Computation

1 Programs

1.1 Logical Instructions

Define $\mathcal{L}$; an ordered set of logical operations s_i

$$\mathcal{L} := \{s_1, s_2, ..., s_N\}$$

1.2 Memory

Define Memory $\mathcal{M}$; a set of elements, magnitudes, or sets b_i

$$\mathcal{M} := \{b_1, b_2, ..., b_M\}$$

1.3 State |

Define state; the memory utilized to perform program P

$$P := \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M\} =$$
$$\{s_1, s_2, ..., s_N, b_1, b_2, ..., b_M\}$$

1.4 Boolean Programs

Define a boolean program; boolean programs can represent functions with inputs x_i, input set C, and boolean output y_o

$$X = \{x_1, ..., x_n, C\}; \;\; C = \{u_1, u_2, ..., u_c\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, y_o\} =$$
$$P[X] \rightarrow y_o \in \{\mathbb{T}, \mathbb{F}\}$$

1.5 Void Programs

Define a void program; a program with inputs x_i, input set C, and no output

$$X = \{x_1, ..., x_n, C\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M\}$$

1.6 Numerical Programs

Define a numerical program; a program with inputs x_i, input set C, and real, rational output y_o

$$X = \{x_1, ..., x_n, C\}$$

$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, y_o\} =$$

$$P[X] \to y_o \in \mathbb{Q} \;\; y_o \geqslant 0$$

1.7 System Programs

Define a system program; a program with inputs x_i, input set C, and real, output set Y_o

$$X = \{x_1, ..., x_n, C\}$$

$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, Y_o\} =$$

$$P[X] \to Y_o = \{y_1, y_2, ..., y_K\}$$

1.8 Mathematical Programs

Define a mathematical program; a program with inputs x_i, input set C and numerical output y_o

$$X = \{x_1, ..., x_n, C\}$$

$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, y_o\} =$$

$$P[X] \to y_o \in \mathbb{Q}$$

2 No-op ;

2.1 Definition

$$; := \varnothing$$

2.2 Property of No-op

No-op can be inserted into any set with equality

$$S = \{s_1, s_2, ..., s_N\}$$

$$S_; = insert[S, ; , i]$$

$$S_; = S_1 \ \ \forall i$$

$$|S_;| = |S| \ \ \forall i$$

2.3 Proof

by definition of magnitude of null = 0 with Set And

3 Decision Problems

3.1 Definition

Define decision problem; a function with inputs x_i and boolean output "answer" a_o

$$X_i = \{x_1, ..., x_n, C\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

4 General Solutions

4.1 Definition

Program P is a general solution s^+ to decision problem D if

1. P outputs answer a_o for all inputs X_i $\forall i$
and
2. $s^+[X_i]$ is a subset of $s^+[\hat{X}_i]$

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_n, x_{n+1}, C\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$
$$P[X_i] = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\}$$
$$s^+ = P[X_i] = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} \quad \forall X_i$$

4.1.1 Property of No-op ;

No-op ; can be added to any solution S_i without modifying the output y_o

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$\hat{s}^+ \to \hat{y}_o = insert[s^+, ;, k]$$
$$\hat{y}_o = y_o \;\; \forall k$$

4.2 Definition of S^+

Define S^+; the set of solutions to decision problem D

$$X_i = \{x_1, ..., x_n, C\}; \ \ \hat{X}_i = \{x_1, ..., x_n, x_{n+1}, C\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s_j^+ = s_j^+[X_i] := P :$$
$$(P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$
$$S^+ := \{s_j^+, ...\} \quad \forall j$$

4.3 Definition of Solvable

Define solvable

$$X_i = \{x_1, ..., x_n, C\}; \ \ \hat{X}_i = \{x_1, ..., x_n, x_{n+1}, C\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$solvable := solvable[D] \to b_o \in \{\mathbb{T}, \mathbb{F}\} =$$
$$\exists P : (P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

5 The set of all Decision Problems $\mathbb{D}$

5.1 Definition

Define the set of decision problems $\mathbb{D}$

$$X_i = \{x_1, ..., x_n, C\}$$
$$D_j := f_j[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$\mathbb{D} := \{D_j, ...\} \quad \forall j$$

6 Instruction and Memory Notation

Define $\mathcal{L}$ a set of logical operations
Define $\mathcal{M}$ a set of memory elements, magnitudes, and sets

$$X_i = \{x_1, ..., x_n, C\};$$
$$P[X_i] \to y_o = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$\mathcal{L} := \{s_1, s_2, ..., s_{O_T[n]}\}$$
$$\mathcal{M} := \{b_1, b_2, ..., b_{O_S[n]}\}$$
$$P[X_i] = \{\mathcal{L}, \mathcal{M}, y_o\}$$

7 Complexity

7.1 Time Complexity of a Decision Problem $O_T[n]$

Define Time Complexity $O_T[n]$ of solution s^+ to Decision Problem D as the total number of logical operations

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$

$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_T[n] := |\mathcal{L}| = N$$

7.2 Space Complexity $O_S[n]$

Define Space Complexity $O_S[n]$ of solution s^+ to Decision Problem D as the total number of memory elements

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$

$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_S[n] := |\mathcal{M}| + |y_o|^* = M + 1$$

*It is convention to reserve one memory element for output y_o.
Void programs do not require the y_o memory element for output

8 Definition of Complexity

Define Complexity $O[n]$ as a vector of dimension Y

$$\mathbf{O}[n] := \; < O_T[n], O_S[n], O_3[n], O_4[n] ..., O_V[n] >$$

9 Total Complexity

$$O[n] := O_T[n] + O_S[n] + \sum_{i=3}^{V} O_i[n]$$

10 Simple Computational Complexity

The remainder of this document assumes simple computational complexity of dimension 2

10.1 Definition

Define simple computational complexity of dimension 2

$$\mathbf{O}[n] := \;\; < O_T[n], O_S[n] >$$

10.2 Time Complexity

Restate definition of Time Complexity $O_T[n]$ of solution s^+

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_T[n] := |\mathcal{L}| = N$$

10.3 Space Complexity

Restate definition of Time Complexity $O_S[n]$ of solution s^+

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_S[n] := |\mathcal{M}| + |y_o| = M + 1$$

