Computation

1 Programs

1.1 Logical Instructions

Define $\mathcal{L}$; an ordered set of logical operations s_i

$$\mathcal{L} := \{s_1, s_2, ..., s_N\}$$

1.2 Memory

Define Memory $\mathcal{M}$; a set of elements, magnitudes, or sets b_i

$$\mathcal{M} := \{b_1, b_2, ..., b_M\}$$

1.3 State |

Define state; the memory b_i utilized to perform program P

$$P := \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M\} =$$
$$\{s_1, s_2, ..., s_N, b_1, b_2, ..., b_M\}$$

1.4 Boolean Programs

Define a boolean program; boolean programs can represent functions with inputs x_i, input set C, and boolean output y_o

$$X = \{x_1, ..., x_n, C\}; \;\; C = \{u_1, u_2, ..., u_c\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, y_o\} =$$
$$P[X] \rightarrow y_o \in \{\mathbb{T}, \mathbb{F}\}$$

1.5 Void Programs

Define a void program; a program with inputs x_i, input set C, and no output

$$X = \{x_1, ..., x_n, C\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M\}$$

1.6 Numerical Programs

Define a numerical program; a program with inputs x_i, input set C, and real, rational output y_o

$$X = \{x_1, ..., x_n, C\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, y_o\} =$$
$$P[X] \to y_o \in \mathbb{Q} \;\; y_o \geqslant 0$$

1.7 System Programs

Define a system program; a program with inputs x_i, input set C, and real, output set Y_o

$$X = \{x_1, ..., x_n, C\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, Y_o\} =$$
$$P[X] \to Y_o = \{y_1, y_2, ..., y_K\}$$

1.8 Mathematical Programs

Define a mathematical program; a program with inputs x_i, input set C and numerical output y_o

$$X = \{x_1, ..., x_n, C\}$$
$$P = P[X] := \{s_1, s_2, ..., s_N \mid b_1, b_2, ..., b_M, y_o\} =$$
$$P[X] \to y_o \in \mathbb{Q}$$

2 No-op ;

2.1 Definition

$$; := \varnothing$$

2.2 Property of No-op

No-op can be inserted into any set with equality

$$S = \{s_1, s_2, ..., s_N\}$$

$$S_; = insert[S, ;, i]$$

$$S_; = S_1 \ \ \forall i$$

$$|S_;| = |S| \ \ \forall i$$

2.3 Proof

by definition of magnitude of null = 0 with Set And

3 Problem Definition

Also denoted as a "Question"

$$X_i = \{x_1, ..., x_n\}$$
$$Q := f[X_i] = Y_o \subseteq \Omega \quad \forall X_i$$

3.1 Set of Questions

Define $\mathbb{Q}$; the set of questions

$$\mathbb{Q} := \{Q_1, Q_2, ...\} :$$
$$Q_i = f[X_j] = Y_o \subseteq \Omega \;\; \forall X_j, i$$

3.2 Decision Questions / Decision Problems

3.2.1 Definition

Define decision problem; a function with inputs x_i and boolean output "answer" a_o

$$X_i = \{x_1, ..., x_n\}$$
$$D := f[X_i] = a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

3.3 Numerical Questions / Numerical Problems

3.3.1 Definition

Define numerical problem; a function with inputs x_i and numerical output y_o

$$X_i = \{x_1, ..., x_n\}$$
$$Q := f[X_i] = y_o \in \mathbb{R} \quad \forall X_i$$

3.4 System Questions / System Problems

3.4.1 Definition

Define system problem; a function with inputs x_i and outputs y_j

$$X_i = \{x_1, ..., x_n\}$$
$$Q := f[X_i] = Y_o = \{y_1, ..., y_m\} \quad \forall X_i$$

4 General Solutions

4.1 Definition

Program P is a general solution s^+ to decision problem D if

1. P outputs answer a_o for all inputs X_i $\forall i$
and
2. $s^+[X_i]$ is a subset of $s^+[\hat{X}_i]$

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_n, x_{n+1}, C\}$$
$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$P[X_i] = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\}$$

$$s^+ = P[X_i] = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} \quad \forall X_i$$

4.1.1 Property of No-op ;

No-op ; can be added to any solution S_i without modifying the output y_o

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$\hat{s}^+ \rightarrow \hat{y}_o = insert[s^+, ;, k]$$
$$\hat{y}_o = y_o \;\; \forall k$$

4.2 Definition of S^+

Define S^+; the set of solutions to decision problem D

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_n, x_{n+1}, C\}$$
$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$s_j^+ = s_j^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$
$$S^+ := \{s_j^+, ...\} \quad \forall j$$

4.3 Definition of Solvable

Define solvable

$$X_i = \{x_1, ..., x_n, C\}; \quad \hat{X}_i = \{x_1, ..., x_n, x_{n+1}, C\}$$
$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$solvable := solvable[D] \rightarrow b_o \in \{\mathbb{T}, \mathbb{F}\} =$$
$$\exists P : (P[X_i] \rightarrow y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

