EECS351 Discussion 7 Problems, 10/26/16

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1 Linear Constant Coefficient Difference Equations (LCCDE)

LCCDE equations are a very important part of DSP. These equations give us a lot of intuition about our system in a short amount of time. We can use our knowledge of the DTFT to analyze difference equations, but we will also introduce the z-transform which is a simply a more general transform.

LCCDE equations have the form:

$$\sum_{k=0}^{N} \alpha_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

This might look complicated at first sight, but let's write the above expression in an expanded fashion:

$\alpha_0 y[n-0] + \alpha_1 y[n-1] + \alpha_2 y[n-2] + ... \alpha_N y[n-N] = b_0 x[n-0] + b_1 x[n-1] + b_2 x[n-2] + ... b_M y[n-M]$

2 Finding the Transfer Function from LCCDE

Using our knowledge of the DTFT, we can find an expression for H(ω):

$$Y(\omega) = X(\omega)H(\omega)$$

$$Y(\omega)/X(\omega) = H(\omega)$$

In the next problem, we will explore why it is so easy to find Y(ω)/X(ω) for LCCDE.

3 Finding Transfer Functions

Solve for H(ω) for the following LCCDE by first taking the DTFT of the left and right hand side respectively:

(a) y[n] = x[n-1]
(b) 2y[n] + 4y[n-4] = x[n] + 3x[n-1]
(c) $\alpha_0 y[n-0] + \alpha_1 y[n-1] + ... \alpha_N y[n-N] = b_0 x[n-0] + b_1 x[n-1] + b_2 x[n-2] + ... b_M y[n-M]$

4 Z-transform equations

$$X(z) = \sum_{\tau=-\infty}^{\infty} x[n]z^{-n}$$

$$x[n] = 1/(2\pi j)\int_C X(z)z^{n-1}dz$$

Note:
The inverse z-transform can be calculated by computing the integral above over any closed contour within the Region of Convergence containing the origin. Typically we never calculate the inverse z-transform this way. Most often, you will use an inverse z-transform pair, or calculate the partial fraction expansion to obtain the x[n].

Find H(z) for the equations from part 3. HINT: Z(x[n-n_0]) has a z-transform of z^{-n_0}X(z)
(a) y[n] = x[n-1]
(b) 2y[n] + 4y[n-4] = x[n] + 3x[n-1]
(c) $\alpha_0 y[n-0]+\alpha_1 y[n-1]+...\alpha_N y[n-N] = b_0 x[n-0]+b_1 x[n-1]+b_2 x[n-2]+...b_M y[n-M]$

5 Region of Convergence (ROC)

$$X(z) = \sum_{\tau=-\infty}^{\infty} x[n]z^{-n}$$

For X(z) to be a valid transform, we need:

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^n < \infty$$

The above equations specify what the Region of Convergence is, but typically we can determine the region of convergence using LCCDE equation rules:

1. The ROC is a connected region with circular symmetry. If z0 is in the ROC, then so is any z such that |z| = |z0|.
2. The ROC for a finite-support sequence is the entire complex plane. There are possible exceptions that zero or infinity may not be included in the ROC.
3. The ROC for a causal sequence extends out to infinity. This is also true in general for a "right-sided sequence," one whose support begins at some finite value and continues to the right.
4. The ROC for an anticausal sequence is a disk in the complex plane. This is also true in general for a "left-sided sequence," one whose support begins at negative infinity and ends at some finite value.
5. The ROC for a stable system includes the unit circle.

6 ROC problem using partial fractions

Recall H(ω) = Y(ω)/X(ω). In this problem, we willl see why the partial fraction expansion is a nifty tool for calculating the impulse response from an inverse Z-transform. Recall the method of partial fractions expansion:

$$\frac{A_0}{(1-a_1z^{-1})(1-a_2z^{-1})...(1-a_nz^{-1})}$$

We want to split the above term into its partial fractions, such that sum of the partial fractions equals our original term

$$\frac{A_1}{(1-a_1z^{-1})} + \frac{A_2}{(1-a_2z^{-1})} + ...\frac{A_n}{(1-a_nz^{-1})} = \frac{A_0}{(1-a_1z^{-1})(1-a_2z^{-1})...(1-a_nz^{-1})}$$

Using partial fractions expansion, determine the impulse response h[n] associated with the following Z-transform. Also draw the pole-zero plot and the ROC:

$$\text{H(z)} = \frac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})} \qquad |\text{z}| > 1/2$$

7 Connection to the Fourier Transform

The DTFT is not always defined. Recall one of the conditions for a discrete sequence to have a DTFT:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Unfortunately x[n] is not always absolutely summable, so we need something general that yields a valid transform for a greater variety of systems.

$$X(\omega) = \sum_{\tau=-\infty}^{\infty} x[n]e^{-j\omega n}$$
$$X(z) = \sum_{\tau=-\infty}^{\infty} x[n]z^{-n}$$

Notice, if we let z = $e^{j\omega}$ then X(z) = X(ω). This means that we are choosing z to be N equally spaced points around the unit circle. And to calculate the inverse z transform, we use the unit circle as our contour in the inverse equation.

8 Extra Problems

(a) Why is a system stable when the unit circle is contained in the ROC? Does a valid DTFT exist when the unit circle is in the ROC?
(b) Solve for the partial fraction expansion for the following:

$$\frac{s+3}{s(s+2)^2(s+5)}$$

(c) Show graphically why: δ[n] - h$_{lp}$[n] = h$_{hp}$[n]. Hint: What is the DTFT of δ[n]