DIFFERENCE TERM OPERATIONS FOR FINITE 2-GENERATED IDEMPOTENT FREE ALGEBRAS

Abstract. This note provides some tools that should enable us to prove the following: if $\mathbf{A}$ is a finite idempotent algebra with a difference term operation, then the 2-generated free algebra in $\mathbb{V}(\mathbf{A})$ has a difference term operation.

1. Definitions and Preliminary Facts

Let $\mathcal{V}$ be a variety (equational class) of algebras. A ternary term d in the language of $\mathcal{V}$ is called a *difference term for* $\mathcal{V}$ if it satisfies the following: for all $\mathbf{A} = \langle A, \dots \rangle \in \mathcal{V}$, for all $a, b \in A$, for every congruence $\theta \in \operatorname{Con} \mathbf{A}$ containing (a, b), we have

$$a \; [\theta, \theta] \; d^{\mathbf{A}}(a, b, b), \quad \text{and} \tag{1.1}$$

$$d^{\mathbf{A}}(a, a, b) = b, \tag{1.2}$$

where $[\cdot, \cdot]$ denotes the (term condition) commutator (see also [HM88] or [KK13]). (By the monotonicity of the commutator, we could replace θ in the definition by $\operatorname{Cg}^{\mathbf{A}}(a, b)$.)

Notice that part of the definition of a difference term involves one of the identities satisfied by a Malcev term, namely (1.2). Below we refer to this as the "strong (difference term) identity," and we refer to (1.1) as the "weak (difference term) relation." We refer to (1.1) and (1.2) collectively as the "difference term realtions."

Suppose $\mathbf{A}$ is an algebra and for all $a, b \in A$ the operation $d^{\mathbf{A}}$ satisfies the difference term relations when $\theta = \operatorname{Cg}^{\mathbf{A}}(a, b)$, then we call $d^{\mathbf{A}}$ a *difference term operation* for $\mathbf{A}$ (whether or not d is a difference term for the whole variety inhabited by $\mathbf{A}$).

If d is idempotent, it may be convenient to present the difference term relations as follows:

$$d^{\mathbf{A}}(a, a, a) \; [\theta, \theta] \; d^{\mathbf{A}}(a, b, b), \quad \text{and} \tag{1.3}$$

$$d^{\mathbf{A}}(a, a, b) = d^{\mathbf{A}}(b, b, b). \tag{1.4}$$

In Lemma 1.1 we gather some easy facts that will be useful below. We use $[\![\theta]\!]$ as a convenient shorthand for $[\theta, \theta]$.

Lemma 1.1. *Let $\mathcal{V}$ be an idempotent variety and suppose $\mathbf{F} = \langle F, \dots \rangle = \mathbf{F}_{\mathcal{V}}(x, y)$ is finite. Let $\mathbf{R} = \langle R, \dots \rangle = \operatorname{Sg}^{\mathbf{F}^2}\{(x, x), (x, y), (y, y)\}$. Then the following hold:*

(1) $\operatorname{Cg}^{\mathbf{F}}(x, y) = 1_F$.

(2) If $[1_F, 1_F] = 1_F$ (i.e., 1_F is "neutral"), then $\mathbf{F}$ has a difference term operation.

(3) The quotient algebra $\mathbf{F}/[\![\operatorname{Cg}^{\mathbf{F}}(x, y)]\!] = \mathbf{F}/[1_F, 1_F]$ is abelian.

(4) $\mathbf{R}$ is a subdirect product of $\mathbf{F} \times \mathbf{F}$.

(5) $\big(\{x\} \times F\big) \cup \big(F \times \{y\}\big) \subseteq R$.

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Proof.

(1) In fact, $\operatorname{Cg}^{\mathbf{B}}(a,b) = 1_B$ holds for every 2-generated idempotent algebra, $\mathbf{B} = \operatorname{Sg}^{\mathbf{B}}(a,b)$, with generators a and b. If $(c,c') \in B^2$, then there are terms s and t such that $s(a,b) = c$ and $t(a,b) = c'$. Therefore, $(c,b) = (s(a,b), s(b,b))$ belongs to $\operatorname{Cg}^{\mathbf{B}}(a,b)$; likewise $(a,c') = (t(a,a), t(a,b))$ belongs to $\operatorname{Cg}^{\mathbf{B}}(a,b)$, so $c \operatorname{Cg}^{\mathbf{F}}(x,y)\ b \operatorname{Cg}^{\mathbf{F}}(x,y)\ a \operatorname{Cg}^{\mathbf{F}}(x,y)\ c'$, as desired.

(2) If $[1_F, 1_F] = [\![\operatorname{Cg}^{\mathbf{F}}(x,y)]\!] = \operatorname{Cg}^{\mathbf{F}}(x,y) = 1_F$, then every term trivially satisfies the condition $t^{\mathbf{F}}(x,y,y)\ [\![\operatorname{Cg}^{\mathbf{F}}(x,y)]\!]\ x$. Therefore, the third projection $d^{\mathbf{F}}(x,y,z) = z$ is a difference term operation for $\mathbf{F}$.

(3) This holds because $[\![\operatorname{Cg}^{\mathbf{F}}(x,y)]\!] = [1_F, 1_F]$ and $\mathsf{C}(1_F, 1_F; [1_F, 1_F])$ implies $\mathsf{C}(1_{F/\delta}, 1_{F/\delta}; 0_{F/\delta})$, where $\delta := [1_F, 1_F]$.

(4) It is obvious. (We denote this by $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{F}_0 \times \mathbf{F}_1$.)

(5) Since $\mathbf{F}$ is idempotent, $\{x\}$ is a subuniverse of $\mathbf{F}$, and since $\{x,y\}$ generates $\mathbf{F}$, we see that $\{x\} \times \mathbf{F} \leqslant \operatorname{Sg}^{\mathbf{F}^2}\{(x,x),(x,y)\} \leqslant \mathbf{R}$. Similarly, $\mathbf{F} \times \{y\} \leqslant \operatorname{Sg}^{\mathbf{F}^2}\{(x,y),(y,y)\} \leqslant \mathbf{R}$. Therefore, $\big(\{x\} \times F\big) \cup \big(F \times \{y\}\big) \subseteq R$.

□

It is sometimes convenient to have indexed copies of $\mathbf{F}$; viz., let $\mathbf{F}_i \cong \mathbf{F}$ denote the "i-th copy" of $\mathbf{F}$. For instance, in the next lemma $\mathbf{F}_0 \times \mathbf{F}_1$ denotes the product of two copies of $\mathbf{F}$.

Lemma 1.2. *Let $\mathcal{V}$ be an idempotent variety and assume $\mathbf{F} = \langle F, \dots\rangle = \mathbf{F}_{\mathcal{V}}(x,y)$ is finite. If $\mathbf{R} = \operatorname{Sg}^{\mathbf{F}^2}\{(x,x),(x,y),(y,y)\}$, and if η_i denotes the kernel of the i-th projection $\mathbf{R} \twoheadrightarrow \mathbf{F}_i$, then $\eta_0 \vee \eta_1 = 1_R$ and $\eta_0 \circ \eta_1 \circ \eta_0 = \eta_0 \vee \eta_1 = \eta_1 \circ \eta_0 \circ \eta_1$.*

Proof. By Lemma 1.1 (5), for every $b \in F$ the pair (x,b) lies in R, and for every $a \in F$ the pair (a,y) lies in R. It follows that the projection kernels η_0 and η_1 3-permute and join to 1_R. Indeed, if (a,b) and (a',b') are arbitrary pairs in R, then $(a,b)\ \eta_0\ (a,y)\ \eta_1\ (a',y)\ \eta_0\ (a',b')$ and $(a,b)\ \eta_1\ (x,b)\ \eta_0\ (x,b')\ \eta_1\ (a',b')$. □

1.1. Some useful facts about abelian algebras.

An very useful property of abelian algebras is that they are absorption-free. We record this fact for future reference.

Lemma 1.3. *Finite idempotent abelian algebras are absorption-free.*

A proof of Lemma 1.3 appeared in [BKS15, Lem 4.1]; we include a similar proof in Appendix Section B.2 for easy reference and to keep the paper somewhat self-contained. Another well-known result, whose proof also appears in the appendix (see Section B.1) is the following useful fact:

Lemma 1.4. *An algebra $\mathbf{A}$ is abelian if and only if there is some $\theta \in \operatorname{Con}(\mathbf{A}^2)$ that has the diagonal $0_A := \{(a,a) : a \in A\}$ as a congruence class.*

1.2. Alternate description of the commutator.

The description of the commutator presented in this section can be useful when we need to establish that a specific pair of elements belongs to the commutator in question. This description is similar to the one in [Kea95, p. 930].

