Math

1 Definition of a line $\mathbb{L}$

1.1 Definition

$$\mathbb{L} := \{l_i, ...$$
$$\exists l_i \in \mathbb{L} \quad \forall i$$

2 Whole Numbers $\mathbb{W}$

2.1 Definition

Define the positive number line $\mathbb{W}$, the set of whole numbers

$$\mathbb{W} :=$$
$$|S_i| \in \mathbb{W} \quad \forall i$$

3 Injection (one-to-one)

3.1 Definition[1]

$$S_1 = \{a_1, a_2, ..., a_n\}; \;\; S_2 = \{b_1, b_2, ..., b_m\} \;\; m \geqslant n$$
$$injective[S_1, S_2] \rightarrow (\forall a_i \in S_1 \;\; \exists b_j \in S_2 : f[a_i] \rightarrow b_j)$$
$$\forall a_i, a_j \in S_1; f[a_i] = f[a_j] \Longleftrightarrow a_i = a_j \;\; [2]$$

4 Surjection (onto)

4.1 Definition [2]

$$S_1 = \{a_1, a_2, ..., a_n\}; \;\; S_2 = \{b_1, b_2, ..., b_m\} \;\; m \geqslant n$$
$$surjective[S_1, S_2] \rightarrow (\forall a_i \in S_1 \;\; \exists b_j \in S_2 : f[a_i] \rightarrow b_j)$$

5 Bijection (One-to-one and onto)

Also known as invertible

5.1 Definition [3]

$$S_1 = \{a_1, a_2, ..., a_n\};\ \ S_2 = \{b_1, b_2, ..., b_n\}$$
$$Invertible[S_1, S_2] \rightarrow$$
$$(\forall a_i \in S_1\ \ \exists b_j \in S_2 : f[a_i] \rightarrow b_j) \wedge (\forall b_j \in S_2\ \ \exists a_i \in S_1 : g[b_j] \rightarrow a_i)$$

6 Hierarchy of Bijections Surjections and Injections

7 Vector

Define vector as a set of more than one cardinalities

7.1 Definition

$$\vec{v} :=$$
$$|S_1| = x_1; |S_2| = x_2; ...; |S_N| = x_N$$
$$\vec{v} = \{|S_1|, |S_2|, ..., |S_N|\} =< x_1, x_2, ..., x_N >$$

7.2 Dimensionality

$$dim[\vec{v}] = |\vec{v}| :=$$
$$\vec{v} = \{|S_1|, |S_2|, ..., |S_N|\} =< x_1, x_2, ..., x_N >$$
$$|\vec{v}| = N$$

8 Vector Space

Define Vector Space **V** as a set of more than one lines

8.1 Definition

$$\mathbf{V} := \{\mathbb{L}_1, \mathbb{L}_2, ..., \mathbb{L}_N\}$$

9 Spans

A function/system mapping to a vector space

9.1 Bijective Span

9.2 Injective Span

9.3 Surjective Span

10 Summation Notation

Define summation $\sum$; the notation for successive additions

$$\textstyle\sum_{i=1}^{N} a :=$$
$$\textstyle\sum_{i=1}^{N} a = a + \sum_{i=2}^{N} a =$$
$$\textstyle a + a + \sum_{i=3}^{N} a = ... = a + a + ... + a = a * N$$

11 Definition of Series

Define series

$$series :=$$
$$\textstyle\sum_{i=1}^{N} x_i = x_1 + x_2 + ... + x_N$$

11.1 Discrete Derivative

Define Derivative for discrete function f[n]; commonly denoted as a difference function

$$\Delta_n^1 f[n] := f[n+1] - f[n]$$

We will use the above definition for the remainder of this document

11.1.1 Left Hand Derivative Definition

$$\Delta_{n_l}^1 f[n] = f[n] - f[n-1]$$

11.2 Zero Order Derivative

$$\Delta_n^0 f[n] := f[n]$$

11.3 Kth Discrete Derivative

Define the K^{th} derivative of discrete function $f[n]$

$$\Delta_n^K f[n] := \Delta_n^{K-1} f[n+1] - \Delta_n^{K-1} f[n]$$

11.4 Kth Discrete Derivative as an Alternating Sum

$$\Delta_n^K f[n] := \Delta_n^{K-1} f[n+1] - \Delta_n^{K-1} f[n]$$

$$= (\Delta_n^{K-2} f[n+2] - \Delta_n^{K-2} f[n+1]) - (\Delta_n^{K-2} f[n+1] - \Delta_n^{K-2} f[n])$$

$$= (\Delta_n^{K-2} f[n+2] - 2\Delta_n^{K-2} f[n+1] - \Delta_n^{K-2} f[n])$$

$$= \textstyle\sum_{i=0}^{K} (-1)^j \ ({}_K C_j) \ \Delta_n^0 f[n+j]$$

$$= \textstyle\sum_{i=0}^{K} (-1)^j \ ({}_K C_j) \ f[n+j]$$

11.5 Z Transform

Define the Z Transform for discrete function f[n]

