Computation

1 Programs

1.1 Logical Instructions

Define $\mathcal{L}$; an ordered set of operations s_i

$$\mathcal{L} := \{s_1, s_2, ..., s_N\}$$

1.2 Memory

Define Memory $\mathcal{M}$; a set of elements, magnitudes, or sets b_i

$$\mathcal{M} := \{b_1, b_2, ..., b_M\}$$

1.3 Program

Define Program; a set of logical instructions and memory

$$P := \{s_1, s_2, ..., s_N, b_1, b_2, ..., b_M\}$$

1.3.1 State Notation

Define State Notation; separating the final operation s_N and the first memory element b_1

$$\begin{aligned} P &:= \{s_1, s_2, ..., s_N, b_1, b_2, ..., b_M\} \\ &= \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M\} \end{aligned}$$

1.4 Boolean Programs

Define a boolean program; boolean programs can represent functions with inputs x_i, input set C, and boolean output y_o

$$X = \{x_1, ..., x_n, C\}; \;\; C = \{u_1, u_2, ..., u_c\}$$

$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, y_o\} =$$

$$P[X] \to y_o \in \{\mathbb{T}, \mathbb{F}\}$$

1.5 Void Programs

Define a void program; a program with inputs x_i, input set C, and no output

$$X = \{x_1, ..., x_n, C\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M\}$$

1.6 Numerical Programs

Define a numerical program; a program with inputs x_i, input set C, and real, rational output y_o

$$X = \{x_1, ..., x_n, C\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, y_o\} =$$
$$P[X] \to y_o \in \mathbb{Q} \;\; y_o \geqslant 0$$

1.7 System Programs

Define a system program; a program with inputs x_i, input set C, and real, output set Y_o

$$X = \{x_1, ..., x_n, C\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, Y_o\} =$$
$$P[X] \to Y_o = \{y_1, y_2, ..., y_K\}$$

1.8 Mathematical Programs

Define a mathematical program; a program with inputs x_i, input set C and numerical output y_o

$$X = \{x_1, ..., x_n, C\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, y_o\} =$$
$$P[X] \to y_o \in \mathbb{Q}$$

2 Problem Definition

Also denoted as a "Question"

$$X_i = \{x_1, ..., x_n\}$$
$$Q := f[X_i] = Y_o \subseteq \Omega \quad \forall X_i$$

2.1 Set of Questions

Define $\mathbb{Q}$; the set of questions

$$\mathbb{Q} := \{Q_1, Q_2, ...\} :$$
$$Q_i = f[X_j] = Y_o \subseteq \Omega \;\; \forall X_j, i$$

2.2 Decision Questions / Decision Problems

2.2.1 Definition

Define decision problem; a function with inputs x_i and boolean output "answer" a_o

$$X_i = \{x_1, ..., x_n\}$$
$$D := f[X_i] = a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

2.3 Numerical Questions / Numerical Problems

2.3.1 Definition

Define numerical problem; a function with inputs x_i and numerical output y_o

$$X_i = \{x_1, ..., x_n\}$$
$$Q := f[X_i] = y_o \in \mathbb{R} \quad \forall X_i$$

2.4 System Questions / System Problems

2.4.1 Definition

Define system problem; a function with inputs x_i and outputs y_j

$$X_i = \{x_1, ..., x_n\}$$
$$Q := f[X_i] = Y_o = \{y_1, ..., y_m\} \quad \forall X_i$$

3 General Solutions

3.1 Definition

Program P is a general solution s^+ to decision problem D if

1. P outputs answer a_o for all inputs X_i $\forall i$
and
2. $s^+[X_i]$ is a subset of $s^+[\hat{X}_i]$

$$X_i = \{x_1, ..., x_n, C\}; \ \ \hat{X}_i = \{x_1, ..., x_n, x_{n+1}, C\}$$
$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$P[X_i] = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\}$$

$$s^+ = P[X_i] = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} \quad \forall X_i$$

3.1.1 Property of No-op ;

No-op ; can be added to any solution S_i without modifying the output y_o or memory b_i

$$; := \varnothing$$
$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$\hat{s}^+ = \{s_1, s_2, ..., \ ; \ , ..., s_{O_T[n]+1}, \hat{b}_1, \hat{b}_2, ..., \hat{b}_{O_S[n]}, \hat{y}_o\}$$
$$\hat{y}_o = y_o \quad \forall k$$

3.2 Definition of S^+

Define S^+; the set of solutions to decision problem D

$$X_i = \{x_1, ..., x_n, C\}; \ \ \hat{X}_i = \{x_1, ..., x_n, x_{n+1}, C\}$$
$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s_j^+ = s_j^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$
$$S^+ := \{s_j^+, ...\} \quad \forall j$$

3.3 Definition of Solvable

Define solvable

$$X_i = \{x_1, ..., x_n, C\}; \quad \hat{X}_i = \{x_1, ..., x_n, x_{n+1}, C\}$$
$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$solvable := solvable[D] \to b_o \in \{\mathbb{T}, \mathbb{F}\} =$$
$$\exists P : (P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

4 The set of all Decision Problems $\mathbb{D}$

4.1 Definition

Define the set of decision problems $\mathbb{D}$

$$X_i = \{x_1, ..., x_n, C\}$$
$$D_j := f_j[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$\mathbb{D} := \{D_j, ...\} \quad \forall j$$

5 Instruction and Memory Notation

Define $\mathcal{L}$ a set of logical operations
Define $\mathcal{M}$ a set of memory elements, magnitudes, and sets

$$X_i = \{x_1, ..., x_n, C\};$$
$$P[X_i] \to y_o = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$\mathcal{L} := \{s_1, s_2, ..., s_{O_T[n]}\}$$
$$\mathcal{M} := \{b_1, b_2, ..., b_{O_S[n]}\}$$
$$P[X_i] = \{\mathcal{L}, \mathcal{M}, y_o\}$$

6 Complexity

6.1 Time Complexity of a Decision Problem $O_T[n]$

Define Time Complexity $O_T[n]$ of solution s^+ to Decision Problem D as the total number of logical operations

$$X_i = \{x_1, ..., x_n, C\};\ \ \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$

