Deciding existence of difference terms

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***Abstract*—We consider the following practical question: given a finite algebra $\mathbf{A}$ in a finite language, can we efficiently decide whether the variety generated by $\mathbf{A}$ has a difference term? To address this question we review some useful definitions and known facts about difference terms, prove some new results, and then describe a polynomial-time algorithm that exploits these results and decides whether a given finite idempotent algebra generates a variety with a difference term.**

I. Introduction

TODO: improve the introduction; add some motivation; e.g., say why diff terms are important.

A *difference term* for a variety $\mathcal{V}$ is a ternary term d in the language of $\mathcal{V}$ that satisfies the following: if $\mathbf{A} = \langle A, \dots \rangle \in \mathcal{V}$, then for all $a, b \in A$ we have

$$d^{\mathbf{A}}(a, a, b) = b \quad \text{and} \quad d^{\mathbf{A}}(a, b, b)\ [\theta, \theta]\ a, \tag{I.1}$$

where θ is any congruence containing (a, b) and $[\cdot, \cdot]$ denotes the *commutator*. When the relations in (I.1) hold we call $d^{\mathbf{A}}$ a *difference term operation* for $\mathbf{A}$. Roughly speaking, having a difference term is slightly stronger than having a Taylor term and slightly weaker than having a Mal'cev term. Note that if $\mathbf{A}$ is an *abelian* algebra, which means that $[1_A, 1_A] = 0_A$, then, by the monotonicity of the commutator, $[\theta, \theta] = 0_A$ for all $\theta \in \operatorname{Con} \mathbf{A}$, in which case (I.1) says that $d^{\mathbf{A}}$ is a Mal'cev term operation.

The question motivating this effort is the following:

Problem 1. Is there a polynomial-time algorithm to decide for a finite, idempotent algebra $\mathbf{A}$ if $\mathbb{V}(\mathbf{A})$ has a difference term.

Let $\mathbf{p} = (p_0, p_1, \dots, p_n)$ be an $(n+1)$-tuple of ternary terms, where $p_0(x, y, z) \approx x$ and $p_n(x, y, z) \approx z$, the first and third ternary projections, respectively. Let $\mathbf{A} = \langle A, \dots \rangle$ be an algebra. In [1], Valeriote and Willard define an $\mathbf{A}$*-triple for* $\mathbf{p}$ to be a triple (a, b, i) such that $a, b \in A$ and $p_i(a, b, b) = p_{i+1}(a, a, b)$. They use this to define a "local Hagemann-Mitschke sequence" on which they base an efficient algorithm for deciding for a given n whether an idempotent variety is n-permutable. Taking this as our inspiration, we developed a similar construct, called a "local difference term," and we use it as the basis of a polynomial-time algorithm for deciding, given a finite idempotent algebra $\mathbf{A}$, whether the variety generated by $\mathbf{A}$ has a difference term.

II. Local Difference Terms

For the most part we use standard notation, definitions, and results of universal algebra, such as those found in [2]. However, we make a few exceptions for notational simplicity. For example, if $\mathbf{A} = \langle A, \dots \rangle$ is an algebra with elements $a, b \in A$, then we use $\theta(a, b)$ to denote the congruence of $\mathbf{A}$ generated by a and b.

Let $\mathbf{A} = \langle A, \dots \rangle$ be an algebra, fix $a, b \in A$ and $i \in \{0, 1\}$. A *local difference term for* (a, b, i) is a ternary term p satisfying the following:

$$\begin{aligned} &\text{if } i = 0, \text{ then } a\ [\theta(a, b), \theta(a, b)]\ p(a, b, b); \\ &\text{if } i = 1, \text{ then } p(a, a, b) = b. \end{aligned} \tag{II.1}$$

If p satisfies (II.1) for all triples in some subset $S \subseteq A \times A \times \{0, 1\}$, then we call p a *local difference term for* S.