5 The set of all Decision Problems $\mathbb{D}$

5.1 Definition

Define the set of decision problems $\mathbb{D}$

$$X_i = \{x_1, ..., x_n, C\}$$
$$D_j := f_j[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$
$$\mathbb{D} := \{D_j, ...\} \quad \forall j$$

6 Instruction and Memory Notation

Define $\mathcal{L}$ a set of logical operations
Define $\mathcal{M}$ a set of memory elements, magnitudes, and sets

$$X_i = \{x_1, ..., x_n, C\};$$
$$P[X_i] \rightarrow y_o = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$\mathcal{L} := \{s_1, s_2, ..., s_{O_T[n]}\}$$
$$\mathcal{M} := \{b_1, b_2, ..., b_{O_S[n]}\}$$
$$P[X_i] = \{\mathcal{L}, \mathcal{M}, y_o\}$$

7 Complexity

7.1 Time Complexity of a Decision Problem $O_T[n]$

Define Time Complexity $O_T[n]$ of solution s^+ to Decision Problem D as the total number of logical operations

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$

$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_T[n] := |\mathcal{L}| = N$$

7.2 Space Complexity $O_S[n]$

Define Space Complexity $O_S[n]$ of solution s^+ to Decision Problem D as the total number of memory elements

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$

$$D := f[X_i] \rightarrow a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_S[n] := |\mathcal{M}| + |y_o|^* = M + 1$$

*It is convention to reserve one memory element for output y_o.
Void programs do not require the y_o memory element for output

8 Definition of Complexity

Define Complexity $O[n]$ as a vector of dimension Y

$$\mathbf{O}[n] := \; < O_T[n], O_S[n], O_3[n], O_4[n] ..., O_V[n] >$$

9 Total Complexity

$$O[n] := O_T[n] + O_S[n] + \sum_{i=3}^{V} O_i[n]$$

10 Simple Computational Complexity

The remainder of this document assumes simple computational complexity of dimension 2

10.1 Definition

Define simple computational complexity of dimension 2

$$\mathbf{O}[n] := \ < O_T[n], O_S[n] >$$

10.2 Time Complexity

Restate definition of Time Complexity $O_T[n]$ of solution s^+

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_T[n] := |\mathcal{L}| = N$$

10.3 Space Complexity

Restate definition of Time Complexity $O_S[n]$ of solution s^+

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O_S[n] := |\mathcal{M}| + |y_o| = M + 1$$

10.4 Total Complexity

$$O[n] := O_T[n] + O_S[n]$$

$$= |\mathcal{L}| + |\mathcal{M}| + |y_o| = N + M + 1$$

10.5 $O_S[n] > 0$

10.5.1 Proof

Assume $O_S[n] = 0$

$$O_S[n] := |\mathcal{M}| + |y_o|$$

$$O_S[n] = 0 \Rightarrow \mathcal{M} = y_o = \varnothing$$

$$y_o = \varnothing; \ \ y_o \in \{\mathbb{T}, \mathbb{F}\} \text{ by definition of } s^+$$

$\therefore O_S[n] = 0$ contradicts the definition of solution s^+ of a decision problem

$$O_S[n] \geqslant 0 \text{ by definition of magnitude}$$
$$\therefore O_S[n] > 0$$

10.6 $O_T[n] > 0$

10.6.1 Proof

Assume $O_T[n] = 0$

$$O_T[n] := |\mathcal{L}|$$
$$O_T[n] = 0 \Rightarrow y_o \notin \{\mathbb{T}, \mathbb{F}\}$$
$$y_o \notin \{\mathbb{T}, \mathbb{F}\};\ \ y_o \in \{\mathbb{T}, \mathbb{F}\} \text{ by definition of } s^+$$

$\therefore O_T[n] = 0$ contradicts the definition of solution s^+ of a decision problem

$$O_T[n] \geqslant 0 \text{ by definition of magnitude}$$
$$\therefore O_T[n] > 0$$

10.7 O[n] $>$ 0

10.7.1 Proof

$$O[n] := O_T[n] + O_S[n]$$
$$O_T[n] > 0;\ \ O_S[n] > 0$$
$$\therefore O[n] > 0$$

