DSGE Model I

An Introduction to Stochastic Dynamic Programming

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Outline

Introduction

- Stochastic growth models are useful for two related reasons.
  - A range of interesting growth problems involve either aggregate uncertainty or nontrivial individual-level uncertainty interacting with investment decisions and the growth process.
  - The stochastic neoclassical growth model has a wide range of applications in macroeconomics and in other areas of dynamic economic analysis.
- Before the baseline neoclassical growth model (with complete markets) augmented with stochastic productivity shocks (Brock and Mirman (1972)),
  - Which is not only an important generalization of the baseline neoclassical growth, but also provides the starting point of the influential **Real Business Cycle** models, which are used extensively for the study of a range of short- and medium-run macroeconomic questions.
- It's necessary to learn how to program the stochastic dynamic.
- We use **Markov Chains** to represent uncertainty.

Stochastic Dynamic Programming

Dynamic Programming with Expectations

- Let us first introduce the *stochastic* (random) variable $z(t) \in \mathcal{Z} \equiv \{z_1, \ldots, z_N\}$, with $z_1 < z_2 < \cdots < z_N$. Note that the set $\mathcal{Z}$ is finite and thus compact. Let the instantaneous payoff at time t be

$$U(x(t), x(t+1), z(t)) \tag{1}$$

  where $x(t) \in X \subset \mathbb{R}^K$ for some $K \geq 1$ and $U : X \times X \times \mathcal{Z} \to \mathbb{R}$.
- Payoffs: a function of the stochastic variable $z(t)$.
- As usual, returns are discounted by some discount factor $\beta \in (0, 1)$.
- $x(t)$ denotes the *state variables* (state vector), and $x(t+1)$ the *control variables* (control vector) at time t.
- The initial values of the state vector, $x(0)$, and of stochastic variable, $z(0)$, are taken as given.

Cont.

- The constraint also incorporates the stochastic variable $z(t)$ and is written as
$$x(t+1) = G(x(t), z(t))$$
where $G(x,z)$ is a set-valued mapping (correspondence):
$$G : X \times \mathcal{Z} \rightrightarrows X$$
- Suppose that the stochastic variable $z(t)$ follows a (first-order) **Markov chain**. The important property implied by the Markov chain assumption is that the current value of $z(t)$ only depends on its value from the last period, $z(t-1)$. Mathematically, this can be expressed as
$$\Pr[z(t) = z_j \mid z(0), \ldots, z(t-1)] = \Pr[z(t) = z_j \mid z(t-1)]$$

Cont.

- The Markov property not only simplifies the mathematical structure of economic models but also allows us to use relatively simple notation for the probability distribution of the random variable $z(t)$. We can also represent a Markov chain as

$$\Pr[z(t) = z_j \mid z(t-1) = z_{j'}] \equiv q_{jj'}$$

For any $j, j' = 1, \ldots, N$, where $q_{ij} \geq 0$ for all j, j', and

$$\sum_{j=1}^{N} q_{jj'} = 1 \quad \text{for each } j' = 1, \ldots, N.$$

- $q_{jj'}$ is also referred to as a *transition probability*, meaning the probability of the stochastic state z transitioning from $z_{j'}$ to z_j.

An Example

Recall the optimal growth problem, where the objective is to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c(t)).$$

The expectations operator $\mathbb{E}_0$ *stands for expectations conditional on information available at (the beginning of) time* $t = 0$.

Expectations are necessary here because the future values of consumption per capita are stochastic (as they depend on the realizations of future z *values).*

Cont.

- The production function:

$$y(t) = f(k(t), z(t))$$

- The most natural interpretation of $z(t)$ in this context is as a stochastic **TFP** term.
- The resource constraint:

$$k(t+1) = f(k(t), z(t)) + (1-\delta)k(t) - c(t) \tag{2}$$

  It implies that when $c(t)$ is chosen, the random variable $z(t)$ has been realized. Thus $c(t)$ depends on the realization of $z(t)$, and in fact on the entire history of $z(t)$.
- We define

$$z^t \equiv (z(1), \ldots, z(t))$$

  as the history of $z(t)$ up to date t. As a convention, this history does not include $z(0)$, which is taken as given, to ensure z^t has t elements.

Cont.

