LOCAL DIFFERENCE TERMS

Let $\mathbf{p} = (p_0, p_1, \ldots, p_n)$ be an $(n+1)$-tuple of ternary terms, where $p_0(x, y, z) \approx x$ and $p_n(x, y, z) \approx z$, the first and third ternary projections, respectively. Let $\mathbf{A} = \langle A, \ldots \rangle$ be an algebra. In [VW14], Valeriote and Willard define an $\mathbf{A}$-*triple for* $\mathbf{p}$ to be a triple (a, b, i) such that $a, b \in A$ and $p_i(a, b, b) = p_{i+1}(a, a, b)$. They use this to define a "local HagemannMitschke sequence" on which they base an efficient algorithm for deciding for a given n whether an idempotent variety is n-permutable. This inspired us to develop a related construct, called a "local difference term," that we expect will be useful for the purpose of deciding, given a finite idempotent algebra $\mathbf{A}$, whether the variety generated by $\mathbf{A}$ has a difference term.

Let $\mathbf{A} = \langle A, \ldots \rangle$ be an algebra, fix $a, b \in A$ and $i \in \{0, 1\}$. An $\mathbf{A}$-*local difference term for* (a, b, i) is a ternary term p satisfying the following:

$$\begin{aligned} &\text{if } i = 0, \text{ then } a\ [\mathrm{Cg}^{\mathbf{A}}(a, b), \mathrm{Cg}^{\mathbf{A}}(a, b)]\ p(a, b, b); \\ &\text{if } i = 1, \text{ then } p(a, a, b) = b. \end{aligned} \tag{0.1}$$

When often drop the $\mathbf{A}$ when the algebra is clear from context. For example, we write Cg in place of $\mathrm{Cg}^{\mathbf{A}}$, and call the term p above a local difference term for (a, b, i). If p satisfies (0.1) for all triples in some subset $S \subseteq A \times A \times \{0, 1\}$, then we call p a *local difference term for* S.

Let $\mathcal{S} = A \times A \times \{0, 1\}$ and suppose that every pair $((a_0, b_0, \chi_0), (a_1, b_1, \chi_1))$ in $\mathcal{S}^2$ has a local difference term. That is, for each pair $((a_0, b_0, \chi_0), (a_1, b_1, \chi_1))$, there exists p such that for each $i \in \{0, 1\}$ we have

$$\begin{aligned} a_i\ [\mathrm{Cg}(a_i, b_i), \mathrm{Cg}(a_i, b_i)]\ p(a_i, b_i, b_i),&\ \text{ if } \chi_i = 0, \text{ and} \\ p(a_i, a_i, b_i) = b_i,&\ \text{ if } \chi_i = 1. \end{aligned} \tag{0.2}$$

Under these hypothesis we will prove that every subset $S \subseteq \mathcal{S}$ has a local difference term. That is, there is a single term p that works (i.e., satisfies (0.2)) for all $(a_i, b_i, \chi_i) \in S$. The statement and proof of this new result follows.

Theorem (cf. [VW14, Theorem 2.2])**.** Let $\mathcal{V}$ be an idempotent variety and $\mathbf{A} \in \mathcal{V}$. Define $\mathcal{S} = A \times A \times \{0, 1\}$ and suppose that every pair $((a_0, b_0, \chi_0), (a_1, b_1, \chi_1)) \in \mathcal{S}^2$ has a local difference term. Then every subset $S \subseteq \mathcal{S}$, has a local difference term.

Proof. The proof is by induction on the size of S. In the base case, $|S| = 2$, the claim holds by assumption. Fix $n > 2$ and assume that every subset of $\mathcal{S}$ of size $2 \leqslant k \leqslant n$ has a local

Date: January 8, 2017.

difference term. Let $S = \{(a_0, b_0, \chi_0), (a_1, b_1, \chi_1), \dots, (a_n, b_n, \chi_n)\} \subseteq \mathcal{S}$, so that $|S| = n + 1$. We prove S has a local difference term.

Since $|S| \geqslant 3$ and $\chi_i \in \{0, 1\}$ for all i, there must exist indices $i \neq j$ such that $\chi_i = \chi_j$. Assume without loss of generality that one of these indices is $j = 0$. Define the set $S' = S \setminus \{(a_0, b_0, \chi_0)\}$. Since $|S'| < |S|$, the set S' has a local difference term p. We split the remainder of the proof into two cases. In the first case $\chi_0 = 0$ and in the second $\chi_0 = 1$.

<u>Case 1:</u> $\chi_0 = 0$.

Assume $\chi_0 = 0$ and, without loss of generality, suppose $\chi_1 = \chi_2 = \cdots = \chi_k = 1$, and $\chi_{k+1} = \chi_{k+2} = \cdots = \chi_n = 0$. Define

$$T = \{(a_0, p(a_0, b_0, b_0), 0), (a_1, b_1, 1), (a_2, b_2 1), \dots, (a_k, b_k, 1)\}.$$

Let t be a local difference term for T. Define

$$d(x, y, z) = t(x, p(x, y, y), p(x, y, z)).$$

Since $\chi_0 = 0$, we want $d(a_0, b_0, b_0)\ [\mathrm{Cg}(a_0, b_0), \mathrm{Cg}(a_0, b_0)]\ a_0$. We have

$$d(a_0, b_0, b_0) = t(a_0, p(a_0, b_0, b_0), p(a_0, b_0, b_0))\ [\tau, \tau]\ a_0, \tag{0.3}$$

where $\tau := \mathrm{Cg}(a_0, p(a_0, b_0, b_0))$. Notice that

$$(a_0, p(a_0, b_0, b_0)) = (p(a_0, a_0, a_0), p(a_0, b_0, b_0)) \in \mathrm{Cg}(a_0, b_0),$$

so $\tau \leqslant \mathrm{Cg}(a_0, b_0)$. Therefore, $[\tau, \tau] \leqslant [\mathrm{Cg}(a_0, b_0), \mathrm{Cg}(a_0, b_0)]$, by monotonicity of the commutator. It follows from this and (0.3) that $d(a_0, b_0, b_0)\ [\mathrm{Cg}(a_0, b_0), \mathrm{Cg}(a_0, b_0)]\ a_0$, as desired.

