $\rho \notin L$, then for some $k < \omega$ there exists a partial operation $f\colon X^k \rightharpoonup X$ that is compatible with L and incompatible with ρ.*

Proof. First we focus on the relations in L that are above ρ. Let $\rho^{\uparrow} \cap L = \{\gamma \in L \mid \gamma \geq \rho\}$. Since $\rho \notin L$, we have $\gamma > \rho$ for all $\gamma \in \rho^{\uparrow} \cap L$. Now, $\rho^{\uparrow} \cap L$ has a least element $\rho^* = \bigwedge(\rho^{\uparrow} \cap L)$. Clearly $\rho^* \geq \rho$ and since $\rho^* \in L$ we have $\rho^* \neq \rho$, so $\rho^* > \rho$. Therefore, there exists $(u, v) \in \rho^* - \rho$.

Next consider the elements of L that are not above ρ. For each such $\alpha_i \in L - \rho^{\uparrow}$ there exists $(x_i, y_i) \in \rho - \alpha_i$. Let $(x_1, y_1), \dots, (x_k, y_k)$ be the list of all unique such pairs (i.e., each pair appears in the list exactly once). Define the partial function $f\colon X^k \rightharpoonup X$ at only two points of X^k; specifically, let

$$f(x_1, \dots, x_k) = u \quad \text{and} \quad f(y_1, \dots, y_k) = v.$$

Then, since $(\forall i)(x_i, y_i) \in \rho$ and $(u, v) \notin \rho$, f is incompatible with ρ. On the other hand, $(u, v) \in \rho^* = \bigwedge(\rho^{\uparrow} \cap L)$, so $(u, v) \in \gamma$ for every $\gamma \in \rho^{\uparrow} \cap L$, so f is compatible with every $\gamma \in \rho^{\uparrow} \cap L$.

Finally, for each $\alpha_i \in L$ not above ρ there is at least one pair $(x_i, y_i) \notin \alpha_i$. Therefore, it is impossible for f to be incompatible with any such α_i. □

Theorem 2.2. *Let X be a finite set and let $L \leq \mathrm{Eq}(X)$ be a 0-1-sublattice. Then there exists a finite partial algebra $\mathbb{X} = \langle X, F\rangle$ with $\mathrm{Con}(\mathbb{X}) = L$.*

Proof. By the lemma, for each $\rho \in \mathrm{Eq}(X) - L$, there exists $k < \omega$ and $f_\rho\colon X^k \rightharpoonup X$ such that f_ρ is compatible with every relation in L and incompatible with ρ. Let $\mathcal{R}$ be the set $\mathrm{Eq}(X) - L$ of all equivalence relations on X that do not belong to L. Define, $F = \{f_\rho \mid \rho \in \mathcal{R}\}$. Evidently, $\mathrm{Con}\langle X, F\rangle = L$. □

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3. Generalities

3.1. **Closure systems, closure operators, and Moore families.** First we recall some standard definitions. A ***closure system*** on a set X is a collection $\mathcal{C}$ of subsets of X that is closed under arbitrary intersection (including the empty intersection, so $\bigcap \emptyset = X \in \mathcal{C}$). Thus a closure system is a complete meet semilattice with respect to subset inclusion ordering. Since every complete meet semilattice is automatically a complete lattice (see [4, Theorem 2.5]), the closed sets of a closure system form a complete lattice.

Examples of closure systems that are especially relevant for our work are the following:

- order ideals of an ordered set
- subalgebras of an algebra
- equivalence relations on a set
- congruence relations of an algebra

Let $\mathbf{P} = \langle P, \leq \rangle$ be a poset. An function $\mathsf{C}\colon P \to P$ is called a ***closure operator*** on $\mathbf{P}$ if it satisfies the following axioms for all $x, y \in P$.

(1) $x \leq \mathsf{C}\, x$ (extensivity)
(2) $x \leq y$ implies $\mathsf{C}\, x \leq \mathsf{C}\, y$ (monotonicity)
(3) $\mathsf{C}\,\mathsf{C}\, x = \mathsf{C}\, x$ (idempotence)

Thus, a closure operator is an extensive idempotent poset endomorphism, and the definition above is equivalent to the single axiom $(\forall\, x, y \in P)(x \leq \mathsf{C}\, y \longleftrightarrow \mathsf{C}\, x \leq \mathsf{C}\, y)$.

Example 3.1. Let X be a set and let $Y \subseteq X$. Define $\mathsf{C}_Y\colon \mathcal{P}(X) \to \mathcal{P}(X)$ by $\mathsf{C}_Y(W) = W \cup Y$. Then C_Y is a closure operator on $\langle \mathcal{P}(X), \subseteq \rangle$.

