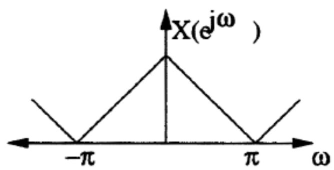
Solution for problem 4.53 from O&S, problem #5 on hmwk 6

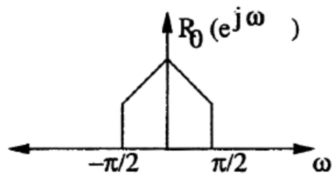
(a)

If $X(e^{j\omega})$ and $H_0(e^{j\omega})$ are as shown in Figure P4.53-2, sketch (to within a scale factor) $X_0(e^{j\omega})$, $G_0(e^{j\omega})$, $Y_0(e^{j\omega})$.

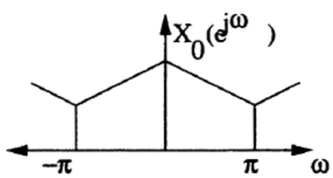
First, $X(e^{j\omega})$ is plotted.



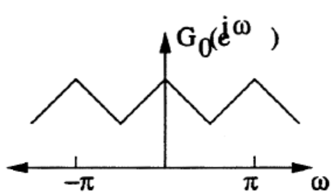
The lowpass filter cuts off at $\frac{\pi}{2}$.



The downsampler expands the frequency axis. Since $R_0(e^{j\omega})$ is bandlimited to $\frac{\pi}{M}$, no aliasing occurs.



The upsampler compresses the frequency axis by a factor of 2.



The lowpass filter cuts off at $\frac{\pi}{2} \Rightarrow Y_0(e^{j\omega}) = R_0(e^{j\omega})$ as sketched above.

(b)

Write a general expression for $G_0(e^{j\omega})$ in terms of $X(e^{j\omega})$ and $H_0(e^{j\omega})$. Do *not* assume that $X(e^{j\omega})$ and $H_0(e^{j\omega})$ are as shown in Figure 4.53-2.

$$\begin{aligned}
R_0(e^{j\omega}) &= X(e^{j\omega})H_0(e^{j\omega}) \\
X_0(e^{j\omega}) &= \frac{1}{2}\left(R_0\left(e^{j\frac{\omega}{2}}\right) + R_0\left(e^{j\left(\frac{\omega}{2}-\pi\right)}\right)\right) \\
G_0(e^{j\omega}) &= X_0(e^{j2\omega}) \\
&\ \text{So combining these together...} \\
X_0(e^{j\omega}) &= \frac{1}{2}\left(X_0\left(e^{j\frac{\omega}{2}}\right)H_0\left(e^{j\frac{\omega}{2}}\right) + X_0\left(e^{j\left(\frac{\omega}{2}-\pi\right)}\right)H_0\left(e^{j\left(\frac{\omega}{2}-\pi\right)}\right)\right) \\
G_0(e^{j\omega}) &= \frac{1}{2}\left(X_0\left(e^{j\omega}\right)H_0\left(e^{j\omega}\right) + X_0\left(e^{j(\omega-\pi)}\right)H_0\left(e^{j(\omega-\pi)}\right)\right)
\end{aligned}$$

(c)

Determine a set of conditions on $H_0(e^{j\omega})$ that is as general as possible and that will guarantee that $y[n]$ is proportional to $x[n-n_d]$ for any stable input $x[n]$.

NOTE: If you have a particular version of 2nd edition, the wording is different: $|Y(e^{j\omega})| \propto |X(e^{j\omega})|$. The final reconstruction is also a difference rather than a sum: $y[n] = y_0[n] - y_1[n]$. This will be discussed at the end.

$$\begin{aligned}
Y_0(e^{j\omega}) &= H_0(e^{j\omega})G_0(e^{j\omega}) \\
&= \frac{1}{2}X(e^{j\omega})H_0^2(e^{j\omega}) + X(e^{j(\omega-\pi)})H_0(e^{j\omega})H_0(e^{j(\omega-\pi)})
\end{aligned}$$

Similarly,

$$\begin{aligned}
Y_1(e^{j\omega}) &= \frac{1}{2}X(e^{j\omega})H_1^2(e^{j\omega}) + X(e^{j(\omega-\pi)})H_1(e^{j\omega})H_1(e^{j(\omega-\pi)}) \\
Y(e^{j\omega}) &= Y_0(e^{j\omega}) + Y_1(e^{j\omega}) \\
&= \frac{1}{2}X(e^{j\omega})\left[H_0^2(e^{j\omega}) + H_1^2(e^{j\omega})\right] + \frac{1}{2}X(e^{j(\omega-\pi)})\left[H_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + H_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right]
\end{aligned}$$

