

THE COMMUTATOR AS FIXED POINT OF A CLOSURE OPERATOR

WILLIAM DEMEO

ABSTRACT. In this note we elaborate on the following remark of Keith Kearnes [Kea95, p. 930] giving an alternate description of the commutator (here β is a congruence of \mathbf{A} and $\mathbf{A} \times_{\beta} \mathbf{A}$ denotes the subalgebra of \mathbf{A}^2 with universe β):

“Let $\Delta_{\beta, \alpha}$ be the congruence on $\mathbf{A} \times_{\beta} \mathbf{A}$ generated by

$$\{ \langle (x, x), (y, y) \rangle \mid (x, y) \in \alpha \}.$$

Call a subset $G \subseteq A^2$ Δ -closed if

$$\Delta_{\beta, \alpha} \circ G \circ \Delta_{\beta, \alpha} \subseteq G.$$

When α and β are reflexive, compatible relations, then $[\alpha, \beta]$ is the smallest subset $\gamma \subseteq A^2$ such that (i) γ is a congruence of \mathbf{A} and (ii) γ is Δ -closed.”

1. INTRODUCTION

To be honest, the genesis of these notes was my futile attempt to interpret the expression $\Delta_{\beta, \alpha} \circ G \circ \Delta_{\beta, \alpha}$. Noting that $\Delta_{\beta, \alpha} \subseteq (A \times A)^2$ while $G \subseteq A^2$, I wasn't aware of a standard means of composing relations of such different arities. In this note I try to reconcile this by giving an alternative “closure” operation that achieves the same end and verifies Keith's assertion that “there is a useful alternate description of $[\alpha, \beta]$.” Indeed, the commutator is a fixed point of some closure operator (i.e., closed set), and the closure operator involves the relation $\Delta_{\beta, \alpha}$ defined above.

1.1. Definitions. For an algebra \mathbf{A} with congruence relations $\alpha, \beta \in \text{Con } \mathbf{A}$, let β denote the subalgebra of $\mathbf{A} \times \mathbf{A}$ with universe β . Let $D = \{(a, a) \mid a \in A\}$ and $D_{\alpha}^2 = \{((a, a), (b, b)) \in D^2 \mid a \alpha b\}$. Finally, let $\Delta_{\beta, \alpha} = \text{Cg}^{\beta}(D_{\alpha}^2)$ be the congruence on β generated by the set D_{α}^2 . The congruence class of $\Delta_{\beta, \alpha}$ that contains (b, b') is denoted and defined as follows:

$$(b, b')/\Delta_{\beta, \alpha} = \{(a, a') \in \beta \mid (a, a') \Delta_{\beta, \alpha} (b, b')\}.$$

Let $\Phi_{\beta, \alpha}: \mathcal{P}(\beta) \rightarrow \mathcal{P}(\beta)$ be the function that takes each $B \subseteq \beta$ to

$$(1.1) \quad \Phi_{\beta, \alpha}(B) = \bigcup_{(b, b') \in B} (b, b')/\Delta_{\beta, \alpha}.$$

2. FIXED POINT LEMMA

Lemma 2.1. *Let \mathbf{A} be an algebra with $\alpha, \beta \in \text{Con}(\mathbf{A})$. If $\Phi := \Phi_{\beta, \alpha}$ is defined as in (1.1), then*

- (i) Φ is a closure operator on $\mathcal{P}(\beta)$;
- (ii) $[\alpha, \beta]$ is the least fixed point of Φ .

Date: March 1, 2017.

