**Notes # 5.**

## Discrete uncertainty principle

$\boxed{1}$ *Discrete Fourier transform:*
$y = \{y_k\}_0^{N-1} \in \mathbb{C}^N; \quad \mathcal{F}_N : y \mapsto Y = \{Y_n\}_0^{N-1} = \mathcal{F}_N y,$

$$Y_N = \frac{1}{N} \sum_{k=0}^{N-1} y_k \omega_N^{-kn}, \text{ where } \omega_N = e^{2i\pi/N}.$$

$\boxed{2}$ *Parseval identity:*

$$\sum_{k=0}^{N-1} |y_k|^2 = N \sum_{n=0}^{N-1} |Y_n|^2.$$

$\boxed{3}$ *Support of a signal:*

$$\text{supp } y = \{k, y_k \neq 0\}, \ |\text{supp } y| = \text{number of elements of supp } y$$

$\boxed{4}$ *Formulation of the uncertainty principle:*

$$|\text{supp } y||\text{supp } Y| \geq N.$$

$\boxed{5}$ *Proof:*

For each $n = 0, 1, \ldots, N-1$

$$|Y_n| \leq \frac{1}{N} \sum_{k \in \text{supp } y} |y_k| \underbrace{\leq}_{\text{Schwarz inqty}} \frac{1}{N} \left( \sum_{k \in \text{supp } y} |y_k|^2 \right)^{1/2} (|\text{supp } y|)^{1/2}$$

$$\underbrace{=}_{\text{Parseval idnty}} \frac{1}{N} \left( N \sum_{l=0}^{N-1} |Y_l|^2 \right)^{1/2} (|\text{supp } y|)^{1/2} =$$

$$\left(\frac{1}{N}\right)^{1/2} \left( \sum_{l \in \text{supp } Y} |Y_l|^2 \right)^{1/2} (|\text{supp } y|)^{1/2}.$$

So, for each $n$,

$$|Y_n|^2 \leq \frac{1}{N} |\text{supp } y| \sum_{l \in \text{supp } Y} |Y_l|^2.$$

Now sum this with respect to all $n \in \text{supp } Y$ taking into account that the right-hand side is independent of $n$:

$$\left( \sum_{n \in \text{supp } Y} |Y_n|^2 \right) \leq \frac{1}{N} \left( \sum_{l \in \text{supp } Y} |Y_l|^2 \right) |\text{supp } y||\text{supp } Y|.$$

This yields the desired inequality.