For $1 \leqslant i \leqslant k$ we have $\chi_i = 1$, so we want $d(a_i, a_i, b_i) = b_i$. Indeed,

$$d(a_i, a_i, b_i) = t(a_i, p(a_i, a_i, a_i), p(a_i, a_i, b_i)) = t(a_i, a_i, b_i) = b_i. \tag{0.4}$$

The first equality in (0.4) holds by definition of d; the second, because p is an idempotent local difference term for S'; the third, because t is a local difference term for T.

The remaining triples in our original set S have index j satisfying $k < j \leqslant n$ and $\chi_j = 0$. Thus, for these triples we want $d(a_j, b_j, b_j)\ [\mathrm{Cg}(a_j, b_j), \mathrm{Cg}(a_j, b_j)]\ a_j$. Indeed, this holds since p is a local difference term for S', whence $p(a_j, b_j, b_j))\ [\mathrm{Cg}(a_j, b_j), \mathrm{Cg}(a_j, b_j)]\ a_j$, so

$$d(a_j, b_j, b_j) = t(a_j, p(a_j, b_j, b_j), p(a_j, b_j, b_j))\ [\mathrm{Cg}(a_j, b_j), \mathrm{Cg}(a_j, b_j)]\ t(a_j, a_j, a_j) = a_j,$$

as desired.

<u>Case 2:</u> $\chi_0 = 1$.

Assume $\chi_0 = 1$ and, without loss of generality, suppose $\chi_1 = \chi_2 = \cdots = \chi_k = 0$, and

$\chi_{k+1} = \chi_{k+2} = \cdots = \chi_n = 1$. Define

$$T = \{(p(a_0, a_0, b_0), b_0, 1), (a_1, b_1, 0), (a_2, b_2 0), \ldots, (a_k, b_k, 0)\}.$$

Let t be a local difference term for T. Define

$$d(x, y, z) = t(p(x, y, z), p(y, y, z), z).$$

Since $\chi_0 = 1$, we want $d(a_0, a_0, b_0) = b_0$. By the definition of d,

$$d(a_0, a_0, b_0) = t(p(a_0, a_0, b_0), p(a_0, a_0, b_0), b_0) = b_0.$$

The last equality holds since t is a local difference term for T, thus, for $(p(a_0, a_0, b_0), b_0, 1)$.

For $1 \leqslant i \leqslant k$ we have $\chi_i = 0$, so we want $d(a_i, b_i, b_i)\ [\mathrm{Cg}(a_i, b_i), \mathrm{Cg}(a_i, b_i)]\ a_i$. Again, starting from the definition of d and using idempotence of p, we have

$$d(a_i, b_i, b_i) = t(p(a_i, b_i, b_i), p(b_i, b_i, b_i), b_i) = t(p(a_i, b_i, b_i), b_i, b_i). \tag{0.5}$$

Next, since p is a local difference term for S', we have

$$t(p(a_i, b_i, b_i), b_i, b_i)\ [\mathrm{Cg}(a_i, b_i), \mathrm{Cg}(a_i, b_i)]\ t(a_i, b_i, b_i). \tag{0.6}$$

Finally, since t is a local difference term for T, hence for (a_i, b_i, b_i) $(1 \leqslant i \leqslant k)$, we have $t(a_i, b_i, b_i)\ [\mathrm{Cg}(a_i, b_i), \mathrm{Cg}(a_i, b_i)]\ a_i$. Combining this with (0.5) and (0.6) yields

$$d(a_i, b_i, b_i)\ [\mathrm{Cg}(a_i, b_i), \mathrm{Cg}(a_i, b_i)]\ a_i,$$

as desired.

The remaining elements of our original set S have indices j satisfying $k < j \leqslant n$ and $\chi_j = 1$. For these we want $d(a_j, a_j, b_j) = b_j$. Since p is a local difference term for S', we have $p(a_j, a_j, b_j) = b_j$, and this along with idempotence of t yields

$$d(a_j, a_j, b_j) = t(p(a_j, a_j, b_j), p(a_j, a_j, b_j), b_i) = t(b_j, b_j, b_j) = b_j,$$

as desired. □

Corollary. Let $\mathbf{A}$ be a finite idempotent algebra and suppose every pair $((a, b, i), (a', b', i'))$ in $(A \times A \times \{0, 1\})^2$ has a local difference term. Then $\mathbf{A}$ has a difference term operation.

Proof. By the Theorem, the whole set $(A \times A \times \{0, 1\})^2$ has a single local difference term, d, and this is a difference term operation for $\mathbf{A}$. Indeed, for all $a, b \in A$, d is a local difference term for $(a, b, 0)$, so $a\ [\mathrm{Cg}(a, b), \mathrm{Cg}(a, b)]\ d(a, b, b)$, and d is also a local difference term for $(a, b, 1)$, so $d(a, a, b) = b$. □

References

[VW14] M. Valeriote and R. Willard. Idempotent n-permutable varieties. *Bull. Lond. Math. Soc.*, 46(4):870–880, 2014. URL: `http://dx.doi.org/10.1112/blms/bdu044`, `doi:10.1112/blms/bdu044`.

University of Hawaii
E-mail address: `williamdemeo@gmail.com`
E-mail address: `ralph@math.hawaii.edu`