- In particular, let $\mathcal{Z}^t = \mathcal{Z} \times \cdots \times \mathcal{Z}$ (the t-times product), so that $z^t \in \mathcal{Z}^t$.
- For given $k(0)$, the consumption at time t can be written as

$$c(t) = \tilde{c}[z^t]$$

  which states that consumption at time t is a function of the entire sequence of random variables observed up to that point.
- In terms of (1), here $x(t) = k(t)$, so that

$$\begin{aligned} x(t+1) &= k(t+1) \\ &= f(k(t), z(t)) + (1-\delta)k(t) - \tilde{c}[z^t] \\ &\equiv \tilde{k}[z^t] \end{aligned}$$

  by definition, $k(t+1)$ depends only on the history of the stochastic shocks up to time t and not on $z(t+1)$.

Cont.

- In addition, from the resource constraint we have:

$$\tilde{k}[z^t] = f(\tilde{k}[z^{t-1}], z(t)) + (1-\delta)\tilde{k}[z^{t-1}] - \tilde{c}[z^t] \qquad (3)$$

for all $z^{t-1} \in \mathcal{Z}^{t-1}$ and $z(t) \in \mathcal{Z}$.

- The maximization problem can then be expressed as:

$$\max_{\{\tilde{c}[z^t], \tilde{k}[z^t]\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(\tilde{c}[z^t])$$

subject to (3), $\tilde{c}[z^t] \geq 0$ and $\tilde{k}[z^t] \geq 0$ and starting with the initial conditions $\tilde{k}[z^{-1}] = k(0)$ and $z(0)$.

Cont.

- This maximization problem can also be written using the instantaneous payoff function $U(x(t), x(t+1), z(t))$ introduced in (1). Then, the maximization problem takes the form

$$\max_{\{\tilde{k}[z^t]\}_{t=0}^{\infty}} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t U(\tilde{k}[z'^{t-1}], \tilde{k}[z^t], z(t)) \tag{4}$$

  where we have

$$U(\tilde{k}[z'^{t-1}], \tilde{k}[z^t], z(t)) = u\left(f(k(t), z(t)) - k(t+1) + (1-\delta)k(t)\right)$$

- This example can also be used to show how the same maximization problem can be represented recursively.

Cont.

- Since $z(t)$ follows a Markov chain, the current value of $z(t)$ contains both information about the available resources for consumption and future capital stock and information regarding the stochastic distribution of $z(t+1)$. Thus we might naturally expect the policy function determining the capital stock at the next date to take the form

$$k(t+1) = \pi(k(t), z(t)) \tag{5}$$

- The recursive characterization would take the form

$$V(k,z) = \sup_{y \in [0, f(k,z)+(1-\delta)k]} \left\{ u(f(k,z) + (1-\delta)k - y) + \beta \mathbb{E}[V(y,z') \mid z] \right\} \tag{6}$$

This expectation is different from that in (4). In (4), the expectation is over the entire set of future values of z, whereas in (6), it is over next period's value of z, z'.

Solution

- Let us suppose that this program has a solution, meaning that there exists a feasible plan that achieves the value $V(k,z)$ starting with capital-labor ratio k and stochastic variable z.
- Then the set of the next date's capital stock that achieves this maximum value can be represented by a correspondence $\Pi(k,z)$ for $k \in \mathbb{R}_+$, and $z \in \mathcal{Z}$. For any $\pi(k,z) \in \Pi(k,z)$, we have

  $$V(k,z) = u(f(k,z) + (1-\delta)k - \pi(k,z)) + \beta\mathbb{E}[V(\pi(k,z), z') \mid z]$$

  When the correspondence $\Pi(k,z)$ is single valued, then $\pi(k,z)$ is uniquely defined and the optimal choice of next period's capital stock can be represented as in (5).
- **This example indicates how a stochastic optimization problem can be written in sequence form and also gives us a hint about how to express such a problem recursively.**

Problem 1

- Let's do this more systematically.
- Let a **plan** be denoted by $\tilde{x}[z^t]$. It specifies the value of the vector $x \in \mathbb{R}^K$ for time $t+1$ (i.e., $x(t+1) = \tilde{x}[z^t]$) for any $z^t \in \mathcal{Z}^t$. The sequence problem takes the form
Problem 1

$$V^*(x(0), z(0)) = \sup_{\{\tilde{x}[z^t]\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(\tilde{x}[z^{t-1}], \tilde{x}[z^t], z(t))$$

subject to

$$\tilde{x}[z^t] \in G(\tilde{x}[z^{t-1}], z(t)) \quad \text{for all } t \geq 0,$$

$$\tilde{x}[z^{-1}] = x(0) \text{ given.}$$

Problem 2

- **Problem 2**
  Similar to (6) in our first example, the functional equation corresponding to the recursive formulation of this problem can be written as follows.

