EVERY FINITE LATTICE IS THE CONGRUENCE LATTICE OF A FINITE PARTIAL ALGEBRA

1. Introduction

This note begins with a proof in Section 2 of the result stated in the title. Bill Lampe pointed out that this result has been known for a long time and subsequently explained to DeMeo a more general viewpoint. In Section 3 we try to describe what Lampe explained, but surely our presentation is imperfect. Hopefully it will elicit comments and criticisms that we will use to improve it.

2. A well known result

In this section we give a straight-forward proof of the fact that every finite lattice is the congruence lattice of a finite partial algebra.

Lemma 2.1. *Let X be a finite set, and let $\operatorname{Eq}(X)$ denote the lattice of equivalence relations on X. If $L \leq \operatorname{Eq}(X)$ is a 0-1-sublattice, and $\rho \in \operatorname{Eq}(X)$ and $\rho \notin L$, then for some $k < \omega$ there exists a partial operation $f \colon X^k \rightharpoonup X$ that is compatible with L and incompatible with ρ.*

Proof. First we focus on the relations in L that are above ρ. Let $\rho^\uparrow \cap L = \{\gamma \in L \mid \gamma \geq \rho\}$. Since $\rho \notin L$, we have $\gamma > \rho$ for all $\gamma \in \rho^\uparrow \cap L$. Now, $\rho^\uparrow \cap L$ has a least element $\rho^* = \bigwedge(\rho^\uparrow \cap L)$. Clearly $\rho^* \geq \rho$ and since $\rho^* \in L$ we have $\rho^* \neq \rho$, so $\rho^* > \rho$. Therefore, there exists $(u, v) \in \rho^* - \rho$.

Next consider the elements of L that are not above ρ. For each such $\alpha_i \in L - \rho^\uparrow$ there exists $(x_i, y_i) \in \rho - \alpha_i$. Let $(x_1, y_1), \ldots, (x_k, y_k)$ be the list of all unique such pairs (i.e., each pair appears in the list exactly once). Define the partial function $f \colon X^k \rightharpoonup X$ at only two points of X^k; specifically, let

$$f(x_1, \ldots, x_k) = u \quad \text{and} \quad f(y_1, \ldots, y_k) = v.$$

Then, since $(\forall i)(x_i, y_i) \in \rho$ and $(u, v) \notin \rho$, f is incompatible with ρ. On the other hand, $(u, v) \in \rho^* = \bigwedge(\rho^\uparrow \cap L)$, so $(u, v) \in \gamma$ for every $\gamma \in \rho^\uparrow \cap L$, so f is compatible with every $\gamma \in \rho^\uparrow \cap L$.

Finally, for each $\alpha_i \in L$ not above ρ there is at least one pair $(x_i, y_i) \notin \alpha_i$. Therefore, it is impossible for f to be incompatible with any such α_i. ☐

Theorem 2.2. *Let X be a finite set and let $L \leq \operatorname{Eq}(X)$ be a 0-1-sublattice. Then there exists a finite partial algebra $\mathbb{X} = \langle X, F \rangle$ with $\operatorname{Con}(\mathbb{X}) = L$.*

Proof. By the lemma, for each $\rho \in \operatorname{Eq}(X) - L$, there exists $k < \omega$ and $f_\rho \colon X^k \rightharpoonup X$ such that f_ρ is compatible with every relation in L and incompatible with ρ. Let $\mathcal{R}$ be the set $\operatorname{Eq}(X) - L$ of all equivalence relations on X that do not belong to L. Define, $F = \{f_\rho \mid \rho \in \mathcal{R}\}$. Evidently, $\operatorname{Con}\langle X, F \rangle = L$. ☐

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3. Generalities

First we recall some standard definitions. A *closure system* on a set X is a collection $\mathcal{C}$ of subsets of X that is closed under arbitrary intersection (including the empty intersection, so $\bigcap \emptyset = X \in \mathcal{C}$). Thus a closure system is a complete meet semilattice with respect to subset inclusion ordering. Since every complete meet semilattice is automatically a complete lattice (see [1, Theorem 2.5]), the closed sets of a closure system form a complete lattice.

For us, the key examples of closure systems are the

- order ideals of an ordered set
- subalgebras of an algebra
- equivalence relations on a set
- congruence relations of an algebra

Let $\mathbf{P} = \langle P, \leq \rangle$ be a poset. An function $\mathrm{c}\colon P \to P$ is called a *closure operator* if it satisfies the following axioms for all $x, y \in P$.

(1) $x \leq \mathrm{c}\, x$ (extensivity)
(2) $x \leq y$ implies $\mathrm{c}(x) \leq \mathrm{c}(y)$ (monotonicity)
(3) $\mathrm{c}\,\mathrm{c}\, x = \mathrm{c}\, x$ (idempotence)

More concisely, a closure operator is an extensive idempotent poset endomorphism, and the definition above is equivalent to the single axiom

$$(\forall x \in P)(\forall y \in P)(x \leq \mathrm{c}\, y \quad \longleftrightarrow \quad \mathrm{c}\, x \leq \mathrm{c}\, y)$$

Example 3.1. Let X be a set and let $Y \subseteq X$. Define $\mathrm{c}_Y\colon \mathcal{P}(X) \to \mathcal{P}(X)$ by $\mathrm{c}_Y(W) = W \cup Y$. Then c_Y is a closure operator on the poset $\langle \mathcal{P}(X), \subseteq \rangle$.