Proof. (i) Fix $B \subseteq \beta$. We must prove the following: (a) $B \subseteq \Phi(B)$; (b) $B \subseteq C \Rightarrow \Phi(B) \subseteq \Phi(C)$; and (c) $\Phi(\Phi(B)) = \Phi(B)$. If $(b, b') \in B$, then $(b, b') \in \Phi(B)$ since the operation (1.1) does not discard any of the pairs that were already in B . As for (b), if $(a, a') \in \Phi(B)$, then there exists $(b, b') \in B \subseteq C$ such that $(a, a') \Delta_{\beta, \alpha} (b, b')$. Since (b, b') belongs to C we have $(a, a') \in \Phi(C)$ as well. As for (c), it clearly follows from (a) and (b) that $\Phi(B) \subseteq \Phi(\Phi(B))$, so we prove the reverse inclusion. Let $(d, d') \in \Phi(\Phi(B))$. Then $(c, c') \Delta_{\beta, \alpha} (d, d')$ for some $(c, c') \in \Phi(B)$, which implies $(b, b') \Delta_{\beta, \alpha} (c, c')$ for some $(b, b') \in B$. By transitivity of $\Delta_{\beta, \alpha}$ we conclude that $(d, d') \in \Phi(B)$, as desired.

(ii) Since $[\alpha, \beta] \in \mathcal{P}(\beta)$ we have $[\alpha, \beta] \subseteq \Phi([\alpha, \beta])$, by (i). We prove the reverse inclusion. If $(c, c') \in \Phi([\alpha, \beta])$, then (1.1) implies there exists $(b, b') \in [\alpha, \beta]$ such that

$$(2.1) \quad (b, b') \Delta_{\beta, \alpha} (c, c').$$

From the definition of $\Delta_{\beta, \alpha}$ and Mal'tsev's theorem on congruence generation, (2.1) holds if and only if $\exists z_i \beta z'_i$ ($0 \leq i \leq n$), $\exists x_i \alpha y_i$ ($0 \leq i < n$), $\exists f_i \in \text{Pol}_1(\mathbf{A} \times \mathbf{A})$ ($0 \leq i < n$) such that $(b, b') = (z_0, z'_0)$ and $(z_n, z'_n) = (c, c')$, and

$$(2.2) \quad \{(b, b'), (z_1, z'_1)\} = \{f_0(x_0, x_0), f_0(y_0, y_0)\}$$

$$(2.3) \quad \{(z_1, z'_1), (z_2, z'_2)\} = \{f_1(x_1, x_1), f_1(y_1, y_1)\}$$

⋮

$$\{(z_{n-1}, z'_{n-1}), (c, c')\} = \{f_{n-1}(x_{n-1}, x_{n-1}), f_{n-1}(y_{n-1}, y_{n-1})\}$$

For each $(0 \leq i < n)$, $f_i \in \text{Pol}_1(\mathbf{A} \times \mathbf{A})$, which means

$$f_i(x, x') = g_i^\beta((x, x'), (a_1, a'_1), \dots, (a_k, a'_k)) = (g_i^\mathbf{A}(x, \mathbf{a}), g_i^\mathbf{A}(x, \mathbf{a}')),$$

for some k , $g_i \in \text{Clo}_{k+1}(\mathbf{A})$, and constants tuples $\mathbf{a} = (a_1, \dots, a_k)$ and $\mathbf{a}' = (a'_1, \dots, a'_k)$ such that $a_i \beta a'_i$ ($1 \leq i \leq k$). By (2.2), either

$$(b, b') = (g_0(x_0, \mathbf{a}), g_0(x_0, \mathbf{a}')) \quad \text{and} \quad (z_1, z'_1) = (g_0(y_0, \mathbf{a}), g_0(y_0, \mathbf{a}')),$$

or vice-versa. Since $x_0 \alpha y_0$ and $a_i \beta a'_i$ ($1 \leq i \leq k$), the α, β -term condition entails

$$g_0(x_0, \mathbf{a}) [\alpha, \beta] g_0(x_0, \mathbf{a}') \iff g_0(y_0, \mathbf{a}) [\alpha, \beta] g_0(y_0, \mathbf{a}').$$

This and (2.2) yield $(b, b') \in [\alpha, \beta]$ iff $(z_1, z'_1) \in [\alpha, \beta]$. Similarly (2.3) and $x_1 \alpha y_1$ imply $(z_1, z'_1) \in [\alpha, \beta]$ iff $(z_2, z'_2) \in [\alpha, \beta]$. Inductively, we arrive at $(b, b') \in [\alpha, \beta]$ iff $(c, c') \in [\alpha, \beta]$, as desired.