10.4 Total Complexity

$$O[n] := O_T[n] + O_S[n]$$

$$= |\mathcal{L}| + |\mathcal{M}| + |y_o| = N + M + 1$$

10.5 $O_S[n] > 0$

10.5.1 Proof

Assume $O_S[n] = 0$

$$O_S[n] := |\mathcal{M}| + |y_o|$$

$$O_S[n] = 0 \Rightarrow \mathcal{M} = y_o = \varnothing$$

$$y_o = \varnothing; \;\; y_o \in \{\mathbb{T}, \mathbb{F}\} \text{ by definition of } s^+$$

$\therefore O_S[n] = 0$ contradicts the definition of solution s^+ of a decision problem

$$O_S[n] \geqslant 0 \text{ by definition of magnitude}$$

$$\therefore O_S[n] > 0$$

10.6 $O_T[n] > 0$

10.6.1 Proof

Assume $O_T[n] = 0$

$$O_T[n] := |\mathcal{L}|$$

$$O_T[n] = 0 \Rightarrow y_o \notin \{\mathbb{T}, \mathbb{F}\}$$

$$y_o \notin \{\mathbb{T}, \mathbb{F}\}; \ \ y_o \in \{\mathbb{T}, \mathbb{F}\} \text{ by definition of } s^+$$

$\therefore O_T[n] = 0$ contradicts the definition of solution s^+ of a decision problem

$$O_T[n] \geqslant 0 \text{ by definition of magnitude}$$

$$\therefore O_T[n] > 0$$

10.7 O[n] $>$ 0

10.7.1 Proof

$$O[n] := O_T[n] + O_S[n]$$

$$O_T[n] > 0; \ \ O_S[n] > 0$$

$$\therefore O[n] > 0$$

10.8 $O[n] > O_T[n]$

10.8.1 Proof

$$O[n] := O_T[n] + O_S[n]$$

$$O_S[n] > 0$$

$$\therefore O[n] > O_T[n]$$

10.9 $O[n] > O_S[n]$

10.9.1 Proof

$$O[n] := O_T[n] + O_S[n]$$

$$O_T[n] > 0$$

$$\therefore O[n] > O_S[n]$$

10.10 $O[n+1] \geqslant O[n]$

10.10.1 Proof

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$

$$O[n] = |\ s^+[X_i]|$$

$$O[n+1] = \hat{O}[n] = |\ s^+[\hat{X}_i]|$$

For general solutions s^+

$$s^+[\hat{X}_i] \supseteq s^+[X_i]$$

$$\Rightarrow |s^+[\hat{X}_i]| \geqslant |s^+[X_i]|$$

$$\therefore \hat{O}[n] = O[n+1] \geqslant O[n]$$

11 Polynomial Complexity

11.1 Definition

Decision problem D with solution s^+ has polynomial total complexity $O[n]$ if

$$\exists K, C, \lambda_1 ... \lambda_K \quad :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

11.2 Polynomial Problems

Define $\mathbb{P}$, the set of Decision Problems that can be solved with Polynomial Complexity

$$\mathbb{P} := \{D_1, D_2, ...\} :$$

$$\exists K, C, \lambda_1 ... \lambda_K :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, D_i \in \mathbb{P}$$

11.3 Polynomial Order of Complexity

Solution s^+ with total complexity $O[n]$ is said to be of order n^K

$$O[n] \sim n^K$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

11.4 Property of Polynomial Complexity 1

Solutions with polynomial complexity have convergent complexity

$$lim_{n \to \infty} \frac{O[n+1]}{O[n]} = 1$$

11.4.1 Proof

$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$$
$$O[n+1] = (\lambda_K (n+1))^K + (\lambda_{K-1} (n+1))^{K-1} ... + \lambda_1 (n+1) + C$$
$$= (\lambda_K n)^K + (\tilde{\lambda}_{K-1} n)^{K-1} ... + \tilde{\lambda_1} n + \tilde{C}$$

$$lim_{n \to \infty} \frac{O[n+1]}{O[n]}$$

$$= lim_{n \to \infty} \frac{(\lambda_K n)^K + (\tilde{\lambda}_{K-1} n)^{K-1} ... + \tilde{\lambda_1} n + \tilde{C}}{(\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C}$$

$$= lim_{n\to\infty}\frac{(\lambda_K n)^K}{(\lambda_K n)^K+(\lambda_{K-1}n)^{K-1}...+\lambda_1 n+C}+\frac{(\tilde{\lambda}_{K-1}n)^{K-1}}{(\lambda_K n)^K+(\lambda_{K-1}n)^{K-1}...+\lambda_1 n+C}+...+$$
$$\frac{\tilde{\lambda}_1 n}{(\lambda_K n)^K+(\lambda_{K-1}n)^{K-1}...+\lambda_1 n+C}+\frac{\tilde{C}}{(\lambda_K n)^K+(\lambda_{K-1}n)^{K-1}...+\lambda_1 n+C}$$
$$= 1 = lim_{n\to\infty}\frac{O[n+1]}{O[n]}$$

11.5 Property of Polynomial Complexity 2

$$\exists K, \hat{C}, \hat{\lambda}_1, ..., \hat{\lambda}_{K-1} :$$
$$O[n+1] - O[n] = f_{n+1}[n] = (\hat{\lambda}_{K-1}n)^{K-1}... + \hat{\lambda}_1 n + \hat{C} \quad \forall n$$

11.5.1 Proof

$$O[n] = (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C$$
$$O[n+1] = (\lambda_K(n+1))^K + (\lambda_{K-1}(n+1))^{K-1}... + \lambda_1(n+1) + C$$
$$= (\lambda_K n)^K + (\tilde{\lambda}_{K-1}n)^{K-1}... + \tilde{\lambda_1}n + \tilde{C}$$
$$O[n+1] - O[n] = ((\tilde{\lambda}_{K-1} - \lambda_{K-1})n)^{K-1}... + (\tilde{\lambda_1} - \lambda_1)n + (\tilde{C} - C)$$
$$O[n+1] - O[n] = (\hat{\lambda}_{K-1}n)^{K-1}... + \hat{\lambda}_1 n + \hat{C}$$

11.6 Total Polynomial Complexity Implies Time bounded Polynomial Complexity

$$D \in \mathbb{P} \Longrightarrow O_T[n] < (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C$$

