Solutions for homework 4, problems #2,3

Problem 2, 3.25 O&S 3rd Ed.

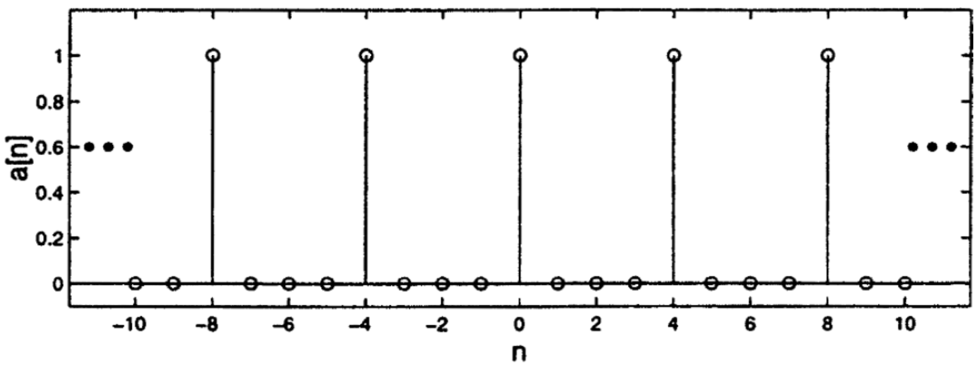
Sketch each of the following sequences and determine their z-transforms, including the region of convergence.

(a)

$a[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k]$

$$\begin{aligned}
A(z) &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta[n-4k] z^{-n} \\
&= \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} \delta[n-4k] z^{-n} \right) \\
\text{from the sifting property,} \quad &= \sum_{k=-\infty}^{\infty} z^{-4k} \\
&= \sum_{k=0}^{\infty} (z^{-4})^k + \sum_{k=-\infty}^{-1} (z^{-4})^k \\
&= \underbrace{\sum_{k=0}^{\infty} (z^{-4})^k}_{\alpha} + \underbrace{\sum_{k=1}^{\infty} (z^{4})^k}_{\beta}
\end{aligned}$$

Note that α converges only if $|z^{-4}| < 1$ and β converges only if $|z^4| < 1$. There is no complex number z that satisfies both of these conditions. Since the ROC of $A(z)$ is the intersection of the ROC of $\alpha \cap$ ROC of β, ROC of $A(z)$ is empty, $\{\}$.



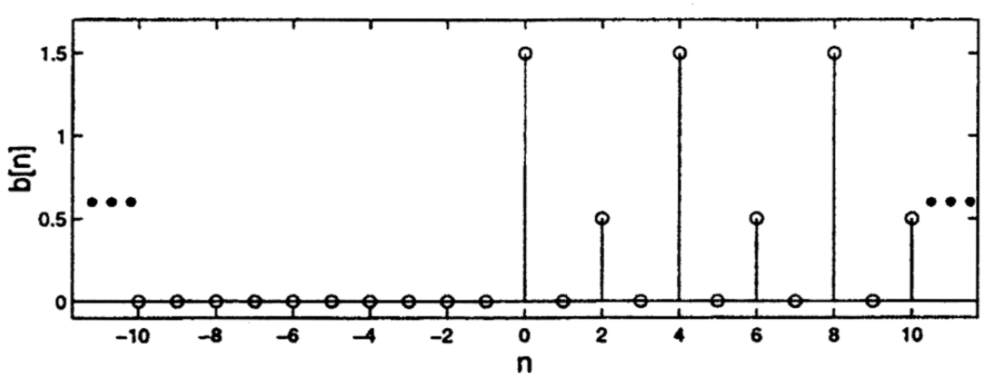
(b)

$b[n] = \frac{1}{2}\left[e^{j\pi n} + cos\left(\frac{\pi}{2}n\right) + sin\left(\frac{\pi}{2} + 2\pi n\right)\right] u[n]$

First, let us simplify $x[n]$.

$$\begin{aligned} b[n] &= \frac{1}{2}\left[e^{j\pi n} + cos\left(\frac{\pi}{2}n\right) + sin\left(\frac{\pi}{2} + 2\pi n\right)\right] u[n] \\ &= \frac{1}{2}\left[(-1)^n + cos\left(\frac{\pi}{2}n\right) + sin\left(\frac{\pi}{2}\right)\right] u[n] \\ &= \frac{1}{2}\left[(-1)^n + cos\left(\frac{\pi}{2}n\right) + 1\right] u[n] \\ &= \begin{cases} \frac{3}{2}, & n = 4k, k \geq 0 \\ \frac{1}{2}, & n = 4k+2, k \geq 0 \\ 0, & otherwise \end{cases} \end{aligned}$$

$$
\begin{aligned}
B(z) &= \sum_{n=-\infty}^{\infty} b[n]z^{-n} \\
&= \sum_{\substack{n=-\infty \\ n=4k\geq 0}}^{\infty} b[n]z^{-n} + \sum_{\substack{n=-\infty \\ n=2+4k\geq 0}}^{\infty} b[n]z^{-n} + \sum_{\substack{n=-\infty \\ n \text{ odd or } n<0}}^{\infty} b[n]z^{-n} \\
&= \sum_{\substack{n=-\infty \\ n=4k\geq 0}}^{\infty} \frac{3}{2}z^{-n} + \sum_{\substack{n=-\infty \\ n=2+4k\geq 0}}^{\infty} \frac{1}{2}z^{-n} + \sum_{\substack{n=-\infty \\ n \text{ odd or } n<0}}^{\infty} 0\cdot z^{-n} \\
&= \sum_{k=0}^{\infty}\frac{3}{2}z^{-4k} + \sum_{k=0}^{\infty}\frac{1}{2}z^{-(2+4k)} \\
&= \left(\frac{3}{2}+\frac{1}{2}z^{-2}\right)\sum_{k=0}^{\infty}(z^{-4})^k \\
&= \left(\frac{3}{2}+\frac{1}{2}z^{-2}\right)\left(\frac{1}{1-z^{-4}}\right) \text{ for } |z^{-4}|<1, \text{ or equivalently } |z|>1 \\
&= \frac{\frac{3}{2}+\frac{1}{2}z^{-2}}{1-z^{-1}} \text{ with ROC } |z|>1
\end{aligned}
$$



Problem 3, 3.31 O&S 3rd Ed.