For an algebra $\mathbf{A}$ with congruence relations $\alpha, \beta \in \operatorname{Con}\mathbf{A}$, let $\underline{\boldsymbol{\beta}}$ denote the subalgebra of $\mathbf{A} \times \mathbf{A}$ with universe β. Let $0_A = \{(a,a) \mid a \in A\}$ and let

$$D^2_\alpha := (\alpha \otimes \alpha) \cap D^2 = \{((a,a),(b,b)) \in D^2 \mid a\ \alpha\ b\}, \text{ and} \tag{1.5}$$

$$\Delta_{\beta,\alpha} := \operatorname{Cg}^{\underline{\boldsymbol{\beta}}}\big(D^2_\alpha\big). \tag{1.6}$$

As usual, the congruence class of $\Delta_{\beta,\alpha}$ that contains (b,b') is denoted and defined by

$$(b,b')/\Delta_{\beta,\alpha} = \{(a,a') \in \beta \mid (a,a')\ \Delta_{\beta,\alpha}\ (b,b')\}.$$

In our application below, the most important special case will concern a finite 2-generated free algebra $\mathbf{F} := \mathbf{F}_{\mathcal{V}}(x,y)$ over an idempotent variety $\mathcal{V}$. When the context makes the meaning clear, we sometimes write 1 instead of 1_F or $F \times F$. As noted above in Lemma 1.1, $\mathrm{Cg}^{\mathbf{F}}(x,y) = 1$, so we will be especially interested in

$$\Delta_{1,1} := \mathrm{Cg}^{F\times F}\{\big((x,x),(y,y)\big) \mid x,y \in F\}. \tag{1.7}$$

Recall that $\mathbf{F}$ is abelian if and only if $\exists\,\theta \in \mathrm{Con}(\mathbf{F}^2)$ with the diagonal $0_F := \{(z,z) \mid z \in F\}$ as a congruence class. Thus if $\mathbf{F}$ is nonabelian, then there exist $u, v \in F$ such that $u \neq v$ and $(u,v)\ \Delta_{1,1}\ (x,x)$, for all $x \in F$. (Otherwise, 0_F is a congruence class of $\Delta_{1,1}$.)

Let $\Phi_{\beta,\alpha} \colon \mathcal{P}(\beta) \to \mathcal{P}(\beta)$ be the function that takes each $B \subseteq \beta$ to

$$\Phi_{\beta,\alpha}(B) = \bigcup_{(b,b')\in B} (b,b')/\Delta_{\beta,\alpha}. \tag{1.8}$$

Let $\Psi_{\beta,\alpha}\colon \mathrm{Eq}(\beta) \to \mathrm{Eq}(\beta)$ be the function that takes each $R \in \mathrm{Eq}(\beta)$ to the equivalence relation

$$\begin{aligned}\Psi_{\beta,\alpha}(R) &= \Delta_{\beta,\alpha} \circ (R \otimes R) \circ \Delta_{\beta,\alpha}\\ &= \{\big((x,y),(x',y')\big) \mid \big(\exists (r,r') \in R\big)\ \big((x,y)\ \Delta_{\beta,\alpha}\ (r,r) \ \text{ and } \ (r',r')\ \Delta_{\beta,\alpha}\ (x'y')\big)\}\\ &= \{\big((x,y),(x',y')\big) \mid \big(\exists (r,r')\big)\ (x,y)\ \Delta_{\beta,\alpha}\ (r,r)\ (R\otimes R)\ (r',r')\ \Delta_{\beta,\alpha}\ (x'y')\}.\end{aligned}$$

We are finally ready to give the promised alternative description of the commutator $[\alpha,\beta]$. (Spoiler alert! It is the least fixed point of $\Phi_{\beta,\alpha}$.)

Lemma 1.5. *Let $\mathbf{A}$ be an algebra with α, $\beta \in \mathrm{Con}(\mathbf{A})$. If $\Phi := \Phi_{\beta,\alpha}$ is defined as in (1.8), then*
(i) Φ is a closure operator on $\mathcal{P}(\beta)$;
(ii) $[\alpha,\beta]$ is the least fixed point of Φ.

(A proof of Lemma 1.5 appears in Appendix Section B.3 below.)

2. Main Result

Our goal is to prove Theorem 2.2 below. Our proof will use the following lemma.

Lemma 2.1. *If d is a difference term operation for $\mathbf{A}$, then $d(x,x,y) \approx y$ holds in $\mathcal{V} = \mathbb{V}(\mathbf{A})$. In particular, interpreted in the 2-generated free algebra $\mathbf{F} := \mathbf{F}_{\mathcal{V}}(x,y)$, we have $d^{\mathbf{F}}(x,x,y) = y$.*

Proof. $d(x,x,y) \approx y$ is an identity in $\mathcal{V} = \mathsf{HSP}(\mathbf{A})$ since $d^{\mathbf{A}}(a,a,b) = b$ holds $\forall a, b \in A$. $\square$

Theorem 2.2. *Let $\mathbf{A} = \langle A, \dots\rangle$ be a finite idempotent algebra. Assume $\mathbf{A}$ has a difference term operation and $\mathcal{V} = \mathbb{V}(\mathbf{A})$ has a Taylor term. Then $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x,y)$ has a difference term operation.*

Proof. Let $\mathbf{F}_0 \times \mathbf{F}_1$ be the product of two copies of $\mathbf{F}$, and let $\mathbf{R} \leqslant_{\mathrm{sd}} \mathbf{F}_0 \times \mathbf{F}_1$ be the subalgebra generated by the set $\{(x,x),(x,y),(y,y)\}$ (as above). Recall from Lemma 1.1 (1) that $[\![\mathrm{Cg}^{\mathbf{F}}(x,y)]\!] = [1_F, 1_F]$. Therefore, if we can prove

$$R \cap \big(\{y\} \times (x/[1_{F_1}, 1_{F_1}])\big) \neq \emptyset, \tag{2.1}$$

then it will follow that there is a term t satisfying

$$t^{\mathbf{F}^2}((x,x),(x,y),(y,y)) = \big(t^{\mathbf{F}}(x,x,y), t^{\mathbf{F}}(x,y,y)\big) \in \{y\} \times \big(x/[\![\mathrm{Cg}^{\mathbf{F}}(x,y)]\!]\big).$$

That is, $(t^{\mathbf{F}}(x,y,y), x) \in [\![\mathrm{Cg}^{\mathbf{F}}(x,y)]\!]$ and $t^{\mathbf{F}}(x,x,y) = y$, which means that $t^{\mathbf{F}}$ is a difference term operation for $\mathbf{F}$. In short, to prove the theorem it suffices to establish (2.1).

Observe that $\mathbf{R}/\eta_i \cong \mathbf{F}_i$ for $i \in \{0,1\}$. Let $\rho_1 \in \mathrm{Con}(\mathbf{R})$ be the congruence (above η_1) corresponding to $[1_{F_1}, 1_{F_1}]$ via the projection homomorphism $\mathbf{R} \twoheadrightarrow \mathbf{F}_1$. Thus,

$$\mathbf{R}/\rho_1 \cong (\mathbf{R}/\eta_1)/(\rho_1/\eta_1) \cong \mathbf{F}_1/[1_{F_1}, 1_{F_1}],$$

so $\mathbf{R}/\rho_1$ is abelian (by Lemma 1.1 (3)). Moreover, $\mathbf{R}/(\eta_0 \wedge \rho_1)$ is a subdirect product of $\mathbf{R}/\eta_0 \cong \mathbf{F}_0$ and $\mathbf{R}/\rho_1 \cong \mathbf{F}_1/[1_{F_1}, 1_{F_1}]$. That is,

$$\mathbf{R}/(\eta_0 \wedge \rho_1) \leqslant_{\text{sd}} \mathbf{F}_0 \times \big(\mathbf{F}_1/[1_{F_1}, 1_{F_1}]\big).$$

Let $\tau_0 := \eta_0/(\eta_0 \wedge \rho_1)$ be the kernel of the projection of $\mathbf{R}/(\eta_0 \wedge \rho_1)$ onto $\mathbf{F}_0$, and $\tau_1 := \rho_1/(\eta_0 \wedge \rho_1)$ the kernel of the projection of $\mathbf{R}/(\eta_0 \wedge \rho_1)$ onto $\mathbf{F}_1/[1_{F_1}, 1_{F_1}]$. Then,

$$\begin{aligned}\big(\mathbf{R}/(\eta_0 \wedge \rho_1)\big)/\tau_0 &= \big(\mathbf{R}/(\eta_0 \wedge \rho_1)\big)/\big(\eta_0/(\eta_0 \wedge \rho_1)\big) \cong \mathbf{R}/\eta_0 \cong \mathbf{F}_0\\ \big(\mathbf{R}/(\eta_0 \wedge \rho_1)\big)/\tau_1 &= \big(\mathbf{R}/(\eta_0 \wedge \rho_1)\big)/\big(\rho_1/(\eta_0 \wedge \rho_1)\big) \cong \mathbf{R}/\rho_1 \cong \mathbf{F}_1/[1_{F_1}, 1_{F_1}].\end{aligned}$$

Next observe that $\tau_0 \vee \tau_1 = 1_{R/(\eta_0 \wedge \rho_1)}$. (This is clear since $\rho_1 \geqslant \eta_1$, so $\eta_0 \vee \rho_1 \geqslant \eta_0 \vee \eta_1 = 1_R$.) Thus, by the Absorption Theorem (Thm. B.2 below), exactly one of the following holds:

(1) $\mathbf{F}_0$ has a proper minimal absorbing subuniverse;
(2) $\mathbf{F}_1/[1_{F_1}, 1_{F_1}]$ has a proper minimal absorbing subuniverse;
(3) $\mathbf{R}/(\eta_0 \wedge \rho_1) = \mathbf{F}_0 \times \big(\mathbf{F}_1/[1_{F_1}, 1_{F_1}]\big)$.

(From now on we drop the subscripts on $\mathbf{F}$, since distinguishing between the first and second factors of the product is no longer helpful.) If (3) holds, then we're done since in that case $(y, x/[1_F, 1_F]) \in R/(\eta_0 \wedge \rho_1)$, which implies $(y, z) \in R$ for some $z \; [1_F, 1_F] \; x$. That is, (B.9) holds. We can rule out case (2) because $\mathbf{F}/[1_F, 1_F]$ is abelian and is therefore absorption free (see Appendix Section B.2 below).

