Z- Transform Properties

① Linearity.

If $x_1[n] \overset{z}{\longleftrightarrow} X_1(z)$, R_{x_1}

$x_2[n] \overset{z}{\longleftrightarrow} X_2(z)$, R_{x_2}

then

$$a x_1[n] + b x_2[n] \overset{z}{\longleftrightarrow} a X_1(z) + b X_2(z)$$

ROC contains $R_{x_1} \cap R_{x_2}$

② Time-shifting Property

$$x[n-n_0] \overset{z}{\longleftrightarrow} z^{-n_0} X(z)$$

ROC = R_x except for possibly at $z=0$ and $z=\infty$.

③ Multiplication by an exponential sequence (z_0^n)

$$z_0^n x[n] \overset{z}{\longleftrightarrow} X\left(\frac{z}{z_0}\right)$$

$$\text{ROC} = |z_0| R_x$$

④ Differentiation of $X(z)$

$$n x[n] \overset{z}{\longleftrightarrow} \left(\frac{d}{dz} X(z)\right)(-z)$$

$$\text{ROC} = R_x$$

⑤ Time Reversal

Real sequences $x[n]$:

$$x[-n] \overset{z}{\longleftrightarrow} X(1/z)$$

$$\text{ROC} = \frac{1}{R_x}$$

General (real or complex) $x[n]$:

$$x^*[-n] \overset{z}{\longleftrightarrow} X^*(1/z^*)$$

$$\text{ROC} = \frac{1}{R_x}$$

⑥ Convolution Property

$$x_1[n] * x_2[n] \longleftrightarrow X_1(z)X_2(z)$$

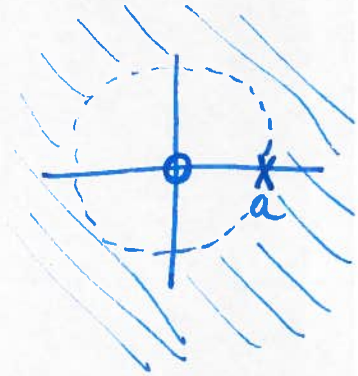
ROC contains $R_{x_1} \cap R_{x_2}$

Example

$$x[n] = a^n u[n] - a^n u[n-n_0].$$

What is the z-transform of $a^n u[n]$?

$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$



what about $a^n u[n-n_0]$?

$$x_2[n] = a^n u[n-n_0]$$

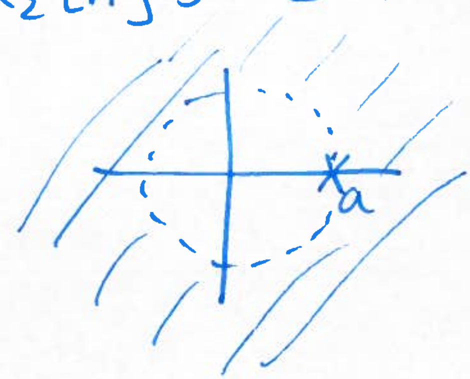
$$X_2(z) = \sum_{k=-\infty}^{\infty} a^k u[k-n_0] z^{-k}$$

$$= \sum_{k=n_0}^{\infty} (az^{-1})^k \qquad l = k - n_0$$

$$= \sum_{l=0}^{\infty} (az^{-1})^{n_0} (az^{-1})^l$$

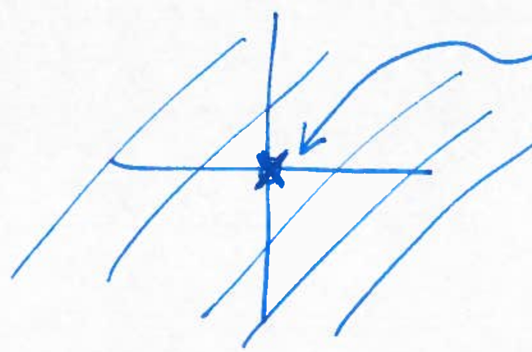
$$= (az^{-1})^{n_0} \frac{z}{z-a}, \quad |z| > |a|$$

ROC for $x_2[n]$'s Z-transform
is also



In this case, the intersection of the ROC's is also $|z|>|a|$.

But our output sequence is finite duration, so its ROC is the entire z-plane (except possibly 0 or ∞).



Why? $\sum_{k=0}^{n_0} c_k z^{-k} = \infty$ for $z=0$.

Example 3.14 in the Text.

Consider the z-transform

$$X(z) = \frac{1}{z - \frac{1}{4}}, \quad \underset{\text{ROC}}{|z| > \frac{1}{4}}$$

$$X(z) = \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}}$$

Equation 3.39

$$X(z) = -4 + \frac{4}{1 - \frac{1}{4}z^{-1}} = -4 + 4\left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right)$$

$$\boxed{x[n] = -4\,\delta[n] + 4\left(\tfrac{1}{4}\right)^n u[n].}$$

Alternatively,

$$X(z) = \frac{z^{-1}}{1-\frac{1}{4}z^{-1}}$$

$$= z^{-1}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)$$

$$\boxed{x[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1]}$$

Example 3.16 in Text

$$X(z) = \log(1+az^{-1}), \quad |z|>|a|$$

$$\frac{dX(z)}{dz} = \frac{1}{1+az^{-1}}(-az^{-2})$$

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1+az^{-1}}$$

$$nx[n] = a(-a)^{n-1}u[n-1]$$

$$\Rightarrow x[n] = \frac{a}{n}(-a)^{n-1}u[n-1].$$