Ch. 5 Computation

1 Programs

1.1 Logical Instructions

Define $\mathcal{L}$; an ordered set of logical operations s_i

$$\mathcal{L} := \{s_1, s_2, ..., s_N\}$$

1.2 Memory

Define Memory $\mathcal{M}$; a set of elements, magnitudes, or sets b_i

$$\mathcal{M} := \{b_1, b_2, ..., b_M\}$$

1.3 State |

Define state; the memory utilized to perform program P

$$P := \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M\} =$$
$$\{s_1, s_2, ..., s_N, b_1, b_2, ..., b_M\}$$

1.4 Boolean Programs

Define a boolean program; boolean programs can represent functions with inputs x_i, input set C, and boolean output y_o

$$X = \{x_1, ..., x_n, C\}; \;\; C = \{u_1, u_2, ..., u_c\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, y_o\} =$$
$$P[X] \rightarrow y_o \in \{\mathbb{T}, \mathbb{F}\}$$

1.5 Void Programs

Define a void program; a program with inputs x_i, input set C, and no output

$$X = \{x_1, ..., x_n, C\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M\}$$

1.6 Numerical Programs

Define a numerical program; a program with inputs x_i, input set C, and real, rational output y_o

$$X = \{x_1, ..., x_n, C\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, y_o\} =$$
$$P[X] \to y_o \in \mathbb{Q} \;\; y_o \geqslant 0$$

1.7 System Programs

Define a system program; a program with inputs x_i, input set C, and real, output set Y_o

$$X = \{x_1, ..., x_n, C\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, Y_o\} =$$
$$P[X] \to Y_o = \{y_1, y_2, ..., y_K\}$$

1.8 Mathematical Programs

Define a mathematical program; a program with inputs x_i, input set C and numerical output y_o

$$X = \{x_1, ..., x_n, C\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, y_o\} =$$
$$P[X] \to y_o \in \mathbb{Q}$$

2 No-op ;

2.1 Definition

$$; := \varnothing$$

2.2 Property of No-op

No-op can be inserted into any set with equality

$$S = \{s_1, s_2, ..., s_N\}$$

$$S_; = insert[S, ;, i]$$

$$S_; = S_1 \ \ \forall i$$

$$|S_;| = |S| \ \ \forall i$$

2.3 Proof

by definition of magnitude of null = 0 with Set And

3 Decision Problems

3.1 Definition

Define decision problem; a function with inputs x_i and boolean output "answer" a_o

$$X_i = \{x_1, ..., x_n, C\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

4 General Solutions

4.1 Definition

Program P is a general solution s^+ to decision problem D if

1. P outputs answer a_o for all inputs X_i $\forall i$
and
2. $s^+[X_i]$ is a subset of $s^+[\hat{X}_i]$

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_n, x_{n+1}, C\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$P[X_i] = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\}$$
$$s^+ = P[X_i] = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} \quad \forall X_i$$

4.1.1 Property of No-op ;

No-op ; can be added to any solution S_i without modifying the output y_o

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$\hat{s}^+ \to \hat{y}_o = insert[s^+, ;, k]$$
$$\hat{y}_o = y_o \;\; \forall k$$

4.2 Definition of S^+

Define S^+; the set of solutions to decision problem D

$$X_i = \{x_1, ..., x_n, C\}; \quad \hat{X}_i = \{x_1, ..., x_n, x_{n+1}, C\}$$
$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s_j^+ = s_j^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$
$$S^+ := \{s_j^+, ...\} \quad \forall j$$

4.3 Definition of Solvable

Define solvable

$$X_i = \{x_1, ..., x_n, C\}; \quad \hat{X}_i = \{x_1, ..., x_n, x_{n+1}, C\}$$
$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$solvable := solvable[D] \rightarrow b_o \in \{\mathbb{T}, \mathbb{F}\} =$$
$$\exists P : (P[X_i] \rightarrow y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

5 The set of all Decision Problems $\mathbb{D}$

5.1 Definition

Define the set of decision problems $\mathbb{D}$

$$X_i = \{x_1, ..., x_n, C\}$$
$$D_j := f_j[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$\mathbb{D} := \{D_j, ...\} \quad \forall j$$

6 Instruction and Memory Notation

Define $\mathcal{L}$ a set of logical operations
Define $\mathcal{M}$ a set of memory elements, magnitudes, and sets

$$X_i = \{x_1, ..., x_n, C\};$$
$$P[X_i] \rightarrow y_o = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$\mathcal{L} := \{s_1, s_2, ..., s_{O_T[n]}\}$$
$$\mathcal{M} := \{b_1, b_2, ..., b_{O_S[n]}\}$$
$$P[X_i] = \{\mathcal{L}, \mathcal{M}, y_o\}$$

7 Complexity

7.1 Time Complexity of a Decision Problem $O_T[n]$

Define Time Complexity $O_T[n]$ of solution s^+ to Decision Problem D as the total number of logical operations

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$

$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$s^+[X_i] := P :$$
$$(P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_T[n] := |\mathcal{L}| = N$$

7.2 Space Complexity $O_S[n]$

Define Space Complexity $O_S[n]$ of solution s^+ to Decision Problem D as the total number of memory elements

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$

$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$s^+[X_i] := P :$$
$$(P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_S[n] := |\mathcal{M}| + |y_o|^* = M + 1$$

*It is convention to reserve one memory element for output y_o.
Void programs do not require the y_o memory element for output

8 Definition of Complexity

Define Complexity $O[n]$ as a vector of dimension Y

$$\mathbf{O}[n] := \ < O_T[n], O_S[n], O_3[n], O_4[n]..., O_V[n] >$$

9 Total Complexity

$$O[n] := O_T[n] + O_S[n] + \sum_{i=3}^{V} O_i[n]$$

10 Simple Computational Complexity

The remainder of this chapter assumes simple computational complexity of dimension 2

10.1 Definition

Define simple computational complexity of dimension 2

$$\mathbf{O}[n] := \; < O_T[n], O_S[n] >$$

10.2 Time Complexity

Restate definition of Time Complexity $O_T[n]$ of solution s^+

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_T[n] := |\mathcal{L}| = N$$

10.3 Space Complexity

Restate definition of Time Complexity $O_S[n]$ of solution s^+

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_S[n] := |\mathcal{M}| + |y_o| = M + 1$$

10.4 Total Complexity

$$O[n] := O_T[n] + O_S[n]$$

$$= |\mathcal{L}| + |\mathcal{M}| + |y_o| = N + M + 1$$

10.5 $O_S[n] \neq 0$

10.5.1 Proof

By definition of decision problem; Proof by contradiction; y_o must be set to TF by definition; Suppose yo = 0; then yo is empty set; contradicts definition of D

