COMPUTING THE TAME CONGRUECE THEORY TYPE OF AN ALGEBRA

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1. Introduction

In [1] J. Berman, E. Kiss, P. Prőhle and Á. Szendrei gave a polynomial time algorithm, here referred to as the BKPS algorithm, for computing typ($\mathbf{A}$), the set of tame congruence theory types that occur in a finite algebra $\mathbf{A}$. The bound given by those authors in [1] for computing the type of a covering $\beta \succ \alpha$ in $\mathbf{Con}\,(\mathbf{A})$ is (a constant times) $||\mathbf{A}||^4$, the fourth power of the input size of $\mathbf{A}$, which is defined by

$$||\mathbf{A}|| = \sum_{i=0}^{r} k_i n^i$$

where $n = |\mathbf{A}|$, r the largest arity, and k_i the number of operations of arity i. The algorithm, after finding a subtrace $\{a, b\}$, computes the images of $\{a, b\}^2$ under two variable polynomials, which is the subalgebra, $\mathbf{T}_{a,b}$, of $\mathbf{A}^4$ generated by (a, a, b, b) and (a, b, a, b) and the constant 4-tuples. This is the bottleneck of the algorithm and it is the size of $\mathbf{T}_{a,b}$ that determines the running time. Of course $\mathbf{T}_{a,b}$ can have at most n^4 elements (and in some cases does have this many), so $||\mathbf{A}||^4$ is an upper bound.

The main result of this paper is that the type of a prime quotient can be determined after finding at most n^3 elements of $\mathbf{T}_{a,b}$, and thus the running time of the algorithm is at most (a constant times) $||\mathbf{A}||^3$. The key to this result is the following theorem which gives bounds on $|\mathbf{T}_{a,b}|$ for the various types.

Theorem 1. *Let $\mathbf{A}$ be a finite algebra with n elements. Let $\beta \succ 0$ be an atom of $\mathbf{Con}\,(\mathbf{A})$ and let $\{a, b\}$ be two elements of a $0 - \beta$ trace. The maximum size of $\mathbf{T}_{a,b}$ depending on the type of β over 0 is*

1. *type **1** or **2**: n^3.*
2. *type **3**: n^4.*
3. *type **4**: $n^4/12 + n^3/3 + 5n^2/12 + n/6$.*
4. *type **5**: $n^3/3 + n^2/2 + n/6$.*

These bounds all obtain infinitely often.

Incidently we conjecture that in the case of type **1** the size of $\mathbf{T}_{a,b}$ is at most $n^{1+\log_2 3}$ ($\approx n^{2.6}$). We know that $n^{1+\log_2 3}$ occurs infinitely often.

The Universal Algebra Calculator [5] uses the BKPS algorithm to calculate types and it was used in producing most of the examples in this paper.

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2. The BKPS algorithm

In this section we review the BKPS algorithm. Throughout $\mathbf{A}$ will be a finite algebra. Every covering pair of $\mathbf{Con\,A}$ has a type associated with it and the *type set of* $\mathbf{A}$ is the collection of these types; see [6]. By Lemma 6.2 of [6], if two covering pairs are projective (connected by a sequence of transposes) then they have the same type. So we need only consider one representative from each projectivity class. In particular we need only consider coverings of the form $\beta_* \prec \beta$, β a join irreducible congruence, where β_* donotes the unique lower cover of β.

An $\langle\alpha,\beta\rangle$*–subtrace*, for $\alpha \prec \beta$, is a subset of an $\langle\alpha,\beta\rangle$–trace. See [1, 2] for the basic facts about subtraces. Here we are only interested in two elment subtraces and use the term subtrace to mean a two element subtrace. If β is join irreducible, a β–subtrace means a $\langle\beta_*,\beta\rangle$–subtrace.