10.8 $O[n] > O_T[n]$

10.8.1 Proof

$$O[n] := O_T[n] + O_S[n]$$
$$O_S[n] > 0$$
$$\therefore O[n] > O_T[n]$$

10.9 $O[n] > O_S[n]$

10.9.1 Proof

$$O[n] := O_T[n] + O_S[n]$$

$$O_T[n] > 0$$

$$\therefore O[n] > O_S[n]$$

10.10 $O[n+1] \geqslant O[n]$

10.10.1 Proof

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$

$$O[n] = |\; s^+[X_i]|$$

$$O[n+1] = \hat{O}[n] = |\; s^+[\hat{X}_i]|$$

For general solutions s^+

$$s^+[\hat{X}_i] \supseteq s^+[X_i]$$

$$\Rightarrow |s^+[\hat{X}_i]| \geqslant |s^+[X_i]|$$

$$\therefore \hat{O}[n] = O[n+1] \geqslant O[n]$$

11 Polynomial Complexity

11.1 Definition

Decision problem D with solution s^+ has polynomial total complexity $O[n]$ if

$$\exists K, C, \lambda_1 ... \lambda_K \quad :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

11.2 Polynomial Problems

Define $\mathbb{P}$, the set of Decision Problems that can be solved with Polynomial Complexity

$$\mathbb{P} := \{D_1, D_2, ...\} :$$

$$\exists K, C, \lambda_1 ... \lambda_K :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, D_i \in \mathbb{P}$$

11.3 Polynomial Order of Complexity

Solution s^+ with total complexity $O[n]$ is said to be of order n^K

$$O[n] \sim n^K$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

11.4 Property of Polynomial Complexity 1

Solutions with polynomial complexity have convergent complexity

$$lim_{n \to \infty} \frac{O[n+1]}{O[n]} = 1$$

11.4.1 Proof

$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C$$
$$O[n+1] = (\lambda_K (n+1))^K + (\lambda_{K-1} (n+1))^{K-1} ... + \lambda_1 (n+1) + C$$
$$= (\lambda_K n)^K + (\tilde{\lambda}_{K-1} n)^{K-1} ... + \tilde{\lambda_1} n + \tilde{C}$$

$$lim_{n \to \infty} \frac{O[n+1]}{O[n]}$$

$$= lim_{n \to \infty} \frac{(\lambda_K n)^K + (\tilde{\lambda}_{K-1} n)^{K-1} ... + \tilde{\lambda_1} n + \tilde{C}}{(\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C}$$

$$= lim_{n\to\infty}\frac{(\lambda_K n)^K}{(\lambda_K n)^K+(\lambda_{K-1}n)^{K-1}...+\lambda_1 n+C} + \frac{(\tilde{\lambda}_{K-1}n)^{K-1}}{(\lambda_K n)^K+(\lambda_{K-1}n)^{K-1}...+\lambda_1 n+C} + ... + \frac{\tilde{\lambda}_1 n}{(\lambda_K n)^K+(\lambda_{K-1}n)^{K-1}...+\lambda_1 n+C} + \frac{\tilde{C}}{(\lambda_K n)^K+(\lambda_{K-1}n)^{K-1}...+\lambda_1 n+C}$$

$$= 1 = lim_{n\to\infty}\frac{O[n+1]}{O[n]}$$

11.5 Property of Polynomial Complexity 2

$$\exists K, \hat{C}, \hat{\lambda}_1, ..., \hat{\lambda}_{K-1} :$$
$$O[n+1] - O[n] = f_{n+1}[n] = (\hat{\lambda}_{K-1}n)^{K-1}... + \hat{\lambda}_1 n + \hat{C} \quad \forall n$$

11.5.1 Proof

$$O[n] = (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C$$
$$O[n+1] = (\lambda_K(n+1))^K + (\lambda_{K-1}(n+1))^{K-1}... + \lambda_1(n+1) + C$$
$$= (\lambda_K n)^K + (\tilde{\lambda}_{K-1}n)^{K-1}... + \tilde{\lambda_1} n + \tilde{C}$$
$$O[n+1] - O[n] = ((\tilde{\lambda}_{K-1} - \lambda_{K-1})n)^{K-1}... + (\tilde{\lambda_1} - \lambda_1)n + (\tilde{C} - C)$$
$$O[n+1] - O[n] = (\hat{\lambda}_{K-1}n)^{K-1}... + \hat{\lambda}_1 n + \hat{C}$$

11.6 Total Polynomial Complexity Implies Time bounded Polynomial Complexity

$$D \in \mathbb{P} \Longrightarrow O_T[n] < (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C$$

11.6.1 Proof

$$O[n] = (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C \ \ \forall n$$
$$O[n] := O_T[n] + O_S[n]; \ \ O_S[n] > 0$$
$$\therefore O_T[n] < (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C \ \ \forall n$$

11.7 Total Polynomial Complexity Implies Space bounded Polynomial Complexity

$$D \in \mathbb{P} \Longrightarrow O_S[n] < (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C$$

11.7.1 Proof

$$O[n] = (\lambda_K n)^K + (\lambda_{K-1}n)^{K-1}... + \lambda_1 n + C \ \ \forall n$$

$$O[n] := O_T[n] + O_S[n];\;\; O_T[n] > 0$$
$$\therefore O_S[n] < (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C\;\; \forall n$$

