DISCUSSION 9, NOVEMBER 27

1. (10 points, 5 points each sub-part.) The continuous-time signal

$$x_c(t) = \sin(21\pi t) + \cos(35\pi t)$$

is sampled with a sampling period T to obtain the discrete-time signal

$$x[n] = \sin\left(\frac{\pi n}{15}\right) + \cos\left(\frac{\pi n}{9}\right) \ .$$

(a) Determine a choice for T consistent with this information.
Hint: Write an equation that relates $x[n]$ with $x_c(t)$ through the sampling period T.

(b) Is your choice for T in Part (a) unique? If so, explain why. If not, specify another choice of T consistent with the information given.
Hints:
   i. We notice that $x_c(t)$ is periodic. What is the definition of periodicity for continuous time signals? If there exists some positive value, T_P, s.t. for all t, $x_c(t) = \ldots$
   ii. What is the period of $x_c(t)$?
   iii. What effect did we notice with the wagon wheel demonstration in class? How does that relate to the period? It may help to draw a simple sine and simulate sampling on it.

2. (20 points total, see point values below.) The given sequence $x_1[n]$ is periodic with period $N = 4$. We would like to convolve it with the $h[n]$ given.

$$x_1[n] = \begin{cases} 3, & n \bmod 4 = 0 \\ 1, & n \bmod 4 = 1 \text{ or } 3 \\ 0, & n \bmod 4 = 2 \end{cases} \qquad h[n] = \{\underline{1}, -1\}$$

(a) (5 points) What is the result of the length-$N = 4$ periodic convolution between these two sequences?
Hint: Write the mathematical of periodic convolution. What does the tilde indicate?

(b) (5 points) Now consider $x_2[n]$, which is zero for all indices not shown. What is the result of the length-$N = 4$ circular convolution of $x_2[n]$ and $h[n]$?

$$x_2[n] = \{\underline{3}, 1, 0, 1\}$$

Hint: Write the mathematical definition of circular convolution. How does it differ from the definition of periodic convolution in part (a)? Over what values of n is this defined?

(c) (10 points) Now consider that we would like to implement the following LTI system, where $H(\omega)$ is the DTFT of $h[n]$ given above.

$$x_2[n] \rightarrow \boxed{H(\omega)} \rightarrow y_2[n]\ .$$

Instead we need to implement it on a computer, so we'll use the DFT as follows.

$$h[n] \rightarrow \boxed{?\ \#1} \rightarrow g[n] \rightarrow \boxed{\text{M-point DFT}} \rightarrow G[k]$$

$$x_2[n] \rightarrow \boxed{?\ \#2} \rightarrow w[n] \rightarrow \boxed{\text{M-point DFT}} \rightarrow \boxed{G[k]} \rightarrow \boxed{\text{M-point IDFT}} \rightarrow y_2[n]\ .$$

i. What is the minimum value M can be to guarantee we get the linear convolution?
Hint: what is the length of linear convolution?

ii. Describe what we must do to $x_2[n]$ and/or $h[n]$ (i.e. what should we do at the "?" boxes) to guarantee that

$$y_2[n] = x_2[n] * h[n],$$

i.e. the output is the linear convolution of $x_2[n]$ and $h[n]$.
Hint: We are not asking about alternate ways to compute $y_2[n]$- just what needs to go in ?#1 and ?#2. Think about part(i).

iii. Suppose we took $M = 8$. In this case, describe how $G[k]$ relates to $H[k]$, the $N = 4$-point DFT of $h[n]$. (You do not have to solve for $H[k]$ and $G[k]$ explicitly, just describe how they relate.)
Hints:
A. Write down how $H[k]$ and $G[k]$ mathematically relate to $H(e^{j\omega})$.
B. Knowing one of the two DFTs can tell you the other, but not vice versa. Why?

