Solution for problem 2.78 form O&S, problem #3 on hmwk 2

Let $x[n]$ and $X(\omega)$ represent a sequence and its Fourier transform, respectively. Determine, in terms of $X(\omega)$, the transforms of $y_s[n]$, $y_d[n]$, $y_e[n]$. In each case, sketch $Y(\omega)$ for $X(\omega)$ as shown in Figure P278-1.

(a) Sampler:

$$y_s[n] = \begin{cases} x[n], & n \text{ even} \\ 0, & n \text{ even} \end{cases}$$

Note that $y_s[n] = \frac{1}{2}\{x[n] + (-1)^n x[n]\}$ and $-1 = e^{j\pi}$.

$$\begin{aligned} y_s[n] &= x[n]\left(\frac{1+(-1)^n}{2}\right) \text{ from hint} \\ &= \frac{1}{2}x[n] + \frac{1}{2}(-1)^n x[n] \\ &= \frac{1}{2}x[n] + \frac{1}{2}(e^{j\pi})^n x[n] \\ Y(\omega) &= \frac{1}{2}X(\omega) + \frac{1}{2}X(\omega - \pi) \text{ from hint} \\ & \quad \text{from linearity of IDTFT and frequency shift theorem} \end{aligned}$$

Insert picture here- it should look like a constant value of 1/2.

(b) Compressor:

$$y_d[n] = x[2n]$$

$$
\begin{aligned}
Y_d(\omega) &= \sum_{n=-\infty}^{\infty} y_d[n] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} x[2n] e^{-j\omega n} \\
& \text{Let } m = 2n \\
&= \sum_{\substack{m=-\infty \\ m \text{ even}}}^{\infty} x[m] e^{-j\omega m/2} \text{ because } y_d[n] \text{ is only defined on integer } n \\
&= \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m/2} \left(\frac{1 + (-1)^m}{2} \right) \text{ from hint in part (a)} \\
&= \frac{1}{2} \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m/2} \left(1 + e^{j\pi m} \right) \\
& \text{Let } \tilde{\omega} = \omega/2 \\
&= \frac{1}{2} \sum_{m=-\infty}^{\infty} x[m] e^{-j\tilde{\omega} m} \left(1 + e^{j\pi m} \right) \\
&= \frac{1}{2} \sum_{m=-\infty}^{\infty} x[m] e^{-j\tilde{\omega} m} + \frac{1}{2} \sum_{m=-\infty}^{\infty} x[m] e^{-j(\tilde{\omega} - \pi) m} \\
&= \frac{1}{2} X(\tilde{\omega}) + \frac{1}{2} X(\tilde{\omega} - \pi) \\
&= \frac{1}{2} X\left(\frac{\omega}{2} \right) + \frac{1}{2} X\left(\frac{\omega}{2} - \pi \right)
\end{aligned}
$$

Insert picture here- it should also look like a constant value of 1/2.

(c) Expander:

$$
y_e[n] = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}
$$

$$
\begin{aligned}
Y_e(\omega) &= \sum_{n=-\infty}^{\infty} y_e[n]e^{-j\omega n} \\
&= \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} y_e[n]e^{-j\omega n} + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} y_e[n]e^{-j\omega n} \\
&= \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} x[n/2]e^{-j\omega n} + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} 0\ e^{-j\omega n} \\
&= \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} x[n/2]e^{-j\omega n} \\
&\quad \text{Let } m = n/2 \text{ for even } n \\
&= \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega 2m} \\
&= \sum_{m=-\infty}^{\infty} x[m]e^{-j(2\omega)m} \\
&\quad \text{Let } \tilde{\omega} = 2\omega \\
&= \sum_{m=-\infty}^{\infty} x[m]e^{-j\tilde{\omega}m} \\
&= X(\tilde{\omega}) \\
&= X(2\omega)
\end{aligned}
$$

Expanding in the time domain yields compression in the frequency domain. The picture should look like a squished version of $X(\omega)$.