Let $\mathcal{S} = A \times A \times \{0, 1\}$ and suppose that every pair $((a_0, b_0, \chi_0), (a_1, b_1, \chi_1))$ in $\mathcal{S}^2$ has a local difference term. That is, for each pair $((a_0, b_0, \chi_0), (a_1, b_1, \chi_1))$, there exists p such that for each $i \in \{0, 1\}$ we have

$$a_i\ [\theta(a_i, b_i), \theta(a_i, b_i)]\ p(a_i, b_i, b_i), \ \text{ if } \chi_i = 0, \text{ and} \tag{II.2}$$

$$p(a_i, a_i, b_i) = b_i, \ \text{ if } \chi_i = 1. \tag{II.3}$$

Under these hypothesis we will prove that every subset $S \subseteq \mathcal{S}$ has a local difference term. That is, there is a single term p that works (i.e., satisfies (II.2) and (II.3)) for all $(a_i, b_i, \chi_i) \in S$. The statement and proof of this new result follows.

Theorem II.1 (cf. [1, Theorem 2.2])**.** *Let $\mathcal{V}$ be an idempotent variety and $\mathbf{A} \in \mathcal{V}$. Define $\mathcal{S} = A \times A \times \{0, 1\}$ and suppose that every pair $((a_0, b_0, \chi_0), (a_1, b_1, \chi_1)) \in \mathcal{S}^2$ has a local difference term. Then every subset $S \subseteq \mathcal{S}$, has a local difference term.*

Proof. The proof is by induction on the size of S. In the base case, $|S| = 2$, the claim holds by assumption. Fix $n > 2$ and assume that every subset of $\mathcal{S}$ of size $2 \leqslant k \leqslant n$ has a local difference term. Let $S = \{(a_0, b_0, \chi_0), (a_1, b_1, \chi_1), \dots, (a_n, b_n, \chi_n)\} \subseteq \mathcal{S}$, so that $|S| = n + 1$. We prove S has a local difference term.

Since $|S| \geqslant 3$ and $\chi_i \in \{0, 1\}$ for all i, there must exist indices $i \neq j$ such that $\chi_i = \chi_j$. Assume without loss of generality that one of these indices is $j = 0$. Define the set $S' = S \setminus \{(a_0, b_0, \chi_0)\}$. Since $|S'| < |S|$, the set S' has a local difference term p. We split the remainder of the proof into two cases. In the first case $\chi_0 = 0$ and in the second $\chi_0 = 1$.

Case 1: $\chi_0 = 0$. Without loss of generality, suppose that $\chi_1 = \cdots = \chi_k = 1$, and $\chi_{k+1} = \cdots = \chi_n = 0$. Define $T = \{(a_0, p(a_0, b_0, b_0), 0), (a_1, b_1, 1), (a_2, b_2, 1), \dots, (a_k, b_k, 1)\}$, and note that $|T| < |S|$. Let t be a local difference term for T. Define

$$d(x, y, z) = t(x, p(x, y, y), p(x, y, z)).$$

Since $\chi_0 = 0$, we need to show $(a_0, d(a_0, b_0, b_0))$ belongs to $[\theta(a_0, b_0), \theta(a_0, b_0)]$. We have

$$d(a_0, b_0, b_0) = t(a_0, p(a_0, b_0, b_0), p(a_0, b_0, b_0)) \; [\tau, \tau] \; a_0, \tag{II.4}$$

where we have used τ to denote $\theta(a_0, p(a_0, b_0, b_0))$. Note that $(a_0, p(a_0, b_0, b_0)) = (p(a_0, a_0, a_0), p(a_0, b_0, b_0))$ belongs to $\theta(a_0, b_0)$, so $\tau \leqslant \theta(a_0, b_0)$. Therefore, by monotonicity of the commutator, $[\tau, \tau] \leqslant [\theta(a_0, b_0), \theta(a_0, b_0)]$. It follows from this and (II.4) that

$$d(a_0, b_0, b_0) \; [\theta(a_0, b_0), \theta(a_0, b_0)] \; a_0,$$

as desired.

For the indices $1 \leqslant i \leqslant k$ we have $\chi_i = 1$, so we wish to prove $d(a_i, a_i, b_i) = b_i$ for such i. Observe,

$$\begin{aligned} d(a_i, a_i, b_i) &= t(a_i, p(a_i, a_i, a_i), p(a_i, a_i, b_i)) && \text{(II.5)} \\ &= t(a_i, a_i, b_i) && \text{(II.6)} \\ &= b_i. && \text{(II.7)} \end{aligned}$$

Equation (II.5) holds by definition of d, (II.6) because p is an idempotent local difference term for S', and (II.7) because t is a local difference term for T.

