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ON THE COMPLEXITY OF DIFFERENCE TERM EXISTENCE

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We consider the following practical question: given a finite algebra $\mathbf{A}$ in a finite language, can we efficiently decide whether the variety generated by $\mathbf{A}$ has a difference term? In [4] "local difference terms" were defined and used to solve a related but easier problem—namely, it was shown that there is a polynomial-time algorithm for deciding whether any finite idempotent algebra has a difference term operation. In the present paper, we continue to build on the ideas in [14] and complete the project started [4]. More specifically, we define "global-local difference terms" which we use to devise an efficient algorithm for deciding whether the variety generated by a finite idempotent algebra has a difference term.

Keywords: difference term; finite idempotent algebra; polynomial-time algorithm

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1. Introduction

Let $\mathcal{V}$ be a variety (equational class) of algebras. A ternary term d in the language of $\mathcal{V}$ is called a *difference term for* $\mathcal{V}$ if it satisfies the following: for all $\mathbf{A} = \langle A, \ldots \rangle \in \mathcal{V}$ and $a, b \in A$ we have

$$d^{\mathbf{A}}(a, a, b) = b \quad \text{and} \quad d^{\mathbf{A}}(a, b, b) \; [\theta, \theta] \; a, \tag{1.1}$$

where θ is any congruence containing (a, b) and $[\cdot, \cdot]$ denotes the (term condition) commutator. (See [7] or [12] for definitions.) When the relations in (1.1) hold we will call $d^{\mathbf{A}}$ a *difference term operation* for $\mathbf{A}$.

Difference terms are studied extensively in the universal algebra literature. (See, for example, [7,9,10,11,12,13].) There are many reasons to study difference terms,

but perhaps the most obvious is that knowing a variety has a difference term allows us to deduce many useful properties of the algebras inhabiting that variety. (Very roughly speaking, having a difference term is slightly stronger than having a Taylor term and slightly weaker than having a Mal'tsev term. Note that if $\mathbf{A}$ is an *abelian* algebra—that is, $[1_A, 1_A] = 0_A$—then by the monotonicity of the commutator we have $[\theta, \theta] = 0_A$ for all $\theta \in \operatorname{Con}\mathbf{A}$, in which case (1.1) says that $d^{\mathbf{A}}$ is a Mal'tsev term operation.)

Digital computers have turned out to be invaluable tools for exploring and understanding algebras and the varieties they inhabit, and this is largely due to the fact that researchers have found ingenious ways to get computers to solve abstract decision problems—such as whether a variety is congruence-modular ([6]) or congruence-n-permutable ([14])—and to do so efficiently. The contribution of the present paper is to present a solution to the following:

Problem 1. Is there a polynomial-time algorithm that takes a finite idempotent algebra $\mathbf{A}$ as input and decides whether the variety generated by $\mathbf{A}$ has a difference term?

By solving Problem 1 we complete the project started in [4]; in the latter, we solved the following easier problem:

Problem 2. Is there a polynomial-time algorithm that takes a finite idempotent algebra $\mathbf{A}$ as input and decides whether $\mathbf{A}$ has a difference term operation?

The rest of the paper is organized as follows: Section 2 introduces notation and definitions and some of the background that we expect the reader to have. In [11] it was shown that a locally finite idempotent variety $\mathcal{V}$ has a difference term if and only if $\mathsf{HSP}(\mathbf{F}_{\mathcal{V}}(2))$ has a difference term (where $\mathbf{F}_{\mathcal{V}}(2)$ denotes the 2-generated free algebra in $\mathcal{V}$). In Section 3 we extend this result by showing that this is also equivalent to the free algebra $\mathbf{F}_{\mathcal{V}}(2)$ itself having a difference term operation. In [14], Valeriote and Willard define a "local Hagemann-Mitschke sequence" which they use as the basis of an efficient algorithm for deciding for a given n whether an idempotent variety is n-permutable. In Section 4 we devise a similar construct, called a "local difference term," that we use to develop a polynomial-time algorithm for deciding the existence of a ("global-local") difference term operation for $\mathbf{A}$. In Section 5 we introduce "global-local difference terms" which we use to devise a polynomial-time algorithm for deciding whether the variety generated by a finite idempotent algebra has a difference term.

2. Background, Notation, Definitions

Our arguments depend on some basic results of universal algebra that we now review. For the most part we use standard notation such as those found in [1]. However, we make the following exception for notational simplicity: if $\mathbf{A} = \langle A, \ldots \rangle$ is an algebra with elements $a, b \in A$, then we use $\theta(a, b)$ to denote the congruence of $\mathbf{A}$ generated by a and b.

Let $\mathbf{A} = \langle A, F^{\mathbf{A}} \rangle$ be an algebra. A reflexive, symmetric, compatible binary relation $T \subseteq A^2$ is called a *tolerance of* $\mathbf{A}$. Given a pair $(\mathbf{u}, \mathbf{v}) \in A^m \times A^m$ of m-tuples of A, we write $\mathbf{u}\ \mathbf{T}\ \mathbf{v}$ just in case $\mathbf{u}(i)\ T\ \mathbf{v}(i)$ for all $0 \leqslant i < m$. We state a number of definitions in this section using tolerance relations, but the definitions don't change when the tolerance in question happens to be a congruence relation (i.e., a transitive tolerance).

Suppose S and T are tolerances on $\mathbf{A}$. An *S,T-matrix* is a 2×2 array of the form

$$\begin{bmatrix} t(\mathbf{a}, \mathbf{u})\ t(\mathbf{a}, \mathbf{v}) \\ t(\mathbf{b}, \mathbf{u})\ t(\mathbf{b}, \mathbf{v}) \end{bmatrix},$$

where t, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$, $\mathbf{v}$ have the following properties:

(i) $t \in \mathsf{Clo}_{\ell+m}(\mathbf{A})$,
(ii) $(\mathbf{a}, \mathbf{b}) \in A^\ell \times A^\ell$ and $\mathbf{a}\ \mathbf{S}\ \mathbf{b}$,
(iii) $(\mathbf{u}, \mathbf{v}) \in A^m \times A^m$ and $\mathbf{u}\ \mathbf{T}\ \mathbf{v}$.

