Boolean Functions

1 Identity

1.1 Definition

$$I[b] \overset{def}{\rightarrow} b$$

1.2 Explicit Definition

b ∈ 𝕋

$$I[b] \rightarrow b \in \mathbb{T}$$

b ∈ 𝔽

$$I[b] \rightarrow b \in \mathbb{F}$$

1.3 Domain

$$\{b \mid b \equiv bool\}$$
$$\overset{def}{=}$$
$$\{b \mid |b| \in \{\mathbb{T}, \mathbb{F}\}\}$$

2 Not ¬

2.1 Definition

$$\neg[b] \overset{def}{\rightarrow} b_\perp$$

2.2 Explicit Definition

b ∈ 𝕋

$$\neg[b] \rightarrow b_\perp \in \mathbb{F}$$

b ∈ 𝔽

$$\neg[b] \rightarrow b_\perp \in \mathbb{T}$$

2.3 Domain

$$\{b \mid b \equiv bool\}$$
$$\overset{def}{=}$$
$$\{b \mid |b| \in \{\mathbb{T}, \mathbb{F}\}\}$$

Logical Functions

3 Buffer

3.1 Definition

$$\rhd[|b|] \overset{def}{\to} |b|$$

3.2 Explicit Definition

b $\in \mathbb{T}$

$$\rhd[|b|] \to t$$

b $\in \mathbb{F}$

$$\rhd[|b|] \to f$$

3.3 Domain

$$\{b \mid b \equiv bool\}$$
$$\overset{def}{=}$$
$$\{b \mid |b| \in \{\mathbb{T}, \mathbb{F}\}\}$$

4 Inverter !

4.1 Definition

$$!\,[|b|] \overset{def}{\to} |b_{\perp}|$$

4.2 Explicit Definition

b $\in \mathbb{T}$

$$! \, [|b|] \rightarrow f$$

b $\in \mathbb{F}$

$$! \, [|b|] \rightarrow t$$

4.3 Domain

$$\{b \mid b \equiv bool\} \stackrel{def}{=} \{b \mid |b| \in \{\mathbb{T}, \mathbb{F}\}\}$$

5 Comparisons

5.1 Definition of a Boolean Comparison

$$f[x_1, x_2] :$$
$$f[x_1, x_2] \rightarrow b \in \{t, f\} \quad \forall x_1, x_2 \in \mathbb{D}_f$$

5.2 Abstraction Notation

$$f[input_1, input_2] :$$
$$(f \equiv function) \wedge (f.input_1, f.input_2 \in \mathbb{D}_f) \wedge (f.out \equiv bool)$$

6 ==

6.1 Definition

$$(a == b) = (== [a, b]) :=$$

a = b

$$a == b \rightarrow \mathbb{T}$$

a $\neq$ b

$$a == b \rightarrow \mathbb{F}$$

6.2 Domain

$$\{a \mid a \in \{\mathbb{T}, \mathbb{F}\}\}$$
$$\{b \mid b \in \{\mathbb{T}, \mathbb{F}\}\}$$

7 $\vee$

7.1 Definition

$$a \vee b = \vee[a, b] :=$$

a = b = $\mathbb{T}$

$$a \vee b = \mathbb{T}$$

a = b = $\mathbb{F}$

$$a \vee b = \mathbb{F}$$

$\mathbb{T} = a \neq b = \mathbb{F}$

$$a \vee b = \mathbb{T}$$

$\mathbb{F} = a \neq b = \mathbb{T}$

$$a \vee b = \mathbb{T}$$

7.2 Domain

$$\{a \mid a \in \{\mathbb{T}, \mathbb{F}\}\}$$
$$\{b \mid b \in \{\mathbb{T}, \mathbb{F}\}\}$$

8 $\wedge$

8.1 Definition

$$a \wedge b = \wedge[a, b] :=$$

a = b = $\mathbb{T}$

$$a \wedge b = \mathbb{T}$$

a = b = $\mathbb{F}$

$$a \wedge b = \mathbb{F}$$

$\mathbb{T} = a \neq b = \mathbb{F}$

$$a \wedge b = \mathbb{F}$$

$\mathbb{F} = a \neq b = \mathbb{T}$

$$a \wedge b = \mathbb{F}$$

8.2 Domain

$$\{a \mid a \in \{\mathbb{T}, \mathbb{F}\}\}$$
$$\{b \mid b \in \{\mathbb{T}, \mathbb{F}\}\}$$

