

Problems for Discussion 5, 10/23/13

Compiled by Mai Le, some problems from Prof. Fessler, Daehyun Yoon

1 Alternative Sampling Theorem

Generalize the ideal impulse sampling theory to the more realistic case to account for finite detector size:

$$g_d[n] = \frac{1}{T} \int_{(n-\frac{1}{2})T}^{(n+\frac{1}{2})T} g_c(t) dt$$

The main results of Shannon/Nyquist's sampling theorem (for ideal sampling) addresses the following three questions:

- Q1: Can we recover a bandlimited signal from its samples?
A1: Yes, if it is bandlimited and sampled correctly.
- Q2: How "fast" must we sample?
A2: The required sampling frequency is twice the highest signal frequency.
- Q3: How do we recover the analog signal from its samples?
A3: In the frequency domain, $G_a(\Omega) = G_s(\Omega) \text{rect}(\Omega T)$, or equivalently since interpolation in the time domain.

For this more realistic sampling method, we are integrating over the time between samples. This is equivalent to filtering the signal with a rect function before sampling. Let's call that filtered signal $f(t)$.

$$\begin{aligned} \text{Let } r(t) &= \frac{1}{T} \text{rect}\left(\frac{t}{T}\right) \\ f(t) = r(t) * g_c(t) &= \int_{-\infty}^{\infty} g_c(\tau) r(t - \tau) d\tau \\ &= \int_{-\infty}^{\infty} g_c(\tau) \frac{1}{T} \text{rect}\left(\frac{t - \tau}{T}\right) d\tau \\ &= \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} g_c(\tau) d\tau \end{aligned}$$

Now consider samples of $f(t)$, called $f_d[n]$. Also let $f_s(n) = \text{comb}\left(\frac{t}{T}\right) \cdot f(t)$.

$$\begin{aligned} f_d[n] &= f(nT) = f_s(nT) \\ &= \frac{1}{T} \int_{nT-\frac{T}{2}}^{nT+\frac{T}{2}} g_c(\tau) d\tau \\ &= \frac{1}{T} \int_{(n-\frac{1}{2})T}^{(n+\frac{1}{2})T} g_c(t) dt = g_d[n] \end{aligned}$$

Thus I have shown that the samples $g_d[n]$ acquired with this more realistic sensor are equivalent to the ideal samples $f_d[n]$ of the filtered signal $f(t) = r(t) * g_c(t)$. This means we can use the results from the Shannon-Nyquist sampling theorem for ideal sampling!

But first, we need to see how this filter changes the CTFT of $g_c(t)$, $G_c(\Omega)$. Assuming $G_c(\Omega)$ is bandlimited with maximum frequency Ω_c , then is $F(\Omega)$ also bandlimited? If so, what is its maximum frequency?

From the convolution theorem,

$$F(\Omega) = G_c(\Omega)R(\Omega)$$

and from known Fourier Transform pairs,

$$R(\Omega) = \text{sinc}(T\Omega).$$

Assuming T is small, this is a pretty wide sinc, with its first set of zeros far out, at $\pm\frac{1}{T}$. So the maximum frequency for $F(\Omega)$ is $\min(\frac{1}{T}, \Omega_c)$. In most cases, T will be large enough that the maximum frequency for $F(\Omega)$ is Ω_c , same as for $G_c(\Omega)$.

So now we re-address the three questions answered in the sampling theorem:

- Q1: Can we recover a bandlimited signal ($g_c[n]$) from its samples?
A1: Yes, if $g_c[n]$ is bandlimited and sampled correctly.
- Q2: How "fast" must we sample?
A2: The required sampling frequency is twice the highest signal frequency of $F(\Omega)$, $\min(\frac{1}{T}, \Omega_c)$.
- Q3: How do we recover the analog signal from its samples?
A3: In the frequency domain, $F(\Omega) = F_s(\Omega)\text{rect}(\Omega T) = G_c(\Omega)R(\Omega)$ or $G_c(\Omega) = \frac{F_s(\Omega)\text{rect}(\Omega T)}{R(\Omega)}$. In other words, we need to do low-pass filtering by $\frac{1}{R(\Omega)}$ to undo the distortion from $R(\Omega)$.

2 Nyquist Rate for Communications Channel

A seismic signal with amplitude range $[-1,1]$ volts is sampled and quantized with a 12 bit A/D converter before transmitting it over a communication channel which can support a maximum bit rate of 240 Kbits/sec. What is the maximum sampling frequency the system can support? What is the maximum frequency that can be present in the original seismic signal in order to avoid any loss of information?

We can sample at $240000/12 = 20000\text{Hz}$. In order to avoid aliasing, the Nyquist rate of the signal should be no longer than 20000 Hz. So the maximum possible frequency is 10000 Hz.