We are left with case (1). Let B be a proper minimal absorbing subuniverse of $\mathbf{F}_0$. Then the product algebra $\mathbf{B} \times \big(\mathbf{F}/[1_F, 1_F]\big)$ is absorbtion free. Since $\mathbf{R}/(\eta_0 \wedge \rho_1)$ is subdirect, the intersection $R/(\eta_0 \wedge \rho_1) \cap \big(B \times (F/[1_F, 1_F])\big)$ is non-empty. Therefore, by the Absorption Theorem (Thm. B.2), together with [BD16, Lemmas 4.7, 4.9, 4.11, 4.12], we have

$$B \times \big(F/[1_F, 1_F]\big) \subseteq R/(\eta_0 \wedge \rho_1). \tag{2.2}$$

We split the remainder of the proof into two subcases.

<u>Subcase 1:</u> $(y \in B)$

In this case we're done, since (2.2) implies $y \times \big(x/[1_F, 1_F]\big) \in R$.

<u>Subcase 2:</u> $(x \in B)$

In this case, we have $x \in B \lll \mathbf{F}$, so instead of $\mathbf{R}$ we consider $\mathbf{R}' = \operatorname{Sg}^{\mathbf{F}^2}\{(y, y), (y, x), (x, x)\}$. We wish to show that there is a term t satisfying

$$t^{\mathbf{R}'}\big((y, y), (y, x), (x, x)\big) = \big(t^{\mathbf{F}}(y, y, x), t^{\mathbf{F}}(y, x, x)\big) \in \{x\} \times \big(y/[1_F, 1_F]\big).$$

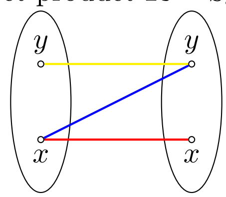
Applying (2.2) again, we have $x \times \big(y/[1_F, 1_F]\big) \in R'$.

<u>Subcase 3:</u> $(x \notin B$ and $y \notin B)$

TODO: finish proof!!!

$\square$

FIGURE 1. The subdirect product $\mathbf{R} = \mathrm{Sg}^{\mathbf{F}^2}\{(x,x),(x,y),(y,y)\}$.



(lines represent generators of R)

3. Other potentially useful observations

3.1. A special case. Let $\mathbf{A}$ be a finite idempotent algebra, let $\mathcal{V} = \mathbb{V}(\mathbf{A})$, and let

$$\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x,y) \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1},$$

where $n \leqslant |A|^2$ and each $\mathbf{A}_i$ is a 2-generated subalgebra of $\mathbf{A}$. Let $\mathbf{R} = \mathrm{Sg}^{\mathbf{F}^2}\{(x,x),(x,y),(y,y)\}$, and recall that $\mathbf{R}$ is a subdirect product of $\mathbf{F} \times \mathbf{F}$.

Let's see what we can prove under the following simplifying assumptions:

(1) Assume $\mathbf{F} \leqslant_{\mathrm{sd}} \mathbf{A}_0 \times \mathbf{A}_1 \times \mathbf{A}_2$,
(2) For each $i \in \{0,1,2\}$ assume $\mathbf{A}_i = \mathrm{Sg}^{\mathbf{A}}(a_i, b_i)$ where $x = (a_0,a_1,a_2)$ and $y = (b_0,b_1,b_2)$ are the generators of $\mathbf{F}$.
(3) Let d be a difference term operation when interpreted in each of the following algebras:
   (a) $\mathbf{A}_i$ for each $i \in \{0,1,2\}$;
   (b) $\mathbf{A}_i \times \mathbf{A}_j$ for all $i \neq j$ in $\{0,1,2\}$;
   (c) $\mathbf{A}_0 \times \mathbf{A}_1 \times \mathbf{A}_2$.

Recall from Lemma 1.1 (1) that $\mathrm{Cg}^{\mathbf{F}}(x,y) = F \times F = 1_F$ and $\mathrm{Cg}^{\mathbf{A}_i}(a_i,b_i) = A_i \times A_i = 1_{A_i}$. It's clear that $d^{\mathbf{F}}$ satisfies the strong difference term identity, so our goal is to prove that $d^{\mathbf{F}}$ also satisfies the weak difference term relation,

$$d^{\mathbf{F}}(x,y,y)\ [1_F, 1_F]\ x. \tag{3.1}$$

This is equivalent to $d^{\mathbf{A}_0 \times \mathbf{A}_1 \times \mathbf{A}_2}\big((a_0,a_1,a_2),(b_0,b_1,b_2),(b_0,b_1,b_2)\big)\ [1_F,1_F]\ (a_0,a_1,a_2)$. Equivalently,

$$\big(d^{\mathbf{A}_0}(a_0,b_0,b_0), d^{\mathbf{A}_1}(a_1,b_1,b_1), d^{\mathbf{A}_2}(a_2,b_2,b_2)\big)\ [1_F,1_F]\ (a_0,a_1,a_2).$$

Assumption (3c) says $d^{\mathbf{A}_0 \times \mathbf{A}_1 \times \mathbf{A}_2}$ is a difference term operation for $\mathbf{A}_0 \times \mathbf{A}_1 \times \mathbf{A}_2$, so

$$d^{\mathbf{A}_0 \times \mathbf{A}_1 \times \mathbf{A}_2}\big((a_0,a_1,a_2),(b_0,b_1,b_2),(b_0,b_1,b_2)\big)\ [1_{A_0\times A_1\times A_2}, 1_{A_0\times A_1\times A_2}]\ (a_0,a_1,a_2). \tag{3.2}$$

Observe that (3.1) is still stronger than (3.2) because $\mathbf{F}$ is a subalgebra of $\mathbf{A}_0 \times \mathbf{A}_1 \times \mathbf{A}_2$, so $F \times F = 1_F$ is a subuniverse of $(\mathbf{A}_0 \times \mathbf{A}_1 \times \mathbf{A}_2)^2$. The latter has universe $1_{A_0\times A_1\times A_2}$, so $1_F \subseteq 1_{A_0\times A_1\times A_2}$, so by monotonicity of the commutator, $[1_F,1_F] \subseteq [1_{A_0\times A_1\times A_2}, 1_{A_0\times A_1\times A_2}]$ holds.

Our goal is to prove (3.1), namely, $d^{\mathbf{F}}(x,y,y)\ [1_F,1_F]\ x$. To do so, we will exploit the alternate description of the commutator from Section 1.2. In the present context, $\underline{\boldsymbol{\beta}}$ is represented by 1_F, the universe of the algebra $\mathbf{F} \times \mathbf{F}$. Let

$$D = \{(x,x) \mid x \in F\}, \quad D^2_{1_F} = \{\big((a,a),(b,b)\big) \in D^2 \mid a,b \in F\}, \quad \Delta = \Delta_{1_F,1_F} = \mathrm{Cg}^{\mathbf{F}^2}(D^2_{1_F}).$$

Then $D^2_{1_F} = D \times D$, so $\Delta = \mathrm{Cg}^{\mathbf{F}^2}(D \times D)$.

By Lemma 1.4, $\mathbf{F}$ is abelian iff $\mathrm{Con}(\mathbf{F} \times \mathbf{F})$ contains a congruence that has the diagonal set D as a class. For example, suppose for all a, b, $b' \in F$ that $(a,a)\ \Delta\ (b,b')$ iff $b = b'$. Then D is a class of Δ and $\mathbf{F}$ is abelian. In that case, $\mathbf{F}$ is absorption free (see Lemma 1.3) and the Absorption Theorem implies $\mathbf{R} = \mathbf{F} \times \mathbf{F}$. Our earlier goal (2.1) is trivially satisfied in this case. Therefore, we proceed under the assumption that $\mathbf{F}$ is nonabelian, or equivalently, that no congruence of $\mathbf{F}^2$ has a class that is exactly the diagonal set, $D = \{(x,x) \mid x \in F\}$.

Let $\Phi\colon \mathcal{P}(F\times F) \to \mathcal{P}(F \times F)$ be the function that takes each $B \subseteq F\times F$ to

$$\Phi(B) = \bigcup_{(b,b')\in B} (b,b')/\Delta \tag{3.3}$$

We now prove that $\big(d^{\mathbf{F}}(x,y,y),x\big) \in \Phi[1_F, 1_F]$. This will complete the proof of (3.1), since $\Phi[1_F,1_F] \subseteq [1_F,1_F]$ by Lemma 1.5. By (3.3), $(d^{\mathbf{F}}(x,y,y),x) \in \Phi[1_F,1_F]$ iff $(d^{\mathbf{F}}(x,y,y),x)\ \Delta\ (b,b')$ for some $(b,b') \in [1_F,1_F]$.

To be continued...

