## Problems for Discussion 3, 09/25/13

Compiled by Mai Le, some problems from Prof. Yagle, Prof. Fessler

# 1 Using DTFT Properties

Let $x[n] = \{1, 4, 3, 2, 5, 7, \underline{-45}, 7, 5, 2, 3, 4, 1\}$. Let $X(\omega) = DTFT(x[n])$. Compute the following without actually computing $X(\omega)$. Use DTFT properties!

## 1.1 (a) $X(\pi)$

$X(\pi) = \sum\limits_{n=-\infty}^{\infty} x[n]e^{-j\pi n} = \sum\limits_{n=-\infty}^{\infty} x[n](-1)^n$

$= 1 - 4 + 3 - 2 + 5 - 7 + -45 - 7 + 5 - 2 + 3 - 4 + 1 = -53.$

## 1.2 (b) arg$X(\omega)$

$x[n] = x[-n]$ so $X(\omega) = X^*(\omega)$. This implies that $X(\omega)$ is real.

$Re\{X(\omega)\} = Re\{\sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}\} = \sum_{n=-\infty}^{\infty} Re\{x[n]e^{j\omega n}\}$

Each term $Re\{x[n]e^{j\omega n}\} = Re\{x[n]cos(\omega n) + x[n]i \cdot sin(\omega n)\} = x[n]cos(\omega n) \leq x[n]$.

So, $Re\{X(\omega)\} = X(\omega) \leq \sum_{n=-\infty}^{\infty} x[n] = -45 + 2(1 + 4 + 3 + 2 + 5 + 7) = -1 < 0$

Therefore, $argX(\omega) = \pi$ for all $\omega$.

## 1.3 (c) $\int_{-\pi}^{\pi} X(\omega)d\omega$

We will use the following theorem, found on line 9 of table 2.2 (Fourier Transform Theorems):

$\sum\limits_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2}\int_{-\pi}^{\pi} X(\omega)Y^*(\omega)d\omega$

$\int_{-\pi}^{\pi} X(\omega)d\omega = \int_{-\pi}^{\pi} X(\omega)1(\omega)d\omega = \int_{-\pi}^{\pi} X(\omega)1^*(\omega)d\omega.$

We know that $1(\omega) = 1^*(\omega)$ has the IDTFT $\delta[n]$, so $\int_{-\pi}^{\pi} X(\omega)d\omega = 2\pi \sum\limits_{n=-\infty}^{\infty} x[n]\delta^*[n] =$

$2\pi \sum\limits_{n=-\infty}^{\infty} x[n]\delta[n]$

$= 2\pi x[0] = -90\pi$

### 1.4 (d) $\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$

Use Parseval's Theorem!

Parseval's: $\sum\limits_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$.

So, $\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega = 2\pi \sum\limits_{n=-\infty}^{\infty} |x[n]|^2$

$= 2\pi \left(1^2 + 4^2 + 3^2 + 2^2 + 5^2 + 7^2 + (-45)^2 + 7^2 + 5^2 + 2^2 + 3^2 + 4^2 + 1^2\right) = 4466\pi$

## 2 Computing the Z-Transform

Compute the z-transform and the corresponding ROCs of the following sequences.

**(a)** $x[n] = \left(\frac{1}{5}\right)^n u[n]$

From $a^n u[n] \to \frac{z}{z-a}$, we get $(1/5)^n u[n] \to \frac{z}{z-1/5}$.

ROC: $|z| > |a| = 1/5$

**(b)** $x[n] = 2^n u[-n] + \left(\frac{1}{3}\right)^n u[n]$

From $-a^n u[-n-1] \to \frac{z}{z-a}$, and manipulating $2^n u[-n] = 2 \cdot 2^{n-1} u[-(n-1)-1]$, we get $2^n u[-n] \to -\frac{2z}{z-2} z^{-1} = -\frac{1}{z-2}$. ROC: $|z| < |a| = 2$.

From $a^n u[n] \to \frac{z}{z-a}$, we get $(1/3)^n u[n] \to \frac{z}{z-1/3}$. ROC: $|z| > |a| = 1/3$

$X(z) = \frac{z}{z-1/3} - \frac{1}{z-2}$.

ROC: $1/3 < |z| < 2$.

**(c)** $x[n] = \{-1, \underline{0}, 1\}$

$X(z) = -1 \cdot z + 0 \cdot 1 + 1 \cdot z^{-1} = -z + z^{-1}$

ROC: $z \neq 0$

**(d)** $x[n] = \sum_{k=-\infty}^{n} 3^k u[n]$

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Using the change of variables $l = n - k$, $x[n] = \sum\limits_{l=0}^{\infty} 3^{n-l} u[n] = 3^n u[n] \sum\limits_{l=0}^{\infty} \left(\frac{1}{3}\right)^l$

Using the geometric series formula, $x[n] = 3^n u[n] \left(\frac{1}{1-\frac{1}{3}}\right) = \frac{3}{2} 3^n u[n]$.

Now we can apply the known z-transform pair to get $X(z) = \frac{3}{2} \frac{1}{1-3z^{-1}}$ with ROC $|z| > 3$.

## 3 Z-Transform Manipulation

Express the z-transform of $y[n] = \sum\limits_{k=-\infty}^{n} x[k]$ in terms of $X(z)$.

$$
\begin{aligned}
y[n] &= x[n] + x[n-1] + x[n-2] + \cdots \\
Y(z) &= X(z) + z^{-1}X(z) + z^{-2}X(z) + \cdots \text{ using the time shifting property of z-transform} \\
&= X(z)\left(1 + z^{-1} + z^{-2} + \cdots\right) \\
&\quad \text{use the geometric series formula ...} \\
&= \frac{X(z)}{1 - z^{-1}} \text{ if } |z| > 1
\end{aligned}
$$

ROC: {ROC of x} $\cap \{|z| > 1\}$

## 4 Inverse z-transform strategies

Consider the z-transform $X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$. We can rewrite $X(z)$ into a friendlier form $X(z) = B(z) + \frac{A_1}{D_1} + \frac{A_2}{D_2}$, where $B(z)$, $A_1(z)$, $A_2(z)$, $D_1(z)$, and $D_2(z)$ are polynomials of $z$ (or rather $z^{-1}$).

Use long division to find $B(z)$ and partial fraction expansion to find the remaining $A_1(z)$, $A_2(z)$, $D_1(z)$, and $D_2(z)$.

To get $B(z)$, we divide $1 + 2z^{-1} + z^{-2}$ by $1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}$ until the polynomial order of the remainder is less than the polynomial order of the denominator (2 in this case).

Doing this, we get $B = 2$ with a remainder of $-1+5z^{-1}$, so $X(z) = 2 + \frac{-1+5z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$.

Next we do partial fraction expansion of $\frac{-1+5z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$ to determine $A_1(z)$, $A_2(z)$, $D_1(z)$, and $D_2(z)$.

$\frac{-1+5z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = \frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$, so $D_1(z) = 1 - \frac{1}{2}z^{-1}$ and $D_2(z) = 1 - z^{-1}$.

Lastly, to find $A_1(z)$ and $A_2(z)$, we use equation 3.43:

$A_1(z) = (1 - \frac{1}{2}z^{-1})\frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}\bigg|_{z=1/2} = -9$

$A_2(z) = (1 - z^{-1})\frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}\bigg|_{z=1} = 8$

Finally, we have $X(z) = 2 + \frac{-9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}}$. This formulation is easier for finding the inverse z-transform because it is the sum of known z-transforms. Its ROC will be the intersection of the ROCs of each term.

Note: this example was covered in the textbook on pages 120-121 if you would like to read more details.