3 Sinc interpolation

A signal $x(t)$ has a triangular spectrum, with maximum frequencies $\pm\Omega_c$ and height A . $x(t)$ is sampled at a rate of $f_s = \frac{1}{T_s} = \frac{\Omega_s}{2\pi}$, producing the samples $\{x(nT_s)\}_{n=-\infty}^{\infty}$.

Let $y(t)$ be the signal constructed from these samples as follows:

$$y(t) = \frac{\Omega_c T_s}{\pi} \sum_{n=-\infty}^{\infty} x(nT_s) \text{sinc} \left(\frac{\Omega_c}{\pi} (t - nT_s) \right)$$

How is $y(t)$ related to $x(t)$ when:

(a) $\Omega_c = \frac{\Omega_s}{4}$

$$\begin{aligned} y(t) &= \frac{\omega_c T_s}{\pi} \sum_{n=-\infty}^{\infty} x(nT_s) \text{sinc} \left(\frac{\Omega_c}{\pi} (t - nT_s) \right) \\ &= \left(\sum_n x(nT_s) \delta(t - nT_s) \right) * \left(\text{sinc} \left(\frac{\Omega_c t}{\pi} \right) \frac{\Omega_c T_s}{\pi} \right) \\ Y(\Omega) &= \frac{1}{T_s} \left(\sum_{n=-\infty}^{\infty} X(\Omega - \Omega_s n) \right) H(\Omega) \end{aligned}$$

where $H(\Omega) = T_s \text{rect}(\frac{\Omega}{2\Omega_C})$.

Thus, we can interpret the CTFT of $y(t)$ as a low-pass filter applied to a series of replicas of $X(\Omega)$.

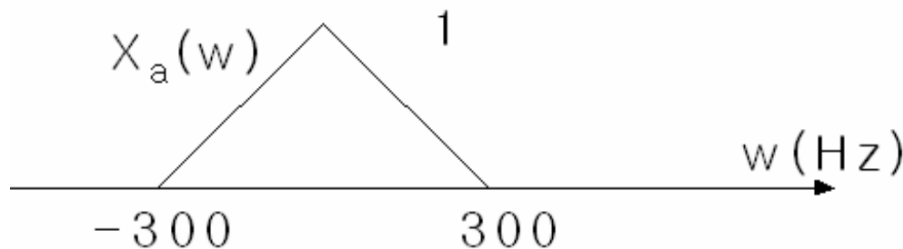
So for $\Omega_c = \frac{\Omega_s}{4}$, the replicas have width $2\Omega_c$ and are placed $4\Omega_c$ apart, with space of $2\Omega_c$ in between. There is no aliasing, so we are able to perfectly recover the central replica, $Y(\Omega) = X(\Omega)$, and thus recover the original signal $y[n] = x[n]$.

(b) $\Omega_c = \Omega_s$?

This sampling rate does not satisfy the Nyquist rate. The replicas are spaced only Ω_c apart, each pair of replicas having an overlap region of width Ω_c . In fact, the entire domain Ω suffers from aliasing, and our attempt to recover the original signal results in $Y(\Omega) = \text{Arect}\left(\frac{\Omega}{2\Omega_c}\right) \neq X(\Omega)$.

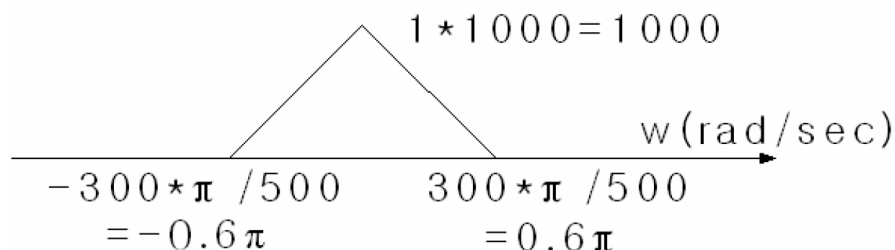
4 Relating CTFT and DTFT

Suppose the CTFT of the continuous time signal $x_a(t)$ is as following.



Determine the DTFT of the sampled signal $x[n]$, when the sampling period is 1 ms. What happens if the sampling period is 2 ms?

First we compute the baseband by multiplying $2\pi T_s = 2\pi 0.001 = \frac{\pi}{500}$. Next we scale the amplitude by $\frac{1}{T_s} = 1000$.



Next we create copies of the above baseband signal by shifting it to every multiple of 2π . For example, shift it by 2π for one copy and shift it by 4π to get another copy and so on. Summing these copies together we get $X_d(\omega)$:

