LOCAL DIFFERENCE TERMS

Let $\mathbf{p} = (p_0, p_1, \ldots, p_n)$ be an $(n+1)$-tuple of ternary terms, where $p_0(x, y, z) \approx x$ and $p_n(x, y, z) \approx z$, the first and third ternary projections, respectively. Let $\mathbf{A} = \langle A, \ldots \rangle$ be an algebra. In [VW14], Valeriote and Willard define an $\mathbf{A}$-*triple for* $\mathbf{p}$ to be a triple (a, b, i) such that $a, b \in A$ and $p_i(a, b, b) = p_{i+1}(a, a, b)$. They use this to derive methods for deciding for a given k whether an idempotent variety is k-permutable. This inspired us to develop the concept of a "local difference term" that we hope will be useful for the purpose of deciding, given a finite idempotent algebra $\mathbf{A}$, whether the variety generated by $\mathbf{A}$ has a difference term.

Let $\mathbf{A} = \langle A, \ldots \rangle$ be an algebra, fix $a, b \in A$ and $i \in \{0, 1\}$. An $\mathbf{A}$-*local difference term for* (a, b, i) is a ternary term p satisfying the following:

(0.1)
$$\text{if } i = 0, \text{ then } a \; [\mathrm{Cg}^{\mathbf{A}}(a, b), \mathrm{Cg}^{\mathbf{A}}(a, b)] \; p(a, b, b);$$
$$\text{if } i = 1, \text{ then } p(a, a, b) = b.$$

When often drop the $\mathbf{A}$ when the algebra is clear from context. For example, we write Cg in place of $\mathrm{Cg}^{\mathbf{A}}$, and call the term p above a local difference term for (a, b, i). If p satisfies (0.1) for all triples in some subset $S \subseteq A \times A \times \{0, 1\}$, then we call p a *local difference term for* S.

Let $\mathcal{S} = A \times A \times \{0, 1\}$ and suppose that every pair $((a_0, b_0, \chi_0), (a_1, b_1, \chi_1))$ in $\mathcal{S}^2$ has a local difference term. That is, for each pair $((a_0, b_0, \chi_0), (a_1, b_1, \chi_1))$, there exists p such that for each $i \in \{0, 1\}$ we have

(0.2)
$$a_i \; [\mathrm{Cg}(a_i, b_i), \mathrm{Cg}(a_i, b_i)] \; p(a_i, b_i, b_i), \text{ if } \chi_i = 0, \text{ and}$$
$$p(a_i, a_i, b_i) = b_i, \text{ if } \chi_i = 1.$$

Under these hypothesis we will prove that every subset $S \subseteq \mathcal{S}$ has a local difference term. That is, there is a single term p that works (i.e., satisfies (0.2)) for all $(a_i, b_i, \chi_i) \in S$. The statement and proof of this new result follows.

Lemma. Let $\mathcal{V}$ be an idempotent variety and $\mathbf{A} \in \mathcal{V}$. Define $\mathcal{S} = A \times A \times \{0, 1\}$, and suppose that every pair $((a_0, b_0, \chi_0), (a_1, b_1, \chi_1)) \in \mathcal{S}^2$ has a local difference term. Then every subset $S \subseteq \mathcal{S}$, has a local difference term.

Proof. The proof is by induction on the size of S. In the base case, $|S| = 2$, the claim holds by assumption. Fix $n > 2$ and assume that every subset of $\mathcal{S}$ of size $2 \leqslant k \leqslant n$ has a local difference term. Let $S = \{(a_0, b_0, \chi_0), (a_1, b_1, \chi_1), \ldots, (a_n, b_n, \chi_n)\} \subseteq \mathcal{S}$, so that $|S| = n + 1$. We prove S has a local difference term.

Since $|S| \geqslant 3$ and $\chi_i \in \{0, 1\}$ for all i, there must exist indices $i \neq j$ such that $\chi_i = \chi_j$. Assume without loss of generality that one of these indices is $j = 0$. Define the set $S' = S \setminus \{(a_0, b_0, \chi_0)\}$. Since $|S'| < |S|$, the set S' has a local difference term p. We split the remainder of the proof into two cases. In the first case $\chi_0 = 0$ and in the second $\chi_0 = 1$.

Case 1: $\chi_0 = 0$.

