Richardson extrapolation can also be considered as extrapolation using an interpolating polynomial for $F(h)$. We conclude the section by demonstrating this in a special case.

Assume that

$$F(h) = a_0 + a_1h^2 + a_2h^4 + a_3h^6 + \dots,$$

and that the values $F(4h_0), F(2h_0)$ and $F(h_0)$ have been computed. It is natural to approximate F by the polynomial of the form $c_0 + c_1h^2 + c_2h^4$ which interpolates F in the three known points. The value of this polynomial for $h = 0$ is used to approximate $F(0)$. It is convenient to use Neville's method here (see Section 5.5). If we put $h^2 = x$, then we shall compute the value, for $x = 0$, of the polynomial $p(x) = c_0 + c_1x + c_2x^2$ through the points $((4h_0)^2, F(4h_0)), ((2h_0)^2, F(2h_0))$ and $(h_0^2, F(h_0))$. We get the scheme

$0-x$	x	$p(x)$		
$-(4h_0)^2$	$(4h_0)^2$	F_{00}		
			F_{11}	
$-(2h_0)^2$	$(2h_0)^2$	F_{10}		F_{22}
			F_{21}	
$-h_0^2$	h_0^2	F_{20}		

Here $F_{i0} = F(2^{2-i}h_0)$, $i = 0, 1, 2$ are known. The other values are computed according to the rules of Neville's method:

$$F_{11} = \frac{1}{(2h_0)^2 - (4h_0)^2} \begin{vmatrix} -(4h_0)^2 & F_{00} \\ -(2h_0)^2 & F_{10} \end{vmatrix} = F_{10} + \frac{1}{3}(F_{10} - F_{00}),$$

$$F_{21} = \frac{1}{h_0^2 - (2h_0)^2} \begin{vmatrix} -(2h_0)^2 & F_{10} \\ -h_0^2 & F_{20} \end{vmatrix} = F_{20} + \frac{1}{3}(F_{20} - F_{10}),$$

$$F_{22} = \frac{1}{h_0^2 - (4h_0)^2} \begin{vmatrix} -(4h_0)^2 & F_{11} \\ -h_0^2 & F_{21} \end{vmatrix} = F_{21} + \frac{1}{15}(F_{21} - F_{11}).$$

We see that we obtain exactly the same values as we did earlier in Richardson extrapolation.

There are also extrapolation methods based on interpolation with rational functions. In some applications, they give better approximations to $F(0)$ than methods based on interpolation by polynomials.

Exercises

1. a) Show that the truncation error in the approximation of $f'(x)$ by the central difference formula is $h^2 f^{(3)}(\xi)/6$, where $\xi \in (x-h, x+h)$.
   b) Show that the total error in the approximation in a) is minimized for $h = \sqrt[3]{3\epsilon/|f^{(3)}(\xi)|}$, where $\xi \in (x-h, x+h)$, and ϵ is an upper bound for the absolute error in the function values. (Here we neglect that ξ depends on h.)
   c) Compute an optimal step length h for approximating $f'(1)$, when $f(x) = e^x$, and the function values can be computed with a relative error smaller than or equal to the unit roundoff 2^{-27}. Estimate the total error in the approximation of the derivative. Compare the result to Example 6.3.3.

2. Show that the truncation error in the approximation

$$\frac{1}{2h^3}(f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h))$$

to $f^{(3)}(x)$ is $R_T = a_1h^2 + a_2h^4 + a_3h^6 + \dots$.

3. The following table of a function $f(x)$ is given:

x	$f(x)$
0.6	0.564642
0.8	0.717356
0.9	0.783327
1	0.841471
1.1	0.891207
1.2	0.932039
1.4	0.985450

Compute approximations to $f'(1)$ and $f''(1)$ using repeated Richardson extrapolation. (We have chosen $f(x) = \sin x$ in this example. How well do your results match the exact values?)

4. The expression

$$\frac{1}{12h}(f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h))$$

approximates $f'(x)$ with truncation error $R_T = a_4h^4 + a_6h^6 + a_8h^8 + \dots$. Assume that we know the following correctly rounded function values, and want to compute $f'(0.5)$ as accurately as possible.

x	$f(x)$
0.1	0.000167
0.3	0.004480
0.4	0.010582
0.425	0.012679
0.450	0.015034
0.475	0.017662
0.5	0.020574
0.525	0.023787
0.550	0.027313
0.575	0.031165
0.6	0.035358
0.7	0.055782
0.9	0.116673

Use the above difference approximation and Richardson extrapolation to approximate $f'(0.5)$. Estimate the error in the approximations. Why does it not pay off to use Richardson extrapolation in this case?

5. A car starts from standstill, and its speedometer is read every second during acceleration, giving the following readings.

$t(s)$	0	1	2	3	4	5	6	7	8	9
$v(m/s)$	0	3.61	7.22	10.10	12.50	14.62	16.60	18.06	19.54	20.28

What is the acceleration after $4s$? We assume that the speedometer has an error of less than 1%. Use the table and a central difference to compute an approximation with as little total error as possible.

7 Numerical Integration

7.1 Introduction

Numerical integration or quadrature is the computation of approximations to $\int_a^b f(x)\,dx$. We shall derive methods for this.

When are such methods needed? If the integrand f is known only in discrete points, of course. But even if an explicit formula for f is given, it may be impossible to compute the integral exactly because a primitive function cannot be found. This is the case for $\int_0^1 e^{-x^2}\,dx$ and $\int_0^{\pi/2}\sqrt{1+\cos^2 x}\;dx$.

Finally—even if a primitive function is known, the computation of this function may be so costly, that approximate, numerical methods for computing the integral are to be preferred.

We construct methods for numerical integration by approximating f by a function that can easily be integrated, a polynomial. There are two possible ways to obtain good accuracy:

a) approximating f by a single interpolation polynomial of high degree;
b) approximating f by different polynomials of low degree in small subintervals of the interval of integration.

We shall see that method b) is to be preferred.

7.2 Newton–Cote's Quadrature Formulas

Newton–Cote's quadrature formulas are derived by replacing the integrand