Ch. 5 Computation

1 Programs

1.1 Logical Instructions

Define $\mathcal{L}$; an ordered set of logical operations s_i

$$\mathcal{L} := \{s_1, s_2, ..., s_N\}$$

1.2 Memory

Define Memory $\mathcal{M}$; a set of elements

$$\mathcal{M} := \{b_1, b_2, ..., b_M\}$$

1.3 State |

Define state; the memory utilized to perform program P

$$P := \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M\} =$$
$$\{s_1, s_2, ..., s_N, b_1, b_2, ..., b_M\}$$

1.4 Boolean Programs

Define a boolean program; boolean programs can represent functions with inputs x_i and boolean output y_o

$$X = \{x_1, ..., x_n\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, y_o\} =$$
$$P[X] \to y_o \in \{\mathbb{T}, \mathbb{F}\}$$

1.5 Void Programs

Define a void program; a program with inputs x_i and no output

$$X = \{x_1, ..., x_n\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M\}$$

1.6 Numerical Programs

Define a numerical program; a program with inputs x_i and real, rational output y_o

$$X = \{x_1, ..., x_n\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, y_o\} =$$
$$P[X] \rightarrow y_o \in \mathbb{Q} \;\; y_o \geqslant 0$$

1.7 System Programs

Naming convention to be formalized; a program that outputs one or more elements

$$X = \{x_1, ..., x_n\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, Y_o\} =$$
$$P[X] \rightarrow Y_o = \{y_1, y_2, ..., y_K\}$$

1.8 Mathematical Programs

Define a mathematical program; a program with inputs x_i and numerical output y_o

$$X = \{x_1, ..., x_n\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, y_o\} =$$
$$P[X] \rightarrow y_o \in \mathbb{Q}$$

2 No-op ;

2.1 Definition

$$; := \varnothing$$

2.2 Property of No-op

No-op can be inserted into any set with equality

$$S = \{s_1, s_2, ..., s_N\}$$
$$S_; = insert[S, ;, i]$$
$$S_; = S_1 \ \ \forall i$$
$$|S_;| = |S| \ \ \forall i$$

2.3 Proof

by definition of magnitude of null = 0 with Set And

3 Decision Problems

3.1 Definition

Define decision problem; a function with inputs x_i and boolean output "answer" a_o

$$X_i = \{x_1, ..., x_n\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

4 Solutions

4.1 Definition

Program P is a solution s^+ if P outputs answer a_o for all inputs X_i $\forall i$
s^+ is a function of the number of inputs n

$$X_i = \{x_1, ..., x_n\}; \quad \hat{X}_i = \{x_1, ..., x_n, x_{n+1}\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ = s^+[n] := P :$$
$$(P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$
$$P[X_i] = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\}$$
$$s^+ = P[X_i] = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} \quad \forall X_i$$

4.1.1 Property of No-op ;

No-op ; can be added to any solution S_i and remain a solution for all i anywhere in the order for all j

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$\hat{s}^+ = insert[s^+, ;, k]$$
$$\hat{s}^+ = s^+ \quad \forall k$$

4.2 Definition of S^+

Define S^+; the set of solutions to decision problem D

$$X_i = \{x_1, ..., x_n\}; \quad \hat{X}_i = \{x_1, ..., x_n, x_{n+1}\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$s_j^+ = s_j^+[n] := P :$$
$$(P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$
$$S^+ := \{s_j^+, ...\} \quad \forall j$$

4.3 Definition of Solvable

Define solvable

$$X_i = \{x_1, ..., x_n\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$solvable := solvable[D] \to b_o \in \{\mathbb{T}, \mathbb{F}\} =$$
$$\exists P : (P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

5 The set of all Decision Problems $\mathbb{D}$

5.1 Definition

Define the set of decision problems $\mathbb{D}$

$$X_i = \{x_1, ..., x_n\}$$
$$D_j := f_j[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$\mathbb{D} := \{D_j, ...\} \quad \forall j$$

6 Instruction and Memory Notation

Define $\mathcal{L}$ a set of logical operations
Define $\mathcal{M}$ a set of bits "memory"

$$P[X_i] \to y_o = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$\mathcal{L} := \{s_1, s_2, ..., s_{O_T[n]}\}$$
$$\mathcal{M} := \{b_1, b_2, ..., b_{O_S[n]}\}$$
$$P[X_i] = \{\mathcal{L}, \mathcal{M}, y_o\}$$

7 Complexity

7.1 Time Complexity of a Decision Problem $O_T[n]$

Define Time Complexity $O_T[n]$ of Decision Problem D with solution s^+

$$X_i = \{x_1, ..., x_n\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ := P[X_i] \to y_o : y_o = a_o \quad \forall X_i =$$
$$\{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$
$$O_T[n] := |\mathcal{L}| = N$$

7.2 Space Complexity $O_S[n]$

Define Space Complexity $O_S[n]$ of Decision Problem D with solution s^+

$$X_i = \{x_1, ..., x_n\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ := P[X_i] \to y_o : y_o = a_o \quad \forall X_i =$$
$$\{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$
$$O_S[n] := |\mathcal{M}| + |y_o| = M$$

8 Definition of Complexity

Define Complexity $O[n]$ as a vector of dimension C

$$\mathbf{O}[n] := \; < O_T[n], O_S[n], O_3[n], O_4[n]..., O_C[n] >$$

9 Total Complexity

$$O[n] := O_T[n] + O_S[n] + \sum_{i=3}^{C} O_i[n]$$

10 Simple Computational Complexity

The remainder of this chapter assumes simple computational complexity of dimension 2

10.1 Definition

Define simple computational complexity of dimension 2

$$\mathbf{O}[n] := \; < O_T[n], O_S[n] >$$

10.2 Time Complexity

Restate definition of Time Complexity $O_T[n]$

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_T[n] := |\mathcal{L}| = N$$

10.3 Space Complexity

Restate definition of Time Complexity $O_S[n]$

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_S[n] := |\mathcal{M}| + |y_o| = M$$

10.4 Total Complexity

$$O[n] := O_T[n] + O_S[n]$$

$$= |\mathcal{L}| + |\mathcal{M}| + |y_o| = N + M$$

10.5 $O_S[n] \neq 0$

10.5.1 Proof

By definition of decision problem; Proof by contradiction; y_o must be set to TF by definition; Suppose yo = 0; then yo is empty set; contradicts definition of D

10.6 $O_T[n] \neq 0$

10.6.1 Proof

By definition of decision problem; Proof by contradiction; y_o must be set to TF by definition; Suppose $|\mathrm{L}| = 0$; yo <- TF cap L is null by definition of empty set; implies yo emptyset (doesnt exist)

10.7 O[n] $= O_T[n] + O_S[n] \neq \mathbf{0}$

10.7.1 Proof

10.8 $O[n] > O_T[n]$

10.8.1 Proof

10.9 $O[n] > O_S[n]$

10.9.1 Proof

10.10 $O[n+1] \geqslant O[n]$

11 Optimal Complexity

11.1 Definition

Define Optimal Complexity; the minimum total complexity required to solve a decision problem

$$O_{opt}[n] :=$$
$$\nexists \hat{O}[n] : \hat{O}[n] < O_{opt}[n] \;\; \forall n$$

11.2 Proof of Existence

Prove the existence of at least one $O_{min}[n]$ by induction/contradiction

12 Optimal solution

Define an optimal solution s^+_{opt}

12.1 Definition

$$X_i = \{x_1, ..., x_n\}$$
$$D_j := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ := P[X_i] \rightarrow y_o : y_o = a_o \quad \forall X_i$$
$$s^+_{opt} := s^+ :$$
$$\nexists \hat{O}[n] < O_{opt}[n] \quad \forall n, \; s^+ \in S^+_j$$

12.2 Optimal Time Complexity Solution

$$X_i = \{x_1, ..., x_n\}$$
$$D_j := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ := P[X_i] \rightarrow y_o : y_o = a_o \quad \forall X_i =$$
$$\{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, X_i, y_o\} = \{\mathcal{L}, \mathcal{M}, X_i, y_o\}$$
$$O_T[n] := |\mathcal{L}| = N$$
$$s^+_T := s^+ :$$
$$\nexists \hat{O_T}[n] < O_T[n] \quad \forall n, \; s^+ \in S^+_j$$

12.3 Optimal Space Complexity Solution

$$X_i = \{x_1, ..., x_n\}$$
$$D_j := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ := P[X_i] \to y_o : y_o = a_o \quad \forall X_i =$$
$$\{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, X_i, y_o\} = \{\mathcal{L}, \mathcal{M}, X_i, y_o\}$$
$$O_S[n] := |\mathcal{M}| = M$$
$$s_S^+ := s^+ :$$
$$\nexists \hat{O}_S[n] < O_S[n] \quad \forall n, \ s^+ \in S_j^+$$