Determine the inverse z-transform of each of the following. In Parts (a)-(c), use the methods specified. In Part (d), use any method you prefer.

(a) Long division:

$x[n]$ is right-sided and $X(z) = \frac{1-\frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}}$

$$
\begin{array}{r|llll}
 & 1 & -\frac{2}{3}z^{-1} & +\frac{2}{9}z^{-2} & +\ldots \\ \hline
1+\frac{1}{3}z^{-1} & 1 & -\frac{1}{3}z^{-1} & & \\
 & 1 & -\frac{1}{3}z^{-1} & & \\ \cline{2-3}
 & & -\frac{2}{3}z^{-1} & & \\
 & & -\frac{2}{3}z^{-1} & -\frac{2}{9}z^{-2} & \\ \cline{3-4}
 & & & +\frac{2}{9}z^{-2} &
\end{array}
$$

$1 - \frac{2}{3}z^{-1} + \frac{2}{9}z^{-2} - \frac{2}{27}z^{-3} + \cdots = 2\left(1 - \frac{1}{3}z^{-1} + \frac{1}{9}z^{-2} - \frac{1}{27}z^{-3} + \cdots\right) - 1$

So we can take the inverse z-transform by inspection:

$x[n] = 2\{\underline{1}, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots\} - \{\underline{1}, 0, 0, \ldots\} = 2\left(-\frac{1}{3}\right)^n u[n] - \delta[n]$

(b) Partial fraction:

$X(z) = \frac{3}{z-\frac{1}{4}-\frac{1}{8}z^{-1}}$ and $x[n]$ is stable

First, we factor to find the denominator terms for the partial fraction expansion: $X(z) = \frac{3}{z-\frac{1}{4}-\frac{1}{8}z^{-1}} = \frac{3z^{-1}}{1-\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}} = \frac{3z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}$

So now we know that it can be written in the form $X(z) = \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1+\frac{1}{4}z^{-1}}$. To find A_1 and A_2, we use Equation 3.41:

$A_1 = (1 - \frac{1}{2}z^{-1})\left(\frac{3z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}\right)\Big|_{z=1/2} = \frac{3z^{-1}}{1+\frac{1}{4}z^{-1}}\Big|_{z=1/2} = \frac{6}{1+\frac{1}{2}} = 4$

$A_2 = (1 + \frac{1}{4}z^{-1})\left(\frac{3z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}\right)\Big|_{z=-1/4} = \frac{3z^{-1}}{1-\frac{1}{2}z^{-1}}\Big|_{z=-1/4} = \frac{-12}{1+2} = -4$

Thus, $X(z) = \frac{4}{1-\frac{1}{2}z^{-1}} + \frac{-4}{1+\frac{1}{4}z^{-1}}$

Since $x[n]$ is stable, the ROC must include the unit circle. It also has poles at $\frac{1}{2}$ and $-\frac{1}{4}$, so the ROC must be $|z| > \frac{1}{2}$, and $x[n]$ must be causal (or right-sided).

Therefore, $x[n] = 4\left(\frac{1}{2}\right)^n u[n] - 4\left(-\frac{1}{4}\right)^n u[n]$

(c) Power series:

$X(z) = ln(1-4z)$ and $|z| < \frac{1}{4}$

Using the formula on page 117 (also covered in lecture), $X(z) = ln(1-4z) = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}(-4z)^n}{n} = -\sum\limits_{n=1}^{\infty} \frac{(4z)^n}{n}$

Note that the above series converges only if $|-4z| < 1$, but this is consistent with $|z| < \frac{1}{4}$.

Let $m = -n$ and let's change variables. $X(z) = \sum\limits_{m=-\infty}^{-1} \frac{4^{-m}}{m} z^{-m}$

Hey, this looks like the definition of the z-transform. Since we know $x[n]$ is left-sided, we have $X(z) = \sum\limits_{m=-\infty}^{\infty} \left(\frac{4^{-m}}{m} u[-m-1]\right) z^{-m}$, and by inspection we arrive at $x[n] = \frac{1}{n}(4)^{-n}u[-n-1]$.

(d)

$X(z) = \frac{1}{1-\frac{1}{3}z^{-3}}$, $|z| > (3)^{-1/3}$

We choose to analyze this with long division:

$$\begin{array}{rllll}
 & 1 & +\frac{1}{3}z^{-3} & +\frac{1}{9}z^{-6} & +\dots \\ \cline{2-5}
1-\frac{1}{3}z^{-3} & \multicolumn{1}{|l}{1} & & & \\
 & 1 & -\frac{1}{3}z^{-3} & & \\ \cline{2-3}
 & & +\frac{1}{3}z^{-3} & & \\
 & & +\frac{1}{3}z^{-3} & -\frac{1}{9}z^{-6} & \\ \cline{3-4}
 & & & +\frac{1}{9}z^{-6} &
\end{array}$$

Because the ROC extends outward, we know that $x[n]$ is right-sided. By inspection we see that this is $x[n] = \begin{cases} \left(\frac{1}{3}\right)^{\frac{n}{3}}, & n = 0, 3, 6, \dots \\ 0, & otherwise \end{cases}$