EVERY FINITE LATTICE IS THE CONGRUENCE LATTICE OF A FINITE PARTIAL ALGEBRA

Lemma 0.1. *Let X be a finite set, and let $\mathrm{Eq}(X)$ denote the lattice of equivalence relations on X. If $L \leq \mathrm{Eq}(X)$ is a 0-1-sublattice, and $\rho \in \mathrm{Eq}(X)$ and $\rho \notin L$, then for some $k < \omega$ there exists a partial operation $f\colon X^k \rightharpoonup X$ that is compatible with L and incompatible with ρ.*

Proof. First we focus on the relations in L that are above ρ. Let $\rho^{\uparrow} \cap L = \{\gamma \in L \mid \gamma \geq \rho\}$. Since $\rho \notin L$, we have $\gamma > \rho$ for all $\gamma \in \rho^{\uparrow} \cap L$. Now, $\rho^{\uparrow} \cap L$ has a least element $\rho^* = \bigwedge(\rho^{\uparrow} \cap L)$. Clearly $\rho^* \geq \rho$ and since $\rho^* \in L$ we have $\rho^* \neq \rho$, so $\rho^* > \rho$. Therefore, there exists $(u, v) \in \rho^* - \rho$.

Next consider the elements of L that are not above ρ. For each such $\alpha_i \in L - \rho^{\uparrow}$ there exists $(x_i, y_i) \in \rho - \alpha_i$. Let $(x_1, y_1), \dots, (x_k, y_k)$ be the list of all unique such pairs (i.e., each pair appears in the list exactly once). Define the partial function $f\colon X^k \rightharpoonup X$ at only two points of X^k; specifically, let

$$f(x_1, \dots, x_k) = u \quad \text{and} \quad f(y_1, \dots, y_k) = v.$$

Then, since $(\forall i)(x_i, y_i) \in \rho$ and $(u, v) \notin \rho$, f is incompatible with ρ. On the other hand, $(u, v) \in \rho^* = \bigwedge(\rho^{\uparrow} \cap L)$, so $(u, v) \in \gamma$ for every $\gamma \in \rho^{\uparrow} \cap L$, so f is compatible with every $\gamma \in \rho^{\uparrow} \cap L$.

Finally, for each $\alpha_i \in L$ not above ρ there is at least one pair $(x_i, y_i) \notin \alpha_i$. Therefore, it is impossible for f to be incompatible with any such α_i. □

Theorem 0.2. *Let X be a finite set and let $L \leq \mathrm{Eq}(X)$ be a 0-1-sublattice. Then there exists a finite partial algebra $\mathbb{X} = \langle X, F \rangle$ with $\mathrm{Con}(\mathbb{X}) = L$.*

Proof. By the lemma, for each $\rho \in \mathrm{Eq}(X) - L$, there exists $k < \omega$ and $f_\rho\colon X^k \rightharpoonup X$ such that f_ρ is compatible with every relation in L and incompatible with ρ. Let $\mathcal{R}$ be the set $\mathrm{Eq}(X) - L$ of all equivalence relations on X that do not belong to L. Define, $F = \{f_\rho \mid \rho \in \mathcal{R}\}$. Evidently, $\mathrm{Con}\langle X, F \rangle = L$. □

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Date: October 13, 2016.