$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$s^+[X_i] := P :$$
$$(P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_T[n] := |\mathcal{L}| = N$$

6.2 Space Complexity $O_S[n]$

Define Space Complexity $O_S[n]$ of solution s^+ to Decision Problem D as the total number of memory elements

$$X_i = \{x_1, ..., x_n, C\};\ \ \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$

$$D := f[X_i] \to a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$s^+[X_i] := P :$$
$$(P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_S[n] := |\mathcal{M}| + |y_o|^* = M + 1$$

*It is convention to reserve one memory element for output y_o.
Void programs do not require the y_o memory element for output

7 Definition of Complexity

Define Complexity $O[n]$ as a vector of dimension V

$$\mathbf{O}[n] := \ < O_T[n], O_S[n], O_3[n], O_4[n] ..., O_V[n] >$$

8 Simple Computational Complexity

The remainder of this document assumes simple computational complexity of dimension 2

8.1 Definition

Define simple computational complexity of dimension 2

$$\mathbf{O}[n] := \; < O_T[n], O_S[n] >$$

8.2 Total Complexity

Define Total Complexity of solution s^+

$$O[n] := |s^+[X_n]| = |\{\mathcal{L}, \mathcal{M}, y_o\}|$$

$$= |\mathcal{L}| + |\mathcal{M}| + |y_o| = N + M + 1$$

8.3 Time Complexity

Restate definition of Time Complexity $O_T[n]$ of solution s^+

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_T[n] := |\mathcal{L}| = N$$

8.4 Space Complexity

Restate definition of Time Complexity $O_S[n]$ of solution s^+

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_S[n] := |\mathcal{M}| + |y_o| = M + 1$$

8.5 Total Complexity as a Function of Time and Space Complexity

$$O[n] := |s^+[X_n]| = |\{\mathcal{L}, \mathcal{M}, y_o\}|$$

$$= |\mathcal{L}| + |\mathcal{M}| + |y_o|$$

$$= O_T[n] + O_S[n]$$

8.6 $O_S[n] > 0^*$

Assuming Program is not void

8.6.1 Proof

Assume $O_S[n] = 0$

$$O_S[n] := |\mathcal{M}| + |y_o|$$
$$O_S[n] = 0 \Rightarrow \mathcal{M} = y_o = \varnothing$$
$$y_o = \varnothing;\;\; y_o \in \{\mathbb{T}, \mathbb{F}\} \text{ by definition of } s^+$$

$\therefore O_S[n] = 0$ contradicts the definition of solution s^+ of a decision problem

$$O_S[n] \geqslant 0 \text{ by definition of magnitude}$$
$$\therefore O_S[n] > 0$$

8.7 $O_T[n] > 0^*$

Assuming Program is not void

8.7.1 Proof

Assume $O_T[n] = 0$

$$O_T[n] := |\mathcal{L}|$$
$$O_T[n] = 0 \Rightarrow y_o \notin \{\mathbb{T}, \mathbb{F}\}$$
$$y_o \notin \{\mathbb{T}, \mathbb{F}\};\;\; y_o \in \{\mathbb{T}, \mathbb{F}\} \text{ by definition of } s^+$$

$\therefore O_T[n] = 0$ contradicts the definition of solution s^+ of a decision problem

$$O_T[n] \geqslant 0 \text{ by definition of magnitude}$$
$$\therefore O_T[n] > 0$$

8.8 $O[n] > 0^*$

Assuming Program is not void

8.8.1 Proof

$$O[n] := O_T[n] + O_S[n]$$

$$O_T[n] > 0;\ \ O_S[n] > 0$$

$$\therefore O[n] > 0$$

8.9 $O[n] > O_T[n]*$

Assuming Program is not void

8.9.1 Proof

$$O[n] := O_T[n] + O_S[n]$$

$$O_S[n] > 0$$

$$\therefore O[n] > O_T[n]$$

8.10 $O[n] > O_S[n]^*$

Assuming Program is not void

8.10.1 Proof

$$O[n] := O_T[n] + O_S[n]$$

$$O_T[n] > 0$$

$$\therefore O[n] > O_S[n]$$

8.11 $O[n+1] \geqslant O[n]$

8.11.1 Proof

$$X_i = \{x_1, ..., x_n, C\};\ \ \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$

$$O[n] = |\ s^+[X_i]|$$
$$O[n+1] = \hat{O}[n] = |\ s^+[\hat{X}_i]|$$

For general solutions s^+

$$s^+[\hat{X}_i] \supseteq s^+[X_i]$$
$$\Rightarrow |s^+[\hat{X}_i]| \geqslant |s^+[X_i]|$$
$$\therefore \hat{O}[n] = O[n+1] \geqslant O[n]$$

Computational Operators

9 No-op ;

9.1 Define No-op ;

Define void program No-op ;

$$;[\varnothing] := \{s_; = \varnothing\} = \{\mathcal{L}\}$$

9.2 Time Complexity of No-op

$$O_T[n] := |\mathcal{L}| = 1$$

9.3 Space Complexity of No-op

$$O_S[n] := |\mathcal{M}| = 0$$

9.4 Total Complexity of No-op

$$O[n] = O_T[n] + O_S[n] = 0$$

9.5 Property of No-op ;

Prove No-op can be added to any Program P without modifying the output y_o or memory b_i

$$P = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$

Let

$$P_; = \{s_1, s_2, ..., s_j, s_;, s_{j+1}, ..., s_{O_T[n]}, \hat{b}_1, \hat{b}_2, ..., \hat{b}_{O_S[n]}, \hat{y}_o\}$$

Suppose

$$\hat{y}_o \neq y_o$$

10 Inductive Functions

10.1 Inductive Function f_{n+1}

$$O[n] = O_T[n] + O_S[n]$$
$$O[n+1] = O_T[n+1] + O_S[n+1]$$

$$f_{n+1}[n] := O[n+1] - O[n]$$

10.2 Inductive Space and Time Formulas

$$f_{n+1}^T[n] := O_T[n+1] - O_T[n]$$
$$O_T[n+1] = O_T[n] + f_{n+1}^T[n]$$
$$f_{n+1}^S[n] := O_S[n+1] - O_S[n]$$
$$O_S[n+1] = O_S[n] + f_{n+1}^S[n]$$