12.4 Conjecture of Optimal Solutions

$O_{T_{min}}$ subject to $O_{S_{opt}} = 1$

$$\exists s_{opt}^+ :$$
$$O_{opt}[n] = 1 + O_T[n] = O_{S_{opt}} + O_T[n] \quad \forall s^+ \in S^+$$

12.4.1 Proof

12.5 Theorem of Optimal Solutions

Alternate way of expressing $O_{opt}[n]$ possibly with efficiency function and equivalence functions?
$f_{S\to T}[n, O_S[n]]$ as a function of space complexity order K?
Efficiency function ($f_{S\to T}[n, O_S[n]]$ as a function of space complexity order K) is strictly decreasing for Polynomial Functions
Or there's an inflection point
Efficiency function might have a general pattern for all problems in D

13 Polynomial Complexity

13.1 Definition

Decision problem D with solution s^+ has polynomial total complexity $O[n]$ if

$$\exists K, C, \lambda_1 ... \lambda_K \quad :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

13.2 Polynomial Problems

Define $\mathbb{P}$, the set of Decision Problems that can be solved with Polynomial Complexity

$$\mathbb{P} := \{D_1, D_2, ...\} :$$
$$\exists K, C, \lambda_1 ... \lambda_K :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, D_i \in \mathbb{P}$$

13.3 Polynomial Order of Complexity

Total complexity $O[n]$ is said to be of order K_{opt}

$$O[n] \sim K_{opt}$$
$$O_{opt}[n] := O[n] :$$
$$\nexists \hat{O}[n] < O_{opt}[n] \;\; \forall n$$
$$O_{opt}[n] < (\lambda_{K_{opt}} n)^{K_{opt}} + (\lambda_{K_{opt}-1} n)^{K_{opt}-1} ... + \lambda_1 n + C \quad \forall n$$
$$K_{opt} := K :$$
$$\nexists \hat{K} : O_T[n] < (\lambda_{\hat{K}} n)^{\hat{K}} + (\lambda_{\hat{K}-1} n)^{\hat{K}-1} ... + \lambda_1 n + C \quad \forall n, \; \hat{K} < K$$

13.4 Corrolary of Optimal Complexity

$$\nexists s^+ \in S^+ :$$
$$O_T[n] < (\lambda_{\hat{K}} n)^{\hat{K}} + (\lambda_{\hat{K}-1} n)^{\hat{K}-1} ... + \lambda_1 n + C \quad \forall n, \; \hat{K} < K_{opt}$$

13.4.1 Proof

Proof by contradiction; definition of optimal complexity

13.5 Property of Polynomial Complexity 1

$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} = 1$$

13.5.1 Proof WIP

FIX!!! Show there exists no constant satisfying the decreasing limit condition

$$O[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$$
$$O[n+1] < (\lambda_K (n+1))^K + (\lambda_{K-1} (n+1))^{K-1} ... + \lambda_1 (n+1) + C$$
$$O[n] \sim (\lambda n)^K;\;\; O[n+1] \sim (\lambda n)^K$$
$$lim_{n\to\infty}\frac{(\lambda n)^K}{(\lambda n)^K} = 1$$

13.6 Property of Polynomial Complexity 2

$$\exists K, C, \lambda_1, ..., \lambda_K :$$
$$(O[n+1] - O[n]) = f_{n+1}[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

13.6.1 Proof FIX!!!

$$O[n+1] < (\lambda_K (n+1))^K + (\lambda_{K-1} (n+1))^{K-1} ... + \lambda_1 (n+1) + C$$
$$O[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$$

13.7 Total Polynomial Complexity Implies Time bounded Polynomial Complexity

$$D \in \mathbb{P} \Longrightarrow O_T[n] < ...$$

13.7.1 Proof FIX!!!

$$O[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \;\; \forall n$$
$$O[n] := O_T[n] + O_S[n];\;\; O_T[n] < O[n]$$
$$\therefore O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \;\; \forall n$$

13.8 Total Polynomial Complexity Implies Space bounded Polynomial Complexity

$$D \in \mathbb{P} \Longrightarrow O_S[n] < ...$$

13.8.1 Proof FIX!!!

$$O[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \ \ \forall n$$

$$O[n] := O_T[n] + O_S[n]; \ \ O_S[n] < O[n]$$

$$\therefore O_S[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \ \ \forall n$$

13.9 Total Polynomial Complexity iff Time and Space bounded by Polynomial Complexity

Use limit definition

13.10 Order of Complexity

ERROR in second condition
Total Complexity is said to be on the order of K_{max}

$$O[n] \sim K_{max}$$

$$K_{max} := K :$$

$$O[n] < (\lambda_{K_{max}} n)^{K_{max}} + (\lambda_{K_{max}-1} n)^{K_{max}-1} ... + \lambda_1 n + C \quad \forall n$$

$$\nexists O[n] < (\lambda_{\hat{K}_{max}} n)^{\hat{K}_{max}} + (\lambda_{\hat{K}_{max}-1} n)^{\hat{K}_{max}-1} ... + \lambda_1 n + C \quad \forall n, \hat{K} < K_{max}$$

13.11 Theorem Either OT or OS is on the order of Oopt

Proof by contradiction

14 Polynomial Time Complexity

14.1 Definition

Decision problem D with (optimal) Time Complexity $O_T[n]$ is bounded by polynomial time complexity if

$$\exists K, C, \lambda_1 ... \lambda_K \quad :$$
$$O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

14.2 Polynomial Time Solutions

Define $\mathbb{S}^+_{time}$, the set of solutions that can be solved with polynomial time complexity

$$\mathbb{S}^+_{time} := \{s^+_1, s^+_2, ...\} :$$
$$\exists K, C, \lambda_1 ... \lambda_K :$$
$$O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, s_i \in \mathbb{S}^+_{time}$$

14.3 Property of Polynomial Time Complexity 1

$$lim_{n \to \infty} \frac{O_T[n+1]}{O_T[n]} = 1$$

14.3.1 Proof

14.4 Property of Polynomial Time Complexity 2

$$\exists K, C, \lambda_1, ..., \lambda_K :$$
$$(O_T[n+1] - O_T[n]) < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \ \forall n$$

14.4.1 Proof

14.5 Order of Complexity

Time complexity $O_T[n]$ is said to be on the order of K_{max}

$$O_T[n] < (\lambda_{K_{max}} n)^{K_{max}} + (\lambda_{K_{max}-1} n)^{K_{max}-1} ... + \lambda_1 n + C$$
$$O_T[n] \sim K_{max}$$

14.6 Proof of the existence of $O_{T_{opt}}$

15 Polynomial Space Complexity

15.1 Defintion

Decision problem D with (optimal) Time Complexity $O_S[n]$ is bounded by polynomial time complexity if

$$\exists K, C, \lambda_1 ... \lambda_K \quad :$$
$$O_S[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

15.2 Polynomial Space Problems

Define $\mathbb{S}^+_{space}$, the set of solutions that can be solved with polynomial time complexity

$$\mathbb{S}^+_{space} := \{s_1, s_2, ...\} :$$
$$\exists K, C, \lambda_1 ... \lambda_K :$$
$$O_S[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, s_i \in \mathbb{S}^+_{time}$$

15.3 Total Polynomial Complexity Implies Space bounded Polynomial Complexity

15.4 Space Bounded Polynomial Complexity Implies Total Polynomial Complexity

15.5 Polynomial Space Complexity iff Polynomial Complexity

15.6 Property of Polynomial Space Complexity 1

$$lim_{n \to \infty} \frac{O_S[n+1]}{O_S[n]} = 1$$

15.6.1 Proof

15.7 Property of Polynomial Space Complexity 2

$$\exists K, C, \lambda_1, ..., \lambda_K :$$
$$(O_S[n+1] - O_S[n]) < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \ \forall n \ \forall n$$

15.7.1 Proof

15.8 Order of Complexity

Space complexity $O_S[n]$ is said to be on the order of K_{max}

$$O_S[n] < (\lambda_{K_{max}} n)^{K_{max}} + (\lambda_{K_{max}-1} n)^{K_{max}-1} ... + \lambda_1 n + C$$
$$O_S[n] \sim K_{max}$$