Let δ be a congruence relation of $\mathbf{A}$. If the entries of every S,T-matrix satisfy

$$t(\mathbf{a}, \mathbf{u})\ \delta\ t(\mathbf{a}, \mathbf{v}) \quad \Longleftrightarrow \quad t(\mathbf{b}, \mathbf{u})\ \delta\ t(\mathbf{b}, \mathbf{v}), \tag{2.1}$$

then we say that S *centralizes* T *modulo* δ and we write $\mathsf{C}(S, T; \delta)$. That is, $\mathsf{C}(S, T; \delta)$ means that (2.1) holds *for all* ℓ, m, t, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$, $\mathbf{v}$ satisfying properties (i)–(iii).

The *commutator* of S and T, denoted by $[S, T]$, is the least congruence δ such that $\mathsf{C}(S, T; \delta)$ holds. Note that $\mathsf{C}(S, T; 0_A)$ is equivalent to $[S, T] = 0_A$, and this is sometimes called the *S,T-term condition*; when it holds we say that S *centralizes* T. A tolerance T is called *abelian* if $[T, T] = 0_A$. An algebra $\mathbf{A}$ is called *abelian* if 1_A is abelian (i.e., $[1_A, 1_A] = 0_A$).

Here are some properties of the centralizer relation that are well-known and not too hard to prove (see [7, Prop 3.4] or [12, Thm 2.19]).

Lemma 2.1. *Let* $\mathbf{A}$ *be an algebra and suppose* $\mathbf{B}$ *is a subalgebra of* $\mathbf{A}$*. Let* α*,* β*,* γ*,* δ*,* α_i β_j*,* γ_k *be congruences of* $\mathbf{A}$*, for all* $i \in I$*,* $j \in J$*,* $k \in K$*. Then the following hold:*

(1) $\mathsf{C}(\alpha, \beta; \alpha \wedge \beta)$*;*
(2) if $\mathsf{C}(\alpha, \beta; \gamma_k)$ *for all* $k \in K$*, then* $\mathsf{C}(\alpha, \beta; \bigwedge_K \gamma_k)$*;*
(3) if $\mathsf{C}(\alpha_i, \beta; \gamma)$ *for all* $i \in I$*, then* $\mathsf{C}(\bigvee_I \alpha_i, \beta; \gamma)$*;*
(4) if $\mathsf{C}(\alpha, \beta; \gamma)$ *and* $\alpha' \leqslant \alpha$*, then* $\mathsf{C}(\alpha', \beta; \gamma)$*;*
(5) if $\mathsf{C}(\alpha, \beta; \gamma)$ *and* $\beta' \leqslant \beta$*, then* $\mathsf{C}(\alpha, \beta'; \gamma)$*;*
(6) if $\mathsf{C}(\alpha, \beta; \gamma)$ *in* $\mathbf{A}$*, then* $\mathsf{C}(\alpha \cap B^2, \beta \cap B^2; \gamma \cap B^2)$ *in* $\mathbf{B}$*;*
(7) if $\gamma \leqslant \delta$*, then* $\mathsf{C}(\alpha, \beta; \delta)$ *in* $\mathbf{A}$ *if and only if* $\mathsf{C}(\alpha/\gamma, \beta/\gamma; \delta/\gamma)$ *in* $\mathbf{A}/\gamma$*.*

Remark 2.2. By (1), if $\alpha \wedge \beta = 0_A$, then $[\beta, \alpha] = 0_A = [\alpha, \beta]$.

Before proceeding, we collect some facts about the commutator that are sometimes useful when reasoning about difference terms.

Lemma 2.3. *Let* $\mathbf{A}$ *be an algebra with congruences* α, α', β, β' *satisfying* $\alpha \leqslant \alpha'$ *and* $\beta \leqslant \beta'$. *Then* $[\alpha, \beta] \leqslant [\alpha', \beta']$.

Proof. For every $\delta \in \operatorname{Con}\mathbf{A}$, $\mathsf{C}(\alpha', \beta'; \delta)$ implies $\mathsf{C}(\alpha, \beta; \delta)$, since $\alpha \leqslant \alpha'$ and $\beta \leqslant \beta'$. In particular, $\mathsf{C}(\alpha', \beta'; [\alpha', \beta'])$ implies $\mathsf{C}(\alpha, \beta; [\alpha', \beta'])$, so $[\alpha, \beta] \leqslant [\alpha', \beta']$. □

Lemma 2.4. *Let* $\mathbf{A}$ *be an algebra with congruences* α_i *and* β_i *for all* $i \in I$. *Then*

$$[\bigwedge \alpha_i, \bigwedge \beta_i] \leqslant \bigwedge [\alpha_i, \beta_i] \qquad \text{and} \qquad \bigvee [\alpha_i, \beta_i] \leqslant [\bigvee \alpha_i, \bigvee \beta_i] \,.$$

Proof. By Lemma 2.3, $[\bigwedge \alpha_i, \bigwedge \beta_i] \leqslant [\alpha_i, \beta_i] \leqslant [\bigvee \alpha_i, \bigvee \beta_i]$, for all $i \in I$. □

Lemma 2.5 ([11, Theorem 2.10]). *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be algebras of the same similarity type and suppose* $\phi : \mathbf{A} \to \mathbf{B}$ *is a surjective homomorphism. If* $\alpha, \beta \in \operatorname{Con}\mathbf{A}$, *then* $\phi([\alpha, \beta]) \subseteq [\phi(\alpha), \phi(\beta)]$. *Moreover, if there exists a homomorphism* $\psi : \mathbf{B} \to \mathbf{A}$ *such that* $\phi \circ \psi = \mathrm{id}_B$ *and if* $\rho, \sigma \in \operatorname{Con}\mathbf{B}$, *then* $\psi^{-1}\{[\psi(\rho), \psi(\sigma)]\} = \phi\big([\psi(\rho), \psi(\sigma)]\big) = [\rho, \sigma]$.

