Review of Topics for Discussion 1, 09/11/13

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1 Properties of Discrete Signals

1.1 Periodicity

As signal $x[n]$ is periodic if there exists a number n_0 such that $x[n] = x[n - n_0]$ for all n. The period of the signal is n_0.

For sinusoidal functions (i.e. $cos\,(\omega n + \phi)$), the function is periodic iff ω is a rational multiple of π, i.e. $\omega = \frac{M}{N}\pi$ for integers M and N.

1.2 Boundedness

A signal is bounded if there exists a positive, finite number B such that $|x[n]| \leq B$ for all n.

1.3 Causality

A signal $x[n]$ is causal if $x[n] = 0$ for $n < 0$.

1.4 Symmetry

A signal $x[n]$ is symmetric if $x[n] = x[-n]$. Signals with symmetry are also called "even".

If instead a signal has the property that $-x[n] = x[-n]$, then it is called "odd".

2 Properties of Discrete-Domain Systems

2.1 Causality

A system $\mathcal{T}$ is causal if $y = [n_0] = \mathcal{T}\{x[n]\}$ depends only on $x[n]$ for $n \leq n_0$. In other words, it depends only on the present and past values of $x[n]$ and not the future values.

2.2 Linearity

A system $\mathcal{T}$ is linear iff for any inputs $x_1[n]$ and $x_2[n]$ and any scalars a_1 and a_2, $\mathcal{T}\{a_1x_1[n]+a_2x_2[n]\} = a_1\mathcal{T}\{x_1[n]\} + a_2\mathcal{T}\{x_2[n]\}$.

2.3 Shift-Invariance (a.k.a. Time-Invariance)

Let $y[n] = \mathcal{T}\{x[n]\}$. Then the system $\mathcal{T}$ is shift-invariant if $\mathcal{T}\{x[n-n_0]\} = y[n-n_0]$. In other words, the output of the delayed signal is the same as the delay of the output signal.

2.4 Bounded Input Bounded Output (BIBO) Stability

A system $\mathcal{T}$ is called BIBO stable if any bounded input signal $x[n]$ results in a bounded output signal $y[n]$. If a bound B_x exists for bounded $x[n]$, then there exists a bound B_y such that $|y[n]| = |\mathcal{T}\{x[n]\}| \leq B_y$ for all n.

3 Kronecker Delta (a.k.a. Delta Sequence, Unit Sample Sequence)

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

3.1 Decomposition of Discrete Signals with Shifted Deltas

Any sequence $x[n]$ can be written as $x[n] = \sum\limits_{k=-\infty}^{\infty} x[k]\delta[n-k]$

3.2 Sampling Property

$\delta[n-n_0]x[n] = \delta[n-n_0]x[n_0]$

note: The result is a sequence!

3.3 Sifting Property

$\sum\limits_{n=-\infty}^{\infty} \delta[n-n_0]x[n] = x[n_0]$

note: The result is a scalar!

3.4 Scaling Property

$\delta[2n] = \delta[n]$ Note that there is no scaling factor!

4 Dirac Delta (a.k.a. Unit Impulse Function)

A loose definition:

$\delta(t) = \begin{cases} +\infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$ s.t. $\int_{-\infty}^{\infty} \delta(t)dt = 1.$

A more precise definition:

$\delta(t) = \lim\limits_{\Delta \to 0} \frac{1}{\Delta} rect(\frac{t}{\Delta})$

4.1 Decomposition of Continuous Signals with Shifted Deltas

Any continuous-domain function $x(t)$ can be written as $x(t) = \int\limits_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

4.2 Sampling Property

$\delta(t-t_0)x(t) = \delta(t-t_0)x(t_0)$

note: The result is a function!

4.3 Sifting Property

$\int\limits_{-\infty}^{\infty} \delta(t-t_0)x(t) = x(t_0)$

note: The result is a scalar!

4.4 Scaling Property

$\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)$ for any scalar α

4.5 Kronecker vs. Dirac

What's the deal with these different delta functions?

The Kronecker delta, denoted $\delta[n]$, is a sequence indexed over a discrete domain, in other words, $n \in \mathbb{Z}$. The Dirac delta, denoted $\delta(t)$ or drawn as an upward pointing arrow, is a function over a continuous domain, in other words $t \in \mathbb{R}$. In this class, we're dealing exclusively with discrete-time signals, so in the time domain, we'll have Kronecker deltas. However, when we take the DTFT of a discrete-time signal, we get a result in the continuous frequency domain (ω), so if we have an impulse in the continuous frequency domain, it will be a Dirac delta, $\delta(\omega)$.

Spiritually, these functions are the same though.

$\delta(t) = 0$ for all $t \neq 0$, and $\delta[n] = 0$ for all $n \neq 0$. $\int_{-\infty}^{\infty} \delta(t)dt = 1$ and $\sum\limits_{n=-\infty}^{\infty} \delta[n] = 1$.

5 Complex Numbers and Euler's Formula

A complex number z can be represented as $z = a + jb$ or as $z = re^{j\theta}$. The magnitude and phase are related to a and b: $r = \sqrt{a^2 + b^2}$ and $tan(\theta) = \frac{b}{a}$.

Euler's formula: $e^{j\theta} = cos(\theta) + jsin(\theta)$

which you can use to derive these other useful equations:

$cos(x) = \frac{1}{2}\left[e^{jx} + e^{-jx}\right]$

$sin(x) = \frac{1}{2j}\left[e^{jx} - e^{-jx}\right]$

Other equivalences you should get comfortable with:

- $e^{\frac{\pi}{2}j} = j$
- $e^{\pi j} = -1$ Fun fact: this equation is known as Euler's Identity and is renowned for its mathematical beauty.
- $e^{-\frac{\pi}{2}j} = e^{\frac{3\pi}{2}j} = -j$

Note: you can also easily remember these values by imagining the unit circle on the real vs. imaginary plane. $e^{\frac{-\pi}{2}j}$ is the value of the unit circle at angle $\frac{-\pi}{2}$ radians, which is along the negative imaginary axis, so the value is $-j$.