A ***fixpoint*** of a function $\mathsf{C}\colon P \to P$ is an $x \in P$ satisfying $\mathsf{C}\, x = x$. A fixpoint of a closure operator is called ***closed***. If the poset $\mathbf{P} = \langle P, \leq \rangle$ happens to be a complete lattice, then by extensivity the largest element $\top = \bigvee P$ is a fixpoint of every closure operator on $\mathbf{P}$. Also, the collection of fixpoints of a closure operator is closed under arbitrary meets. (Proof: If $\mathcal{A}$ is a set of fixpoints of C and $a \in \mathcal{A}$, then $\bigwedge \mathcal{A} \leq a$, so by monotonicity $\mathsf{C}\big(\bigwedge \mathcal{A}\big) \leq \mathsf{C}\, a = a$. Since a was arbitrary, $\mathsf{C}\big(\bigwedge \mathcal{A}\big) \leq \bigwedge \mathcal{A}$. By extensivity, $\bigwedge \mathcal{A} \leq \mathsf{C}\big(\bigwedge \mathcal{A}\big)$. Therefore, $\mathsf{C}\big(\bigwedge \mathcal{A}\big) = \bigwedge \mathcal{A}$.) The set of closure operators on $\mathbf{P}$ themselves form a complete lattice under the pointwise order: $\mathsf{C}_1 \leq \mathsf{C}_2$ iff $\mathsf{C}_1\, x \leq \mathsf{C}_2\, x$ for all $x \in P$.

Some of these observations can be restated as follows: if $\mathbf{P} = \langle P, \leq \rangle$ is a complete lattice, then the set $\mathcal{C} \subseteq P$ of fixedpoints of a closure operator is a ***Moore family*** on $\mathbf{P}$—that is, $\bigvee P \in \mathcal{C}$ and every nonempty subset of $\mathcal{C}$ is closed under arbitrary meets.

Conversely, if we are given a Moore family $\mathcal{C}$ on $\mathbf{P}$, and if we define $\mathsf{C}\colon P \to P$ by

$$\mathsf{C}\, a = \bigwedge \{ b \in \mathcal{C} \mid a \leq b \},$$

then C satisfies conditions (1)–(3) above, making it a closure operator. To summarize, $\mathcal{C} \subseteq P$ is the set of fixpoints (i.e., closed elements) of a closure operator on $\mathbf{P}$ if and only if $\mathcal{C}$ is a Moore family on $\mathbf{P}$.

Every Moore family on $\mathbf{P}$ is itself the universe of a complete lattice with the order inherited from $\mathbf{P}$, though the join may differ from the join of $\mathbf{P}$.

3.1.1. *More Moore families.* The name "closure system" is typically reserved for the special case in which $\mathbf{P}$ happens to be the powerset Boolean algebra of a set X—that is, $\mathbf{P} = \langle \mathcal{P}(X), \subseteq \rangle$; in that case, a Moore family on $\mathbf{P}$ is called a closure system on X.

$$
\begin{tikzcd}
L \arrow[rr, "\phi"] \arrow[dr, "\sigma_{u,v}"'] & \arrow[d, phantom, "\circlearrowleft"] & K \\
& L_{u,v} \arrow[ur, dashed, "\exists\,\psi"'] &
\end{tikzcd}
$$

FIGURE 1. Every join-homomorphism collapsing u and v is divisible by $\sigma_{u,v}$.

Example 3.2. Let X be a set, let $\mathsf{R}(X) = \bigcup_{n<\omega} \mathcal{P}(X^n)$ be the set of all finitary relations on X, and let $\mathsf{O}(X) = \bigcup_{n<\omega} X^{X^n}$ be the set of all finitary operations on X. Define $F\colon \mathcal{P}(\mathsf{R}(X)) \to \mathcal{P}(\mathsf{O}(X))$ and $G\colon \mathcal{P}(\mathsf{O}(X)) \to \mathcal{P}(\mathsf{R}(X))$ as follows: if $A \subseteq \mathsf{R}(X)$ and $B \subseteq \mathsf{O}(X)$, then

$$F(A) = \{f \in \mathsf{O}(X) \mid f \text{ is compatible with every relation in } A\},$$

$$G(B) = \{\rho \in \mathsf{R}(X) \mid \rho \text{ is compatible with every operation in } B\}.$$

Then $G \circ F$ is a closure operator on the lattice of all relations on X.