10.6 $O_T[n] \neq 0$

10.6.1 Proof

By definition of decision problem; Proof by contradiction; y_o must be set to TF by definition; Suppose |L| = 0; yo <- TF cap L is null by definition of empty set; implies yo emptyset (doesnt exist)

10.7 **O[n]** $= O_T[n] + O_S[n] \neq \mathbf{0}$

10.7.1 Proof

10.8 $O[n] > O_T[n]$

10.8.1 Proof

10.9 $O[n] > O_S[n]$

10.9.1 Proof

10.10 $O[n+1] \geqslant O[n]$

11 Polynomial Complexity

11.1 Definition

Decision problem D with solution s^+ has polynomial total complexity $O[n]$ if

$$\exists K, C, \lambda_1 ... \lambda_K \quad :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

11.2 Polynomial Problems

Define $\mathbb{P}$, the set of Decision Problems that can be solved with Polynomial Complexity

$$\mathbb{P} := \{D_1, D_2, ...\} :$$
$$\exists K, C, \lambda_1 ... \lambda_K :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, D_i \in \mathbb{P}$$

11.3 Polynomial Order of Complexity

Solution s^+ with total complexity $O[n]$ is said to be of order n^K

$$O[n] \sim n^K$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

11.4 Property of Polynomial Complexity 1

Solutions with polynomial complexity have convergent complexity

$$lim_{n \to \infty} \frac{O[n+1]}{O[n]} = 1$$

11.4.1 Proof WIP

FIX!!! Show there exists no constant satisfying the decreasing limit condition

$$O[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$$
$$O[n+1] < (\lambda_K (n+1))^K + (\lambda_{K-1}(n+1))^{K-1} ... + \lambda_1 (n+1) + C$$
$$O[n] \sim (\lambda n)^K; \;\; O[n+1] \sim (\lambda n)^K$$
$$lim_{n \to \infty} \frac{(\lambda n)^K}{(\lambda n)^K} = 1$$

11.5 Property of Polynomial Complexity 2

$$\exists K, C, \lambda_1, ..., \lambda_K :$$
$$(O[n+1] - O[n]) = f_{n+1}[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

11.5.1 Proof FIX!!!

$$O[n+1] < (\lambda_K (n+1))^K + (\lambda_{K-1}(n+1))^{K-1} ... + \lambda_1 (n+1) + C$$
$$O[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$$

11.6 Total Polynomial Complexity Implies Time bounded Polynomial Complexity

$$D \in \mathbb{P} \Longrightarrow O_T[n] < ...$$

11.6.1 Proof FIX!!!

$$O[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \;\; \forall n$$
$$O[n] := O_T[n] + O_S[n]; \;\; O_T[n] < O[n]$$
$$\therefore O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \;\; \forall n$$

11.7 Total Polynomial Complexity Implies Space bounded Polynomial Complexity

$$D \in \mathbb{P} \Longrightarrow O_S[n] < ...$$

11.7.1 Proof FIX!!!

$$O[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \;\; \forall n$$
$$O[n] := O_T[n] + O_S[n]; \;\; O_S[n] < O[n]$$
$$\therefore O_S[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \;\; \forall n$$

11.8 Total Polynomial Complexity iff Time and Space bounded by Polynomial Complexity

Use limit definition

11.9 Order of Complexity

ERROR in second condition
Total Complexity is said to be on the order of K_{max}

$$O[n] \sim K_{max}$$

$$K_{max} := K :$$

$$O[n] < (\lambda_{K_{max}} n)^{K_{max}} + (\lambda_{K_{max}-1} n)^{K_{max}-1} ... + \lambda_1 n + C \quad \forall n$$

$$\nexists O[n] < (\lambda_{\hat{K}_{max}} n)^{\hat{K}_{max}} + (\lambda_{\hat{K}_{max}-1} n)^{\hat{K}_{max}-1} ... + \lambda_1 n + C \quad \forall n, \hat{K} < K_{max}$$

11.10 Theorem Either OT or OS is on the order of Oopt

Proof by contradiction

12 Polynomial Time Complexity

12.1 Definition

Decision problem D with (optimal) Time Complexity $O_T[n]$ is bounded by polynomial time complexity if

$$\exists K, C, \lambda_1 ... \lambda_K \quad :$$
$$O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

12.2 Polynomial Time Solutions

Define $\mathbb{S}^+_{time}$, the set of solutions that can be solved with polynomial time complexity

$$\mathbb{S}^+_{time} := \{s^+_1, s^+_2, ...\} :$$
$$\exists K, C, \lambda_1 ... \lambda_K :$$
$$O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, s_i \in \mathbb{S}^+_{time}$$

12.3 Property of Polynomial Time Complexity 1

$$lim_{n \to \infty} \frac{O_T[n+1]}{O_T[n]} = 1$$

12.3.1 Proof

12.4 Property of Polynomial Time Complexity 2

$$\exists K, C, \lambda_1, ..., \lambda_K :$$
$$(O_T[n+1] - O_T[n]) < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \ \forall n$$

12.4.1 Proof

12.5 Order of Complexity

Time complexity $O_T[n]$ is said to be on the order of K_{max}

$$O_T[n] < (\lambda_{K_{max}} n)^{K_{max}} + (\lambda_{K_{max}-1} n)^{K_{max}-1} ... + \lambda_1 n + C$$
$$O_T[n] \sim K_{max}$$

12.6 Proof of the existence of $O_{T_{opt}}$

13 Polynomial Space Complexity

13.1 Defintion

Decision problem D with (optimal) Time Complexity $O_S[n]$ is bounded by polynomial time complexity if

$$\exists K, C, \lambda_1 ... \lambda_K \quad :$$
$$O_S[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

13.2 Polynomial Space Problems

Define $\mathbb{S}^+_{space}$, the set of solutions that can be solved with polynomial time complexity

$$\mathbb{S}^+_{space} := \{s_1, s_2, ...\} :$$
$$\exists K, C, \lambda_1 ... \lambda_K :$$
$$O_S[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, s_i \in \mathbb{S}^+_{time}$$

13.3 Total Polynomial Complexity Implies Space bounded Polynomial Complexity

13.4 Space Bounded Polynomial Complexity Implies Total Polynomial Complexity

13.5 Polynomial Space Complexity iff Polynomial Complexity

13.6 Property of Polynomial Space Complexity 1

$$lim_{n \to \infty} \frac{O_S[n+1]}{O_S[n]} = 1$$

13.6.1 Proof

13.7 Property of Polynomial Space Complexity 2

$$\exists K, C, \lambda_1, ..., \lambda_K :$$
$$(O_S[n+1] - O_S[n]) < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \ \ \forall n \ \ \forall n$$