We define $\{a,b\} \to \{c,d\}$ for two element subsets of A if $f(\{a,b\}) = \{c,d\}$ for some $f \in \mathrm{Pol}_1\,\mathbf{A}$. This relation on the two elements subsets of A is transitive and relexive, so it is a quasiorder. It is denoted $\mathbf{G}$ or $\mathbf{G}(\mathbf{A})$. We let $\mathbf{G}_{a,b}$ denote the restriction of $\mathbf{G}$ to $\{\{c,d\} : \mathrm{Cg}\,(c,d) = \mathrm{Cg}\,(a,b)\}$. (**Note:** this differs from $\mathbf{G}_{a,b}$ defined in [1].) If $\theta = \mathrm{Cg}\,(a,b)$ is a principal congruence we let $\mathbf{G}_\theta = \mathbf{G}_{a,b}$.

For distinct a and $b \in A$ let $\sigma(a,b)$ be the equivalence relation generated by

$$S = \big\{(c,d) : \{a,b\} \to \{c,d\} \text{ but } \{c,d\} \not\to \{a,b\}\big\}. \tag{1}$$

It is shown in [1, 2] that $\sigma(a,b)$ is a congruence relation, $\sigma(a,b) \le \mathrm{Cg}\,(a,b)$, and $\{a,b\}$ is a subtrace if and only if $\sigma(a,b) < \mathrm{Cg}\,(a,b)$. Note that if $\{a,b\} \in \mathbf{G}_{a,b}$ is not minimal then there is a pair $(c,d) \in S$ with $\mathrm{Cg}\,(c,d) = \mathrm{Cg}\,(a,b)$. So $\{a,b\}$ is not a subtrace. If $\beta = \mathrm{Cg}\,(a,b)$ is join irreducible and $\{a,b\}$ is a minimal element of $\mathbf{G}_\beta$, then all the elements of S are below β_* and hence $\{a,b\}$ is a subtrace. Thus we have the following lemma.

Lemma 2. *If $\{a,b\}$ is a subtrace then $\{a,b\}$ is a minimal element of $\mathbf{G}_{a,b}$. If β is a join irreducible congruence on $\mathbf{A}$ then $\{a,b\}$ is a β–subtrace if and only if it is a minimal element of $\mathbf{G}_\beta$.*

Example 3. Let $\mathbf{S}$ be a meet semilattice. If $f \in \mathrm{Pol}_1\,\mathbf{S}$ then $f(x) \le x$ and from this it follows that $\{a,b\}$ is the only element of $\mathbf{G}_{a,b}$ and hence is a minimal element of $\mathbf{G}_{a,b}$. Now $\mathrm{Cg}\,(a,b)$ is join irreducible if and only if $a \prec b$ or $b \prec a$. Also $\{a,b\}$ is a subtrace if and only if $a < b$ or $b < a$ because $\sigma(a,b) < \mathrm{Cg}\,(a,b)$ if and only if $a < b$ or $b < a$. Thus $\{a,b\}$ can be a subtrace without $\mathrm{Cg}\,(a,b)$ being join irreducible. Moreover, if a and b are incomparable, then $\{a,b\}$ is not a subtrace even though $\{a,b\}$ is a minimal element of $\mathbf{G}_{a,b}$.

The next lemma follows from Theorem 2.20 of [2].

Lemma 4. *Suppose β be a join irreducible congruence on $\mathbf{A}$, $\{a,b\}$ is a β–subtrace, and δ is a congruence with $\beta \not\le \beta_* \vee \delta$. Then $\{a/\delta, b/\delta\}$ is a subtrace in $\mathbf{A}/\delta$ and the type of $\{a/\delta, b/\delta\}$ is the same as that of $\{a,b\}$.*

Finding a subtrace. By Lemma 4 we may assume $\beta_* = 0$. Throughout the algorithm a lookup table is maintained which keeps track of which elements of $\mathbf{G}_{a,b}$ have been visited. Initially it has only $\{a,b\}$. Let

$$\Delta_2 = \{(v,v) : v \in A\}.$$

Now

$$\{(f(a), f(b)) : f \in \mathrm{Pol}_1 \mathbf{A}\} = \mathrm{Sg}_{\mathbf{A}^2}(\{(a, b)\} \cup \Delta_2) \tag{2}$$

We let $\mathbf{S}_{a,b}$ be this subalgebra of $\mathbf{A}^2$. The algorithm starts generating the universe of $\mathbf{S}_{a,b}$ by forming the closure in the usual way. Suppose an element (c, d) is generated. There are three cases. If $(c, d) \in \beta_*$ (and so $c = d$ by our assumption) we ignore it since it is in Δ_2 and we proceed to generate another element of the universe. If $c \neq d$ then the unordered pair $\{c, d\}$ is in $\mathbf{G}_{a,b}$. If $\{c, d\}$ has been visited before, we add (c, d) to the universe if it is not already there and again proceed to generate another element of the universe. But if $\{c, d\}$ has not been visited, we abandon our partially calculated universe and start over with (c, d), generating the universe of $\mathbf{S}_{c,d}$.