3. Equivalent conditions for existence of a difference term

The main result proved in this section is Theorem 3.1, which is a slightly improved version of the observation in [11] stating that a variety $\mathcal{V}$ has a difference term if and only if $\mathsf{HSP}(\mathbf{F}_{\mathcal{V}}(2))$ has a difference term. The forward implication of this claim is trivial; the argument for the converse goes as follows: assume that $d(x, y, z)$ is a difference term for $\mathsf{HSP}(\mathbf{F})$. Choose $\mathbf{A} \in \mathcal{V}$ and $a, b \in A$. Let $\mathbf{B} = \mathrm{Sg}^{\mathbf{A}}(\{a, b\})$. Since $\mathbf{B}$ is 2-generated, $B \in \mathsf{HSP}(\mathbf{F})$. Hence $d(x, y, z)$ interprets as a difference term in $\mathbf{B}$. This means that $d^{\mathbf{A}}(a, a, b) = d^{\mathbf{B}}(a, a, b) = b$. Furthermore,

$$d^{\mathbf{A}}(a, b, b) = d^{\mathbf{B}}(a, b, b)\ [\theta^{\mathbf{B}}(a, b), \theta^{\mathbf{B}}(a, b)]\ a.$$

But $[\theta^{\mathbf{B}}(a, b), \theta^{\mathbf{B}}(a, b)] \subseteq [\theta, \theta]$ for any congruence $\theta \in \operatorname{Con}\mathbf{A}$ for which $(a, b) \in \theta$. Consequently $d^{\mathbf{A}}(a, b, b)\ [\theta, \theta]\ a$ as desired.

Considering the goal of the present project, it seems natural to begin by trying to prove that the existence of a difference term for $\mathcal{V}$ is equivalent to the existence of a difference term *operation* for a specific algebra in $\mathcal{V}$. This is achieved in Theorem 3.1, which will play a key role in our main complexity argument in Section 5.

Theorem 3.1. *Let* $\mathcal{V}$ *be a variety and* $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(2)$, *the 2-generated free algebra in* $\mathcal{V}$. *The following are equivalent:*

(i) $\mathcal{V}$ *has a difference term;*
(ii) $\mathsf{HSP}(\mathbf{F})$ *has a difference term;*
(iii) $\mathbf{F}$ *has a difference term operation.*

Proof. The implications (i) ⇒ (ii) ⇒ (iii) are obvious. We prove (iii) ⇒ (i) by contraposition. Suppose $\mathcal{V}$ has no difference term. (We show $\mathbf{F}$ has no difference term operation.) Let $d(x, y, z)$ be a ternary term of $\mathcal{V}$. Let $\mathbf{A} \in \mathcal{V}$ be such that $d^{\mathbf{A}}(x, y, z)$ is not a difference term operation in $\mathbf{A}$. Choose $a, b \in A$ witnessing this fact. Then either

(1) $d^{\mathbf{A}}(a, a, b) \neq b$, or
(2) $(d^{\mathbf{A}}(a, b, b), a) \notin [\theta^{\mathbf{A}}(a, b), \theta^{\mathbf{A}}(a, b)]$.

Let $\mathbf{B} = \mathrm{Sg}^{\mathbf{A}}(\{a, b\})$. In case (1), $d^{\mathbf{B}}(a, a, b) = d^{\mathbf{A}}(a, a, b) \neq b$, so $d^{\mathbf{B}}(x, y, z)$ is not a difference term operation for $\mathbf{B}$. In case (2), observe that the pair $(d^{\mathbf{B}}(a, b, b), a)$ is equal to the pair $(d^{\mathbf{A}}(a, b, b), a)$ which does not belong to $[\theta^{\mathbf{A}}(a, b), \theta^{\mathbf{A}}(a, b)]$. But $[\theta^{\mathbf{B}}(a, b), \theta^{\mathbf{B}}(a, b)] \subseteq [\theta^{\mathbf{A}}(a, b), \theta^{\mathbf{A}}(a, b)]$, so

$$(d^{\mathbf{B}}(a, b, b), a) \notin [\theta^{\mathbf{B}}(a, b), \theta^{\mathbf{B}}(a, b)],$$

and again we conclude that $d^{\mathbf{B}}(x, y, z)$ is not a difference term operation for $\mathbf{B}$. Now, since there is a surjective homomorphism from $\mathbf{F}$ to $\mathbf{B}$, it follows that $d^{\mathbf{F}}(x, y, z)$ cannot be a difference term operation for $\mathbf{F}$. Finally, recall that $d(x, y, z)$ was an arbitrary ternary term of $\mathcal{V}$, so $\mathbf{F}$ has no difference term operation whatsoever. □

4. Local difference terms

In [14], Valeriote and Willard define a "local Hagemann-Mitschke sequence" which they use as the basis of an efficient algorithm for deciding for a given n whether an idempotent variety is n-permutable. Inspired by that work, we devise a similar construct, called a "local difference term," that we use to develop a polynomial-time algorithm for deciding the existence of a (global) difference term operation.

Let $\mathbf{A} = \langle A, \ldots \rangle$ be an algebra, fix $a, b \in A$ and $i \in \{0, 1\}$. A *local difference term for* (a, b, i) is a ternary term d satisfying the following:

$$\begin{aligned} &\text{if } i = 0, \text{ then } a\ [\theta(a, b), \theta(a, b)]\ d(a, b, b); \\ &\text{if } i = 1, \text{ then } d(a, a, b) = b. \end{aligned} \tag{4.1}$$

If d satisfies (4.1) for all triples in some subset $S \subseteq A \times A \times \{0, 1\}$, then we call d a *local difference term for* S.