9 $>$

9.1 Definition

$$\begin{aligned} a > b \ &= \ > [a, b] := \\ 0 > 1 \ &= \ > [0, 1] \to \mathbb{F} \\ 1 > 0 \ &= \ > [1, 0] \to \mathbb{T} \end{aligned}$$

9.2 Domain

$$\{a \mid a \in \{0, 1\}\}$$
$$\{b \mid b \in \{0, 1\}\}$$

10 $<$

10.1 Definition

$$
\begin{aligned}
a < b &= < [a, b] := \\
0 < 1 &= < [0, 1] \rightarrow \mathbb{T} \\
1 < 0 &= < [1, 0] \rightarrow \mathbb{F}
\end{aligned}
$$

10.2 Domain

$$
\{a \mid a \in \{0, 1\}\}
$$
$$
\{b \mid b \in \{0, 1\}\}
$$

11 Prove all comparators can be expressed with $==,<,>$

12 $\cup$

12.1 Definition

$$a \cup b = \cup[a, b] :=$$

$a = b \neq \varnothing$

$$a \cup a = b \cup b \to \{a\} = \{b\}$$

$\varnothing \neq a \neq b \neq \varnothing$

$$a \cup b \to \{a, b\}$$

$\varnothing = a \neq b$

$$\varnothing \cup b \to \{b\}$$

$a \neq b = \varnothing$

$$a \cup \varnothing \to \{a\}$$

$a = b = \varnothing$

$$\varnothing \cup \varnothing \to \varnothing$$

12.2 Domain

$$\{a \mid a \in \Omega\}$$
$$\{b \mid b \in \Omega\}$$

13 $\setminus$

13.1 Definition

$$a \setminus b = \setminus[a, b] :=$$

$a \neq b = \varnothing$

$$a \setminus b \to a$$

$a = b$

$$a \setminus b \to \varnothing$$

$\varnothing = a \neq b$

$$\text{Undefined}$$

13.2 Domain

$$\{a \mid a \in \Omega\}$$

$$\{b \mid b \subseteq a\}$$

14 $\cap$

14.1 Definition

$$a \cap b = \cap[a, b] :=$$

a = b

$$a \cap a = b \cap b \to \{a\} = \{b\}$$

$a \neq b$

$$a \cap b \to \varnothing$$

14.2 Domain

$$\{a \mid a \in \Omega\}$$

$$\{b \mid b \in \Omega\}$$

15 Cardinality | |

15.1 Definition

$$|S| = |\ |\text{[S]:=}$$

$S = \varnothing$

$$|S| \rightarrow 0$$

$S = \{s_1\}$

$$|S| = |\{s_1\}| \rightarrow 1$$

$S = \{s_1, s_2, ..., s_N\}$

$$|S| = |\{s_1, s_2, ..., s_N\}| \rightarrow N$$

15.2 Domain

$$\{S \mid S \subset \Omega\}$$

16 Definition of Get

$$"get" = get[a] :=$$

$\exists \doteq a$

$$get[a] \rightarrow a$$

16.1 Domain

$$\{a \mid a \in \Omega\}$$

17 Definition of Assign

$$"set" = set[a] :=$$

$\exists \doteq a$

$$set[a]$$

17.1 Domain

$$\{a \mid a \in \Omega\}$$

18 Definition of Return

19 Definition of New

20 Computational Operations

20.1 Set of Canonical Functions

Define $\mathbb{C}$; the set of canonical functions

$$\mathbb{C} = \{get, \neg, \vee, \wedge, ==, >, <, \cup, \cap, ||, ...\}$$

20.2 Definition of Assignment

Define assignment $\leftarrow$

$$a \leftarrow b \ = \leftarrow [a, b] :=$$
$$a = \varnothing \cup b$$

20.3 Definition of Canonical Instruction

Define canonical instruction c as the assignment of a canonical program $l[X_n]$ to element a