15.9 Proof of the existence of $O_{S_{opt}}$

16 Inductive Functions

16.1 Inductive Function f_{n+1}

$$O[n] := O_T[n] + O_S[n]$$
$$O[n+1] = O_T[n+1] + O_S[n+1]$$

$$f_{n+1}[n] := f[n] :$$
$$O[n+1] = f[n] + O[n] \quad \forall n$$

16.1.1 Proof of existence

Algebraic Proof

16.2 Inductive Space and Time Formulas

$$f^T_{n+1} := O_T[n+1] - O_T[n]$$
$$O_T[n+1] = O_T[n] + f^T_{n+1}$$
$$f^S_{n+1} := O_S[n+1] - O_S[n]$$
$$O_S[n+1] = O_S[n] + f^S_{n+1}$$

16.2.1 Proof of existence

Algebraic Proof

16.3 Inductive Function Expressions

Relate $f_{n+1}[n]$ to equivalence functions

$$D \in \mathbb{P}$$
$$O[n] := O_T[n] + O_S[n]$$
$$O[n+1] = O_T[n+1] + O_S[n+1] = O[n] + f_{n+1}[n]$$
$$O_T[n] = O[n] - O_S[n]$$
$$O_S[n] = O[n] - O_T[n]$$

$$f_{n+1} = O[n+1] - O[n]$$
$$f_{n+1} = O_T[n+1] + O_S[n+1] - O[n]$$

$$f_{n+1} = O_T[n+1] - O_T[n] + O_S[n+1] - O_S[n]$$
$$f_{n+1} = O[n+1] - O_T[n] - O_S[n]$$
$$f_{n+1}[n] = f^T_{n+1}[n] + f^S_{n+1}[n]$$

16.4 Zero Order Inductive Function

$$Let\ \ O_S[n] \sim n^0$$
$$f_{n+1} = O_T[n+1] - O_T[n] + O_S[n+1] - O_S[n] = O_T[n+1] - O_T[n]$$

16.5 Property of Polynomial Complexity

$f_{n+1}[n]$ has order less than O[n]
$f_{n+1}[n]$ is bound by K_{max} - 1

16.5.1 Proof

Proof by contradiction; limit doesn't converge

17 Duality of $O_T[n]$, $O_S[n]$?

17.0.1 O_T to O_S

Define equivalence function $f_{T \to S}$; a function converting logical operations into memory elements

$$f_{T \to S} := f :$$
$$O_S[n] = f[n, O_T[n]] \quad \forall n, s^+ \in S^+$$

17.0.2 O_S to O_T

Define equivalence function $f_{S \to T}$; a function converting memeory elements into logical operations

$$f_{S \to T} := f :$$
$$O_T[n] = f[n, O_S[n]] \quad \forall n, s^+ \in S^+$$

17.0.3 Invertibility?

17.0.4 Polynomial Bounded?

17.1 Efficiency Function?

Function relating the decrease in O[n] as $O_S[n]$ increases in order
$f_{S \to T}[n, O_S[n]]$ as a function of space complexity order K?

18 Theorem of Computational Duality?

For all Problems in P there exists a duality function
Formally define dynamic programming, Optimal polynomial complexity minimizes the difference between time and space complexity order

$$D \in \mathbb{P}$$
$$O[n] := O_T[n] + O_S[n]$$
$$limit_{n \to \infty} \frac{O[n+1]}{O[n]} = 1 \quad \forall s^+ \in S^+_{\mathbb{P}}$$

$$O_T[n] = f_{S \to T}[n, O_S[n]]$$
$$O_S[n] = f_{T \to S}[n, O_T[n]]$$

$$limit_{n\to\infty}\frac{O_T[n+1]+O_S[n+1]}{O_T[n]+O_S[n]} = 1 \quad \forall s^+ \in S^+_{\mathbb{P}}$$

$$limit_{n\to\infty}\frac{f_{S\to T}[n+1, O_S[n+1]]+O_S[n+1]}{f_{S\to T}[n, O_S[n]]+O_S[n]} = 1 \quad \forall s^+ \in S^+_{\mathbb{P}}$$

$$limit_{n\to\infty}\frac{O_T[n+1]+f_{T\to S}[n+1, O_T[n+1]]}{O_T[n]+f_{T\to S}[n, O_T[n]]} = 1 \quad \forall s^+ \in S^+_{\mathbb{P}}$$

19 Subfunctions

19.1 Restate the subfunction condition of solutions

19.2 Show O[n] is a strictly increasing function

19.3 Theorem of Polynomial Subfunctions

19.4 Theorem of Divergent Subfunctions

20 Sum to N Problem with 2 integers

20.1 State formal definition of Sum to N : $x_i + x_j == N$

$$X_i = \{x_1, ..., x_n, N\}$$
$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ = P :$$
$$(P[X_i] \rightarrow y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$
$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$D = f[X_i] = \exists x_j, x_k \in X_i : x_j + x_k == N$$

20.2 Express a formal solution : $O_S[n] \sim n^0$

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$
$$s_1 = y_o \leftarrow \mathbb{F};$$
$$\forall i, j > i$$
$$s_2, s_3, s_8, s_9, ..., s_{3ij-4}, s_{3ij-3} ..., s_{3n(n-1)-4}, s_{3n(n-1)-3} = b_1 \leftarrow x_i + x_j$$
$$s_4, s_5, s_{10}, s_{11}, ..., s_{3ij-2}, s_{3ij-1} ..., s_{3n(n-1)-2}, s_{3n(n-1)-1} = b_1 \leftarrow b_1 == N$$
$$s_6, s_7, s_{12}, s_{13} ..., s_{3ij}, s_{3ij+1} ..., s_{3n(n-1)}, s_{3n(n-1)+1} = y_o \leftarrow y_o \vee b_1$$
$$s^+ = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\}$$

20.3 Show s^+ satisfies the subfunction condition of solutions: $P[\hat{X}_i] \supseteq P[X_i] \quad \forall \hat{X}_i, X_i$

$$X_i = \{x_1, ..., x_n, N\}; \quad \hat{X}_i = \{x_1, ..., x_n, x_{n+1}, N\}$$
$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$
$$s^+_{n+1} = s^+ \cup \hat{s}^+$$
$$s_1 = y_o \leftarrow \mathbb{F};$$
$$\forall i, j > i$$
$$s_2, s_3, s_8, s_9, ..., s_{3ij-4}, s_{3ij-3} ..., s_{3n(n-1)-4}, s_{3n(n-1)-3} = b_1 \leftarrow x_i + x_j$$
$$s_4, s_5, s_{10}, s_{11}, ..., s_{3ij-2}, s_{3ij-1} ..., s_{3n(n-1)-2}, s_{3n(n-1)-1} = b_1 \leftarrow b_1 == N$$

$$s_6, s_7, s_{12}, s_{13}..., s_{3ij}, s_{3ij+1}..., s_{3n(n-1)}, s_{3n(n-1)+1} = y_o \leftarrow y_o \vee b_1$$

$$\forall k$$

$$s... = b_1 \leftarrow x_k + x_{n+1}$$

$$s... = b_1 \leftarrow b_1 == N$$

$$s... = y_o \leftarrow y_o \vee b_1$$

$$s^+ = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\}$$

$$\hat{s}^+ = \{y_o \leftarrow y_o \vee (x_k + x_{n+1} == N) \ \ \forall k < n+1 | \ b_1, y_o\}$$

$$s^+_{n+1} = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\} \ \cup$$
$$\{y_o \leftarrow y_o \vee (x_k + x_{n+1} == N) \ \ \forall k < n+1 | \ b_1, y_o\}$$

$$s^+_{n+1} = P[\hat{X}_i] \supseteq P[X_i] = s^+$$

20.4 Determine $O[n], O_S[n], O_T[n], \hat{O}[n], \hat{O}_T[n], \hat{O}_S[n]$ for the above solution

$$O_S[n] = |y_o| + |b_1| = 2$$
$$O_T[n] = 3n(n-1) + 1 = 3n(n-1) - 1 + O_S[n]$$
$$O[n] = 3n(n-1) + 3 = 3n^2 - 3n + 3$$

$$\hat{O}_S[n] = 0$$
$$\hat{O}_T[n] = 6n$$
$$\hat{O}[n] = \hat{O}_S[n] + \hat{O}_T[n]$$

20.5 Verify $O[n+1] = O[n] + \hat{O}[n]$

$$O[n+1] = O[n] + \hat{O}[n]$$
$$3(n+1)^2 - 3(n+1) + 3 = 3n^2 - 3n + 3 + 6n$$
$$3n^2 + 6n + 3 - 3n - 3 + 3 = 3n^2 + 3n + 3$$
$$3n^2 + 3n + 3 = 3n^2 + 3n + 3$$

