

Ch. 5 Computation

1 Memory

1.1 Definition

Define Memory; a set of elements

$$\mathcal{M} := \{b_1, b_2, \dots, b_M\}$$

2 Logical Operations

Working list; possibly any set operation/function

+

-

/

exp

\odot

\leftarrow

delete

remove

insert

append

if

==

!

3 Program

3.1 Logical Instructions

Define \mathcal{L} ; an ordered set of logical operations s_i

$$\mathcal{L} := \{s_1, s_2, \dots, s_N\}$$

3.2 State |

Define state; the memory utilized to perform program P

$$P := \{s_1, s_2, \dots, s_N | b_1, b_2, \dots, b_M\} = \\ \{s_1, s_2, \dots, s_N, b_1, b_2, \dots, b_M\}$$

3.3 Boolean Programs

Define a boolean program; boolean programs can represent functions with inputs x_i and boolean output y_o

$$X = \{x_1, \dots, x_n\} \\ P = P[X] := \{s_1, s_2, \dots, s_N | b_1, b_2, \dots, b_M, y_o\} = \\ P[X] \rightarrow y_o \in \{\mathbb{T}, \mathbb{F}\}$$

3.4 Void Programs

Define a void program; a program with inputs x_i and no output

$$X = \{x_1, \dots, x_n\} \\ P = P[X] := \{s_1, s_2, \dots, s_N | b_1, b_2, \dots, b_M\}$$

3.5 Numerical Programs

Define a numerical program; a program with inputs x_i and real, rational output y_o

$$X = \{x_1, \dots, x_n\} \\ P = P[X] := \{s_1, s_2, \dots, s_N | b_1, b_2, \dots, b_M, y_o\} = \\ P[X] \rightarrow y_o \in \mathbb{Q} \quad y_o \geq 0$$

3.6 System Programs

Naming convention to be formalized; a program that outputs one or more elements

$$X = \{x_1, \dots, x_n\} \\ P = P[X] := \{s_1, s_2, \dots, s_N | b_1, b_2, \dots, b_M, Y_o\} = \\ P[X] \rightarrow Y_o = \{y_1, y_2, \dots, y_K\}$$

3.7 Mathematical Programs

Define a mathematical program; a program with inputs x_i and numerical output y_o

$$\begin{aligned} X &= \{x_1, \dots, x_n\} \\ P = P[X] &:= \{s_1, s_2, \dots, s_N \mid b_1, b_2, \dots, b_M, y_o\} = \\ P[X] &\rightarrow y_o \in \mathbb{Q} \end{aligned}$$

4 No-op ;

4.1 Definition

$$; := \emptyset$$

4.2 Property of No-op

No-op can be inserted into any set with equality

$$S = \{s_1, s_2, \dots, s_N\}$$

$$S_; = insert[S, ;, i]$$

$$S_; = S_1 \quad \forall i$$

$$|S_;| = |S| \quad \forall i$$

4.3 Proof

by definition of magnitude of null = 0 with Set And

5 For Loop \mathbb{C}

$\mathbb{C}[\text{startindex}, \text{endindex}, \text{condition}]$

5.1 Definition

6 Nested For Loop \mathbb{C}^n

$\mathbb{C}[\text{startindex}, \text{endindex}, \text{condition1}, \dots, \text{condition n}]$

6.1 Definition

7 Decision Problems

7.1 Definition

Define decision problem; a function with inputs x_i and boolean output "answer" a_o

$$X_i = \{x_1, \dots, x_n\}$$

$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

8 Solutions

8.1 Definition

Program P is a solution s^+ if P outputs answer a_o for all inputs $X_i \quad \forall i$
 s^+ is a function of the number of inputs n

$$X_i = \{x_1, \dots, x_n\}$$

$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$s^+ = s^+[n] := P[X_i] \rightarrow y_o : y_o = a_o \quad \forall X_i$$

$$P[X_i] = \{s_1, s_2, \dots, s_N | b_1, b_2, \dots, b_M, y_o\}$$

$$s^+ = P[X_i] = \{s_1, s_2, \dots, s_{O_T[n]}, b_1, b_2, \dots, b_{O_S[n]}, y_o\} \quad \forall X_i$$

8.1.1 Property of No-op ;

No-op ; can be added to any solution S_i and remain a solution for all i
 anywhere in the order for all j

$$s^+ = \{s_1, s_2, \dots, s_{O_T[n]}, b_1, b_2, \dots, b_{O_S[n]}, y_o\}$$

$$\hat{s}^+ = insert[s^+, ;, k]$$

$$\hat{s}^+ = s^+ \quad \forall k$$

8.2 Definition of S^+

Define S^+ ; the set of solutions to decision problem D

$$X_i = \{x_1, \dots, x_n\}$$

$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$s_j^+ := P[X_i] \rightarrow y_o : y_o = a_o \quad \forall X_i$$

$$S^+ := \{s_j^+, \dots\} \quad \forall j$$

8.3 Definition of Solvable

Define solvable

$$\begin{aligned} X_i &= \{x_1, \dots, x_n\} \\ D &:= f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i \\ \text{solvable} &:= \text{solvable}[D] \rightarrow b_o \in \{\mathbb{T}, \mathbb{F}\} = \\ \exists s^+ : s^+ &= P[X_i] \rightarrow y_o : y_o = a_o \quad \forall X_i \end{aligned}$$