$$c := a \leftarrow l[X_n]$$

20.4 Definition of Computational Instruction

Define computational instruction s as a set of canonical instructions with output a_{2M-1}

$$X_n = \{x_1, x_2, ..., x_n\};$$

$$s[X_n] := \{c_1, c_3, c_5, ..., c_{2M-1}\} \rightarrow a_1 \mid$$
$$c_{2i-1} = a_i \leftarrow l_i[\hat{X}_i] \quad l_i \in \mathbb{C} \quad \forall i \leqslant M$$

$$= \{\{a_1 \leftarrow l[\hat{X}_1]\}, \{a_2 \leftarrow l[\hat{X}_2]\}, ..., \{a_M \leftarrow l[\hat{X}_M]\}\} \rightarrow a_M \mid$$
$$l_i \in \mathbb{C} \quad \forall i$$

Operators

21 Get

21.1 Definition

$$get[b] :=$$
$$get[b] \to b$$

21.2 Set Union $\cup$

21.2.1 Canonical Union $\cup$

Restate the definion of Union $\cup$ for elements $a; b$

$$a \cup b = \cup[a, b] \quad a \in \Omega \;\; b \in \Omega :=$$

$a = b \neq \varnothing$

$$a \cup a = b \cup b \to \{a\} = \{b\}$$

$\varnothing \neq a \neq b \neq \varnothing$

$$a \cup b \to \{a, b\}$$

$\varnothing = a \neq b$

$$\varnothing \cup b \to \{b\}$$

$a \neq b = \varnothing$

$$a \cup \varnothing \to \{a\}$$

$a = b = \varnothing$

$$\varnothing \cup \varnothing \to \varnothing$$

21.2.2 Domain of Canonical Union $\cup$

$$\{a \mid a \in \Omega\}$$
$$\{b \mid b \in \Omega\}$$

21.2.3 Set Theory Definition of Set Union $\cup$

Define the overloaded symbol $\cup$; Set Union

$$A = \{a_1, a_2, ..., a_N\} : a_i \in \Omega \quad \forall a_i \in A$$
$$B = \{b_1, b_2, ..., b_M\} : b_j \in \Omega \quad \forall b_j \in A$$

$$A \cup B := \tilde{A} = \{\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_K\} :$$
$$\tilde{a}_i \in A \vee \tilde{a}_i \in B \quad \forall \tilde{a}_i \in \tilde{A}$$

21.2.4 Domain Set Union

$$\{A \mid A \subseteq \Omega\}$$
$$\{B \mid B \subseteq \Omega\}$$

21.2.5 Overloaded $\cup$

Union is defined distinctly according to the domain of inputs

21.2.6 Computational Defintion of Set Union $\cup$

21.3 Set Minus $\setminus$

21.3.1 Formal Definition

21.3.2 Computational Defintion of Set Minus $\setminus$

21.3.3 Domain

21.4 Set is equal to ==

21.4.1 Formal Definition

21.4.2 Computational Defintion of Set is equal to ==

21.4.3 Domain

21.5 Plus +

21.5.1 Formal Definition

$$N + M :=$$
$$|\{a_1, a_2, ..., a_N\} \cup \{b_1, b_2, ..., b_M\}|$$
$$= |a_1 \cup a_2 \cup ... \cup a_N \cup b_1 \cup b_2 ... \cup b_M|$$

Let $\hat{a}_i = a_i \quad \forall i \leqslant N; \quad \hat{a}_{N+j} = b_j \quad \forall j \leqslant M;$

$$= |\{\hat{a}_1\} \cup \{\hat{a}_2\} \cup ... \cup \{\hat{a}_N\} \cup \{\hat{a}_{N+1}\} \cup \{\hat{a}_{N+2}\} \cup ... \cup \{\hat{a}_{N+M}\}|$$