20.6 Show s^+ has Polynomial Complexity by the definition of Total Polynomial Complexity

$$O[n] = 3n^2 - 3n + 3$$

20.7 Show s^+ has Polynomial Complexity by showing $\mathbf{limit}_{n\to\infty}\frac{O[n+1]}{O[n]}$ = 1

$$limit_{n\to\infty}\frac{O[n+1]}{O[n]} =$$

$$limit_{n\to\infty}\frac{3n^2+3n+3}{3n^2-3n+3} =$$

$$limit_{n\to\infty}(\frac{3n^2-3n+3}{3n^2-3n+3} + \frac{6n}{3n^2-3n+3}) =$$

$$limit_{n\to\infty}(1 + \frac{6n}{3n^2-3n+3}) = 1$$

21 Divergent Problems

21.1 Definition

$$\hat{\mathcal{D}} := \{\hat{D}_1, \hat{D}_2, ...\} :$$
$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} \ \ diverges \ \ \forall \hat{D} \in \hat{\mathcal{D}}$$

21.2 Theorem of Divergent Subfunctions

If an (inductive) subfunction of s^+ diverges, the solution is divergent

$$f_{n+1} = \sum g_{n+1}$$
$$limit\frac{g[n+1]}{g[n]}diverges$$

$$\exists g_{n+1} : limit\frac{g_{n+1}}{O[n]}diverges \Longrightarrow limit\frac{O[n+1]}{O[n]}diverges$$

21.3 Proof

21.4 The Set of Polynomial Solutions and the Set of Divergent Solutions are disjoint

$$\mathbb{P} \cap \hat{D} = \varnothing$$

21.5 Proof

Proof by contradiction; Let $s^+ \in \mathbb{P}, \hat{D}; s^+ \in \mathbb{P} \cap \hat{D}$

$$X_i = \{x_1, ..., x_n\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$Let \ \ D \in \mathbb{P}$$
$$s^+ := P[X_i] \to y_o : y_o = a_o \quad \forall X_i$$
$$O[n] = O_T[n] + O_S[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

Traveling Salesman Problem of Dimension 2

22 Proof of the existence of $\hat{\mathcal{D}}$

22.1 The Traveling Salesman Problem of Dimension 2

English description

22.2 Formal Definition

$$X_i = \{l_1, l_2, ..., l_n, C\}$$

$$l_i = \{x_i, y_i\} \;\; \forall i$$

l_i denotes the 2D coordinates of location i

$$C = \{d_{proposed}, p_{decimal}\}$$

$d_{proposed}$ denotes the suggested shortest distance

$p_{decimal}$ is the decimal precision

$$L[l_i, l_j] := \sqrt{(y_j - y_i)^2 + (x_j - x_i)^2}$$

Let $L[l_i, l_j]$ denote the distance between location l_i and l_j

$$\tilde{L}[l_i, l_j] := d_{trunc} : -p_{decimal} < d_{trunc} - L[l_i, l_j] < p_{decimal}$$

Let $\tilde{L}[l_i, l_j]$ denote a truncated decimal representation of $L[l_i, l_j]$

$$R_i := \{r_1, r_2, ..., r_n, r_1\} \;\; : \;\; r_i \in X_i \;\; \forall i; \;\; r_i \neq r_j$$

Let R_i denote route i

$$L_{Total}[R_i] := (\textstyle\sum_{i=1}^{n-1} \tilde{L}[r_i, r_{i+1}]) + \tilde{L}[r_n, r_1]$$

Let $L_{Total}[R_i]$ denote the sum of truncated lengths of route R_i

$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \;\; \forall X_i$$

$$a_o =$$

$$(\exists R_k : L_{total}[R_k] == d_{proposed}) \cap (\nexists R_j : L_{total}[R_j] < d_{proposed})$$

Traveling Salesman Problem of Dimension 2

22.3 Define subpath, subpath distance, subpath storage

$\tilde{L}[l_i, l_j]$ denotes "the distance of a subpath of length 1"

$$\begin{aligned}\tilde{L}[l_i, l_j] &:= d_{trunc} : -p_{decimal} < d_{trunc} - L[l_i, l_j] < p_{decimal} \\ &= abs(d_{trunc} - L[l_i, l_j]) < p_{decimal}\end{aligned}$$

$\tilde{R}$ denotes a subpath of length k

$$\tilde{R} = \{\tilde{r}_1, \tilde{r}_2, ..., \tilde{r}_k\} \ \ : \ \ \tilde{r}_i \in X_i \ \ \forall i, r_i \neq r_j$$

$\tilde{L}_k[\tilde{R}]$ denotes "the distance of a subpath of length k"

$$\tilde{L}_k[\tilde{R}] := \textstyle\sum_{i=1}^{k} \tilde{L}[\tilde{r}, \tilde{r}_{i+1}]$$

Let $\mathcal{M}_1$ denote the memory reserved for subpaths distances of length 1

$$\mathcal{M}_1 = \{\hat{b}_{1;1}, \hat{b}_{1;2}, \hat{b}_{1;3}, ..., \hat{b}_{startindex;finishindex}, ..., \hat{b}_{n-1;n}\}^*$$
$$\mathcal{M} \supseteq \mathcal{M}_1$$

* Note $\hat{b}_{i;j} = \hat{b}_{j;i}$
$\sqrt{(y_j - y_i)^2 + (x_j - x_i)^2} = \sqrt{(y_i - y_j)^2 + (x_i - x_j)^2}$

22.4 Define the following functions

22.4.1 $sqrt[x, p_{decimal}] = \sqrt{x}$ [1]

22.4.2 $pow[x, 2, p_{decimal}] = x^2$ [2]

22.5 Define the following subfunctions

22.5.1 loadM1Subpaths$[X]$

// Computes all subpaths of length 1 and stores in $\mathcal{M}_1 = \{\hat{b}_{1;1}, \hat{b}_{1,;2}, ..., \hat{b}_{n-1;n}\}$

$$//X_i = \{l_1, l_2, ..., l_n, C\}$$
$$//l_i = \{x_i, y_i\} \ \ \forall i$$

$$//\mathcal{M} = \{b_1, b_2, ..., b_M, \hat{b}_{1;1}, \hat{b}_{1,;2}, ..., \hat{b}_{n-1;n}, y_o\} = \{b_1, b_2, ..., b_M, \mathcal{M}_1, y_o\} = \{\mathcal{M}, \mathcal{M}_1, y_o\}$$

$$\forall i, j > i$$

$$b_3 \leftarrow y_i - y_j$$

$$b_4 \leftarrow x_i - x_j$$

$$b_3 \leftarrow b_3^2$$

$$b_4 \leftarrow b_4^2$$

$$b_3 \leftarrow b_3 + b_4$$

$$\hat{b}_{i;j} \leftarrow \sqrt{b_3}^*$$

$^*\hat{b}_{i;j} = \tilde{L}[l_i, l_j]$

22.5.2 computeAllRoutes$[X]$

// Computes all complete routes, checks for a route $==$ $d_{proposed}$, sets y_o to false if the current route is shorter than $d_{proposed}$

$$\forall i, j \neq i, k \neq i, j ..., q \neq i, j, ..., m$$

$$b_3 \leftarrow \hat{b}_{1;j} + \hat{b}_{j;k}$$

$$b_3 \leftarrow b_3 + \hat{b}_{k;l}$$

$$...$$

$$b_3 \leftarrow b_3 + \hat{b}_{m;q}$$

$$b_3 \leftarrow b_3 + \hat{b}_{q;1}$$

$$b_4 \leftarrow b_3 == b_2$$

$$b_1 \leftarrow b_1 \vee b_4$$

$$b_4 \leftarrow b_2 \leqslant b_3$$

$$y_o \leftarrow y_o \wedge b_4$$

22.6 Express a solution using subfunctions, storing subpaths of length 1 in memory

// $d_{proposed}$ is the shortest path

$$y_o \leftarrow \mathbb{T}$$

$$//d_{proposed} \text{ exists as a path}$$
$$b_1 \leftarrow \mathbb{F}$$

$$// \text{ shortest path register}$$
$$b_2 \leftarrow d_{proposed}$$

$$loadM1Subpaths[X]$$
$$computeAllRoutes[X]$$

22.7 Show each subfunction satisfies the subfunction condition of solutions : $P[\hat{X}_i] \supseteq P[X_i] \quad \forall \hat{X}_i, X_i, \quad \hat{X}_i \supseteq X_i$

Let

$$\mathcal{M}_0 = \{b_1, b_2, b_3, b_4, y_o\}$$
$$\mathcal{M}_1 = \{\hat{b}_{1;1}, \hat{b}_{1,;2}, ..., \hat{b}_{n-1;n}\}$$