9 The set of all Decision Problems \mathbb{D}

9.1 Definition

Define the set of decision problems \mathbb{D}

$$\begin{aligned} X_i &= \{x_1, \dots, x_n\} \\ D_j &:= f_j[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i \\ \mathbb{D} &:= \{D_j, \dots\} \quad \forall j \end{aligned}$$

10 Instruction and Memory Notation

Define \mathcal{L} a set of logical operations

Define \mathcal{M} a set of bits "memory"

$$\begin{aligned} P[X_i] \rightarrow y_o &= \{s_1, s_2, \dots, s_{O_T[n]}, b_1, b_2, \dots, b_{O_S[n]}, y_o\} \\ \mathcal{L} &:= \{s_1, s_2, \dots, s_{O_T[n]}\} \\ \mathcal{M} &:= \{b_1, b_2, \dots, b_{O_S[n]}\} \\ P[X_i] &= \{\mathcal{L}, \mathcal{M}, y_o\} \end{aligned}$$

11 Complexity

11.1 Time Complexity of a Decision Problem $O_T[n]$

Define Time Complexity $O_T[n]$ of Decision Problem D with solution s^+

$$\begin{aligned}
X_i &= \{x_1, \dots, x_n\} \\
D &:= f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i \\
s^+ &:= P[X_i] \rightarrow y_o : y_o = a_o \quad \forall X_i = \\
\{s_1, s_2, \dots, s_{O_T[n]}, b_1, b_2, \dots, b_{O_S[n]}, y_o\} &= \{\mathcal{L}, \mathcal{M}, y_o\} \\
O_T[n] &:= |\mathcal{L}| = N
\end{aligned}$$

11.2 Space Complexity $O_S[n]$

Define Space Complexity $O_S[n]$ of Decision Problem D with solution s^+

$$\begin{aligned}
X_i &= \{x_1, \dots, x_n\} \\
D &:= f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i \\
s^+ &:= P[X_i] \rightarrow y_o : y_o = a_o \quad \forall X_i = \\
\{s_1, s_2, \dots, s_{O_T[n]}, b_1, b_2, \dots, b_{O_S[n]}, y_o\} &= \{\mathcal{L}, \mathcal{M}, y_o\} \\
O_S[n] &:= |\mathcal{M}| + |y_o| = M
\end{aligned}$$

12 Definition of Complexity

Define Complexity $O[n]$ as a vector of dimension C

$$\mathbf{O}[n] := \langle O_T[n], O_S[n], O_3[n], O_4[n], \dots, O_C[n] \rangle$$

13 Total Complexity

$$O[n] := O_T[n] + O_S[n] + \sum_{i=3}^C O_i[n]$$

14 Simple Computational Complexity

The remainder of this chapter assumes simple computational complexity of dimension 2

14.1 Definition

Define simple computational complexity of dimension 2

$$\mathbf{O}[n] := \langle O_T[n], O_S[n] \rangle$$

14.2 Time Complexity

Restate definition of Time Complexity $O_T[n]$

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_T[n] := |\mathcal{L}| = N$$

14.3 Space Complexity

Restate definition of Time Complexity $O_S[n]$

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_S[n] := |\mathcal{M}| + |y_o| = M$$

14.4 Total Complexity

$$\begin{aligned} O[n] &:= O_T[n] + O_S[n] \\ &= |\mathcal{L}| + |\mathcal{M}| + |y_o| = N + M \end{aligned}$$

14.5 $O_S[n] \neq 0$

14.5.1 Proof

By definition of decision problem; Proof by contradiction; y_o must be set to TF by definition; Suppose $y_o = 0$; then y_o is empty set; contradicts definition of D

14.6 $O_T[n] \neq 0$

14.6.1 Proof

By definition of decision problem; Proof by contradiction; y_o must be set to TF by definition; Suppose $|L| = 0$; $y_o < \text{TF} \cap L$ is null by definition of empty set; implies y_o emptyset (doesnt exist)

14.7 $O[n] = O_T[n] + O_S[n] \neq 0$

14.7.1 Proof

14.8 $O[n] > O_T[n]$

14.8.1 Proof

14.9 $O[n] > O_S[n]$

14.9.1 Proof

14.10 $O_{opt}[n] > O_{T_{opt}}[n]$

14.10.1 Proof

14.11 $O_{opt}[n] > O_{S_{opt}}[n]$

14.11.1 Proof

14.12 $O_{opt}[n] < O_{S_{opt}}[n] + O_{T_{opt}}[n]$

14.12.1 Proof

15 Optimal Complexity

15.1 Definition

Define Optimal Complexity; the minimum total complexity required to solve a decision problem

$$O_{opt}[n] := \# \hat{O}[n] : \hat{O}[n] < O_{opt}[n] \quad \forall n$$

15.2 Proof of Existence

Prove the existence of at least one $O_{min}[n]$ by induction/contradiction

16 Optimal solution

Define an optimal solution s_{opt}^+

16.1 Definition

$$\begin{aligned} X_i &= \{x_1, \dots, x_n\} \\ D_j &:= f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i \\ s^+ &:= P[X_i] \rightarrow y_o : y_o = a_o \quad \forall X_i \\ s_{opt}^+ &:= s^+ : \\ \# \hat{O}[n] &< O_{opt}[n] \quad \forall n, s^+ \in S_j^+ \end{aligned}$$

16.2 Optimal Time Complexity Solution

$$\begin{aligned} X_i &= \{x_1, \dots, x_n\} \\ D_j &:= f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i \\ s^+ &:= P[X_i] \rightarrow y_o : y_o = a_o \quad \forall X_i = \\ \{s_1, s_2, \dots, s_{O_T[n]}, b_1, b_2, \dots, b_{O_S[n]}, X_i, y_o\} &= \{\mathcal{L}, \mathcal{M}, X_i, y_o\} \\ O_T[n] &:= |\mathcal{L}| = N \\ s_T^+ &:= s^+ : \\ \# \hat{O}_T[n] &< O_T[n] \quad \forall n, s^+ \in S_j^+ \end{aligned}$$

16.3 Optimal Space Complexity Solution

$$\begin{aligned}
X_i &= \{x_1, \dots, x_n\} \\
D_j &:= f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i \\
s^+ &:= P[X_i] \rightarrow y_o : y_o = a_o \quad \forall X_i = \\
\{s_1, s_2, \dots, s_{O_T[n]}, b_1, b_2, \dots, b_{O_S[n]}, X_i, y_o\} &= \{\mathcal{L}, \mathcal{M}, X_i, y_o\} \\
O_S[n] &:= |\mathcal{M}| = M \\
s_S^+ &:= s^+ : \\
\# \hat{O}_S[n] &< O_S[n] \quad \forall n, s^+ \in S_j^+
\end{aligned}$$

16.4 Conjecture of Optimal Solutions

$O_{T_{min}}$ subject to $O_{S_{opt}} = 1$

$$\begin{aligned}
&\exists s_{opt}^+ : \\
O_{opt}[n] &= 1 + O_T[n] = O_{S_{opt}} + O_T[n] \quad \forall s^+ \in S^+
\end{aligned}$$

16.4.1 Proof

16.5 Theorem of Optimal Solutions

Alternate way of expressing $O_{opt}[n]$ possibly with efficiency function and equivalence functions?

$f_{S \rightarrow T}[n, O_S[n]]$ as a function of space complexity order K?

Efficiency function ($f_{S \rightarrow T}[n, O_S[n]]$ as a function of space complexity order K) is strictly decreasing for Polynomial Functions

Or there's an inflection point

Efficiency function might have a general pattern for all problems in D

17 Polynomial Complexity

17.1 Definition

Decision problem D with solution s^+ has (optimal) total complexity $O[n]$ bounded by polynomial complexity if

$$\begin{aligned} & \exists K, C, \lambda_1 \dots \lambda_K : \\ & O_{opt}[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C \quad \forall n \end{aligned}$$

17.2 Polynomial Problems

Define \mathbb{P} , the set of Decision Problems that can be solved with Polynomial Complexity

$$\begin{aligned} \mathbb{P} &:= \{D_1, D_2, \dots\} : \\ & \exists K, C, \lambda_1 \dots \lambda_K : \\ & O_{opt}[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C \quad \forall n, D_j \in \mathbb{P} \end{aligned}$$

17.3 Polynomial Order of Complexity

Total complexity $O[n]$ is said to be of order K_{opt}

$$\begin{aligned} & O[n] \sim K_{opt} \\ & O_{opt}[n] := O[n] : \\ & \# \hat{O}[n] < O_{opt}[n] \quad \forall n \\ & O_{opt}[n] < (\lambda_{K_{opt}} n)^{K_{opt}} + (\lambda_{K_{opt}-1} n)^{K_{opt}-1} \dots + \lambda_1 n + C \quad \forall n \\ & K_{opt} := K : \\ & \# \hat{K} : O_T[n] < (\lambda_{\hat{K}} n)^{\hat{K}} + (\lambda_{\hat{K}-1} n)^{\hat{K}-1} \dots + \lambda_1 n + C \quad \forall n, \hat{K} < K \end{aligned}$$

17.4 Corrolary of Optimal Complexity

$$\begin{aligned} & \# s^+ \in S^+ : \\ & O_T[n] < (\lambda_{\hat{K}} n)^{\hat{K}} + (\lambda_{\hat{K}-1} n)^{\hat{K}-1} \dots + \lambda_1 n + C \quad \forall n, \hat{K} < K_{opt} \end{aligned}$$

17.4.1 Proof

Proof by contradiction; definition of optimal complexity

17.5 Total Polynomial Complexity Implies Time bounded Polynomial Complexity

$$D \in \mathbb{P} \implies \exists s^+ \in \mathbb{S}_{time}^+$$

17.5.1 Proof

$$\begin{aligned} O[n] &< (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C \quad \forall n \\ O[n] &:= O_T[n] + O_S[n]; \quad O_T[n] < O[n] \\ \therefore O_T[n] &< (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C \quad \forall n \end{aligned}$$

17.6 Total Polynomial Complexity Implies Space bounded Polynomial Complexity

$$D \in \mathbb{P} \implies \exists s^+ \in \mathbb{S}_{space}^+$$

17.6.1 Proof

$$\begin{aligned} O[n] &< (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C \quad \forall n \\ O[n] &:= O_T[n] + O_S[n]; \quad O_S[n] < O[n] \\ \therefore O_S[n] &< (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C \quad \forall n \end{aligned}$$

17.7 Total Polynomial Complexity iff Time and Space bounded by Polynomial Complexity

Use limit definition

17.8 Property of Polynomial Complexity 1

$$\lim_{n \rightarrow \infty} \frac{O[n+1]}{O[n]} = 1$$

17.8.1 Proof WIP

Show there exists no constant satisfying the decreasing limit condition

$$\begin{aligned} O[n] &< (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C \\ O[n+1] &< (\lambda_K (n+1))^K + (\lambda_{K-1} (n+1))^{K-1} \dots + \lambda_1 (n+1) + C \\ O[n] &\sim (\lambda n)^K; \quad O[n+1] \sim (\lambda n)^K \\ \lim_{n \rightarrow \infty} \frac{(\lambda n)^K}{(\lambda n)^K} &= 1 \end{aligned}$$