13.7.1 Proof

13.8 Order of Complexity

Space complexity $O_S[n]$ is said to be on the order of K_{max}

$$O_S[n] < (\lambda_{K_{max}} n)^{K_{max}} + (\lambda_{K_{max}-1} n)^{K_{max}-1} ... + \lambda_1 n + C$$

$$O_S[n] \sim K_{max}$$

13.9 Proof of the existence of $O_{S_{opt}}$

14 Non-Polynomial Complexity

14.1 Definition

Decision problem $\tilde{D}$ with solution s^+ has non-polynomial total complexity $O[n]$ if

$$\nexists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

14.2 Non-Polynomial Problems

Define $\mathcal{N}$, the set of Decision Problems that cannot be solved with Polynomial Complexity

$$\mathcal{N} := \{\tilde{D}_1, \tilde{D}_2, ...\} :$$
$$\nexists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, \tilde{D} \in \mathcal{N}$$

14.3 $\mathbb{P}$ and $\mathcal{N}$ are disjoint

$$\mathbb{P} \cap \mathcal{N} = \varnothing$$

14.3.1 Proof

Proof by contradiction

15 Divergent Complexity

15.1 Defintion

Decision problem $\hat{D}$ with solution s^+ has divergent total complexity $O[n]$ if

$$lim_{n \to \infty} \frac{O[n+1]}{O[n]} \ \ diverges \quad \forall n$$

15.2 Divergent Problems

$$\hat{\mathcal{D}} := \{\hat{D}_1, \hat{D}_2, ...\} :$$
$$lim_{n \to \infty} \frac{O[n+1]}{O[n]} \ \ diverges \quad \forall \hat{D} \in \hat{\mathcal{D}}$$

15.3 Derivative Property of Divergent Solutions

$$lim_{n\to\infty}\ O[n+1]-O[n] \text{ diverges}$$

15.3.1 Proof

15.4 The Set of Polynomial Solutions and the Set of Divergent Solutions are disjoint

$$\mathbb{P}\cap\hat{D}=\varnothing$$

15.5 Proof

Proof by contradiction; Let $s^+\in\mathbb{P},\hat{D};s^+\in\mathbb{P}\cap\hat{D}$

$$X_i=\{x_1,...,x_n\}$$

$$D:=f[X_i]\to a_o\in\{\mathbb{T},\mathbb{F}\}\quad\forall X_i$$

$$Let\ \ D\in\mathbb{P}$$

$$s^+:=P[X_i]\to y_o:y_o=a_o\quad\forall X_i$$

$$O[n]=O_T[n]+O_S[n]<(\lambda_K n)^K+(\lambda_{K-1}n)^{K-1}...+\lambda_1 n+C\quad\forall n$$

16 Inductive Functions

16.1 Inductive Function f_{n+1}

$$O[n] := O_T[n] + O_S[n]$$
$$O[n+1] = O_T[n+1] + O_S[n+1]$$

$$f_{n+1}[n] := f[n] :$$
$$O[n+1] = f[n] + O[n] \quad \forall n$$

16.1.1 Proof of existence

Algebraic Proof

16.2 Inductive Space and Time Formulas

$$f^T_{n+1} := O_T[n+1] - O_T[n]$$
$$O_T[n+1] = O_T[n] + f^T_{n+1}$$
$$f^S_{n+1} := O_S[n+1] - O_S[n]$$
$$O_S[n+1] = O_S[n] + f^S_{n+1}$$

16.2.1 Proof of existence

Algebraic Proof

16.3 Inductive Function Expressions

Relate $f_{n+1}[n]$ to equivalence functions

$$D \in \mathbb{P}$$

$$O[n] := O_T[n] + O_S[n]$$
$$O[n+1] = O_T[n+1] + O_S[n+1] = O[n] + f_{n+1}[n]$$
$$O_T[n] = O[n] - O_S[n]$$
$$O_S[n] = O[n] - O_T[n]$$

$$f_{n+1} = O[n+1] - O[n]$$
$$f_{n+1} = O_T[n+1] + O_S[n+1] - O[n]$$

$$f_{n+1} = O_T[n+1] - O_T[n] + O_S[n+1] - O_S[n]$$
$$f_{n+1} = O[n+1] - O_T[n] - O_S[n]$$
$$f_{n+1}[n] = f^T_{n+1}[n] + f^S_{n+1}[n]$$

16.4 Zero Order Inductive Function

$$Let\ \ O_S[n] \sim n^0$$
$$f_{n+1} = O_T[n+1] - O_T[n] + O_S[n+1] - O_S[n] = O_T[n+1] - O_T[n]$$

16.5 Property of Polynomial Complexity

$f_{n+1}[n]$ has order less than O[n]
$f_{n+1}[n]$ is bound by K_{max} - 1

16.5.1 Proof

Proof by contradiction; limit doesn't converge

17 Subfunctions

17.1 Restate the subfunction condition of general solutions

Recall the definition of general solution s^+

$$X_i = \{x_1, ..., x_n, C\}; \ \ \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \ \ \forall X_i) \ \ \cap \ \ (P[\hat{X}_i] \supseteq P[X_i] \ \ \forall X_i, \hat{X}_i)$$

The subfunction condition is one of two conditions for a general solution

$$P[\hat{X}_i] \supseteq P[X_i] \ \ \forall X_i, \hat{X}_i$$

17.2 Prove O[n] is a non-decreasing function

Consider solution s^+ with complexity $O[n]$

$$X_i = \{x_1, ..., x_n, C\}; \ \ \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \ \ \forall X_i) \ \ \cap \ \ (P[\hat{X}_i] \supseteq P[X_i] \ \ \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O[n] := O_T[n] + O_S[n]$$

$$O_T[n] := |\mathcal{L}| = N$$
$$O_S[n] := |\mathcal{M}| + |y_o| = M + 1$$

O[n+1] denotes the total complexity for solution $s^+[\hat{X}_i]$

$$s^+[\hat{X}_i] = \hat{s}^+$$

Let

$$O[n+1] < O[n]$$
$$\Rightarrow \hat{N} + \hat{M} < N + M$$
$$\hat{s}^+ = \{s_1, s_2, ..., s_{\hat{N}} | b_1, b_2, ..., b_{\hat{M}}, y_o\}$$

$$\Rightarrow \hat{s}^+ \not\supseteq s^+$$
$$P[\hat{X}_i] \not\supseteq P[X_i] \quad \forall X_i, \hat{X}_i$$

$$\therefore O[n+1] < O[n] \text{ contradicts the definition of solution } s^+$$
$$O[n+1] \geqslant O[n]$$

17.3 Definition of Subfunction

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub[X_i] := S = \{s_j, ... | b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \quad \forall s_j, b_k \in S$$

17.3.1 $s^+[X_i]$ is a subfunction of $s^+[\hat{X}_i]$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$\hat{s}^+ = \{s_1, s_2, ..., s_N, ..., s_{\hat{N}} | b_1, b_2, ..., b_M, ..., b_{\hat{M}}, y_o\}; \;\; \hat{N} + \hat{M} \geqslant N + M$$

