EECS351 Discussion 2, 1/17/17

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1 Periodicity of mulitiple sinusoids

Suppose x[n] is the summation of m sinusoids with different fundamental periods:

$$x[n] = cos[\frac{n\pi}{4}] + sin[\frac{n\pi}{32}] + ...$$

Assuming the i^{th} sinusoid has fundamental period T_i, the fundamental period of x[n] is the Least Common Multiple of $(T_1, T_2, ..., T_m)$

$$\text{Let } x[n] = cos[\frac{n\pi}{4}] + sin[\frac{n\pi}{32}]$$

$T_1 = 8$, $T_2 = 64$, thus the Fundamental Period of x[n] is LCM(8,64) = 64

2 What is a basis?

A basis for a class of signals is a collection of M signals in the class that have the property that any other signal in that class can be written as a weighted sum of those signals.

$$y[n] = \sum_{m=1}^{M} a_m x_m[n]$$

Example:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

can represent the following vector using a weighted sum

$$\begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$$

using the weights $a_1 = 1$, $a_2 = 4$, and $a_3 = 9$. In fact, the three vectors form a basis for any signal of length three. These are called the standard basis or canonical basis vectors for $\mathbb{R}^3$.

3 Strategies for identifying a basis

One way to characterize a basis for signals of length N:

$$\text{y[n]} = \sum_{m=1}^{N-1} \beta_m y_m[n] = 0 \text{ implies that } \beta_k = 0 \text{ for all m} = 0, \ldots, \text{N - 1}.$$

If the condition above is true, then the vectors $y_1, ..., y_{N-1}$ are *linearly independent*. If any combination of weighted sum of y_m add up to 0, then the vectors $y_1, ..., y_{N-1}$ are *linearly dependent*

Example:

$$\begin{bmatrix} 1 & 3 & -4 \\ 1 & 9 & -10 \\ 0 & 8 & -8 \end{bmatrix}$$

If the i^{th} column represents the vector y_i, do these vectors represent a valid basis?
NO! using the weights $\beta_1 = \beta_2 = \beta_3 = 1$, we can see:

$$\text{y[n]} = \sum_{m=1}^{N-1} \beta_m y_m[n] = 0$$

4 Identifying a basis by forming the canonical basis

A collection of signals consitute a basis for $\mathbb{R}^n$ if one can form the n canonical basis vectors as a weighted sum of the collection.

Example:
Do the columns of A constitute a basis for $\mathbb{R}^3$?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

We must show that we can form the 3 canonical basis vectors for $\mathbb{R}^3$ as a weighted sum of the columns of A. Recall the canonical basis vectors for $\mathbb{R}^3$:

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Solution:

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = 1 * column1 + 0 * column2 + (-1) * column3$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix} = (-2) * column1 + 1 * column2 + 0 * column3$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix} = 0 * column1 + 0 * column2 + 1 * column3$$

5 Problems

5.1 Basis Problem

Consider the Hilbert Space $\mathbb{R}^3$ with real scalars. Find a basis for $\mathbb{R}^3$ that includes the following two vectors:

$$\begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\4\\-2\end{bmatrix}$$

5.2 Show vectors form a basis

Show that these four vectors form a basis for $\mathbb{R}^4$:

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

5.3 Orthonormal Basis

Show that these four vectors form an orthonormal basis for $\mathbb{R}^4$ (these are called the Bell basis):

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

5.4 Finding vectors that DONT form a Basis

What values of α make the following vectors an invalid basis?

$$\begin{bmatrix} \alpha \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \alpha \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \alpha \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ \alpha \end{bmatrix}$$

5.5 Functions as a basis

Consider the Hilbert space of continuous-time real-valued polynomials of degree 3 defined on the interval [a,b]. Show that the polynomials

$$x_0(t) = 1, x_1(t) = t, x_2(t) = t^2, x_3(t) = t^3$$

form a basis for this space. Is the orthonormal basis where we use the same inner product as L_2 and a = -1, b = -1?

6 Representing the same signal with different bases

Let x[n] =[$\underline{1}$, 1, 2, 3, 5]. Represent x[n] in each of the following bases.

6.1 Standard Basis

6.2 Unit Step Sequence

Recall $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$. Let $\mathcal{S}_u = \{u[n - n_0] | n_0 \in \mathbb{Z}\}$ (unit steps shifted by any integer n_0) be your basis sequences. Represent $x[n]$ in $\mathcal{S}_u$.

In other words, show you can write $x[n] = \sum_{k=-\infty}^{\infty} c_k u[n-k]$ by finding the values for c_k.

6.3 Three-Tap Rectangles Basis

Let $r[n] = \{\underline{1}, 1, 1\}$ and $\mathcal{S}_r = \{r[n-n_0] | n_0 \in \mathbb{Z}\}$. Represent $x[n]$ in $\mathcal{S}_r$.

7 Vector Spaces, Hilbert Spaces, and Inner Products

7.1 Inner Products

Prove the Pythagorean theorem for norms defined by inner products. That is, if $\mathrm{z} = \mathrm{x} + \mathrm{y}$ and $\mathrm{x} \perp \mathrm{y}$, then $||\mathrm{x}||^2 + ||\mathrm{y}||^2 = ||\mathrm{z}||^2$