Solutions for Discussion 1, 09/11/13

Compiled by Mai Le

1 Alternate Expressions for Sequences

Let $x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \text{ nonnegative multiple of } 4 \\ -\left(\frac{1}{2}\right)^n, & n \text{ nonnegative multiple of 2, but not a nonnegative multiple of } 4 \\ 0, & \text{otherwise} \end{cases}$

Express $x[n]$ mathematically in three different ways.

1. $x[n] = \{\underline{1}, 0, \frac{-1}{4}, 0, \frac{1}{16}, 0, \frac{-1}{64}, ...\}$

2. $x[n] = \delta[n] - \frac{1}{4}\delta[n-2] + \frac{1}{16}\delta[n-4] - \frac{1}{64}\delta[n] + ...$

3. $x[n] = \sum\limits_{k=0}^{\infty}(-1)^k(\frac{1}{4})^k\delta[b-2k]$

4. $x[n] = u[n]cos\left(\frac{\pi}{2}n\right)\left(\frac{1}{2}\right)^n$

2 Nonlinear Systems

Give an example of a system that is nonlinear but satisfies $\mathcal{T}\{\alpha x[n]\} = \alpha\mathcal{T}\{x[n]\}$ for all sequences $x[n]$ and for all scalars $\alpha \in \mathbb{R}$.

$y[n] = \begin{cases} \frac{x^2[n]}{x[n-1]}, & x[n-1] \neq 0 \\ x[n], & x[n-1] = 0 \end{cases}$

Then

$$\begin{aligned} \mathcal{T}\{\alpha x[n]\} &= \begin{cases} \frac{\alpha^2 x^2[n]}{\alpha x[n-1]}, & x[n-1] \neq 0 \\ \alpha x[n], & x[n-1] = 0 \end{cases} \\ &= \begin{cases} \frac{\alpha x^2[n]}{x[n-1]}, & x[n-1] \neq 0 \\ \alpha x[n], & x[n-1] = 0 \end{cases} \\ &= \alpha \begin{cases} \frac{x^2[n]}{x[n-1]}, & x[n-1] \neq 0 \\ x[n], & x[n-1] = 0 \end{cases} \\ &= \alpha\mathcal{T}\{x[n]\} \end{aligned}$$

but the system fails the addition property:

$\mathcal{T}\{x_1[n] + x_2[n]\} \neq \mathcal{T}\{x_1[n]\} + \mathcal{T}\{x_2[n]\}$

example: Let $x_1[n] = \delta[n]$ and $x_2[n] = \delta[n-1]$. Then $\mathcal{T}\{x_1[n]\} = \delta[n]$ and $\mathcal{T}\{x_2[n]\} = \delta[n-1]$. But $\mathcal{T}\{x_1[n] + x_2[n]\} = \mathcal{T}\{\delta[n] + \delta[n-1]\} =$

3 Length of Convolution

Let $x[n]$ be non-zero only over $N_1 \leq n \leq N_2$ and $h[n]$ be non-zero only over $M_1 \leq n \leq M_2$. Let $y[n] = x[n] * h[n]$. Then $y[n]$ is only non-zero over $L_1 \leq n \leq L_2$. Define L_1 and L_2 in terms of N_1, N_2, M_1, and M_2.

For convenience, let $h[n]$ be shorter than $x[n]$. To answer this, we'll envision the "flip and slide", where we flip $h[n]$ and slide it over $x[n]$. Then it goes through three stages.

For $M_1 - M_2$ steps, the reversed h is entering into $x[n]$. For $N_1 - N_2 - (M_1 - M_2) - 1$ steps, $h[n]$ moves within/across $x[n]$. Finally, $h[n]$ takes $M_1 - M_2$ steps to exit $x[n]$.

The total length, $L_1 - L_2 = (M_1 - M_2) + (N_1 - N_2 - (M_1 - M_2) - 1) + (M_1 - M_2) = N_1 - N_2 + M_1 - M_2 - 1$.

More specifically, L_1 is when $h[n]$ is just beginning to enter $x[n]$, so $L_1 = N_1 + M_1$, and L_2 is when $h[n]$ has finished exiting $x[n]$ so $L_2 = N_2 + M_2 + 1$.

4 Distributivity of Convolution

Prove the distributive property of convolution.

$$
\begin{aligned}
x_1[n] * (x_2[n] + x_3[n]) &= \sum_{k=-\infty}^{\infty} x_1[k]\,(x_2[n-k] + x_3[n-k]) \text{ from definition of convolution} \\
&= \sum_{k=-\infty}^{\infty} x_1[k]\,(x_2[n-k]) + \sum_{k=-\infty}^{\infty} x_1[k]\,(x_3[n-k]) \\
&= x_1[n] * x_2[n] + x_1[n] * x_3[n]
\end{aligned}
$$

5 Computing Discrete Convolution

Let $y[n] = x[n] * h[n]$. Find an expression for $y[n]$.