10.3 Inductive Function Expressions

Relate $f_{n+1}[n]$ to equivalence functions

$$O[n] = O_T[n] + O_S[n]$$
$$O[n+1] = O_T[n+1] + O_S[n+1] = O[n] + f_{n+1}[n]$$
$$O_T[n] = O[n] - O_S[n]$$
$$O_S[n] = O[n] - O_T[n]$$

$$f_{n+1}[n] = O[n+1] - O[n]$$
$$f_{n+1}[n] = O_T[n+1] + O_S[n+1] - O[n]$$
$$f_{n+1}[n] = O_T[n+1] - O_T[n] + O_S[n+1] - O_S[n]$$
$$f_{n+1}[n] = O[n+1] - O_T[n] - O_S[n]$$
$$f_{n+1}[n] = f_{n+1}^T[n] + f_{n+1}^S[n]$$

10.4 Zero Order Space Inductive Function

$$Let\ \ O_S[n] \sim n^0$$
$$f_{n+1}[n] = O_T[n+1] - O_T[n] + O_S[n+1] - O_S[n] = O_T[n+1] - O_T[n]$$

11 Polynomial Complexity

11.1 Definition

Decision problem D with solution s^+ has polynomial total complexity $O[n]$ if

$$\exists K, C, \lambda_1 ... \lambda_K \quad :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

11.2 Polynomial Problems

Define $\mathbb{P}$, the set of Decision Problems that can be solved with Polynomial Complexity

$$\mathbb{P} := \{D_1, D_2, ...\} :$$

$$\exists K, C, \lambda_1 ... \lambda_K :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, D_i \in \mathbb{P}$$

11.3 Polynomial Order of Complexity

Solution s^+ with total complexity $O[n]$ is said to be of order n^K

$$O[n] \sim n^K$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

11.4 Property of Polynomial Complexity 1

Solutions with polynomial complexity have convergent complexity

$$lim_{n \to \infty} \frac{O[n+1]}{O[n]} = 1$$

11.4.1 Proof

$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$$
$$O[n+1] = (\lambda_K (n+1))^K + (\lambda_{K-1} (n+1))^{K-1} ... + \lambda_1 (n+1) + C$$
$$= (\lambda_K n)^K + (\tilde{\lambda}_{K-1} n)^{K-1} ... + \tilde{\lambda_1} n + \tilde{C}$$

$$lim_{n \to \infty} \frac{O[n+1]}{O[n]}$$

$$= lim_{n \to \infty} \frac{(\lambda_K n)^K + (\tilde{\lambda}_{K-1} n)^{K-1} ... + \tilde{\lambda_1} n + \tilde{C}}{(\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C}$$

$$= lim_{n\to\infty} \frac{(\lambda_K n)^K}{(\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C} + \frac{(\tilde{\lambda}_{K-1} n)^{K-1}}{(\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C} + ... + \frac{\tilde{\lambda}_1 n}{(\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C} + \frac{\tilde{C}}{(\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C}$$

$$= 1 = lim_{n\to\infty} \frac{O[n+1]}{O[n]}$$

11.5 Property of Polynomial Complexity 2

$$\exists K, \hat{C}, \hat{\lambda}_1, ..., \hat{\lambda}_{K-1} :$$
$$O[n+1] - O[n] = f_{n+1}[n] = (\hat{\lambda}_{K-1} n)^{K-1} ... + \hat{\lambda}_1 n + \hat{C} \quad \forall n$$

11.5.1 Proof

$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$$
$$O[n+1] = (\lambda_K (n+1))^K + (\lambda_{K-1} (n+1))^{K-1} ... + \lambda_1 (n+1) + C$$
$$= (\lambda_K n)^K + (\tilde{\lambda}_{K-1} n)^{K-1} ... + \tilde{\lambda_1} n + \tilde{C}$$
$$O[n+1] - O[n] = ((\tilde{\lambda}_{K-1} - \lambda_{K-1}) n)^{K-1} ... + (\tilde{\lambda_1} - \lambda_1) n + (\tilde{C} - C)$$
$$O[n+1] - O[n] = (\hat{\lambda}_{K-1} n)^{K-1} ... + \hat{\lambda}_1 n + \hat{C}$$

11.6 Property of Polynomial Complexity 3

$$limit_{n\to\infty} \frac{f_{n+1}[n]}{O[n]} = 0$$

11.6.1 Proof

$$limit_{n\to\infty} \frac{O[n+1]}{O[n]} = 1$$
$$limit_{n\to\infty} \frac{O[n] + f_{n+1}[n]}{O[n]} = 1$$
$$limit_{n\to\infty} \frac{O[n]}{O[n]} + \frac{f_{n+1}[n]]}{O[n]} = 1$$
$$limit_{n\to\infty} 1 + \frac{f_{n+1}[n]]}{O[n]} = 1$$
$$limit_{n\to\infty} \frac{f_{n+1}[n]]}{O[n]} = 0$$

11.7 Total Polynomial Complexity Implies Time bounded Polynomial Complexity

$$D \in \mathbb{P} \Longrightarrow O_T[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$$

11.7.1 Proof

$$O[n] = (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C \ \ \forall n$$

$$O[n] := O_T[n] + O_S[n]; \ \ O_S[n] > 0$$

$$\therefore O_T[n] < (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C \ \ \forall n$$

11.8 Total Polynomial Complexity Implies Space bounded Polynomial Complexity

$$D \in \mathbb{P} \Longrightarrow O_S[n] < (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C$$

11.8.1 Proof

$$O[n] = (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C \ \ \forall n$$

$$O[n] := O_T[n] + O_S[n]; \ \ O_T[n] > 0$$

$$\therefore O_S[n] < (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C \ \ \forall n$$