11.6.1 Proof

$$O[n] = (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C \;\; \forall n$$
$$O[n] := O_T[n] + O_S[n]; \;\; O_S[n] > 0$$
$$\therefore O_T[n] < (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C \;\; \forall n$$

11.7 Total Polynomial Complexity Implies Space bounded Polynomial Complexity

$$D \in \mathbb{P} \Longrightarrow O_S[n] < (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C$$

11.7.1 Proof

$$O[n] = (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C \;\; \forall n$$

$$O[n] := O_T[n] + O_S[n];\ \ O_T[n] > 0$$
$$\therefore O_S[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C\ \ \forall n$$

12 Non-Polynomial Complexity

12.1 Definition

Decision problem $\tilde{D}$ with solution s^+ has non-polynomial total complexity $O[n]$ if

$$\nexists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

12.2 Non-Polynomial Problems

Define $\mathcal{N}$, the set of Decision Problems that cannot be solved with Polynomial Complexity

$$\mathcal{N} := \{\tilde{D}_1, \tilde{D}_2, ...\} :$$
$$\nexists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, \ s^+ \in S_i^+, \ \tilde{D}_i \in \mathcal{N}$$

12.3 $\mathbb{P}$ and $\mathcal{N}$ are disjoint

$$\mathbb{P} \cap \mathcal{N} = \varnothing$$

12.3.1 Proof

Let D $\in \mathcal{N}$

$$\nexists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

Assume D $\in \mathbb{P}$

$$\exists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

Contradicts the definition of $\mathcal{N}$

$$\therefore D \in \mathcal{N} \Rightarrow D \notin \mathbb{P}$$

Let D $\in \mathbb{P}$

$$\exists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

Assume D $\in \mathcal{N}$

$$\nexists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

Contradicts the definition of $\mathbb{P}$

$$\therefore D \in \mathbb{P} \Rightarrow D \notin \mathcal{N}$$

$$D \in \mathcal{N} \Rightarrow D \notin \mathbb{P}; D \in \mathbb{P} \Rightarrow D \notin \mathcal{N}$$
$$\therefore \mathbb{P} \cap \mathcal{N} = \varnothing$$

13 Divergent Complexity

13.1 Defintion

Decision problem $\hat{D}$ with solution s^+ has divergent total complexity $O[n]$ if

$$lim_{n \to \infty} \tfrac{O[n+1]}{O[n]} \ \ diverges \ \ \forall n$$

13.2 Divergent Problems

$$\hat{\mathcal{D}} := \{\hat{D}_1, \hat{D}_2, ...\} :$$
$$lim_{n \to \infty} \tfrac{O[n+1]}{O[n]} \ \ diverges \ \ \forall s^+ \in S_i^+, \ \hat{D}_i \in \hat{\mathcal{D}}$$

13.3 The Set of Polynomial Solutions and the Set of Divergent Solutions are disjoint

$$\mathbb{P} \cap \hat{D} = \varnothing$$

13.4 Proof

Let D $\in \hat{\mathcal{D}}$

$$lim_{n \to \infty} \tfrac{O[n+1]}{O[n]} \ \ diverges \text{ by definition}$$

Assume D $\in \mathbb{P}$

$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} = 1$$

$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} = 1 \text{ contradicts the definition of Divergent Problems}$$

$$\therefore D \in \hat{\mathcal{D}} \Rightarrow D \notin \mathbb{P}$$

Let D $\in \mathbb{P}$

$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} = 1 \text{ by property of Polynomial complexity}$$

Assume D $\in \hat{D}$

$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} \text{ diverges}$$

$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} \text{ diverges contradicts a property of Polynomial complexity}$$

$$\therefore D \in \mathbb{P} \Rightarrow D \notin \hat{\mathcal{D}}$$

$$\therefore \mathbb{P} \cap \hat{\mathcal{D}} = \varnothing$$

14 Inductive Functions

14.1 Inductive Function f_{n+1}

$$O[n] := O_T[n] + O_S[n]$$
$$O[n+1] = O_T[n+1] + O_S[n+1]$$

$$f_{n+1}[n] := O[n+1] - O[n]$$

14.2 Inductive Space and Time Formulas

$$f_{n+1}^T[n] := O_T[n+1] - O_T[n]$$
$$O_T[n+1] = O_T[n] + f_{n+1}^T[n]$$
$$f_{n+1}^S[n] := O_S[n+1] - O_S[n]$$
$$O_S[n+1] = O_S[n] + f_{n+1}^S[n]$$

14.3 Inductive Function Expressions

Relate $f_{n+1}[n]$ to equivalence functions

$$D \in \mathbb{P}$$
$$O[n] := O_T[n] + O_S[n]$$
$$O[n+1] = O_T[n+1] + O_S[n+1] = O[n] + f_{n+1}[n]$$
$$O_T[n] = O[n] - O_S[n]$$
$$O_S[n] = O[n] - O_T[n]$$

$$f_{n+1} = O[n+1] - O[n]$$
$$f_{n+1} = O_T[n+1] + O_S[n+1] - O[n]$$
$$f_{n+1} = O_T[n+1] - O_T[n] + O_S[n+1] - O_S[n]$$
$$f_{n+1} = O[n+1] - O_T[n] - O_S[n]$$
$$f_{n+1}[n] = f_{n+1}^T[n] + f_{n+1}^S[n]$$

14.4 Zero Order Inductive Function

$$\textit{Let } O_S[n] \sim n^0$$
$$f_{n+1} = O_T[n+1] - O_T[n] + O_S[n+1] - O_S[n] = O_T[n+1] - O_T[n]$$

14.5 Property of Polynomial Complexity

$$limit_{n\to\infty}\frac{f_{n+1}[n]}{O[n]} = 0$$

14.5.1 Proof

$$limit_{n\to\infty}\frac{O[n+1]}{O[n]} = 1$$

$$limit_{n\to\infty}\frac{O[n]+f_{n+1}[n]}{O[n]} = 1$$

$$limit_{n\to\infty}\frac{O[n]}{O[n]} + \frac{f_{n+1}[n]]}{O[n]} = 1$$

$$limit_{n\to\infty}1 + \frac{f_{n+1}[n]]}{O[n]} = 1$$

$$limit_{n\to\infty}\frac{f_{n+1}[n]]}{O[n]} = 0$$

15 Subfunctions

15.1 Restate the subfunction condition of general solutions

Recall the definition of general solution s^+

$$X_i = \{x_1, ..., x_n, C\}; \ \ \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \ \ \forall X_i) \ \ \cap \ \ (P[\hat{X}_i] \supseteq P[X_i] \ \ \forall X_i, \hat{X}_i)$$