$$\mathcal{Z}(f[n]) := \textstyle\sum_{n=0}^{\infty} f[n] z^{-n}$$

11.6 Z Transform of 0 Order Derivative

$$\Delta_n^0 f[n] := f[n]$$

$$\mathcal{Z}(\Delta_n^0 f[n]) = \mathcal{Z}(f[n])$$

11.7 Z Transform of 1st Derivative

$$\Delta_n^1 f[n] := f[n+1] - f[n]$$

$$\mathcal{Z}(\Delta_n^1 f[n]) = \mathcal{Z}(f[n+1] - f[n])$$

$$= \textstyle\sum_{n=0}^{\infty} (f[n+1] - f[n]) z^{-n}$$

$$= \textstyle\sum_{n=0}^{\infty} (f[n+1] z^{-n} - f[n] z^{-n})$$

$$= \textstyle\sum_{n=0}^{\infty} f[n+1] z^{-n} - \sum_{n=0}^{\infty} f[n] z^{-n}$$

$$= \textstyle\sum_{m=0}^{\infty} f[m+1] z^{-m} - \sum_{n=0}^{\infty} f[n] z^{-n}$$

Let

$$\hat{m} = m + 1; \;\; m = \hat{m} - 1$$

$$= \sum_{m=0}^{\infty} f[\hat{m}] z^{-(\hat{m}-1)} - \mathcal{Z}(f[n])$$

$$= z^1 \sum_{\hat{m}=1}^{\infty} f[\hat{m}] z^{-\hat{m}} - \mathcal{Z}(f[n])$$

$$= z^1 \sum_{\hat{m}=1}^{\infty} f[\hat{m}] z^{-\hat{m}} + f[0] - f[0] - \mathcal{Z}(f[n])$$

$$= z^1 \sum_{\hat{m}=0}^{\infty} f[\hat{m}] z^{-\hat{m}} - f[0] - \mathcal{Z}(f[n])$$

$$= z^1 \mathcal{Z}(f[n]) - f[0] - \mathcal{Z}(f[n])$$

$$\mathcal{Z}(\Delta_n^1 f[n]) = \mathcal{Z}(f[n])(z^1 - 1) - f[0]$$

11.8 Z Transform of Kth Derivative

$$\mathcal{Z}(f[n]) := \sum_{n=0}^{\infty} f[n] z^{-n}$$

$$\mathcal{Z}(\Delta_n^K f[n]) = \sum_{n=0}^{\infty} \Delta_n^K f[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{K} (-1)^j \; ({}_K C_j) \; f[n+j] z^{-n}$$

$$= \sum_{n=0}^{\infty} (f[n+K] - ({}_K C_1) f[n+K-1] + ({}_K C_2) f[n+K-2] - ... \pm f[n]) z^{-n}$$

$$= \sum_{n=0}^{\infty} f[n+K] z^{-n} - ({}_K C_1) f[n+K-1] z^{-n} + ({}_K C_2) f[n+K-2] z^{-n} - ... \pm f[n] z^{-n}$$

$$= z^K \mathcal{Z}(f[n]) + \sum_{i=0}^{K-1} f[i] - ({}_K C_1) z^{K-1} \mathcal{Z}(f[n]) - \sum_{j=0}^{K-2} f[j] + ({}_K C_2) z^{K-2} \mathcal{Z}(f[n]) + \sum_{k=0}^{K-3} f[k] - ... \pm \mathcal{Z}(f[n])$$

When K is odd

$$\mathcal{Z}(\Delta_n^K f[n]) = (z-1)^K \mathcal{Z}(f[n]) + \sum_{i=0}^{\frac{n+1}{2}} f[2i] \quad K > 0$$

When K is even

$$\mathcal{Z}(\Delta_n^K f[n]) = (z-1)^K \mathcal{Z}(f[n]) + \sum_{j=0}^{\frac{n}{2}} f[2j+1] \quad K > 0$$

12 Convergent Functions

12.1 Definition of Converges to

$$f[n] \; converges \; to \; C = convergent[f[n], C] = a_o; \; a_o \in \{\mathbb{T}, \mathbb{F}\} =$$

$$|C - f[n+1]| < |C - f[n]| \;\; \forall n$$
$$\wedge$$
$$\nexists K : |C - f[\hat{n}]| > K \quad \forall n; K > 0$$

12.1.1 Notation

C is commonly denoted by a limit

$$C = lim_{n\to\infty} f[n]$$

12.2 Definition of General Convergence

$$f[n] \text{ is } convergent = convergent[f[n]] = a_o;\; a_o \in \{\mathbb{T}, \mathbb{F}\} =$$

$$\exists C :$$
$$convergent[f[n], C] == \mathbb{T}$$

Alternatively

$$f[n] \text{ is } convergent = convergent[f[n]] = a_o \in \{\mathbb{T}, \mathbb{F}\} =$$
$$\exists C :$$
$$f[n] \; converges \; to \; C$$