(a)

$$x[n] = \begin{cases} 1, & n = -2, 0, 1 \\ 2, & n = -1 \\ 0, & otherwise \end{cases}$$
$$h[n] = \delta[n] - \delta[n-1] + \delta[n-4] + \delta[n-5]$$

cry

(b)

$$x[n] = u[n+1] - u[n-4] - \delta[n-5]$$
$$h[n] = (u[n+2] - u[n-3])\,(3 - |n|)$$

cry

6 Convolution and Signal Energy

Let $y[n] = x[n] * h[n]$. Prove that $\left(\sum_{n=-\infty}^{\infty} y[n]\right) = \left(\sum_{n=-\infty}^{\infty} x[n]\right)\left(\sum_{n=-\infty}^{\infty} h[n]\right)$.

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
\left(\sum_{n=-\infty}^{\infty} y[n]\right) &= \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right) \\
&= \sum_{k=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} h[k]x[n-k]\right) \text{ switch order of sums} \\
&= \sum_{k=-\infty}^{\infty} h[k]\left(\sum_{n=-\infty}^{\infty} x[n-k]\right) \text{ part in parentheses has same value for all } k \\
&= \left(\sum_{k=-\infty}^{\infty} h[k]\right)\left(\sum_{m=-\infty}^{\infty} x[m]\right) \text{ change of indexing variable for } x,\ m = n-k
\end{aligned}$$

NOT REALLY ENERGY

7 Impulse Response of a BIBO Stable System

Let $h[n]$ be the impulse response of a BIBO stable system. (Remember that impulse responses are only defined for LSI systems, so this system is also LSI.) What must hold true for $h[n]$?

Let $x[n]$ be a bounded input with bound B_x. Then $|x[n]| \leq B_x$ for all n.

$$\begin{aligned} |y[n]| &= \Big|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\Big| \leq \sum_{k=-\infty}^{\infty} \big|h[k]x[n-k]\big| \text{ from triangle inequality} \\ \sum_{k=-\infty}^{\infty} \big|h[k]x[n-k]\big| &\leq \sum_{k=-\infty}^{\infty} \big|h[k]\big| \, \big|x[n-k]\big| \leq \sum_{k=-\infty}^{\infty} \big|h[k]\big| B_x = B_x \left(\sum_{k=-\infty}^{\infty} \big|h[k]\big| \right) = B_y \end{aligned}$$

We choose $B_x \left(\sum_{k=-\infty}^{\infty} \big|h[k]\big| \right)$ as our bound B_y. Since $B_x < \infty$, we can ensure $B_y < \infty$ if $\sum_{k=-\infty}^{\infty} \big|h[k]\big| < \infty$. In other words, if $h[n]$ is absolutely summable (the sum of its absolute values is finite), the output of a bounded input is also bounded. Thus, a system is BIBO stable if its impulse response is absolutely summable. (It is also true that a LSI system with an absolutely summable impulse response is BIBO stable.)

8 Impulse Response of a Causal System

Let $h[n]$ be the impulse response of a causal system. What must hold true for $h[n]$?

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{-1} h[k]x[n-k] \\ &\quad + \; h[0]x[n-0] \\ &\quad + \sum_{k=1}^{\infty} h[k]x[n-k] \\ &= \sum_{k=1}^{\infty} h[-k]x[n+k] \quad \text{future values of } x[n] \\ &\quad + \; h[0]x[n-0] \quad \text{present value of } x[n] \\ &\quad + \sum_{k=1}^{\infty} h[k]x[n-k] \quad \text{past values of } x[n] \end{aligned}$$

The first term is a sum of future values of $x[n]$. For the system to be causal, $y[n]$ cannot depend on any of these values- we want each term in that sum to be zero, regardless of the input $x[n]$. We can guarantee that by choosing $h[n] = 0$ for $n < 0$. Thus, a causal system has a causal impulse response. (It is also true that an LSI system with a causal impulse response is a causal system.)

9 Impulse Response of an Invertible System

A system $\mathcal{T}_1$ is invertible if there exists a system $\mathcal{T}_2$ such that $\mathcal{T}_2\{\mathcal{T}_1\{x[n]\}\} = x[n]$ for all $n \in \mathbb{Z}$. Let $h[n]$ be the impulse response of an invertible system. What must hold true for $h[n]$?