22.7.1 $loadM1Subpaths[X] \rightarrow \mathcal{M}_1$

Let

$$//X = \{l_1, l_2, ..., l_n, C\}; \ \ \hat{X} = \{l_1, l_2, ..., l_n, l_{n+1}, C\}$$

$$loadM1Subpaths[X, \mathcal{M}] \rightarrow \mathcal{M}_1 = Sub_1[X, \mathcal{M}] \rightarrow \mathcal{M}_1$$

$$Sub_1[X, \mathcal{M}] = \{\mathcal{L}, \mathcal{M}\}$$
$$= \{\hat{b}_{i;j} \leftarrow \tilde{L}[l_i, l_j] \ \ \forall i, j > i | b_3, b_4, \hat{b}_{1;1}, \hat{b}_{1;2}, ..., \hat{b}_{n-1;n}\}$$
$$Sub_1[X_i, \mathcal{M}] = \{\hat{b}_{i;j} \leftarrow \tilde{L}[l_i, l_j] \ \ \forall i, j > i | b_3, b_4, \mathcal{M}_1\}$$

$$Sub_1[\hat{X}, \mathcal{M}] = \{\hat{\mathcal{L}}, \hat{\mathcal{M}}\}$$
$$= \{\hat{b}_{i;j} \leftarrow \tilde{L}[l_i, l_j] \ \ \forall i, j > i | b_3, b_4, \hat{b}_{1;1}, \hat{b}_{1;2}, ..., \hat{b}_{n;n+1}\}$$
$$= \{\mathcal{L}, \hat{b}_{i;j} \leftarrow \tilde{L}[l_i, l_j] \ \ \forall i, j = n+1 | \mathcal{M}, \hat{b}_{1;n+1}, \hat{b}_{2;n+1}, ..., \hat{b}_{n;n+1}\}$$
$$Sub_1[\hat{X}, \mathcal{M}] = \{\mathcal{L}, \mathcal{L}_{n+1} | \mathcal{M}, \mathcal{M}_{n+1}\}$$
$$Sub_1[\hat{X}, \mathcal{M}] = \{\mathcal{L}, \mathcal{L}_{n+1} | \mathcal{M}, \mathcal{M}_{n+1}\} \supseteq \{\mathcal{L} | \mathcal{M}\} = Sub_1[X, \mathcal{M}]$$

22.7.2 $computeAllRoutes[X]$

Let

$$computeAllRoutes[X] = Sub_2[X]$$

$$Sub_2[X] = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$= \{\hat{b}_{1;i_2} + \hat{b}_{i_2;i_3} + \hat{b}_{i_3;i_4} + ... + \hat{b}_{i_n;1} \ \ \forall i_2, i_3 \neq i_2, i_4 \neq i_2, i_3 ... i_n \neq i_2, i_3 ..., i_{n-1}$$
$$|b_1, b_2, b_3, b_4, \mathcal{M}_1, y_o\}$$

$$Sub_2[\hat{X}] = \{\hat{\mathcal{L}}, \hat{\mathcal{M}}, y_o\}$$

$$= \{\hat{b}_{1;i_2} + \hat{b}_{i_2;i_3} + \hat{b}_{i_3;i_4} + ... + \hat{b}_{i_{n+1};1} \ \ \forall i_2, i_3 \neq i_2, i_4 \neq i_2, i_3 ... i_{n+1} \neq i_2, i_3 ..., i_n$$
$$|\mathcal{M}, \mathcal{M}_{n+1}, y_o\}$$

Let

$$insert\_subpath[\mathcal{L}] =$$

$$Sub_2[\hat{X}] = \{insert\_subpath[\mathcal{L}, \hat{b}_{i_{n+1};j}, j] \ \ \forall j \neq n+1 | \mathcal{M}, \mathcal{M}_{n+1}, y_o\}$$

22.7.3 Show the overall solution storing subpaths of length 1 satisfies the subfunction condition of solutions : $P[\hat{X}_i] \supseteq P[X_i] \ \ \forall \hat{X}_i, X_i$

22.8 Express $O[n]$ in terms of subfunction complexities

$$O_{sub1}[n] = O_{T_{sub1}}[n] + O_{S_{sub1}}[n]$$
$$O_{sub2}[n] = O_{T_{sub2}}[n] + O_{S_{sub2}}[n]$$
$$O[n] = O_{sub1}[n] + O_{sub2}[n] + 3$$

22.9 Find an expression for $O_+[n]$:= the number of $\tilde{L}[l_i, l_j] + \tilde{L}[l_j, l_k]$ length 1 subpath additions

$$O_+[n] =$$

22.10 Prove $O_+[n]$ is a subfunction of all s^+ by contradiction

22.11 Express the solution that stores subpaths of length 1 in memory in terms of $O_+[n]$

22.12 Express $O[n], O_T[n], O_S[n], f_{n+1}$ in terms of subfunction complexities including $O_+[n]$ as a subfunction

22.13 Show $O_+[n]$ diverges

22.14 Prove D diverges by the theorem of divergent subfunctions

23 Universal Bound of Computation?

maybe

$$O[n] < n^n \quad \forall s^+$$

23.1 Show Polynomial Union Divergent Solutions represent the universe of solutions

23.2 Show Polynomial Solutions are bounded by n^n

23.3 Show Divergent Solutions are bounded by n^n?

24 Theorem of Prime Numbers "Riemann Hypothesis"

Riemann Zeta Function

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} \quad [2]$$

"The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2." [3]

Prove the problem is divergent
There fore it can only be proven to a certain degree
The limit as n approaches infinity implies a real part of one half
Connection with the real and imaginary part of O $[n]$

24.1 Determine a duality function for the Riemann Hypothesis

24.2 Determine an expression for O[n+1] as a function of O[n]

24.3 Prove O_{opt} is performing O_{opt} recursively for the ints less than square root of n

Testing the primes less than sqrt(n)? double check
1. Optimal solution for n=1,2,3, everything else is a recursive optimal proof by induction
Time Complexity seems to be on the order of n log n... implies divergence or lack of bound? Add in the complexity of division.. probably approaches n^n

24.4 Since divergent, no s^+ exists.. only rules

Express as a limit

24.5 Show that the limit as n $\rightarrow \infty$ implies the real part is 1/2

1/2 ± 14.134725 i 1/2 ± 21.022040 i 1/2 ± 25.010858 i 1/2 ± 30.424876 i 1/2 ± 32.935062 i 1/2 ± 37.586178 i

$Z = \zeta(1/2 + it)$

24.6 Notation, real imaginary parts of the problem

Even numbers and numbers ending in 5 are automatically convergent
Testing numbers ending in 1,3,7,9 results in divergent expression
we can continue to add rules to a certain degree

25 Divergent Problems

Define $\hat{\mathbb{D}}$ the set of decision problems with no convergent?/finite? solution $\hat{D}_j$

$$\hat{\mathbb{D}} := \{\hat{D}_j, ...\}$$
$$\hat{D}_j \in \mathbb{D} \quad \forall j$$
$$\nexists \hat{s}^+ \in S^+ : \hat{s}^+ \text{ solves } \hat{D}_j \quad \forall \hat{D}_j \in \hat{\mathbb{D}} \Longleftrightarrow$$
$$\nexists s^+ \in S^+ : O_j[n] < n^n \quad \forall n, j$$

There exists no such solution such that O[n] $< n^n$, but there is a right and wrong answer
Either here or in the next chapter we'll prove you can only solve to a certain degree
!!! There exists no such solution such that O[n] $< n^n$ $\quad \forall$ n

25.1 Definition

$$\hat{O}[n] := n^n$$
$$limit_{n\to\infty} \frac{O_{opt}[n]}{\hat{O}[n]} \ diverges$$
$$diverges[\ O_{opt}[n], \hat{O}[n]\ ] \to \mathbb{T}$$

25.2 Theorem of Divergent Programs

Prove that Divergent implies not in polynomial (trivial)
Prove that Divergent implies Non-polynomial (trial after proving above)
Show that there exists at least one member of Divergent

26 Properties of Solvable and Divergent problems

26.1 Solvable and Divergent are disjoint

Prove by contradiction

27 "Theorem of Divergent Programs"

27.1 Divergence Test

1. Let $d_j \in D$
2. $d_j = (d_j \in \hat{\mathcal{D}}) \cup (d_j \in$ set of solvable problems) by disjoint condition of solvable and divergent
3. Let $O_{opt}[n]$, the optimal complexity of d_j
4. $\rightarrow s_i^+$ that solve d_j have larger complexity $\forall i$
5. 2 implies $O_{opt}[n]$ is either bounded by n^n or not
6. $\hat{O}[n] \equiv n^n$
7. Easy Suppose $d_j \in$solvable$limit_{n \to \infty} \frac{O_{opt}[n]}{\hat{O}[n]} = 0$
8.Suppose $d_j \in \hat{\mathcal{D}} limit_{n \to \infty} \frac{O_{opt}[n]}{\hat{O}[n]} \neq 0$ (by disjoint condition)

$$limit_{n \to \infty} \frac{O_{opt}[n]}{\hat{O}[n]} = 1$$

28 Connection to verification in polynomial time

29 Fundamental Theorem of Computation

n^n or $\lambda n^n + C$ the universal bound to solvable computational complexity $(\lambda n)^n$ + C ?