The algorithm ends with some pair (c, d) where all of $\mathbf{S}_{c,d}$ is calculated without finding an unvisited pair. Then $\{c, d\}$ is a minimal element of $\mathbf{G}_\beta$ and so a $\langle \beta_*, \beta \rangle$–subtrace. Also note that by (2) there is an $f \in \mathrm{Pol}_1(\mathbf{A})$ with $f(c) = d$ and $f(d) = c$ if and only if $(d, c) \in \mathbf{S}_{c,d}$. This information is useful in calculating the type since having such an involution rules out types **4** and **5** and not having one rules out types **2** and **3**.

Finding the type. Suppose $\{a, b\}$ is a $\langle \beta_*, \beta \rangle$–subtrace. Again we assume $\beta_* = 0$. Similar to what we did before, we let

$$\Delta_4 = \{(v, v, v, v) : v \in A\}.$$

and define

$$\begin{aligned} \mathbf{T}_{a,b} &= \{(h(a, a), h(a, b), h(b, a), h(b, b)) : h \in \mathrm{Pol}_2 \mathbf{A}\} \\ &= \mathrm{Sg}_{\mathbf{A}^4}(\{(a, a, b, b), (a, b, a, b)\} \cup \Delta_4) \end{aligned}$$

We may think of the elements of $\mathbf{T}_{a,b}$ as 2×2 tables, like

$$\begin{array}{c|cc} & a & b \\ \hline a & u & v \\ b & w & t \end{array}$$

Lemma 5. *Suppose $\beta \succ 0$ and $\{a, b\}$ is a subtrace. If $\mathbf{T}_{a,b}$ contains the element*

$$\begin{array}{c|cc} & a & b \\ \hline a & u & v \\ b & w & w \end{array} \quad \text{or} \quad \begin{array}{c|cc} & a & b \\ \hline a & u & w \\ b & v & w \end{array} \tag{3}$$

where $u \neq v$, then it contains

$$\begin{array}{c|cc} & a & b \\ \hline a & a & b \\ b & b & b \end{array} \tag{4}$$

Proof. Suppose the first table occurs and let $h \in \mathrm{Pol}_2(\mathbf{A})$ be the polynomial giving this table. Since $a \mathrel{\beta} b$, u, v and w all lie in a single β class. By Theorem 2.8(4) of [6] there is an $f \in \mathrm{Pol}_1(\mathbf{A})$ which maps A to a $\langle 0, \beta \rangle$ minimal set U and $f(u) \neq f(v)$. Since u, v and w are β-related, their images all lie in a single trace N of U. So the elements of the fh table all lie in N. Changing notation, we assume that u, v and w all lie in N.

Our table shows that the β, β-term condition fails and so the type of β is either **3**, **4** or **5** and thus $|N| = 2$ and so $\{a, b\}$ is a trace, $\{u, v\}$ is a trace, and $w = u$ or $w = v$. Traces are polynomially isomorphic by Proposition 2.12 of [3], so there is an element in $\mathrm{Pol}_1(\mathbf{A})$ mapping $\{u, v\}$ onto $\{a, b\}$. Applying this to our table we see that we get one of the following tables

$$\begin{array}{c|cc} & a & b \\ \hline a & a & b \\ b & b & b \end{array} \qquad \begin{array}{c|cc} & a & b \\ \hline a & a & b \\ b & a & a \end{array} \qquad \begin{array}{c|cc} & a & b \\ \hline a & b & a \\ b & b & b \end{array} \qquad \begin{array}{c|cc} & a & b \\ \hline a & b & a \\ b & a & a \end{array}$$

