Ch. 5 Computation

1 Definition of State

Something like operation s_i | state "memory" m_i
Operations previously defined
m_i is a single bit 0/1

2 Definition of a Decision Problem

A decision problem is a function whos output "answer" a_o is True/False
$\forall$ inputs x_i, f[$\{x_i, ...\}$] -> True or False

$$d \equiv f[\{x_i, ...\}] \rightarrow a_{out} \in \{True, False\} \quad \forall x_i$$

3 Definition of Program

Program p is defined as an ordered set of logical operations s_i

$$p \equiv \{s_1, s_2, ..., s_N\} \qquad (Definition 3)$$

3.1 Function Notation

$$p[x_{in}] = \{s_1, s_2, ..., s_{n-1}, y_o | m_1, m_2, ..., m_L\}$$
$$p[x_{in}] \rightarrow y_o, \quad \forall x_{in}$$

Note: m_i represents the memory required to complete program p

3.2 System Notation

$p[\{X_{in}\}] \rightarrow Y_{out} = p[\{x_i, ...\}] \rightarrow Y_{out} = \{s_1, s_2, ..., s_{n-1}, Y_{out} | m_1, m_2, ..., m_L\}$

4 Definition of No-op

Define ";" , the no-op (The C representation of a null statement or no-op)

4.1 Properties of No-op

1. Can be added to any solution S_i and remain a solution for all i anywhere in the order for all j

5 Boolean Program

Define boolean program p, sometimes called a boolean function

$$p[\{X_{in}\}] = \{s_1, s_2, ..., s_{n-1}, y_o\} \rightarrow y_o \in \{True, False\}, \quad \forall x_{in}$$

6 Definition of Solution

Define the solution s^+ to decision problem d
Program p is said to be a solution s^+, if s^+ outputs the correct answer a_o for all inputs x_i

$$d \equiv f[\{x_i, ...\}] \rightarrow a_{out} \in \{True, False\} \quad \forall x_i$$
$$s^+ \equiv p[\{x_i, ...\}] \rightarrow y_o : y_o == a_{out} \quad \forall x_i$$

6.1 Define the set of all Solutions

6.2 Definition of Solvable

6.3 Definition of Valid Problem

By definnition decision problem has a true or false output for all inputs

7 Define the set of all Solvable Decision Problems

Complexity

7.1 Time Complexity of a Decision Problem

The Time Complexity $O_T[n]$ of Decision Problem d

$$O_T[n] <= n(d)(Theorem1)$$

7.2 Time Complexity of a Proper Decision Problem

The Time Complexity $O_T[n]$ of Proper Decision Problem D

$$O_T[n] = n(d)(Theorem2)$$

7.3 Definition of a Divergent Problem

An expressed problem that has no solution, does not exist in D

8 Definition of Complexity

Define Complexity $O[n]$ as a Tensor of dimension N

$$\mathbf{O}[n] \equiv \ < O_T[n], O_S[n], O_3[n], O_4[n] ..., O_N[n] > \quad (Definition 1)$$

8.1 Time Complexity

Define Time Complexity O_T as the maximum number of logical operations in a Program P

$$O_T[n] \equiv |p[\{X_{in}\}]| \quad (Definition 2)$$

8.2 Space Complexity

Define Time Complexity O_T as the maximum number of bits required to complete Program P

8.3 Definition of Solution

9 Total Complexity

Define O[n], the total complexity

9.1 Definition

O[n] = $\lambda_T O_T[n]$ + $\lambda_S O_S[n]$ + $\sum_{i=3}^{N} \lambda_i O_i[n]$

9.2 Proof of Existence

10 Theorem of Optimal Complexity

Proof the necessity of at least one $O_{min}[n]$

11 Definition of Polynomial Time Complexity

Decision problem d with Time Complexity $O_T[n]$ can be solved with Polynomial Time Complexity if

$$\exists K, C \ : \ O_T[n] < n^K + C, \quad \forall n$$

11.1 Definition of Polynomial Problems

Define P, the set of Decision Problems that can be solved with Polynomial Time Complexity

$$P \equiv \{d_1, d_2, ...\}$$
$$\exists K, C, \lambda_1 ... \lambda_K :$$
$$O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C, \quad \forall n, \ \forall d_j \in P$$

