Ch. 5 Computation

1 Memory

1.1 Definition

Define Memory; a set of elements

$$\mathcal{M} := \{b_1, b_2, ..., b_M\}$$

1.2 Silicon Computation

In silicon based computation memory is represented with bits 0/1

$$\mathcal{M} := \{b_1, b_2, ..., b_m\}$$
$$b_i \in \{0, 1\} \ \ \forall i$$

2 Logical Operations

Working list; possibly any set operation/function

$+$
-

/
exp
$\circlearrowright$
$\leftarrow$
delete
remove
insert
append
if
$==$
!

3 Program

3.1 Logical Instructions

Define $\mathcal{L}$; an ordered set of logical operations s_i

$$\mathcal{L} := \{s_1, s_2, ..., s_N\}$$

3.2 State |

Define state; the memory required to perform program P

$$P := \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M\} =$$
$$\{s_1, s_2, ..., s_N, b_1, b_2, ..., b_M\}$$

3.3 Boolean Programs

Define a boolean program; boolean programs can represent functions with inputs x_i and boolean output y_o

$$X = \{x_1, ..., x_n\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, X_i, y_o\} =$$
$$P[X] \to y_o \in \{\mathbb{T}, \mathbb{F}\}$$

3.4 Void Programs

Define a void program; a program with inputs x_i and no output

$$X = \{x_1, ..., x_n\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, X_i\}$$

3.5 Numerical Programs

Define a numerical program; a program with inputs x_i and real, rational output y_o

$$X = \{x_1, ..., x_n\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, X_i, y_o\} =$$
$$P[X] \to y_o \in \mathbb{Q} \;\; y_o \geqslant 0$$

3.6 System Programs

Naming convention to be formalized; a program that outputs one or more elements

$$X = \{x_1, ..., x_n\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, X_i, Y_o\} =$$
$$P[X] \to Y_o = \{y_1, y_2, ..., y_K\}$$

3.7 Mathematical Programs

Define a mathematical program; a program with inputs x_i and numerical output y_o

$$X = \{x_1, ..., x_n\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, X_i, y_o\} =$$
$$P[X] \to y_o \in \mathbb{Q}$$

4 No-op ;

4.1 Definition

$$; := \varnothing$$

4.2 Property of No-op

No-op can be inserted into any set with equality

$$S = \{s_1, s_2, ..., s_N\}$$

$$S_; = insert[S, ;, i]$$

$$S_; = S_1 \ \ \forall i$$

$$|S_;| = |S| \ \ \forall i$$

4.3 Proof

by definition of magnitude of null = 0 with Set And

5 For Loop $\circlearrowright$

$\circlearrowright$[startindex, endindex, condition]

5.1 Definition

6 Nested For Loop $\circlearrowright^n$

$\circlearrowright$[startindex, endindex, condition1,...condition n]

6.1 Definition

7 Decision Problems

7.1 Definition

Define decision problem; a function with inputs x_i and boolean output "answer" a_o

$$X_i = \{x_1, ..., x_n\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

8 Solutions

8.1 Definition

Program P is a solution s^+ if P outputs answer a_o for all inputs X_i $\forall i$
s^+ is a function of the number of inputs n

$$X_i = \{x_1, ..., x_n\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ = s^+[n] := P[X_i] \to y_o : y_o = a_o \quad \forall X_i$$
$$P[X_i] = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, X_i, y_o\}$$
$$s^+ = P[X_i] = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, X_i, y_o\} \quad \forall X_i$$

8.1.1 Property of No-op ;

No-op ; can be added to any solution S_i and remain a solution for all i anywhere in the order for all j

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, X_i, y_o\}$$
$$\hat{s}^+ = insert[s^+, ;, k]$$
$$\hat{s}^+ = s^+ \quad \forall k$$

8.2 Definition of S^+

Define S^+; the set of solutions to decision problem D

$$X_i = \{x_1, ..., x_n\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s_j^+ := P[X_i] \to y_o : y_o = a_o \quad \forall X_i$$

$$S^+ := \{s_j^+, ...\} \quad \forall j$$

8.3 Definition of Solvable

Define solvable

$$X_i = \{x_1, ..., x_n\}$$
$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$solvable := solvable[D] \rightarrow b_o \in \{\mathbb{T}, \mathbb{F}\} =$$
$$\exists s^+ : s^+ = P[X_i] \rightarrow y_o : y_o = a_o \quad \forall X_i$$

9 The set of all Decision Problems $\mathbb{D}$

9.1 Definition

Define the set of decision problems $\mathbb{D}$

$$X_i = \{x_1, ..., x_n\}$$
$$D_j := f_j[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$\mathbb{D} := \{D_j, ...\} \quad \forall j$$

10 Instruction and Memory Notation

Define $\mathcal{L}$ a set of logical operations
Define $\mathcal{M}$ a set of bits "memory"

$$P[X_i] \rightarrow y_o = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, X_i, y_o\}$$
$$\mathcal{L} := \{s_1, s_2, ..., s_{O_T[n]}\}$$
$$\mathcal{M} := \{b_1, b_2, ..., b_{O_S[n]}\}$$
$$P[X_i] = \{\mathcal{L}, \mathcal{M}, X_i, y_o\}$$

11 Complexity

11.1 Time Complexity of a Decision Problem $O_T[n]$

Define Time Complexity $O_T[n]$ of Decision Problem D with solution s^+

$$X_i = \{x_1, ..., x_n\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ := P[X_i] \to y_o : y_o = a_o \quad \forall X_i =$$
$$\{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, X_i, y_o\} = \{\mathcal{L}, \mathcal{M}, X_i, y_o\}$$
$$O_T[n] := |\mathcal{L}| = N$$

11.2 Space Complexity $O_S[n]$

Define Space Complexity $O_S[n]$ of Decision Problem D with solution s^+

$$X_i = \{x_1, ..., x_n\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ := P[X_i] \to y_o : y_o = a_o \quad \forall X_i =$$
$$\{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, X_i, y_o\} = \{\mathcal{L}, \mathcal{M}, X_i, y_o\}$$
$$O_S[n] := |\mathcal{M}| = M$$

12 Definition of Complexity

Define Complexity $O[n]$ as a vector of dimension C

$$\mathbf{O}[n] := \quad < O_T[n], O_S[n], O_3[n], O_4[n]..., O_C[n] >$$

13 Total Complexity

$$O[n] := O_T[n] + O_S[n] + \sum_{i=3}^{C} O_i[n]$$

14 Simple Computational Complexity

The remainder of this chapter assumes simple computational complexity of dimension 2

14.1 Definition

Define simple computational complexity of dimension 2

$$\mathbf{O}[n] := \; < O_T[n], O_S[n] >$$

14.2 Time Complexity

Restate definition of Time Complexity $O_T[n]$

$$s^+ = \{\mathcal{L}, \mathcal{M}, X_i, y_o\}$$

$$O_T[n] := |\mathcal{L}| = N$$

14.3 Space Complexity

Restate definition of Time Complexity $O_S[n]$

$$s^+ = \{\mathcal{L}, \mathcal{M}, X_i, y_o\}$$
$$O_S[n] := |\mathcal{M}| = M$$

14.4 Total Complexity

$$O[n] := O_T[n] + O_S[n]$$
$$= |\mathcal{L}| + |\mathcal{M}| = N + M$$

14.5 Axiom O[n] $\neq$ 0

14.6 Theorem $O_T[n] + O_S[n] \neq \mathbf{0}$

14.6.1 Proof

14.7 Theorem $O[n] \geqslant O_T[n]$

14.7.1 Proof

14.8 Theorem $O[n] \geqslant O_S[n]$

14.8.1 Proof

15 Optimal Complexity

15.1 Definition

Define Optimal Complexity; the minimum total complexity required to solve a decision problem

$$O_{opt}[n] :=$$
$$\nexists \hat{O}[n] : \hat{O}[n] < O_{opt}[n] \;\; \forall n$$

15.2 Proof of Existence

Prove the existence of at least one $O_{min}[n]$ by induction/contradiction

16 Optimal solution

Define an optimal solution s^+_{opt}

16.1 Definition

$$X_i = \{x_1, ..., x_n\}$$
$$D_j := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ := P[X_i] \to y_o : y_o = a_o \quad \forall X_i$$
$$s^+_{opt} := s^+ :$$
$$\nexists \hat{O}[n] < O_{opt}[n] \quad \forall n, \; s^+ \in S^+_j$$

16.2 Optimal Time Complexity Solution

$$X_i = \{x_1, ..., x_n\}$$
$$D_j := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ := P[X_i] \to y_o : y_o = a_o \quad \forall X_i =$$
$$\{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, X_i, y_o\} = \{\mathcal{L}, \mathcal{M}, X_i, y_o\}$$
$$O_T[n] := |\mathcal{L}| = N$$
$$s^+_T := s^+ :$$
$$\nexists \hat{O_T}[n] < O_T[n] \quad \forall n, \; s^+ \in S^+_j$$

16.3 Optimal Space Complexity Solution

$$X_i = \{x_1, ..., x_n\}$$

$$D_j := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$s^+ := P[X_i] \to y_o : y_o = a_o \quad \forall X_i =$$

$$\{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, X_i, y_o\} = \{\mathcal{L}, \mathcal{M}, X_i, y_o\}$$

$$O_S[n] := |\mathcal{M}| = M$$

$$s_S^+ := s^+ :$$

$$\nexists \hat{O}_S[n] < O_S[n] \quad \forall n,\ s^+ \in S_j^+$$

17 Polynomial Complexity

17.1 Definition

Decision problem D with solution s^+ has (optimal) total complexity $O[n]$ bounded by polynomial complexity if

$$\exists K, C, \lambda_1 ... \lambda_K \quad :$$
$$O_{opt}[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

17.2 Polynomial Problems

Define $\mathbb{P}$, the set of Decision Problems that can be solved with Polynomial Complexity

$$\mathbb{P} := \{D_1, D_2, ...\} :$$
$$\exists K, C, \lambda_1 ... \lambda_K :$$
$$O_{opt}[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, D_j \in \mathbb{P}$$

17.3 Polynomial Order of Complexity

Total complexity $O[n]$ is said to be of order K_{opt}

$$O[n] \sim K_{opt}$$
$$O_{opt}[n] := O[n] :$$
$$\nexists \hat{O}[n] < O_{opt}[n] \;\; \forall n$$
$$O_{opt}[n] < (\lambda_{K_{opt}} n)^{K_{opt}} + (\lambda_{K_{opt}-1} n)^{K_{opt}-1} ... + \lambda_1 n + C \quad \forall n$$
$$K_{opt} := K :$$
$$\nexists \hat{K} : O_T[n] < (\lambda_{\hat{K}} n)^{\hat{K}} + (\lambda_{\hat{K}-1} n)^{\hat{K}-1} ... + \lambda_1 n + C \quad \forall n, \; \hat{K} < K$$

17.4 Corrolary of Optimal Complexity

$$\nexists s^+ \in S^+ :$$
$$O_T[n] < (\lambda_{\hat{K}} n)^{\hat{K}} + (\lambda_{\hat{K}-1} n)^{\hat{K}-1} ... + \lambda_1 n + C \quad \forall n, \; \hat{K} < K_{opt}$$

17.4.1 Proof

Proof by contradiction; definition of optimal complexity

17.5 Property of Polynomial Complexity 1

$$lim_{n \to \infty} \frac{O[n+1]}{O[n]} = 1$$

17.5.1 Proof WIP

Show there exists no constant satisfying the decreasing limit condition

$$O[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$$

$$O[n+1] < (\lambda_K (n+1))^K + (\lambda_{K-1} (n+1))^{K-1} ... + \lambda_1 (n+1) + C$$

$$O[n] \sim (\lambda n)^K; \;\; O[n+1] \sim (\lambda n)^K$$

$$lim_{n \to \infty} \frac{(\lambda n)^K}{(\lambda n)^K} = 1$$

17.6 Property of Polynomial Complexity 2

$$lim_{n \to \infty} (O[n+1] - O[n]) \;\; diverges$$

17.6.1 Proof

$$O[n+1] < (\lambda_K (n+1))^K + (\lambda_{K-1} (n+1))^{K-1} ... + \lambda_1 (n+1) + C$$

$$O[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$$

17.7 Order of Complexity

Total Complexity is said to be on the order of K_{max}

$$O[n] < (\lambda_{K_{max}} n)^{K_{max}} + (\lambda_{K_{max}-1} n)^{K_{max}-1} ... + \lambda_1 n + C$$

$$O[n] \sim K_{max}$$

18 Polynomial Time Complexity

18.1 Definition

Decision problem D with (optimal) Time Complexity $O_T[n]$ is bounded by polynomial time complexity if

$$\exists K, C, \lambda_1 ... \lambda_K \quad :$$
$$O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

18.2 Polynomial Time Problems

Define $\mathbb{P}_{time}$, the set of Decision Problems that can be solved with polynomial time complexity

$$\mathbb{P}_{time} := \{D_1, D_2, ...\} :$$
$$\exists K, C, \lambda_1 ... \lambda_K :$$
$$O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, D_j \in \mathbb{P}_{time}$$

18.3 Total Polynomial Complexity Implies Time bounded Polynomial Complexity

$$D \in \mathbb{P} \implies D \in \mathbb{P}_{time}$$

18.3.1 Proof

$$O[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \ \ \forall n$$
$$O[n] := O_T[n] + O_S[n]; \ \ O_T[n] \leqslant O[n]$$
$$\therefore O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \ \ \forall n$$

18.4 Time bounded Polynomial Complexity implies Total Polynomial Complexity?

18.5 Polynomial Time Complexity iff Polynomial Complexity?

18.6 Property of Polynomial Time Complexity 1

$$lim_{n \to \infty} \frac{O_T[n+1]}{O_T[n]} = 1$$

18.6.1 Proof

18.7 Property of Polynomial Time Complexity 2

$$lim_{n \to \infty}(O_T[n+1] - O_T[n]) \;\; diverges$$

18.7.1 Proof

18.8 Order of Complexity

Time complexity $O_T[n]$ is said to be on the order of K_{max}

$$O_T[n] < (\lambda_{K_{max}} n)^{K_{max}} + (\lambda_{K_{max}-1} n)^{K_{max}-1} ... + \lambda_1 n + C$$