Let $\mathcal{S} = A \times A \times \{0, 1\}$ and suppose that every pair $((a_0, b_0, \chi_0), (a_1, b_1, \chi_1))$ in $\mathcal{S}^2$ has a local difference term. That is, for each pair $((a_0, b_0, \chi_0), (a_1, b_1, \chi_1))$, there exists d such that for each $i \in \{0, 1\}$ we have

$$a_i\ [\theta(a_i, b_i), \theta(a_i, b_i)]\ d(a_i, b_i, b_i), \text{ if } \chi_i = 0, \text{ and} \tag{4.2}$$

$$d(a_i, a_i, b_i) = b_i, \text{ if } \chi_i = 1. \tag{4.3}$$

Under these hypothesis we will prove that every subset $S \subseteq \mathcal{S}$ has a local difference term. That is, there is a single term d that works (i.e., satisfies (4.2) and (4.3)) for all $(a_i, b_i, \chi_i) \in S$. The statement and proof of this new result follows.

Theorem 4.1. *Let $\mathcal{V}$ be an idempotent variety and $\mathbf{A} \in \mathcal{V}$. Define $\mathcal{S} = A \times A \times \{0, 1\}$*

and suppose that every pair $((a_0, b_0, \chi_0), (a_1, b_1, \chi_1)) \in \mathcal{S}^2$ *has a local difference term. Then every subset* $S \subseteq \mathcal{S}$*, has a local difference term.*

Proof. The proof is by induction on the size of S. In the base case, $|S| = 2$, the claim holds by assumption. Fix $n \geqslant 2$ and assume that every subset of $\mathcal{S}$ of size $2 \leqslant k \leqslant n$ has a local difference term. Let

$$S = \{(a_0, b_0, \chi_0), (a_1, b_1, \chi_1), \ldots, (a_n, b_n, \chi_n)\} \subseteq \mathcal{S},$$

so that $|S| = n + 1$. We prove S has a local difference term.

Since $|S| \geqslant 3$ and $\chi_i \in \{0, 1\}$ for all i, there must exist indices $i \neq j$ such that $\chi_i = \chi_j$. Assume without loss of generality that one of these indices is $j = 0$. Define the set $S' = S \setminus \{(a_0, b_0, \chi_0)\}$. Since $|S'| < |S|$, the set S' has a local difference term p. We split the remainder of the proof into two cases.

Case $\chi_0 = 0$: Without loss of generality, suppose that $\chi_1 = \cdots = \chi_k = 1$, and $\chi_{k+1} = \cdots = \chi_n = 0$. Define

$$T = \{(a_0, p(a_0, b_0, b_0), 0), (a_1, b_1, 1), (a_2, b_2, 1), \ldots, (a_k, b_k, 1)\},$$

and note that $|T| < |S|$. Let t be a local difference term for T. Define

$$d(x, y, z) = t(x, p(x, y, y), p(x, y, z)).$$

Since $\chi_0 = 0$, we need to show $(a_0, d(a_0, b_0, b_0))$ belongs to $[\theta(a_0, b_0), \theta(a_0, b_0)]$. We have

$$d(a_0, b_0, b_0) = t(a_0, p(a_0, b_0, b_0), p(a_0, b_0, b_0))\ [\tau, \tau]\ a_0, \tag{4.4}$$

where we have used τ to denote $\theta(a_0, p(a_0, b_0, b_0))$. Note that

$$(a_0, p(a_0, b_0, b_0)) = (p(a_0, a_0, a_0), p(a_0, b_0, b_0)) \in \theta(a_0, b_0),$$

so $\tau \leqslant \theta(a_0, b_0)$. Therefore, by monotonicity of the commutator we have $[\tau, \tau] \leqslant [\theta(a_0, b_0), \theta(a_0, b_0)]$. It follows from this and (4.4) that

$$d(a_0, b_0, b_0)\ [\theta(a_0, b_0), \theta(a_0, b_0)]\ a_0,$$

as desired.

For the indices $1 \leqslant i \leqslant k$ we have $\chi_i = 1$, so we prove $d(a_i, a_i, b_i) = b_i$ for such i. Observe,

$$\begin{aligned} d(a_i, a_i, b_i) &= t(a_i, p(a_i, a_i, a_i), p(a_i, a_i, b_i)) && (4.5)\\ &= t(a_i, a_i, b_i) && (4.6)\\ &= b_i. && (4.7) \end{aligned}$$

Equation (4.5) holds by definition of d, (4.6) because p is an idempotent local difference term for S', and (4.7) because t is a local difference term for T.

The remaining triples in our original set S have indices satisfying $k < j \leqslant n$ and $\chi_j = 0$. Thus, for these triples we want $d(a_j, b_j, b_j)\ [\theta(a_j, b_j), \theta(a_j, b_j)]\ a_j$. By definition,

$$d(a_j, b_j, b_j) = t(a_j, p(a_j, b_j, b_j), p(a_j, b_j, b_j)). \tag{4.8}$$

Since p is a local difference term for S', the pair $(p(a_j, b_j, b_j), a_j)$ belongs to $[\theta(a_j, b_j), \theta(a_j, b_j)]$. This and (4.8) imply that $(d(a_j, b_j, b_j), t(a_j, a_j, a_j))$ belongs to $[\theta(a_j, b_j), \theta(a_j, b_j)]$. Finally, by idempotence of t we have

$$d(a_j, b_j, b_j)\ [\theta(a_j, b_j), \theta(a_j, b_j)]\ a_j,$$

as desired.

Case $\chi_0 = 1$: Without loss of generality, suppose $\chi_1 = \chi_2 = \cdots = \chi_k = 0$, and $\chi_{k+1} = \chi_{k+2} = \cdots = \chi_n = 1$. Define

$$T = \{(p(a_0, a_0, b_0), b_0, 1), (a_1, b_1, 0), (a_2, b_2 0), \ldots, (a_k, b_k, 0)\},$$

and note that $|T| < |S|$. Let t be a local difference term for T and define $d(x, y, z) = t(p(x, y, z), p(y, y, z), z)$. Since $\chi_0 = 1$, we want $d(a_0, a_0, b_0) = b_0$. By the definition of d,

$$d(a_0, a_0, b_0) = t(p(a_0, a_0, b_0), p(a_0, a_0, b_0), b_0) = b_0.$$

The last equality holds since t is a local difference term for T, thus, for $(p(a_0, a_0, b_0), b_0, 1)$.