3.2. **Observation about commutator in subalgebras.**

By assumption (3a) $d^{\mathbf{A}_i}(a_i,b_i,b_i)\ [1_{A_i},1_{A_i}]\ a_i$ for each $i \in \{0,1,2\}$, so

$$\big(d^{\mathbf{A}_0}(a_0,b_0,b_0), d^{\mathbf{A}_1}(a_1,b_1,b_1), d^{\mathbf{A}_2}(a_2,b_2,b_2)\big)\ [1_{A_0},1_{A_0}] \times [1_{A_1},1_{A_1}] \times [1_{A_2},1_{A_2}]\ (a_0,a_1,a_2), \tag{3.4}$$

but we can do better. Indeed, by assumption (3b),

$$\big(d^{\mathbf{A}_i}(a_i,b_i,b_i), d^{\mathbf{A}_j}(a_j,b_j,b_j)\big) = d^{\mathbf{A}_i\times\mathbf{A}_j}\big((a_i,a_j),(b_i,b_j),(b_i,b_j)\big)\ [1_{A_i\times A_j},1_{A_i\times A_j}]\ (a_i,a_j)$$

for all $i \neq j$ in $\{0,1,2\}$. Combining this with assumption (3a) we have

$$\big(d^{\mathbf{A}_0\times\mathbf{A}_1}((a_0,a_1),(b_0,b_1),(b_0,b_1)), d^{\mathbf{A}_2}(a_2,b_2,b_2)\big)\ [1_{A_0\times A_1},1_{A_0\times A_1}] \times [1_{A_2},1_{A_2}]\ \big((a_0,a_1),a_2\big).$$

and

$$\big(d^{\mathbf{A}_0}(a_0,b_0,b_0), d^{\mathbf{A}_1\times\mathbf{A}_2}((a_1,a_2),(b_1,b_2),(b_1,b_2))\big)\ [1_{A_0},1_{A_0}] \times [1_{A_1\times A_2},1_{A_1\times A_2}]\ \big(a_0,(a_1,a_2)\big).$$

This improves upon (3.4), but we still haven't made full use of our assumptions. Indeed, (3c) says $d^{\mathbf{A}_0\times\mathbf{A}_1\times\mathbf{A}_2}$ is a difference term operation for $\mathbf{A}_0\times\mathbf{A}_1\times\mathbf{A}_2$, so

$$d^{\mathbf{A}_0\times\mathbf{A}_1\times\mathbf{A}_2}\big((a_0,a_1,a_2),(b_0,b_1,b_2),(b_0,b_1,b_2)\big)\ [1_{A_0\times A_1\times A_2},1_{A_0\times A_1\times A_2}]\ (a_0,a_1,a_2). \tag{3.5}$$

Observe that (3.1) is still stronger than (3.5) because $\mathbf{F}$ is a subalgebra of $\mathbf{A}_0\times\mathbf{A}_1\times\mathbf{A}_2$, so $F\times F = 1_F$ is a subuniverse of $(\mathbf{A}_0\times\mathbf{A}_1\times\mathbf{A}_2)^2$. The latter has universe $1_{A_0\times A_1\times A_2}$, so $1_F \subseteq 1_{A_0\times A_1\times A_2}$, so by monotonicity of the commutator, $[1_F,1_F] \subseteq [1_{A_0\times A_1\times A_2},1_{A_0\times A_1\times A_2}]$ holds.

Question. How do the following congruence relations compare?

(1) $[1_{A_0},1_{A_0}] \times [1_{A_1\times A_2},1_{A_1\times A_2}]$
(2) $[1_{A_0\times A_1},1_{A_0\times A_1}] \times [1_{A_2},1_{A_2}]$
(3) $[1_{A_0\times A_1\times A_2},1_{A_0\times A_1\times A_2}]$
(4) $[1_F,1_F]$

$$[\![1_{A_0}]\!] \times [\![1_{A_1\times A_2}]\!] = \{(\mathbf{x},\mathbf{y}) \mid x_0\ [\![1_{A_0}]\!]\ y_0 \ \text{ and }\ (x_1,x_2)\ [\![1_{A_1\times A_2}]\!]\ (y_1,y_2)\}.$$
$$[\![1_{A_0\times A_1}]\!] \times [\![1_{A_2}]\!] = \{(\mathbf{x},\mathbf{y}) \mid (x_0,x_1)\ [\![1_{A_0\times A_1}]\!]\ (y_0,y_1) \ \text{ and }\ x_2\ [\![1_{A_2}]\!]\ y_2\}.$$

4. More potentially useful observations

A very useful result of Kearnes and Kiss that holds in varieties that satisfy a nontrivial idempotent Mal'tsev condition is the following:

Theorem 4.1 ([KK13, Thm 3.27])**.** *Suppose α and β are congruences of a Taylor algebra. Then $\mathsf{C}(\alpha,\alpha;\alpha\wedge\beta)$ if and only if $\mathsf{C}(\alpha\vee\beta,\alpha\vee\beta;\beta)$.*

Observation 1. That $\mathbf{R}/\rho_1 \cong \mathbf{F}/[1_F, 1_F]$ is abelian means $\mathsf{C}(1_{R/\rho_1}, 1_{R/\rho_1}; 0_{R/\rho_1})$ holds, and this is equivalent to $\mathsf{C}(1_R, 1_R; \rho_1)$. By Theorem 4.1 below, the latter holds if and only if $\mathsf{C}(\eta_0, \eta_0; \eta_0 \wedge \rho_1)$. It follows that $[\eta_0, \eta_0] \leqslant \eta_0 \wedge \rho_1$.

Observation 2. For each $i \in \{0, 1, 2\}$, let $\eta_i = \ker(\mathbf{F} \twoheadrightarrow \mathbf{A}_i)$ be the kernel of the projection of $\mathbf{F}$ onto the i-th coordinate. Then $\mathbf{F}/\eta_i \cong \mathbf{A}_i$ and by the correspondence theorem there exists $\varphi_i \in \operatorname{Con} \mathbf{F}$ corresponding to $[1_{A_i}, 1_{A_i}] \in \operatorname{Con} \mathbf{A}_i$ such that

$$\mathbf{F}/\varphi_i \cong \big(\mathbf{F}/\eta_i\big)\big(\varphi_i/\eta_i\big) \cong \mathbf{A}_i/[1_{A_i}, 1_{A_i}].$$

Appendix A. Product Congruences

It is important, especially in this section, to be clear about what we mean by the product of two equivalence relations. Let A and B be sets and let $\alpha \in \operatorname{Eq}(A)$ and $\beta \in \operatorname{Eq}(B)$ be equivalence relations on A and B, respectively. We denote by $\alpha \otimes \beta$ the *pairwise product* relation, which we define as follows:

$$\alpha \otimes \beta = \{\big((a,b),(a',b')\big) \in (A \times B)^2 \mid a \;\alpha\; a',\, b \;\beta\; b'\}. \tag{A.1}$$

In contrast, the Cartesian product of α and β is

$$\alpha \times \beta = \{\big((a,a'),(b,b')\big) \in A^2 \times B^2 \mid a \;\alpha\; a',\, b \;\beta\; b'\}. \tag{A.2}$$

Notice that $\alpha \otimes \beta$ defines an equivalence relation on the set $A \times B$, whereas $\alpha \times \beta$ is generally not even a binary relation on a set. The equivalence class of $\alpha \otimes \beta$ containing the pair (a,b) is

$$(a,b)/(\alpha \otimes \beta) = a/\alpha \times b/\beta = \{(a',b') \in A \times B \mid a \;\alpha\; a',\, b \;\beta\; b'\},$$

This is simply the Cartesian product of the sets a/α and b/β. The collection of all such equivalence classes is also a Cartesian product, namely,

$$(A \times B)/(\alpha \otimes \beta) = A/\alpha \times B/\beta = \{(a,b)/(\alpha \otimes \beta) \mid a \in A,\, b \in B\}.$$

Let $\mathbf{L}$ be a sublattice of the lattice $\operatorname{Eq}(A)$ of all equivalence relations on a finite set A. Fix $\alpha \in L$ and consider the lattice $\operatorname{Eq}(\alpha)$ of equivalence relations on $\alpha \subseteq A^2$. The bottom and top elements of $\operatorname{Eq}(\alpha)$ are, respectively,

$$0_\alpha = \{\big((x,y),(x,y)\big) \mid x \;\alpha\; y\}, \quad 1_\alpha = \{\big((u,v),(x,y)\big) \mid u \;\alpha\; v,\, x \;\alpha\; y\} = \alpha \times \alpha.$$

For each $\beta \in L$, define the following equivalence relations on α:

$$\beta_0 = \{\big((u,v),(x,y)\big) \in \alpha \times \alpha \mid u \;\beta\; x\} = \{\big((u,v),(x,y)\big) \mid v \;\alpha\; u \;\beta\; x \;\alpha\; y\};$$

$$\beta_1 = \{\big((u,v),(x,y)\big) \in \alpha \times \alpha \mid v \;\beta\; y\} = \{\big((u,v),(x,y)\big) \mid u \;\alpha\; v \;\beta\; y \;\alpha\; x\}.$$

We make the following important exception to these notational conventions: when it comes to the least equivalence relation, $0_A = \{(a,a) \mid a \in A\}$, instead of writing $(0_A)_0$ and $(0_A)_1$ for the first and second projection kernels, we will denote these realtions by η_0 and η_1, respectively; that is,

$$\eta_0 = \{\big((u,v),(x,y)\big) \in \alpha \times \alpha \mid u = x\} = \{\big((u,v),(x,y)\big) \mid v \;\alpha\; u = x \;\alpha\; y\};$$

$$\eta_1 = \{\big((u,v),(x,y)\big) \in \alpha \times \alpha \mid v = y\} = \{\big((u,v),(x,y)\big) \mid u \;\alpha\; v = y \;\alpha\; x\}.$$

Notice that $\alpha_0 = \alpha_1$ since

$$\alpha_0 = \{\big((u,v),(x,y)\big) \mid v \;\alpha\; u \;\alpha\; x \;\alpha\; y\} = \{\big((u,v),(x,y)\big) \mid u \;\alpha\; v \;\alpha\; y \;\alpha\; x\} = \alpha_1.$$

In fact, for each $\gamma \geqslant \alpha$ we have $\gamma_0 = \gamma_1$. Indeed,

$$\begin{aligned}
\gamma_0 &= \{\big((u,v),(x,y)\big) \in \alpha \times \alpha \mid u \;\gamma\; x\} = \{\big((u,v),(x,y)\big) \mid y \;\alpha\; x \;\gamma\; u \;\alpha\; v\} \\
&= \{\big((u,v),(x,y)\big) \in \alpha \times \alpha \mid v \;\gamma\; u \;\gamma\; x \;\gamma\; y\} \qquad \text{(assuming } \gamma \geqslant \alpha\text{)} \\
&= \{\big((u,v),(x,y)\big) \in \alpha \times \alpha \mid v \;\gamma\; y\} = \gamma_1.
\end{aligned}$$

Let $\mathbf{K}_\alpha$ be the sublattice of $\operatorname{Eq}(\alpha)$ generated by the set $X = \{\beta_i \mid \beta \in L, i = 0,1\}$. In X there is a pair β_0, β_1 for each $\beta \in L$.