$$O_T[n] \sim K_{max}$$

19 Polynomial Space Complexity

19.1 Defintion

19.2 Polynomial Space Problems

19.3 Total Polynomial Complexity Implies Space bounded Polynomial Complexity

19.4 Space Bounded Polynomial Complexity Implies Total Polynomial Complexity

19.5 Polynomial Space Complexity iff Polynomial Complexity

19.6 Property of Polynomial Space Complexity 1

$$lim_{n\to\infty}\frac{O_S[n+1]}{O_S[n]} = 1$$

19.7 Property of Polynomial Space Complexity 2

$$lim_{n\to\infty}(O_S[n+1] - O_S[n])\ \ diverges$$

19.8 Order of Complexity

Space complexity $O_S[n]$ is said to be on the order of K_{max}

$$O_S[n] < (\lambda_{K_{max}}n)^{K_{max}} + (\lambda_{K_{max}-1}n)^{K_{max}-1}... + \lambda_1 n + C$$

$$O_S[n] \sim K_{max}$$

20 Polynomial Duality

20.1 Proof of the existence of $O_{S_{opt}}$

20.2 Proof of the existence of $O_{T_{opt}}$

20.3 Theorem Either OT or OS is on the order of Oopt

Proof by contradiction

20.4 Duality Functions

$$
\begin{gathered}
X_i = \{x_1, ..., x_n\} \\
D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i \\
s^+ = s^+[n] := P[X_i] \to y_o : y_o = a_o \quad \forall X_i \\
s^+ = \{\mathcal{L}, \mathcal{M}, X_i, y_o\}
\end{gathered}
$$

$$
\begin{gathered}
f_{\mathcal{L}\to\mathcal{M}} := f[\mathcal{L}, \mathcal{M}] \to \hat{\mathcal{L}}, \hat{\mathcal{M}} : \\
s^+_{\mathcal{L}\to\mathcal{M}} = \{f_{\mathcal{L}\to\mathcal{M}}[\mathcal{L}, \mathcal{M}], X_i, y_o\} = \\
\{\hat{\mathcal{L}}, \hat{\mathcal{M}}, X_i, y_o\} \;\; \forall s^+ \in S^+; \;\; \hat{\mathcal{M}} \subseteq \mathcal{M}; \;\; \mathcal{L} \subseteq \hat{\mathcal{L}}
\end{gathered}
$$

$$
\begin{gathered}
f_{\mathcal{M}\to\mathcal{L}} := f[\mathcal{L}, \mathcal{M}] \to \hat{\mathcal{L}}, \hat{\mathcal{M}} : \\
s^+_{\mathcal{M}\to\mathcal{L}} = \{f_{\mathcal{L}\to\mathcal{M}}[\mathcal{L}, \mathcal{M}], X_i, y_o\} = \\
\{\hat{\mathcal{L}}, \hat{\mathcal{M}}, X_i, y_o\} \;\; \forall s^+ \in S^+; \;\; \hat{\mathcal{L}} \subseteq \mathcal{L}; \;\; \mathcal{M} \subseteq \hat{\mathcal{M}}
\end{gathered}
$$

$$
\begin{gathered}
s^+_{\mathcal{L}\to\mathcal{M}}[n] := P_{\mathcal{L}\to\mathcal{M}}[X_i] \to y_o : y_o = a_o \quad \forall X_i \\
s^+_{\mathcal{M}\to\mathcal{L}}[n] := P_{\mathcal{M}\to\mathcal{L}}[X_i] \to y_o : y_o = a_o \quad \forall X_i
\end{gathered}
$$

20.5 Inductive Function O[n+1]

System of equations? Might be able to tie back to O[n]

$$
\begin{gathered}
O[n] := O_T[n] + O_S[n] \\
O[n+1] = O_T[n+1] + O_S[n+1]
\end{gathered}
$$

20.5.1 Connection to duality functions

System of equations? Might be able to tie back to O[n]

$$O[n] := O_T[n] + O_S[n]$$
$$O[n+1] = O_T[n+1] + O_S[n+1]$$
$$O_T[n] := |\mathcal{L}| = N; \;\; O_S[n] := |\mathcal{M}| = M$$

$$s^+_{\mathcal{L}\to\mathcal{M}}[n] := P_{\mathcal{L}}[X_i] \to y_o : y_o = a_o \quad \forall X_i =$$
$$\{f_{\mathcal{L}\to\mathcal{M}}[\mathcal{L}, \mathcal{M}], X_i, y_o\} = \{\hat{\mathcal{L}}, \hat{\mathcal{M}}, X_i, y_o\} \quad \forall s^+ \in S^+; \quad \hat{\mathcal{M}} \subseteq \mathcal{M}; \;\; \mathcal{L} \subseteq \hat{\mathcal{L}}$$

$$s^+_{\mathcal{M}\to\mathcal{L}}[n] := P_{\mathcal{M}}[X_i] \to y_o : y_o = a_o \quad \forall X_i$$
$$\{f_{\mathcal{M}\to\mathcal{L}}[\mathcal{L}, \mathcal{M}], X_i, y_o\} = \{\hat{\mathcal{L}}, \hat{\mathcal{M}}, X_i, y_o\} \quad \forall s^+ \in S^+; \quad \hat{\mathcal{L}} \subseteq \mathcal{L}; \;\; \mathcal{M} \subseteq \hat{\mathcal{M}}$$