By definition of solution

$$\hat{s}^+ = P[\hat{X}_i] \supseteq P[X_i] = s^+ \quad \forall X_i, \hat{X}_i$$
$$\Rightarrow s_j, b_k \in \hat{s}^+ \;\; \forall s_j, b_k \in s^+$$

17.4 Subfunction Decomposition of Solutions

Solutions s^+ can be written as the union of subfunctions $Sub_k[X_i]$

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \;\; \forall X_i) \;\; \cap \;\; (P[\hat{X}_i] \supseteq P[X_i] \;\; \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$s^+ = Sub_1[X_i] \cup Sub_2[X_i] \cup ... \cup Sub_z[X_i]$$
$$= \{\mathcal{L}_1 | \mathcal{M}_1, y_o\} \cup \{\mathcal{L}_2 | \mathcal{M}_2, y_o\} \cup ... \cup \{\mathcal{L}_z | \mathcal{M}_z, y_o\} :$$
$$\mathcal{L}_j \cap \mathcal{L}_k = \varnothing \;\; \forall j, k \neq j$$

$$s^+ = \{s_1^1, ..., s_{N_1}^1 | b_1^1, ..., y_o\} \cup \{s_1^2, ..., s_{N_2}^2 | b_1^2, ..., y_o\} \cup ... \cup \{s_1^z, ..., s_{N_z}^z | b_1^z, ..., y_o\} :$$
$$\textstyle\sum_{l=1}^{z} N_l = N = O_T[n]$$

18 Subfunction Complexity

18.1 Disjoint Subfunction Operations

$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \;\; \forall i, j \neq i$$

18.2 Shared Subfunction Memory

$$|\mathcal{M}_i \cap \mathcal{M}_j| \geqslant 0 \;\; \forall i, j \neq i$$

18.2.1 Time Complexity of Subfunctions

Subfunction time complexity is additive

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub_i[X] := S_i = \{s_j, ... | b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \;\;\; \forall s_j, b_k \in S_i$$

$$s^+ = \{\mathcal{L}_1 | \mathcal{M}_1, y_o\} \cup \{\mathcal{L}_2 | \mathcal{M}_2, y_o\} \cup ... \cup \{\mathcal{L}_z | \mathcal{M}_z, y_o\} :$$
$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \;\; \forall i, j \neq i$$

$$\mathcal{L} = \cup_{i=1}^{z} \mathcal{L}_i$$

$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$$

$$O_T[n] = |\mathcal{L}| = N$$

$$O_T[n] = |\cup_{i=1}^{z} \mathcal{L}_i| = \textstyle\sum_{i=1}^{z} |\mathcal{L}_i|^* = |\mathcal{L}_1| + |\mathcal{L}_2| + ... + |\mathcal{L}_z|$$
$$= O_{T_1}[n] + O_{T_2}[n] + ... + O_{T_z}[n] = N_1 + N_2 + ... + N_z$$

*Due to the disjoint condition of subfunction operations $\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$

18.2.2 Space Complexity of Subfunctions

Subfunctions can access the full memory $\mathcal{M}$ with no added space complexity

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub_i[X] := S_i = \{s_j, ...|b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \quad \forall s_j, b_k \in S_i$$

$$s^+ = \{\mathcal{L}_1|\mathcal{M}_1, y_o\} \cup \{\mathcal{L}_2|\mathcal{M}_2, y_o\} \cup ... \cup \{\mathcal{L}_z|\mathcal{M}_z, y_o\} :$$
$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$$

$$\mathcal{M} = \cup_{i=1}^{z} \mathcal{M}_i = \cup_{i=1}^{z} \mathcal{M}$$

$$O_S[n] = |\mathcal{M}| = M$$
$$O_S[n] = |\cup_{i=1}^{z} \mathcal{M}_i| = |\cup_{i=1}^{z} \mathcal{M}| = M$$

18.2.3 Shared State Notation

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub_i[X] := S_i = \{s_j, ...|b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \quad \forall s_j, b_k \in S_i$$

$$s^+ = \{\mathcal{L}_1|\mathcal{M}, y_o\} \cup \{\mathcal{L}_2|\mathcal{M}, y_o\} \cup ... \cup \{\mathcal{L}_z|\mathcal{M}, y_o\} :$$
$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$$

19 Subfunction Theorems

Might be for subfunctions of polynomial and divergent

19.1 Restate Definition of Subfunction

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \;\; \forall X_i) \;\; \cap \;\; (P[\hat{X}_i] \supseteq P[X_i] \;\; \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub[X_i] := S = \{s_j, ... | b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \;\; \forall s_j, b_k \in S$$

19.2 The Union of Two Converging Subfunctions is Convergent

Let

$$Sub_1[X] \text{ with total complexity } O_1[n]$$
$$O_1[n] :$$
$$limit_{n \rightarrow \infty} \frac{O_1[n+1]}{O_1[n]} = c_1$$

$$Sub_2[X] \text{ with total complexity } O_2[n]$$
$$O_2[n] :$$
$$limit_{n \rightarrow \infty} \frac{O_2[n+1]}{O_2[n]} = c_2$$

$$Sub_{1;2}[X] = Sub_1[X] \cup Sub_2[X] \text{ with complexity } O[n]$$
$$limit_{n \rightarrow \infty} \frac{O[n+1]}{O[n]} = c$$

19.2.1 Proof

Show

$$limit_{n\to\infty}\frac{O[n+1]}{O[n]} = c$$

$$Sub_{1;2}[X] = Sub_1[X] \cup Sub_2[X] \text{ with complexity } O[n]$$
$$O_1[n] = O_{T_1}[n] + O_S[n]$$
$$O_2[n] = O_{T_2}[n] + O_S[n]$$
$$O[n] = O_{T_1}[n] + O_{T_2}[n] + O_S[n]$$

$$\frac{O[n+1]}{O[n]} = \frac{O_{T_1}[n+1]+O_{T_2}[n+1]+O_S[n+1]}{O_{T_1}[n]+O_{T_2}[n]+O_S[n]}$$
$$\frac{O[n+1]}{O[n]} = \frac{O_{T_1}[n+1]+O_S[n+1]}{O_{T_1}[n]+O_{T_2}[n]+O_S[n]} + \frac{O_{T_2}[n+1]}{O_{T_1}[n]+O_{T_2}[n]+O_S[n]}$$