The subfunction condition is one of two conditions for a general solution

$$P[\hat{X}_i] \supseteq P[X_i] \ \ \forall X_i, \hat{X}_i$$

15.2 Prove O[n] is a non-decreasing function

Consider solution s^+ with complexity $O[n]$

$$X_i = \{x_1, ..., x_n, C\}; \ \ \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \ \ \forall X_i) \ \ \cap \ \ (P[\hat{X}_i] \supseteq P[X_i] \ \ \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O[n] := O_T[n] + O_S[n]$$
$$O_T[n] := |\mathcal{L}| = N$$
$$O_S[n] := |\mathcal{M}| + |y_o| = M + 1$$

O[n+1] denotes the total complexity for solution $s^+[\hat{X}_i]$

$$s^+[\hat{X}_i] = \hat{s}^+$$

Let

$$O[n+1] < O[n]$$
$$\Rightarrow \hat{N} + \hat{M} < N + M$$
$$\hat{s}^+ = \{s_1, s_2, ..., s_{\hat{N}} | b_1, b_2, ..., b_{\hat{M}}, y_o\}$$

$$\Rightarrow \hat{s}^+ \not\supseteq s^+$$
$$P[\hat{X}_i] \not\supseteq P[X_i] \quad \forall X_i, \hat{X}_i$$

$$\therefore O[n+1] < O[n] \text{ contradicts the definition of solution } s^+$$
$$O[n+1] \geqslant O[n]$$

15.3 Definition of Subfunction

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub[X_i] := S = \{s_j, ... | b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \quad \forall s_j, b_k \in S$$

15.3.1 $s^+[X_i]$ is a subfunction of $s^+[\hat{X}_i]$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$\hat{s}^+ = \{s_1, s_2, ..., s_N, ..., s_{\hat{N}} | b_1, b_2, ..., b_M, ..., b_{\hat{M}}, y_o\}; \;\; \hat{N} + \hat{M} \geqslant N + M$$

By definition of solution

$$\hat{s}^+ = P[\hat{X}_i] \supseteq P[X_i] = s^+ \quad \forall X_i, \hat{X}_i$$
$$\Rightarrow s_j, b_k \in \hat{s}^+ \;\; \forall s_j, b_k \in s^+$$

15.4 Subfunction Decomposition of Solutions

FIX Double check conditions!!! Solutions s^+ can be written as the union of subfunctions $Sub_k[X_i]$

$$X_i = \{x_1, ..., x_n, C\}; \ \ \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \ \ \forall X_i) \ \ \cap \ \ (P[\hat{X}_i] \supseteq P[X_i] \ \ \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$s^+ = Sub_1[X_i] \cup Sub_2[X_i] \cup ... \cup Sub_z[X_i]$$
$$= \{\mathcal{L}_1 | \mathcal{M}_1, y_o\} \cup \{\mathcal{L}_2 | \mathcal{M}_2, y_o\} \cup ... \cup \{\mathcal{L}_z | \mathcal{M}_z, y_o\} :$$
$$\mathcal{L}_j \cap \mathcal{L}_k = \varnothing \ \ \forall j, k \neq j$$

$$s^+ = \{s_1^1, ..., s_{N_1}^1 | b_1^1, ..., y_o\} \cup \{s_1^2, ..., s_{N_2}^2 | b_1^2, ..., y_o\} \cup ... \cup \{s_1^z, ..., s_{N_z}^z | b_1^z, ..., y_o\} :$$
$$\textstyle\sum_{l=1}^{z} N_l = N = O_T[n]$$

16 Subfunction Complexity

16.1 Disjoint Subfunction Operations

$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$$

16.2 Shared Subfunction Memory

$$|\mathcal{M}_i \cap \mathcal{M}_j| \geqslant 0 \ \ \forall i, j \neq i$$

16.2.1 Time Complexity of Subfunctions

Subfunction time complexity is additive

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub_i[X] := S_i = \{s_j, ... | b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \ \ \ \forall s_j, b_k \in S_i$$

$$s^+ = \{\mathcal{L}_1 | \mathcal{M}_1, y_o\} \cup \{\mathcal{L}_2 | \mathcal{M}_2, y_o\} \cup ... \cup \{\mathcal{L}_z | \mathcal{M}_z, y_o\} :$$
$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$$

$$\mathcal{L} = \cup_{i=1}^{z} \mathcal{L}_i$$

$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$$

$$O_T[n] = |\mathcal{L}| = N$$

$$O_T[n] = | \cup_{i=1}^{z} \mathcal{L}_i| = \textstyle\sum_{i=1}^{z} |\mathcal{L}_i|^* = |\mathcal{L}_1| + |\mathcal{L}_2| + ... + |\mathcal{L}_z|$$

$$= O_{T_1}[n] + O_{T_2}[n] + ... + O_{T_z}[n] = N_1 + N_2 + ... + N_z$$

*Due to the disjoint condition of subfunction operations $\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$

16.2.2 Space Complexity of Subfunctions

Subfunctions can access the full memory $\mathcal{M}$ with no added space complexity

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub_i[X] := S_i = \{s_j, ...|b_k, ..., y_o\} :$$

$$s_j, b_k \in s^+ \quad \forall s_j, b_k \in S_i$$

$$s^+ = \{\mathcal{L}_1|\mathcal{M}_1, y_o\} \cup \{\mathcal{L}_2|\mathcal{M}_2, y_o\} \cup ... \cup \{\mathcal{L}_z|\mathcal{M}_z, y_o\} :$$

$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$$

$$s^+ = \{\mathcal{L}_1|\mathcal{M}, y_o\} \cup \{\mathcal{L}_2|\mathcal{M}, y_o\} \cup ... \cup \{\mathcal{L}_z|\mathcal{M}, y_o\} :$$