21.5.2 Computational Instruction Definition of Plus $+$

$$s[X_n] := \{c_1, c_3, c_5, ..., c_{2M-1}\} \rightarrow a_1 \mid$$

$$c_{2i-1} = a_i \leftarrow l_i[\hat{X}_i] \quad l_i \in \mathbb{C} \quad \forall i \leqslant M$$

$$+[N, M] :=$$

$$c_1 = a_1 \leftarrow I_1[1]$$

$$c_3 = a_2 \leftarrow I_2[1]$$

$$c_5 = a_1 \leftarrow a_1 \cup a_2$$

$$c_7 = a_3 \leftarrow I_3[1]$$

$$c_9 = a_1 \leftarrow a_1 \cup a_3$$

$$...$$

$$c_{4(M+N-2)+3} = a_{M+N} \leftarrow I_{M+N}[1]$$

$$c_{4(M+N-2)+5} = a_1 \leftarrow a_1 \cup a_{M+N}$$

$$c_{4(M+N-1)+3} = a_1 \leftarrow |a_1|$$

21.5.3 Domain

$$\{N \mid N \in \mathbb{R}\}$$

$$\{M \mid M \in \mathbb{R}\}$$

21.6 Minus -

21.6.1 Formal Definition

$$N - M := |\{a_1 \cup a_2 \cup ... \cup a_N\} \backslash \{a_1 \cup a_2 \cup ... \cup a_M\}|$$

$$= |a_1 \cup a_2 \cup ... \cup a_N \setminus a_1 \setminus a_2 ... \setminus a_M|$$

$$= |a_{M+1} \cup a_{M+2} \cup ... \cup a_N|$$

Let $\hat{a}_i = a_{M+i} \quad M < i \leqslant N$

$$= |\hat{a}_1 \cup \hat{a}_2 \cup ... \cup \hat{a}_{N-M}|$$

21.6.2 Computational Instruction Definition of Minus -

$$s[X_n] := \{c_1, c_3, c_5, ..., c_{2M-1}\} \rightarrow a_1 \mid$$
$$c_{2i-1} = a_i \leftarrow l_i[\hat{X}_i] \quad l_i \in \mathbb{C} \quad \forall i \leqslant M$$

$$-[N, M] :=$$
$$c_1 = a_1 \leftarrow I_1[1]$$
$$c_3 = a_2 \leftarrow I_2[1]$$
$$c_5 = a_1 \leftarrow a_1 \cup a_2$$
$$c_7 = a_3 \leftarrow I_3[1]$$
$$c_9 = a_1 \leftarrow a_1 \cup a_3$$
$$...$$
$$c_{4(N-2)+3} = a_N \leftarrow I_N[1]$$
$$c_{4(N-2)+5} = a_1 \leftarrow a_1 \cup a_N$$
$$c_{4(N-1)+3} = a_{N+1} \leftarrow I_1[1]$$
$$c_{4(N-1)+5} = a_1 \leftarrow a_1 \setminus a_{N+1}$$
$$c_{4N+3} = a_{N+2} \leftarrow I_2[1]$$
$$c_{4N+5} = a_1 \leftarrow a_1 \setminus a_{N+2}$$
$$...$$
$$c_{4(M+N-2)+3} = a_M \leftarrow I_M[1]$$
$$c_{4(M+N-2)+5} = a_1 \leftarrow a_1 \setminus a_M$$
$$c_{4(M+N-1)+3} = a_1 \leftarrow |a_1|$$

21.6.3 Domain

$$\{a \mid a \in \mathbb{R}\}$$
$$\{b \mid b \in \mathbb{R} \ \wedge b \leqslant a\}$$

22 Compound Instruction Theorem

Any Instruction that can be defined as a set of computational instructions "Compound Instruction" is also a computational instruction