If $1 \leqslant i \leqslant k$, then $\chi_i = 0$, so for these indices we prove that $(a_i, d(a_i, b_i, b_i))$ belongs to $[\theta(a_i, b_i), \theta(a_i, b_i)]$. Again, starting from the definition of d and using idempotence of p, we have

$$\begin{aligned} d(a_i, b_i, b_i) &= t(p(a_i, b_i, b_i), p(b_i, b_i, b_i), b_i) \\ &= t(p(a_i, b_i, b_i), b_i, b_i). \end{aligned} \tag{4.9}$$

Next, since p is a local difference term for S', we have

$$t(p(a_i, b_i, b_i), b_i, b_i)\ [\theta(a_i, b_i), \theta(a_i, b_i)]\ t(a_i, b_i, b_i). \tag{4.10}$$

Finally, since t is a local difference term for T, hence for (a_i, b_i, b_i), we have $t(a_i, b_i, b_i)\ \ [\theta(a_i, b_i), \theta(a_i, b_i)]\ \ a_i$. Combining this with (4.9) and (4.10) yields $d(a_i, b_i, b_i)\ [\theta(a_i, b_i), \theta(a_i, b_i)]\ a_i$, as desired.

The remaining elements of our original set S have indices j satisfying $k < j \leqslant n$ and $\chi_j = 1$. For these we want $d(a_j, a_j, b_j) = b_j$. Since p is a local difference term for S', we have $p(a_j, a_j, b_j) = b_j$, and this along with idempotence of t yields

$$\begin{aligned} d(a_j, a_j, b_j) &= t(p(a_j, a_j, b_j), p(a_j, a_j, b_j), b_j) \\ &= t(b_j, b_j, b_j) = b_j, \end{aligned}$$

as desired. □

Corollary 4.2. *A finite idempotent algebra* $\mathbf{A}$ *has a difference term operation if and only if each pair* $((a,b,i),(a',b',i')) \in (A \times A \times \{0,1\})^2$ *has a local difference term.*

Proof. One direction is clear, since a difference term operation for $\mathbf{A}$ is obviously a local difference term for the whole set $A \times A \times \{0,1\}$. For the converse, suppose each pair in $(A \times A \times \{0,1\})^2$ has a local difference term. Then, by Theorem 4.1, there is a single local difference term for the whole set $A \times A \times \{0,1\}$, and this is a difference term operation for $\mathbf{A}$. Indeed, if d is a local difference term for $A \times A \times \{0,1\}$, then for all $a, b \in A$, we have $a\ [\theta(a,b), \theta(a,b)]\ d(a,b,b)$, since d is a local difference term for $(a,b,0)$, and we have $d(a,a,b) = b$, since d is also a local difference term for $(a,b,1)$. □

Algorithm 1: existence of difference term operations

Corollary 4.3. *There is a polynomial-time algorithm that takes as input any finite idempotent algebra* $\mathbf{A}$ *and decides whether* $\mathbf{A}$ *has a difference term operation.*

Proof. We describe an efficient algorithm for deciding, given a finite idempotent algebra $\mathbf{A}$, whether every pair $((a,b,i),(a',b',i')) \in (A \times A \times \{0,1\})^2$ has a local difference term. By Corollary 4.2, this will prove we can decide in polynomial-time whether $\mathbf{A}$ has a difference term operation.

Fix a pair $((a,b,i),(a',b',i'))$ in $(A \times A \times \{0,1\})^2$. If $i = i' = 0$, then the first projection is a local difference term. If $i = i' = 1$, then the third projection is a local difference term. The two remaining cases to consider are (1) $i = 0$ and $i' = 1$, and (2) $i = 1$ and $i' = 0$. Since these are completely symmetric, we only handle the first case. Assume the given pair of triples is $((a,b,0),(a',b',1))$. By definition, a term t is local difference term for this pair iff

$$a\ [\theta(a,b), \theta(a,b)]\ t^{\mathbf{A}}(a,b,b) \ \text{ and } \ t^{\mathbf{A}}(a',a',b') = b'.$$

We can rewrite this condition more compactly by considering

$$t^{\mathbf{A}\times\mathbf{A}}((a,a'),(b,a'),(b,b')) = (t^{\mathbf{A}}(a,b,b), t^{\mathbf{A}}(a',a',b')).$$

Clearly t is a local difference term for $((a,b,0),(a',b',1))$ iff

$$t^{\mathbf{A}\times\mathbf{A}}((a,a'),(b,a'),(b,b')) \in a/\delta \times \{b'\},$$

where $\delta = [\theta(a,b), \theta(a,b)]$ and a/δ denotes the δ-class containing a. (Observe that $a/\delta \times \{b'\}$ is a subalgebra of $\mathbf{A} \times \mathbf{A}$ by idempotence.) It follows that the pair $((a,b,0),(a',b',1))$ has a local difference term iff the subuniverse of $\mathbf{A} \times \mathbf{A}$ generated by $\{(a,a'),(b,a'),(b,b')\}$ intersects nontrivially with the subuniverse $a/\delta \times \{b'\}$.

Thus, the algorithm takes as input $\mathbf{A}$ and, for each $((a,a'),(b,a'),(b,b'))$ in $(A \times A)^3$, computes $\delta = [\theta(a,b), \theta(a,b)]$, computes the subalgebra $\mathbf{S}$ of $\mathbf{A} \times \mathbf{A}$ generated by $\{(a,a'),(b,a'),(b,b')\}$, and then tests whether $S \cap (a/\delta \times \{b'\})$ is empty.

If we find an empty intersection at any point, then $\mathbf{A}$ has a difference term operation. Otherwise the algorithm halts without witnessing an empty intersection, in which case $\mathbf{A}$ has a difference term operation.

Most of the operations carried out by this algorithm are well known to be polynomial-time. For example, that the running time of subalgebra generation is polynomial has been known for a long time (see [8]). The time complexity of congruence generation is also known to be polynomial (see [5]). The only operation whose tractability might be questionable is the commutator, but there is a straight-forward algorithm for computing it which, after the congruences have been computed, simply involves generating more subalgebras.

