The series to the right converges so slowly that in this case it is better to use another method.

Another type of difficulty occurs if the interval of integration is infinite. Sometimes a change of variables can be used to transform the interval into a finite one. E.g., the interval $(0, \infty)$ is transformed into $(1, 0)$ by $t = 1/(x+1)$ or $t = e^{-x}$.

In some cases an infinite interval of integration can be approximated by a finite interval. If, e.g., $\int_0^\infty e^{-x^2}\,dx$ is approximated by $\int_0^a e^{-x^2}\,dx$, we make a truncation error

$$R_{\mathrm{T}} = \int_a^\infty e^{-x^2}\,dx < \frac{1}{a}\int_a^\infty x e^{-x^2}\,dx = \frac{1}{2a}e^{-a^2}.$$

For $a = 5$, we have $R_{\mathrm{T}} < 1.4 \cdot 10^{-12}$.

A third difficulty occurs if the integrand is oscillating heavily. Special methods have been developed for this case.

7.5 Adaptive Quadrature

In many cases, our goal is to approximate a given integral to a certain accuracy using as few function evaluations as possible. If the integral varies rapidly in part of the interval of integration, then a small step length is needed to get a satisfactory accuracy in this subinterval, while in the rest of the interval it would be possible to get the same accuracy with a much larger step. To avoid making unnecessarily many function evaluations, we must adjust the step length to the behaviour of the integrand. This is done in adaptive quadrature.

For instance, we can use the following principle: approximations I_1 and I_2 of the integral are computed using Simpson's rule with step lengths $(b-a)/2$ and $(b-a)/4$. Then I_3 is computed using extrapolation. If $|I_3 - I_2|/|I_2| < \epsilon$, where ϵ is a given tolerance, then I_3 is accepted as an approximation of the integral. Otherwise the interval (a, b) is partitioned in two parts (a, m) and (m, b), where $m = (a+b)/2$. The integrals over these subintervals are then computed independently. With this procedure, a fine partitioning is made only where it is needed to obtain the required accuracy. Using a recursive function, this can be programmed very easily.

Consider, as an example,

$$\int_0^4 \frac{32}{1 + 1024x^2}\,dx.$$

The integrand has a maximum at the origin and decreases very rapidly to zero. Using a program for adaptive quadrature, working according to the principles outlined above, 33 function evaluations were needed to compute the integral with three correct decimals. The table below shows the smallest step length that was used in successive subintervals.

Interval		Step Length
0	0.0625	1/128
0.0625	0.125	1/64
0.125	0.25	1/32
0.25	0.5	1/16
0.5	1	1/8
1	2	1/4
2	4	1/2

We see that very small step lengths were needed close to the origin. To compute the integral with three correct decimals using Romberg's method, 257 evaluations of the integrand were needed (in fact, the best value was obtained with the trapezoidal rule for $h = 1/64$, without extrapolation.)

Exercises

1. Compute $\int_0^1 t^3\,dt$ using Romberg's method. Use the step lengths $h = 1$ and $h = \frac{1}{2}$. Explain why the result agrees with the exact value.

2. A car is started in cold weather, and a flux meter is used to measure the consumption of gasoline. The following values are obtained.

x	0	1.25	2.5	3.75	5	6.25	7.5	8.75	10
$f(x)$	0.260	0.208	0.172	0.145	0.126	0.113	0.104	0.097	0.092

(x is the distance (km), $f(x)$ is the instantaneous flux (liters/km).) The values of $f(x)$ are assumed to be correctly rounded. Compute the gas consumption for 10 km with an absolute error smaller than 10^{-2}.

3. Consider the integral

$$F(a) = \int_0^1 e^{-a^2x^2}\,dx.$$

a) Compute $F(1)$ with an absolute error not larger than 10^{-5}.

b) Assume that $a = 1 \pm \epsilon$, where $\epsilon \ll 1$. Give a bound for the error that arises because we use the approximation $\bar{a} = 1$ for the computation of $F(a)$.

4. Compute

$$I = \int_0^4 \frac{dx}{1 + 5xe^{x^2}}$$

with two correct decimals. Sketch the integrand, and use Romberg's method on separate subintervals.

5. To compute the integral

$$\int_0^1 \frac{\sqrt{x}}{4 - x^2}\, dx,$$

we used a program for Romberg's method based on formula (7.3.1). The following results were obtained.

h	$T(h)$			
0.5	0.17761420			
0.25	0.18353680	0.18551100		
0.125	0.18622450	0.18712040	0.18722770	
0.0625	0.18732990	0.18769830	0.18773690	0.18774500

a) Explain the slow convergence.
b) Compute a better approximation to the integral using the values in the table.

6. Suggest different methods for the numerical computation of

$$\int_0^1 \frac{\sin x}{\sqrt{1 - x^2}}\, dx.$$

7. Suggest a method for the numerical computation of

$$\int_0^\infty \frac{e^{-x}}{1 + xe^{-x}}\, dx.$$

References

For a more extensive treatment of extrapolation methods for numerical integration, we refer to

J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis*, Springer Verlag, 1983.

A recursive Pascal program for adaptive quadrature is given in

W. Gander, *Computermathematik*, Birkhäuser Verlag, 1985.