For all non-decreasing functions f[n], g[n]

$$f_{n+1} \text{ goes to 0 faster}$$

19.3 The Union of Two Divergent Subfunctions is Divergent

Let

$$Sub_1[X] \text{ with total complexity } O_1[n]$$
$$O_1[n]:$$
$$limit_{n\to\infty}\frac{O_1[n+1]}{O_1[n]} diverges$$

$$Sub_2[X] \text{ with total complexity } O_2[n]$$
$$O_2[n]:$$
$$limit_{n\to\infty}\frac{O_2[n+1]}{O_2[n]} diverges$$

$$Sub_{1;2}[X] = Sub_1[X] \cup Sub_2[X] \text{ with complexity } O[n]$$
$$limit_{n\to\infty}\frac{O[n+1]}{O[n]} diverges$$

19.3.1 Proof

Show

$$limit_{n\to\infty}\frac{O[n+1]}{O[n]} diverges$$

19.4 The Union of a convergent and divergent subfunction is Divergent

Let

$$Sub_1[X] \text{ with total complexity } O_1[n]$$
$$O_1[n]:$$
$$limit_{n\to\infty}\frac{O_1[n+1]}{O_1[n]} = c_1$$

$$Sub_2[X] \text{ with total complexity } O_2[n]$$
$$O_2[n]:$$
$$limit_{n\to\infty}\frac{O_2[n+1]}{O_2[n]} = diverges$$

$$Sub_{1;2}[X] = Sub_1[X] \cup Sub_2[X] \text{ with complexity } O[n]$$

Show

$$limit_{n\to\infty}\frac{O[n+1]}{O[n]} diverges$$

19.4.1 Proof

19.5 Theorem of Divergent Subfunctions

19.5.1 $limit_{n\to\infty}\frac{O[n+1]}{O[n]}$ diverges $\Rightarrow$ $\exists Sub_h[X_i] : limit_{n\to\infty}\frac{O_h[n+1]}{O_h[n]}$ diverges

If any subfunction of s^+ diverges, then O[n+1]/O[n] diverges, f_{n+1}/O[n] diverges Consider solution s^+ with polynomial total complexity O[n] containing z subfunctions $Sub_k[X_i]$ k = 1..z
FIX!!! concerns about OS memory complexity; $c_h = c_{T_h} + c_{S_h}$; c_{S_h} is the same for all subfuncrions

$$X_i = \{x_1, ..., x_n, C\};\ \ \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P:$$
$$(P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub_h[X_i] := S_h = \{s_j, ...|b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \quad \forall s_j, b_k \in S_h$$

$$s^+ = Sub_1[X_i] \cup Sub_2[X_i] \cup ... \cup Sub_z[X_i]$$

$$O[n] = O_{T_1}[n] + O_{T_2}[n] + ... + O_{T_z}[n] + |O_{S_1}[n] \cup O_{S_2}[n] \cup ... \cup O_{S_z}[n]|$$
$$= O_{T_1}[n] + O_{T_2}[n] + ... + O_{T_z}[n] + O_S[n]$$

By defintion of divergent complexity

$$limit_{n\to\infty} \frac{O[n+1]}{O[n]} \text{ diverges}$$

Suppose there does not exist a diverging subfunction $Sub_h[X_i]$ for all h

$$\nexists Sub_h[X_i] :$$
$$limit_{n\to\infty} \frac{O_h[n+1]}{O_h[n]} \text{ diverges} \quad \forall h$$
$$\Rightarrow limit_{n\to\infty} \frac{O_h[n+1]}{O_h[n]} = c_h \ \ \forall h$$
$$limit_{n\to\infty} \frac{O_1[n+1]+O_2[n+1]+...+O_z[n+1]}{O_1[n]+O_2[n]+...+O_z[n]}$$

Let

$$g_h[n] = \textstyle\sum_{i\neq h} O_i[n] \geqslant 0^*$$
$$\Rightarrow 0 \leqslant limit_{n\to\infty} \frac{O_h[n+1]}{O_h[n]+g_h[n]} \leqslant c_h$$
$$limit_{n\to\infty} \frac{O_1[n+1]}{O_1[n]+g_1[n]} + \frac{O_2[n+1]}{O_2[n]+g_2[n]} + ... + \frac{O_z[n+1]}{O_1[n]+g_z[n]}$$
$$0 \leqslant limit_{n\to\infty} \frac{O_1[n+1]}{O_1[n]+g_1[n]} + \frac{O_2[n+1]}{O_2[n]+g_2[n]} + ... + \frac{O_z[n+1]}{O_1[n]+g_z[n]} \leqslant \textstyle\sum_{i=1}^{z} c_i$$
$$\Rightarrow limit_{n\to\infty} \frac{O_1[n+1]}{O_1[n]+g_1[n]} + \frac{O_2[n+1]}{O_2[n]+g_2[n]} + ... + \frac{O_z[n+1]}{O_1[n]+g_z[n]} = \tilde{C}$$
$$0 \leqslant \tilde{C} \leqslant \textstyle\sum_{i=1}^{z} c_i$$

*O$_i[n] \geqslant 0$ is a non-decreasing function

Assuming

$$\nexists Sub_h[X_i]:$$
$$limit_{n\to\infty}\frac{O_h[n+1]}{O_h[n]}\text{ diverges}\quad\forall h$$
$$\Rightarrow limit_{n\to\infty}\frac{O_[n+1]}{O_1[n]}=\tilde{C}$$

Contradicting the definition of divergent solution

$$\therefore \exists Sub_h[X_i]:$$
$$limit_{n\to\infty}\frac{O_h[n+1]}{O_h[n]}\text{ diverges}$$

19.5.2 $\exists Sub_h[X_i] : limit_{n\to\infty}\frac{O_h[n+1]}{O_h[n]}$ diverges $\Rightarrow$ $limit_{n\to\infty}\frac{O[n+1]}{O[n]}$ diverges

FIX!!! SPACE OS portion

$$X_i = \{x_1, ..., x_n, C\};\ \ \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub_h[X_i] := S_h = \{s_j, ... | b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \quad \forall s_j, b_k \in S_h$$

$$s^+ = Sub_1[X_i] \cup Sub_2[X_i] \cup ... \cup Sub_z[X_i]$$
$$O[n] = O_{T_1}[n] + O_{T_2}[n] + ... + O_{T_z}[n] + |O_{S_1}[n] \cup O_{S_2}[n] \cup ... \cup O_{S_z}[n]|$$
$$= O_{T_1}[n] + O_{T_2}[n] + ... + O_{T_z}[n] + O_S[n]$$