12.2.1 Notation

$$f[n] \text{ is } convergent = convergent[f[n]] = a_o;\; a_o \in \{\mathbb{T}, \mathbb{F}\} =$$

$$\exists lim_{n\to\infty} f[n] :$$
$$f[n] \; converges \; to \; lim_{n\to\infty} f[n]$$

12.3 Increasing Convergence

For strictly increasing functions

$$f[n] = \sum_{i=1}^{n} x_i$$
$$\forall f[n] : convergent[f[n]] \to \mathbb{T}$$
$$\lim_{n\to\infty} f[n] = C :=$$
$$C > f[n] \quad \forall n$$

$$f[n+1] > f[n] \ \ \forall n$$

$$\nexists K : C - \sum_{i=1}^{n} x_i > K \quad \forall n; K > 0$$

12.3.1 Prove Increasing Convergence has General Convergence

12.4 Decreasing Convergence

For strictly decreasing functions

$$f[n] = \sum_{i=1}^{n} x_i$$

$$\forall f[n] : convergent[f[n]] \to \mathbb{T}$$

$$\lim_{n\to\infty} f[n] = C :=$$

$$f[n] > C \quad \forall n$$

$$f[n+1] < f[n] \ \ \forall n$$

$$\nexists K : \sum_{i=1}^{n} x_i - C > K \quad \forall n; K > 0$$

12.4.1 Prove Decreasing Convergence has General Convergence

12.5 Transient Convergence

For alternating functions

12.5.1 Prove Transient Convergence has General Convergence

13 Chaotic Convergence

$$f[n] \ is \ chaotic \ convergent = chaotic \ convergent[f[n]] =$$

$$\exists C, \hat{n} :$$

$$|C - f[n]| > |C - f[n+\hat{n}]| \ \ \forall n$$

13.1 Prove General Convergence implies Chaotic Convergence

13.2 Prove Choatic Convergence does not necessarily imply General Convergence

13.3 Definition of Divergent Function

14 Divergence

14.1 Definition of Divergence

$$diverges[f[n]] = \neg converges[f[n]] = d_o;\ d_o \in \{\mathbb{T}, \mathbb{F}\}$$

$$:= \nexists C : convergent[f[n], C] == \mathbb{T}$$

14.2 Alternate Definition of Divergence

$$diverges[f[n]] = \neg converges[f[n]] = d_o;\ d_o \in \{\mathbb{T}, \mathbb{F}\}$$

$$= convergent[f[n], C] == \mathbb{F} \quad \forall C$$

14.2.1 Proof of Equivalence; Alternate Definition of Divergence

14.3 Necessary and Sufficient Criteria 1 For Divergence

Function f[n] diverges if and only if the K^{th} derivative of f[n] is strictly increasing

$$diverges[f[n]] := \nexists C : convergent[f[n], C] == \mathbb{T}$$

$$\Longleftrightarrow$$

$$\Delta_n^{K+1} f[n] > \Delta_n^K f[n] \quad \forall K$$

Alternatively

$$\Delta_n^{K+1} f[n] - \Delta_n^K f[n] > 0 \quad \forall K$$

$$\Delta_n^{K+2} > 0 \ \forall K$$

14.3.1 Criteria 1; Proof of Necessity and Sufficiency

$$diverges[f[n]] = d_o;\ d_o \in \{\mathbb{T}, \mathbb{F}\}$$

$$= \nexists C : convergent[f[n], C] == \mathbb{T}$$

Let

$$f[n] :$$
$$\Delta_n^{K+2} > 0 \quad \forall K$$

14.4 Necessary and Sufficient Criteria 2 For Divergence

$$f[n]\ is\ Divergent = Divergent[f[n]] = a_o \in \{\mathbb{T}, \mathbb{F}\} :=$$

$$\nexists c : lim_{n \to \infty} \Delta_n^K f[n] = c$$

14.4.1 Criteria 2; Proof of Necessity and Sufficiency

$$diverges[f[n]] = d_o;\ d_o \in \{\mathbb{T}, \mathbb{F}\}$$

$$= \nexists C : convergent[f[n], C] == \mathbb{T}$$

14.5 Verbal Expressions

$$f[n]\ diverges = f[n]\ is\ divergent =$$
$$f[n]\ is\ not\ convergent = f[n]\ does\ not\ converge$$

Citations

[1] $https://en.wikipedia.org/wiki/Bijection,\_injection\_and\_surjection\#Injection$
[2] $https://en.wikipedia.org/wiki/Bijection,\_injection\_and\_surjection\#Surjection$
[3] $https://en.wikipedia.org/wiki/Bijection,\_injection\_and\_surjection\#Bijection$
[4] $https://www.wolframalpha.com$