Proposition A.1. *Let $\mathbf{A}$ be a finite algebra with congruence lattice $\mathbf{L}$, and let $\alpha \in L$. In the sublattice $\mathbf{K}_\alpha \leqslant \operatorname{Eq}(\alpha)$ described above, the intervals $[\![\eta_0, 1_\alpha]\!]$ and $[\![\eta_1, 1_\alpha]\!]$ are isomorphic to $\mathbf{L}$.*

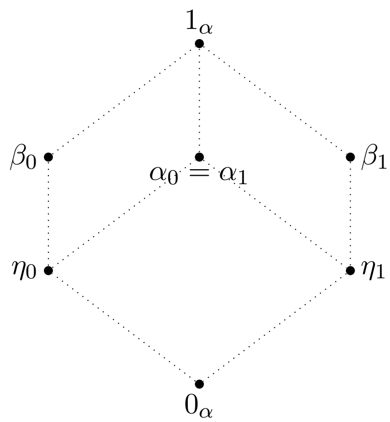


FIGURE 2. Some relations in the lattice $\mathrm{Eq}(\alpha)$.

Proof. Let $\mathbf{A}$ be an algebra on A with congruence lattice $\mathbf{L}$. Recall that $\mathbf{L} = \mathrm{Eq}(A) \cap \mathrm{Sub}(\mathbf{A}^2)$, so each $\alpha \in L$ is, in particular, a subalgebra of $\mathbf{A}^2$.

Fix $\alpha \in L$, and define the function $\varphi : \alpha \to A$ by $\varphi((x, y)) = x$, for all $(x, y) \in \alpha$. Then, φ is clearly an epimorphism from $\boldsymbol{\alpha}$ to $\mathbf{A}$ with kernel η_0, so the correspondence theorem yields $\mathrm{Con}(\boldsymbol{\alpha}/\eta_0) \cong \mathbf{L}$. Finally, note that $\mathrm{Con}(\boldsymbol{\alpha}/\eta_0)$ is precisely the interval above η_0 in the sublattice $\mathbf{K}_\alpha$ of $\mathrm{Eq}(\alpha)$. ☐

APPENDIX B. MISCELLANEOUS THEOREMS

We collect here the statement and proofs of results that we used above but whose proofs were omitted from the main body above so as not to iterrupt the flow.

B.1. **Abelian algebras have a congruence with a "diagonal" class.**

Lemma (Lem. 1.4 above)**.** An algebra $\mathbf{A}$ is abelian if and only if there is some $\theta \in \mathrm{Con}(\mathbf{A}^2)$ that has the diagonal $D(A) := \{(a, a) : a \in A\}$ as a congruence class.

Proof. ($\Leftarrow$) Assume Θ is such a congruence. Fix $k < \omega$, $t^{\mathbf{A}} \in \mathsf{Clo}_{k+1}(\mathbf{A})$, $u, v \in A$, and $\mathbf{x}, \mathbf{y} \in A^k$. We must prove the implication

$$t^{\mathbf{A}}(\mathbf{x}, u) = t^{\mathbf{A}}(\mathbf{y}, u) \quad \Longrightarrow \quad t^{\mathbf{A}}(\mathbf{x}, v) = t^{\mathbf{A}}(\mathbf{y}, v).$$

Since $D(A)$ is a class of Θ, we have $(u, u)\ \Theta\ (v, v)$, and since Θ is a reflexive relation, we have $(x_i, y_i)\ \Theta\ (x_i, y_i)$ for all i. Therefore,

$$t^{\mathbf{A}\times\mathbf{A}}((x_1, y_1), \dots, (x_k, y_k), (u, u))\ \Theta\ t^{\mathbf{A}\times\mathbf{A}}((x_1, y_1), \dots, (x_k, y_k), (v, v)). \tag{B.1}$$

since $t^{\mathbf{A}\times\mathbf{A}}$ is a term operation of $\mathbf{A} \times \mathbf{A}$. Note that (B.1) is equivalent to

$$(t^{\mathbf{A}}(\mathbf{x}, u), t^{\mathbf{A}}(\mathbf{y}, u))\ \Theta\ (t^{\mathbf{A}}(\mathbf{x}, v), t^{\mathbf{A}}(\mathbf{y}, v)). \tag{B.2}$$

If $t^{\mathbf{A}}(\mathbf{x}, u) = t^{\mathbf{A}}(\mathbf{y}, u)$ then the first pair in (B.2) belongs to the Θ-class $D(A)$, so the second pair must also belong this Θ-class. That is, $t^{\mathbf{A}}(\mathbf{x}, v) = t^{\mathbf{A}}(\mathbf{y}, v)$, as desired.

($\Rightarrow$) Assume $\mathbf{A}$ is abelian. We show $\mathrm{Cg}^{\mathbf{A}^2}(D(A)^2)$ has $D(A)$ as a block. Assume

$$((x, x), (c, c')) \in \mathrm{Cg}^{\mathbf{A}^2}(D(A)^2). \tag{B.3}$$

It suffices to prove that $c = c'$. Recall, Mal'tsev's congruence generation theorem states that (B.3) holds iff

$$\exists\,(z_0, z_0'), (z_1, z_1'), \dots, (z_n, z_n') \in A^2$$
$$\exists\,((x_0, x_0'), (y_0, y_0')), ((x_1, x_1'), (y_1, y_1')), \dots, ((x_{n-1}, x_{n-1}'), (y_{n-1}, y_{n-1}')) \in D(A)^2$$
$$\exists\, f_0, f_1, \dots, f_{n-1} \in F^*_{\mathbf{A}^2}$$

such that

$$\{(x,x), (z_1, z_1')\} = \{f_0(x_0, x_0'), f_0(y_0, y_0')\} \tag{B.4}$$
$$\{(z_1, z_1'), (z_2, z_2')\} = \{f_1(x_1, x_1'), f_1(y_1, y_1')\}$$
$$\vdots$$
$$\{(z_{n-1}, z_{n-1}'), (c, c')\} = \{f_{n-1}(x_{n-1}, x_{n-1}'), f_{n-1}(y_{n-1}, y_{n-1}')\} \tag{B.5}$$

The notation $f_i \in F^*_{\mathbf{A}^2}$ means

$$\begin{aligned} f_i(x, x') &= g_i^{\mathbf{A}^2}((a_1, a_1'), (a_2, a_2'), \dots, (a_k, a_k'), (x, x')) \\ &= (g_i^{\mathbf{A}}(a_1, a_2, \dots, a_k, x), g_i^{\mathbf{A}}(a_1', a_2', \dots, a_k', x')), \end{aligned}$$

for some $g_i^{\mathbf{A}} \in \mathsf{Clo}_{k+1}(\mathbf{A})$ and some constants $\mathbf{a} = (a_1, \dots, a_k)$ and $\mathbf{a}' = (a_1', \dots, a_k')$ in A^k. Now, $((x_i, x_i'), (y_i, y_i')) \in D(A)^2$ implies $x_i = x_i'$, and $y_i = y_i'$, so in fact we have

$$\{(z_i, z_i'), (z_{i+1}, z_{i+1}')\} = \{f_i(x_i, x_i), f_i(y_i, y_i)\} \quad (0 \leqslant i < n).$$

Therefore, by Equation (B.4) we have either

$$(x, x) = (g_i^{\mathbf{A}}(\mathbf{a}, x_0), g_i^{\mathbf{A}}(\mathbf{a}', x_0)) \quad \text{or} \quad (x, x) = (g_i^{\mathbf{A}}(\mathbf{a}, y_0), g_i^{\mathbf{A}}(\mathbf{a}', y_0)).$$

Thus, either $g_i^{\mathbf{A}}(\mathbf{a}, x_0) = g_i^{\mathbf{A}}(\mathbf{a}', x_0)$ or $g_i^{\mathbf{A}}(\mathbf{a}, y_0) = g_i^{\mathbf{A}}(\mathbf{a}', y_0)$. By the abelian assumption, if one of these equations holds, then so does the other. This and and Equation (B.4) imply $z_1 = z_1'$. Applying the same argument inductively, we find that $z_i = z_i'$ for all $1 \leqslant i < n$ and so, by (B.5) and the abelian property, we have $c = c'$. □

Lemma B.1. *Suppose $\rho \colon A_0 \to A_1$ is a bijection and suppose the graph $\{(x, \rho x) \mid x \in A_0\}$ is a block of some congruence $\beta \in \mathrm{Con}(A_0 \times A_1)$. Then both $\mathbf{A}_0$ and $\mathbf{A}_1$ are abelian.*

Proof. Define the relation $\alpha \subseteq (A_1 \times A_1)^2$ as follows: for $((a, a'), (b, b')) \in (A_1 \times A_1)^2$,

$$(a, a')\ \alpha\ (b, b') \quad \Longleftrightarrow \quad (a, \rho a')\ \beta\ (b, \rho b')$$

We prove that the diagonal $D(A_1)$ is a block of α by showing that $(a, a)\ \alpha\ (b, b')$ implies $b = b'$. Indeed, if $(a, a)\ \alpha\ (b, b')$, then $(a, \rho a)\ \beta\ (b, \rho b')$, which means that $(b, \rho b')$ belongs to the block and $(a, \rho a)/\beta = \{(x, \rho x) \mid x \in A_1\}$. Therefore, $\rho b = \rho b'$, so $b = b'$ since ρ is injective. This proves that $\mathbf{A}_1$ is abelian.