The first is the desired table. If any of the other three occur, we see there is a $g \in \mathrm{Pol}_1(\mathbf{A})$ which interchanges a and b. Using g, the other tables can be transformed into the first one. □

We call the table (4) a *join* and the dual is called a *meet*. As the algorithm proceeds we look at the each table we get while generating $\mathbf{T}_{a,b}$. If the first row of the table has distinct elements but the second doesn't or the first column has distinct elements but the second doesn't, we record that the trace has a join. If the dual situation occurs we record that it has a meet. If the trace has an involution, we quit as soon as we find either a join or a meet, since the type must be **3** in this case. If the elements on the main diagonal of the table are equal but different from one of the off-diagonal elements, or this same situation holds for the sinister diagonal, we record that the trace has a *one-snag*. A one-snag means the trace cannot be of type **1**; see Theorem 7.2 of [6].

If we find both a join and a meet (and there is no involution) we quit since the type must be **4**. In the other cases we must generate all of $\mathbf{T}_{a,b}$. If a join or a meet has been found, the type is **5**. Otherwise if a one-snag has been found the type is **2** and in the remaining case it is **1**.

Now if the type of $\mathrm{typ}(\beta_*, \beta)$ is either **3** or **4** then the algorithm does not compute all of $\mathbf{T}_{a,b}$. So we investigate $\mathbf{T}_{a,b}$ (and how big it can be) when the type is **2** or **5**.

3. Type 5

In this section we assume $\beta \succ 0$ for convenience.

Lemma 6. *If the type of β is not* **1***, then $\mathbf{G}_\beta$ has a unique minimal block ($\equiv$–class).*

Proof. Suppose $\{a, b\}$ and $\{c, d\}$ are both minimal in $\mathbf{G}_\beta$. Then by Lemma 2 each is a subtrace. Let N and N' be traces with a, $b \in N$ and c, $d \in N'$. By Proposition 2.12 of [3], $f : N \simeq N'$ for some $f \in \mathrm{Pol}_1 A$. If the type of β is **3**, **4**, or **5** then $|N| = |N'| = 2$ and so there an edge $\{a, b\} \to \{c, d\}$ in $\mathbf{G}_\beta$ and vise versa. If the type of β is **2**, then the traces are polynomially equivalent to one dimensional vector spaces over a field and so the polynomials are doubly transitive on the trace. This again implies there are edges in each direction connecting $\{a, b\}$ and $\{c, d\}$. □

A cyclic permutation acting on a 5 element set gives a simple unary algebra with more than one minimal block in $\mathbf{G}_\beta$.

The next lemma states some well known facts about traces that we will need.

Lemma 7. *Suppose $\beta \succ 0$.*

(1) *If $|x/\beta| > 1$, then there is a β–trace $N \subseteq x/\beta$ containing x.*
(2) *If $x \mathrel{\beta} y$, $x \neq y$, and N is a β–trace, then there is an $f \in \mathrm{Pol}_1(\mathbf{A})$ mapping x/β into N such that $f(x) \neq f(y)$. Moreover, if the type of β is not* **1***, then we can find such an f mapping onto N.*

Now we assume that the type of β is **4** or **5**. In this case there is no involution so for each $x \neq y$ at most one of (x, y) and (y, x) is in $\mathbf{S}_{a,b}$. So this defines a directed graph on A without edges in both directions between any pair of elements. But more is true: the graph is acyclic so its transitive closure is an admissible (partial) order on A. We denote this order using $\leq$. The connected components of $\leq$ are precisely the β-blocks. These facts are proved in Theorem 5.26 of [6].

Under this order, if $(x, y, u, v) \in \mathbf{T}_{a,b}$, then

$$x \leq y \leq v \quad \text{and} \quad x \leq u \leq v$$

since this holds for the generators. Also observe that if $(x, y, u, v) \in \mathbf{T}_{a,b}$ then there is an $h \in \mathrm{Pol}_2(\mathbf{A})$ such that

$$\begin{array}{c|cc} h & a & b \\ \hline a & x & y \\ b & u & v \end{array}$$

and so (x, y), (x, u), (y, v), (u, v) and (x, v) are all in $\mathbf{S}_{a,b}$ and thus traces unless they are in Δ_2.