$$limit_{n\to\infty} \frac{O[n+1]}{O[n]} = 1$$

20.6 Theorem of (Polynomial?) Duality

For all Problems in P there exists a duality function
Formally define dynamic programming, Optimal polynomial complexity minimizes the difference between time and space complexity order

$$D \in \mathbb{P}$$
$$O[n] := O_T[n] + O_S[n]$$
$$O_{opt}[n] := O[n] :$$
$$\nexists \hat{O}[n] < O[n] \quad \forall n$$

$$O^+_T[n] := |\mathcal{L}| = N :$$
$$\nexists \hat{O}_T[n] < O^+_T[n] \quad \forall n$$
$$O^+_S[n] := |\mathcal{M}| = M$$
$$\nexists \hat{O}_S[n] < O^+_S[n] \quad \forall n$$

20.7 Proof

Prove that Order can be subtracted from Os or Ot and added to the other; double check cauchy schwartz inequal

$$O[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, D_j \in \mathbb{P}$$

$$O[n] := O_T[n] + O_S[n]$$

$$O_T[n] + O_S[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, D_j \in \mathbb{P}$$

20.8 There exists an optimal OT and OS on the order of Oopt

20.9 Even ordered decision problems

M-N = N-M = 0

20.10 Odd ordered decision problems

N = M + 1 or M = N + 1

20.11 N Sum Problem

20.11.1 Restate formal definition

$$X_i = \{x_1, ..., x_n\}$$
$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ = s^+[n] := P[X_i] \rightarrow y_o : y_o = a_o \quad \forall X_i$$
$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, X_i, y_o\} = \{\mathcal{L}, \mathcal{M}, X_i, y_o\}$$

$$D = f[X_i] = \exists x_j, x_k \in X_i : x_j + x_k = N$$

20.11.2 Express a formal solution

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, X_i, y_o\} = \{\mathcal{L}, \mathcal{M}, X_i, y_o\}$$
$$b_1 = s_{O_S[n]} = y_o \leftarrow \mathbb{F};$$
$$s_1 ... s_{O_T[n]} = s_1 ... s_{\frac{n(n-1)}{2}} = y_o \leftarrow y_o \cup (x_i + x_j == N) \quad \forall i, j > i$$
$$s^+ = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \cup (x_i + x_j == N) \quad \forall i, j > i\}$$

20.11.3 Express the order of complexity, order of time complexity, order of space complexity

$$O_T[n] = \tfrac{n(n-1)}{2} \sim n^2$$
$$O_S[n] = 1 \sim n^0$$
$$O[n] = \tfrac{n(n-1)}{2} + 1 \sim n^2$$

20.11.4 Express O[n+1] in terms of O[n]

$$O[n] = n^2 - n + 2$$
$$O[n+1] = (n+1)^2 - n - 1 + 2 = n^2 + 2n + 1 - n - 1 + 2 = O[n] + 2n$$

20.11.5 Prove $|\mathbb{P}| > 0$

$$O[n] = n^2 - n + 2$$
$$O[n] = n^2 - n + 2 < n^2 - n + 3 \quad \forall n$$
$$\therefore D \in \mathbb{P}$$

20.11.6 N Sum alternate solution

$$s_1 = (b_1 = y_o) \leftarrow \mathbb{F}$$
$$s_2, s_4, ..., s_{2n} = s_{2i} = b_{i+1} \leftarrow N - x_i \quad i = 1...n$$
$$s_3, s_5...s_{2n+1} = s_{2i+1} = y_o \leftarrow y_o \cup (x_i \in \mathcal{M})) \quad i = 1...n$$
$$O_S[n] = n + 1; O_T[n] = n(1 + O_s[n])$$
$$O[n] := O_T[n] + O_S[n] = n(1 + O_S[n]) + n + 1 = n(1 + n + 1) + n + 1$$
$$O[n] = n^2 + 3n + 1$$

20.11.7 Find a dual function of solution s^+

$$O_S[n] = n + 1; O_T[n] = n(1 + O_S[n])$$
$$O[n] = O_T[n] + O_S[n] = n(1 + O_S[n]) + O_S[n] = n + O_S[n](1 + n)$$

20.11.8 Find an inductive function for $O_S[n]$

$$O_S[n] = n + 1; O_S[n + 1] = n + 2$$
$$O_S[n + 1] = n + 2 = (n + 1) + 1 = O_S[n] + 1$$

