THE COMMUTATOR AS FIXED POINT OF A CLOSURE OPERATOR

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Abstract. In this note we elaborate on the following remark of Keith Kearnes [Kea95, p. 930] giving an alternate description of the commutator (here β is a congruence of $\mathbf{A}$ and $\mathbf{A} \times_\beta \mathbf{A}$ denotes the subalgebra of $\mathbf{A}^2$ with universe β):

"Let $\Delta_{\beta,\alpha}$ be the congruence on $\mathbf{A} \times_\beta \mathbf{A}$ generated by

$$\{\langle (x,x),(y,y)\rangle \mid (x,y) \in \alpha\}.$$

Call a subset $G \subseteq A^2$ **Δ-closed** if

$$\Delta_{\beta,\alpha} \circ G \circ \Delta_{\beta,\alpha} \subseteq G.$$

When α and β are reflexive, compatible relations, then $[\alpha,\beta]$ is the smallest subset $\gamma \subseteq A^2$ such that (i) γ is a congruence of $\mathbf{A}$ and (ii) γ is Δ-closed."

1. Introduction

The genesis of these notes was my futile attempt to interpret the expression $\Delta_{\beta,\alpha} \circ G \circ \Delta_{\beta,\alpha}$. Noting that $\Delta_{\beta,\alpha} \subseteq (A\times A)^2$ while $G \subseteq A^2$, I wasn't aware of a standard means of composing relations of such different arities. In this note I try to reconcile this by giving an alternative "closure" operation that acheives the same end and verifies Keith's assertion that "there is a useful alternate description of $[\alpha,\beta]$." Indeed, the commutator is a fixed point of some closure operator (i.e., closed set), and the closure operator involves the relation $\Delta_{\beta,\alpha}$ defined above.

1.1. **Definitions.** If α and β are binary relations on A, we let $\alpha \otimes \beta$ denote the following binary relation on A^2:

$$\alpha \otimes \beta := \{\big((x,y),(x',y')\big) \mid x \mathrel{\alpha} x' \text{ and } y \mathrel{\beta} y'\}.$$

We reserve $\alpha \times \beta$ for the usual Cartesian product

$$\alpha \times \beta := \{\big((x,x'),(y,y')\big) \mid x \mathrel{\alpha} x' \text{ and } y \mathrel{\beta} y'\}.$$

For an algebra $\mathbf{A}$ with congruence relations α, $\beta \in \operatorname{Con}\mathbf{A}$, let $\underset{\sim}{\boldsymbol{\beta}}$ denote the subalgebra of $\mathbf{A} \times \mathbf{A}$ with universe β. Let $0_A = \{(a,a) \mid a \in A\}$, let

$$D_\alpha := (\alpha \otimes \alpha) \cap (0_A \times 0_A) = \{((a,a),(b,b)) \in 0_A \times 0_A \mid a \mathrel{\alpha} b\}, \text{ and let} \tag{1.1}$$

$$\Delta_{\beta,\alpha} := \operatorname{Cg}^{\underset{\sim}{\boldsymbol{\beta}}}(D_\alpha). \tag{1.2}$$

As usual, the congruence class of $\Delta_{\beta,\alpha}$ containing (b,b') is denoted and defined by

$$(b,b')/\Delta_{\beta,\alpha} = \{(a,a') \in \beta \mid (a,a') \mathrel{\Delta_{\beta,\alpha}} (b,b')\}.$$

In our application below, the most important special case will concern a finite 2-generated free algebra $\mathbf{F} := \mathbf{F}_{\mathcal{V}}(x,y)$ over an idempotent variety $\mathcal{V}$. When the context makes the

Date: March 1, 2017.

meaning clear, we sometimes write 1 instead of 1_F or $F \times F$. As noted above in Lemma **??**, $\mathrm{Cg}^{\mathbf{F}}(x, y) = 1$, so we will be especially interested in

$$\Delta_{1,1} := \mathrm{Cg}^{F\times F}\{\big((x,x),(y,y)\big) \mid x, y \in F\}. \tag{1.3}$$

Recall that $\mathbf{F}$ is abelian if and only if $\exists\,\theta \in \mathrm{Con}(\mathbf{F}^2)$ with the diagonal $0_F := \{(z, z) \mid z \in F\}$ as a congruence class. Thus if $\mathbf{F}$ is nonabelian, then there exist $u, v \in F$ such that $u \neq v$ and $(u, v)\ \Delta_{1,1}\ (x, x)$, for all $x \in F$. (Otherwise, 0_F is a congruence class of $\Delta_{1,1}$.)