17.9 Property of Polynomial Complexity 2

$$\begin{aligned} & \exists K, C, \lambda_1, \dots, \lambda_K : \\ & (O[n+1] - O[n]) < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C \quad \forall n \end{aligned}$$

17.9.1 Proof

$$\begin{aligned} O[n+1] & < (\lambda_K (n+1))^K + (\lambda_{K-1} (n+1))^{K-1} \dots + \lambda_1 (n+1) + C \\ O[n] & < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C \end{aligned}$$

17.10 Order of Complexity

ERROR in second condition

Total Complexity is said to be on the order of K_{max}

$$\begin{aligned} O[n] & \sim K_{max} \\ K_{max} & := K : \\ O[n] & < (\lambda_{K_{max}} n)^{K_{max}} + (\lambda_{K_{max}-1} n)^{K_{max}-1} \dots + \lambda_1 n + C \quad \forall n \\ \#O[n] & < (\lambda_{\hat{K}_{max}} n)^{\hat{K}_{max}} + (\lambda_{\hat{K}_{max}-1} n)^{\hat{K}_{max}-1} \dots + \lambda_1 n + C \quad \forall n, \hat{K} < K_{max} \end{aligned}$$

18 Polynomial Time Complexity

18.1 Definition

Decision problem D with (optimal) Time Complexity $O_T[n]$ is bounded by polynomial time complexity if

$$\begin{aligned} & \exists K, C, \lambda_1 \dots \lambda_K : \\ & O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C \quad \forall n \end{aligned}$$

18.2 Polynomial Time Solutions

Define \mathbb{S}_{time}^+ , the set of solutions that can be solved with polynomial time complexity

$$\begin{aligned} & \mathbb{S}_{time}^+ := \{s_1^+, s_2^+, \dots\} : \\ & \exists K, C, \lambda_1 \dots \lambda_K : \\ & O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C \quad \forall n, s_i \in \mathbb{S}_{time}^+ \end{aligned}$$

18.3 Property of Polynomial Time Complexity 1

$$\lim_{n \rightarrow \infty} \frac{O_T[n+1]}{O_T[n]} = 1$$

18.3.1 Proof

18.4 Property of Polynomial Time Complexity 2

$$\begin{aligned} & \exists K, C, \lambda_1, \dots, \lambda_K : \\ & (O_T[n+1] - O_T[n]) < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C \quad \forall n \end{aligned}$$

18.4.1 Proof

18.5 Order of Complexity

Time complexity $O_T[n]$ is said to be on the order of K_{max}

$$\begin{aligned} & O_T[n] < (\lambda_{K_{max}} n)^{K_{max}} + (\lambda_{K_{max}-1} n)^{K_{max}-1} \dots + \lambda_1 n + C \\ & O_T[n] \sim K_{max} \end{aligned}$$

19 Polynomial Space Complexity

19.1 Defintion

Decision problem D with (optimal) Time Complexity $O_S[n]$ is bounded by polynomial time complexity if

$$\begin{aligned} & \exists K, C, \lambda_1 \dots \lambda_K : \\ & O_S[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C \quad \forall n \end{aligned}$$

19.2 Polynomial Space Problems

Define \mathbb{S}_{space}^+ , the set of solutions that can be solved with polynomial time complexity

$$\begin{aligned} & \mathbb{S}_{space}^+ := \{s_1, s_2, \dots\} : \\ & \exists K, C, \lambda_1 \dots \lambda_K : \\ & O_S[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C \quad \forall n, s_i \in \mathbb{S}_{time}^+ \end{aligned}$$

19.3 Total Polynomial Complexity Implies Space bounded Polynomial Complexity

19.4 Space Bounded Polynomial Complexity Implies Total Polynomial Complexity

19.5 Polynomial Space Complexity iff Polynomial Complexity

19.6 Property of Polynomial Space Complexity 1

$$\lim_{n \rightarrow \infty} \frac{O_S[n+1]}{O_S[n]} = 1$$

19.6.1 Proof

19.7 Property of Polynomial Space Complexity 2

$$\begin{aligned} & \exists K, C, \lambda_1, \dots, \lambda_K : \\ & (O_S[n+1] - O_S[n]) < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C \quad \forall n \quad \forall n \end{aligned}$$

19.7.1 Proof

19.8 Order of Complexity

Space complexity $O_S[n]$ is said to be on the order of K_{max}

$$O_S[n] < (\lambda_{K_{max}} n)^{K_{max}} + (\lambda_{K_{max}-1} n)^{K_{max}-1} \dots + \lambda_1 n + C$$

$$O_S[n] \sim K_{max}$$

20 Polynomial Duality

20.1 Proof of the existence of $O_{S_{opt}}$

20.2 Proof of the existence of $O_{T_{opt}}$

20.3 Theorem Either OT or OS is on the order of Oopt

Proof by contradiction

20.4 Inductive Function f_{n+1}

$$\begin{aligned} O[n] &:= O_T[n] + O_S[n] \\ O[n+1] &= O_T[n+1] + O_S[n+1] \\ f_{n+1} &:= f : \\ O[n+1] &= f[n] + O[n] \quad \forall n \end{aligned}$$

