(1)

(a). Per diferències dividides, calculem $p(x)$,

$$\begin{array}{llllll}
a & f_a & & & & \\
 & & f'_a & & & \\
a & f_a & & \frac{f_b-f_a}{(b-a)^2}-\frac{f'_a}{b-a} & & \\
 & & \frac{f_b-f_a}{b-a} & & \frac{f'_b}{(b-a)^2}-2\frac{f_b-f_a}{(b-a)^3}+\frac{f'_a}{(b-a)^2} & \\
b & f_b & & \frac{f'(b)}{b-a}-\frac{f_b-f_a}{(b-a)^2} & & \\
 & & f'_b & & & \\
b & f_b & & & &
\end{array}$$

$$\Rightarrow p(x) = f_a + f'_a(x-a) + \left[\frac{f_b-f_a}{(b-a)^2}-\frac{f'_a}{b-a}\right](x-a)^2 + \left[\frac{f'_b+f'_a}{(b-a)^2} - 2\frac{f_b-f_a}{(b-a)^3}\right](x-a)^2(x-b)$$

Avaluem $p(x)$ en $x = z = \frac{a+b}{2}$ i usem $b-a=h$, $z-a=\frac{h}{2}$; $z-b=-\frac{h}{2}$:

$$p(z) = f_a + f'_a\frac{h}{2} + \left[\frac{f_b-f_a}{h^2}-\frac{f'_a}{h}\right]\frac{h^2}{4} + \left[\frac{f'_b+f'_a}{h^2}-2\frac{f_b-f_a}{h^3}\right]\left(-\frac{h^3}{8}\right) =$$

$$= \left[f_a + \frac{f_b-f_a}{4} + 2\frac{f_b-f_a}{8}\right] + \frac{h}{2}\left[f'_a - \frac{1}{2}f'_a - \frac{1}{4}(f'_b+f'_a)\right] =$$

$$= \left[f_a\underbrace{\left(1-\frac{1}{4}-\frac{2}{8}\right)}_{\frac{1}{2}} + f_b\underbrace{\left(\frac{1}{4}+\frac{2}{8}\right)}_{\frac{1}{2}}\right] + \frac{h}{2}\left[f'_a\underbrace{\left(1-\frac{1}{2}-\frac{1}{4}\right)}_{\frac{1}{4}} - \frac{1}{4}f'_b\right] = \boxed{\frac{1}{2}(f_a+f_b) + \frac{h}{8}(f'_a-f'_b)}$$

Ok!

(b) $$f(x)-p(x) = \frac{f^{(4)}(\xi(x))}{4!}(x-a)^2(x-b)^2 \quad \forall x$$

Ho apliquem en $x=z$ i usem $z-a=\frac{h}{2}$, $z-b=-\frac{h}{2}$, $|f^{(4)}(\xi)| \le 5M$:

$$|f(z)-p(z)| \le \frac{5M}{4!}\left(\frac{h}{2}\right)^4 = \boxed{\frac{5M\cdot h^4}{384}}$$

(c) Si $f(x) = x\cdot e^x$ llavors $f'(x) = (1+x)e^x$, $f''(x) = (2+x)e^x$, $f'''(x) = (3+x)e^x$, etc.

Quan $x\in[0,1]$, es verifica la suposició de (b) prenent $M=e$. A més, $h=1$.