20.11.9 Find an expression for O[n+1] as a function of O[n]

$$O[n] = O_T[n] + O_S[n] = n(1 + O_S[n]) + O_S[n]$$
$$O[n + 1] = n + 1(1 + O_S[n]) + O_S[n]$$
$$O[n] + inductive[n] = O[n + 1]$$
$$n(1 + O_S[n]) + O_S[n] + inductive[n] = (n + 1)(1 + O_S[n + 1]) + O_S[n + 1]$$
$$inductive[n] = (n + 1)(1 + O_S[n + 1]) + O_S[n + 1] - n(1 + O_S[n]) - O_S[n]$$
$$inductive[n] = (n + 1)(1 + O_S[n] + 1) + O_S[n] + 1 - n(1 + O_S[n]) - O_S[n]$$
$$inductive[n] = (n + 1)(O_S[n] + 2) + 1 - n(1 + O_S[n])$$
$$inductive[n] = nO_S[n] + 2n + O_S[n] + 2 + 1 - n - nO_S[n]$$
$$inductive[n] = 2n + O_s[n] - n + 3 = 2n + n + 1 - n + 3 = 2n + 4$$

$$O[n + 1] = O[n] + inductive[n] = O[n] + 2n + 4$$

20.11.10 Show the $\text{limit}_{n\to\infty}\frac{O[n+1]}{O[n]} = 1$

$$limit_{n\to\infty}\frac{O[n+1]}{O[n]} =$$

$$limit_{n\to\infty}\frac{O[n]+2n+4}{O[n]} =$$

$$limit_{n\to\infty}1+\frac{2n+4}{O[n]}$$

$$\nexists K : 1-1+\frac{2n+4}{O[n]} > K \quad \forall n, K > 0$$

$$\therefore limit_{n\to\infty}\frac{O[n+1]}{O[n]} = 1$$

20.11.11 Prove $|\mathbb{P}| > 0$

$$D \in \mathbb{P} \iff limit_{n\to\infty}\frac{O[n+1]}{O[n]} = 1$$
$$\therefore D \in \mathbb{P}$$

20.11.12 Criticism on interpretation of hashing solutions

In the below solution to N sum problem, the solution is typically considered to be $O_T[n] \sim n$. However, the line "if element in M:" requires a search through an (indexed) dictionary. During element x_i; |M| = i . If M is pre-allocated to the total number of elements the search requires n lookups each iteration where n = $|int\_list|$. Additional proofs required for optimized indexing. Prove the cost of storing and querying yields the same or different optimal order of complexity.

```
def SumToN(int_list,N):
output = False;
M = {};
for element in int_list:
if element in M:
output = True;
else:
M[N - element] = True;
output = output;
return output;
```

21 Definition of Non-Polynomial Problems

Define $\mathcal{N}$, the set of Decision Problems that cannot be solved with Polynomial Time Complexity

$$\mathcal{N} := \{D_1, D_2, ...\}$$
$$\nexists K, C, \lambda_1 ... \lambda_K :$$
$$O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, D_j \in \mathcal{N}$$

22 Divergent Problems

22.1 Definition

$$\hat{\mathcal{D}} := \{\hat{D}_1, \hat{D}_2, ...\}$$
$$lim_{n \to \infty} \frac{\hat{O}[n+1]}{\hat{O}[n]} \;\; diverges$$
$$\forall s^+_{\hat{D}}$$

22.2 Property of Divergent Problem Complexity 1

$$lim_{n \to \infty} \frac{\hat{O}[n+1]}{\hat{O}[n]} \;\; diverges$$

22.2.1 Proof

22.3 Property of Divergent Problem Complexity 2

$$lim_{n \to \infty} (\hat{O}[n+1] - \hat{O}[n]) \;\; diverges$$

22.3.1 Proof

22.4 Divergent Duality?

22.5 Divergent Induction Functions?

23 Fundamental Theorem of Computation

Some solutions for Polynomial Problems are divergent; There exist no solution for divergent problems with polynomial bound

$$\mathbb{P} = \{D_1, D_2, ...\}$$

$$S^+_{\mathbb{P}} = \{s_1, s_2, ...\}$$

$$\nexists s \in S^+_{\mathbb{P}} : O[n] \geqslant n^n \quad \forall s_i \in S^+_{\mathbb{P}}$$

$$\mathbb{P} \cap \hat{D} = \varnothing$$

$$limit_{n \to \infty} \circlearrowright^n [s^+[n]] = n^n \quad \forall s_i \in S^+_{\mathbb{P}}$$

23.1 Proof

$$X_i = \{x_1, ..., x_n\}$$

$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$Let \;\; D \in \mathbb{P}$$

$$s^+ := P[X_i] \to y_o : y_o = a_o \quad \forall X_i$$

$$O[n] = O_T[n] + O_S[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

23.2 Non-Polynomial implies Divergent

24 Proof of the existence of $\hat{\mathcal{D}}$

Non-trivial; Formalize the traveling salesman problem as a decision problem (any optimization problem)

24.1 The Traveling Salesman Problem

English description

24.2 Formal Definition

$$X_i = \{c_1, c_2, ..., c_n\} :$$
$$dim[c_i] = C > 1$$

$$\bar{X}_i = \{c_1, c_2, ..., c_n, \bar{P}, \bar{f}[c_i, c_j]\}$$
$$\bar{P} := \{c_k, ...\} :$$
$$\exists c_k \in \bar{P} \ \ \forall c_k \in X_i$$