$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$$

$$\mathcal{M} = \cup_{i=1}^{z} \mathcal{M}_i = \cup_{i=1}^{z} \mathcal{M}$$

$$O_S[n] = |\mathcal{M}| = M$$

$$O_S[n] = | \cup_{i=1}^{z} \mathcal{M}_i| = M$$

16.2.3 Shared State Notation

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub_i[X] := S_i = \{s_j, ...|b_k, ..., y_o\} :$$

$$s_j, b_k \in s^+ \quad \forall s_j, b_k \in S_i$$

$$s^+ = \{\mathcal{L}_1|\mathcal{M}, y_o\} \cup \{\mathcal{L}_2|\mathcal{M}, y_o\} \cup ... \cup \{\mathcal{L}_z|\mathcal{M}, y_o\} :$$

$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \;\; \forall i, j \neq i$$

17 Polynomial Solution Subfunction Properties

17.1 Restate Definition of Subfunction

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \;\; \forall X_i) \;\; \cap \;\; (P[\hat{X}_i] \supseteq P[X_i] \;\; \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub[X_i] := S = \{s_j, ... | b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \;\; \forall s_j, b_k \in S$$

17.2 Property of Polynomial Solution Subfunctions

Let

$$D \in \mathbb{P}$$

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \;\; \forall X_i) \;\; \cap \;\; (P[\hat{X}_i] \supseteq P[X_i] \;\; \forall X_i, \hat{X}_i)$$

$$\exists K, C, \lambda_1 ... \lambda_K \;\; :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \;\; \forall n$$

$$s^+ = Sub_1[X_i] \cup Sub_2[X_i] \cup ... \cup Sub_z[X_i]$$

$$limit_{n\rightarrow\infty} \frac{O[n+1]}{O[n]} = 1$$
$$= limit_{n\rightarrow\infty} \frac{O_T^1[n+1] + O_T^2[n+1] + ... + O_T^z[n+1] + O_S[n+1]}{O_T^1[n] + O_T^2[n] + ... + O_T^z[n] + O_S[n]}$$
$$=$$
$$limit_{n\rightarrow\infty} \frac{O_T^1[n] + O_T^2[n] + ... + O_T^z[n] + O_S[n] + f^1_{T_{n+1}}[n] + f^2_{T_{n+1}}[n+1] + ... + f^z_{T_{n+1}}[n] + f_{S_{n+1}}[n]}{O_T^1[n] + O_T^2[n] + ... + O_T^z[n] + O_S[n]}$$

$$= limit_{n\rightarrow\infty} 1 + \frac{f^1_{T_{n+1}}[n] + f^2_{T_{n+1}}[n+1] + ... + f^z_{T_{n+1}}[n] + f_{S_{n+1}}[n]}{O_T^1[n] + O_T^2[n] + ... + O_T^z[n] + O_S[n]} = 1$$

$$\Rightarrow limit_{n\to\infty}\frac{f^1_{T_{n+1}}[n]+f^2_{T_{n+1}}[n+1]+...+f^z_{T_{n+1}}[n]+f_{S_{n+1}}[n]}{O^1_T[n]+O^2_T[n]+...+O^z_T[n]+O_S[n]} = 0^*$$

$$\Rightarrow limit_{n\to\infty}\frac{f^i_{T_{n+1}}[n]+f_{S_{n+1}}[n]}{O^1_T[n]+O^2_T[n]+...+O^z_T[n]+O_S[n]} = 0 \;\; \forall i$$

$$limit_{n\to\infty}\frac{f^i_{n+1}[n]}{O[n]} = 0 \;\; \forall i$$

* O[n] is a positive, non-decreasing function

18 Sum to N Problem with 2 integers

18.1 State formal definition of Sum to N : $x_i + x_j == N$

$$X_i = \{x_1, ..., x_n, N\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ = P :$$
$$(P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$
$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$D = f[X_i] = \exists x_j, x_k \in X_i : x_j + x_k == N$$

18.2 Express a formal solution : $O_S[n] \sim n^0$

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$
$$s_1 = y_o \leftarrow \mathbb{F};$$
$$\forall i, j > i$$
$$s_2, s_3, s_8, s_9, ..., s_{3ij-4}, s_{3ij-3} ..., s_{3n(n-1)-4}, s_{3n(n-1)-3} = b_1 \leftarrow x_i + x_j$$
$$s_4, s_5, s_{10}, s_{11}, ..., s_{3ij-2}, s_{3ij-1} ..., s_{3n(n-1)-2}, s_{3n(n-1)-1} = b_1 \leftarrow b_1 == N$$
$$s_6, s_7, s_{12}, s_{13} ..., s_{3ij}, s_{3ij+1} ..., s_{3n(n-1)}, s_{3n(n-1)+1} = y_o \leftarrow y_o \vee b_1$$
$$s^+ = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\}$$

18.3 Show s^+ satisfies the subfunction condition of solutions: $P[\hat{X}_i] \supseteq P[X_i] \quad \forall \hat{X}_i, X_i$

$$X_i = \{x_1, ..., x_n, N\}; \quad \hat{X}_i = \{x_1, ..., x_n, x_{n+1}, N\}$$
$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$
$$s^+_{n+1} = s^+ \cup \hat{s}^+$$
$$s_1 = y_o \leftarrow \mathbb{F};$$
$$\forall i, j > i$$
$$s_2, s_3, s_8, s_9, ..., s_{3ij-4}, s_{3ij-3} ..., s_{3n(n-1)-4}, s_{3n(n-1)-3} = b_1 \leftarrow x_i + x_j$$
$$s_4, s_5, s_{10}, s_{11}, ..., s_{3ij-2}, s_{3ij-1} ..., s_{3n(n-1)-2}, s_{3n(n-1)-1} = b_1 \leftarrow b_1 == N$$

$$s_6, s_7, s_{12}, s_{13}..., s_{3ij}, s_{3ij+1}..., s_{3n(n-1)}, s_{3n(n-1)+1} = y_o \leftarrow y_o \vee b_1$$

$$\forall k$$

$$s... = b_1 \leftarrow x_k + x_{n+1}$$

$$s... = b_1 \leftarrow b_1 == N$$

$$s... = y_o \leftarrow y_o \vee b_1$$

$$s^+ = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\}$$

$$\hat{s}^+ = \{y_o \leftarrow y_o \vee (x_k + x_{n+1} == N) \ \forall k < n+1 | \ b_1, y_o\}$$

$$s^+_{n+1} = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\} \ \cup$$
$$\{y_o \leftarrow y_o \vee (x_k + x_{n+1} == N) \ \forall k < n+1 | \ b_1, y_o\}$$

$$s^+_{n+1} = P[\hat{X}_i] \supseteq P[X_i] = s^+$$

18.4 Determine $O[n], O_S[n], O_T[n], \hat{O}[n], \hat{O}_T[n], \hat{O}_S[n]$ for the above solution

$$O_S[n] = |y_o| + |b_1| = 2$$
$$O_T[n] = 3n(n-1) + 1 = 3n(n-1) - 1 + O_S[n]$$
$$O[n] = 3n(n-1) + 3 = 3n^2 - 3n + 3$$

$$\hat{O}_S[n] = 0$$
$$\hat{O}_T[n] = 6n$$
$$\hat{O}[n] = \hat{O}_S[n] + \hat{O}_T[n]$$