TODO: insert more details about complexity of commutator.

□

More details on the complexity of operations carried out by the algorithm, as well as many other algebraic operations, can be found in the references mentioned, as well as [3,2,6].

5. Global-local difference terms

The methods from the previous section can be lifted up to universes, as we now describe. Let $\mathcal{V}$ be a variety, let $\mathbf{A} = \langle A, \ldots \rangle \in \mathcal{V}$ and $i \in \{0, 1\}$. We call a term d a *global-local (or "glocal") difference term for* (A, i) provided for all $a, b \in A$ we have

$$\text{if } i = 0, \text{ then } a\ [\theta(a, b), \theta(a, b)]\ d^{\mathbf{A}}(a, b, b); \tag{5.1}$$

$$\text{if } i = 1, \text{ then } d^{\mathbf{A}}(a, a, b) = b. \tag{5.2}$$

Let $\mathcal{V}$ be a variety and let $\mathcal{A}$ be a collection of algebras that belong to $\mathcal{V}$. Let $\mathcal{U}(\mathcal{A})$ be the collection of all pairs (A, i) where A is the universe of some algebra in $\mathcal{A}$ and $i \in \{0, 1\}$. That is,

$$\mathcal{U}(\mathcal{A}) = \{(A, i) \mid \langle A, \ldots \rangle \in \mathcal{A} \text{ and } i \in \{0, 1\}\}.$$

Given a sequence

$$S = ((A_0, \chi_0), (A_1, \chi_1), \ldots, (A_{n-1}, \chi_{n-1})) \in \mathcal{U}(\mathcal{A})^n,$$

(or a subset $S \subseteq \mathcal{U}(\mathcal{A})$), a term d is called a *glocal difference term for* S if it is a glocal difference term for every element (A_i, χ_i) of S. Besides these definitions, in the proof of the next theorem we use $|S|$ to denote the *length of the sequence* S.

Theorem 5.1. *Let $\mathcal{V}$ be a variety. Let $\mathcal{A}$ be a collection of finite idempotent algebras in $\mathcal{V}$. Fix $n \geqslant 2$ and let $S = ((A_0, \chi_0), (A_1, \chi_1), \ldots, (A_{n-1}, \chi_{n-1})) \in \mathcal{U}(\mathcal{A})^n$. Then there exists a term that is a glocal difference term for S if and only if each 2-element subsequence $((A_i, \chi_i), (A_j, \chi_j))$ of S has a glocal difference term.*

Proof. One direction is clear; if d is a glocal difference term for every element (A_i, χ_i) of S, then every pair $((A_i, \chi_i), (A_j, \chi_j))$ of elements of S also has a glocal difference term—namely, d.

For the converse, suppose that for each pair $((A_i, \chi_i), (A_j, \chi_j))$ of elements of S there exists a term p_{ij} that is a glocal difference term for both (A_i, χ_i) and (A_j, χ_j). We will prove by induction on the length of S that there exists a term d that is a glocal difference term for every (A_i, χ_i) in S.

In the base case, $n = |S| = 2$, the claim holds by assumption. Fix $n \geqslant 2$ and assume for every $2 \leqslant k \leqslant n$ that every sequence in $\mathcal{U}(\mathcal{A})^k$ has a glocal difference term. Let $S = ((A_0, \chi_0), (A_1, \chi_1), \ldots, (A_n, \chi_n)) \in \mathcal{U}(\mathcal{A})^{n+1}$. We prove S has a glocal difference term.

Since $|S| \geqslant 3$ and $\chi_i \in \{0, 1\}$ for all i, there must exist indices $i \neq j$ such that $\chi_i = \chi_j$. Assume without loss of generality that one of these indices is $j = 0$. Define the subsequence $S' = ((A_1, \chi_1), \ldots, (A_n, \chi_n))$ of S. Since $|S'| = n$, the sequence S' has a glocal difference term p. Thus, for all $1 \leqslant i \leqslant n$, for all $a, b \in A_i$ we have

$$\begin{aligned} &\text{if } \chi_i = 0, \text{ then } a \; [\theta(a,b), \theta(a,b)] \; d(a,b,b); \\ &\text{if } \chi_i = 1, \text{ then } d(a,a,b) = b. \end{aligned}$$

We split the remainder of the proof into two cases.

Case $\chi_0 = 0$: Without loss of generality, suppose that $\chi_1 = \chi_2 = \cdots = \chi_k = 1$, and $\chi_{k+1} = \chi_{k+2} = \cdots = \chi_n = 0$. Define

$$T = ((A_0, 0), (A_1, 1), (A_2, 1), \ldots, (A_k, 1)).$$

Note that $|T| < |S|$. Let t be a glocal difference term for T. We will prove that the term $d(x, y, z) = t(x, p(x, y, y), p(x, y, z))$ is a glocal difference term for S.

The first element of S is $(A_0, 0)$, so we need to show for all a, $b \in A_0$ that

$$d(a, b, b) \; [\theta(a, b), \theta(a, b)] \; a.$$

Fix $a, b \in A_0$. By definition of d, and since t is a glocal difference term for $(A_0, 0)$, we have

$$d(a, b, b) = t(a, c, c) \; [\theta(a, c), \theta(a, c)] \; a, \tag{5.3}$$

where $c = p(a, b, b)$. Now, $(a, c) = (p(a, a, a), p(a, b, b)) \in \theta(a, b)$, therefore, $\theta(a, c) \leqslant \theta(a, b')$. It follows from this and monotonicity of the commutator that $[\theta(a, c), \theta(a, c)] \leqslant [\theta(a, b), \theta(a, b)]$, This and (5.3) imply $d(a, b, b) \; [\theta(a, b), \theta(a, b)] \; a$, as desired.