Theorem 8. *Suppose $\beta \succ 0$ and the type of β is* **4** *or* **5** *and $\{a, b\}$ is a trace for β. Suppose $\mathbf{T}_{a,b}$ contains a meet. Then the type of β is* **5** *if and only if (x, y, u, v) and $(x', y, u, v') \in \mathbf{T}_{a,b}$ imply $x = x'$.*

Moreover, if the type of β is **5** *and $(x, y, u, v) \in \mathbf{T}_{a,b}$, then x is a maximal element of $\{z : z \leq y,\ z \leq u\}$.*

Proof. If the type is **4**, $(a, b, b, b) \in \mathbf{T}_{a,b}$. Since (b, b, b, b) is also in $\mathbf{T}_{a,b}$, we see the condition fails, proving one direction.

For the converse suppose $(x, y, u, v) \in \mathbf{T}_{a,b}$. All four of these elements lie in a single β-class C. First suppose there is an element w with $x < w \leq y$, u. We may assume w covers x and so $(x, w) \in \mathbf{S}_{a,b}$ and so is a trace. Since $N = \{x, v\}$ is also a trace, there is an $f \in \mathrm{Pol}_1(\mathbf{A})$ mapping C onto N with $f(x) \neq f(w)$ by Lemma 7. Since f must preserve the order, $f(x) = x$ and $f(w) = v$. Since $w \leq y$, $f(y) = v$ and similarly $f(u) = v$. So $(f(x), f(y), f(u), f(v)) = (x, v, v, v) \in \mathbf{T}_{a,b}$, giving a join by Lemma 5. This proves the last statement of the theorem.

Now suppose both (x, y, u, v) and $(x', y, u, v') \in \mathbf{T}_{a,b}$ and that $x \neq x'$. By the above, x and x' must be incomparable. Since x and x' are below y and (x, y) and (x', y) are both in β, $(x, x') \in \beta$. Thus there is an $f \in \mathrm{Pol}_1(\mathbf{A})$ mapping C onto N with $f(x) \neq f(x')$. First suppose $f(x) = v$ and $f(x') = x$. Then, since $x \leq y$ and u, $f(y) = f(u) = f(v') = v$. So $(f(x'), f(y), f(u), f(v')) = (x, v, v, v)$, giving a join.

Finally if $f(x) = x$ and $f(x') = v$ then as before $(f(x), f(y), f(u), f(v)) = (x, v, v, v)$, again giving a join. Hence $x = x'$. □

Example 9. Let $\mathbf{A} = (\{a, b, c, d, 1\}, \cdot, f, g, h)$ be a five element algebra of similarity type $(2, 1, 1, 1)$. The binary operation is defined by the rules (applied in this order) $1 \cdot 1 = 1$, $x \cdot 1 = b$, $1 \cdot y = d$ and $x \cdot y = a$. The first unary is defined by $f(b) = d$, $f(d) = b$ and $f(x) = x$ otherwise. We define $g(a) = d$, $g(b) = 1$, and $g(x) = x$ otherwise. For h we define $h(a) = h(b) = h(c) = c$ and $h(d) = h(1) = d$.

One can check this algebra is simple of type **5**. Its ordered set is given in Figure 1.

$\mathbf{T}_{a,b}$ contains $(a, b, d, 1)$ (but not $(c, b, d, 1)$, of course). This shows that it is not necessarily true that $(x, y, u, v) \in \mathbf{T}_{a,b}$ implies x is the greatest lower bound of y and u.

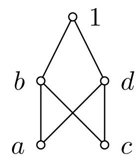

FIGURE 1

In [6], the authors show there is an admissible order also contained in β, here denoted $\sqsubseteq$, extending $\leq$. This order is defined by $x \sqsubseteq y$ if $x \ \beta \ y$ and, for all $f \in \mathrm{Pol}_1(\mathbf{A})$ mapping x/β onto a β–trace, $f(x) \leq f(y)$. Note that by Lemma 6, if $x \neq y$ and $x \ \beta \ y$ there is an $f \in \mathrm{Pol}_1(\mathbf{A})$ such that $f\{x, y\}) = \{a, b\}$. If all such f map x to a and y to b, then $x \sqsubseteq y$ and conversely.