Using the given relationship that $H_1(e^{j\omega}) = H_0(e^{j(\omega-\pi)})$,

$$
\begin{aligned}
Y(e^{j\omega}) &= \frac{1}{2}X(e^{j\omega})\left[H_0^2(e^{j\omega}) + H_0^2(e^{j(\omega-\pi)})\right] + \frac{1}{2}X(e^{j(\omega-\pi)})\left[H_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + H_0(e^{j(\omega-\pi)})H_0(e^{j\omega})\right] \\
&= \frac{1}{2}X(e^{j\omega})\underbrace{\left[H_0^2(e^{j\omega}) + H_0^2(e^{j(\omega-\pi)})\right]}_{A(\omega)} + X(e^{j(\omega-\pi)})\underbrace{\left[H_0(e^{j\omega})H_0(e^{j(\omega-\pi)})\right]}_{B(\omega)}
\end{aligned}
$$

Our goal is to have $y[n] \propto x[n - n_d]$, or in the frequency domain, $Y(e^{j\omega}) = ce^{j\omega n_d}X(e^{j\omega})$. Note that this is the same as the 3rd edition wording because $|Y(e^{j\omega})| = |ce^{j\omega n_d}X(e^{j\omega})| = |c| \cdot |X(e^{j\omega})|$. Thus, we want the aliasing term $X(e^{j(\omega-\pi)})$ to go away completely. Therefore we choose $B(\omega) = 0,\ \forall\omega$. Additionally, we desire to have $A(\omega) = ce^{j\omega n_d}$.

Analyzing $B(\omega)$

Supposing we have $H_0(e^{j\omega})$ with lower cutoff frequency ω_L and upper cutoff frequency ω_R, then $B(\omega) = 0,\ \forall\omega$ can be enforced if $H_0(e^{j\omega})$ and $H_0(e^{j(\omega-\pi)})$ do not overlap, i.e. $\pi + \omega_L > \omega_R$. Rearranging terms we have $\pi > -\omega_L + \omega_R$, or the width of the low-pass filter must be less than π.

Analyzing $A(\omega)$

Case 1: $n_d = 0$

For case of $n_d = 0$, our goal simplifies to $Y(e^{j\omega}) = cX(e^{j\omega})$, so we want $A(\omega) = c_1$, some constant c_1, for all $\{\omega | X(e^{j\omega}) \neq 0\}$ (the support of $X(e^{j\omega})$). However, to make this system general for all signals (even ones that aren't bandlimited), let us say that $A(\omega) = c_1\ \forall\omega$.

$$
H_0^2(e^{j\omega}) + H_0^2(e^{j(\omega-\pi)}) = c_1
$$

Combined with the fact that $B(\omega) = 0$,

$$
\begin{aligned}
H_0^2(e^{j\omega}) &= c_1,\ \omega \in \{\omega_L, \omega_R\} \\
H_0^2(e^{j(\omega-\pi)}) &= c_1,\ \omega \in \{\pi + \omega_L, \pi + \omega_R\} \text{ redundant, so I will drop this line of reasoning} \\
H_0(e^{j\omega}) &= \sqrt{c_1},\ \omega \in \{\omega_L, \omega_R\}
\end{aligned}
$$

Choosing $c_1 = 2c$, we satisfy the necessary condition. Therefore, $H_0(e^{j\omega})$ should have constant height over its support and zero phase. This is equivalent to an ideal low pass filter.

Case 2: $n_d \neq 0$

For the case of $n_d \neq 0$,

$$\begin{aligned}
\frac{1}{2}\left[H_0^2(e^{j\omega}) + H_0^2(e^{j(\omega-\pi)})\right] &= ce^{j\omega n_d} \\
\frac{1}{2}\left[|H_0(e^{j\omega})|e^{j\angle H_0(e^{j\omega})}\right]^2 + \frac{1}{2}\left[|H_0(e^{j(\omega-\pi)})|e^{j\angle H_0(e^{j(\omega-\pi)})}\right]^2 &= ce^{j\omega n_d} \\
\frac{1}{2}|H_0(e^{j\omega})|^2 e^{2j\angle H_0(e^{j\omega})} + \frac{1}{2}|H_0(e^{j(\omega-\pi)})|^2 e^{2j\angle H_0(e^{j(\omega-\pi)})} &= ce^{j\omega n_d}
\end{aligned}$$

Now we consider two cases: (a) $\omega \in \{\omega_L, \omega_R\}$ (in the support of $H_0(e^{j\omega})$) and (b) $\omega > \omega_R$ or $\omega < \omega_L$ (in the support of $H_0(e^{j(\omega-\pi)})$.