Suppose

$$\exists Sub_h[X_i] : limit_{n\to\infty}\frac{O_h[n+1]}{O_h[n]} \text{ diverges}$$

$$\frac{O_h[n+1]}{O_h[n]} \geqslant 1^* \quad \forall h$$

$^*O_h[n]$ is a positive non-decreasing function

$$limit_{n\to\infty}\frac{O[n+1]}{O[n]}$$
$$= limit_{n\to\infty}\frac{O_1[n+1]+O_2[n+1]+...+O_z[n+1]}{O_1[n]+O_2[n]+...+O_z[n]}$$
$$limit_{n\to\infty}\frac{O_1[n+1]}{O[n]} + ... + \frac{O_h[n+1]}{O[n]} + ... + \frac{O_z[n+1]}{O[n]}$$

$$limit_{n\to\infty}\frac{O_h[n+1]}{O[n]} = limit_{n\to\infty}\frac{O_h[n+1]}{O_h[n]+g_h[n]}$$

$$= limit_{n\to\infty}(\frac{O_h[n+1]}{O_h[n]]} - \frac{g_h[n]O_h[n+1]}{O_h[n](O_h[n]+g_h[n])})$$

$$= limit_{n\to\infty}(\frac{O_h[n+1]}{O_h[n]]} - \frac{(O[n]-O_h[n])(O_h[n]+f_{n+1}[n])}{O_h[n]O[n]})$$

$$= limit_{n\to\infty}(\frac{O_h[n+1]}{O_h[n]]} + \frac{-O_h[n]O[n]-f_{n+1}[n]O[n]+O_h^2[n]+f_{n+1}O_h[n]}{O_h[n]O[n]})$$

$$= limit_{n\to\infty}(\frac{O_h[n+1]}{O_h[n]]} - 1 - \frac{f_{n+1}^h[n]}{O_h[n]} + \frac{O_h[n]}{O[n]} + \frac{f_{n+1}^h[n]}{O[n]})$$

19.6 Sum of convergent, divergent, and constant subfunctions

Let

$$s^+ = \cup_{i=1}^z Sub_i[X]$$

20 Sum to N Problem with 2 integers

20.1 State formal definition of Sum to N : $x_i + x_j == N$

$$X_i = \{x_1, ..., x_n, N\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ = P :$$
$$(P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$
$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$D = f[X_i] = \exists x_j, x_k \in X_i : x_j + x_k == N$$

20.2 Express a formal solution : $O_S[n] \sim n^0$

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$
$$s_1 = y_o \leftarrow \mathbb{F};$$
$$\forall i, j > i$$
$$s_2, s_3, s_8, s_9, ..., s_{3ij-4}, s_{3ij-3}..., s_{3n(n-1)-4}, s_{3n(n-1)-3} = b_1 \leftarrow x_i + x_j$$
$$s_4, s_5, s_{10}, s_{11}, ..., s_{3ij-2}, s_{3ij-1}..., s_{3n(n-1)-2}, s_{3n(n-1)-1} = b_1 \leftarrow b_1 == N$$
$$s_6, s_7, s_{12}, s_{13}..., s_{3ij}, s_{3ij+1}..., s_{3n(n-1)}, s_{3n(n-1)+1} = y_o \leftarrow y_o \vee b_1$$
$$s^+ = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\}$$

20.3 Show s^+ satisfies the subfunction condition of solutions: $P[\hat{X}_i] \supseteq P[X_i] \quad \forall \hat{X}_i, X_i$

$$X_i = \{x_1, ..., x_n, N\}; \quad \hat{X}_i = \{x_1, ..., x_n, x_{n+1}, N\}$$
$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$
$$s^+_{n+1} = s^+ \cup \hat{s}^+$$
$$s_1 = y_o \leftarrow \mathbb{F};$$
$$\forall i, j > i$$
$$s_2, s_3, s_8, s_9, ..., s_{3ij-4}, s_{3ij-3}..., s_{3n(n-1)-4}, s_{3n(n-1)-3} = b_1 \leftarrow x_i + x_j$$
$$s_4, s_5, s_{10}, s_{11}, ..., s_{3ij-2}, s_{3ij-1}..., s_{3n(n-1)-2}, s_{3n(n-1)-1} = b_1 \leftarrow b_1 == N$$

$$s_6, s_7, s_{12}, s_{13}..., s_{3ij}, s_{3ij+1}..., s_{3n(n-1)}, s_{3n(n-1)+1} = y_o \leftarrow y_o \lor b_1$$

$$\forall k$$

$$s... = b_1 \leftarrow x_k + x_{n+1}$$

$$s... = b_1 \leftarrow b_1 == N$$

$$s... = y_o \leftarrow y_o \lor b_1$$

$$s^+ = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \lor (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\}$$

$$\hat{s}^+ = \{y_o \leftarrow y_o \lor (x_k + x_{n+1} == N) \ \ \forall k < n+1 | \ b_1, y_o\}$$

$$s^+_{n+1} = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \lor (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\} \ \cup$$
$$\{y_o \leftarrow y_o \lor (x_k + x_{n+1} == N) \ \ \forall k < n+1 | \ b_1, y_o\}$$

$$s^+_{n+1} = P[\hat{X}_i] \supseteq P[X_i] = s^+$$

20.4 Determine $O[n], O_S[n], O_T[n], \hat{O}[n], \hat{O}_T[n], \hat{O}_S[n]$ for the above solution

$$O_S[n] = |y_o| + |b_1| = 2$$
$$O_T[n] = 3n(n-1) + 1 = 3n(n-1) - 1 + O_S[n]$$
$$O[n] = 3n(n-1) + 3 = 3n^2 - 3n + 3$$

$$\hat{O}_S[n] = 0$$
$$\hat{O}_T[n] = 6n$$
$$\hat{O}[n] = \hat{O}_S[n] + \hat{O}_T[n]$$

20.5 Verify $O[n+1] = O[n] + \hat{O}[n]$

$$O[n+1] = O[n] + \hat{O}[n]$$
$$3(n+1)^2 - 3(n+1) + 3 = 3n^2 - 3n + 3 + 6n$$
$$3n^2 + 6n + 3 - 3n - 3 + 3 = 3n^2 + 3n + 3$$
$$3n^2 + 3n + 3 = 3n^2 + 3n + 3$$

20.6 Show s^+ has Polynomial Complexity by the definition of Total Polynomial Complexity

$$O[n] = 3n^2 - 3n + 3$$

20.7 Show s^+ has Polynomial Complexity by showing $\textbf{limit}_{n\to\infty}\frac{O[n+1]}{O[n]}$ $= \mathbf{1}$

$$limit_{n\to\infty}\frac{O[n+1]}{O[n]} =$$

$$limit_{n\to\infty}\frac{3n^2+3n+3}{3n^2-3n+3} =$$

$$limit_{n\to\infty}(\frac{3n^2-3n+3}{3n^2-3n+3} + \frac{6n}{3n^2-3n+3}) =$$

$$limit_{n\to\infty}(1 + \frac{6n}{3n^2-3n+3}) = 1$$

Traveling Salesman Problem of Dimension 2

21 Proof of the existence of $\hat{\mathcal{D}}$

21.1 The Traveling Salesman Problem of Dimension 2

English description

21.2 Formal Definition

$$X_i = \{l_1, l_2, ..., l_n, C\}$$

$$l_i = \{x_i, y_i\} \ \ \forall i$$

l_i denotes the 2D coordinates of location i

$$C = \{d_{proposed}, p_{decimal}\}$$

$d_{proposed}$ denotes the suggested shortest distance

$p_{decimal}$ is the decimal precision

$$L[l_i, l_j] := \sqrt{(y_j - y_i)^2 + (x_j - x_i)^2}$$

Let $L[l_i, l_j]$ denote the distance between location l_i and l_j

$$\tilde{L}[l_i, l_j] := d_{trunc} : -p_{decimal} < d_{trunc} - L[l_i, l_j] < p_{decimal}$$

