EECS351 Discussion 1 with MATLAB demo SOLUTIONS, 09/08/16

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4 Basis Examples

4.1 Basis Problem

Consider the Hilbert Space $\mathbb{R}^3$ with real scalars. Find a basis for $\mathbb{R}^3$ that includes the following two vectors:

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

Solution:
We try the following vector as the third basis vector

$$\begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$$

We can verify these vectors as a valid basis if the matrix

$$\begin{bmatrix} 1 & 1 & -2 \\ -2 & 4 & -2 \\ 1 & -2 & 2 \end{bmatrix}$$

can be row-reduced to the identity matrix. This is equivalent to writing the canonical basis as a linear combination of basis vectors. You can also plug the above matrix into MATLAB. The function rank(matrix) returns the number of linearly independent columns.

4.2 Show vectors form a basis

Show that these four vectors form a basis for $\mathbb{R}^4$:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution:
Similar to 4.1, it is easy to show that these vectors row reduce to the identity matrix, or canonical basis. Thus, these vectors represent a valid basis for $\mathbb{R}^4$

4.3 Orthonormal Basis

Similar to 4.1 and 4.2, these vectors can be row-reduced to the identity matrix, so we know these form a valid basis for $\mathbb{R}^4$. To satisfy orthonormality, we just need to verify that each vector satisfies the following properties:

$$v_i{}^T v_j{}^T = 0 \text{ when i} \neq \text{j.}$$

$$||v_i|| = \sqrt{v_i^T v_i} = 1 \text{ for all i}$$

4.4 Finding vectors that DON'T form a Basis

Any value of α will make this a valid basis except for $\alpha = 1$. Note, $\alpha =$ -1 makes these vectors orthogonal.

4.5 Functions as a basis

Consider the Hilbert space of continuous-time real-valued polynomials of degree 3 defined on the interval [a,b]. Show that the polynomials

$$x_0(t) = 1, x_1(t) = t, x_2(t) = t^2, x_3(t) = t^3$$

form a basis for this space.

Solution: For this to be a basis, we require linear independece between the vectors. This means that we need to show that uniformily *for all* tϵ[a,b], the only B_i that satisfy the equation

$$\beta_0 x_0(t) + \beta_1 x_1(t) + \beta_2 x_2(t) + \beta_3 x_3(t) = 0 \quad \text{(for all t over the interval)}$$

are $B_i = 0$ for all i.

We can do this by choosing four distinct values of t and then showing that the resulting system of equations has only one solution with $B_i = 0$ for all i. Feel free to pick four values and verify the solution for B_i, but these x_i do in fact represent a basis.

Representing the same signal with different bases

5.2 Standard Basis

$c = \{\underline{1}, 1, 2, 3, 5\}$

5.2 Unit Step Sequences

Recall $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$. Let $\mathcal{S}_u = \{u[n - n_0] | n_0 \in \mathbb{Z}\}$ (unit steps shifted by any integer n_0) be your basis sequences. Represent $x[n]$ in $\mathcal{S}_u$.

In other words, show you can write $x[n] = \sum_{k=-\infty}^{\infty} c_k u[n-k]$ by finding the values for c_k.

$c = \{\underline{1}, 0, 1, 1, 2, -5\}$

5.3 Three-Tap Rectangles

Let $r[n] = \{\underline{1}, 1, 1\}$ and $\mathcal{S}_r = \{r[n - n_0] | n_0 \in \mathbb{Z}\}$. Represent $x[n]$ in $\mathcal{S}_r$.

$c = \{\underline{1}, 0, 1, 2, 0, 2, -4, 2, 2, -4, 2, 2, -4, \ldots\}$

Vector Space, Hilbert Spaces, and Inner Products

6.1 Inner Product

Prove the Pythagorean theorem for norms defined by inner products. That is, if z = x + y and x ⊥ y, then $||\mathrm{x}||^2 + ||\mathrm{y}||^2 = ||\mathrm{z}||^2$

Solution:

$$||\mathrm{z}||^2$$

$$= ||\mathrm{x} + \mathrm{y}||^2$$

$$= (\mathrm{x}+\mathrm{y})(\mathrm{x}+\mathrm{y})$$

$$= x^2 + 2xy + y^2$$

$$\mathrm{xy} = 0 \text{ since } \mathrm{x} \perp \mathrm{y}$$

$$= x^2 + y^2 = ||\mathrm{x}||^2 + ||\mathrm{y}||^2$$

7.1 Complex Space

Nate needs a new solution to this one. Post your answers to Piazza :)