Using the $\sqsubseteq$ order, there is a nicer formulation of Theorem 8. The next theorem follows easily from the results of Kearnes and Szendrei [8] and is closely related to the concept of rectangularization as presented in Chapter 5 of the monograph of Kearnes and Kiss [7].

Theorem 10. *Suppose $\beta \succ 0$, the type of β is **4** or **5**, $\{a, b\}$ is a trace for β and $\mathbf{T}_{a,b}$ contains a meet. Then the type of β is **5** if and only if $(x, y, u, v) \in \mathbf{T}_{a,b}$ implies x is the greatest lower bound of y and u in the $\sqsubseteq$ order.*

Proof. Suppose $(x, y, u, v) \in \mathbf{T}_{a,b}$. If $x = v$ then all four entries are equal and the result is trivial, so we may assume $x < v$. Now suppose there is an element $w \neq x$ with $w \sqsubseteq y$ and $w \sqsubseteq u$. Since $x \ \beta \ w$ there is an $f \in \mathrm{Pol}_1(\mathbf{A})$ mapping x/β onto $\{x, v\}$ with $f(x) \neq f(w)$. If all such maps send x to v and w to x, then $w \sqsubseteq x$, as desired. So there must exist such an f with $f(x) = x$ and $f(w) = v$. Since $w \sqsubseteq y$ and $w \sqsubseteq u$, $f(y) = f(u) = v$. So applying f to (x, y, u, v) gives $(x, v, v, v) \in \mathbf{T}_{a,b}$, giving a join and thus showing the type is **4**.

The proof of the converse is the same as in Theorem 8. $\square$

Of course Theorem 8 implies $|\mathbf{T}_{a,b}|$ is bounded by a constant times $|\mathbf{A}|^3$ but we can get the exact bound given in Theorem 1.

Theorem 11. *Suppose $\beta \succ 0$ and the type of β is **5** and $\{a, b\}$ is a trace for β. Let $n = |\mathbf{A}|$. Then $|\mathbf{T}_{a,b}| \leq n^3/3 + n^2/2 + n/6$. Moreover, for each $n \geq 2$ there is a simple algebra of type **5** with n elements which achieves this bound.*

Proof. Let $x_1, \ldots, x_n$ be a linear extension of the order on $\mathbf{A}$. If $(x_r, y_s, u_t, x_k) \in \mathbf{T}_{a,b}$ then s, $t \leq k$ and r is determined by the others. So there are at most k^2 elements of $\mathbf{T}_{a,b}$ whose last coordinate is x_k. Summing on k gives the desired bound.

To see this bound is achieved, we take the n element chain as a meet semilattice. We add 3 unary operations: the first moves every element to its lower cover (and fixes the least element). The second moves every element to its upper cover (and fixes the greatest element). The third fixes the least element and moves everything else to the greatest element. We leave it as an exercise to show the bound is achieved. $\square$

We record a corollary to Theorem 8 which can be used to recognize type **4** after what is known as a two-snag has been found.

Corollary 12. *Suppose that $\beta \succ 0$ and that $\{a,b\}$ is a $(0,\beta)$–subtrace. Assume $\mathbf{T}_{a,b}$ has a meet and that $(b,a) \notin \mathbf{S}_{a,b}$ (so there is no involution and the type of β is either* **4** *or* **5***). If $\mathbf{T}_{a,b}$ contains*

$$\begin{array}{c|cc} & a & b \\ \hline a & u & v \\ b & v & w \end{array}$$

with $u \neq v$, then the type of β is **4***.*

4. Type 2

As before we assume $\beta \succ 0$, but now we assume its type is either **1** or **2**. This implies $[\beta,\beta] = 0$. In order to show that $|\mathbf{T}_{a,b}| \leq |\mathbf{A}|^3$, we show that each β block can be embedded into an Abelian group such that if $(x,y,u,v) \in \mathbf{T}_{a,b}$ then $x - y - u + v = 0$, which of course does it.

If $\mathbf{A}$ happens to lie in a modular variety, then the commutator theory for modular varieties shows each β block is affine; see, for example, Corollary 5.8 of [4].