20.4.1 Proof of existence

Proof by contradiction

20.5 Equivalence Functions

20.5.1 O_T to O_S

Define equivalence function $f_{T \rightarrow S}$; a function converting logical operations into memory elements

$$\begin{aligned} f_{T \rightarrow S} &:= f : \\ O_S[n] &= f[n, O_T[n]] \quad \forall n, s^+ \in S^+ \end{aligned}$$

20.5.2 O_S to O_T

Define equivalence function $f_{S \rightarrow T}$; a function converting memory elements into logical operations

$$\begin{aligned} f_{S \rightarrow T} &:= f : \\ O_T[n] &= f[n, O_S[n]] \quad \forall n, s^+ \in S^+ \end{aligned}$$

20.5.3 Proof of existence

Proof by contradiction

20.5.4 Invertibility

20.5.5 Polynomial Bounded

20.6 Efficiency Function

Function relating the decrease in $O[n]$ as $O_S[n]$ increases in order $f_{S \rightarrow T}[n, O_S[n]]$ as a function of space complexity order K ?

20.7 Theorem of Computational Duality

For all Problems in P there exists a duality function

Formally define dynamic programming, Optimal polynomial complexity minimizes the difference between time and space complexity order

$$D \in \mathbb{P}$$

$$O[n] := O_T[n] + O_S[n]$$

$$\lim_{n \rightarrow \infty} \frac{O[n+1]}{O[n]} = 1 \quad \forall s^+ \in S_{\mathbb{P}}^+$$

$$O_T[n] = f_{S \rightarrow T}[n, O_S[n]]$$

$$O_S[n] = f_{T \rightarrow S}[n, O_T[n]]$$

$$\lim_{n \rightarrow \infty} \frac{O_T[n+1] + O_S[n+1]}{O_T[n] + O_S[n]} = 1 \quad \forall s^+ \in S_{\mathbb{P}}^+$$

$$\lim_{n \rightarrow \infty} \frac{f_{S \rightarrow T}[n+1, O_S[n+1]] + O_S[n+1]}{f_{S \rightarrow T}[n, O_S[n]] + O_S[n]} = 1 \quad \forall s^+ \in S_{\mathbb{P}}^+$$

$$\lim_{n \rightarrow \infty} \frac{O_T[n+1] + f_{T \rightarrow S}[n+1, O_T[n+1]]}{O_T[n] + f_{T \rightarrow S}[n, O_T[n]]} = 1 \quad \forall s^+ \in S_{\mathbb{P}}^+$$

20.8 Theorem of Computational Duality part2

Relate $f_{n+1}[n]$ to equivalence functions

$$D \in \mathbb{P}$$

$$O[n] := O_T[n] + O_S[n]$$

$$O[n+1] = O_T[n+1] + O_S[n+1] = O[n] + f_{n+1}[n]$$

$$O_T[n] = f_{S \rightarrow T}[n, O_S[n]]$$

$$O_S[n] = f_{T \rightarrow S}[n, O_T[n]]$$

$$\begin{aligned}
f_{S \rightarrow T}[n+1, O_S[n+1]] + O_S[n+1] &= O[n] + f_{n+1}[n] \\
f_{S \rightarrow T}[n+1, O_S[n+1]] + O_S[n+1] - O[n] &= f_{n+1}[n] \\
O_T[n+1] + f_{T \rightarrow S}[n+1, O_T[n+1]] - O[n] &= f_{n+1}[n]
\end{aligned}$$

20.9 Property of Polynomial Complexity

$f_{n+1}[n]$ has order less than $O[n]$
 $f_{n+1}[n]$ is bound by $K_{max} - 1$

20.9.1 Proof

Proof by contradiction; limit doesn't converge

21 Sum to N Problem with 2 integers

21.1 State formal definition of Sum to N : $x_i + x_j == N$

$$\begin{aligned}
 X_i &= \{x_1, \dots, x_n, N\} \\
 D &:= f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i \\
 s^+ &= s^+[n] := P[X_i] \rightarrow y_o : y_o = a_o \quad \forall X_i \\
 s^+ &= \{s_1, s_2, \dots, s_{O_T[n]}, b_1, b_2, \dots, b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\} \\
 D &= f[X_i] = \exists x_j, x_k \in X_i : x_j + x_k == N
 \end{aligned}$$

21.2 Express a formal solution : $O_S[n] \sim n^0$

$$\begin{aligned}
 s^+ &= \{s_1, s_2, \dots, s_{O_T[n]}, b_1, b_2, \dots, b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\} \\
 s_1 &= y_o \leftarrow \mathbb{F}; \\
 \forall i, j &> i \\
 s_2, s_3, s_8, s_9, \dots, s_{3ij-4}, s_{3ij-3}, \dots, s_{3n(n-1)-4}, s_{3n(n-1)-3} &= b_1 \leftarrow x_i + x_j \\
 s_4, s_5, s_{10}, s_{11}, \dots, s_{3ij-2}, s_{3ij-1}, \dots, s_{3n(n-1)-2}, s_{3n(n-1)-1} &= b_1 \leftarrow b_1 == N \\
 s_6, s_7, s_{12}, s_{13}, \dots, s_{3ij}, s_{3ij+1}, \dots, s_{3n(n-1)}, s_{3n(n-1)+1} &= y_o \leftarrow y_o \vee b_1 \\
 s^+ &= \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\}
 \end{aligned}$$

21.3 Determine $O[n], O_S[n], O_T[n]$ for the above solution

$$\begin{aligned}
 O_S[n] &= |y_o| + |b_1| = 2 \\
 O_T[n] &= 3n(n-1) + 1 = 3n(n-1) - 1 + O_S[n] \\
 O[n] &= 3n(n-1) + 3 = 3n^2 - 3n + 3
 \end{aligned}$$

