Stealing from the best: Why having a minority is not local

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All of this is reconstructed from what Dmitriy Zhuk showed me in Nashville. However, my memory is imperfect and I've tried to beautify the example as much as possible. I therefore take responsibility for any and all errors.

We will need to employ a minority operation m on $[n]$. To keep our example concrete we pick

$$m(x, y, z) = \begin{cases} x & y = z \\ y & x = z \\ z & \text{else.} \end{cases}$$

We denote by $\oplus$ the addition in $\mathbb{Z}_2$, ie. $1 \oplus 1 = 0$. Note that $2a + 2b \equiv 2(a \oplus b) \pmod 4$. We will evaluate $\oplus$ before $+$ in expressions.

For each $n \in \mathbb{Z}$ there are two unique numbers $n_0, n_1 \in \{0, 1\}$ such that $n \equiv n_0 + 2n_1 \pmod 4$. We denote the function that assigns to n the number n_1 by $\operatorname{carry}(n)$. Alternative definition: $\operatorname{carry}(n)$ is the function from $\mathbb{Z}$ to $\{0, 1\}$ defined as

$$\operatorname{carry}(n) = \begin{cases} 0 & n \equiv 0, 1 \pmod 4 \\ 1 & n \equiv 2, 3 \pmod 4. \end{cases}$$

Putting $n = a_1 - a_2 + a_3$ for some $a_1, a_2, a_3 \in \{0, 1\}$, we get

$$a_1 \oplus a_2 \oplus a_3 + 2 \operatorname{carry}(a_1 - a_2 + a_3) \equiv a_1 - a_2 + a_3 \pmod 4.$$

This will be useful later.

Pick your favorite $n \in \mathbb{N}$ (the larger, the better). Consider the algebra $\mathbf{A}$ with $A = [n] \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and the n basic operations $t_i(x, y, z)$ defined as follows: We let

$$t_i(i, b_1, c_1), (i, b_2, c_2), (i, b_3, c_3)) = (i, b_1 \oplus b_2 \oplus b_3, c_1 \oplus c_2 \oplus c_3 \oplus \text{carry}(b_1 - b_2 + b_3)).$$

Note that the first coordinate of all three inputs needs to be exactly i, i.e.. the index of t_i, for this case to occur.

In all other cases we let:

$$t_i((a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3)) = (m(a_1, a_2, a_3), b_1 \oplus b_2 \oplus b_3, c_1 \oplus c_2 \oplus c_3)$$

Example: We have

$$\begin{aligned} t_4((4,1,1),(4,0,1),(4,1,0)) &= (4,0,1) \\ t_4((4,1,1),(3,0,1),(4,1,0)) &= (4,0,0) \\ t_3((4,1,1),(4,0,1),(4,1,0)) &= (4,0,0). \end{aligned}$$

Claim 1. *For every $n-1$-element subset E of $\mathbf{A}$, there is an operation of $\mathbf{A}$ that acts as a minority on E.*

Proof. Pick i such that no element of E has its first coordinate equal to i. Then t_i will be the local minority. $\square$

Let us define the $3n$-ary relation R on A as follows: R contains exactly all the tuples

$$\begin{pmatrix} (1, a, x_1) \\ (2, a, x_2) \\ \vdots \\ (n, a, x_n) \\ (1, b, x_{n+1}) \\ (2, b, x_{n+2}) \\ \vdots \\ (n, b, x_{2n}) \\ (1, c, x_{2n+1}) \\ (2, c, x_{2n+2}) \\ \vdots \\ (n, c, x_{3n}) \end{pmatrix}$$

where $a + b + c + 2 \bigoplus_{i=1}^{3n} x_i \equiv 2 \pmod 4$.