To prove $\mathbf{A}_2$ is abelian, we reverse the roles of A_1 and A_2 in the foregoing argument. If $\{(x, \rho x) \mid x \in A_1\}$ is a block of β, then $\{(\rho^{-1}(\rho x), \rho x) \mid \rho x \in A_2\}$ is a block of β; that is, $\{(\rho^{-1} y, y) \mid y \in A_2\}$ is a block of β. Define the relation $\alpha \subseteq (A_2 \times A_2)^2$ as follows: for $((a, a'), (b, b')) \in (A_2 \times A_2)^2$,

$$(a, a')\ \alpha\ (b, b') \quad \Longleftrightarrow \quad (\rho^{-1} a, \rho a')\ \beta\ (\rho^{-1} b, \rho b').$$

As above, we can prove that the diagonal $D(A_2)$ is a block of α by using the injectivity of ρ^{-1} to show that $(a, a)\ \alpha\ (b, b')$ implies $b = b'$. □

B.2. **Abelian algebras are absorption free.** An very useful property of abelian algebras is that they are absorption-free. A proof of this appears in [BKS15, Lem 4.1], but we include a proof here for easy reference and to keep the paper somewhat self-contained. First we require an elementary fact about functions on finite sets.

Fact. If $f\colon X \to X$ is a (unary) function on a finite set X, then there is a natural number $k \geqslant 1$ such that the k-fold composition of f with itself is the same function as the $2k$-fold composition. That is, for all $x \in X$, $f^{2k}(x) = f^k(x)$.

Lemma (Lem. 1.3 above)**.** Finite idempotent abelian algebras are absorption-free.

Proof. Suppose $\mathbf{A}$ is a finite idempotent abelian algebra with $\mathbf{B} \triangleleft_t \mathbf{A}$. We show $\mathbf{B} = \mathbf{A}$. If t is unary, then by idempotence t is the identity function and absorption in this case means $t[A] \subseteq B$. It follows that $A = B$ and we're done. So assume t has arity $k > 1$. We will show that there must also be a $(k-1)$-ary term operation $s \in \mathsf{Clo}(\mathbf{A})$ such that $\mathbf{B} \triangleleft_s \mathbf{A}$. It follows inductively that there must also be a unary absorbing term operation. Since a unary idempotent operation is the identity function, this will complete the proof.

Define a sequence of terms $t_0, t_1, \dots$ as follows: for each $\mathbf{x} = (x_1, \dots, x_{k-1}) \in A^{k-1}$ and $y \in A$,

$$\begin{aligned}
t_0(\mathbf{x}, y) &= t(\mathbf{x}, y),\\
t_1(\mathbf{x}, y) &= t(\mathbf{x}, t_0(\mathbf{x}, y)) = t(\mathbf{x}, t(\mathbf{x}, y)),\\
t_2(\mathbf{x}, y) &= t(\mathbf{x}, t_1(\mathbf{x}, y)) = t(\mathbf{x}, t(\mathbf{x}, t(\mathbf{x}, y))),\\
&\ \ \vdots\\
t_m(\mathbf{x}, y) &= t(\mathbf{x}, t_{m-1}(\mathbf{x}, y)) = t(\mathbf{x}, \dots, t(\mathbf{x}, t(\mathbf{x}, t(\mathbf{x}, y)))\cdots)).
\end{aligned}$$

It is easy to see that $\mathbf{B}$ is absorbing in $\mathbf{A}$ with respect to t_m, that is, $\mathbf{B} \triangleleft_{t_m} \mathbf{A}$.

For each $\mathbf{x}_i \in A^{k-1}$, define $p_i : A \to A$ by $p_i(y) = t(\mathbf{x}_i, y)$. Then, $p_i^m(y) = t_m(\mathbf{x}_i, y)$, so by Fact A.1 there exists an $m_i \geqslant 1$ such that $p_i^{2m_i} = p_i^{m_i}$. That is, $t_{m_i}(\mathbf{x}_i, t_{m_i}(\mathbf{x}_i, y)) = t_{m_i}(\mathbf{x}_i, y)$. Let m be the product of all the m_i as $\mathbf{x}_i$ varies over A^{k-1}. Then, for all $\mathbf{x}_i \in A^{k-1}$, we have $p_i^{2m} = p_i^m$. Therefore, for all $\mathbf{x} \in A^{k-1}$, we have $t_m(\mathbf{x}, t_m(\mathbf{x}, y)) = t_m(\mathbf{x}, y)$.

We now show that the $(k-1)$-ary term operation s, defined for all $x_1, \dots, x_{k-1} \in A$ by

$$s(x_1, \dots, x_{k-2}, x_{k-1}) = t(x_1, \dots, x_{k-2}, x_{k-1}, x_{k-1})$$

is absorbing for $\mathbf{B}$, that is, $\mathbf{B} \triangleleft_s \mathbf{A}$. It suffices to prove that $s[B \times \cdots \times B \times A] \subseteq B$. (For if the factor involving A occurs earlier, we appeal to absorption with respect to t.) So, for $\mathbf{b} \in B^{k-2}$ and $a \in A$, we will show $s(\mathbf{b}, a) = t_m(\mathbf{b}, a, a) \in B$. For all $b \in B$, we have

$$t_m(\mathbf{b}, b, a) = t_m(\mathbf{b}, b, t_m(\mathbf{b}, b, a)).$$

Therefore, if we apply (at the $(k-1)$-st coordinate) the fact that $\mathbf{A}$ is abelian, then we have

$$t_m(\mathbf{b}, a, a) = t_m(\mathbf{b}, a, t_m(\mathbf{b}, b, a)). \tag{B.6}$$

By absorption, $t_m(\mathbf{b}, b, a)$ belongs to B, thus so does the entire expression on the right of (B.6). This proves that $s(\mathbf{b}, a) = t_m(\mathbf{b}, a, a) \in B$, as desired. ☐

B.3. **Fixed Point Lemma.**

Lemma (Lem. 1.5 above)**.** Let $\mathbf{A}$ be an algebra with α, $\beta \in \mathrm{Con}(\mathbf{A})$. If $\Phi := \Phi_{\beta,\alpha}$ is defined as in (1.8), then

(i) Φ is a closure operator on $\mathcal{P}(\beta)$;
(ii) $[\alpha, \beta]$ is the least fixed point of Φ.

Proof. (i) Fix $B \subseteq \beta$. We must prove the following: (a) $B \subseteq \Phi(B)$; (b) $B \subseteq C \Rightarrow \Phi(B) \subseteq \Phi(C)$; and (c) $\Phi(\Phi(B)) = \Phi(B)$. If $(b, b') \in B$, then $(b, b') \in \Phi(B)$ since the operation (1.8) does not discard any of the pairs that were already in B. As for (b), if $(a, a') \in \Phi(B)$, then there exists $(b, b') \in B \subseteq C$ such that $(a, a')\ \Delta_{\beta,\alpha}\ (b, b')$. Since (b, b') belongs to C we have $(a, a') \in \Phi(C)$ as well. As for (c), it clearly follows from (a) and (b) that $\Phi(B) \subseteq \Phi(\Phi(B))$, so we prove the reverse inclusion. Let $(d, d') \in \Phi(\Phi(B))$. Then $(c, c')\ \Delta_{\beta,\alpha}\ (d, d')$ for some $(c, c') \in \Phi(B)$, which implies $(b, b')\ \Delta_{\beta,\alpha}\ (c, c')$ for some $(b, b') \in B$. By transitivity of $\Delta_{\beta,\alpha}$ we conclude that $(d, d') \in \Phi(B)$, as desired.