12 Non-Polynomial Complexity

12.1 Definition

Decision problem $\tilde{D}$ with solution s^+ has non-polynomial total complexity $O[n]$ if

$$\nexists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

12.2 Non-Polynomial Problems

Define $\mathcal{N}$, the set of Decision Problems that cannot be solved with Polynomial Complexity

$$\mathcal{N} := \{\tilde{D}_1, \tilde{D}_2, ...\} :$$
$$\nexists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n, \ s^+ \in S_i^+, \ \tilde{D}_i \in \mathcal{N}$$

12.3 $\mathbb{P}$ and $\mathcal{N}$ are disjoint

$$\mathbb{P} \cap \mathcal{N} = \varnothing$$

12.3.1 Proof

Let D $\in \mathcal{N}$

$$\nexists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

Assume D $\in \mathbb{P}$

$$\exists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

Contradicts the definition of $\mathcal{N}$

$$\therefore D \in \mathcal{N} \Rightarrow D \notin \mathbb{P}$$

Let D $\in \mathbb{P}$

$$\exists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

Assume D $\in \mathcal{N}$

$$\nexists K, C, \lambda_1 ... \lambda_K \ :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \quad \forall n$$

Contradicts the definition of $\mathbb{P}$

$$\therefore D \in \mathbb{P} \Rightarrow D \notin \mathcal{N}$$

$$D \in \mathcal{N} \Rightarrow D \notin \mathbb{P}; D \in \mathbb{P} \Rightarrow D \notin \mathcal{N}$$
$$\therefore \mathbb{P} \cap \mathcal{N} = \varnothing$$

13 Divergent Complexity

13.1 Defintion

Decision problem $\hat{D}$ with solution s^+ has divergent total complexity $O[n]$ if

$$lim_{n \to \infty} \frac{O[n+1]}{O[n]} \ \ diverges \quad \forall n$$

13.2 Divergent Problems

$$\hat{\mathcal{D}} := \{\hat{D}_1, \hat{D}_2, ...\} :$$
$$lim_{n \to \infty} \frac{O[n+1]}{O[n]} \ \ diverges \quad \forall s^+ \in S_i^+, \ \hat{D}_i \in \hat{\mathcal{D}}$$

13.3 The Set of Polynomial Solutions and the Set of Divergent Solutions are disjoint

$$\mathbb{P} \cap \hat{D} = \varnothing$$

13.4 Proof

Let D $\in \hat{\mathcal{D}}$

$$lim_{n \to \infty} \frac{O[n+1]}{O[n]} \ \ diverges \text{ by definition}$$

Assume D $\in \mathbb{P}$

$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} = 1$$

$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} = 1 \text{ contradicts the definition of Divergent Problems}$$

$$\therefore D \in \hat{\mathcal{D}} \Rightarrow D \notin \mathbb{P}$$

Let D $\in \mathbb{P}$

$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} = 1 \text{ by property of Polynomial complexity}$$

Assume D $\in \hat{D}$

$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} \text{ diverges}$$

$$lim_{n\to\infty}\frac{O[n+1]}{O[n]} \text{ diverges contradicts a property of Polynomial complexity}$$

$$\therefore D \in \mathbb{P} \Rightarrow D \notin \hat{\mathcal{D}}$$

$$\therefore \mathbb{P} \cap \hat{\mathcal{D}} = \varnothing$$

14 Inductive Functions

14.1 Inductive Function f_{n+1}

$$O[n] := O_T[n] + O_S[n]$$
$$O[n+1] = O_T[n+1] + O_S[n+1]$$

$$f_{n+1}[n] := O[n+1] - O[n]$$

14.2 Inductive Space and Time Formulas

$$f^T_{n+1}[n] := O_T[n+1] - O_T[n]$$
$$O_T[n+1] = O_T[n] + f^T_{n+1}[n]$$
$$f^S_{n+1}[n] := O_S[n+1] - O_S[n]$$
$$O_S[n+1] = O_S[n] + f^S_{n+1}[n]$$

14.3 Inductive Function Expressions

Relate $f_{n+1}[n]$ to equivalence functions

$$D \in \mathbb{P}$$
$$O[n] := O_T[n] + O_S[n]$$
$$O[n+1] = O_T[n+1] + O_S[n+1] = O[n] + f_{n+1}[n]$$
$$O_T[n] = O[n] - O_S[n]$$
$$O_S[n] = O[n] - O_T[n]$$

$$f_{n+1} = O[n+1] - O[n]$$
$$f_{n+1} = O_T[n+1] + O_S[n+1] - O[n]$$
$$f_{n+1} = O_T[n+1] - O_T[n] + O_S[n+1] - O_S[n]$$
$$f_{n+1} = O[n+1] - O_T[n] - O_S[n]$$
$$f_{n+1}[n] = f^T_{n+1}[n] + f^S_{n+1}[n]$$