Let $\tilde{L}[l_i, l_j]$ denote a truncated decimal representation of $L[l_i, l_j]$

$$R_i := \{r_1, r_2, ..., r_n, r_1\} \ \ : \ \ r_i \in X_i \ \ \forall i; \ \ r_i \neq r_j$$

Let R_i denote route i

$$L_{Total}[R_i] := (\textstyle\sum_{i=1}^{n-1} \tilde{L}[r_i, r_{i+1}]) + \tilde{L}[r_n, r_1]$$

Let $L_{Total}[R_i]$ denote the sum of truncated lengths of route R_i

$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \ \ \forall X_i$$

$$a_o =$$

$$(\exists R_k : L_{total}[R_k] == d_{proposed}) \cap (\nexists R_j : L_{total}[R_j] < d_{proposed})$$

Traveling Salesman Problem of Dimension 2

21.3 Define subpath, subpath distance, subpath storage

$\tilde{L}[l_i, l_j]$ denotes "the distance of a subpath of length 1"

$$\tilde{L}[l_i, l_j] := d_{trunc} : -p_{decimal} < d_{trunc} - L[l_i, l_j] < p_{decimal}$$
$$= abs(d_{trunc} - L[l_i, l_j]) < p_{decimal}$$

$\tilde{R}$ denotes a subpath of length k

$$\tilde{R} = \{\tilde{r}_1, \tilde{r}_2, ..., \tilde{r}_k\} \ : \ \tilde{r}_i \in X_i \ \ \forall i, r_i \neq r_j$$

$\tilde{L}_k[\tilde{R}]$ denotes "the distance of a subpath of length k"

$$\tilde{L}_k[\tilde{R}] := \textstyle\sum_{i=1}^{k} \tilde{L}[\tilde{r}, \tilde{r}_{i+1}]$$

Let $\mathcal{M}_1$ denote the memory reserved for subpaths distances of length 1

$$\mathcal{M}_1 = \{\hat{b}_{1;1}, \hat{b}_{1;2}, \hat{b}_{1;3}, ..., \hat{b}_{startindex;finishindex}, ..., \hat{b}_{n-1;n}\}^*$$
$$\mathcal{M} \supseteq \mathcal{M}_1$$

* Note $\hat{b}_{i;j} = \hat{b}_{j;i}$
$\sqrt{(y_j - y_i)^2 + (x_j - x_i)^2} = \sqrt{(y_i - y_j)^2 + (x_i - x_j)^2}$

21.4 Define the following functions

21.4.1 $sqrt[x, p_{decimal}] = \sqrt{x}$ [1]

21.4.2 $pow[x, 2, p_{decimal}] = x^2$ [2]

21.5 Define the following subfunctions

21.5.1 loadM1Subpaths$[X]$

// Computes all subpaths of length 1 and stores in $\mathcal{M}_1 = \{\hat{b}_{1;1}, \hat{b}_{1,;2}, ..., \hat{b}_{n-1;n}\}$

$$//X_i = \{l_1, l_2, ..., l_n, C\}$$
$$//l_i = \{x_i, y_i\} \ \ \forall i$$

$$//\mathcal{M} = \{b_1, b_2, ..., b_M, \hat{b}_{1;1}, \hat{b}_{1,;2}, ..., \hat{b}_{n-1;n}, y_o\} = \{b_1, b_2, ..., b_M, \mathcal{M}_1, y_o\} = \{\mathcal{M}, \mathcal{M}_1, y_o\}$$

$$\forall i, j > i$$

$$b_3 \leftarrow y_i - y_j$$

$$b_4 \leftarrow x_i - x_j$$

$$b_3 \leftarrow b_3^2$$

$$b_4 \leftarrow b_4^2$$

$$b_3 \leftarrow b_3 + b_4$$

$$\hat{b}_{i;j} \leftarrow \sqrt{b_3}^*$$

$^*\hat{b}_{i;j} = \tilde{L}[l_i, l_j]$

21.5.2 computeAllRoutes$[X]$

// Computes all complete routes, checks for a route $==$ $d_{proposed}$, sets y_o to false if the current route is shorter than $d_{proposed}$

$$\forall i, j \neq i, k \neq i, j..., q \neq i, j, ..., m$$

$$b_3 \leftarrow \hat{b}_{1;j} + \hat{b}_{j;k}$$

$$b_3 \leftarrow b_3 + \hat{b}_{k;l}$$

$$...$$

$$b_3 \leftarrow b_3 + \hat{b}_{m;q}$$

$$b_3 \leftarrow b_3 + \hat{b}_{q;1}$$

$$b_4 \leftarrow b_3 == b_2$$

$$b_1 \leftarrow b_1 \vee b_4$$

$$b_4 \leftarrow b_2 \leqslant b_3$$

$$y_o \leftarrow y_o \wedge b_4$$

21.6 Express a solution using subfunctions, storing subpaths of length 1 in memory

// $d_{proposed}$ is the shortest path

$$y_o \leftarrow \mathbb{T}$$

$$//d_{proposed} \text{ exists as a total path length}$$
$$b_1 \leftarrow \mathbb{F}$$

$$// \text{ shortest path register}$$
$$b_2 \leftarrow d_{proposed}$$

$$loadM1Subpaths[X]$$
$$computeAllRoutes[X]$$

21.7 Show each subfunction satisfies the subfunction condition of solutions : $P[\hat{X}_i] \supseteq P[X_i] \quad \forall \hat{X}_i, X_i, \quad \hat{X}_i \supseteq X_i$

Let

$$\mathcal{M}_0 = \{b_1, b_2, b_3, b_4, y_o\}$$
$$\mathcal{M}_1 = \{\hat{b}_{1;1}, \hat{b}_{1,;2}, ..., \hat{b}_{n-1;n}\}$$