Example 3.3. Let X be a set, let $\mathrm{Eq}(X)$ denote the lattice of equivalence relations on X, and let X^X be the set of all unary operations on X. Define $F_1\colon \mathcal{P}(\mathrm{Eq}(X)) \to \mathcal{P}(X^X)$ and $G_1\colon \mathcal{P}(X^X) \to \mathcal{P}(\mathrm{Eq}(X))$ as follows: if $A \subseteq \mathrm{Eq}(X)$ and $B \subseteq X^X$, then

$$F_1(A) = \{f \in X^X \mid f \text{ is compatible with every relation in } A\},$$

$$G_1(B) = \{\rho \in \mathrm{Eq}(X) \mid \rho \text{ is compatible with every operation in } B\}.$$

Then $G_1 \circ F_1$ is a closure operator on the lattice $\mathrm{Eq}(X)$ of all equivalence relations on X.

Example 3.4 (Pudlák-Tůma [5])**.** Let $\mathbf{L} = \langle L, \vee, \wedge \rangle$ be a lattice, $u, v \in L$, and $u \leq v$. Define a subset $L_{u,v}$ of L by $L_{u,v} = \{x \in L \mid v \leq x \text{ or } u \nleq x\}$. The partial order relation of the lattice $\mathbf{L}$ induces a lattice order on $L_{u,v}$. Denote the resulting lattice by $\mathbf{L}_{u,v}$. Then the meet of $\mathbf{L}_{u,v}$ is that of $\mathbf{L}$, whereas the join of $\mathbf{L}_{u,v}$ is

$$x \vee_{u,v} y = \begin{cases} x \vee y, & \text{if } u \nleq x \vee y, \\ x \vee y \vee v, & \text{if } u \leq x \vee y. \end{cases}$$

Define a mapping $\sigma_{u,v}\colon L \to L_{u,v}$ as follows:

$$\sigma_{u,v}(x) = \begin{cases} x, & \text{if } u \nleq x, \\ x \vee v, & \text{if } u \leq x. \end{cases}$$

Then $\sigma_{u,v}$ is a surjective join-homomorphism. In fact, every join-homomorphism $\phi\colon L \to K$ satisfying $\phi(u) = \phi(v)$ splits as $\phi = \psi \circ \sigma_{u,v}$ for some $\psi\colon L_{u,v} \to K$. (See the commutative diagram in Figure 1.)

As a mapping from $\mathbf{L}$ to itself, $\sigma_{u,v}$ does not preserve joins. However, $\sigma_{u,v}\colon L \to L$ is a closure operator and $L_{u,v}$ is the set of fixpoints of $\sigma_{u,v}$ (the closed sets).

3.2. **Algebraicity.** A subset D of an ordered set P is called ***up-directed*** if for every $x, y \in D$ there exists $z \in D$ such that $x \leq z$ and $y \leq z$. A closure operator $\mathsf{C}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is called ***algebraic*** provided, for all $A \subseteq X$,

$$\mathsf{C}\, A = \bigcup \{\mathsf{C}\, F \mid F \subseteq A \text{ and } F \text{ finite } \}.$$

The collection $\mathcal{C}$ of closed sets of an algebraic closure operator is called an ***algebraic closure system***.

Theorem 3.5 (cf. [4] Thm 3.1)**.** *Let $\mathcal{C}$ be the closure system of fixed points of the closure operator $\mathsf{C}\colon \mathcal{P}(X) \to \mathcal{P}(X)$. The following are equivalent:*

(1) C is an algebraic closure operator
(2) $\mathcal{C}$ is an algebraic closure system
(3) If $D \subseteq \mathcal{C}$ is up-directed and $C \subseteq D$ is a chain, then $\bigcup C$ is closed.
(4) If $D \subseteq \mathcal{C}$ is up-directed, then $\bigcup D$ is closed.
(5) If $C \subseteq \mathcal{C}$ is a chain, then $\bigcup C$ is closed.

For an algebra $\mathbf{A}$, the subalgebra generation operator $\mathrm{Sg}^{\mathbf{A}}$ is an algebraic closure operator (on the poset $\langle \mathcal{P}(A), \subseteq \rangle$) whose fixpoints are subalgebras of $\mathbf{A}$. Thus the lattice $\langle \mathrm{Sub}(\mathbf{A}), \vee, \wedge \rangle$ of subalgebras of $\mathbf{A}$ is an algebraic closure system. Conversely, given an algebraic closure system $\mathcal{S}$, we can construct an algebra $\mathbf{A} = \langle A, F \rangle$ so that $\mathcal{S} = \mathrm{Sub}(\mathbf{A})$. Here is how: let $\mathsf{C}_{\mathcal{S}}$ denote the corresponding closure operator. Let $a_0, a_1, \dots, a_{n-1}, b \in A$ be such that $b \in \mathsf{C}_{\mathcal{S}}\big(\{a_0, a_1, \dots, a_{n-1}\}\big)$. Define an n-ary operation $f_{\mathbf{a},b}$ so that $f_{\mathbf{a},b}(a_0, a_1, \dots, a_{n-1}) = b$ and for all other tuples $f_{\mathbf{a},b}(c_0, c_1, \dots, c_{n-1}) = c_0$. Do this for each $\mathbf{a} = (a_0, a_1, \dots, a_{n-1})$. Then define