29.1 Time Complexity Argument

Suppose decision problem d with optimal time complexity $O_{T_{min}}[n]$ and solution s^+, an arbitrary decision problem in P with polynomial complexity
Assumptions
1. $d \in P$, $s^+ \in S^+$
Assertions
2. $\exists K, C, \lambda_1 ... \lambda_K \;\; : \;\; O_{T_{min}}[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C, \quad \forall n$
3. Define $f[K, C, \lambda_1, ... \lambda_K] \equiv (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$
4. $\exists K, C, \lambda_1 ... \lambda_K : O_{T_{min}}[n] < f[K, C, \lambda_1, ... \lambda_K] \quad \forall n$
5. Let $\hat{s}^+ \equiv \circlearrowright^n[s^+]$
6. $O_T[n] <= \hat{O}_T[n]$ (by defition of nested loop)

7. $\hat{O}_{T_{min}}[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$
8. $\hat{O}_{T_{min}}[n] < limit_{n \to \infty} \circlearrowright^n[s^+]$ (by definition of limit + definition of nested loop, expand to show full derivation, valid because this is a series, probably need to show limit applies)
9. $\therefore O_{T_{min}}[n] < \hat{O}_{T_{min}}[n] < n^n = limit_{n \to \infty} \circlearrowright^n[s^+]$
I want to say forall n but seems refutable for n= 1,2.. but as n approach infinity it's a contradiction to say a solvable problem in P $\hat{O}_{T_{min}} = n^n \quad \forall n$
10. For "sufficiently large n"

$$\nexists \hat{s}^+ \in S^+ : |\hat{s}^+| \equiv O_{T_{min}}[n] < n^n, \quad \forall n$$
$$\hat{O}[n] \equiv n^n$$

29.2 Space Argument

Similar but additional notation required?

30 Divergence Criterion

Necessary condition for divergent program, iff
or you can show there exists no lambda, C such that O $[n]$ is n^n is bounded by $\lambda n^n + C$ for all n

$limit_{n\to\infty}$ div / solvable > 1

Assumptions

1. Define the "Null Space of $\mathcal{D}$" or "Null Set" $O_\perp$

$$O_\perp = \{\hat{d}_1, \hat{d}_2, ..., \hat{d}_j\}, \quad j > 0$$
$$\hat{O}_j[n] \equiv (O[n])^n, \forall j$$

2. $O_P \cup O_\mathcal{N} = \mathcal{D}$ (by definition)

Assertions

3. $O_P \cap O_\perp = \varnothing$
4. Let $O_\mathcal{N} \cap O_\perp = \hat{O} = \{\hat{O}_i, ...\}$, $i > 0$
5. Consider $D_j \in O_\mathcal{N}$
6. D_j has finite complexity by definition

$$O_j[n] = C$$

7. D_j has at least one optimal solution by the necessity of optimal solution (theorem Z)

$$O_j[n] = C$$

31 Proof of "P $\neq$ NP"

31.1 Proof N implies D

Is trivial by implication of Theorem x and Theorem y

$$\mathcal{N} \equiv \{d_j, ..\} \quad \forall j, \mathcal{N} \in D$$
$$\nexists K, C, \lambda_1 ... \lambda_K :$$
$$O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C, \quad \forall n,\ \forall d_j \in \mathcal{N}$$

1. $\rightarrow \mathcal{P} \cap \mathcal{N} = \varnothing$ by definition of P,N
2. $d_i \in \hat{D} \vee d_i \in$solvable
3. $1 \rightarrow d_i \notin$ solvable
4. $\therefore d_i \in \hat{D} \quad \forall i$ (theorem y) Show that Definition of Non-Polynomial Problems automatically implies Divergent

1. We've proven Solvable Union are disjoint and complete set P 2. N not in P by definition 3. therefore N in divergence by set theory

Currently we have only defined solvable problems and divergent problems
Additionally polynomial problem which the existance of is trivial
Plus we defined non-polynomial complexity
Prove the existence of $\mathcal{N}$ the set of non polynomial problems

31.2 Proof that D implies N

31.3 D iff N

Show $O[n]$ in the $\varnothing$ the set of problems with $n^n > O[n] > n^k + c$
Proving there's Polynomial and Divergent, in the set of all decision problems

A neat follow up, tie in the definition of $\mathcal{N}$ implies membership to divergent problems

32 Prove the existance of D = N,The Traveling Salesman Problem

Define the traveling salesman problem, prove it is divergent and has the same solution as current approaches
Consider proving with both definition and necessary condition

32.1 Compute every sub path or recursive subpaths in memory

Trade off between time and space, $O_{salesman}[n]$ diverges with a polynomial $O_P[n]$

33 Prove Polynomial and Divergent problems are Complements

Implied by the previous sections

34 Solvable Union Divergent = all decision problems

Trivial as a result of the previous section by definition of Ω

$$\mathbb{P} \cup \hat{\mathbb{D}} = \mathbb{D}$$

35 Theorem of Prime Numbers "Riemann Hypothesis"

Riemann Zeta Function

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} \quad [2]$$

"The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2." [2]

Prove the problem is divergent
There fore it can only be proven to a certain degree
The limit as n approaches infinity implies a real part of one half
Connection with the real and imaginary part of O $[n]$

35.1 Prove O_{opt} is performing O_{opt} recursively for the ints less than square root of n

Testing the primes less than sqrt(n)? double check
1. Optimal solution for n=1,2,3, everything else is a recursive optimal proof by induction
Time Complexity seems to be on the order of n log n... implies divergence or lack of bound? Add in the complexity of division.. probably approaches n^n

35.2 Since divergent, no s^+ exists.. only rules

Express as a limit

35.3 Show that the limit as n $\rightarrow \infty$ implies the real part is 1/2

1/2 ± 14.134725 i 1/2 ± 21.022040 i 1/2 ± 25.010858 i 1/2 ± 30.424876 i 1/2 ± 32.935062 i 1/2 ± 37.586178 i

Z $= \zeta(1/2 + it)$

35.4 Notation, real imaginary parts of the problem

Even numbers and numbers ending in 5 are automatically convergent
Testing numbers ending in 1,3,7,9 results in divergent expression
we can continue to add rules to a certain degree

Citations

[1] $chatgpt$
[2] $https://stackoverflow.com/questions/3518973/floating-point-exponentiation-without-power-function$
[3] $https://stackoverflow.com/questions/27086195/linear-index-upper-triangular-matrix$
[3] $http://www.math.uchicago.edu/ may/VIGRE/VIGRE2011/REUPapers/Riffer-Reinert.pdf$

Appendix

Traveling Salesman Problem of Dimension 2

35.5 Express a formal solution with $\mathcal{M}_{n+1}$

35.6 Show the $\mathcal{M}_{n+1}$ solution satisfies the subfunction condition of solutions: $P[\hat{X}_i] \supseteq P[X_i] \quad \forall \hat{X}_i, X_i$

35.7 Show the $\mathcal{M}_{n+1}$ solution has additions greater than or equal to $O_+[n]$

35.8 Express the $\mathcal{M}_{n+1}$ solution with $O_+[n]$ as a subfunction

35.9 Express O[n] for all solutions with $O_+[n]$ as a subfunction

Traveling Salesman Problem of Dimension 2

35.10 **Express a formal, general solution** $O_S[n] \sim n^0$

35.11 **Express the zero order Space Complexity Solution as the union of subfunctions**

35.12 **Show the** $O_S[n] \sim n^0$ **solution satisfies the subfunction condition of solutions**: $P[\hat{X}_i] \supseteq P[X_i] \quad \forall \hat{X}_i, X_i$

35.13 **Find** $O[n], O_T[n], O_S[n], f_{n+1}$ **for the zero order Space Complexity Solution subfunctions**

$$O_S[n] = |y_o| + |\{b_1, b_2, b_3\}| = 4$$
$$O_T[n] =$$

35.14 **Find** $O[n], O_T[n], O_S[n], f_{n+1}$ **for the overall zero order Space Complexity Solution**