Next, consider the (possibly empty) set of indices $\{i \mid 1 \leqslant i \leqslant k\}$. For such indices $\chi_i = 1$, so we will prove for all a, $b \in A_i$ that $d(a, a, b) = b$. Fix $a, b \in A_i$ and observe that

$$\begin{aligned} d(a, a, b) &= t(a, p(a, a, a), p(a, a, b)) && (5.4) \\ &= t(a, a, b) && (5.5) \\ &= b. && (5.6) \end{aligned}$$

Equation (5.4) holds by definition of d, (5.5) because p is an idempotent glocal difference term for S', and (5.6) because t is a glocal difference term for T.

The indices of the remaining elements of S belong to the set $\{j \mid k < j \leqslant n\}$ (which is nonempty since we assumed $\chi_0 = \chi_i = 0$ for some $i > 0$). For such indices we have $\chi_j = 0$. Thus, fixing $a, b \in A_j$, we check that $d(a,b,b)\ [\theta(a,b), \theta(a,b)]\ a$. By definition,

$$d(a,b,b) = t(a, p(a,b,b), p(a,b,b)). \tag{5.7}$$

Also, $p(a,b,b)\ [\theta(a,b), \theta(a,b)]\ a$, since p is a glocal difference term for S'. This and (5.7) imply that $d(a,b,b)\ [\theta(a,b), \theta(a,b)]\ t(a,a,a))$. Finally, by idempotence of t we have $d(a,b,b)\ [\theta(a,b), \theta(a,b)]\ a$, as desired.

Case $\chi_0 = 1$: Without loss of generality, suppose $\chi_1 = \chi_2 = \cdots = \chi_k = 0$, and $\chi_{k+1} = \chi_{k+2} = \cdots = \chi_n = 1$. Define

$$T = ((A_0, 1), (A_1, 0), (A_2, 0), \ldots, (A_k, 0)).$$

and note that $|T| < |S|$, so T has a glocal difference term t. We will prove that the term $d(x,y,z) = t(p(x,y,z), p(y,y,z), z)$ is a glocal difference term for S.

The first pair in S is $(A_0, 1)$, so we want to show for all $a, b \in A_0$ that $d(a,a,b) = b$. Fix $a, b \in A_0$. By definition of d, we have $d(a,a,b) = t(p(a,a,b), p(a,a,b), b) = b$. The last equality holds since t is a glocal difference term for T, in particular, for $(A_0, 1)$.

Next, consider the (possibly empty) set of indices $\{i \mid 1 \leqslant i \leqslant k\}$. For such indices $\chi_i = 0$, so we will prove for all $a, b \in A_i$ that

$$d(a,b,b)\ [\theta(a,b), \theta(a,b)]\ a.$$

Fix $a, b \in A_i$. By definition of d and idempotence of p, we have

$$\begin{aligned} d(a,b,b) &= t(p(a,b,b), p(b,b,b), b) \\ &= t(p(a,b,b), b, b). \end{aligned} \tag{5.8}$$

Next, since p is a glocal difference term for S', hence for $(A_i, 0)$, we have

$$t(p(a,b,b), b, b)\ [\theta(a,b), \theta(a,b)]\ t(a,b,b). \tag{5.9}$$

Finally, since t is a glocal difference term for T, hence for $(A_i, 0)$, we have

$$t(a,b,b)\ [\theta(a,b), \theta(a,b)]\ a.$$

Combining this with (5.8) and (5.9) yields $d(a,b,b)\ [\theta(a,b), \theta(a,b)]\ a$, as desired.

The indices of the remaining elements of S belong to the set $\{j \mid k < j \leqslant n\}$ (which is nonempty since we assumed $\chi_0 = \chi_i = 1$ for some $i > 0$). For such indices we have $\chi_j = 1$. Thus, fixing $a, b \in A_j$, we check that $d(a,a,b) = b$. Indeed, $p(a,a,b) = b$, since p is a glocal difference term for S'; this, along with idempotence of t, yields $d(a,a,b) = t(p(a,a,b), p(a,a,b), b) = t(b,b,b) = b$. □

Recall, if $\mathcal{A}$ is a collection of algebras belonging to a variety $\mathcal{V}$, we let

$$\mathcal{U}(\mathcal{A}) = \{(A, i) \mid \langle A, \ldots\rangle \in \mathcal{A} \text{ and } i \in \{0, 1\}\}.$$

Corollary 5.2. *Let $\mathcal{V}$ be a variety. Let $\mathcal{A}$ be a collection of finite idempotent algebras in $\mathcal{V}$. Then there exists a term d that interprets as a difference term operation in every algebra in $\mathcal{A}$ if and only if each pair $((A, i), (B, j)) \in \mathcal{U}(\mathcal{A})^2$ has a glocal difference term.*

Proof. One direction is clear, since a term that is a difference term operation for every $\mathbf{A} \in \mathcal{A}$ is obviously a glocal difference term for every $(A, i) \in \mathcal{U}(\mathcal{A})$. For the converse, suppose each pair in $\mathcal{U}(\mathcal{A})^2$ has a glocal difference term. Then, by Theorem 5.1, there is a single term d that is a glocal difference term for every $(A, i) \in \mathcal{U}(\mathcal{A})$, and therefore d interprets as a difference term operation in every $\mathbf{A} \in \mathcal{A}$. To see this, choose an arbitrary $\mathbf{A} = \langle A, \ldots\rangle \in \mathcal{A}$ and fix $a, b \in A$. Then $a\ [\theta(a, b), \theta(a, b)]\ d^{\mathbf{A}}(a, b, b)$, since d is a glocal difference term for $(A, 0)$, and $d^{\mathbf{A}}(a, a, b) = b$, since d is a glocal difference term for $(A, 1)$. □

Algorithm 2: existence of difference terms

Corollary 5.3. *There is a polynomial-time algorithm that takes as input any finite idempotent algebra $\mathbf{A}$ and decides whether the variety $\mathbb{V}(\mathbf{A})$ that it generates has a difference term operation.*