14.4 Zero Order Inductive Function

$$\textit{Let } O_S[n] \sim n^0$$
$$f_{n+1} = O_T[n+1] - O_T[n] + O_S[n+1] - O_S[n] = O_T[n+1] - O_T[n]$$

14.5 Property of Polynomial Complexity

$$limit_{n\to\infty}\frac{f_{n+1}[n]}{O[n]} = 0$$

14.5.1 Proof

$$limit_{n\to\infty}\frac{O[n+1]}{O[n]} = 1$$

$$limit_{n\to\infty}\frac{O[n]+f_{n+1}[n]}{O[n]} = 1$$

$$limit_{n\to\infty}\frac{O[n]}{O[n]} + \frac{f_{n+1}[n]]}{O[n]} = 1$$

$$limit_{n\to\infty}1 + \frac{f_{n+1}[n]]}{O[n]} = 1$$

$$limit_{n\to\infty}\frac{f_{n+1}[n]]}{O[n]} = 0$$

15 Subfunctions

15.1 Restate the subfunction condition of general solutions

Recall the definition of general solution s^+

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

The subfunction condition is one of two conditions for a general solution

$$P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i$$

15.2 Prove O[n] is a non-decreasing function

Consider solution s^+ with complexity $O[n]$

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$O[n] := O_T[n] + O_S[n]$$
$$O_T[n] := |\mathcal{L}| = N$$
$$O_S[n] := |\mathcal{M}| + |y_o| = M + 1$$

O[n+1] denotes the total complexity for solution $s^+[\hat{X}_i]$

$$s^+[\hat{X}_i] = \hat{s}^+$$

Let

$$O[n+1] < O[n]$$
$$\Rightarrow \hat{N} + \hat{M} < N + M$$
$$\hat{s}^+ = \{s_1, s_2, ..., s_{\hat{N}} | b_1, b_2, ..., b_{\hat{M}}, y_o\}$$

$$\Rightarrow \hat{s}^+ \nsupseteq s^+$$
$$P[\hat{X}_i] \nsupseteq P[X_i] \quad \forall X_i, \hat{X}_i$$

$$\therefore O[n+1] < O[n] \text{ contradicts the definition of solution } s^+$$
$$O[n+1] \geqslant O[n]$$

15.3 Definition of Subfunction

$$X_i = \{x_1, ..., x_n, C\}; \quad \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \to y_o == a_o \quad \forall X_i) \quad \cap \quad (P[\hat{X}_i] \supseteq P[X_i] \quad \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub[X_i] := S = \{s_j, ... | b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \quad \forall s_j, b_k \in S$$

15.3.1 $s^+[X_i]$ is a subfunction of $s^+[\hat{X}_i]$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$\hat{s}^+ = \{s_1, s_2, ..., s_N, ..., s_{\hat{N}} | b_1, b_2, ..., b_M, ..., b_{\hat{M}}, y_o\}; \quad \hat{N} + \hat{M} \geqslant N + M$$

By definition of solution

$$\hat{s}^+ = P[\hat{X}_i] \supseteq P[X_i] = s^+ \quad \forall X_i, \hat{X}_i$$
$$\Rightarrow s_j, b_k \in \hat{s}^+ \;\; \forall s_j, b_k \in s^+$$

15.4 Subfunction Decomposition of Solutions

FIX Double check conditions!!! Solutions s^+ can be written as the union of subfunctions $Sub_k[X_i]$

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \;\; \forall X_i) \;\; \cap \;\; (P[\hat{X}_i] \supseteq P[X_i] \;\; \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$s^+ = Sub_1[X_i] \cup Sub_2[X_i] \cup ... \cup Sub_z[X_i]$$
$$= \{\mathcal{L}_1 | \mathcal{M}_1, y_o\} \cup \{\mathcal{L}_2 | \mathcal{M}_2, y_o\} \cup ... \cup \{\mathcal{L}_z | \mathcal{M}_z, y_o\} :$$
$$\mathcal{L}_j \cap \mathcal{L}_k = \varnothing \;\; \forall j, k \neq j$$

$$s^+ = \{s_1^1, ..., s_{N_1}^1 | b_1^1, ..., y_o\} \cup \{s_1^2, ..., s_{N_2}^2 | b_1^2, ..., y_o\} \cup ... \cup \{s_1^z, ..., s_{N_z}^z | b_1^z, ..., y_o\} :$$
$$\textstyle\sum_{l=1}^{z} N_l = N = O_T[n]$$