(ii) Since $[\alpha, \beta] \in \mathcal{P}(\beta)$ we have $[\alpha, \beta] \subseteq \Phi([\alpha, \beta])$, by (1). We prove the reverse inclusion. If $(c, c') \in \Phi([\alpha, \beta])$, then (1.8) implies there exists $(b, b') \in [\alpha, \beta]$ such that

$$(b, b')\ \Delta_{\beta,\alpha}\ (c, c'). \tag{B.7}$$

From the definition of $\Delta_{\beta,\alpha}$ and Mal'tsev's theorem on congruence generation, (B.7) holds if and only if $\exists\, z_i\ \beta\ z_i'$ $(0 \leqslant i \leqslant n)$, $\exists\, x_i\ \alpha\ y_i$ $(0 \leqslant i < n)$, $\exists\, f_i \in \mathrm{Pol}_1(\mathbf{A} \times \mathbf{A})$ $(0 \leqslant i < n)$ such that $(b, b') = (z_0, z_0')$ and $(z_n, z_n') = (c, c')$, and

$$\{(b, b'), (z_1, z_1')\} = \{f_0(x_0, x_0), f_0(y_0, y_0)\} \tag{B.8}$$

$$\{(z_1, z_1'), (z_2, z_2')\} = \{f_1(x_1, x_1), f_1(y_1, y_1)\} \tag{B.9}$$

$$\vdots$$

$$\{(z_{n-1}, z_{n-1}'), (c, c')\} = \{f_{n-1}(x_{n-1}, x_{n-1}), f_{n-1}(y_{n-1}, y_{n-1})\}$$

For each $(0 \leqslant i < n)$, $f_i \in \mathrm{Pol}_1(\mathbf{A} \times \mathbf{A})$, which means

$$f_i(x, x') = g_i^{\boldsymbol{\beta}}((x, x'), (a_1, a_1'), \dots, (a_k, a_k')) = (g_i^{\mathbf{A}}(x, \mathbf{a}), g_i^{\mathbf{A}}(x, \mathbf{a}')),$$

for some k, $g_i \in \mathsf{Clo}_{k+1}(\mathbf{A})$, and constants tuples $\mathbf{a} = (a_1, \dots, a_k)$ and $\mathbf{a}' = (a_1', \dots, a_k')$ such that $a_i\ \beta\ a_i'$ $(1 \leqslant i \leqslant k)$. By (B.8), either

$$(b, b') = \big(g_0(x_0, \mathbf{a}), g_0(x_0, \mathbf{a}')\big) \quad \text{and} \quad (z_1, z_1') = \big(g_0(y_0, \mathbf{a}), g_0(y_0, \mathbf{a}')\big),$$

or vice-versa. Since $x_0\ \alpha\ y_0$ and $a_i\ \beta\ a_i'$ $(1 \leqslant i \leqslant k)$, the α, β-term condition entails

$$g_0(x_0, \mathbf{a})\ [\alpha, \beta]\ g_0(x_0, \mathbf{a}') \quad \Longleftrightarrow \quad g_0(y_0, \mathbf{a})\ [\alpha, \beta]\ g_0(y_0, \mathbf{a}').$$

This and (B.8) yield $(b, b') \in [\alpha, \beta]$ iff $(z_1, z_1') \in [\alpha, \beta]$. Similarly (B.9) and $x_1\ \alpha\ y_1$ imply $(z_1, z_1') \in [\alpha, \beta]$ iff $(z_2, z_2') \in [\alpha, \beta]$. Inductively, we arrive at $(b, b') \in [\alpha, \beta]$ iff $(c, c') \in [\alpha, \beta]$, as desired.

We have thus proved $[\alpha, \beta]$ is a fixed point of Φ. In other words, $[\alpha, \beta]$ is a "Φ-closed" subset of β. (A set $B \subseteq \beta$ is called Φ-*closed* provided $\Phi(B) \subseteq B$.) Recall, if f is a monotone increasing function defined on a complete poset $\langle P, \leqslant \rangle$, then the least fixed point of f is $\bigwedge\{p \in P \mid fp \leqslant p\}$. Thus, Lemma 1.5 (ii) asserts that

$$[\alpha, \beta] = \bigwedge\{B \subseteq \beta \mid \Phi(B) \subseteq B\}.$$

We already proved $[\alpha, \beta]$ is Φ-closed, so it remains to check for every Φ-closed subset $B \subseteq \beta$ that $[\alpha, \beta] \subseteq B$. Fix a Φ-closed subset $B \subseteq \beta$. It suffices to prove $\mathsf{C}(\alpha, \beta; \Phi(B))$, since this implies $[\alpha, \beta] \subseteq \Phi(B) \subseteq B$. Thus, our goal is to establish the α, β-term condition.

Let $p \in \mathrm{Pol}_{k+1}(\mathbf{A})$ and $a\ \alpha\ a'$ and $c_i\ \beta\ c_i'$ $(1 \leqslant i \leqslant k)$; suppose $p(a, \mathbf{c})\ \Phi(B)\ p(a, \mathbf{c}')$. We prove that these hypotheses entail the following relation:

$$p(a', \mathbf{c})\ \Phi(B)\ p(a', \mathbf{c}'). \tag{B.10}$$

By definition of Φ, (B.10) is equivalent to the existence of some pair $(b, b') \in B$ such that $(b, b')\ \Delta_{\beta,\alpha}$ $(p(a', \mathbf{c}), p(a', \mathbf{c}'))$. Notice that the pair $(p(a, \mathbf{c}), p(a, \mathbf{c}'))$ belongs to B since $(p(a, \mathbf{c}), p(a, \mathbf{c}')) \in$

$\Phi(B) \subseteq B$. Also, $c_i \mathrel{\beta} c_i'$ $(0 \leqslant i < k)$ implies

$$((a,a),(c_1,c_1'),(c_1,c_1'),\dots,(c_k,c_k')) \in \beta^{k+1} \quad \text{and}$$
$$((a',a'),(c_1,c_1'),(c_1,c_1'),\dots,(c_k,c_k')) \in \beta^{k+1}.$$

Therefore,

$$p^{\beta}((a,a),(c_1,c_1'),(c_1,c_1'),\dots,(c_k,c_k')) = (p^{\mathbf{A}}(a,\mathbf{c}), p^{\mathbf{A}}(a,\mathbf{c}')) \in \beta \quad \text{and} \tag{B.11}$$
$$p^{\beta}((a',a'),(c_1,c_1'),(c_1,c_1'),\dots,(c_k,c_k')) = (p^{\mathbf{A}}(a',\mathbf{c}), p^{\mathbf{A}}(a',\mathbf{c}')) \in \beta. \tag{B.12}$$

Finally, $a \mathrel{\alpha} a'$ implies $p(a,\mathbf{c}) \mathrel{\alpha} p(a',\mathbf{c})$, and this—together with (B.11) and (B.12)—proves the pair $\big((p(a,\mathbf{c}),p(a,\mathbf{c}')),(p(a',\mathbf{c}),p(a',\mathbf{c}'))\big)$ belongs to $\Delta_{\beta,\alpha}$. Since $(p(a,\mathbf{c}),p(a,\mathbf{c}')) \in B$, this proves $(p(a',\mathbf{c}),p(a',\mathbf{c}')) \in \Phi(B)$, completing the proof. ☐

B.4. **Absorption Theorem (Barto and Kozik).**

Theorem B.2 (Absorption Theorem [BK12, Thm 2.3])**.** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be finite idempotent algebras with Taylor terms; let* $\mathbf{R}$ *be a subdirect product of* $\mathbf{A} \times \mathbf{B}$*, and let* η_A *(*η_B*, resp.) be the kernel of the projection* $\mathbf{R} \twoheadrightarrow \mathbf{A}$ *(*$\mathbf{R} \twoheadrightarrow \mathbf{B}$*, resp.). If* $\eta_A \vee \eta_B = 1_R$*, then either* $\mathbf{R} = \mathbf{A} \times \mathbf{B}$*, or* $\mathbf{A}$ *has a proper absorbing subuniverse, or* $\mathbf{B}$ *has a proper absorbing subuniverse.*

Theorem B.3 ([BKS15, Theorem 3.2])**.** *Let* $\mathcal{V}$ *be a locally finite variety generated by a set* $\mathcal{A}$ *of idempotent "hereditarily absorption free" algebras. If* $\mathcal{V}$ *has a Taylor term, then it has a Mal'tsev term.*

Proof. Let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x,y)$ be the 2-generated free algebra in the variety $\mathcal{V}$. Since it is finite, $\mathbf{F}$ lies in the pseudovariety generated by $\mathcal{A}$, and thus it is absorption-free (see [BKS15, Proposition 2.1.(1)]). Let $\mathbf{R}$ be the subalgebra of $\mathbf{F}^2$ generated by $(x,y),(x,x),(y,x)$. It is subdirect in $\mathbf{F}^2$. Since $\mathbf{F}$ is idempotent, $\eta_A \vee \eta_B = 1_R$ (as observed in Lemma 1.1 (B.12)). Consequently, by Theorem B.2, $\mathbf{R} = \mathbf{F}^2$, and there is a term m witnessing that $(y,y) \in R$, i.e., a term m satisfying $m((x,y),(x,x),(y,x)) = (y,y)$. This term is a Mal'tsev term for $\mathcal{V}$. ☐

Remark. Let π_0 and π_1 be the first and second projections of $\mathbf{R}$ onto $\mathbf{F}$. Since $\mathbf{F}$ is idempotent, $\{x\}$ is a subalgebra of $\mathbf{F}$, hence the inverse image of $\{x\}$ under π_0 is a subuniverse of $\mathbf{R}$, call it S_0. Since (x,y) and (x,x) lie in S_0, and $\{x,y\}$ generates $\mathbf{F}$, we see that $\{x\} \times F$ is contained in S_0. In particular, for every a_1, the pair (x,a_1) must lie in R. Similarly, there is a subalgebra S_1 containing $F \times \{x\}$. Finally, for any (a_0,a_1) and (a_0',a_1') in R we have

$$(a_0,a_1) \mathrel{\eta_1} (x,a_1) \mathrel{\eta_0} (x,a_1') \mathrel{\eta_1} (a_0',a_1').$$

Appendix C. Odds and Ends

C.1. **Skew congruences of idempotent algebras.**
Let $\mathcal{V}$ be a variety and let $\mathbf{A}$ and $\mathbf{B}$ be idempotent algebras in $\mathcal{V}$. Recall the following standard notation: if $\alpha \in \operatorname{Con}(\mathbf{A})$ and $\beta \in \operatorname{Con}(\mathbf{B})$, then $\alpha \times \beta$ denotes the set of pairs $((a,b),(a',b'))$ satisfying $a \mathrel{\alpha} a'$ and $b \mathrel{\beta} b'$. The relation $\alpha \times \beta$ is clearly a congruence of $\mathbf{A} \times \mathbf{B}$.