We have thus proved $[\alpha, \beta]$ is a fixed point of Φ . In other words, $[\alpha, \beta]$ is a “ Φ -closed” subset of β . (A set $B \subseteq \beta$ is called Φ -closed provided $\Phi(B) \subseteq B$.) Recall, if f is a monotone increasing function defined on a complete poset $\langle P, \leq \rangle$, then the least fixed point of f is $\bigwedge \{p \in P \mid fp \leq p\}$. Thus, Lemma 2.1 (ii) asserts that

$$(2.4) \quad [\alpha, \beta] = \bigwedge \{B \subseteq \beta \mid \Phi(B) \subseteq B\}.$$

We already proved $[\alpha, \beta]$ is Φ -closed, so it remains to check for every Φ -closed subset $B \subseteq \beta$ that $[\alpha, \beta] \subseteq B$. Fix a Φ -closed subset $B \subseteq \beta$. It suffices to prove $\mathbf{C}(\alpha, \beta; \Phi(B))$, since this implies $[\alpha, \beta] \subseteq \Phi(B) \subseteq B$. Thus, our goal is to establish the α, β -term condition.

Let $p \in \text{Pol}_{k+1}(\mathbf{A})$ and $a \alpha a'$ and $c_i \beta c'_i$ ($1 \leq i \leq k$); suppose $p(a, \mathbf{c}) \Phi(B) p(a, \mathbf{c}')$. We prove that these hypotheses entail the following relation:

$$(2.5) \quad p(a', \mathbf{c}) \Phi(B) p(a', \mathbf{c}').$$

By definition of Φ , (2.5) is equivalent to the existence of some pair $(b, b') \in B$ such that $(b, b') \Delta_{\beta, \alpha} (p(a', \mathbf{c}), p(a', \mathbf{c}'))$. Notice that the pair $(p(a, \mathbf{c}), p(a, \mathbf{c}'))$ belongs to B since $(p(a, \mathbf{c}), p(a, \mathbf{c}')) \in \Phi(B) \subseteq B$. Also, $c_i \beta c'_i$ ($0 \leq i < k$) implies

$$\begin{aligned} ((a, a), (c_1, c'_1), (c_1, c'_1), \dots, (c_k, c'_k)) &\in \beta^{k+1} \quad \text{and} \\ ((a', a'), (c_1, c'_1), (c_1, c'_1), \dots, (c_k, c'_k)) &\in \beta^{k+1}. \end{aligned}$$

Therefore,

$$(2.6) \quad p^\beta((a, a), (c_1, c'_1), (c_1, c'_1), \dots, (c_k, c'_k)) = (p^{\mathbf{A}}(a, \mathbf{c}), p^{\mathbf{A}}(a, \mathbf{c}')) \in \beta \quad \text{and}$$

$$(2.7) \quad p^\beta((a', a'), (c_1, c'_1), (c_1, c'_1), \dots, (c_k, c'_k)) = (p^{\mathbf{A}}(a', \mathbf{c}), p^{\mathbf{A}}(a', \mathbf{c}')) \in \beta.$$

Finally, $a \alpha a'$ implies $p(a, \mathbf{c}) \alpha p(a', \mathbf{c})$, and this—together with (2.6) and (2.7)—proves the pair $((p(a, \mathbf{c}), p(a, \mathbf{c}')), (p(a', \mathbf{c}), p(a', \mathbf{c}')))$ belongs to $\Delta_{\beta, \alpha}$. Since $(p(a, \mathbf{c}), p(a, \mathbf{c}')) \in B$, this proves $(p(a', \mathbf{c}), p(a', \mathbf{c}')) \in \Phi(B)$, completing the proof. \square

REFERENCES

- [Kea95] Keith A. Kearnes. Varieties with a difference term. *J. Algebra*, 177(3):926–960, 1995. URL: <http://dx.doi.org/10.1006/jabr.1995.1334>, doi:10.1006/jabr.1995.1334.

UNIVERSITY OF HAWAII

E-mail address: williamdemeo@gmail.com