16 Subfunction Complexity

16.1 Disjoint Subfunction Operations

$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \;\; \forall i, j \neq i$$

16.2 Shared Subfunction Memory

$$|\mathcal{M}_i \cap \mathcal{M}_j| \geqslant 0 \;\; \forall i, j \neq i$$

16.2.1 Time Complexity of Subfunctions

Subfunction time complexity is additive

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub_i[X] := S_i = \{s_j, ... | b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \;\;\; \forall s_j, b_k \in S_i$$

$$s^+ = \{\mathcal{L}_1 | \mathcal{M}_1, y_o\} \cup \{\mathcal{L}_2 | \mathcal{M}_2, y_o\} \cup ... \cup \{\mathcal{L}_z | \mathcal{M}_z, y_o\} :$$
$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \;\; \forall i, j \neq i$$

$$\mathcal{L} = \cup_{i=1}^{z} \mathcal{L}_i$$

$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$$

$$O_T[n] = |\mathcal{L}| = N$$
$$O_T[n] = |\cup_{i=1}^{z} \mathcal{L}_i| = \sum_{i=1}^{z} |\mathcal{L}_i|^* = |\mathcal{L}_1| + |\mathcal{L}_2| + ... + |\mathcal{L}_z|$$
$$= O_{T_1}[n] + O_{T_2}[n] + ... + O_{T_z}[n] = N_1 + N_2 + ... + N_z$$

*Due to the disjoint condition of subfunction operations $\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$

16.2.2 Space Complexity of Subfunctions

Subfunctions can access the full memory $\mathcal{M}$ with no added space complexity

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub_i[X] := S_i = \{s_j, ...|b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \quad \forall s_j, b_k \in S_i$$

$$s^+ = \{\mathcal{L}_1|\mathcal{M}_1, y_o\} \cup \{\mathcal{L}_2|\mathcal{M}_2, y_o\} \cup ... \cup \{\mathcal{L}_z|\mathcal{M}_z, y_o\} :$$
$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$$

$$s^+ = \{\mathcal{L}_1|\mathcal{M}, y_o\} \cup \{\mathcal{L}_2|\mathcal{M}, y_o\} \cup ... \cup \{\mathcal{L}_z|\mathcal{M}, y_o\} :$$
$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$$

$$\mathcal{M} = \cup_{i=1}^{z} \mathcal{M}_i = \cup_{i=1}^{z} \mathcal{M}$$

$$O_S[n] = |\mathcal{M}| = M$$
$$O_S[n] = |\cup_{i=1}^{z} \mathcal{M}_i| = M$$

16.2.3 Shared State Notation

$$s^+ = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub_i[X] := S_i = \{s_j, ...|b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \quad \forall s_j, b_k \in S_i$$

$$s^+ = \{\mathcal{L}_1|\mathcal{M}, y_o\} \cup \{\mathcal{L}_2|\mathcal{M}, y_o\} \cup ... \cup \{\mathcal{L}_z|\mathcal{M}, y_o\} :$$

$$\mathcal{L}_i \cap \mathcal{L}_j = \varnothing \ \ \forall i, j \neq i$$

17 Polynomial Solution Subfunction Properties

17.1 Restate Definition of Subfunction

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \;\; \forall X_i) \;\; \cap \;\; (P[\hat{X}_i] \supseteq P[X_i] \;\; \forall X_i, \hat{X}_i)$$

$$s^+ = \{s_1, s_2, ..., s_N | b_1, b_2, ..., b_M, y_o\} = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\}$$
$$= \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$Sub[X_i] := S = \{s_j, ... | b_k, ..., y_o\} :$$
$$s_j, b_k \in s^+ \;\; \forall s_j, b_k \in S$$

17.2 Property of Polynomial Solution Subfunctions

Let

$$D \in \mathbb{P}$$

$$X_i = \{x_1, ..., x_n, C\}; \;\; \hat{X}_i = \{x_1, ..., x_{n+1}, C\}$$
$$s^+ = s^+[X_i] := P :$$
$$(P[X_i] \rightarrow y_o == a_o \;\; \forall X_i) \;\; \cap \;\; (P[\hat{X}_i] \supseteq P[X_i] \;\; \forall X_i, \hat{X}_i)$$

$$\exists K, C, \lambda_1 ... \lambda_K \;\; :$$
$$O[n] = (\lambda_K n)^K + (\lambda_{K-1} n)^{K-1} ... + \lambda_1 n + C \;\; \forall n$$

$$s^+ = Sub_1[X_i] \cup Sub_2[X_i] \cup ... \cup Sub_z[X_i]$$

$$limit_{n \rightarrow \infty} \frac{O[n+1]}{O[n]} = 1$$
$$= limit_{n \rightarrow \infty} \frac{O_T^1[n+1] + O_T^2[n+1] + ... + O_T^z[n+1] + O_S[n+1]}{O_T^1[n] + O_T^2[n] + ... + O_T^z[n] + O_S[n]}$$
$$=$$
$$limit_{n \rightarrow \infty} \frac{O_T^1[n] + O_T^2[n] + ... + O_T^z[n] + O_S[n] + f_{T_{n+1}}^1[n] + f_{T_{n+1}}^2[n+1] + ... + f_{T_{n+1}}^z[n] + f_{S_{n+1}}[n]}{O_T^1[n] + O_T^2[n] + ... + O_T^z[n] + O_S[n]}$$