11.2 Proof of the existence of P

Trivial

11.3 Definition of Non-Polynomial Problems

Define $\mathcal{N}$, the set of Decision Problems that cannot be solved with Polynomial Time Complexity

$$\mathcal{N} \equiv \{d_1, d_2, ..\}$$
$$\nexists K, C, \lambda_1 ... \lambda_K :$$
$$O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C, \quad \forall n, \ \forall d_i \in \mathcal{N}$$

11.4 Proof of the existence of $\mathcal{N}$

Non-trivial

11.5 Definition of Divergent Programs

A program is a function that solves

12 $O_\perp$ the "Null Set" or "Null Space" of $\mathcal{D}$

Contradictory or divergent?

$$O_\perp \equiv \bar{D} - \mathcal{D}$$

13 Fundamental Theorem of Computation

n^n or $\lambda n^n + C$ the universal bound to solvable computational complexity $(\lambda n)^n$ + C ?

13.1 Time Complexity Argument

Suppose decision problem d with optimal time complexity $O_{T_{min}}[n]$ and solution s^+, an arbitrary decision problem in P with polynomial complexity
Assumptions
1. $d \in P$, $s^+ \in S^+$
Assertions
2. $\exists K, C, \lambda_1 ... \lambda_K \;\; : \;\; O_{T_{min}}[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C, \quad \forall n$
3. Define $f[K, C, \lambda_1, ... \lambda_K] \equiv (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$
4. $\exists K, C, \lambda_1 ... \lambda_K : O_{T_{min}}[n] < f[K, C, \lambda_1, ... \lambda_K] \quad \forall n$
5. Let $\hat{s}^+ \equiv \circlearrowright^n [s^+]$
6. $O_T[n] <= \hat{O}_T[n]$ (by defition of nested loop)

7. $\hat{O}_{T_{min}}[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$
8. $\hat{O}_{T_{min}}[n] < limit_{n \to \infty} \circlearrowright^n [s^+]$ (by definition of limit + definition of nested loop, expand to show full derivation, valid because this is a series, probably need to show limit applies)
9. $\therefore O_{T_{min}}[n] < \hat{O}_{T_{min}}[n] < n^n = limit_{n \to \infty} \circlearrowright^n [s^+]$
I want to say forall n but seems refutable for n= 1,2.. but as n approach infinity it's a contradiction to say a solvable problem in P $\hat{O}_{T_{min}} = n^n \quad \forall n$
10. For "sufficiently large n"

$$\nexists \hat{s}^+ \in S^+ : |\hat{s}^+| \equiv O_{T_{min}}[n] < n^n, \quad \forall n$$
$$\hat{O}[n] \equiv n^n$$

13.2 Space Argument

Similar but additional notation required?

14 Definition of Divergent Problems

Define $\hat{D}$ the set of decision problems with no finite solution
Let

$$\hat{D} \equiv \{\hat{d}_j, ...\}$$

$$\nexists \hat{s}^+ \in S^+ : \hat{s}^+ \text{ solves } \hat{d}_j, \quad \forall \hat{d}_j \in \hat{D}$$
$$\nexists s^+ \in S^+ : O_j[n] < n^n or (\lambda n)^n \quad \forall n, j$$

A program is a function that solves There exists no such solution such that O[n] $< n^n$, but there is a right and wrong answer
Either here or in the next chapter we'll prove you can only solve to a certain degree
!!! There exists no such solution such that O[n] $< n^n \quad \forall$ n

15 Properties of Solvable and Divergent problems

15.1 Solvable and Divergent are disjoint (Theorem x)

15.2 Solvable Union Divergent = all decision problems (Theorem y)

15.3 What is the connection to verification in polynomial time

16 "Theorem of Divergent Programs"

16.1 Divergence Test

1. Let $d_j \in D$
2. $d_j = (d_j \in \hat{\mathcal{D}}) \cup (d_j \in$ set of solvable problems) by disjoint condition of solvable and divergent
3. Let $O_{opt}[n]$, the optimal complexity of d_j
4. $\rightarrow s_i^+$ that solve d_j have larger complexity $\forall i$
5. 2 implies $O_{opt}[n]$ is either bounded by n^n or not
6. $\hat{O}[n] \equiv n^n$
7. Easy Suppose $d_j \in$solvable$limit_{n \to \infty} \frac{O_{opt}[n]}{\hat{O}[n]} = 0$
8.Suppose $d_j \in \hat{\mathcal{D}} limit_{n \to \infty} \frac{O_{opt}[n]}{\hat{O}[n]} \neq 0$ (by disjoint condition)

$$limit_{n \to \infty} \frac{O_{opt}[n]}{\hat{O}[n]} = 1$$

16.2 Notes

Necessary condition for divergent program, iff
or you can show there exists no lambda, C such that O $[n]$ is n^n is bounded by $\lambda n^n + C$ for all n