21.4 Show s^+ is bounded by Polynomial Complexity by the definition of Total Polynomial Complexity

$$\begin{aligned}
 O[n] &= 3n(n-1) + 3 = 3n^2 - 3n + 3 \\
 O[n] &= 3n^2 - 3n + 3 < 3n^2 - 3n + 4 \quad \forall n \\
 \therefore s^+ &\in S_{\mathbb{P}}^+
 \end{aligned}$$

21.5 Show s^+ is bounded by Polynomial Complexity by showing $\lim_{n \rightarrow \infty} \frac{O[n+1]}{O[n]} = 1$

21.6 Express a formal solution : $O_S[n] \sim n^1$

21.7 Determine $O[n], O_S[n], O_T[n]$ for the above first order space complexity solution

$$O_S[n] = |y_o| + |b_1 \dots b_n| + |b_{n+1}| = n + 2$$

$$O_T[n] = 1 + 2n + \sum_{i=1}^n i = 1 + 2n + 2n(n+1) = 2n^2 + 4n + 1$$

$$O[n] = 2n^2 + 4n + n + 1 + 2 = 2n^2 + 5n + 3$$

21.8 Express equivalence function $f_{S \rightarrow T}[n, O_S[n]]$

$$O_T[n] = O[n] - O_S[n] = 2n^2 + 5n + 3 - O_S[n] = f_{S \rightarrow T}[n, O_S[n]]$$

21.9 Determine inductive function f_{n+1} using equivalence function $f_{S \rightarrow T}[n, O_S[n]]$

$$f_{S \rightarrow T}[n+1, O_S[n+1]] + O_S[n+1] - O[n] = f_{n+1}[n]$$

$$2(n+1)^2 + 5(n+1) + 3 - O_S[n+1] + O_S[n+1] - (2n^2 - 5n - 3) = f_{n+1}[n]$$

$$2(n+1)^2 + 5(n+1) - 2n^2 - 5n = f_{n+1}[n]$$

$$2n^2 + 4n + 2 + 5n + 5 - 2n^2 - 5n = f_{n+1}[n]$$

$$4n + 7 = f_{n+1}[n]$$

21.10 Show s^+ is bounded by Polynomial Complexity by showing $\lim_{n \rightarrow \infty} \frac{O[n+1]}{O[n]} = 1$

By showing the order $f_{n+1}[n]$ is less than $O[n]$

$$\lim_{n \rightarrow \infty} \frac{O[n+1]}{O[n]} =$$

$$\lim_{n \rightarrow \infty} \frac{O[n] + f_{n+1}[n]}{O[n]} =$$

$$\lim_{n \rightarrow \infty} 1 + \frac{f_{n+1}[n]}{O[n]} =$$

$$\lim_{n \rightarrow \infty} 1 + \frac{4n+7}{2n^2+5n+3} =$$

22 Sum to N Problem General Solution

22.1 State formal definition of Sum to N with 3 integers :

$$x_i + x_j + x_k == N$$

22.2 State general definition of Sum to N with q integers :

$$x_1 + x_2 \dots + x_q == N$$

23 Divergent Problems

23.1 Definition

$$\begin{aligned}\mathcal{D} &:= \{\hat{D}_1, \hat{D}_2, \dots\} \\ \lim_{n \rightarrow \infty} \frac{\hat{O}[n+1]}{\hat{O}[n]} &\text{ diverges} \\ &\forall s_{\hat{D}}^+\end{aligned}$$

23.2 Property of Divergent Problem Complexity 1

$$\lim_{n \rightarrow \infty} \frac{\hat{O}[n+1]}{\hat{O}[n]} \text{ diverges}$$

23.2.1 Proof

23.3 Property of Divergent Problem Complexity 2

$$\begin{aligned}\nexists f[n] : \hat{O}[n+1] - \hat{O}[n] &= f[n] \quad \forall n \\ \lim_{n \rightarrow \infty} (\hat{O}[n+1] - \hat{O}[n]) &\text{ diverges}\end{aligned}$$

23.3.1 Proof

23.4 Divergent Duality?

23.5 Divergent Induction Functions?

24 Fundamental Theorem of Computation

Some solutions for Polynomial Problems are divergent; There exist no solution for divergent problems with polynomial bound

$$\mathbb{P} = \{D_1, D_2, \dots\}$$

$$S_{\mathbb{P}}^+ = \{s_1, s_2, \dots\}$$

$$\nexists s \in S_{\mathbb{P}}^+ : O[n] \geq n^n \quad \forall s_i \in S_{\mathbb{P}}^+$$

$$\mathbb{P} \cap \hat{D} = \emptyset$$

$$\lim_{n \rightarrow \infty} C^n[s^+[n]] = n^n \quad \forall s_i \in S_{\mathbb{P}}^+$$

24.1 Proof

$$X_i = \{x_1, \dots, x_n\}$$

$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$\text{Let } D \in \mathbb{P}$$

$$s^+ := P[X_i] \rightarrow y_o : y_o = a_o \quad \forall X_i$$

$$O[n] = O_T[n] + O_S[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C \quad \forall n$$

24.2 Non-Polynomial implies Divergent

25 Proof of the existence of $\hat{\mathcal{D}}$

Non-trivial; Formalize the traveling salesman problem as a decision problem
(any optimization problem)