$$V(x,z) = \sup_{y \in G(x,z)} \left\{ U(x,y,z) + \beta \mathbb{E}[V(y,z') \mid z] \right\} \tag{7}$$

  for all $x \in X$ and $z \in \mathcal{Z}$.
- Here $V : X \times \mathcal{Z} \to \mathbb{R}$ is a real-valued function, and $y \in G(x,z)$ represents the constraint on next period's state vector as a function of the realization of the stochastic variable z.

Cont.

- let us first introduce the set of feasible plans starting with an initial value $x(t)$ and a value of the stochastic variable $z(t)$ as

  $$\Phi(x(t),z(t)) = \{\{\tilde{x}[z^s]\}_{s=t}^{\infty} : \tilde{x}[z^s] \in G(\tilde{x}[z^{s-1}], z(s)) \quad \text{for } s = t, t+1, \ldots\}$$

  We denote a generic element of $\Phi(x(0),z(0))$ by $\mathbf{x} \equiv \{\tilde{x}[z^t]\}_{t=0}^{\infty}$.
- We are interested in using the formulation in *Problem 2* to characterize the solution to *Problem 1*; thus we will investigate **I. when the solution** $V(x,z)$ **to Problem 2 coincides with the solution** $V^*(x,z)$ and **II. when the set of maximizing plans** $\Pi(x,z) \subset \Phi(x,z)$ **also generates an optimal feasible plan for Problem 1** (presuming that both problems have feasible plans attaining their supremums).

Cont.

- Recall that the set of maximizing plans $\Pi(x,z)$ is defined such that for any $\pi(x,z) \in \Pi(x,z)$,

$$V(x,z) = U(x, \pi(x,z), z) + \beta \mathbb{E}[V(\pi(x,z), z') \mid z] \tag{8}$$

Assumptions and Theorems

Assumptions 1

The correspondence $G(x,z)$ is nonempty-valued for all $x \in X$ and $z \in \mathcal{Z}$. Moreover, for all $x(0) \in X$, $z(0) \in \mathcal{Z}$, and $\mathbf{x} \in \Phi(x(0), z(0))$, the limit of expected discounted utility

$$\lim_{n \to \infty} \mathbb{E}\left[\sum_{t=0}^{n} \beta^t U(\tilde{x}[z^{t-1}], \tilde{x}[z^t], z(t)) \mid z(0)\right]$$

exists and is finite.

Cont.

Assumptions 2

X is a compact subset of $\mathbb{R}^K$, and G is nonempty-valued, compact-valued, and continuous. Moreover, let $X_G = \{(x, y, z) \in X \times X \times \mathcal{Z} : y \in G(x, z)\}$, and suppose that $U : X_G \to \mathbb{R}$ is continuous.

Cont.

Theorem 1

(Equivalence of Values) Suppose Assumptions 1 hold. Then for any $x \in X$ and any $z \in \mathcal{Z}$, $V^*(x, z)$ that is a solution to *Problem 1* is also a solution to *Problem 2*. Moreover, any solution $V(x, z)$ to *Problem 2* is also a solution to *Problem 1*, so that $V^*(x, z) = V(x, z)$ for any $x \in X$ and any $z \in \mathcal{Z}$.

Theorem 2

(Principle of Optimality) Suppose *Assumptions 1* hold. For $x(0) \in X$ and $z(0) \in \mathcal{Z}$, let $\mathbf{x}^* \equiv \{\tilde{x}^*[z^t]\}_{t=0}^{\infty} \in \Phi(x(0), z(0))$ be a feasible plan that attains $V^*(x(0), z(0))$ in *Problem 1*. Then we have

$$V^*(\tilde{x}^*[z^{t-1}], z(t)) = U(\tilde{x}^*[z^{t-1}], \tilde{x}^*[z^t], z(t)) + \beta \mathbb{E}[V^*(\tilde{x}^*[z^t], z(t+1)) \mid z(t)] \quad (9)$$

for $t = 0, 1, \ldots$.