$$F = \{f_{\mathbf{a},b} \mid n < \omega,\ \mathbf{a} \in A^n,\ b \in \mathsf{C}_{\mathcal{S}}\, \mathbf{a}\}.$$

Every subalgebra is the union of its finitely generated subalgebras.

4. Strong Congruence Relations

In this section we'll use bold letters like $\mathbf{x}$ to stand for an n-tuple $(x_0, \dots, x_{n-1})$, for some n whose value will always be clear from the context. Let θ be an equivalence relation on a set A and suppose $f\colon A^n \rightharpoonup A$ is a partial function. We call θ and f

- ***weakly compatible*** provided $\forall (a_i, b_i) \in \theta$, if $f(\mathbf{a})$ and $f(\mathbf{b})$ are both defined, then $f(\mathbf{a})\ \theta\ f(\mathbf{b})$.
- ***strongly compatible*** provided $\forall (a_i, b_i) \in \theta$, if $f(\mathbf{a})$ is defined, then $f(\mathbf{b})$ is defined and $f(\mathbf{a})\ \theta\ f(\mathbf{b})$.

Let $\mathbb{A} = \langle A, F \rangle$ be a partial algebra and θ an equivalence relation on A. We call θ a ***strong*** (resp., ***weak***) ***congruence*** of $\mathbb{A}$ if θ is strongly (resp., weakly) compatible with every $f \in F$. Denote by $\mathrm{SCon}(\mathbb{A})$ (resp., $\mathrm{WCon}(\mathbb{A})$) the set of strong (resp., weak) congruences of $\mathbb{A}$.

Here are some elementary observations about strong and weak congruences. (See Berman's thesis [2] and paper [1] for more details and proofs.) If $\mathbb{A} = \langle A, F \rangle$ is a partial algebra then the set $\mathrm{SCon}(\mathbb{A})$ of strong congruence relations of $\mathbb{A}$ forms a sublattice of $\mathrm{Eq}(A)$. The set $\mathrm{WCon}(\mathbb{A})$ of weak congruences also forms a lattice, but it is not a sublattice of $\mathrm{Eq}(A)$ since there may be pairs of relations whose join in $\mathrm{Eq}(A)$ is strictly below their join in $\mathrm{WCon}(\mathbb{A})$. We will see examples below.

A *total algebra*—an algebra for which all operations are defined everywhere—is a special case of a partial algebra, and to indicate that a partial algebra is not total, we may refer to it as a *proper partial algebra*. This simply means that there is at least one operation that is not defined everywhere. It should be obvious that for total algebras strong and weak congruences reduce to the usual defintion of congruence relation on an algebra. In particular, if $\mathbb{A}$ is total, then the largest strong congruence of $\mathbb{A}$ is $1_A := A \times A$.

The converse of the last sentence in the previous paragraph is also true. That is, if the largest strong congruence of $\mathbb{A}$ is 1_A, then $\mathbb{A}$ is total. To see this, suppose $f \in F$ is an n-ary partial operation and denote by $\mathrm{dom}(f)$ the subset of A^n on which f is defined. Let θ be a strong congruence of $\mathbb{A}$. Then the following implication holds:

$$(a_0, a_1, \dots, a_{n-1}) \in \mathrm{dom}(f) \implies [a_0]_\theta \times [a_1]_\theta \times \cdots \times [a_{n-1}]_\theta \subseteq \mathrm{dom}(f).$$

That is, if f is defined at an n-tuple, then f must also be defined on the whole Cartesian product of θ-blocks containing the coordinates of that tuple. In particular, if 1_A is a strong congruence of $\mathbb{A}$, then every operation of $\mathbb{A}$ must be total, while if $\mathbb{A} = \langle A, F\rangle$ is a *proper partial algebra*, then the largest strong congruence of $\mathbb{A}$ must be strictly below 1_A.

To summarize, $\mathbb{A}$ is a total algebra iff $\bigvee \mathrm{SCon}(\mathbb{A}) = 1_A$.

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