3. (15 points total, see point values below) Consider the systems shown below. Suppose that $H_1(\omega)$ is fixed and known.

$$x_1[n] \rightarrow \boxed{\downarrow 2} \rightarrow v[n] \rightarrow \boxed{H_1(\omega)} \rightarrow q[n] \rightarrow \boxed{\uparrow 2} \rightarrow y_1[n] \ ,$$

$$\begin{array}{c} x_2[n] \rightarrow \boxed{?} \rightarrow \oplus \rightarrow \boxed{H_2(\omega)} \rightarrow y_2[n] \\ \uparrow \\ x_2[n] \end{array} \quad .$$

Remember that $\boxed{\uparrow M}$ means we insert $M-1$ zeros between samples and $\boxed{\downarrow M}$ means we take every M^{th} sample.

(a) (5 points) What is $V(\omega)$ in terms of $X_1(\omega)$? What is $Q(\omega)$ in terms of $X_1(\omega)$ and $H_1(\omega)$?
Hint: What is $Q(\omega)$ in terms of $V(\omega)$ and $H_1(\omega)$?

(b) (10 points) Find $H_2(\omega)$, the frequency response of an LTI system and the operation that belongs in the "?" box, such that $y_1[n] = y_2[n]$ when $x_1[n] = x_2[n]$ (in other words, the outputs are the same when the inputs are the same).
Hints:
   i. Start by writing out $Y_1(\omega)$ in terms of $Q(\omega)$ and then in terms of $X_1(\omega)$ and $H_1(\omega)$.
   ii. For the "?" box, use an idea from problem 5.

4. (15 points) For the FFT, we learned how to break a length-8 DFT into two DFTs of length-4. Use the same approach to show how you could break a length-20 DFT into five DFTs of length-4.
   Hint: Start with the definition of the DFT and think about ways to divide the summation into three different interleaved summations.

5. (30 points total, plus 5 points extra credit, see point values below.)

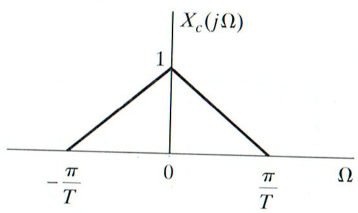
(a) (12 points) Consider the following system.

$$x_c(t) \to \boxed{\text{C/D with period } T} \to x[n] \to \boxed{H(\omega)} \to v[n] \to \boxed{\downarrow L} \to q[n]$$

The filter $H(\omega)$ is defined as

$$H(\omega) = \begin{cases} 1, & |\omega| < \pi/L \\ 0, & \pi/L < |\omega| < \pi \end{cases} .$$

$X_c(\Omega)$ is given in this figure. What do $V(\omega)$ and $Q(\omega)$ look like? Draw them on a plot with clear labels of the axes, the locations where the signal touches the x-axis, and the height of the signal.



Hints:

i. Follow the instructions! Label everything!

ii. Write out the formula that relates $X_c(\Omega)$ to $X(\omega)$ through sampling period T. Pay attention to scaling in both the domain and the amplitude.

(b) (18 points) Now continue the system from part (a) as follows, where $q[n]$ is the output of the system in part (a).

$$q[n] \rightarrow \boxed{\text{Modulator}} \rightarrow y[n] \rightarrow \boxed{\text{D/C with period } T' = LT} \rightarrow y_c(t)$$

The box labeled "Modulator" takes the input sequence $q[n]$ and multiplies it by j^n:

$$y[n] = j^n q[n] \ .$$

Draw $Y(\omega)$, the DTFT of $y[n]$, and $Y_c(\Omega)$, the CTFT of $y_c(t)$. Again be sure to clearly label the axes, locations where the signal touches the x-axis, and the height of the signal.

Hints:

i. j^n can be alternatively expressed as $(?)^n$ (think polar coordinates in the complex plane). What DTFT property is useful here?

ii. Write the formula that relates $Y_c(\Omega)$ to $Y(\omega)$ through the interpolation period T'. (You've already had to use it in this problem.)

(c) (5 points, extra credit) Can we swap the modulator with our LTI filter $H(\omega)$ and still get the same overall system? Why or why not?

Hint: You won't be able to get any extra credits points back but draw some pictures and test it out for yourself. :)