procedure triusolve(k, A, B);

solves the upper triangular system $UX = B$, where U is stored above the main diagonal of A. Finally,

procedure mmul(k, A, B, C);

computes $C := C + AB$. With these procedures, we can write a **parallel block LU decomposition algorithm**:

for $k := 1$ **to** N **do**
  lu(α, A_{kk});
  parallel $k + 1 \le j \le N$ **do** trilsolve(α, A_{kk}, A_{kj});
  parallel $k + 1 \le i \le N$ **do** triusolve(α, A_{kk}, A_{ik});
  parallel $k + 1 \le i, j \le N$ **do** mmul($\alpha, -A_{ik}, A_{kj}, A_{ij}$);

The level 3 Blas routines can be optimized for a certain computer architecture and coded in assembler language. The Fortran programs based on these routines are machine independent, and will perform very well over a wide range of computers.

Exercises

1. Solve the system of equations

$$\begin{aligned} 0.5x_1 \qquad\quad &+ x_3 = 1, \\ x_1 + 2x_2 &- x_3 = 0, \\ x_1 \qquad\quad &+ x_3 = 0, \end{aligned}$$

   using Gaussian elimination and partial pivoting.

2. Determine the permutation matrix P such that

$$P \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_1 \\ x_3 \\ x_2 \\ x_5 \end{pmatrix}.$$

3. a) Solve the system $Ax = b$ with Gaussian elimination and partial pivoting, for

$$A = \begin{pmatrix} 0.8 & 1.4 & 3 \\ 0.6 & 0.9 & 2.8 \\ 2 & 1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 12.6 \\ 10.8 \\ 4 \end{pmatrix}.$$

b) Determine P, L and U in the LU decomposition $PA = LU$.

4. a) Compute a LU decomposition of the matrix

$$A = \begin{pmatrix} 1.4 & 1.42 & 6.5 \\ 2 & 1 & 1 \\ 0.4 & 1.4 & 3.2 \end{pmatrix},$$

using Gaussian elimination and partial pivoting.

b) Use the decomposition from a) to solve the system $Ax = b$, where $b = (39.58, 11, 22)^T$.

5. When solving systems of equations of the type

$$\begin{pmatrix} 10 & -1 & 1 & -1 & 1 \\ -1 & 5 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 0 & 2 \end{pmatrix} x = \begin{pmatrix} 42.8 \\ 1.5 \\ 9.1 \\ 12.5 \\ 4.7 \end{pmatrix},$$

it is useful to exchange rows 1 and 5, and columns 1 and 5 (Why?). Solve the system after having permuted the columns, and determine the LU decomposition of the permuted matrix.

6. a) Compute the LDL^T decomposition of the matrix

$$A = \begin{pmatrix} 4 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{pmatrix}.$$

b) Compute the Cholesky decomposition.

7. Solve the least squares problem $\min \|Ax - b\|$, where

$$A = \begin{pmatrix} 1 & -3 & 1 \\ 3 & 1 & -11 \\ 1 & -2 & -1 \\ 2 & 1 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

8. Fit a second degree polynomial to the following measured values:

x	1	2	3	6	8
$f(x)$	2.2	1.8	1.7	1.3	0.9

using the least squares method.

9. Let B be a positive definite matrix. Show that $\|x\|_B = (x^T Bx)^{1/2}$ is a vector norm.

10. Let $\|\cdot\|$ be a matrix norm corresponding to a vector norm. Show that if D is a diagonal matrix, $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, then $\|D\| = \max |d_i|$.

11. Let $P = I - 2ww^T$, where $\|w\|_2 = 1$.
    a) Show that P is orthogonal, i.e., $P^T P = I$.
    b) Show that $\|Px\|_2 = \|x\|_2$.

12. Given the system of equations $Ax = b$, where
$$A = \begin{pmatrix} 0.5 & 0.4 \\ 0.3 & 0.25 \end{pmatrix},$$
and where we have only an approximation $\bar{b} = (0.200, 1.000)^T$ of the right hand side. Assuming that the approximation is correctly rounded to three decimals, give an estimate for the uncertainty in the solution, $\|\delta x\|_\infty / \|x\|_\infty$.

13. Given
$$A = \begin{pmatrix} 10^{-3} & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 0.117 \\ 0.352 \\ 0.561 \end{pmatrix}.$$
    a) Compute an LU decomposition of A using Gaussian elimination and partial pivoting.
    b) Use the decomposition from a) to compute A^{-1}.
    c) Assume that the components of b are correctly rounded. Give an upper bound for the relative uncertainty in the solution of $Ax = b$ (use maximum norm).

14. a) Compute an LU decomposition of the matrix
$$A = \begin{pmatrix} 20 & 3 & 4 \\ 3 & 40 & 5 \\ 4 & 5 & 60 \end{pmatrix},$$
using the floating point system $(10, 1, -9, 9)$.
    b) The system $Ax = b$, where $b = (15, -360, 420)^T$, has the approximate solution $x^{(1)} = (0.65, -9.8, 7.8)^T$. Compute a better approximation using one step of iterative refinement.