25.1 The Traveling Salesman Problem

English description

25.2 Formal Definition

$$X_i = \{c_1, c_2, \dots, c_n\} :$$

$$\dim[c_i] = C > 1$$

$$\bar{X}_i = \{c_1, c_2, \dots, c_n, \bar{P}, \bar{f}[c_i, c_j]\}$$

$$\bar{P} := \{c_k, \dots\} :$$

$$\exists c_k \in \bar{P} \quad \forall c_k \in X_i$$

25.3 Determine $s^+ \implies O_{S_{opt}}$

25.4 Express $O[n], O_T[n], O_S[n] = O_{S_{opt}}$

25.5 Revisit expressions properties inequalities connecting
 $O_{opt}; O_{T_{opt}}; O_{S_{opt}}$

25.6 Determine an alternate solution storing subpaths

25.7 Express $O[n], O_T[n], O_S[n]$

25.8 Determine a dual function

25.9 Show $\lim_{n \rightarrow \infty} \frac{\hat{O}[n+1]}{\hat{O}[n]}$ diverges

25.10 Let \hat{s}^+ ; a solution with $\lim_{n \rightarrow \infty} \frac{\hat{O}[n+1]}{\hat{O}[n]} = 1$

25.11 Show \hat{s}^+ implies a contradiction

25.12 Determine $s_{S_{opt}}^+$

25.13 Determine equivalence function

25.14 Determine an expression for $\frac{\hat{O}_{opt}[n+1]}{\hat{O}_{opt}[n]}$

26 Proof of " $P \neq NP$ "

27 Theorem of Prime Numbers "Riemann Hypothesis"

Riemann Zeta Function

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} \quad [2]$$

"The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2." [2]

Prove the problem is divergent

There fore it can only be proven to a certain degree

The limit as n approaches infinity implies a real part of one half

Connection with the real and imaginary part of $O[n]$

27.1 Determine a duality function for the Riemann Hypothesis

27.2 Determine an expression for $O[n+1]$ as a function of $O[n]$

27.3 Prove O_{opt} is performing O_{opt} recursively for the ints less than square root of n

Testing the primes less than \sqrt{n} ? double check

1. Optimal solution for $n=1,2,3$, everything else is a recursive optimal proof by induction

Time Complexity seems to be on the order of $n \log n$... implies divergence or lack of bound? Add in the complexity of division.. probably approaches n^n

27.4 Since divergent, no s^+ exists.. only rules

Express as a limit

27.5 Show that the limit as $n \rightarrow \infty$ implies the real part is 1/2

$1/2 \pm 14.134725 i$ $1/2 \pm 21.022040 i$ $1/2 \pm 25.010858 i$ $1/2 \pm 30.424876 i$
 $1/2 \pm 32.935062 i$ $1/2 \pm 37.586178 i$

$$Z = \zeta(1/2 + it)$$

27.6 Notation, real imaginary parts of the problem

Even numbers and numbers ending in 5 are automatically convergent

Testing numbers ending in 1,3,7,9 results in divergent expression

we can continue to add rules to a certain degree

28 Divergent Problems

Define $\hat{\mathbb{D}}$ the set of decision problems with no convergent?/finite? solution \hat{D}_j

$$\begin{aligned}\hat{\mathbb{D}} &:= \{\hat{D}_j, \dots\} \\ \hat{D}_j &\in \mathbb{D} \quad \forall j \\ \nexists \hat{s}^+ \in S^+ : \hat{s}^+ \text{ solves } \hat{D}_j \quad \forall \hat{D}_j \in \hat{\mathbb{D}} &\iff \\ \nexists \hat{s}^+ \in S^+ : O_j[n] < n^n \quad \forall n, j\end{aligned}$$

There exists no such solution such that $O[n] < n^n$, but there is a right and wrong answer

Either here or in the next chapter we'll prove you can only solve to a certain degree

!!! There exists no such solution such that $O[n] < n^n \quad \forall n$

28.1 Definition

$$\begin{aligned}\hat{O}[n] &:= n^n \\ \lim_{n \rightarrow \infty} \frac{O_{opt}[n]}{\hat{O}[n]} &\text{ diverges} \\ \text{diverges}[O_{opt}[n], \hat{O}[n]] &\rightarrow \mathbb{T}\end{aligned}$$

28.2 Theorem of Divergent Programs

Prove that Divergent implies not in polynomial (trivial)

Prove that Divergent implies Non-polynomial (trial after proving above)

Show that there exists at least one member of Divergent

29 Properties of Solvable and Divergent problems

29.1 Solvable and Divergent are disjoint

Prove by contradiction

30 "Theorem of Divergent Programs"

30.1 Divergence Test

1. Let $d_j \in D$
2. $d_j = (d_j \in \hat{\mathcal{D}}) \cup (d_j \in \text{set of solvable problems})$ by disjoint condition of solvable and divergent
3. Let $O_{opt}[n]$, the optimal complexity of d_j
4. $\rightarrow s_i^+$ that solve d_j have larger complexity $\forall i$
5. 2 implies $O_{opt}[n]$ is either bounded by n^n or not
6. $\hat{O}[n] \equiv n^n$
7. Easy Suppose $d_j \in \text{solvable}$ $\lim_{n \rightarrow \infty} \frac{O_{opt}[n]}{\hat{O}[n]} = 0$
8. Suppose $d_j \in \hat{\mathcal{D}}$ $\lim_{n \rightarrow \infty} \frac{O_{opt}[n]}{\hat{O}[n]} \neq 0$ (by disjoint condition)

$$\lim_{n \rightarrow \infty} \frac{O_{opt}[n]}{\hat{O}[n]} = 1$$

31 Connection to verification in polynomial time

32 Fundamental Theorem of Computation

n^n or $\lambda n^n + C$ the universal bound to solvable computational complexity
 $(\lambda n)^n + C$?