24.3 Determine $s^+ \Longrightarrow O_{S_{opt}}$

24.4 Express $O[n], O_T[n], O_S[n] = O_{S_{opt}}$

24.5 Revisit expressions properties inequalities connecting $O_{opt}; O_{T_{opt}}; O_{S_{opt}}$

24.6 Determine an alternate solution storing subpaths

24.7 Express $O[n], O_T[n], O_S[n]$

24.8 Determine a dual function

24.9 Show $limit_{n\to\infty}\frac{\hat{O}[n+1]}{\hat{O}[n]}$ diverges

24.10 Let $\hat{s}^+$; a solution with $limit_{n\to\infty}\frac{\hat{O}[n+1]}{\hat{O}[n]} = 1$

24.11 Show $\hat{s}^+$ implies a contradiction

25 Proof of "P $\neq$ NP"

26 Theorem of Prime Numbers "Riemann Hypothesis"

Riemann Zeta Function

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} \quad [2]$$

"The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2." [2]

Prove the problem is divergent
There fore it can only be proven to a certain degree
The limit as n approaches infinity implies a real part of one half
Connection with the real and imaginary part of O $[n]$

26.1 Determine a duality function for the Riemann Hypothesis

26.2 Determine an expression for O[n+1] as a function of O[n]

26.3 Prove O_{opt} is performing O_{opt} recursively for the ints less than square root of n

Testing the primes less than sqrt(n)? double check
1. Optimal solution for n=1,2,3, everything else is a recursive optimal proof by induction
Time Complexity seems to be on the order of n log n... implies divergence or lack of bound? Add in the complexity of division.. probably approaches n^n

26.4 Since divergent, no s^+ exists.. only rules

Express as a limit

26.5 Show that the limit as n $\rightarrow \infty$ implies the real part is 1/2

1/2 ± 14.134725 i 1/2 ± 21.022040 i 1/2 ± 25.010858 i 1/2 ± 30.424876 i 1/2 ± 32.935062 i 1/2 ± 37.586178 i

$\mathrm{Z} = \zeta(1/2 + it)$

26.6 Notation, real imaginary parts of the problem

Even numbers and numbers ending in 5 are automatically convergent
Testing numbers ending in 1,3,7,9 results in divergent expression
we can continue to add rules to a certain degree

27 Divergent Problems

Define $\hat{\mathbb{D}}$ the set of decision problems with no convergent?/finite? solution $\hat{D}_j$

$$\hat{\mathbb{D}} := \{\hat{D}_j, ...\}$$
$$\hat{D}_j \in \mathbb{D} \quad \forall j$$
$$\nexists \hat{s}^+ \in S^+ : \hat{s}^+ \text{ solves } \hat{D}_j \quad \forall \hat{D}_j \in \hat{\mathbb{D}} \Longleftrightarrow$$
$$\nexists s^+ \in S^+ : O_j[n] < n^n \quad \forall n, j$$

There exists no such solution such that O[n] $< n^n$, but there is a right and wrong answer
Either here or in the next chapter we'll prove you can only solve to a certain degree
!!! There exists no such solution such that O[n] $< n^n$ $\quad \forall$ n

27.1 Definition

$$\hat{O}[n] := n^n$$
$$limit_{n\to\infty} \frac{O_{opt}[n]}{\hat{O}[n]} \;\; diverges$$
$$diverges[\; O_{opt}[n], \hat{O}[n] \;] \to \mathbb{T}$$

27.2 Theorem of Divergent Programs

Prove that Divergent implies not in polynomial (trivial)
Prove that Divergent implies Non-polynomial (trial after proving above)
Show that there exists at least one member of Divergent

28 Properties of Solvable and Divergent problems

28.1 Solvable and Divergent are disjoint

Prove by contradiction

29 "Theorem of Divergent Programs"

29.1 Divergence Test

1. Let $d_j \in D$
2. $d_j = (d_j \in \hat{\mathcal{D}}) \cup (d_j \in$ set of solvable problems) by disjoint condition of solvable and divergent
3. Let $O_{opt}[n]$, the optimal complexity of d_j
4. $\rightarrow s_i^+$ that solve d_j have larger complexity $\forall i$
5. 2 implies $O_{opt}[n]$ is either bounded by n^n or not
6. $\hat{O}[n] \equiv n^n$
7. Easy Suppose $d_j \in$solvable$limit_{n\rightarrow\infty}\frac{O_{opt}[n]}{\hat{O}[n]} = 0$
8.Suppose $d_j \in \hat{\mathcal{D}} limit_{n\rightarrow\infty}\frac{O_{opt}[n]}{\hat{O}[n]} \neq 0$ (by disjoint condition)

$$limit_{n\rightarrow\infty}\frac{O_{opt}[n]}{\hat{O}[n]} = 1$$

30 Connection to verification in polynomial time

31 Fundamental Theorem of Computation

n^n or $\lambda n^n + C$ the universal bound to solvable computational complexity $(\lambda n)^n$ + C ?