Another view of R (this is how Dmitriy would probably prefer to view it; he got this example by thinking about key relations): R is a union of the three relations

$$\begin{pmatrix} (1,0,x_1) \\ (2,0,x_2) \\ \vdots \\ (n,0,x_n) \\ (1,1,x_{n+1}) \\ (2,1,x_{n+2}) \\ \vdots \\ (n,1,x_{2n}) \\ (1,1,x_{2n+1}) \\ (2,1,x_{2n+2}) \\ \vdots \\ (n,1,x_{3n}) \end{pmatrix}, \begin{pmatrix} (1,1,x_1) \\ (2,1,x_2) \\ \vdots \\ (n,1,x_n) \\ (1,0,x_{n+1}) \\ (2,0,x_{n+2}) \\ \vdots \\ (n,0,x_{2n}) \\ (1,1,x_{2n+1}) \\ (2,1,x_{2n+2}) \\ \vdots \\ (n,1,x_{3n}) \end{pmatrix}, \begin{pmatrix} (1,1,x_1) \\ (2,1,x_2) \\ \vdots \\ (n,1,x_n) \\ (1,1,x_{n+1}) \\ (2,1,x_{n+2}) \\ \vdots \\ (n,1,x_{2n}) \\ (1,0,x_{2n+1}) \\ (2,0,x_{2n+2}) \\ \vdots \\ (n,0,x_{3n}) \end{pmatrix},$$

where $x_1 + \cdots + x_{3n} \equiv 0 \pmod 2$, plus the relation

$$\begin{pmatrix} (1,0,x_1) \\ (2,0,x_2) \\ \vdots \\ (n,0,x_n) \\ (1,0,x_{n+1}) \\ (2,0,x_{n+2}) \\ \vdots \\ (n,0,x_{2n}) \\ (1,0,x_{2n+1}) \\ (2,0,x_{2n+2}) \\ \vdots \\ (n,0,x_{3n}) \end{pmatrix},$$

where $x_1 + \cdots + x_{3n} \equiv 1 \pmod 2$.

In particular, R contains the three tuples:

$$\begin{pmatrix}(1,1,0)\\(2,1,0)\\\vdots\\(n,1,0)\\(1,1,0)\\(2,1,0)\\\vdots\\(n,1,0)\\(1,0,0)\\(2,0,0)\\\vdots\\(n,0,0)\end{pmatrix},\begin{pmatrix}(1,1,0)\\(2,1,0)\\\vdots\\(n,1,0)\\(1,0,0)\\(2,0,0)\\\vdots\\(n,0,0)\\(1,1,0)\\(2,1,0)\\\vdots\\(n,1,0)\end{pmatrix},\begin{pmatrix}(1,0,0)\\(2,0,0)\\\vdots\\(n,0,0)\\(1,1,0)\\(2,1,0)\\\vdots\\(n,1,0)\\(1,1,0)\\(2,1,0)\\\vdots\\(n,1,0)\end{pmatrix},$$

but not the tuple:

$$\begin{pmatrix}(1,0,0)\\(2,0,0)\\\vdots\\(n,0,0)\\(1,0,0)\\(2,0,0)\\\vdots\\(n,0,0)\\(1,0,0)\\(2,0,0)\\\vdots\\(n,0,0)\end{pmatrix}.$$

Claim 2. *The relation R defined above is $\mathbf{A}$ invariant (and $\mathbf{A}$ therefore has no global minority).*

Proof. By symmetry of t_i's and R, it is enough to show that t_1 preserves R.

Let us take three arbitrary members of R and apply t_1.

$$\begin{pmatrix} (1, a_1, x_{1,1}) \\ (2, a_1, x_{1,2}) \\ \vdots \\ (n, a_1, x_{1,n}) \\ (1, b_1, x_{1,n+1}) \\ (2, b_1, x_{1,n+2}) \\ \vdots \\ (n, b_1, x_{1,2n}) \\ (1, c_1, x_{1,2n+1}) \\ (2, c_1, x_{1,2n+2}) \\ \vdots \\ (n, c_1, x_{1,3n}) \end{pmatrix}, \begin{pmatrix} (1, a_2, x_{2,1}) \\ (2, a_2, x_{2,2}) \\ \vdots \\ (n, a_2, x_{2,n}) \\ (1, b_2, x_{2,n+1}) \\ (2, b_2, x_{2,n+2}) \\ \vdots \\ (n, b_2, x_{2,2n}) \\ (1, c_2, x_{2,2n+1}) \\ (2, c_2, x_{2,2n+2}) \\ \vdots \\ (n, c_2, x_{2,3n}) \end{pmatrix}, \begin{pmatrix} (1, a_3, x_{3,1}) \\ (2, a_3, x_{3,2}) \\ \vdots \\ (n, a_3, x_{3,n}) \\ (1, b_3, x_{3,n+1}) \\ (2, b_3, x_{3,n+2}) \\ \vdots \\ (n, b_3, x_{3,2n}) \\ (1, c_3, x_{3,2n+1}) \\ (2, c_3, x_{3,2n+2}) \\ \vdots \\ (n, c_3, x_{3,3n}) \end{pmatrix}$$