Assume $\chi_0 = 0$ and, without loss of generality, suppose $\chi_1 = \chi_2 = \cdots = \chi_k = 1$, and $\chi_{k+1} = \chi_{k+2} = \cdots = \chi_n = 0$. Define

$$T = \{(a_0, p(a_0, b_0, b_0), 0), (a_1, b_1, 1), (a_2, b_2 1), \ldots, (a_k, b_k, 1)\}.$$

Let t be a local difference term for T. Define

$$d(x, y, z) = t(x, p(x, y, y), p(x, y, z)).$$

Since $\chi_0 = 0$, we want $d(a_0, b_0, b_0)$ $[\mathrm{Cg}(a_0, b_0), \mathrm{Cg}(a_0, b_0)]$ a_0. We have

$$(0.3) \qquad\qquad d(a_0, b_0, b_0) = t(a_0, p(a_0, b_0, b_0), p(a_0, b_0, b_0))\ [\tau, \tau]\ a_0,$$

where $\tau := \mathrm{Cg}(a_0, p(a_0, b_0, b_0))$. Notice that

$$(a_0, p(a_0, b_0, b_0)) = (p(a_0, a_0, a_0), p(a_0, b_0, b_0)) \in \mathrm{Cg}(a_0, b_0),$$

so $\tau \leqslant \mathrm{Cg}(a_0, b_0)$. Therefore, $[\tau, \tau] \leqslant [\mathrm{Cg}(a_0, b_0), \mathrm{Cg}(a_0, b_0)]$, by monotonicity of the commutator. It follows from this and (0.3) that $d(a_0, b_0, b_0)$ $[\mathrm{Cg}(a_0, b_0), \mathrm{Cg}(a_0, b_0)]$ a_0, as desired.

For $1 \leqslant i \leqslant k$ we have $\chi_i = 1$, so we want $d(a_i, a_i, b_i) = b_i$. Indeed,

$$(0.4) \qquad\qquad d(a_i, a_i, b_i) = t(a_i, p(a_i, a_i, a_i), p(a_i, a_i, b_i)) = t(a_i, a_i, b_i) = b_i.$$

The first equality in (0.4) holds by definition of d; the second, because p is an idempotent local difference term for S'; the third, because t is a local difference term for T.

The remaining triples in our original set S have index j satisfying $k < j \leqslant n$ and $\chi_j = 0$. Thus, for these triples we want $d(a_j, b_j, b_j)$ $[\mathrm{Cg}(a_j, b_j), \mathrm{Cg}(a_j, b_j)]$ a_j. Indeed, since p is a local difference term for S', we have $p(a_j, b_j, b_j))$ $[\mathrm{Cg}(a_j, b_j), \mathrm{Cg}(a_j, b_j)]$ a_j, so

$$d(a_j, b_j, b_j) = t(a_j, p(a_j, b_j, b_j), p(a_j, b_j, b_j)) = [\mathrm{Cg}(a_j, b_j), \mathrm{Cg}(a_j, b_j)]\ t(a_j, a_j, a_j) = a_j,$$

as desired.

Case 2: $\chi_0 = 1$.

Assume $\chi_0 = 1$ and, without loss of generality, suppose $\chi_1 = \chi_2 = \cdots = \chi_k = 0$, and $\chi_{k+1} = \chi_{k+2} = \cdots = \chi_n = 1$. Define

$$T = \{(p(a_0, a_0, b_0), b_0, 1), (a_1, b_1, 0), (a_2, b_2 0), \ldots, (a_k, b_k, 0)\}.$$

Let t be a local difference term for T. Define

$$d(x, y, z) = t(p(x, y, z), p(y, y, z), z).$$

Since $\chi_0 = 1$, we want $d(a_0, a_0, b_0) = b_0$. Indeed, by the definition of d in this case,

$$d(a_0, a_0, b_0) = t(p(a_0, a_0, b_0), p(a_0, a_0, b_0), b_0) = b_0.$$

The last equality holds since t is a local difference term for T, thus, for $(p(a_0, a_0, b_0), b_0, 1)$.

For $1 \leqslant i \leqslant k$ we have $\chi_i = 0$, so we want $d(a_i, b_i, b_i)$ $[\mathrm{Cg}(a_i, b_i), \mathrm{Cg}(a_i, b_i)]$ a_i. Again, starting from the definition of d and using idempotence of p, we have

$$d(a_i, b_i, b_i) = t(p(a_i, b_i, b_i), p(b_i, b_i, b_i), b_i) = t(p(a_i, b_i, b_i), b_i, b_i)$$

Next, since p is a local difference term for S', we have

$$t(p(a_i, b_i, b_i), b_i, b_i)\ [\mathrm{Cg}(a_i, b_i), \mathrm{Cg}(a_i, b_i)]\ t(a_i, b_i, b_i).$$

Finally $t(a_i, b_i, b_i)$ $[\mathrm{Cg}(a_i, b_i), \mathrm{Cg}(a_i, b_i)]$ a_i, as desired.

The remaining elements of our original set S have indices j satisfying $k < j \leqslant n$ and $\chi_j = 1$. For these we want $d(a_j, a_j, b_j) = b_j$. Indeed, since p is a local difference term for S', we have $p(a_j, a_j, b_j) = b_j$, so

$$d(a_j, a_j, b_j) = t(p(a_j, a_j, b_j), p(a_j, a_j, b_j), b_i) = t(b_j, b_j, b_j) = b_j,$$

as desired.   □

References

[VW14] M. Valeriote and R. Willard. Idempotent n-permutable varieties. *Bull. Lond. Math. Soc.*, 46(4):870–880, 2014. URL: http://dx.doi.org/10.1112/blms/bdu044, doi:10.1112/blms/bdu044.

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