22.1 Definition of Compound Instruction

22.2 $!=$

22.3 $\geqslant$

22.4 $\leqslant$

22.5 $\sum$

22.6 Set Intersection $\cap$

22.6.1 Formal Definition

22.6.2 Computational Defintion of $\cap$

22.6.3 Domain

$$\{A \mid A \in \Omega\}$$
$$\{B \mid B \in \Omega\}$$

22.7 *

22.8 Exponentiation a^b

22.9 /

22.9.1 Definition

22.9.2 Prove / is a Computational Instruction

22.10 N^{th} root $\sqrt[n]{a}$

22.10.1 Definition

22.10.2 Prove $\sqrt[n]{a}$ is a Computational Instruction

Appendix

23 Table of Boolean Functions [1]

Generated by chat gpt

<table>
<tr><th>Title</th><th>Expression</th><th>Domain</th><th>Output</th></tr>
<tr><td>Logical NOT</td><td>$\neg A$</td><td>$A \in \{F, T\}$</td><td><table><tr><th>A</th><th>$\neg A$</th></tr><tr><td>F</td><td>T</td></tr><tr><td>T</td><td>F</td></tr></table></td></tr>
<tr><td>Logical Identity</td><td>A</td><td>$A \in \{F, T\}$</td><td><table><tr><th>A</th><th>A</th></tr><tr><td>F</td><td>F</td></tr><tr><td>T</td><td>T</td></tr></table></td></tr>
<tr><td>Logical AND</td><td>$A \wedge B$</td><td>$A, B \in \{F, T\}$</td><td><table><tr><th>A</th><th>B</th><th>$A \wedge B$</th></tr><tr><td>F</td><td>F</td><td>F</td></tr><tr><td>F</td><td>T</td><td>F</td></tr><tr><td>T</td><td>F</td><td>F</td></tr><tr><td>T</td><td>T</td><td>T</td></tr></table></td></tr>
<tr><td></td><td>$A \wedge \neg B$</td><td>$A, B \in \{F, T\}$</td><td><table><tr><th>A</th><th>B</th><th>$A > B$</th></tr><tr><td>F</td><td>F</td><td>F</td></tr><tr><td>F</td><td>T</td><td>F</td></tr><tr><td>T</td><td>F</td><td>T</td></tr><tr><td>T</td><td>T</td><td>F</td></tr></table></td></tr>
<tr><td>$f[A, B]$</td><td>A</td><td>$A, B \in \{F, T\}$</td><td><table><tr><th>A</th><th>B</th><th>A</th></tr><tr><td>F</td><td>F</td><td>F</td></tr><tr><td>F</td><td>T</td><td>F</td></tr><tr><td>T</td><td>F</td><td>T</td></tr><tr><td>T</td><td>T</td><td>T</td></tr></table></td></tr>
</table>

<table>
<tr><td></td><td>$\neg A \wedge B$</td><td>$A, B \in \{F, T\}$</td><td>$\begin{array}{c|c|c} A & B & A < B \\ \hline F & F & F \\ F & T & T \\ T & F & F \\ T & T & F \end{array}$</td></tr>
<tr><td>$f[A, B]$</td><td>B</td><td>$A, B \in \{F, T\}$</td><td>$\begin{array}{c|c|c} A & B & B \\ \hline F & F & F \\ F & T & T \\ T & F & F \\ T & T & T \end{array}$</td></tr>
<tr><td>XOR</td><td>$A \oplus B$</td><td>$A, B \in \{F, T\}$</td><td>$\begin{array}{c|c|c} A & B & A \oplus B \\ \hline F & F & F \\ F & T & T \\ T & F & T \\ T & T & F \end{array}$</td></tr>
<tr><td>OR</td><td>$A \vee B$</td><td>$A, B \in \{F, T\}$</td><td>$\begin{array}{c|c|c} A & B & A \vee B \\ \hline F & F & F \\ F & T & T \\ T & F & T \\ T & T & T \end{array}$</td></tr>
<tr><td>NOT OR (NOR)</td><td>$\neg(A \vee B)$</td><td>$A, B \in \{F, T\}$</td><td>$\begin{array}{c|c|c} A & B & \neg(A \vee B) \\ \hline F & F & T \\ F & T & F \\ T & F & F \\ T & T & F \end{array}$</td></tr>
<tr><td>XNOR</td><td>$A \leftrightarrow B$</td><td>$A, B \in \{F, T\}$</td><td>$\begin{array}{c|c|c} A & B & A \leftrightarrow B \\ \hline F & F & T \\ F & T & F \\ T & F & F \\ T & T & T \end{array}$</td></tr>
</table>