$$= limit_{n \rightarrow \infty} 1 + \frac{f_{T_{n+1}}^1[n] + f_{T_{n+1}}^2[n+1] + ... + f_{T_{n+1}}^z[n] + f_{S_{n+1}}[n]}{O_T^1[n] + O_T^2[n] + ... + O_T^z[n] + O_S[n]} = 1$$

$$\Rightarrow limit_{n\to\infty}\frac{f^1_{T_{n+1}}[n]+f^2_{T_{n+1}}[n+1]+...+f^z_{T_{n+1}}[n]+f_{S_{n+1}}[n]}{O^1_T[n]+O^2_T[n]+...+O^z_T[n]+O_S[n]} = 0^*$$

$$\Rightarrow limit_{n\to\infty}\frac{f^i_{T_{n+1}}[n]+f_{S_{n+1}}[n]}{O^1_T[n]+O^2_T[n]+...+O^z_T[n]+O_S[n]} = 0 \;\; \forall i$$

$$limit_{n\to\infty}\frac{f^i_{n+1}[n]}{O[n]} = 0 \;\; \forall i$$

* O[n] is a positive, non-decreasing function

18 Solution Spaces

18.1 Definition of Solution Space

$$\mathbb{S} = \{c_1^+, c_2^+, ..., c_{C[n]}^+\}$$

$$s^+[X_n] = \vee_{c_i^+ \in \mathbb{S}} \; c_i^+$$

18.2 Existence, Uniqueness, etc.

18.3 Worst Case

19 Fundamental Theorem of Computation

The Fundamental Theorem of Computation relates the complexity of optimal solution to the number of candidate solutions in the Solution Space.

$$\mathbb{S} = \{c_1^+, c_2^+, ..., c_{C[n]}^+\}$$
$$s^+[X_n] = \vee_{c_i^+ \in \mathbb{S}} \; c_i^+$$

$O_{opt}[n]$ has the same order as C[n]

19.1 Proof by Induction

19.2 Proof by Contradiction

20 Sum to N Problem with 2 integers

20.1 State formal definition of Sum to N : $x_i + x_j == N$

$$X_n = \{x_1, ..., x_n\}$$

$$D := f[X_i, N] = a_o \in \{\mathbb{T}, \mathbb{F}\} \quad \forall X_i$$

$$s^+[X_n] = P[X_n] :$$
$$(P[X_i] = y_o == a_o \quad \forall X_i) \quad \cap \quad (P[X_{n+1}] \supseteq P[X_n] \quad \forall X_{n+1})$$

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$

$$D = f[X_i] = \quad \exists x_j, x_k \in X_n \quad j \neq k :$$
$$x_j + x_k == N$$

20.2 Express a formal solution : $O_S[n] \sim n^0$

$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$
$$s_1 = y_o \leftarrow \mathbb{F};$$
$$\forall i < n \;\;,\;\; n \geqslant j > i$$
$$s_2, s_3, s_8, s_9, ..., s_{3ij-4}, s_{3ij-3} ..., s_{3n(n-1)-4}, s_{3n(n-1)-3} = b_1 \leftarrow x_i + x_j$$
$$s_4, s_5, s_{10}, s_{11}, ..., s_{3ij-2}, s_{3ij-1} ..., s_{3n(n-1)-2}, s_{3n(n-1)-1} = b_1 \leftarrow b_1 == N$$
$$s_6, s_7, s_{12}, s_{13} ..., s_{3ij}, s_{3ij+1} ..., s_{3n(n-1)}, s_{3n(n-1)+1} = y_o \leftarrow y_o \vee b_1$$
$$s^+ = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\}$$

20.3 Prove s^+ satisfies the subfunction condition of solutions: $P[X_{n+1}] \supseteq P[X_n] \quad \forall X_{n+1}$

$$X_n = \{x_1, x_2, ..., x_n\}; \quad X_{n+1} = \{x_1, x_2, ..., x_n, x_{n+1}\}$$
$$s^+ = \{s_1, s_2, ..., s_{O_T[n]}, b_1, b_2, ..., b_{O_S[n]}, y_o\} = \{\mathcal{L}, \mathcal{M}, y_o\}$$
$$s^+_{n+1} = s^+ \cup \hat{s}^+$$
$$s_1 = y_o \leftarrow \mathbb{F};$$
$$\forall i < n \;\;,\;\; n \geqslant j > i$$