Moreover, if $\mathbf{x}^* \in \Phi(x(0), z(0))$ satisfies (9), then it attains the optimal value for *Problem 1*.

Cont.

Theorem 3

(Existence of Solutions) Suppose that *Assumptions 1 and 2* hold. Then there exists a unique function $V : X \times \mathcal{Z} \to \mathbb{R}$ that satisfies (7). This function V is continuous and bounded in x for each $z \in \mathcal{Z}$. Moreover, an optimal plan $\mathbf{x}^* \in \Phi(x(0), z(0))$ exists for any $x(0) \in X$ and any $z(0) \in \mathcal{Z}$.

The remaining results, as in their analogues in the **Neoclassical** Models, use further assumptions to establish *concavity*, *monotonicity*, and the *differentiability* of the value function.

Cont.

Assumption 3

U is concave. That is, for any $\alpha \in (0,1)$ and any (x,y,z) and (x',y',z) in X_G, we have

$$U(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y', z) \geq \alpha U(x,y,z) + (1-\alpha)U(x',y',z).$$

Moreover, if $x \neq x'$, then

$$U(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y', z) > \alpha U(x,y,z) + (1-\alpha)U(x',y',z).$$

In addition, $G(x,z)$ is convex in x. That is, for any $z \in \mathcal{Z}$, any $\alpha \in [0,1]$, and any $x, x', y, y' \in X$ such that $y \in G(x,z)$ and $y' \in G(x',z)$, we have

$$\alpha y + (1-\alpha)y' \in G(\alpha x + (1-\alpha)x', z).$$

Cont.

Assumption 4

For each $y \in X$ and $z \in \mathcal{Z}$, $U(\cdot, y, z)$ is strictly increasing in its first K arguments, and G is monotone in x in the sense that $x \leq x'$ implies $G(x, z) \subset G(x', z)$ for each $z \in \mathcal{Z}$.

Assumption 5

$U(x, y, z)$ is continuously differentiable in x in the interior of its domain X_G.

Theorem 4

(Concavity of the Value Function) Suppose that *Assumptions 1–3* hold. Then the unique function V that satisfies (7) is strictly concave in x for each $z \in \mathcal{Z}$. Moreover, the optimal plan can be expressed as $\tilde{x}^*[z^t] = \pi(x^*(t), z(t))$, where the policy function $\pi : X \times \mathcal{Z} \to X$ is continuous.

Cont.

Theorem 5

(Monotonicity of the Value Function I) Suppose that Assumptions 1, 2, and 4 hold, and let $V : X \times \mathcal{Z} \to \mathbb{R}$ be the unique solution to (7). Then for each $z \in \mathcal{Z}$, V is strictly increasing in x.

Theorem 6

(Differentiability of the Value Function) Suppose that *Assumptions 1 2 3 5* hold. Let π be the policy function defined above and assume that $x' \in \operatorname{Int} X$ and $\pi(x', z) \in \operatorname{Int} G(x', z)$ at $z \in \mathcal{Z}$. Then $V(x, z)$ is continuously differentiable at (x', z) with the gradient with respect to x given by

$$D_x V(x', z) = D_x U(x', \pi(x', z), z). \tag{10}$$

Cont.

Above theorems have exact analogues in the Neoclassical programming. Since the value function now depends on the stochastic variable z, an additional *monotonicity* result can also be obtained. To do this, let us introduce the following assumption:

Assumption 6

1. G is monotone in z in the sense that $z \leq z'$ implies $G(x,z) \subseteq G(x,z')$ for each $x \in X$ and $z, z' \in \mathcal{Z}$ such that $z \leq z'$.
2. For each $(x,y,z) \in X_G$, $U(x,y,z)$ is strictly increasing in z.
3. The Markov chain for z is monotone in the sense that for any nondecreasing function $f : \mathcal{Z} \to \mathbb{R}$, $\mathbb{E}[f(z') \mid z]$ is also nondecreasing in z (where z' is next period's value of z).

Cont.

Theorem 7

(Monotonicity of the Value Function II) Suppose that *Assumptions 1 2 6* hold, and let $V : X \times \mathcal{Z} \to \mathbb{R}$ be the unique solution to (7). Then for each $x \in X$, V is strictly increasing in z.

Thank You!