Per tant,

- usant (a): $p(0.5) = \frac{1}{2}(0\cdot e^0 + 1\cdot e^1) + \frac{1}{8}(1\cdot e^0 - 2\cdot e^1) = \frac{e}{2}+\frac{1}{8}(1-2e) = \boxed{\frac{1}{8}+\frac{1}{4}e}$

$\approx 0,80457$

- usant (b): $|f(z)-p(z)| \le \frac{5\cdot e\cdot 1^4}{384} = \boxed{\frac{5e}{384}} \approx 0.03539$

②

(a) Usem els desenvolupaments de Taylor

$$f(a+h)=f(a)+f'(a)h+\frac{1}{2}f''(a)h^2+\frac{1}{6}f'''(a)h^3+O(h^4)$$
$$f(a+2h)=f(a)+2f'(a)h+\frac{4}{2}f''(a)h^2+\frac{8}{6}f'''(a)h^3+O(h^4)$$

Llavors (no escrivim (a) per a simplificar la notació):

$$h\cdot D(h)=A\cdot f+B\left(f+hf'+\frac{1}{2}h^2f''+\frac{1}{6}h^3f'''+O(h^4)\right)+C\left(f+2hf'+\frac{4}{2}h^2f''+\frac{8}{6}h^3f'''+O(h^4)\right)=$$
$$=f\cdot(A+B+C)+h\cdot f'\cdot(B+2C)+\frac{h^2}{2}f''\cdot(B+4C)+\frac{h^3}{6}f'''\cdot(B+8C)+O(h^4)$$

Com que volem $h\cdot D(h)\simeq h\cdot f'(a)+c_2\cdot h^3+O(h^4)$, imposem $\begin{cases}A+B+C=0\\ B+2C=1\\ B+4C=0\end{cases}$

Aquest sistema té solució única $\boxed{A=-\frac{3}{2},\ B=+2;\ C=-\frac{1}{2}}$

A més, per a aquesta solució, és $B+8C=-2$. De manera que

$$D(h)\equiv\frac{1}{h}\left[-\frac{3}{2}f(a)+2f(a+h)-\frac{1}{2}f(a+2h)\right]=f'(a)+\frac{h^2}{6}f'''(a)\cdot(-2)+O(h^3)$$
$$=f'(a)-\frac{f'''(a)}{3}h^2+O(h^3)$$

Per tant, $\boxed{c_2=-\frac{f'''(a)}{3}}$

(b) Prenem $a=0$ i $h=0.1$ i 0.2

$$h=0.1\Rightarrow D(0.1)=\frac{1}{0.1}\left[-\frac{3}{2}(1)+2(0.91)-\frac{1}{2}(0.83)\right]=\boxed{-0.95}$$
$$h=0.2\Rightarrow D(0.2)=\frac{1}{0.2}\left[-\frac{3}{2}(1)+2(0.83)-\frac{1}{2}(0.71)\right]=\boxed{-0.975}$$

Notem que no hem usat $f(0.3)$ enlloc.

Extrapolació: $\left.\begin{array}{l}D(h)=f'(a)+c_2h^2+O_3\\ D(2h)=f'(a)+4c_2h^2+O_3\end{array}\right\}\Rightarrow\boxed{\dfrac{4D(h)-D(2h)}{3}}=f'(a)+O_3$

$\equiv E(h)$ fórmula que cal usar

$$E(0.1)=\frac{4(-0.95)-(-0.975)}{3}=\boxed{-0.941\hat{6}}$$

(c) Calculem

x	$f(x)$
0	0.0000
0.1	0.0001
0.2	0.0016
0.4	0.0256
0.8	0.4096

llavors $\boxed{\begin{array}{l}D(0.1)=-0.006\\ D(0.2)=-0.048\\ D(0.4)=-0.384\end{array}}$ valors aproximats $\equiv$ errors

Efectivament $\frac{D(0.4)}{D(0.2)}=\frac{D(0.2)}{D(0.1)}=8$

Explicació. En aquest cas concret, resulta que $c_2=-\frac{f'''(0)}{3}=0$!

Si es calculen més termes de l'error, resulta $\boxed{D(h)=f'(a)-\frac{f^{IV}(0)}{4}\cdot h^3}$

(3) $I=[0,1]$, $g(x)=\frac{1}{3}(5x^3-7x^2+x+2)$ contínua i derivable tant com calgui.