18.5 Verify $O[n+1] = O[n] + \hat{O}[n]$

$$O[n+1] = O[n] + \hat{O}[n]$$
$$3(n+1)^2 - 3(n+1) + 3 = 3n^2 - 3n + 3 + 6n$$
$$3n^2 + 6n + 3 - 3n - 3 + 3 = 3n^2 + 3n + 3$$
$$3n^2 + 3n + 3 = 3n^2 + 3n + 3$$

18.6 Show s^+ has Polynomial Complexity by the definition of Total Polynomial Complexity

$$O[n] = 3n^2 - 3n + 3$$

18.7 Show s^+ has Polynomial Complexity by showing limit$_{n\to\infty}\frac{O[n+1]}{O[n]}$ = 1

$$limit_{n\to\infty}\frac{O[n+1]}{O[n]} =$$

$$limit_{n\to\infty}\frac{3n^2+3n+3}{3n^2-3n+3} =$$

$$limit_{n\to\infty}(\frac{3n^2-3n+3}{3n^2-3n+3} + \frac{6n}{3n^2-3n+3}) =$$

$$limit_{n\to\infty}(1 + \frac{6n}{3n^2-3n+3}) = 1$$

Traveling Salesman Problem of Dimension 2

19 Proof of the existence of $\mathcal{N}$

19.1 The Traveling Salesman Problem of Dimension 2

English description

19.2 Formal Definition

$$X_i = \{l_1, l_2, ..., l_n, C\}$$

$$l_i = \{x_i, y_i\} \ \ \forall i$$

l_i denotes the 2D coordinates of location i

$$C = \{d_{proposed}, p_{decimal}\}$$

$d_{proposed}$ denotes the suggested shortest distance

$p_{decimal}$ is the decimal precision

$$L[l_i, l_j] := \sqrt{(y_j - y_i)^2 + (x_j - x_i)^2}$$

Let $L[l_i, l_j]$ denote the distance between location l_i and l_j

$$\tilde{L}[l_i, l_j] := d_{trunc} : -p_{decimal} < d_{trunc} - L[l_i, l_j] < p_{decimal}$$

Let $\tilde{L}[l_i, l_j]$ denote a truncated decimal representation of $L[l_i, l_j]$

$$R_i := \{r_1, r_2, ..., r_n, r_1\} \ \ : \ \ r_i \in X_i \ \ \forall i; \ \ r_i \neq r_j$$

Let R_i denote route i

$$L_{Total}[R_i] := (\textstyle\sum_{i=1}^{n-1} \tilde{L}[r_i, r_{i+1}]) + \tilde{L}[r_n, r_1]$$

Let $L_{Total}[R_i]$ denote the sum of truncated lengths of route R_i

$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \ \ \forall X_i$$

$$a_o =$$

$$(\exists R_k : L_{total}[R_k] == d_{proposed}) \cap (\nexists R_j : L_{total}[R_j] < d_{proposed})$$

19.3 Define subpath, subpath distance, subpath storage

$\tilde{L}[l_i, l_j]$ denotes "the distance of a subpath of length 1"

$$\tilde{L}[l_i, l_j] := d_{trunc} : -p_{decimal} < d_{trunc} - L[l_i, l_j] < p_{decimal}$$
$$= abs(d_{trunc} - L[l_i, l_j]) < p_{decimal}$$

$\tilde{R}$ denotes a subpath of length k

$$\tilde{R} = \{\tilde{r}_1, \tilde{r}_2, ..., \tilde{r}_k\} \;\; : \;\; \tilde{r}_i \in X_i \;\; \forall i, r_i \neq r_j$$

$\tilde{L}_k[\tilde{R}]$ denotes "the distance of a subpath of length k"

$$\tilde{L}_k[\tilde{R}] := \textstyle\sum_{i=1}^{k} \tilde{L}[\tilde{r}, \tilde{r}_{i+1}]$$

Let $\mathcal{M}_1$ denote the memory reserved for subpaths distances of length 1

$$\mathcal{M}_1 = \{\hat{b}_{1;1}, \hat{b}_{1;2}, \hat{b}_{1;3}, ..., \hat{b}_{startindex;finishindex}, ..., \hat{b}_{n-1;n}\}^*$$
$$\mathcal{M} \supseteq \mathcal{M}_1$$

* Note $\hat{b}_{i;j} = \hat{b}_{j;i}$
$\sqrt{(y_j - y_i)^2 + (x_j - x_i)^2} = \sqrt{(y_i - y_j)^2 + (x_i - x_j)^2}$

19.4 Define the following functions

19.4.1 $sqrt[x, p_{decimal}] = \sqrt{x}$ [1]

19.4.2 $pow[x, 2, p_{decimal}] = x^2$ [2]