$$s_2, s_3, s_8, s_9, ..., s_{3ij-4}, s_{3ij-3}..., s_{3n(n-1)-4}, s_{3n(n-1)-3} = b_1 \leftarrow x_i + x_j$$
$$s_4, s_5, s_{10}, s_{11}, ..., s_{3ij-2}, s_{3ij-1}..., s_{3n(n-1)-2}, s_{3n(n-1)-1} = b_1 \leftarrow b_1 == N$$
$$s_6, s_7, s_{12}, s_{13}..., s_{3ij}, s_{3ij+1}..., s_{3n(n-1)}, s_{3n(n-1)+1} = y_o \leftarrow y_o \vee b_1$$

$$\forall k < n+1$$
$$s... = b_1 \leftarrow x_k + x_{n+1}$$
$$s... = b_1 \leftarrow b_1 == N$$
$$s... = y_o \leftarrow y_o \vee b_1$$

$$s^+ = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\}$$

$$\hat{s}^+ = \{y_o \leftarrow y_o \vee (x_k + x_{n+1} == N) \;\; \forall k < n+1 | \; b_1, y_o\}$$

$$s^+_{n+1} = \{y_o \leftarrow \mathbb{F}, y_o \leftarrow y_o \vee (x_i + x_j == N) \quad \forall i, j > i \mid b_1, y_o\} \;\; \cup$$
$$\{y_o \leftarrow y_o \vee (x_k + x_{n+1} == N) \;\; \forall k < n+1 | \; b_1, y_o\}$$

$$s^+_{n+1} = s^+ \cup \hat{s}^+ = P[X_{n+1}] \supseteq P[X_n] = s^+$$

20.4 Determine $O[n], O_S[n], O_T[n], f_{n+1}[n], f^T_{n+1}[n], f^S_{n+1}[n]$ for the above solution

$$O_S[n] = |y_o| + |b_1| = 2$$
$$O_T[n] = 3n(n-1) + 1 = 3n(n-1) - 1 + O_S[n]$$
$$O[n] = 3n(n-1) + 3 = 3n^2 - 3n + 3$$

$$f^S_{n+1}[n] = 0$$
$$f^T_{n+1}[n] = 6n$$
$$f_{n+1}[n] = f^S_{n+1}[n] + f^T_{n+1}[n]$$

20.5 Verify $O[n+1] = O[n] + \; f_{n+1}[n]$

$$O[n+1] = O[n] + \hat{O}[n]$$
$$3(n+1)^2 - 3(n+1) + 3 = 3n^2 - 3n + 3 + 6n$$
$$3n^2 + 6n + 3 - 3n - 3 + 3 = 3n^2 + 3n + 3$$

$$3n^2 + 3n + 3 = 3n^2 + 3n + 3$$

20.6 Show s^+ has Polynomial Complexity by the definition of Total Polynomial Complexity

$$O[n] = 3n^2 - 3n + 3$$

20.7 Show the limit $_{n \to \infty} \frac{O[n+1]}{O[n]}$ does not Diverge

$$limit_{n \to \infty} \frac{O[n+1]}{O[n]} =$$

$$limit_{n \to \infty} \frac{3n^2+3n+3}{3n^2-3n+3} =$$

$$limit_{n \to \infty} \left(\frac{3n^2-3n+3}{3n^2-3n+3} + \frac{6n}{3n^2-3n+3}\right) =$$

$$limit_{n \to \infty} \left(1 + \frac{6n}{3n^2-3n+3}\right) = 1$$

21 The Knapsack Problem

21.1 The Knapsack Problem

The Knapsack Problem is a famous problem in computer science which asks if objects can be stored in a knapsack. Typically the problem is designed with two constraints, weight and value. Given objects x_i, each with a respective weight w_i and value v_i, does there exist a combination of objects lighter than input weight W and more valuable than input value V?

21.2 Formal Definition

$$X_n = \{x_1, x_2, ..., x_n\} = \{\{w_1, v_1\}, \{w_2, v_2\}, ..., \{w_n, v_n\}\}$$

$$I = \{i_1, i_2, ..., i_n\} : i_l \in \{0, 1\} \;\; \forall i_l \in I$$

$$D := f[X_n, W, V] = a_o \in \{\mathbb{T}, \mathbb{F}\} = \exists I :$$
$$(\textstyle\sum_{j=1}^{n} i_j w_j < W) \wedge (\textstyle\sum_{j=1}^{n} i_j v_j \geqslant V)$$

21.3 Express a solution s^+ to the Knapsack Problem

21.4 Prove s^+ satisfies the subfunction condition of solutions

21.5 Determine $O[n], O_T[n], O_S[n], f_{n+1}[n]$

21.6 Show $s^+ \notin \mathbb{P}$

21.7 Express the Solution Space $\mathbb{S}$ for The Knapsack Problem

21.8 Prove a lower bound for all solutions $s^+ \in S^+ := O_{lower}[n]$

21.9 Prove $D \notin P$

Citations

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