11.9 Polynomial Complexity in Space and Time Implies Polynomial Total Complexity

$$(O_S[n] == (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + \lambda_0)$$

$$\wedge$$

$$(O_T[n] == (\hat{\lambda}_M n)^M + (\hat{\lambda}_{M-1}n)^{M-1}... + \hat{\lambda}_1 n + \hat{\lambda}_0)$$

$$\Rightarrow D \in \mathbb{P}$$

11.9.1 Proof

$$O_S[n] = \lambda_K n^K + \lambda_{K-1} n^{K-1} + ... + \lambda_1 n + \lambda_0$$

$$O_T[n] = \hat{\lambda}_M n^M + \hat{\lambda}_{M-1} n^{M-1} + ... + \hat{\lambda}_1 n + \hat{\lambda}_0$$

$$O[n] = O_S[n] + O_T[n]$$

$$^*O[n] = (\hat{\lambda}_0 + \lambda_0) + n(\lambda_1 + \hat{\lambda_1}) + ... + n^K(\lambda_K + \hat{\lambda_K}) + \hat{\lambda}_{K+1} n^{K+1} + ... + \hat{\lambda}_M n^M$$

$$\therefore O[n] \text{ has polynomial total complexity by definition}$$

* Assume K < M, similar proof for K=M, K>M

12 Non-Polynomial Complexity

12.1 Definition

Decision problem $\tilde{D}$ with solution s^+ has non-polynomial total complexity $O[n]$ if

$$\nexists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

12.2 Non-Polynomial Problems

Define $\mathcal{N}$, the set of Decision Problems that cannot be solved with Polynomial Complexity

$$\mathcal{N} := \{\tilde{D}_1, \tilde{D}_2, ...\} :$$
$$\nexists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, \ s^+ \in S_i^+, \ \tilde{D}_i \in \mathcal{N}$$

12.3 $\mathbb{P}$ and $\mathcal{N}$ are disjoint

$$\mathbb{P} \cap \mathcal{N} = \varnothing$$

12.3.1 Proof

Let D $\in \mathcal{N}$

$$\nexists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

Assume D $\in \mathbb{P}$

$$\exists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

Contradicts the definition of $\mathcal{N}$

$$\therefore D \in \mathcal{N} \Rightarrow D \notin \mathbb{P}$$

Let D $\in \mathbb{P}$

$$\exists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

Assume D $\in \mathcal{N}$

$$\nexists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

Contradicts the definition of $\mathbb{P}$

$$\therefore D \in \mathbb{P} \Rightarrow D \notin \mathcal{N}$$

$$D \in \mathcal{N} \Rightarrow D \notin \mathbb{P}; D \in \mathbb{P} \Rightarrow D \notin \mathcal{N}$$
$$\therefore \mathbb{P} \cap \mathcal{N} = \varnothing$$

13 Divergent Complexity

13.1 Defintion

Decision problem $\hat{D}$ with solution s^+ has divergent total complexity $O[n]$ if

$$lim_{n \to \infty} \frac{O[n+1]}{O[n]} \ \ diverges$$

13.2 Divergent Problems

$$\hat{\mathcal{D}} := \{\hat{D}_1, \hat{D}_2, ...\} :$$
$$lim_{n \to \infty} \frac{O[n+1]}{O[n]} \ \ diverges \ \ \forall s^+ \in S_i^+, \ \hat{D}_i \in \hat{\mathcal{D}}$$

13.3 The Set of Polynomial Solutions and the Set of Divergent Solutions are disjoint

$$\mathbb{P} \cap \hat{D} = \varnothing$$

13.4 Proof

Let D $\in \hat{\mathcal{D}}$

$$lim_{n \to \infty} \frac{O[n+1]}{O[n]} \ \ diverges \text{ by definition}$$

Assume D $\in \mathbb{P}$

$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} = 1$$

$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} = 1 \text{ contradicts the definition of Divergent Problems}$$

$$\therefore D \in \hat{\mathcal{D}} \Rightarrow D \notin \mathbb{P}$$

Let D $\in \mathbb{P}$

$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} = 1 \text{ by property of Polynomial complexity}$$

Assume D $\in \hat{D}$

$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} \text{ diverges}$$

$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} \text{ diverges contradicts a property of Polynomial complexity}$$

$$\therefore D \in \mathbb{P} \Rightarrow D \notin \hat{\mathcal{D}}$$

$$\therefore \mathbb{P} \cap \hat{\mathcal{D}} = \varnothing$$

14 Subfunctions

14.1 Restate the subfunction condition of general solutions

Recall the definition of general solution s^+

$$X_i = \{x_1, ..., x_n, C\}; \ \ \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \ \ \forall X_i) \ \ \cap \ \ (P[\hat{X}_i] \supseteq P[X_i] \ \ \forall X_i, \hat{X}_i)$$

The subfunction condition is one of two conditions for a general solution

$$P[\hat{X}_i] \supseteq P[X_i] \ \ \forall X_i, \hat{X}_i$$

14.2 Prove O[n] is a non-decreasing function

Consider solution s^+ with complexity $O[n]$

$$X_i = \{x_1, ..., x_n, C\}; \ \ \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \ \ \forall X_i) \ \ \cap \ \ (P[\hat{X}_i] \supseteq P[X_i] \ \ \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O[n] := O_T[n] + O_S[n]$$

$$O_T[n] := |\mathcal{L}| = N$$
$$O_S[n] := |\mathcal{M}| + |y_o| = M + 1$$

O[n+1] denotes the total complexity for solution $s^+[\hat{X}_i]$

$$s^+[\hat{X}_i] = \hat{s}^+$$

Let

$$O[n+1] < O[n]$$
$$\Rightarrow \hat{N} + \hat{M} < N + M$$
$$\hat{s}^+ = \{s_1, s_2, ..., s_{\hat{N}} | b_1, b_2, ..., b_{\hat{M}}, y_o\}$$

$$\Rightarrow \hat{s}^+ \nsupseteq s^+$$
$$P[\hat{X}_i] \nsupseteq P[X_i] \quad \forall X_i, \hat{X}_i$$

$$\therefore O[n+1] < O[n] \text{ contradicts the definition of solution } s^+$$
$$O[n+1] \geqslant O[n]$$