31.1 Time Complexity Argument

Suppose decision problem d with optimal time complexity $O_{T_{min}}[n]$ and solution s^+, an arbitrary decision problem in P with polynomial complexity
Assumptions
1. $d \in P$, $s^+ \in S^+$
Assertions
2. $\exists K, C, \lambda_1 ... \lambda_K \;\; : \;\; O_{T_{min}}[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C, \quad \forall n$
3. Define $f[K, C, \lambda_1, ... \lambda_K] \equiv (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$
4. $\exists K, C, \lambda_1 ... \lambda_K : O_{T_{min}}[n] < f[K, C, \lambda_1, ... \lambda_K] \quad \forall n$
5. Let $\hat{s}^+ \equiv \circlearrowright^n [s^+]$
6. $O_T[n] <= \hat{O}_T[n]$ (by defition of nested loop)

7. $\hat{O}_{T_{min}}[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$
8. $\hat{O}_{T_{min}}[n] < limit_{n \to \infty} \circlearrowright^n [s^+]$ (by definition of limit + definition of nested loop, expand to show full derivation, valid because this is a series, probably need to show limit applies)
9. $\therefore O_{T_{min}}[n] < \hat{O}_{T_{min}}[n] < n^n = limit_{n \to \infty} \circlearrowright^n [s^+]$
I want to say forall n but seems refutable for n= 1,2.. but as n approach infinity it's a contradiction to say a solvable problem in P $\hat{O}_{T_{min}} = n^n \quad \forall n$
10. For "sufficiently large n"

$$\nexists \hat{s}^+ \in S^+ : |\hat{s}^+| \equiv O_{T_{min}}[n] < n^n, \quad \forall n$$
$$\hat{O}[n] \equiv n^n$$

31.2 Space Argument

Similar but additional notation required?

32 Divergence Criterion

Necessary condition for divergent program, iff
or you can show there exists no lambda, C such that O $[n]$ is n^n is bounded by $\lambda n^n + C$ for all n

$limit_{n\to\infty}$ div / solvable > 1

Assumptions

1. Define the "Null Space of $\mathcal{D}$" or "Null Set" $O_\perp$

$$O_\perp = \{\hat{d}_1, \hat{d}_2, ..., \hat{d}_j\}, \quad j > 0$$
$$\hat{O}_j[n] \equiv (O[n])^n, \forall j$$

2. $O_P \cup O_\mathcal{N} = \mathcal{D}$ (by definition)

Assertions

3. $O_P \cap O_\perp = \varnothing$
4. Let $O_\mathcal{N} \cap O_\perp = \hat{O} = \{\hat{O}_i, ...\}$, $i > 0$
5. Consider $D_j \in O_\mathcal{N}$
6. D_j has finite complexity by definition

$$O_j[n] = C$$

7. D_j has at least one optimal solution by the necessity of optimal solution (theorem Z)

$$O_j[n] = C$$

33 Proof of "P $\neq$ NP"

33.1 Proof N implies D

Is trivial by implication of Theorem x and Theorem y

$$\mathcal{N} \equiv \{d_j, ..\} \quad \forall j, \mathcal{N} \in D$$
$$\nexists K, C, \lambda_1 ... \lambda_K :$$
$$O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C, \quad \forall n, \; \forall d_j \in \mathcal{N}$$

1. $\rightarrow \mathcal{P} \cap \mathcal{N} = \varnothing$ by definition of P,N
2. $d_i \in \hat{D} \vee d_i \in$solvable
3. $1 \rightarrow d_i \notin$ solvable
4. $\therefore d_i \in \hat{D} \quad \forall i$ (theorem y) Show that Definition of Non-Polynomial Problems automatically implies Divergent

1. We've proven Solvable Union are disjoint and complete set P 2. N not in P by definition 3. therefore N in divergence by set theory

Currently we have only defined solvable problems and divergent problems
Additionally polynomial problem which the existance of is trivial
Plus we defined non-polynomial complexity
Prove the existence of $\mathcal{N}$ the set of non polynomial problems

33.2 Proof that D implies N

33.3 D iff N

Show $O[n]$ in the $\varnothing$ the set of problems with $n^n > O[n] > n^k + c$
Proving there's Polynomial and Divergent, in the set of all decision problems

A neat follow up, tie in the definition of $\mathcal{N}$ implies membership to divergent problems

34 Prove the existance of D = N,The Traveling Salesman Problem

Define the traveling salesman problem, prove it is divergent and has the same solution as current approaches
Consider proving with both definition and necessary condition

34.1 Compute every sub path or recursive subpaths in memory

Trade off between time and space, $O_{salesman}[n]$ diverges with a polynomial $O_P[n]$

35 Prove Polynomial and Divergent problems are Complements

Implied by the previous sections

36 Solvable Union Divergent = all decision problems

Trivial as a result of the previous section by definition of Ω

$$\mathbb{P} \cup \hat{\mathbb{D}} = \mathbb{D}$$

37 Theorem of Prime Numbers "Riemann Hypothesis"

Riemann Zeta Function

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} \quad [2]$$

"The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2." [2]

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Citations

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