We obtain the tuple:

$$r = \left(\begin{array}{llll} (1, & a_1 \oplus a_2 \oplus a_3, & \bigoplus_{i=1}^{3} x_{i,1} & \oplus \operatorname{carry}(a_1 - a_2 + a_3)) \\ (2, & a_1 \oplus a_2 \oplus a_3, & \bigoplus_{i=1}^{3} x_{i,2}) & \\ & \quad\vdots & & \\ (n, & a_1 \oplus a_2 \oplus a_3, & \bigoplus_{i=1}^{3} x_{i,n}) & \\ (1, & b_1 \oplus b_2 \oplus b_3, & \bigoplus_{i=1}^{3} x_{i,n+1} & \oplus \operatorname{carry}(b_1 - b_2 + b_3)) \\ (2, & b_1 \oplus b_2 \oplus b_3, & \bigoplus_{i=1}^{3} x_{i,n+2}) & \\ & \quad\vdots & & \\ (n, & b_1 \oplus b_2 \oplus b_3, & \bigoplus_{i=1}^{3} x_{i,2n}) & \\ (1, & c_1 \oplus c_2 \oplus c_3, & \bigoplus_{i=1}^{3} x_{i,2n+1} & \oplus \operatorname{carry}(c_1 - c_2 + c_3)) \\ (2, & c_1 \oplus c_2 \oplus c_3, & \bigoplus_{i=1}^{3} x_{i,2n+2}) & \\ & \quad\vdots & & \\ (n, & c_1 \oplus c_2 \oplus c_3, & \bigoplus_{i=1}^{3} x_{i,3n}) & \end{array} \right)$$

We now need to calculate for a while. We know that

$$a_1 + b_1 + c_1 + 2\bigoplus_{i=1}^{3n} x_{i,1} \equiv 2 \pmod 4$$
$$a_2 + b_2 + c_2 + 2\bigoplus_{i=1}^{3n} x_{i,2} \equiv 2 \pmod 4$$
$$a_3 + b_3 + c_3 + 2\bigoplus_{i=1}^{3n} x_{i,3} \equiv 2 \pmod 4.$$

Take the sum of the first and third equation and subtract the second. We get

$$(a_1 - a_2 + a_3) + (b_1 - b_2 + b_3) + (c_1 - c_2 + c_3) + 2 \bigoplus_{i=1,j=1,2,3}^{3n} x_{i,j} \equiv 2 \pmod 4$$

We now use the equality

$$a_1 \oplus a_2 \oplus a_3 + 2\,\mathrm{carry}(a_1 - a_2 + a_3) \equiv a_1 - a_2 + a_3 \pmod 4.$$

Therefore

$$\bigoplus_{i=1}^{3} a_i + \bigoplus_{i=1}^{3} b_i + \bigoplus_{i=1}^{3} c_i + 2\,\mathrm{carry}(a_1 - a_2 + a_3) + 2\,\mathrm{carry}(b_1 - b_2 + b_3) +$$
$$+ 2\,\mathrm{carry}(c_1 - c_2 + c_3) + 2 \bigoplus_{i=1,j=1,2,3}^{3n} x_{i,j} \equiv 2 \pmod 4,$$

which means that the tuple r is indeed in R and we are done. $\square$