Example 3.2 (Pudlák-Tůma [2])**.** Let L be a lattice, $u, v \in L$, and $u \leq v$. Define a subset $L_{u,v}$ of L by

$$L_{u,v} = \{x \in L \mid v \leq x \text{ or } u \nleq x\}.$$

The lattice order of L induces a lattice order on $L_{u,v}$. The meet of $L_{u,v}$ is that of L, whereas the join of $L_{u,v}$ is

$$x \vee_{u,v} y = \begin{cases} x \vee y, & \text{if } u \nleq x \vee y, \\ x \vee y \vee v, & \text{if } u \leq x \vee y. \end{cases}$$

Define a mapping $\sigma_{u,v}\colon L \to L_{u,v}$ as follows:

$$\sigma_{u,v}(x) = \begin{cases} x, & \text{if } u \nleq x, \\ x \vee v, & \text{if } u \leq x. \end{cases}$$

Then $\sigma_{u,v}$ is a surjective join-homomorphism. In fact, every join-homomorphism $\phi\colon L \to K$ satisfying $\phi(u) = \phi(v)$ splits as $\phi = \psi \circ \sigma_{u,v}$ for some $\psi\colon L_{u,v} \to K$, as indicated in the commutative diagram below.

A fixpoint of a closure operator c (i.e., an element $x \in P$ satisfying $cx = x$) is called a *closed element*. A closure operator on a partially ordered set is determined by its closed elements. If x is closed, then $y \leq x$ iff $cy \leq x$.

The collection of fixpoints of a closure operator is closed under arbitrary intersection, hence forms a closure system. Indeed, let $\mathcal{A}$ be a collection of fixpoints of C. For each $A \in \mathcal{A}$

$$\begin{tikzcd}
L \arrow[rr, "\phi"] \arrow[dr, "\sigma_{u,v}"'] & & K \\
& L_{u,v} \arrow[u, phantom, "\circlearrowleft"] \arrow[ur, dashed, "\exists\,\psi"'] &
\end{tikzcd}$$

FIGURE 1. Every join-homomorphism collapsing u and v is divisible by $\sigma_{u,v}$.

we have $\bigcap \mathcal{A} \subseteq A$, so $\mathsf{C}(\bigcap \mathcal{A}) \subseteq A$ by monotonicity. Therefore, $\bigcap \mathcal{A} \subseteq \mathsf{C}(\bigcap \mathcal{A}) \subseteq \bigcap \mathcal{A}$, so $\mathsf{C}(\bigcap \mathcal{A}) = \bigcap \mathcal{A}$; that is, $\bigcap \mathcal{A}$ is a fixpoint of C.

Thus, if $\mathcal{C}$ is the collection of closed sets of the closure operator C, then $\mathcal{C}$ is a closure system. Conversely, given a closure system $\mathcal{C}$ on a set X, define the operator $\mathsf{C}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ as follows

$$\mathsf{C}A = \bigcap \{B \in \mathcal{C} \mid A \subseteq B\}.$$

It is obvious that C so defined satisfies conditions (a)–(c) above, making it a closure operator.

A closure operator C is called *algebraic* iff

$$\mathsf{C}A = \bigcup \{\mathsf{C}F \mid F \subseteq A \text{ and } F \text{ finite } \}.$$

Let $\mathbf{L} = \langle L, \vee, \wedge \rangle$ be a complete lattice and let $\top = \bigvee L$. A subset $M \subseteq L$ is called a *Moore family* on $\mathbf{L}$ if $\top \in M$ and every nonempty subset of M is closed under meet. That is, if $\emptyset \neq S \subseteq M$, then $\bigwedge S \in S$.

If $\mathbf{L} = \langle L, \vee, \wedge \rangle$ is a complete lattice, then a subset $C \subseteq L$ is the set of closed elements for some closure operator on L if and only if C is a *Moore family* on L. Any such set C is itself the universe of a complete lattice $\mathbf{C}$ with the order inherited from $\mathbf{L}$. However, it is important to note that join of $\mathbf{C}$ might differ from that of $\mathbf{L}$. When $\mathcal{L}$ is the powerset Boolean algebra of X—that is, when $\mathcal{L} = \langle \mathcal{P}(X), \subseteq \rangle$—then a Moore family in $\mathcal{L}$ is called a *closure system* on X.

The closure operators on $\mathcal{L}$ form themselves a complete lattice; the order on closure operators is defined by $\mathrm{c}_1 \leq \mathrm{c}_2$ iff $\mathrm{c}_1\, x \leq \mathrm{c}_2\, x$ for all $x \in L$.

References

[1] J. B. Nation. "Notes on Lattice Theory". Unpublished notes. 2007, 2016. URL: `http://www.math.hawaii.edu/~jb/`.

[2] Pavel Pudlák and Jiří Tůma. "Every finite lattice can be embedded in a finite partition lattice". In: *Algebra Universalis* 10.1 (1980), pp. 74–95. ISSN: 0002-5240. DOI: `10.1007/BF02482893`. URL: `http://dx.doi.org/10.1007/BF02482893`.

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