Traveling Salesman Problem of Dimension 2

35.15 Express a formal, general $\mathcal{M}_1$ solution

$$\text{Traveling Salesman Solution} \quad n == N \quad O_S[n] \sim n^0$$

$$X_i = \{x_1, x_2, ..., x_n, C\}$$

$$C = \{d_{proposed}, p_{decimal}\}; \quad |C| = |\{d_{proposed}, p_{decimal}\}| = 2$$

$$y_o \;\; d_{proposed} \text{ is the shortest distance}$$

$$b_1 \;\; d_{proposed} \text{ the proposed shortest distance}$$

$$b_2 \text{ current path length}$$

$$b_3 \text{ intermediate path length or } d_{proposed} < \text{current path}$$

// Set y_o and b_1
$s_1 = b_1 \leftarrow d_{proposed}$
$s_2 = y_o \leftarrow \mathbb{T}$

// Calculate $\mathcal{M}_1$ subpaths := x_i to x_j $\forall i, j > i$
// Calculate (n P (n-1))/2 routes
// Let current route = $\{x_{c1}, x_{c2}, ..., x_{cn}\}$, $x_{ci} \in X_i$ $\forall x_{ci}$
$s, s = b_2 \leftarrow L[x_{c1}, x_{c2}]$
$s, s = b_3 \leftarrow L[x_{c2}, x_{c3}]$
$s, s = b_2 \leftarrow b_2 + b_3$
$s, s = b_3 \leftarrow L[x_{c3}, x_{c4}]$
$s, s = b_2 \leftarrow b_2 + b_3$
...
$s, s = b_3 \leftarrow L[x_{c(n-1)}, x_{cn}]$
$s, s = b_2 \leftarrow b_2 + b_3$
$s, s = b_3 \leftarrow L[x_{cn}, x_{c1}]$
$s, s = b_2 \leftarrow b_2 + b_3$
$s, s = b_3 \leftarrow b_1 < b_2$
$s = y_o \leftarrow y_o \wedge b_3$

35.16 Express the $\mathcal{M}_1$ Solution as the union of subfunctions

35.17 Show the $\mathcal{M}_1$ solution satisfies the subfunction condition of solutions: $P[\hat{X}_i] \supseteq P[X_i] \quad \forall \hat{X}_i, X_i$

35.18 Find $O[n], O_T[n], O_S[n], f_{n+1}$ for the $\mathcal{M}_1$ Solution Subfunctions

$$O_S[n] = |y_o| + |\{b_1, b_2, b_3\}| = 4$$
$$O_T[n] =$$

35.19 Find $O[n], O_T[n], O_S[n], f_{n+1}$ for the $\mathcal{M}_1$ Overall Solution

35.20 Find an expression for the number of additions $x_i + x_j \;\; \forall i, j > 1 \;\; := O_+[n]$ for all solutions s^+

35.21 Prove $O_+[n]$ is a subfunction of all solutions s^+

Proof by contradiction

35.22 Show $O_+[n]$ diverges

35.23 Prove The Traveling Salesman Problem is in $\hat{D}$ using the Theorem of Divergent Subfunctions

36 Sum to N Problem with 2 integers Old

36.1 State formal definition of Sum to N : $x_i + x_j == N$

$$X_i = \{x_1, ..., x_n, N\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ = s^+[n] := P[X_i] \to y_o : y_o = a_o \quad \forall X_i$$
$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$D = f[X_i] = \exists x_j, x_k \in X_i : x_j + x_k == N$$

36.2 Express a formal solution : $O_S[n] \sim n^0$

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$
$$s_1 = y_o \leftarrow \mathbb{F};$$

$$\forall i, j > i$$

$$s_2, s_3, s_8, s_9, ..., s_{3ij-4}, s_{3ij-3}..., s_{3n(n-1)-4}, s_{3n(n-1)-3} = b_1 \leftarrow x_i + x_j$$

$$s_4, s_5, s_{10}, s_{11}, ..., s_{3ij-2}, s_{3ij-1}..., s_{3n(n-1)-2}, s_{3n(n-1)-1} = b_1 \leftarrow b_1 == N$$

$$s_6, s_7, s_{12}, s_{13}..., s_{3ij}, s_{3ij+1}..., s_{3n(n-1)}, s_{3n(n-1)+1} = y_o \leftarrow y_o \vee b_1$$

$$s^+ = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\}$$

36.3 Determine $O[n], O_S[n], O_T[n]$ for the above solution

$$O_S[n] = |y_o| + |b_1| = 2$$

$$O_T[n] = 3n(n-1) + 1 = 3n(n-1) - 1 + O_S[n]$$

$$O[n] = 3n(n-1) + 3 = 3n^2 - 3n + 3$$

36.4 Show s^+ is bounded by Polynomial Complexity by the definition of Total Polynomial Complexity

$$O[n] = 3n(n-1) + 3 = 3n^2 - 3n + 3$$

$$O[n] = 3n^2 - 3n + 3 < 3n^2 - 3n + 4 \ \forall n$$

$$\therefore s^+ \in S^+_{\mathbb{P}}$$

36.5 Show s^+ is bounded by Polynomial Complexity by showing $\text{limit}_{n\to\infty}\frac{O[n+1]}{O[n]} = 1$

$$limit_{n\to\infty}\frac{O[n+1]}{O[n]} =$$

$$limit_{n\to\infty}\frac{3n^2+3n+3}{3n^2-3n+3} =$$

$$limit_{n\to\infty}\left(\frac{3n^2-3n+3}{3n^2-3n+3} + \frac{6n}{3n^2-3n+3}\right) =$$

$$limit_{n\to\infty}\left(1 + \frac{6n}{3n^2-3n+3}\right) = 1$$

36.6 Express a formal solution : $O_S[n] \sim n^1$

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$s_1 = y_o \leftarrow \mathbb{F};$$

$$\forall x_i$$

$$s_2, s_3, ... = b_i \leftarrow N - x_i$$

$$\forall b_j \in \mathcal{M} \;\; j \leqslant i$$

$$s_4, s_5, ... = b_{n+1} \leftarrow b_j == x_i$$

$$s_6, s_7, ... = y_o \leftarrow y_o \vee b_{n+1}$$

36.7 Determine $O[n], O_S[n], O_T[n]$ for the above first order space complexity solution

$$O_S[n] = |y_o| + |b_1 ... b_n| + |b_{n+1}| = n + 2$$

$$O_T[n] = 1 + 2n + 4\sum_{i=1}^{n} i = 1 + 2n + 2n(n+1) = 2n^2 + 4n + 1$$

$$O[n] = 2n^2 + 4n + n + 1 + 2 = 2n^2 + 5n + 3$$

36.8 Determine inductive function f_{n+1} using time and space inductive functions f_{n+1}^T, f_{n+1}^S

$$f_{n+1}[n] = f_{n+1}^T + f_{n+1}^S$$

$$f_{n+1}[n] = O_T[n+1] - O_T[n] + O_S[n+1] - O_S[n]$$

$$f_{n+1}[n] = 1 + 2(n+1) + 4\sum_{i=1}^{n+1} i - 1 - 2n - 4\sum_{i=1}^{n} i + n + 1 + 2 - n - 2$$

$$f_{n+1}[n] = 2(1) + 4\sum_{i=n+1}^{n+1} i + 1 = 4(n+1) + 3 = 4n + 7$$

36.9 Show s^+ is bounded by Polynomial Complexity by showing limit$_{n\to\infty}\frac{O[n+1]}{O[n]} = 1$

By showing the order $f_{n+1}[n]$ is less than O[n]

$$limit_{n\to\infty}\frac{O[n+1]}{O[n]} =$$

$$limit_{n\to\infty}\frac{O[n] + f_{n+1}[n]}{O[n]} =$$

$$limit_{n\to\infty} 1 + \frac{f_{n+1}[n]}{O[n]} =$$

$$limit_{n\to\infty} 1 + \frac{4n+7}{2n^2+5n+3} = 1$$

37 Sum to N Problem: q == 3

37.1 State formal definition of Sum to N with 3 integers : $x_j + x_k + x_l == N$

$$X_i = \{x_1, ..., x_n, N\}$$

$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$s^+ = s^+[n] := P[X_i] \to y_o : y_o = a_o \quad \forall X_i$$

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$D = f[X_i] = \exists x_j, x_k, x_l \in X_i : x_j + x_k + x_l == N$$

37.2 Determine a solution of zero order Space Complexity

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$s_1 = y_o \leftarrow \mathbb{F};$$

$$\forall j, k \neq j, l \neq k, j$$

$$s_2, s_3, ... = b_1 \leftarrow x_j + x_k$$

$$s_4, s_5, ... = b_1 \leftarrow b_1 + x_l$$

$$s_6, s_7, ... = b_1 \leftarrow b_1 == N$$

$$s_8, s_9, ... = y_o \leftarrow y_o \vee b_1$$

$$s^+ = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_j + x_k + x_l == N) \quad \forall j, k \neq j, l \neq k, j \mid b_1, y_o\}$$