Case 2a

For $\omega \in \{\omega_L, \omega_R\}$, $H_0(e^{j(\omega-\pi)}) = 0$, so

$$\begin{aligned}
\frac{1}{2}|H_0(e^{j\omega})|^2 e^{j2\angle H_0(e^{j\omega})} &= ce^{j\omega n_d} \\
\frac{1}{2}|H_0(e^{j\omega})|^2 &= c \quad \rightarrow \quad |H_0(e^{j\omega})| = \sqrt{2c} \\
e^{j2\angle H_0(e^{j\omega})} &= e^{j\omega n_d} \quad \rightarrow \quad \angle H_0(e^{j\omega}) = \frac{1}{2}\omega n_d
\end{aligned}$$

In other words, $H_0(e^{j\omega})$ must have constant magnitude and linear phase.

Case 2b

For $\omega \notin \{\omega_L, \omega_R\}$, $H_0(e^{j\omega}) = 0$, so

$$
\begin{aligned}
\frac{1}{2}|H_0(e^{j(\omega-\pi)})|^2 e^{j2\angle H_0(e^{j(\omega-\pi)})} &= ce^{j\omega n_d} \\
\frac{1}{2}|H_0(e^{j(\omega-\pi)})|^2 &= c \quad \rightarrow \quad |H_0(e^{j(\omega-\pi)})| = \sqrt{2c} \\
e^{j2\angle H_0(e^{j(\omega-\pi)})} &= e^{j\omega n_d} \quad \rightarrow \quad \angle H_0(e^{j(\omega-\pi)}) = \frac{1}{2}\omega n_d
\end{aligned}
$$

Thus, we see that when shifted $H_0(e^{j\omega})$ should still have constant magnitude (which is consistent with what we saw in case 2a). We also see that the phase of shifted $H_0(e^{j\omega})$ is the same as the phase of unshifted $H_0(e^{j\omega})$. This implies that the phase is periodic with period π.

Thus our final conditions for $H_0(e^{j\omega})$ are that:

$|H_0(e^{j\omega})| = \sqrt{2c}$

and

$\angle H_0(e^{j\omega}) = \begin{cases} \frac{1}{2}\omega n_d, & \omega \in \{\omega_L, \omega_R\} \\ \frac{1}{2}(\omega-\pi)n_d, & \omega \in \{\pi+\omega_L, \pi+\omega_R\} \end{cases}.$

Note that if we choose $n_d = 0$, this reduces to our previous conditions.

Alternate version of problem

For the particular version of the 2nd edition/3rd edition:

$$
\begin{aligned}
Y(e^{j\omega}) &= Y_0(e^{j\omega}) - Y_1(e^{j\omega}) \\
&= \frac{1}{2}X(e^{j\omega})\left[H_0^2(e^{j\omega}) - H_1^2(e^{j\omega})\right] + \frac{1}{2}X(e^{j(\omega-\pi)})\left[H_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) - H_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right]
\end{aligned}
$$

Using the given relationship that $H_1(e^{j\omega}) = H_0(e^{j(\omega-\pi)})$,

$$
\begin{aligned}
Y(e^{j\omega}) &= \frac{1}{2}X(e^{j\omega})\left[H_0^2(e^{j\omega}) - H_0^2(e^{j(\omega-\pi)})\right] + \frac{1}{2}X(e^{j(\omega-\pi)})\left[H_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) - H_0(e^{j(\omega-\pi)})H_0(e^{j\omega})\right] \\
&= \frac{1}{2}X(e^{j\omega})\underbrace{\left[H_0^2(e^{j\omega}) + H_0^2(e^{j(\omega-\pi)})\right]}_{A(\omega)}
\end{aligned}
$$

We desire $|Y(e^{j\omega})| \propto |X(e^{j\omega})|$, so

$$\begin{aligned}
|Y(e^{j\omega})| &= c|X(e^{j\omega})| \\
|\frac{1}{2}X(e^{j\omega})A(\omega)| &= c|X(e^{j\omega})| \\
\frac{1}{2}|X(e^{j\omega})| \cdot |A(\omega)| &= c|X(e^{j\omega})| \\
|A(\omega)| = 2c &
\end{aligned}$$

So long as $A(\omega) = H_0^2(e^{j\omega}) + H_0^2(e^{j(\omega-\pi)})$ has constant magnitude, we have $|Y(e^{j\omega})| \propto |X(e^{j\omega})|$. Note that in this version of the problem, we do not have the requirement that $H_0(e^{j\omega})$ not overlap with $H_0(e^{j(\omega-\pi)})$.

Note to graders

Since this problem was different depending the students' textbooks, please be aware that if they had the alternate version of the problem ($y[n] = y_0[n] - y_1[n]$), they should have completed the analysis in the above section for full credit. If they had the first version of the problem ($y[n] = y_0[n] + y_1[n]$), they need to have completed the analysis at least for the $n_d = 0$ case. Any additional analysis is optional.