Proof. Let $\mathcal{V} = \mathbb{V}(\mathbf{A})$ and let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$ be the free algebra in $\mathcal{V}$ generated by x and y. We can assume $\mathbf{F}$ is a subdirect product of $\mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{n-1}$, where $n \leqslant |A|^2$. Let $\mathcal{A} = \{A_0, A_1, \ldots, A_{n-1}\}$ and (as above) let $\mathcal{U}(\mathcal{A})$ denote all pairs (A, i) such that $\mathbf{A} = \langle A, \ldots\rangle \in \mathcal{A}$ and $i \in \{0, 1\}$. We first check that each of the n^2 pairs $((A_i, \chi_i), (A_j, \chi_j)) \in \mathcal{U}(\mathcal{A})^2$ has a glocal difference term. □

6. Local and glocal minority terms

Let $\mathcal{V}$ be a variety. A term m in the language of $\mathcal{V}$ is called a *minority term* if for every $\mathbf{A} \in \mathcal{V}$ and all $a, b \in A$ it is the case that

$$m(a, b, b) = m(b, a, b) = m(b, b, a) = a.$$

If t is a term of $\mathcal{V}$ and $\mathbf{A} = \langle A, \ldots\rangle$ an algebra in $\mathcal{V}$, then we call t a *local minority term* for the triple $(a, b, i) \in A \times A \times \{0, 1, 2\}$ provided

$$\text{if } i = 0, \text{ then } t(a, b, b) = a \tag{6.1}$$

$$\text{if } i = 1, \text{ then } t(b, a, b) = a \tag{6.2}$$

$$\text{if } i = 2, \text{ then } t(b, b, a) = a. \tag{6.3}$$

We call t a *local minority term* for the sequence

$$((a_0, b_0, \chi_0), (a_1, b_1, \chi_1), \ldots, (a_{n-1}, b_{n-1}, \chi_{n-1})) \in (A \times A \times \{0, 1, 2\})^n$$

provided it is a local minority term for each triple in the sequence. We call t a *glocal minority term* for (A, i) provided conditions (6.1), (6.2), and (6.3) hold for all $a, b \in A$. If $\mathcal{A}$ is a collection of algebras in $\mathcal{V}$, and if $S = ((A_0, \chi_0), (A_1, \chi_1), \ldots, (A_{n-1}, \chi_{n-1}))$ is a sequence of pairs $(A_i, \chi_i) \in \mathcal{A} \times \{0, 1, 2\}$, then we call t a *glocal minority term for* S if it is a glocal minority term for every (A_i, χ_i) in the sequence S.

Theorem 6.1. *Let $n \geqslant 3$ and let $S = ((a_0, b_0, \chi_0), \ldots, (a_{n-1}, b_{n-1}, \chi_{n-1})) \in (A \times A \times \{0, 1, 2\})^n$. Suppose for all $0 \leqslant i, j, k < n$ that the subsequence $((a_i, b_i, \chi_i), (a_j, b_j, \chi_j), (a_k, b_k, \chi_k))$ of S has a local minority term. Then S has a local minority term.*

Proof. The proof is by induction on the length, $|S|$, of the sequence. In the base case, $|S| = 3$, the claim is true by assumption. Fix $n \geqslant 3$ and assume the claim holds for all sequences of length $3 \leqslant k \leqslant n$. We prove the claim holds for an arbitrary length-$(n + 1)$ sequence, $S = ((a_0, b_0, \chi_0), (a_1, b_1, \chi_1), \ldots, (a_n, b_n, \chi_n))$. Since $|S| > 3$, there exists $i \neq j$ such that $\chi_i = \chi_j$. Assume without loss of generality that $j = 0$. Let $\sigma' = ((a_1, b_1, \chi_1), \ldots, (a_n, b_n, \chi_n))$, which has a local minority term p by the induction hypothesis.

We split the remainder of the proof into two cases.

Case $\chi_0 = 0$: Without loss of generality, suppose for some $1 \leqslant k \leqslant j < n$ that $\chi_1 = \cdots = \chi_k = 1$, $\chi_{k+1} = \cdots = \chi_j = 2$, and $\chi_{j+1} = \cdots = \chi_n = 0$. Define $T = ((a_0, b_0, \chi_0), (a_1, b_1, \chi_1), \ldots, (a_j, b_j, \chi_j))$. Note that $|T| < |S|$, so T has a local minority term t. Define $m(x, y, z) = t(x, p(x, y, y), p(x, y, z))$. We will prove that m is a local minority term for S.

The first triple in S is $(a_0, b_0, 0)$, so we first check that $m(a_0, b_0, b_0) = a_0$. Indeed,

$$m(a_0, b_0, b_0) = t(a_0, p(a_0, b_0, b_0), p(a_0, b_0, b_0)) = a_0.$$

The second equality holds because t is a local minority term for T.

For indices in the (possibly empty) set $\{i \mid 0 < i \leqslant k\}$, we have $\chi_i = 1$, so we check that $m(b_i, a_i, b_i) = a_i$, as follows:

$$m(b_i, a_i, b_i) = t(b_i, p(b_i, a_i, a_i), p(b_i, a_i, b_i)) = t(b_i, b_i, a_i) = a_i.$$

The second equality holds because p is a local minority term for S', the third holds because t is a local minority term for T.

Next for indices in the (possibly empty) set $\{i \mid k < i \leqslant j\}$, we have $\chi_i = 2$, so we check that $m(b_i, b_i, a_i) = a_i$. Indeed,

$$m(b_i, b_i, a_i) = t(b_i, p(b_i, b_i, b_i), p(b_i, b_i, a_i)) = t(b_i, b_i, a_i) = a_i.$$

The second equality holds by idempotence and because p is a local minority term for S'; the third equality holds since t is a local minority term for T.

Finally, we consider indices in the set $\{i \mid j < i \leqslant n\}$, which we know is nonempty since we assumed $\chi_i = \chi_0 = 0$ for some $i > 0$. For such indices, $\chi_i = 0$ so we check that $m(a_i, b_i, b_i) = a_0$, as follows:

$$m(a_i, b_i, b_i) = t(a_i, p(a_i, b_i, b_i), p(a_i, b_i, b_i)) = t(a_i, a_i, a_i) = a_i.$$

The second equality holds since p is a local minority term for S', and the last equality holds by idempotence.
Case $\chi_0 = 1$:

Case $\chi_0 = 2$: □

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