(a) $g(I)\subset I$?

$$\left.\begin{array}{l} g(0)=\frac{2}{3}\in I \\ g(1)=\frac{1}{3}(5-7+1+2)=\frac{1}{3}\in I \\ g'(x)=\frac{1}{3}(15x^2-14x+1)\ ;\quad g'(x)=0 \Leftrightarrow x=\frac{7\pm\sqrt{34}}{15}=\begin{cases}\approx 0{,}855 \in I \\ \approx 0{,}078 \in I\end{cases} \\ g(0{,}855)\approx 0{,}28765\ldots \in I \quad \text{(mínim relatiu i absolut)} \\ g(0{,}078)\approx 0{,}67926\ldots \in I \quad \text{(màxim relatiu i absolut)} \end{array}\right\} \Rightarrow \text{ok!}$$

(b) Busquem el màxim de $|g'(x)|$ a $x\in[0,1]$, on $g'(x)=\frac{1}{3}(15x^2-14x+1)$

$$\left.\begin{array}{l} g'(0)=\frac{1}{3} \\ g'(1)=\frac{1}{3}(15-14+1)=\frac{2}{3} \\ 0=g''(x)=\frac{1}{3}(30x-14) \Leftrightarrow x=\frac{7}{15}\quad ;\ i\ \ g'(\frac{7}{15})=\frac{1}{3}\left(15\cdot\frac{49}{15^2}-14\cdot\frac{7}{15}+1\right)=-\frac{34}{45} \end{array}\right\}$$

Llavors, $L=\max\{|g'(0)|,|g'(1)|,|g'(\frac{7}{15})|\}=\frac{34}{45}$ ok!

(c) $f(x)=0 \Leftrightarrow x=g(x)$

Usant (a) i (b), g és una contracció en I $\Rightarrow \exists!$ arrel

(d) Usem $|x_n-\alpha|\le\frac{L^n}{1-L}|x_1-x_0|$ amb $L=34/45$

$x_0=\frac{1}{2} \Rightarrow x_1=\frac{1}{3}\left[\frac{5}{8}-\frac{7}{4}+\frac{1}{2}+2\right]=\frac{11}{24} \Rightarrow |x_1-x_0|=\frac{1}{24}$

És suficient imposar $\frac{(34/45)^n}{11/45}\cdot\frac{1}{24}\le\frac{1}{2}10^{-6} \Leftrightarrow \left(\frac{34}{45}\right)^n\le\frac{11\cdot 12}{45}\cdot 10^{-6}$

$\Leftrightarrow n\left(\log_{10}34/45\right)\le\log_{10}\left(\frac{132}{45}\right)-6 \Leftrightarrow n\geqslant 45.448\ldots$

$\boxed{n\geqslant 46}$ és suficient.

Nota. També podem usar $|x_n-\alpha|\le L^n|x_0-\alpha|$ i $|x_0-\alpha|\le\frac{1}{2}$

Imposant $\left(\frac{34}{45}\right)^n\cdot\frac{1}{2}\le\frac{1}{2}10^{-6}$ s'obté $n\geqslant\frac{-6}{\log(34/45)}\approx 49{,}3 \Leftrightarrow \boxed{n\geqslant 50}$

(e)

$$\left.\begin{array}{l} f(x)=5x^3-7x^2-2x+2 \\ f'(x)=15x^2-14x-2 \end{array}\right\}\ x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}$$

$x_0=0{,}5$

$x_1=0.5-\frac{-0.125}{-5.25}=0.476190\ldots\approx\boxed{0.47619} \Rightarrow x_2=0.47619-\frac{0.218466\ldots\times 10^{-3}}{-5.265306\ldots}\approx\boxed{0.47623}$

$\Rightarrow x_3=0.47623-\frac{0.7854\times 10^{-5}}{-5.2652948\ldots}\simeq\boxed{0.47623}$. Efectivament $\tilde{x}_3=\tilde{x}_2$.