Let $\Phi_{\beta,\alpha} \colon \mathcal{P}(\beta) \to \mathcal{P}(\beta)$ be the function that takes each $B \subseteq \beta$ to

$$\Phi_{\beta,\alpha}(B) = \bigcup_{(b,b')\in B} (b, b')/\Delta_{\beta,\alpha}. \tag{1.4}$$

Let $\Psi_{\beta,\alpha} \colon \mathrm{Eq}(\beta) \to \mathrm{Eq}(\beta)$ be the function that takes each $R \in \mathrm{Eq}(\beta)$ to the equivalence relation

$$\begin{aligned}
\Psi_{\beta,\alpha}(R) &= \Delta_{\beta,\alpha} \circ (R \otimes R) \circ \Delta_{\beta,\alpha}\\
&= \{\big((x,y),(x',y')\big) \mid \big(\exists (r,r') \in R\big)\ \big((x,y)\ \Delta_{\beta,\alpha}\ (r,r)\ \text{ and }\ (r',r')\ \Delta_{\beta,\alpha}\ (x'y')\big)\}\\
&= \{\big((x,y),(x',y')\big) \mid \big(\exists (r,r')\big)\ (x,y)\ \Delta_{\beta,\alpha}\ (r,r)\ (R\otimes R)\ (r',r')\ \Delta_{\beta,\alpha}\ (x'y')\}.
\end{aligned}$$

For an algebra $\mathbf{A}$ with congruence relations $\alpha,\ \beta \in \mathrm{Con}\,\mathbf{A}$, let $\underset{\sim}{\boldsymbol{\beta}}$ denote the subalgebra of $\mathbf{A} \times \mathbf{A}$ with universe β. Let $D = \{(a, a) \mid a \in A\}$ and $D^2_\alpha = \{((a, a), (b, b)) \in D^2 \mid a\ \alpha\ b\}$. Finally, let $\Delta_{\beta,\alpha} = \mathrm{Cg}^{\underset{\sim}{\boldsymbol{\beta}}}\big(D^2_\alpha\big)$ be the congruence on $\underset{\sim}{\boldsymbol{\beta}}$ generated by the set D^2_α. The congruence class of $\Delta_{\beta,\alpha}$ that contains (b, b') is denoted and defined as follows:

$$(b, b')/\Delta_{\beta,\alpha} = \{(a, a') \in \beta \mid (a, a')\ \Delta_{\beta,\alpha}\ (b, b')\}.$$

2. Fixed Point Lemma

Lemma 2.1. *Let $\mathbf{A}$ be an algebra with $\alpha,\ \beta \in \mathrm{Con}(\mathbf{A})$. If $\Phi := \Phi_{\beta,\alpha}$ is defined as in (1.4), then*

(i) Φ is a closure operator on $\mathcal{P}(\beta)$;
(ii) $[\alpha, \beta]$ is the least fixed point of Φ.

Proof. (i) Fix $B \subseteq \beta$. We must prove the following: (a) $B \subseteq \Phi(B)$; (b) $B \subseteq C \Rightarrow \Phi(B) \subseteq \Phi(C)$; and (c) $\Phi(\Phi(B)) = \Phi(B)$. If $(b, b') \in B$, then $(b, b') \in \Phi(B)$ since the operation (1.4) does not discard any of the pairs that were already in B. As for (b), if $(a, a') \in \Phi(B)$, then there exists $(b, b') \in B \subseteq C$ such that $(a, a')\ \Delta_{\beta,\alpha}\ (b, b')$. Since (b, b') belongs to C we have $(a, a') \in \Phi(C)$ as well. As for (c), it clearly follows from (a) and (b) that $\Phi(B) \subseteq \Phi(\Phi(B))$, so we prove the reverse inclusion. Let $(d, d') \in \Phi(\Phi(B))$. Then $(c, c')\ \Delta_{\beta,\alpha}\ (d, d')$ for some $(c, c') \in \Phi(B)$, which implies $(b, b')\ \Delta_{\beta,\alpha}\ (c, c')$ for some $(b, b') \in B$. By transitivity of $\Delta_{\beta,\alpha}$ we conclude that $(d, d') \in \Phi(B)$, as desired.