In [9], Kearnes and Szendrei investigate the *linear commutator* of Quackenbush [10] which is denoted $[\alpha,\beta]_\ell$. An unpublished result of Ross Willard shows that $[\beta,\beta]_\ell = 0$, and this implies that the β blocks are quasi-affine which gives us what we want.

Theorem 13 (R. Willard)**.** *If $\mathbf{A}$ is a finite algebra, $\beta \succ 0$ in $\mathbf{Con\ A}$ and $[\beta,\beta] = 0$, then $[\beta,\beta]_\ell = 0$.*

Proof. Recall that $\mathrm{M}(\beta,\beta)$ is the subalgebra of $\mathbf{A}^4$, with elements viewed as two by two matrices, generated by all matrices of the form $\begin{bmatrix} x & y \\ x & y \end{bmatrix}$ and $\begin{bmatrix} x & x \\ y & y \end{bmatrix}$, where $x\ \beta\ y$; see Proposition 3.3 in [4]. For a matrix in $\mathrm{M}(\beta,\beta)$ we call the positions on the main diagonal *positive positions* and positions on the sinister diagonal *negative positions*. By Section 2 of Kearnes and Szendrei's paper [9] if $[\beta,\beta]_\ell \neq 0$ then there is a finite collection of matrices in $\mathrm{M}(\beta,\beta)$ and a matching between the positive and negative positions such that the elements occupying the matched positions are equal except in one case. Let $u \neq v$ be the unequal elements in the matching. If we assume our set of matrices witnessing $[\beta,\beta]_\ell \neq 0$ is minimal, we can argue that all the elements of this collection of matrices lie in a single β class. By Lemma 7, there is an $f \in \mathrm{Pol}_1(\mathbf{A})$ mapping this β class into a β trace N such that $f(u) \neq f(v)$. If we apply f to all the matrices in our collection, the resulting set will still witnesses $[\beta,\beta]_\ell \neq 0$. Moreover all the elements of the matrices lie in N.

Now if the type of β is **2**, then N has an Abelian group structure such that if $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is in $\mathrm{M}(\beta,\beta)$ and x, y, z and $w \in N$, then $x - y - z + w = 0$. Thus substracting the sum of the negative elements of our set of matrices from the sum of the positive elements gives 0. But it is also clear by the matching that this sum is $u - v$ (or $v - u$). This contradiction proves the result when the type of β is **2**.

In the remaining case β has type **1**. The fact that the elements of the matrices lie in a single trace implies these matrices must be of the form $\begin{bmatrix} x & y \\ x & y \end{bmatrix}$ or $\begin{bmatrix} x & x \\ y & y \end{bmatrix}$, and from this it is not hard to see that again $u = v$. □

Since $[\beta,\beta]_\ell = 0$ implies that the β-classes are quasi-affine, Theorem 13 implies that $\mathbf{T}_{a,b} \leq |\mathbf{A}|^3$. Combining this with Theorem 11, we have the following.

Theorem 14. *Suppose $\beta \succ 0$ and the type of β is not* **3** *or* **4**. *Suppose $\{a, b\}$ is a subtrace for β. Then*

$$|\mathbf{T}_{a,b}(\mathbf{A})| \leq |\mathbf{A}|^3$$

From this theorem we see that the time to compute the type of a congruence $\beta \succ 0$ in an algebra $\mathbf{A}$ is bounded by a constant times the cube of the input size (the size of the operation tables) of $\mathbf{A}$; see [1].

5. Type 4

The proof of the bound in Theorem 1 for a type 4 quotient is similar to the proof of Theorem 11. The number of 4-tuples (a, b, c, d) with $1 \leq a \leq b \leq d \leq n$ and $1 \leq a \leq c \leq d \leq n$ is $n^4/12 + n^3/3 + 5n^2/12 + n/6$ so this is an upper bound. The algebra $\mathbf{B}$ we get by adding the join operation to the algebra $\mathbf{A}$ used in Theorem 11 witnesses that the bound can occur for every n. (We can omit the second unary operation of $\mathbf{A}$ from $\mathbf{B}$ and it will still witness the bound, but we cannot omit it from $\mathbf{A}$.)

We leave the details and the proof that the bounds in Theorem 1 for types 2 and 3 can be achieved infinitely often to the reader.

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