$f[A,B]$	$\neg B$	$A,B \in \{F,T\}$	$\begin{array}{c\|c\|c} A & B & \neg B \\ \hline F & F & T \\ F & T & F \\ T & F & T \\ T & T & F \end{array}$
	$A \vee \neg B$	$A,B \in \{F,T\}$	$\begin{array}{c\|c\|c} A & B & A \geqslant B \\ \hline F & F & T \\ F & T & F \\ T & F & T \\ T & T & T \end{array}$
$f[A,B]$	$\neg A$	$A,B \in \{F,T\}$	$\begin{array}{c\|c\|c} A & B & \neg A \\ \hline F & F & T \\ F & T & T \\ T & F & F \\ T & T & F \end{array}$
	$\neg A \vee B$	$A,B \in \{F,T\}$	$\begin{array}{c\|c\|c} A & B & A \leqslant B \\ \hline F & F & T \\ F & T & T \\ T & F & F \\ T & T & T \end{array}$
NOT AND (NAND)	$\neg(A \wedge B)$	$A,B \in \{F,T\}$	$\begin{array}{c\|c\|c} A & B & \neg(A \wedge B) \\ \hline F & F & T \\ F & T & T \\ T & F & T \\ T & T & F \end{array}$
$f[A,B]$	0	$A,B \in \{F,T\}$	$\begin{array}{c\|c\|c} A & B & F \\ \hline F & F & F \\ F & T & F \\ T & F & F \\ T & T & F \end{array}$
$f[A,B]$	1	$A,B \in \{F,T\}$	$\begin{array}{c\|c\|c} A & B & T \\ \hline F & F & T \\ F & T & T \\ T & F & T \\ T & T & T \end{array}$

24 Table of Binary Functions

<table>
<tr><th>Title</th><th>Expression</th><th>Domain</th><th>Output</th></tr>
<tr><td>Is equal to</td><td>$A == B$</td><td>$A, B \in \{0, 1\}$</td><td><table><tr><th>A</th><th>B</th><th>$A == B$</th></tr><tr><td>0</td><td>0</td><td>1</td></tr><tr><td>0</td><td>1</td><td>0</td></tr><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table></td></tr>
<tr><td>Is NOT equal to</td><td>$A \,!= B$</td><td>$A, B \in \{0, 1\}$</td><td><table><tr><th>A</th><th>B</th><th>$A \,!= B$</th></tr><tr><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>1</td></tr><tr><td>1</td><td>1</td><td>0</td></tr></table></td></tr>
<tr><td>Greater than</td><td>$A > B$</td><td>$A, B \in \{0, 1\}$</td><td><table><tr><th>A</th><th>B</th><th>$A > B$</th></tr><tr><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>0</td></tr><tr><td>1</td><td>0</td><td>1</td></tr><tr><td>1</td><td>1</td><td>0</td></tr></table></td></tr>
<tr><td>Greater than or equal to</td><td>$A \geqslant B$</td><td>$A \in \{0, 1\}$</td><td><table><tr><th>A</th><th>B</th><th>$A \geqslant B$</th></tr><tr><td>0</td><td>0</td><td>1</td></tr><tr><td>0</td><td>1</td><td>0</td></tr><tr><td>1</td><td>0</td><td>1</td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table></td></tr>
<tr><td>Less than</td><td>$A < B$</td><td>$A, B \in \{0, 1\}$</td><td><table><tr><th>A</th><th>B</th><th>$A < B$</th></tr><tr><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td><td>0</td></tr></table></td></tr>
<tr><td>Less than or equal to</td><td>$A \leqslant B$</td><td>$A, B \in \{0, 1\}$</td><td><table><tr><th>A</th><th>B</th><th>$A \leqslant B$</th></tr><tr><td>0</td><td>0</td><td>1</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table></td></tr>
</table>

25 Definition of is Boolean

$$is\ boolean(f) \to b \in \{\mathbb{T}, \mathbb{F}\} :=$$
$$(f.out \subseteq \{\mathbb{T}, \mathbb{F}\}) \to b \in \{\mathbb{T}, \mathbb{F}\}$$

25.1 Domain

$$\{f | f \equiv function\}$$

26 Citations

[1] chatgpt.com