19.5 Define the following subfunctions

19.5.1 loadM1Subpaths$[X]$

// Computes all subpaths of length 1 and stores in $\mathcal{M}_1 = \{\hat{b}_{1;1}, \hat{b}_{1,;2}, ..., \hat{b}_{n-1;n}\}$

$$//X_i = \{l_1, l_2, ..., l_n, C\}$$
$$//l_i = \{x_i, y_i\} \;\; \forall i$$

$$//\mathcal{M} = \{b_1, b_2, ..., b_M, \hat{b}_{1;1}, \hat{b}_{1,;2}, ..., \hat{b}_{n-1;n}, y_o\} = \{b_1, b_2, ..., b_M, \mathcal{M}_1, y_o\} = \{\mathcal{M}, \mathcal{M}_1, y_o\}$$

$$\forall i, j > i$$

$$b_3 \leftarrow y_i - y_j$$

$$b_4 \leftarrow x_i - x_j$$

$$b_3 \leftarrow b_3^2$$

$$b_4 \leftarrow b_4^2$$

$$b_3 \leftarrow b_3 + b_4$$

$$\hat{b}_{i;j} \leftarrow \sqrt{b_3}^*$$

${}^*\hat{b}_{i;j} = \tilde{L}[l_i, l_j]$

19.5.2 computeAllRoutes$[X]$

// Computes all complete routes, checks for a route $==$ $d_{proposed}$, sets y_o to false if the current route is shorter than $d_{proposed}$

$$\forall i, j \neq i, k \neq i, j..., q \neq i, j, ..., m$$

$$b_3 \leftarrow \hat{b}_{1;j} + \hat{b}_{j;k}$$

$$b_3 \leftarrow b_3 + \hat{b}_{k;l}$$

$$...$$

$$b_3 \leftarrow b_3 + \hat{b}_{m;q}$$

$$b_3 \leftarrow b_3 + \hat{b}_{q;1}$$

$$b_4 \leftarrow b_3 == b_2$$

$$b_1 \leftarrow b_1 \vee b_4$$

$$b_4 \leftarrow b_2 \leqslant b_3$$

$$y_o \leftarrow y_o \wedge b_4$$

19.6 Express a solution using subfunctions, storing subpaths of length 1 in memory

// $d_{proposed}$ is the shortest path

$$y_o \leftarrow \mathbb{T}$$

$$//d_{proposed} \text{ exists as a total path length}$$
$$b_1 \leftarrow \mathbb{F}$$

$$// \text{ shortest path register}$$
$$b_2 \leftarrow d_{proposed}$$

$$loadM1Subpaths[X]$$
$$computeAllRoutes[X]$$

19.7 Show each subfunction satisfies the subfunction condition of solutions : $P[\hat{X}_i] \supseteq P[X_i] \ \ \forall \hat{X}_i, X_i, \ \ \hat{X}_i \supseteq X_i$

Let

$$\mathcal{M}_0 = \{b_1, b_2, b_3, b_4, y_o\}$$
$$\mathcal{M}_1 = \{\hat{b}_{1;1}, \hat{b}_{1,;2}, ..., \hat{b}_{n-1;n}\}$$

19.7.1 $loadM1Subpaths[X] \rightarrow \mathcal{M}_1$

Let

$$//X = \{l_1, l_2, ..., l_n, C\}; \ \ \hat{X} = \{l_1, l_2, ..., l_n, l_{n+1}, C\}$$

$$loadM1Subpaths[X, \mathcal{M}] \rightarrow \mathcal{M}_1 = Sub_1[X, \mathcal{M}] \rightarrow \mathcal{M}_1$$

$$Sub_1[X, \mathcal{M}] = \{\mathcal{L}, \mathcal{M}\}$$
$$= \{\hat{b}_{i;j} \leftarrow \tilde{L}[l_i, l_j] \ \ \forall i, j > i | b_3, b_4, \hat{b}_{1;1}, \hat{b}_{1;2}, ..., \hat{b}_{n-1;n}\}$$
$$Sub_1[X_i, \mathcal{M}] = \{\hat{b}_{i;j} \leftarrow \tilde{L}[l_i, l_j] \ \ \forall i, j > i | b_3, b_4, \mathcal{M}_1\}$$

$$Sub_1[\hat{X}, \mathcal{M}] = \{\hat{\mathcal{L}}, \hat{\mathcal{M}}\}$$
$$= \{\hat{b}_{i;j} \leftarrow \tilde{L}[l_i, l_j] \ \ \forall i, j > i | b_3, b_4, \hat{b}_{1;1}, \hat{b}_{1;2}, ..., \hat{b}_{n;n+1}\}$$
$$= \{\mathcal{L}, \hat{b}_{i;j} \leftarrow \tilde{L}[l_i, l_j] \ \ \forall i, j = n+1 | \mathcal{M}, \hat{b}_{1;n+1}, \hat{b}_{2;n+1}, ..., \hat{b}_{n;n+1}\}$$
$$Sub_1[\hat{X}, \mathcal{M}] = \{\mathcal{L}, \mathcal{L}_{n+1} | \mathcal{M}, \mathcal{M}_{n+1}\}$$
$$Sub_1[\hat{X}, \mathcal{M}] = \{\mathcal{L}, \mathcal{L}_{n+1} | \mathcal{M}, \mathcal{M}_{n+1}\} \supseteq \{\mathcal{L} | \mathcal{M}\} = Sub_1[X, \mathcal{M}]$$

19.7.2 $computeAllRoutes[X]$

Let

$$computeAllRoutes[X] = Sub_2[X]$$

$$Sub_2[X] = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$= \{\hat{b}_{1;i_2} + \hat{b}_{i_2;i_3} + \hat{b}_{i_3;i_4} + ... + \hat{b}_{i_n;1} \ \ \forall i_2, i_3 \neq i_2, i_4 \neq i_2, i_3 ... i_n \neq i_2, i_3 ..., i_{n-1}$$
$$|b_1, b_2, b_3, b_4, \mathcal{M}_1, y_o\}$$

$$Sub_2[\hat{X}] = \{\hat{\mathcal{L}}, \hat{\mathcal{M}}, y_o\}$$

$$= \{\hat{b}_{1;i_2} + \hat{b}_{i_2;i_3} + \hat{b}_{i_3;i_4} + ... + \hat{b}_{i_{n+1};1} \ \ \forall i_2, i_3 \neq i_2, i_4 \neq i_2, i_3 ... i_{n+1} \neq i_2, i_3 ..., i_n$$
$$|\mathcal{M}, \mathcal{M}_{n+1}, y_o\}$$

Let

$$insert\_subpath[\mathcal{L}] =$$

$$Sub_2[\hat{X}] = \{insert\_subpath[\mathcal{L}, \hat{b}_{i_{n+1};j}, j] \ \ \forall j \neq n+1 | \mathcal{M}, \mathcal{M}_{n+1}, y_o\}$$