14.3 Definition of Subfunction

$$X_i = \{x_1, ..., x_n, C\}; \quad \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub[X_i] := S = \{s_j, ... | b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \quad \forall s_j, b_k \in S$$

14.3.1 $s^+[X_i]$ is a subfunction of $s^+[\hat{X}_i]$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$\hat{s}^+ = \{s_1, s_2, ..., s_N, ..., s_{\hat{N}} | b_1, b_2, ..., b_M, ..., b_{\hat{M}}, y_o\}; \quad \hat{N} + \hat{M} \geqslant N + M$$

By definition of solution

$$\hat{s}^+ = P[\hat{X}_i] \supseteq P[X_i] = s^+ \quad \forall X_i, \hat{X}_i$$
$$\Rightarrow s_j, b_k \in \hat{s}^+ \quad \forall s_j, b_k \in s^+$$

14.4 Subfunction Decomposition of Solutions

FIX Double check conditions!!! Solutions s^+ can be written as the union of subfunctions $Sub_k[X_i]$

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \;\; \forall X_i) \;\; \cap \;\; (P[\hat{X}_i] \supseteq P[X_i] \;\; \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$s^+ = Sub_1[X_i] \cup Sub_2[X_i] \cup ... \cup Sub_z[X_i]$$
$$= \{\mathcal{L}_1 | \mathcal{M}_1, y_o\} \cup \{\mathcal{L}_2 | \mathcal{M}_2, y_o\} \cup ... \cup \{\mathcal{L}_z | \mathcal{M}_z, y_o\} :$$
$$\mathcal{L}_j \cap \mathcal{L}_k = \varnothing \;\; \forall j, k \neq j$$

$$s^+ = \{s^1_1, ..., s^1_{N_1} | b^1_1, ..., y_o\} \cup \{s^2_1, ..., s^2_{N_2} | b^2_1, ..., y_o\} \cup ... \cup \{s^z_1, ..., s^z_{N_z} | b^z_1, ..., y_o\} :$$
$$\textstyle\sum_{l=1}^{z} N_l = N = O_T[n]$$

15 Subfunction Complexity

15.1 Disjoint Subfunction Operations

$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \;\; \forall i, j \neq i$$

15.2 Shared Subfunction Memory

$$|\mathcal{M}_i \cap \mathcal{M}_j| \geqslant 0 \;\; \forall i, j \neq i$$

15.2.1 Time Complexity of Subfunctions

Subfunction time complexity is additive

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub_i[X] := S_i = \{s_j, ... | b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \;\;\; \forall s_j, b_k \in S_i$$

$$s^+ = \{\mathcal{L}_1 | \mathcal{M}_1, y_o\} \cup \{\mathcal{L}_2 | \mathcal{M}_2, y_o\} \cup ... \cup \{\mathcal{L}_z | \mathcal{M}_z, y_o\} :$$
$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \;\; \forall i, j \neq i$$

$$\mathcal{L} = \cup_{i=1}^{z} \mathcal{L}_i$$

$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$$

$$O_T[n] = |\mathcal{L}| = N$$
$$O_T[n] = | \cup_{i=1}^{z} \mathcal{L}_i| = \textstyle\sum_{i=1}^{z} |\mathcal{L}_i|^* = |\mathcal{L}_1| + |\mathcal{L}_2| + ... + |\mathcal{L}_z|$$
$$= O_{T_1}[n] + O_{T_2}[n] + ... + O_{T_z}[n] = N_1 + N_2 + ... + N_z$$

*Due to the disjoint condition of subfunction operations $\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$

15.2.2 Space Complexity of Subfunctions

Subfunctions can access the full memory $\mathcal{M}$ with no added space complexity

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub_i[X] := S_i = \{s_j, ...|b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \quad \forall s_j, b_k \in S_i$$

$$s^+ = \{\mathcal{L}_1|\mathcal{M}_1, y_o\} \cup \{\mathcal{L}_2|\mathcal{M}_2, y_o\} \cup ... \cup \{\mathcal{L}_z|\mathcal{M}_z, y_o\} :$$
$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$$

$$s^+ = \{\mathcal{L}_1|\mathcal{M}, y_o\} \cup \{\mathcal{L}_2|\mathcal{M}, y_o\} \cup ... \cup \{\mathcal{L}_z|\mathcal{M}, y_o\} :$$
$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$$

$$\mathcal{M} = \cup_{i=1}^{z} \mathcal{M}_i = \cup_{i=1}^{z} \mathcal{M}$$

$$O_S[n] = |\mathcal{M}| = M$$
$$O_S[n] = | \cup_{i=1}^{z} \mathcal{M}_i| = M$$

15.2.3 Shared State Notation

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub_i[X] := S_i = \{s_j, ...|b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \quad \forall s_j, b_k \in S_i$$

$$s^+ = \{\mathcal{L}_1|\mathcal{M}, y_o\} \cup \{\mathcal{L}_2|\mathcal{M}, y_o\} \cup ... \cup \{\mathcal{L}_z|\mathcal{M}, y_o\} :$$

$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \;\; \forall i, j \neq i$$

16 Polynomial Solution Subfunction Properties

16.1 Restate Definition of Subfunction

$$X_n = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$

$$s^+ = s^+[X_n] := P :$$
$$(P[X_n] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_n] \supseteq P[X_n] \quad \forall X_n, \hat{X}_n)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub[X_n] := S = \{s_j, ... | b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \quad \forall s_j, b_k \in S$$

