In the same way as in the one-dimensional case, fixed point iterations can be constructed by rewriting the system in the form

$$x = \varphi(x),$$

where $\varphi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x))^T$. Starting from an initial approximation, one iterates

$$x^{k+1} = \varphi(x^k), \qquad k = 0, 1, 2, \ldots.$$

Theorem 4.3.1, which gives sufficient conditions for convergence, can be generalized to this case. Assume that $x^* = \varphi(x^*)$, and that the partial derivatives

$$d_{ij}(x) = \frac{\partial \varphi_i}{\partial x_j}, \qquad 1 \le i, j \le n,$$

exist for $x \in R = \{x \mid \|x - x^*\| < \rho\}$. Let D be a $n \times n$ matrix with elements $d_{ij}(x)$. A sufficient condition for the fixed point iteration to converge for each $x^0 \in R$ is that, for some matrix norm, we have

$$\|D(x)\| \le m < 1, \qquad x \in R.$$

If this is satisfied, then we have linear convergence:

$$\left\|x^{k+1} - x^*\right\| \le m \left\|x^k - x^*\right\|.$$

Example 4.7.3 We shall determine a solution of

$$x_1 = \frac{1}{5}\sin(x_1 + x_2),$$

$$x_2 = \frac{1}{5}\cos(x_1 - x_2),$$

in the neighborhood of $x = (0.1, 0.2)^T$. The matrix D is

$$D = \begin{pmatrix} \frac{1}{5}\cos(x_1 + x_2) & \frac{1}{5}\cos(x_1 + x_2) \\ -\frac{1}{5}\sin(x_1 - x_2) & \frac{1}{5}\sin(x_1 - x_2) \end{pmatrix}.$$

The condition for convergence $\|D\|_\infty \le m < 1$ is satisfied since

$$\|D\|_\infty \le \frac{1}{5} + \frac{1}{5} = 0.4.$$

The iteration

$$x_1^{k+1} = \frac{1}{5}\sin(x_1^k + x_2^k),$$

$$x_2^{k+1} = \frac{1}{5}\cos(x_1^k - x_2^k),$$

gives

k	x_1^k	x_2^k
0	0.1	0.2
1	0.059104	0.199001
2	0.051050	0.198046
3	0.049306	0.197843
4	0.048928	0.197798
5	0.048846	0.197788
6	0.048828	0.197787
7	0.048824	0.197785
8	0.048824	0.197785

Exercises

1. Illustrate graphically the solution of the equation $f(x) = x - e^{-x}$ by the secant method for $x_0 = 0.55$ and $x_1 = 0.575$. Solve the equation also with Regula Falsi, and illustrate the solution graphically.

2. Show that Newton–Raphson's method has the asymptotic error constant $\frac{1}{2}f''(x^*)/f'(x^*)$.

3. Compute the root of the equation

$$f(x) = \sqrt{x} - e^{-x} = 0,$$

with five correct decimals. Use Newton–Raphson's method.

4. Assume that our computer implements the IEEE single precision standard, and that standard functions are computed with the accuracy

$$\mathrm{fl}[\sqrt{\mathrm{x}}] = \sqrt{x}(1 + \epsilon_1),$$
$$\mathrm{fl}[\mathrm{e}^{-\mathrm{x}}] = e^{-x}(1 + \epsilon_2),$$

where $|\epsilon_i| \le \mu$, $i = 1, 2$ (μ denotes the unit roundoff). How accurately can the equation in Exercise 3 be solved?

5. Let

$$f(x) = \cos(\beta x) - x, \quad \beta = 2.7332 \pm 0.5 \cdot 10^{-4}.$$

How accurately can we solve the equation? Determine the root with an absolute error that does not exceed the attainable accuracy by more than 10%.

6. Determine λ so that the fixed point iteration

$$x_{n+1} = \frac{\lambda x_n + 1 - \sin x_n}{1 + \lambda}$$

converges as fast as possible to the root of the equation

$$1 - x - \sin x = 0.$$

Compute the root with six correct decimals.

7. If a processor does not have hardware division, one can implement division by solving the equation $f(x) = 1/x - a = 0$ using Newton–Raphson's method. On some computers (e.g., Cray X-MP) this method is used for the hardware implementation of division. Analyze this algorithm in the same way as the square root algorithm is analyzed in Section 4.6. In particular, investigate how the number of iterations depends on the table size.

8. Compute the positive root of the equation

$$f(x) = \int_0^x \cos t \, dt - 0.5 = 0$$

with five correct decimals. Hint: Solve the equation $p(x) = 0$, where $p(x)$ is a partial sum of the Maclaurin expansion of $\cos t$ integrated term by term. Then use the method–independent error estimate for the original equation.

9. Show that the fixed point iteration $x_{n+1} = \varphi(x_n)$ does not converge if $|\varphi'(x)| > 1$ in an interval around the required root x^*. This does not necessarily mean that the iteration diverges towards infinity. Try the iteration $x_{n+1} = 1 - \lambda x_n^2$, for $\lambda = 0.7$, 0.9 and 2. Illustrate the iterates as a function of n. For $\lambda = 2$, the iteration exhibits a chaotic behavior.

10. Use Newton–Raphson's method to determine the largest root of the equation $f(x) = 16x^3 - 132x^2 - 12x + 99 = 0$. Compute function values and derivatives with Horner's rule. When the largest root has been computed to six significant digits, perform synthetic division and compute the remaining two roots.

11. The equation $f(x) = x^3 - 7.5x^2 + 18x - 14$ has a double root $x^* = 2$. If we apply Newton–Raphson's method to this equation with $x_0 = 2$, we get quite slow convergence

n	x_n	$f(x_n)$	$f'(x_n)$
0	1	−2.5	6
1	1.42	−0.7	2.77
2	1.67	−0.20	1.30
⋮			
9	1.997	$-1.4 \cdot 10^{-5}$	0.0093

a) Show that the following error estimate holds for double roots

$$|\overline{x} - x^*|^2 \le \left| \frac{2f(\overline{x})}{Q} \right|,$$

where $|f''(x)| \ge Q$ close to the root.

b) Estimate the error in the approximation x_9.

c) Show that Newton–Raphson's method converges linearly for double roots.

12. The equation $\sin(xy) = y - x$ defines y implicitly as a function of x. The function $y(x)$ has a maximum for $(x, y) \approx (1, 2)$. Show that the coordinates for this maximum can be found by solving the nonlinear system of equations

$$\sin(xy) - y + x = 0,$$
$$y \cos(xy) + 1 = 0.$$

Compute an approximation of the coordinates of the maximum point by iterating once with Newton–Raphson's method.

References

The classical theory for the solution of equations is given in

A. S. Householder, *The Numerical Treatment of a Single Nonlinear Equation*, McGraw–Hill, New York, 1970.

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R. P. Brent, *Algorithms for Minimization without Derivatives*, Prentice–Hall, Englewood Cliffs, New Jersey, 1972.