Fix (a,b) and (a',b') in $A \times B$. We claim that

$$\operatorname{Cg}^{\mathbf{A}}(a,a') \times \operatorname{Cg}^{\mathbf{B}}(b,b') = \operatorname{Cg}^{\mathbf{A}\times\mathbf{B}}((a,b),(a',b')).$$

Let $\tau := \operatorname{Cg}^{\mathbf{A}}(a,a') \times \operatorname{Cg}^{\mathbf{B}}(b,b')$ and $\theta := \operatorname{Cg}^{\mathbf{A}\times\mathbf{B}}((a,b),(a',b'))$. First note that τ is a product of a congruence of $\mathbf{A}$ with a congruence of $\mathbf{B}$, so τ is a congruence of $\mathbf{A} \times \mathbf{B}$. Moreover, the pair $((a,b),(a',b'))$ clearly belongs to τ, so $\theta \leqslant \tau$. We must prove that $\tau \leqslant \theta$.

Fix $((x,y),(x'y')) \in \tau$. This means that $(x,x') \in \mathrm{Cg}^{\mathbf{A}}(a,a')$ and $(y,y') \in \mathrm{Cg}^{\mathbf{B}}(b,b')$. Therefore, there exist $n > 0$, $m > 0$, $c_0, c_1, \dots, c_n \in A$, $f_0, f_1, \dots, f_{n-1} \in \mathrm{Pol}_1(\mathbf{A})$, $d_0, d_1, \dots, d_n \in B$, and $g_0, g_1, \dots, g_{n-1} \in \mathrm{Pol}_1(\mathbf{B})$ such that

$$x = c_0,\ c_n = x',\ \text{and}\ \ \{c_i, c_{i+1}\} = \{f_i(a), f_i(a')\},\ \text{for all } 0 \leqslant i < n, \text{ and}$$
$$y = d_0,\ d_n = y',\ \text{and}\ \{d_i, d_{i+1}\} = \{g_i(b), g_i(b')\},\ \text{for all } 0 \leqslant i < m.$$

We can assume without loss of generality that $n = m$, since we can insert dummy terms to extend the shorter of the two sequences.

We wish to prove $((x,y),(x'y')) \in \theta := \mathrm{Cg}^{\mathbf{A}\times\mathbf{B}}((a,b),(a',b'))$, which is equivalent to the following: there exist $n > 0$, $(e_0,e_0'), (e_1,e_1'), \dots, (e_n,e_n') \in A \times B$, $h_0, h_1, \dots, h_{n-1} \in \mathrm{Pol}_1(\mathbf{A}\times\mathbf{B})$ such that

$$(x,y) = (e_0,e_0'),\ (e_n,e_n') = (x',y'),\ \text{and for all } 0 \leqslant i < n,$$
$$\{(e_i,e_i'),(e_{i+1},e_{i+1}')\} = \{h_i(a,b), h_i(a',b')\}.$$

Let $f = f_0$. Since $f \in \mathrm{Pol}_1(\mathbf{A})$, for some $\ell > 0$ there exist a term $s^{\mathbf{A}} \in \mathsf{Clo}_{\ell+1}(\mathbf{A})$ and $a_0, a_1, \dots, a_\ell$ in A such that

$$f^{\mathbf{A}}(x) = s^{\mathbf{A}}(a_0, \dots, a_{i-1}, x, a_{i+1}, \dots, a_\ell),\ \text{for some } i.$$

Similarly, letting $g = g_0$, for some $k > 0$ there exist a term $t^{\mathbf{B}} \in \mathsf{Clo}_{k+1}(\mathbf{B})$ and $b_0, b_1, \dots, b_k$ in B such that

$$g^{\mathbf{B}}(y) = t^{\mathbf{B}}(b_0, \dots, b_{j-1}, y, b_{j+1}, \dots, b_k),\ \text{for some } j.$$

Without loss of generality, we can assume $j = i$. (check this)

Consider the polynomial $p \in \mathrm{Pol}_1(\mathbf{A}\times\mathbf{B})$ defined via s as follows:

$$\begin{aligned} p(x,y) &= s^{\mathbf{A}\times\mathbf{B}}((a_0,b),(a_1,b),\dots,(a_{i-1},b),(x,y),(a_{i+1},b),\dots,(a_\ell,b)) \\ &= (s^{\mathbf{A}}(a_0,a_1,\dots,a_{i-1},x,a_{i+1},\dots,a_\ell), s^{\mathbf{B}}(b,\dots,b,y,b,\dots,b)) \\ &= (f^{\mathbf{A}}(x), s^{\mathbf{B}}(b,\dots,b,y,b,\dots,b)). \end{aligned}$$

Consider the polynomial $q \in \mathrm{Pol}_1(\mathbf{A}\times\mathbf{B})$ defined via t as follows:

$$\begin{aligned} q(x,y) &= t^{\mathbf{A}\times\mathbf{B}}((f^{\mathbf{A}}(a),b_0),(f^{\mathbf{A}}(a),b_1),\dots,(f^{\mathbf{A}}(a),b_{i-1}),(x,y),(f^{\mathbf{A}}(a),b_{i+1}),\dots,(f^{\mathbf{A}}(a),b_\ell)) \\ &= (t^{\mathbf{A}}(f^{\mathbf{A}}(a),\dots,f^{\mathbf{A}}(a),x,f^{\mathbf{A}}(a),\dots,f^{\mathbf{A}}(a)), t^{\mathbf{B}}(b_0,\dots,b_{i-1},y,b_{i+1},\dots,b_\ell)) \\ &= (t^{\mathbf{A}}(f^{\mathbf{A}}(a),\dots,f^{\mathbf{A}}(a),x,f^{\mathbf{A}}(a),\dots,f^{\mathbf{A}}(a)), g^{\mathbf{B}}(y)). \end{aligned}$$

Then $p(a,b) = (f^{\mathbf{A}}(a),b)$ and $q(f^{\mathbf{A}}(a),b) = (f^{\mathbf{A}}(a), g^{\mathbf{B}}(b))$. Therefore,

$$(q \circ p)^{\mathbf{A}\times\mathbf{B}}(a,b) = (f^{\mathbf{A}}(a), g^{\mathbf{B}}(b)).$$

We can carry out this construction for each pair (f_i,g_i), $0 \leqslant i < n$, arriving at a sequence $q_i \circ p_i$ of polynomials in $\mathrm{Pol}_1(\mathbf{A}\times\mathbf{B})$ such that

$$(q_i \circ p_i)^{\mathbf{A}\times\mathbf{B}}(a,b) = (f_i^{\mathbf{A}}(a), g_i^{\mathbf{B}}(b)).$$

Unfortunately, evaluating this polynomial at (a',b') yields the less desirable result:

$$(q_i \circ p_i)(a',b') = (t_i^{\mathbf{A}}(f_i^{\mathbf{A}}(a),\dots,f_i^{\mathbf{A}}(a),f_i^{\mathbf{A}}(a'),f_i^{\mathbf{A}}(a),\dots,f_i^{\mathbf{A}}(a)), g_i(s_i^{\mathbf{B}}(b,\dots,b,b',b,\dots,b))).$$

So we try again...

Consider the polynomial $p \in \mathrm{Pol}_1(\mathbf{A}\times\mathbf{B})$ defined via s as follows:

$$\begin{aligned} p(x,y) &= s^{\mathbf{A}\times\mathbf{B}}((a_0,y),\dots,(a_{i-1},y),(x,y),(a_{i+1},y),\dots,(a_\ell,y)) \\ &= (s^{\mathbf{A}}(a_0,\dots,a_{i-1},x,a_{i+1},\dots,a_\ell), s^{\mathbf{B}}(y,\dots,y)) \\ &= (f^{\mathbf{A}}(x), y). \end{aligned}$$

Consider the polynomial $q \in \mathrm{Pol}_1(\mathbf{A} \times \mathbf{B})$ defined via t as follows:

$$\begin{aligned} q(x,y) &= t^{\mathbf{A}\times\mathbf{B}}(p(x,b_0), \dots, p(x,b_{i-1}), p(x,y), p(x,b_{i+1}), \dots, p(x,b_\ell)) \\ &= t^{\mathbf{A}\times\mathbf{B}}((f^{\mathbf{A}}(x),b_0), \dots, (f^{\mathbf{A}}(x),b_{i-1}), (f^{\mathbf{A}}(x),y), (f^{\mathbf{A}}(x),b_{i+1}), \dots, (f^{\mathbf{A}}(x),b_\ell)) \\ &= (f^{\mathbf{A}}(x), g^{\mathbf{B}}(y)). \end{aligned}$$

Then $p(a,b) = (f^{\mathbf{A}}(a), b)$ and $q(f^{\mathbf{A}}(a), b) = (f^{\mathbf{A}}(a), g^{\mathbf{B}}(b))$. Therefore,

$$(q \circ p)^{\mathbf{A}\times\mathbf{B}}(a,b) = (f^{\mathbf{A}}(a), g^{\mathbf{B}}(b)).$$

We can carry out this construction for each pair (f_i, g_i), $0 \leqslant i < n$, arriving at a sequence $q_i \circ p_i$ of polynomials in $\mathrm{Pol}_1(\mathbf{A} \times \mathbf{B})$ such that

$$(q_i \circ p_i)^{\mathbf{A}\times\mathbf{B}}(a,b) = (f_i^{\mathbf{A}}(a), g_i^{\mathbf{B}}(b)).$$

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