32.1 Time Complexity Argument

Suppose decision problem d with optimal time complexity $O_{T_{min}}[n]$ and solution s^+ , an arbitrary decision problem in P with polynomial complexity

Assumptions

1. $d \in P, s^+ \in S^+$

Assertions

2. $\exists K, C, \lambda_1 \dots \lambda_K : O_{T_{min}}[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C, \quad \forall n$
3. Define $f[K, C, \lambda_1, \dots, \lambda_K] \equiv (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C$
4. $\exists K, C, \lambda_1 \dots \lambda_K : O_{T_{min}}[n] < f[K, C, \lambda_1, \dots, \lambda_K] \quad \forall n$
5. Let $\hat{s}^+ \equiv \mathcal{C}^n[s^+]$
6. $O_T[n] \leq \hat{O}_T[n]$ (by definition of nested loop)

7. $\hat{O}_{T_{min}}[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} \dots + \lambda_1 n + C$

8. $\hat{O}_{T_{min}}[n] < \lim_{n \rightarrow \infty} \mathcal{C}^n[s^+]$ (by definition of limit + definition of nested loop, expand to show full derivation, valid because this is a series, probably need to show limit applies)

9. $\therefore O_{T_{min}}[n] < \hat{O}_{T_{min}}[n] < n^n = \lim_{n \rightarrow \infty} \mathcal{C}^n[s^+]$

I want to say for all n but seems refutable for $n = 1, 2, \dots$ but as n approach infinity it's a contradiction to say a solvable problem in P $\hat{O}_{T_{min}} = n^n \quad \forall n$

10. For "sufficiently large n "

$$\nexists \hat{s}^+ \in S^+ : |\hat{s}^+| \equiv O_{T_{min}}[n] < n^n, \quad \forall n$$

$$\hat{O}[n] \equiv n^n$$

32.2 Space Argument

Similar but additional notation required?

33 Divergence Criterion

Necessary condition for divergent program, iff
or you can show there exists no lambda, C such that $O[n]$ is n^n is bounded
by $\lambda n^n + C$ for all n

$$\lim_{n \rightarrow \infty} \text{div} / \text{solvable} > 1$$

Assumptions

1. Define the "Null Space of \mathcal{D} " or "Null Set" O_\perp

$$O_\perp = \{\hat{d}_1, \hat{d}_2, \dots, \hat{d}_j\}, \quad j > 0$$

$$\hat{O}_j[n] \equiv (O[n])^n, \forall j$$

2. $O_P \cup O_N = \mathcal{D}$ (by definition)

Assertions

3. $O_P \cap O_\perp = \emptyset$
4. Let $O_N \cap O_\perp = \hat{O} = \{\hat{O}_i, \dots\}, i > 0$
5. Consider $D_j \in O_N$
6. D_j has finite complexity by definition

$$O_j[n] = C$$

7. D_j has at least one optimal solution by the necessity of optimal solution (theorem Z)

$$O_j[n] = C$$

34 Proof of "P ≠ NP"

34.1 Proof N implies D

Is trivial by implication of Theorem x and Theorem y

$$\begin{aligned} \mathcal{N} &\equiv \{d_j, ..\} \quad \forall j, \mathcal{N} \in D \\ &\quad \nexists K, C, \lambda_1... \lambda_K : \\ O_T[n] &< (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C, \quad \forall n, \forall d_j \in \mathcal{N} \end{aligned}$$

1. $\rightarrow \mathcal{P} \cap \mathcal{N} = \emptyset$ by definition of P,N
 2. $d_i \in \hat{D} \vee d_i \in \text{solvable}$
 3. $1 \rightarrow d_i \notin \text{solvable}$
 4. $\therefore d_i \in \hat{D} \quad \forall i$ (theorem y) Show that Definition of Non-Polynomial Problems automatically implies Divergent
1. We've proven Solvable Union are disjoint and complete set P 2. N not in P by definition 3. therefore N in divergence by set theory

Currently we have only defined solvable problems and divergent problems
Additionally polynomial problem which the existence of is trivial
Plus we defined non-polynomial complexity
Prove the existence of \mathcal{N} the set of non polynomial problems

34.2 Proof that D implies N

34.3 D iff N

Show $O[n]$ in the \emptyset the set of problems with $n^n > O[n] > n^k + c$
Proving there's Polynomial and Divergent, in the set of all decision problems

A neat follow up, tie in the definition of \mathcal{N} implies membership to divergent problems

35 Prove the existence of $D = N$, The Traveling Salesman Problem

Define the traveling salesman problem, prove it is divergent and has the same solution as current approaches

Consider proving with both definition and necessary condition

35.1 Compute every sub path or recursive subpaths in memory

Trade off between time and space, $O_{salesman}[n]$ diverges with a polynomial $O_P[n]$

36 Prove Polynomial and Divergent problems are Complements

Implied by the previous sections

37 Solvable Union Divergent = all decision problems

Trivial as a result of the previous section by definition of Ω

$$\mathbb{P} \cup \hat{\mathbb{D}} = \mathbb{D}$$

38 Theorem of Prime Numbers "Riemann Hypothesis"

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The limit as n approaches infinity implies a real part of one half

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Citations

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