16.2 Property of Polynomial Solution Subfunctions

Let

$$D \in \mathbb{P}$$

$$X_n = \{x_1, ..., x_n, C\}; \;\; \hat{X}_n = \{x_1, ..., x_{n+1}, C\}$$

$$s^+ = s^+[X_n] := P :$$
$$(P[X_i] \to y_o == a_o \quad \forall X_n) \quad \cap \quad (P[\hat{X}_n] \supseteq P[X_n] \quad \forall X_n, \hat{X}_n)$$

$$\exists K, C, \lambda_1 ... \lambda_K \quad :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

$$s^+ = Sub_1[X_n] \cup Sub_2[X_n] \cup ... \cup Sub_z[X_n]$$

$$limit_{n \to \infty} \tfrac{O[n+1]}{O[n]} = 1$$
$$= limit_{n \to \infty} \tfrac{O_T^1[n+1] + O_T^2[n+1] + ... + O_T^z[n+1] + O_S[n+1]}{O_T^1[n] + O_T^2[n] + ... + O_T^z[n] + O_S[n]}$$
$$=$$
$$limit_{n \to \infty} \frac{O_T^1[n] + O_T^2[n] + ... + O_T^z[n] + O_S[n] + f_{T_{n+1}}^1[n] + f_{T_{n+1}}^2[n+1] + ... + f_{T_{n+1}}^z[n] + f_{S_{n+1}}[n]}{O_T^1[n] + O_T^2[n] + ... + O_T^z[n] + O_S[n]}$$

$$= limit_{n\to\infty} 1 + \frac{f^1_{T_{n+1}}[n]+f^2_{T_{n+1}}[n+1]+...+f^z_{T_{n+1}}[n]+f_{S_{n+1}}[n]}{O^1_T[n]+O^2_T[n]+...+O^z_T[n]+O_S[n]} = 1$$

$$\Rightarrow limit_{n\to\infty} \frac{f^1_{T_{n+1}}[n]+f^2_{T_{n+1}}[n+1]+...+f^z_{T_{n+1}}[n]+f_{S_{n+1}}[n]}{O^1_T[n]+O^2_T[n]+...+O^z_T[n]+O_S[n]} = 0^*$$

$$\Rightarrow limit_{n\to\infty} \frac{f^i_{T_{n+1}}[n]+f_{S_{n+1}}[n]}{O^1_T[n]+O^2_T[n]+...+O^z_T[n]+O_S[n]} = 0 \ \ \forall i$$

$$limit_{n\to\infty} \frac{f^i_{n+1}[n]}{O[n]} = 0 \ \ \forall i$$

* O[n] is a positive, non-decreasing function

16.3 Theorem of Polynomial Subfunctions

The Theorem of Polynomial Subfunctions states a solution has polynomial complexity if and only if all of its subfunctions have polynomial complexity

$$|s^+[X_n]| = O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

$$s^+ = Sub_1[X_n] \cup Sub_2[X_n] \cup ... \cup Sub_z[X_n]$$

$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

$$\Longleftrightarrow$$

$$|Sub_i[X_n]| = O_i[n] = (\hat{\lambda}_M n)^M + (\hat{\lambda}_{M-1} n)^{M-1} + ... + \hat{\lambda}_1 n + C \quad \forall i, n$$

16.3.1 Forward Direction

Solution s^+ having polynomial complexity implies all of its subfunctions Sub_i have polynomial complexity

Let

$$|s^+[X_n]| = O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$
$$O[n] = \textstyle\sum_{i=1}^{z} O_i[n] = O_1[n] + O_2[n] + ... + O_z[n]$$

Since $O[n], O_i[n]$ is positive, non-decreasing

$$O_i[n] = (\hat{\lambda}_{M_i} n)^{M_i} + (\hat{\lambda}_{M_i-1} n)^{M_i-1} ... + \hat{\lambda}_1 n + C \quad M_i \leqslant K \ \ \forall i, n$$

$\Rightarrow Sub_i$ has polynomial complexity by definition of polynomial complexity

16.3.2 Backward Direction

Every subfunction Sub_i having polynomial complexity implies solution s^+ has polynomial complexity

Let

$$O_i[n] = (\hat{\lambda}_{M_i} n)^{M_i} + (\hat{\lambda}_{M_i-1} n)^{M_i-1} + ... + \hat{\lambda}_{1_i} n + \hat{\lambda}_{0_i} \quad \forall i, n$$

$$O_{max}[n]^* := \tilde{O}[n] \in \{O_1[n], O_2[n], ...O_z[n]\} :$$

$$lim_{n \to \infty} \frac{\tilde{O}[n]}{\sum_{i=1}^{z} O_i[n]} = c \neq 0$$

$$O_{max}[n] = (\hat{\lambda}_{M_{max}} n)^{M_{max}} + (\hat{\lambda}_{M_{max}1} n)^{M_{max}-1} + ... + \hat{\lambda}_{1_{max}} n + \hat{\lambda}_{0_{max}} \quad \forall i, n$$

$$O[n] = \sum_{i=1}^{z} O_i[n] = O_1[n] + O_2[n] + ... + O_z[n]$$

$$= (\tilde{\lambda}_L n)^L + (\tilde{\lambda}_{L-1} n)^{L-1} + ... + \tilde{\lambda}_1 n + C \quad L = M_{max} \ \forall n$$

$\Rightarrow s^+$ has polynomial complexity by definition of polynomial complexity

* O$_{max}$ is not necessarily unique, but necessarily exists. See appendix for proof

17 Divergent Solution Subfunction Properties

17.1 Restate Definition of Subfunction

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \;\;\; \forall X_i) \;\;\; \cap \;\;\; (P[\hat{X}_i] \supseteq P[X_i] \;\;\; \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub[X_i] := S = \{s_j, ... | b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \;\;\; \forall s_j, b_k \in S$$

17.2 Property of Divergent Subfunctions

Let

$$D \in \hat{\mathcal{D}}$$

$$X_n = \{x_1, ..., x_n\}; \;\; \hat{X}_n = \{x_1, ..., x_{n+1}\}$$

$$s^+ = s^+[X_n] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \;\;\; \forall X_n) \;\;\; \cap \;\;\; (P[\hat{X}_n] \supseteq P[X_n] \;\;\; \forall \hat{X}_n : \hat{X}_n \supset X_n)$$