The remaining triples in our original set S have indices satisfying $k < j \leqslant n$ and $\chi_j = 0$. Thus, for these triples we want $d(a_j, b_j, b_j) \; [\theta(a_j, b_j), \theta(a_j, b_j)] \; a_j$. By definition,

$$d(a_j, b_j, b_j) = t(a_j, p(a_j, b_j, b_j), p(a_j, b_j, b_j)). \tag{II.8}$$

Since p is a local difference term for S', we have

$$(p(a_j, b_j, b_j), a_j) \in [\theta(a_j, b_j), \theta(a_j, b_j)].$$

This and (II.8) imply that $(d(a_j, b_j, b_j), t(a_j, a_j, a_j))$ belongs to $[\theta(a_j, b_j), \theta(a_j, b_j)]$. Finally, by idempotence of t we have $d(a_j, b_j, b_j) \; [\theta(a_j, b_j), \theta(a_j, b_j)] \; a_j$, as desired.

Case 2: $\chi_0 = 1$. Without loss of generality, suppose $\chi_1 = \chi_2 = \cdots = \chi_k = 0$, and $\chi_{k+1} = \chi_{k+2} = \cdots = \chi_n = 1$. Define T to be the set

$$\{(p(a_0, a_0, b_0), b_0, 1), (a_1, b_1, 0), (a_2, b_2 0), \dots, (a_k, b_k, 0)\},$$

and note that $|T| < |S|$. Let t be a local difference term for T and define $d(x, y, z) = t(p(x, y, z), p(y, y, z), z)$. Since $\chi_0 = 1$, we want $d(a_0, a_0, b_0) = b_0$. By the definition of d,

$$d(a_0, a_0, b_0) = t(p(a_0, a_0, b_0), p(a_0, a_0, b_0), b_0) = b_0.$$

The last equality holds since t is a local difference term for T, thus, for $(p(a_0, a_0, b_0), b_0, 1)$.

If $1 \leqslant i \leqslant k$, then $\chi_i = 0$, so for these indices we want $d(a_i, b_i, b_i) \; [\theta(a_i, b_i), \theta(a_i, b_i)] \; a_i$. Again, starting from the definition of d and using idempotence of p, we have

$$\begin{aligned} d(a_i, b_i, b_i) &= t(p(a_i, b_i, b_i), p(b_i, b_i, b_i), b_i) && \text{(II.9)} \\ &= t(p(a_i, b_i, b_i), b_i, b_i). \end{aligned}$$

Next, since p is a local difference term for S', we have

$$t(p(a_i, b_i, b_i), b_i, b_i) \; [\theta(a_i, b_i), \theta(a_i, b_i)] \; t(a_i, b_i, b_i). \tag{II.10}$$

Finally, since t is a local difference term for T, hence for (a_i, b_i, b_i), we have $t(a_i, b_i, b_i) \; [\theta(a_i, b_i), \theta(a_i, b_i)] \; a_i$. Combining this with (II.9) and (II.10) yields

$$d(a_i, b_i, b_i) \; [\theta(a_i, b_i), \theta(a_i, b_i)] \; a_i,$$

as desired.

The remaining elements of our original set S have indices j satisfying $k < j \leqslant n$ and $\chi_j = 1$. For these we want $d(a_j, a_j, b_j) = b_j$. Since p is a local difference term for S', we have $p(a_j, a_j, b_j) = b_j$, and this along with idempotence of t yields

$$\begin{aligned} d(a_j, a_j, b_j) &= t(p(a_j, a_j, b_j), p(a_j, a_j, b_j), b_j) \\ &= t(b_j, b_j, b_j) = b_j, \end{aligned}$$

as desired. ☐

Corollary II.2. A finite idempotent algebra $\mathbf{A}$ has a difference term operation if and only if every pair $((a, b, i), (a', b', i')) \in (A \times A \times \{0, 1\})^2$ has a local difference term.