21.7.1 $loadM1Subpaths[X] \rightarrow \mathcal{M}_1$

Let

$$//X = \{l_1, l_2, ..., l_n, C\}; \quad \hat{X} = \{l_1, l_2, ..., l_n, l_{n+1}, C\}$$

$$loadM1Subpaths[X, \mathcal{M}] \rightarrow \mathcal{M}_1 = Sub_1[X, \mathcal{M}] \rightarrow \mathcal{M}_1$$

$$Sub_1[X, \mathcal{M}] = \{\mathcal{L}, \mathcal{M}\}$$
$$= \{\hat{b}_{i;j} \leftarrow \tilde{L}[l_i, l_j] \ \ \forall i, j > i | b_3, b_4, \hat{b}_{1;1}, \hat{b}_{1;2}, ..., \hat{b}_{n-1;n}\}$$
$$Sub_1[X_i, \mathcal{M}] = \{\hat{b}_{i;j} \leftarrow \tilde{L}[l_i, l_j] \ \ \forall i, j > i | b_3, b_4, \mathcal{M}_1\}$$

$$Sub_1[\hat{X}, \mathcal{M}] = \{\hat{\mathcal{L}}, \hat{\mathcal{M}}\}$$
$$= \{\hat{b}_{i;j} \leftarrow \tilde{L}[l_i, l_j] \ \ \forall i, j > i | b_3, b_4, \hat{b}_{1;1}, \hat{b}_{1;2}, ..., \hat{b}_{n;n+1}\}$$
$$= \{\mathcal{L}, \hat{b}_{i;j} \leftarrow \tilde{L}[l_i, l_j] \ \ \forall i, j = n+1 | \mathcal{M}, \hat{b}_{1;n+1}, \hat{b}_{2;n+1}, ..., \hat{b}_{n;n+1}\}$$
$$Sub_1[\hat{X}, \mathcal{M}] = \{\mathcal{L}, \mathcal{L}_{n+1} | \mathcal{M}, \mathcal{M}_{n+1}\}$$
$$Sub_1[\hat{X}, \mathcal{M}] = \{\mathcal{L}, \mathcal{L}_{n+1} | \mathcal{M}, \mathcal{M}_{n+1}\} \supseteq \{\mathcal{L} | \mathcal{M}\} = Sub_1[X, \mathcal{M}]$$

21.7.2 $computeAllRoutes[X]$

Let

$$computeAllRoutes[X] = Sub_2[X]$$

$$Sub_2[X] = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$= \{\hat{b}_{1;i_2} + \hat{b}_{i_2;i_3} + \hat{b}_{i_3;i_4} + ... + \hat{b}_{i_n;1} \;\; \forall i_2, i_3 \neq i_2, i_4 \neq i_2, i_3 ... i_n \neq i_2, i_3 ..., i_{n-1}$$
$$|b_1, b_2, b_3, b_4, \mathcal{M}_1, y_o\}$$

$$Sub_2[\hat{X}] = \{\hat{\mathcal{L}}, \hat{\mathcal{M}}, y_o\}$$

$$= \{\hat{b}_{1;i_2} + \hat{b}_{i_2;i_3} + \hat{b}_{i_3;i_4} + ... + \hat{b}_{i_{n+1};1} \;\; \forall i_2, i_3 \neq i_2, i_4 \neq i_2, i_3 ... i_{n+1} \neq i_2, i_3 ..., i_n$$
$$|\mathcal{M}, \mathcal{M}_{n+1}, y_o\}$$

Let

$$insert\_subpath[\mathcal{L}] =$$

$$Sub_2[\hat{X}] = \{insert\_subpath[\mathcal{L}, \hat{b}_{i_{n+1};j}, j] \;\; \forall j \neq n+1 | \mathcal{M}, \mathcal{M}_{n+1}, y_o\}$$

21.7.3 Show the overall solution storing subpaths of length 1 satisfies the subfunction condition of solutions : $P[\hat{X}_i] \supseteq P[X_i] \;\; \forall \hat{X}_i, X_i$

21.8 Express $O[n]$ in terms of subfunction complexities

$$O_{sub1}[n] = O_{T_{sub1}}[n] + O_{S_{sub1}}[n]$$

$$O_{sub2}[n] = O_{T_{sub2}}[n] + O_{S_{sub2}}[n]$$

$$O[n] = O_{sub1}[n] + O_{sub2}[n] + 3$$

21.9 $\mathbf{Sub_+[X]}$

Let $Sub_+[X]$ denote a subfunction that adds all subpaths of length 1
Let $O_+[n]$ denote the total complexity of subfunction $Sub_+[X]$

21.9.1 Find an expression for $O_+[n]$:= the number of $\tilde{L}[l_i, l_j] + \tilde{L}[l_j, l_k]$ length 1 subpath additions

$$O_+[n] = (\textstyle\sum_{i=1}^{n} 1)\frac{(nP(n-1)}{2}$$

$$O_+[n] = \frac{n(n-1)!}{2}$$
$$O_+[n] = \frac{n!}{2}$$

21.10 Prove $Sub_+[X]$ is a subfunction of all s^+ by contradiction

suppose not all subpaths are considered
there could exist subpath resulting in an incorrect solution
contradicts definition of solution

21.11 Show the solution storing subpaths of length 1 contains Sub$_+$[X]

$$// \ d_{proposed} \text{ is the shortest path}$$
$$y_o \leftarrow \mathbb{T}$$

$$//d_{proposed} \text{ exists as a total path length}$$
$$b_1 \leftarrow \mathbb{F}$$

$$// \text{ shortest path register}$$
$$b_2 \leftarrow d_{proposed}$$

$$loadM1Subpaths[X]$$
$$computeAllRoutes[X]$$

21.12 Express $O[n]$ in terms of subfunction complexities including $O_+[n]$ as a subfunction complexity

$$O[n] = O_{sub1}[n] + O_{sub2}[n] + 3$$
$$O_{sub2}[n] = O_+[n] + 8\frac{nP(n-1)}{2}$$
$$O_{sub2}[n] = O_+[n] + 8\frac{(n-1)!}{2}$$
$$O[n] = O_{sub1}[n] + O_+[n] + 8\frac{(n-1)!}{2} + 3$$

21.13 Show $limit_{n\to\infty}\frac{O_+[n+1]}{O_+[n]}$ diverges

$$limit_{n\to\infty}\frac{O_+[n+1]}{O_+[n]}$$

$$= limit_{n\to\infty} \frac{(n+1)!}{2} \frac{2}{n!}$$
$$= limit_{n\to\infty}\ n$$

There does not exist ...
therefore $limit_{n\to\infty} \frac{O_+[n+1]}{O_+[n]}$ diverges

21.14 Prove D diverges by the theorem of divergent subfunctions

21.15 Connection to "P $\neq$ NP"

Citations

[1] *chatgpt*

[2] *https://stackoverflow.com/questions/*3518973*/floating-point-exponentiation-without-power-function*

[3] *https://stackoverflow.com/questions/*27086195*/linear-index-upper-triangular-matrix*

[4] *http://www.math.uchicago.edu/ may/VIGRE/VIGRE*2011*/REUPapers/Riffer-Reinert.pdf*