19.7.3 Show the overall solution storing subpaths of length 1 satisfies the subfunction condition of solutions : $P[\hat{X}_i] \supseteq P[X_i] \ \ \forall \hat{X}_i, X_i$

19.8 Express $O[n]$ in terms of subfunction complexities

$$O_{sub1}[n] = O_{T_{sub1}}[n] + O_{S_{sub1}}[n]$$

$$O_{sub2}[n] = O_{T_{sub2}}[n] + O_{S_{sub2}}[n]$$

$$O[n] = O_{sub1}[n] + O_{sub2}[n] + 3$$

19.9 $\mathbf{Sub_+[X]}$

Let $Sub_+[X]$ denote a subfunction that adds all subpaths of length 1
Let $O_+[n]$ denote the total complexity of subfunction $Sub_+[X]$

19.9.1 Find an expression for $O_+[n]$:= the number of $\tilde{L}[l_i, l_j] + \tilde{L}[l_j, l_k]$ length 1 subpath additions

$$O_+[n] = (\textstyle\sum_{i=1}^{n} 1)\frac{(nP(n-1)}{2}$$

$$O_+[n] = \frac{n(n-1)!}{2}$$
$$O_+[n] = \frac{n!}{2}$$

19.10 Prove $Sub_+[X]$ is a subfunction of all s^+ by contradiction

suppose not all subpaths are considered
there could exist subpath resulting in an incorrect solution
contradicts definition of solution

19.11 Show the solution storing subpaths of length 1 contains Sub$_+$[X]

$$//\ d_{proposed} \text{ is the shortest path}$$
$$y_o \leftarrow \mathbb{T}$$

$$//d_{proposed} \text{ exists as a total path length}$$
$$b_1 \leftarrow \mathbb{F}$$

$$//\text{ shortest path register}$$
$$b_2 \leftarrow d_{proposed}$$

$$loadM1Subpaths[X]$$
$$computeAllRoutes[X]$$

19.12 Express $O[n]$ in terms of subfunction complexities including $O_+[n]$ as a subfunction complexity

$$O[n] = O_{sub1}[n] + O_{sub2}[n] + 3$$
$$O_{sub2}[n] = O_+[n] + 8\frac{nP(n-1)}{2}$$
$$O_{sub2}[n] = O_+[n] + 8\frac{(n-1)!}{2}$$
$$O[n] = O_{sub1}[n] + O_+[n] + 8\frac{(n-1)!}{2} + 3$$

19.13 Show $limit_{n\to\infty}\frac{O_+[n+1]}{O_+[n]}$ diverges

$$limit_{n\to\infty}\frac{O_+[n+1]}{O_+[n]}$$

$$= limit_{n\to\infty} \frac{(n+1)!}{2} \frac{2}{n!}$$
$$= limit_{n\to\infty}\, n$$

There does not exist ...
therefore $limit_{n\to\infty} \frac{O_+[n+1]}{O_+[n]}$ diverges

19.14 Prove D is not in $\mathbb{P}$ by the definition of Polynomial Complexity

Let

$Sub_i[X]$ denote all subfunctions of a solution s^+ except $Sub_+[X]$

$$O[n] = O_+[n] + \sum_{i=1}^{z} O_{i_T}[n]$$
$$O[n] = \frac{n!}{2} + \sum_{i=1}^{z} O_{i_T}[n]$$

Assume O[n] satisfies the condition of Polynomial total complexity for $n = \hat{n}$
Let n = K!!!

$$\exists K, C, \lambda_1 ... \lambda_K \quad :$$
$$O[\hat{n}] = (\lambda_K \hat{n})^K + (\lambda_{K-1}\hat{n})^{K-1} ... + \lambda_1 \hat{n} + C$$
$$= \frac{\hat{n}!}{2} + \sum_{i=1}^{z} O_{i_T}[\hat{n}]$$

Let $n = \hat{n} + 1$

$$O[\hat{n}+1] = (\lambda_K \hat{n})^K + (\hat{\lambda_{K-1}}\hat{n})^{K-1} ... + \hat{\lambda_1}\hat{n} + \hat{C} \sim (\lambda_K \hat{n})^K$$
$$\neq \frac{(\hat{n}+1)!}{2} + \sum_{i=1}^{z} O_{i_T}[\hat{n}+1] \sim (\lambda_K \hat{n})^{K+1}$$

$$\therefore D \in \mathbb{P} \text{ is a contradiction}$$

$$\therefore D \notin \mathbb{P} \Rightarrow D \in \mathcal{N}$$

19.15 Prove D is not in $\mathbb{P}$ by limit property of Polynomial Complexity

Let

$Sub_i[X]$ denote all subfunctions of a solution s^+ except $Sub_+[X]$

$$O[n] = O_+[n] + \sum_{i=1}^{z} O_{i_T}[n]$$
$$O[n] = \frac{n!}{2} + \sum_{i=1}^{z} O_{i_T}[n]$$

Assume O[n] satisfies the condition of Polynomial total complexity

$$\exists K, C, \lambda_1 ... \lambda_K \quad :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$$

$$f_{+n+1}[n] = O_+[n+1] - O_+[n] = \frac{(n+1)!}{2} - \frac{n!}{2}$$
$$= \frac{n!(n+1-1)}{2} = \frac{n * n!}{2}$$

$$limit_{n\to\infty} \frac{O[n+1]}{O[n]} = 1$$
$$limit_{n\to\infty} \frac{\frac{(n+1)!}{2} + \sum_{i=1}^{z} O_{i_T}[n+1]}{(\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C}$$
$$limit_{n\to\infty} \frac{\frac{(n+1)!}{2}}{(\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C} + \frac{\sum_{i=1}^{z} O_{i_T}[n+1]}{(\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C}$$

$$limit_{n\to\infty} \frac{\frac{(n+1)!}{2}}{(\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C} \text{ diverges}$$

$O_{i_T}[n+1]$ is a positive, non-decreasing function

$$\therefore limit_{n\to\infty} \frac{O[n+1]}{O[n]} \text{ diverges}$$

$\therefore D \in \mathbb{P}$ contradicts the limit property of Polynomial solutions

$$\therefore D \notin \mathbb{P} \Rightarrow D \in \mathcal{N}$$

Citations

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[4] $http://www.math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/Riffer-Reinert.pdf$