$limit_{n \to \infty}$ div / solvable > 1

Assumptions

1. Define the "Null Space of $\mathcal{D}$" or "Null Set" $O_\perp$

$$O_\perp = \{\hat{d}_1, \hat{d}_2, ..., \hat{d}_j\}, \quad j > 0$$
$$\hat{O}_j[n] \equiv (O[n])^n, \forall j$$

2. $O_P \cup O_\mathcal{N} = \mathcal{D}$ (by definition)

Assertions

3. $O_P \cap O_\perp = \varnothing$
4. Let $O_\mathcal{N} \cap O_\perp = \hat{O} = \{\hat{O}_i, ...\}$, $i > 0$
5. Consider $D_j \in O_\mathcal{N}$
6. D_j has finite complexity by definition

$$O_j[n] = C$$

7. D_j has at least one optimal solution by the necessity of optimal solution (theorem Z)

$$O_j[n] = C$$

17 Proof of "P $\neq$ NP"

17.1 Proof N implies D

Is trivial by implication of Theorem x and Theorem y

$$\mathcal{N} \equiv \{d_j, ..\} \quad \forall j, \mathcal{N} \in D$$
$$\nexists K, C, \lambda_1 ... \lambda_K :$$
$$O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C, \quad \forall n, \ \forall d_j \in \mathcal{N}$$

1. $\rightarrow \mathcal{P} \cap \mathcal{N} = \varnothing$ by definition of P,N
2. $d_i \in \hat{D} \vee d_i \in$solvable
3. $1 \rightarrow d_i \notin$ solvable
4. $\therefore d_i \in \hat{D} \quad \forall i$ (theorem y) Show that Definition of Non-Polynomial Problems automatically implies Divergent

1. We've proven Solvable Union are disjoint and complete set P 2. N not in P by definition 3. therefore N in divergence by set theory

Currently we have only defined solvable problems and divergent problems
Additionally polynomial problem which the existance of is trivial
Plus we defined non-polynomial complexity
Prove the existence of $\mathcal{N}$ the set of non polynomial problems

17.2 Proof that D implies N

17.3 D iff N

Show $O[n]$ in the $\varnothing$ the set of problems with $n^n > O[n] > n^k + c$
Proving there's Polynomial and Divergent, in the set of all decision problems

A neat follow up, tie in the definition of $\mathcal{N}$ implies membership to divergent problems

18 Prove the existance of D = N,The Traveling Salesman Problem

Define the traveling salesman problem, prove it is divergent and has the same solution as current approaches
Consider proving with both definition and necessary condition

19 Theorem of Prime Numbers "Riemann Hypothesis"

Riemann Zeta Function

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} \quad [2]$$

"The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2." [2]

Prove the problem is divergent
There fore it can only be proven to a certain degree
The limit as n approaches infinity implies a real part of one half
Connection with the real and imaginary part of O $[n]$

19.1 Prove O_{opt} is testing the primes less than square root of n by induction

1. Optimal solution for n=1,2,3, everything else is a recursive optimal proof by induction
Time Complexity seems to be on the order of n log n... implies divergence or lack of bound? Add in the complexity of division.. probably approaches n^n

19.2 Show that O_{opt} diverges with n^n, isn't bounded by n^n

Proves O is divergent

19.3 Since divergent, no s^+ exists.. only rules

Express as a limit

19.4 Show that the limit as n $\rightarrow \infty$ implies the real part is 1/2

1/2 $\pm$ 14.134725 i 1/2 $\pm$ 21.022040 i 1/2 $\pm$ 25.010858 i 1/2 $\pm$ 30.424876 i 1/2 $\pm$ 32.935062 i 1/2 $\pm$ 37.586178 i

Z = $\zeta(1/2 + it)$

19.5 Notation, real imaginary parts of the problem

Even numbers and numbers ending in 5 are automatically convergent
Testing numbers ending in 1,3,7,9 results in divergent expression
we can continue to add rules to a certain degree

Citations

[1] $https://www.claymath.org/millennium-problems$
[2] $https://www.claymath.org/sites/default/files/official\_problem\_description.pdf$
[3] $http://www.math.uchicago.edu/ may/VIGRE/VIGRE2011/REUPapers/Riffer-Reinert.pdf$