By Definition of Divergent Problem

$$\nexists c : limit_{n\rightarrow\infty} \frac{O[n+1]}{O[n]} = c$$
$$= limit_{n\rightarrow\infty} \frac{O_T^1[n+1] + O_T^2[n+1] + ... + O_T^z[n+1] + O_S[n+1]}{O_T^1[n] + O_T^2[n] + ... + O_T^z[n] + O_S[n]}$$
$$=$$
$$limit_{n\rightarrow\infty} \frac{O_T^1[n] + O_T^2[n] + ... + O_T^z[n] + O_S[n] + f^1_{T_{n+1}}[n] + f^2_{T_{n+1}}[n+1] + ... + f^z_{T_{n+1}}[n] + f_{S_{n+1}}[n]}{O_T^1[n] + O_T^2[n] + ... + O_T^z[n] + O_S[n]}$$

$$= limit_{n\rightarrow\infty} 1 + \frac{f^1_{T_{n+1}}[n] + f^2_{T_{n+1}}[n+1] + ... + f^z_{T_{n+1}}[n] + f_{S_{n+1}}[n]}{O_T^1[n] + O_T^2[n] + ... + O_T^z[n] + O_S[n]} \neq c$$

$$\Rightarrow limit_{n\rightarrow\infty} \frac{f^1_{T_{n+1}}[n] + f^2_{T_{n+1}}[n+1] + ... + f^z_{T_{n+1}}[n] + f_{S_{n+1}}[n]}{O_T^1[n] + O_T^2[n] + ... + O_T^z[n] + O_S[n]} \neq c^*$$

Prove

$$\exists i : limit_{n\to\infty} \frac{f^i_{T_{n+1}}[n] + f_{S_{n+1}}[n]}{O^1_T[n] + O^2_T[n] + ... + O^z_T[n] + O_S[n]} \;\; diverges$$

* O[n] is a positive, non-decreasing function

17.3 Theorem of Divergent Subfunctions

The Theorem of Divergent Subfunctions states a solution is divergent if and only if at least one of its subfunctions are divergent.

$$lim_{n\to\infty} \frac{O[n+1]}{O[n]} \;\; diverges$$

$$\iff$$

$$\exists i : lim_{n\to\infty} \frac{O_i[n+1]}{O[n]} \;\; diverges$$

17.3.1 Forward Direction

See 18.2

17.3.2 Backward Direction

18 Solution Spaces

18.1 Definition of Solution Space

$$\mathbb{S} = \{c_1^+, c_2^+, ..., c_{C[n]}^+\}$$

$$s^+[X_n] = \vee_{c_i^+ \in \mathbb{S}} \; c_i^+$$

18.2 Existence, Uniqueness, etc.

18.3 Worst Case

19 Fundamental Theorem of Computation

The Fundamental Theorem of Computation relates the complexity of optimal solution to the number of candidate solutions in the Solution Space.

$$\mathbb{S} = \{c_1^+, c_2^+, ..., c_{C[n]}^+\}$$

$$s^+[X_n] = \vee_{c_i^+ \in \mathbb{S}} \; c_i^+$$

$O_{opt}[n]$ has the same order as C[n]

19.1 Proof by Induction

19.2 Proof by Contradiction

20 Combination Notation

20.1 Set Definition

20.2 Numerical Definition

20.3 Properties

20.3.1 Binary Property of Combination

20.3.2 Subfunction Property of Combination

21 Sum to N Problem with 2 integers

21.1 State formal definition of Sum to N : $x_i + x_j == N$

$$X_n = \{x_1, ..., x_n\}$$

$$D := f[X_i, N] = a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$s^+[X_n] = P[X_n] :$$
$$(P[X_i] = y_o == a_o \quad \forall X_i) \quad \cap \quad (P[X_{n+1}] \supseteq P[X_n] \quad \forall X_{n+1})$$

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$D = f[X_i] = \quad \exists x_j, x_k \in X_n \quad j \neq k :$$
$$x_j + x_k == N$$

21.2 Express a formal solution : $O_S[n] \sim n^0$

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$
$$s_1 = y_o \leftarrow \mathbb{F};$$
$$\forall i < n \ \ , \ \ n \geqslant j > i$$
$$s_2, s_3, s_8, s_9, ..., s_{3ij-4}, s_{3ij-3} ..., s_{3n(n-1)-4}, s_{3n(n-1)-3} = b_1 \leftarrow x_i + x_j$$
$$s_4, s_5, s_{10}, s_{11}, ..., s_{3ij-2}, s_{3ij-1} ..., s_{3n(n-1)-2}, s_{3n(n-1)-1} = b_1 \leftarrow b_1 == N$$
$$s_6, s_7, s_{12}, s_{13} ..., s_{3ij}, s_{3ij+1} ..., s_{3n(n-1)}, s_{3n(n-1)+1} = y_o \leftarrow y_o \vee b_1$$
$$s^+ = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\}$$

21.3 Prove s^+ satisfies the subfunction condition of solutions: $P[X_{n+1}] \supseteq P[X_n] \quad \forall X_{n+1}$

$$X_n = \{x_1, x_2, ..., x_n\}; \quad X_{n+1} = \{x_1, x_2, ..., x_n, x_{n+1}\}$$
$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$
$$s^+_{n+1} = s^+ \cup \hat{s}^+$$
$$s_1 = y_o \leftarrow \mathbb{F};$$
$$\forall i < n \ \ , \ \ n \geqslant j > i$$

$$s_2, s_3, s_8, s_9, ..., s_{3ij-4}, s_{3ij-3}..., s_{3n(n-1)-4}, s_{3n(n-1)-3} = b_1 \leftarrow x_i + x_j$$
$$s_4, s_5, s_{10}, s_{11}, ..., s_{3ij-2}, s_{3ij-1}..., s_{3n(n-1)-2}, s_{3n(n-1)-1} = b_1 \leftarrow b_1 == N$$
$$s_6, s_7, s_{12}, s_{13}..., s_{3ij}, s_{3ij+1}..., s_{3n(n-1)}, s_{3n(n-1)+1} = y_o \leftarrow y_o \vee b_1$$

$$\forall k < n+1$$
$$s... = b_1 \leftarrow x_k + x_{n+1}$$
$$s... = b_1 \leftarrow b_1 == N$$
$$s... = y_o \leftarrow y_o \vee b_1$$