37.3 Determine $O[n], O_S[n], O_T[n]$ for the above solution

$$O_S[n] = |y_o| + |b_1| = 2$$

$$O_T[n] = \tfrac{8n(n-1)(n-2)}{6} + 1$$

$$O[n] = \tfrac{8n(n-1)(n-2)}{6} + 3$$

37.4 Find the order of s^+; Show s^+ is bounded by Polynomial Complexity

$$O[n] = \tfrac{8n(n-1)(n-2)}{6} + 3$$

$$O[n] \sim n^3$$

$$O[n] < \frac{8n(n-1)(n-2)}{6} + 4$$

37.5 Determine a solution of first order Space Complexity

37.6 Determine $O[n], O_S[n], O_T[n]$ for the above first order Space Complexity Solution

37.7 Determine inductive functions $f^T_{n+1}[n], f^S_{n+1}[n]$

37.8 Determine inductive function $f_{n+1}[n] = f^T_{n+1}[n] + f^S_{n+1}[n]$

37.9 Show s^+ is bounded by Polynomial Complexity by showing $\text{limit}_{n\to\infty}\frac{O[n+1]}{O[n]} = 1$

Satisfying the criteria that f_{n+1}[n] has lower order than O[n]

38 Sum to N problem: general q

38.1 State general definition of Sum to N with q integers : $x_j + x_k \ldots + x_q == N$

$$X_i = \{x_1, ..., x_n, N\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ = s^+[n] := P[X_i] \to y_o : y_o = a_o \quad \forall X_i$$
$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$D = f[X_i] = \exists x_j, x_k, x_l ..., x_q \in X_i : x_j + x_k + ... + x_q == N$$

38.2 Determine a general solution of zero order Space Complexity

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$s_1 = y_o \leftarrow \mathbb{F};$$

$$\forall j; k \neq j; l \neq k, j; ...; q \neq j, k, ..., p$$

$$s_2, s_3, ... = b_1 \leftarrow x_j + x_k$$

$$s_4, s_5, ... = b_1 \leftarrow b_1 + x_l$$

$$s_6, s_7, ... = b_1 \leftarrow b_1 == N$$

$$s_8, s_9, ... = y_o \leftarrow y_o \vee b_1$$

$$s^+ = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_j + x_k + x_l == N) \quad \forall j, k \neq j, l \neq k, j \mid b_1, y_o\}$$

39 Proof of the existence of $\hat{\mathcal{D}}$

Non-trivial; Formalize the traveling salesman problem as a decision problem (any optimization problem)

39.1 The Traveling Salesman Problem

English description

39.2 Formal Definition

$$X_i = \{c_1, c_2, ..., c_n\} :$$
$$dim[c_i] = C > 1$$

$$\bar{X}_i = \{c_1, c_2, ..., c_n, \bar{P}, \bar{f}[c_i, c_j]\}$$
$$\bar{P} := \{c_k, ...\} :$$
$$\exists c_k \in \bar{P} \ \ \forall c_k \in X_i$$

39.3 Express a formal, general solution $O_S[n] \sim n^0$

$$\text{Traveling Salesman Solution} \quad n == N \quad O_S[n] \sim n^0$$

$$X_i = \{x_1, x_2, ..., x_n, L[x_i, x_j], R\}$$

$$R = \{\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\}$$

$$y_o \quad \text{R is the shortest path}$$

$$b_1 \text{ length of path R}$$

$$b_2 \text{ current path length}$$

$$b_3 \text{ intermediate path length or R} < \text{current path}$$

// Calculate Proposed Route Distance
$s_1, s_2 = b_1 \leftarrow L[\hat{x_1}, \hat{x_2}]$
$s_3, s_4 = b_2 \leftarrow L[\hat{x_2}, \hat{x_3}]$
$s_5, s_6 = b_1 \leftarrow b_1 + b_2$
$s_7, s_8 = b_2 \leftarrow L[\hat{x_3}, \hat{x_4}]$
$s_9, s_{10} = b_1 \leftarrow b_1 + b_2$
...
$s, s = b_2 \leftarrow L[\hat{x}_{n-1}, \hat{x}_n]$
$s, s = b_1 + b_2$
$s, s = b_2 \leftarrow L[\hat{x}_n, \hat{x}_1]$
$s, s = b_1 + b_2$
$s = y_o \leftarrow \mathbb{T}$

// Calculate other (n P (n-1))/2 routes
// Let current route = $\{x_{c1}, x_{c2}, ..., x_{cn}\}, \;\; x_{ci} \in X_i \;\; \forall x_{ci}$
$s, s = b_2 \leftarrow L[x_{c1}, x_{c2}]$
$s, s = b_3 \leftarrow L[x_{c2}, x_{c3}]$
$s, s = b_2 \leftarrow b_2 + b_3$
$s, s = b_3 \leftarrow L[x_{c3}, x_{c4}]$
$s, s = b_2 \leftarrow b_2 + b_3$
...
$s, s = b_3 \leftarrow L[x_{c(n-1)}, x_{cn}]$
$s, s = b_2 \leftarrow b_2 + b_3$
$s, s = b_3 \leftarrow L[x_{cn}, x_{c1}]$
$s, s = b_2 \leftarrow b_2 + b_3$
$s, s = b_3 \leftarrow b_1 < b_2$
$s = y_o \leftarrow y_o \wedge b_3$

39.4 Express the zero order Space Complexity Solution as the union of two subfunctions

39.5 Show the $O_S[n] \sim n^0$ solution satisfies the subfunction condition of solutions: $P[\hat{X}_i] \supseteq P[X_i] \quad \forall \hat{X}_i, X_i$

39.6 Find $O[n], O_T[n], O_S[n], f_{n+1}$ for the zero order Space Complexity Solution

$$O_S[n] = |y_o| + |\{b_1, b_2, b_3\}| = 4$$
$$O_T[n] =$$

39.7 Find a formal expression for the minimum number of additions $O_+[n]$ for all solutions s^+

Proof by contradiction

39.8 Show the $O_S[n] \sim n^0$ solution has additions greater than or equal to $O_+[n]$

39.9 Express the $O_S[n] \sim n^0$ solution with $O_+[n]$ as a subfunction

39.10 Introduce $\mathcal{M}_x$ notation

39.11 Express a formal solution with $\mathcal{M}_{n+1}$

39.12 Show the $\mathcal{M}_{n+1}$ solution satisfies the subfunction condition of solutions: $P[\hat{X}_i] \supseteq P[X_i] \quad \forall \hat{X}_i, X_i$

39.13 Show the $\mathcal{M}_{n+1}$ solution has additions greater than or equal to $O_+[n]$

39.14 Express the $\mathcal{M}_{n+1}$ solution with $O_+[n]$ as a subfunction

39.15 Express O[n] for all solutions with $O_+[n]$ as a subfunction

39.16 Prove The Traveling Salesman Problem is in $\hat{D}$ using the Theorem of Divergent Subfunctions

Previous Material

40 Proof of the existence of $\hat{\mathcal{D}}$

Non-trivial; Formalize the traveling salesman problem as a decision problem (any optimization problem)

40.1 The Traveling Salesman Problem

English description

40.2 Formal Definition

$$X_i = \{c_1, c_2, ..., c_n\} :$$
$$dim[c_i] = C > 1$$

$$\bar{X}_i = \{c_1, c_2, ..., c_n, \bar{P}, \bar{f}[c_i, c_j]\}$$
$$\bar{P} := \{c_k, ...\} :$$
$$\exists c_k \in \bar{P} \ \ \forall c_k \in X_i$$

40.3 Determine $s^+ \Longrightarrow O_{S_{opt}}$

40.4 Express $O[n], O_T[n], O_S[n] = O_{S_{opt}}$

40.5 Revisit expressions properties inequalities connecting $O_{opt}; O_{T_{opt}}; O_{S_{opt}}$

40.6 Determine an alternate solution storing subpaths

40.7 Express $O[n], O_T[n], O_S[n]$

40.8 Determine a dual function

40.9 Show $limit_{n\to\infty}\frac{\hat{O}[n+1]}{\hat{O}[n]}$ **diverges**

40.10 Let $\hat{s}^+$**; a solution with** $limit_{n\to\infty}\frac{\hat{O}[n+1]}{\hat{O}[n]} = \mathbf{1}$

40.11 Show $\hat{s}^+$ **implies a contradiction**

40.12 Determine $s^+_{S_{opt}}$

40.13 Determine equivalence function

40.14 Determine an expression for $\frac{\hat{O_{opt}}[n+1]}{\hat{O_{opt}}[n]}$

40.15 Show inductive function diverges for all orders of $O_S[n]$

41 Proof of "P $\neq$ NP"