(ii) Since $[\alpha, \beta] \in \mathcal{P}(\beta)$ we have $[\alpha, \beta] \subseteq \Phi([\alpha, \beta])$, by (i). We prove the reverse inclusion. If $(c, c') \in \Phi([\alpha, \beta])$, then (1.4) implies there exists $(b, b') \in [\alpha, \beta]$ such that

$$(b, b')\ \Delta_{\beta,\alpha}\ (c, c'). \tag{2.1}$$

From the definition of $\Delta_{\beta,\alpha}$ and Mal'tsev's theorem on congruence generation, (2.1) holds if and only if $\exists\, z_i\ \beta\ z'_i$ $(0 \leqslant i \leqslant n)$, $\exists\, x_i\ \alpha\ y_i$ $(0 \leqslant i < n)$, $\exists\, f_i \in \mathrm{Pol}_1(\mathbf{A} \times \mathbf{A})$ $(0 \leqslant i < n)$ such that $(b, b') = (z_0, z_0')$ and $(z_n, z_n') = (c, c')$, and

$$\{(b, b'), (z_1, z_1')\} = \{f_0(x_0, x_0), f_0(y_0, y_0)\} \tag{2.2}$$

$$\{(z_1, z_1'), (z_2, z_2')\} = \{f_1(x_1, x_1), f_1(y_1, y_1)\} \tag{2.3}$$

$$\vdots$$

$$\{(z_{n-1}, z_{n-1}'), (c, c')\} = \{f_{n-1}(x_{n-1}, x_{n-1}), f_{n-1}(y_{n-1}, y_{n-1})\}$$

For each $(0 \leqslant i < n)$, $f_i \in \mathrm{Pol}_1(\mathbf{A} \times \mathbf{A})$, which means

$$f_i(x, x') = g_i^{\boldsymbol{\beta}}((x, x'), (a_1, a_1'), \dots, (a_k, a_k')) = (g_i^{\mathbf{A}}(x, \mathbf{a}), g_i^{\mathbf{A}}(x, \mathbf{a}')),$$

for some k, $g_i \in \mathsf{Clo}_{k+1}(\mathbf{A})$, and constants tuples $\mathbf{a} = (a_1, \dots, a_k)$ and $\mathbf{a}' = (a_1', \dots, a_k')$ such that $a_i \ \beta \ a_i'$ $(1 \leqslant i \leqslant k)$. By (2.2), either

$$(b, b') = \big(g_0(x_0, \mathbf{a}), g_0(x_0, \mathbf{a}')\big) \quad \text{and} \quad (z_1, z_1') = \big(g_0(y_0, \mathbf{a}), g_0(y_0, \mathbf{a}')\big),$$

or vice-versa. Since $x_0 \ \alpha \ y_0$ and $a_i \ \beta \ a_i'$ $(1 \leqslant i \leqslant k)$, the α, β-term condition entails

$$g_0(x_0, \mathbf{a}) \ [\alpha, \beta] \ g_0(x_0, \mathbf{a}') \quad \Longleftrightarrow \quad g_0(y_0, \mathbf{a}) \ [\alpha, \beta] \ g_0(y_0, \mathbf{a}').$$

This and (2.2) yield $(b, b') \in [\alpha, \beta]$ iff $(z_1, z_1') \in [\alpha, \beta]$. Similarly (2.3) and $x_1 \ \alpha \ y_1$ imply $(z_1, z_1') \in [\alpha, \beta]$ iff $(z_2, z_2') \in [\alpha, \beta]$. Inductively, we arrive at $(b, b') \in [\alpha, \beta]$ iff $(c, c') \in [\alpha, \beta]$, as desired.