$$s^+ = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\}$$

$$\hat{s}^+ = \{y_o \leftarrow y_o \vee (x_k + x_{n+1} == N) \ \forall k < n+1 | \ b_1, y_o\}$$

$$s^+_{n+1} = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\} \ \cup$$
$$\{y_o \leftarrow y_o \vee (x_k + x_{n+1} == N) \ \forall k < n+1 | \ b_1, y_o\}$$

$$s^+_{n+1} = s^+ \cup \hat{s}^+ = P[X_{n+1}] \supseteq P[X_n] = s^+$$

21.4 Determine $O[n], O_S[n], O_T[n], f_{n+1}[n], f^T_{n+1}[n], f^S_{n+1}[n]$ for the above solution

$$O_S[n] = |y_o| + |b_1| = 2$$
$$O_T[n] = 3n(n-1) + 1 = 3n(n-1) - 1 + O_S[n]$$
$$O[n] = 3n(n-1) + 3 = 3n^2 - 3n + 3$$

$$f^S_{n+1}[n] = 0$$
$$f^T_{n+1}[n] = 6n$$
$$f_{n+1}[n] = f^S_{n+1}[n] + f^T_{n+1}[n]$$

21.5 Verify $O[n+1] = O[n] + \ f_{n+1}[n]$

$$O[n+1] = O[n] + \hat{O}[n]$$
$$3(n+1)^2 - 3(n+1) + 3 = 3n^2 - 3n + 3 + 6n$$
$$3n^2 + 6n + 3 - 3n - 3 + 3 = 3n^2 + 3n + 3$$

$$3n^2 + 3n + 3 = 3n^2 + 3n + 3$$

21.6 Show s^+ has Polynomial Complexity by the definition of Total Polynomial Complexity

$$O[n] = 3n^2 - 3n + 3$$

21.7 Show the limit ${}_{n\to\infty}\frac{O[n+1]}{O[n]}$ does not Diverge

$$limit_{n\to\infty}\frac{O[n+1]}{O[n]} =$$

$$limit_{n\to\infty}\frac{3n^2+3n+3}{3n^2-3n+3} =$$

$$limit_{n\to\infty}\left(\frac{3n^2-3n+3}{3n^2-3n+3} + \frac{6n}{3n^2-3n+3}\right) =$$

$$limit_{n\to\infty}\left(1 + \frac{6n}{3n^2-3n+3}\right) = 1$$

22 The Knapsack Problem

22.1 The Knapsack Problem

The Knapsack Problem is a famous problem in computer science which asks if objects can be stored in a knapsack. Typically the problem is designed with two constraints, weight and value. Given objects x_i, each with a respective weight w_i and value v_i, does there exist a combination of objects lighter than input weight W and more valuable than input value V?

22.2 Formal Definition

$$X_n = \{x_1, x_2, ..., x_n\} = \{\{w_1, v_1\}, \{w_2, v_2\}, ..., \{w_n, v_n\}\}$$

$$I = \{i_1, i_2, ..., i_n\} : i_l \in \{0, 1\} \;\; \forall i_l \in I$$

$$D := f[X_n, W, V] = a_o \in \{\mathbb{T}, \mathbb{F}\} = \exists I :$$
$$(\textstyle\sum_{j=1}^{n} i_j w_j < W) \wedge (\sum_{j=1}^{n} i_j v_j \geqslant V)$$

22.3 Solve for C[n]

22.3.1 Expressing I as a binary number

$$I = \{i_1, i_2, ..., i_n\} : i_l \in \{0, 1\} \;\; \forall i_l \in I$$

Valid combinations of I

$$I_{valid} =$$
$$\{\{0, 0, 0, ..., 0, 0, 1\}, \{0, 0, 0, ..., 0, 1, 0\}, \{0, 0, 0, ..., 0, 1, 1\}, ..., \{1, 1, 1, ..., 1, 1, 1\}\}$$
$$C[n] = |I_{valid}[n]| = 2^n - 1$$

22.3.2 Using a sum of combinations of inputs x_i

$$X_n = \{x_1, x_2, ..., x_n\} = \{\{w_1, v_1\}, \{w_2, v_2\}, ..., \{w_n, v_n\}\}$$

Valid combinations of x_i

$$X_{valid}[n] =$$
$$\{x_1\}\cup\{x_2\}\cup...\cup\{x_n\}\cup\{x_1,x_2\}\cup\{x_1,x_3\}\cup...\cup\{x_{n-1},x_n\}\cup...\cup\{x_1,x_2,...,x_n\}$$
$$= \ _{X_n}C_1 \cup \ _{X_n}C_2 \cup ... \cup \ _{X_n}C_n$$

$$C[n] = |X_{valid}[n]| = \textstyle\sum_{j=1}^{n} \ _nC_j$$

22.3.3 Verify consistency

$$C[n] = |X_{valid}[n]| = |I_{valid}[n]|$$
$$= 2^n - 1 = \textstyle\sum_{j=1}^{n} \ _nC_j = {}_nC_1 + {}_nC_2 + ... + {}_nC_n$$
$$= 2^n - 1 = 2^n - 1$$

22.4 Express a solution s^+ of the Knapsack Problem

22.5 Prove s^+ satisfies the subfunction condition of solutions

22.6 Determine $O[n], O_T[n], O_S[n], f_{n+1}[n]$

22.7 Show $s^+ \notin \mathbb{P}$

22.8 Express the Solution Space $\mathbb{S}$ for The Knapsack Problem

22.9 Prove a lower bound for all solutions $s^+ \in S^+ := O_{lower}[n]$

22.10 Prove D has Divergent Complexity

Appendix

23 Existence of $O_{max}[n]$

23.1 Proof

Citations

[1] *chatgpt*

[2] *https://stackoverflow.com/questions/3518973/floating-point-exponentiation-without-power-function*

[3] *https://stackoverflow.com/questions/27086195/linear-index-upper-triangular-matrix*

[4] *http://www.math.uchicago.edu/ may/VIGRE/VIGRE2011/REUPapers/Riffer-Reinert.pdf*