For this problem, you can take for granted that the inverse of an LSI system is also LSI.

If $h[n]$ is the impulse response of an invertible system $\mathcal{T}_1$, then there also exists an LSI system $\mathcal{T}_2$ such that $\mathcal{T}_2\{\mathcal{T}_1\{x[n]\}\} = x[n]$.

$$\begin{aligned}
\mathcal{T}_2\{\mathcal{T}_1\{x[n]\}\} &= x[n] \\
\mathcal{T}_2\{h_1[n] * x[n]\} &= x[n] \\
h_2[n] * (h_1[n] * x[n]) &= x[n] \text{ where } h_2[n] \text{ is the impulse response of } \mathcal{T}_\in \\
(h_2[n] * h_1[n]) * x[n]) &= x[n] \text{ associativity of convolution} \\
(h_2[n] * h_1[n]) * x[n]) &= \delta[n] * x[n] \text{ property of Kronecker delta}
\end{aligned}$$

The Kronecker delta has the property that $\delta[n]*x[n] = x[n]$ for all n. It is also the only discrete signal with that property. Therefore, $(h_2[n] * h_1[n]) = \delta[n]$. In conclusion, if a system is invertible with impulse response $h_1[n]$, then there exists some other sequence $h_2[n]$ such that $(h_2[n] * h_1[n]) = \delta[n]$, and $h_2[n]$ is the impulse response of the inverse system.

10 Eigensequences

A sequence $x[n]$ is an eigensequence of a system $\mathcal{T}$ if $\mathcal{T}\{x[n]\} = \lambda x[n]$ for some non-zero scalar $\lambda \in \mathbb{C}$. What are the eigensequences for the following systems?

10.1 $\mathcal{T}\{x[n]\} = 3x[n]$

Any sequence is an eigensequence of this system!

10.2 $\mathcal{T}\{x[n]\} = x[n]u[n]$

Any causal sequence is an eigensequence of this system.

10.3 causal moving average: $\mathcal{T}\{x[n]\} = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$

Any constant sequence is an eigensequence of this system. Complex exponentials of the form $x[n] = Ae^{j\omega n}$ are also eigenfunctions (because this is an LSI system).

10.4 general LSI system: $\mathcal{T}\{x[n]\} = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

We showed this in lecture 3.

Any sequence of the form $x[n] = Ae^{j\omega n}$ is an eigensequence:

$$
\begin{aligned}
\mathcal{T}\{Ae^{j\omega n}\} &= \sum_{k=-\infty}^{\infty} h[k]Ae^{j\omega(n-k)} \\
&= Ae^{j\omega n}\left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right) \\
&= Ae^{j\omega n}\lambda
\end{aligned}
$$

Since $\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$ is not a function of n, we can rewrite is as a constant λ and see that the output is a scaled version of the input.

11 Geometric Basis for Sequences

Consider the signal $\gamma[n] = a^n u[n]$ for $0 < a < 1$.

(a)

Show that any sequence $x[n]$ can be decomposed as $x[n] = \sum_{n=-\infty}^{\infty} c_k \gamma[n-k]$ and express c_k in terms of $x[n]$.

(b)

Use the properties of linearity and time invariance to express the output $y[n] = \mathcal{T}\{x[n]\}$ in terms of the input $x[n]$ and the signal $g[n] = \mathcal{T}\{y[n]\}$, where $\mathcal{T}$ is an LTI system.

(c)

Express the impulse response $h[n] = \mathcal{T}\{\delta[n]\}$ in terms of $g[n]$.

12 Steady State of Stable Systems

Let $\mathcal{T}$ be an LTI, BIBO stable system. Show if x[n] is bounded and tends to a constant, the corresponding output, $y[n]$, will also tend to a constant.

If $x[n]$ tends towards a constant, then $\lim_{n\to\infty} x[n] = c_x < \infty$. Since the system is LSI, we also know it has an impulse response $h[n]$. Since it is BIBO stable, we also know from Problem 7 that $\sum_{n=-\infty}^{\infty} \left|h[n]\right| < \infty$.

$$\lim_{n\to\infty} y[n] = \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} \lim_{n\to\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]\left(\lim_{n\to\infty} x[n-k]\right) = c_y$$

Since $\lim_{n\to\infty} x[n-k] = \lim_{n\to\infty} x[n] = c_x$ and $\sum_{k=-\infty}^{\infty} h[k] \leq \sum_{k=-\infty}^{\infty} |h[k]| < \infty$, their product, called c_y is also finite. Thus, the corresponding output $y[n]$ also tends toward a constant.