We have thus proved $[\alpha, \beta]$ is a fixed point of Φ. In other words, $[\alpha, \beta]$ is a "Φ-closed" subset of β. (A set $B \subseteq \beta$ is called Φ-*closed* provided $\Phi(B) \subseteq B$.) Recall, if f is a monotone increasing function defined on a complete poset $\langle P, \leqslant \rangle$, then the least fixed point of f is $\bigwedge\{p \in P \mid fp \leqslant p\}$. Thus, Lemma 2.1 (ii) asserts that

$$[\alpha, \beta] = \bigwedge\{B \subseteq \beta \mid \Phi(B) \subseteq B\}. \tag{2.4}$$

We already proved $[\alpha, \beta]$ is Φ-closed, so it remains to check for every Φ-closed subset $B \subseteq \beta$ that $[\alpha, \beta] \subseteq B$. Fix a Φ-closed subset $B \subseteq \beta$. It suffices to prove $\mathsf{C}(\alpha, \beta; \Phi(B))$, since this implies $[\alpha, \beta] \subseteq \Phi(B) \subseteq B$. Thus, our goal is to establish the α, β-term condition.

Let $p \in \mathrm{Pol}_{k+1}(\mathbf{A})$ and $a \ \alpha \ a'$ and $c_i \ \beta \ c_i'$ $(1 \leqslant i \leqslant k)$; suppose $p(a, \mathbf{c}) \ \Phi(B) \ p(a, \mathbf{c}')$. We prove that these hypotheses entail the following relation:

$$p(a', \mathbf{c}) \ \Phi(B) \ p(a', \mathbf{c}'). \tag{2.5}$$

By definition of Φ, (2.5) is equivalent to the existence of some pair $(b, b') \in B$ such that $(b, b') \ \Delta_{\beta,\alpha} \ (p(a', \mathbf{c}), p(a', \mathbf{c}'))$. Notice that the pair $(p(a, \mathbf{c}), p(a, \mathbf{c}'))$ belongs to B since $(p(a, \mathbf{c}), p(a, \mathbf{c}')) \in \Phi(B) \subseteq B$. Also, $c_i \ \beta \ c_i'$ $(0 \leqslant i < k)$ implies

$$((a, a), (c_1, c_1'), (c_1, c_1'), \dots, (c_k, c_k')) \in \beta^{k+1} \quad \text{and}$$
$$((a', a'), (c_1, c_1'), (c_1, c_1'), \dots, (c_k, c_k')) \in \beta^{k+1}.$$

Therefore,

$$p^{\beta}((a, a), (c_1, c_1'), (c_1, c_1'), \dots, (c_k, c_k')) = (p^{\mathbf{A}}(a, \mathbf{c}), p^{\mathbf{A}}(a, \mathbf{c}')) \in \beta \quad \text{and} \tag{2.6}$$

$$p^{\beta}((a', a'), (c_1, c_1'), (c_1, c_1'), \dots, (c_k, c_k')) = (p^{\mathbf{A}}(a', \mathbf{c}), p^{\mathbf{A}}(a', \mathbf{c}')) \in \beta. \tag{2.7}$$

Finally, $a \ \alpha \ a'$ implies $p(a, \mathbf{c}) \ \alpha \ p(a', \mathbf{c})$, and this—together with (2.6) and (2.7)—proves the pair $\big((p(a, \mathbf{c}), p(a, \mathbf{c}')), (p(a', \mathbf{c}), p(a', \mathbf{c}'))\big)$ belongs to $\Delta_{\beta,\alpha}$. Since $(p(a, \mathbf{c}), p(a, \mathbf{c}')) \in B$, this proves $(p(a', \mathbf{c}), p(a', \mathbf{c}')) \in \Phi(B)$, completing the proof. $\square$

References

[Kea95] Keith A. Kearnes. Varieties with a difference term. *J. Algebra*, 177(3):926–960, 1995. URL: `http://dx.doi.org/10.1006/jabr.1995.1334`, `doi:10.1006/jabr.1995.1334`.

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