Proof. One direction is clear, since a difference term operation for $\mathbf{A}$ is obviously a local difference term for the whole set $A \times A \times \{0, 1\}$. For the converse, suppose each pair in $(A \times A \times \{0, 1\})^2$ has a local difference term. Then, by Theorem II.1, there is a single local difference term for the whole set $A \times A \times \{0, 1\}$, and this is a difference term operation for $\mathbf{A}$. Indeed, if d is a local difference term for $A \times A \times \{0, 1\}$, then for all $a, b \in A$, we have $a \; [\theta(a, b), \theta(a, b)] \; d(a, b, b)$, since d is a local difference term for $(a, b, 0)$, and we have $d(a, a, b) = b$, since d is also a local difference term for $(a, b, 1)$. ☐

Corollary II.3. If $\mathcal{V}$ is a variety generated by a finite idempotent algebra, then there is a polynomial-time algorithm for deciding whether or not $\mathcal{V}$ has a difference term.

Proof. Let $\mathbf{A}$ be a finite idempotent algebra and let $\mathcal{V} = \mathbb{V}(\mathbf{A})$. We describe a polynomial-time algorithm for deciding whether the hypothesis of Corollary II.2 holds for $\mathbf{A}$, thereby proving that we can decide in polynomial-time whether there is a difference term operation for $\mathbf{A}$. We will then complete the proof by explaining why $\mathbf{A}$ has a difference term operation iff the variety it generates has a difference term.

Fix a pair $((a, b, i), (a', b', i'))$ in $(A \times A \times \{0, 1\})^2$. If $i = i' = 0$, then the first projection is a local difference term. If $i = i' = 1$, then the third projection is a local difference term. The two remaining cases to consider are (1) $i = 0$ and $i' = 1$, and (2) $i = 1$ and $i' = 0$. Since these are completely symmetric, we only handle the first case. Assume the given pair of triples is $((a, b, 0), (a', b', 1))$. By definition, a term t is local difference term for this pair iff

$$a \; [\theta(a, b), \theta(a, b)] \; t^{\mathbf{A}}(a, b, b) \quad \text{and} \quad t^{\mathbf{A}}(a', a', b') = b'.$$

We can rewrite this condition more compactly by considering $t^{\mathbf{A} \times \mathbf{A}}((a, a'), (b, a'), (b, b')) = (t^{\mathbf{A}}(a, b, b), t^{\mathbf{A}}(a', a', b'))$. Clearly t is a local difference term for $((a, b, 0), (a', b', 1))$ iff

$$t^{\mathbf{A} \times \mathbf{A}}((a, a'), (b, a'), (b, b')) \in a/\delta \times \{b'\},$$

where $\delta = [\theta(a,b), \theta(a,b)]$ and a/δ denotes the δ-class containing a. (Observe that $a/\delta \times \{b'\}$ is a subalgebra of $\mathbf{A}\times\mathbf{A}$ by idempotence.) It follows that the pair $((a,b,0),(a',b',1))$ has a local difference term iff the subuniverse of $\mathbf{A}\times\mathbf{A}$ generated by $\{(a,a'),(b,a'),(b,b')\}$ intersects nontrivially with the subuniverse $a/\delta \times \{b'\}$.

Thus, the algorithm takes as input $\mathbf{A}$ and, for each triple $((a,a'),(b,a'),(b,b'))$ in $(A \times A)^3$, computes $\delta = [\theta(a,b), \theta(a,b)]$, computes the subalgebra $\mathbf{S}$ of $\mathbf{A} \times \mathbf{A}$ generated by $\{(a,a'),(b,a'),(b,b')\}$, and then tests whether $S \cap (a/\delta \times \{b'\})$ is empty. If we find an empty intersection at any point, then the algorithm returns the answer "no difference term operation." Otherwise, $\mathbf{A}$ has a difference term operation.

Finally, we observe that if $\mathbf{A}$ has a difference term operation, then the variety it generates has a difference term. $\square$

TODO: justify the last sentence of the last proof.

III. Conclusion

TODO: the conclusion goes here.

Acknowledgment

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The author would like to thank Ralph Freese for proposing this project and for suggesting that the methods in [1] could be useful for proving the main result of this paper.

References

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