1) a) Sigui $z \in [x_i, x_{i+1}]$

$$|f(z) - P_1(z)| = \left|\frac{f''(\xi)}{2}(z-x_i)(z-x_{i+1})\right| \leq \frac{1}{2}\left[\max_{x\in[1,4]}|f''(x)|\right] \cdot \max_{z\in[x_i,x_{i+1}]}|z-x_i||z-x_{i+1}| = \frac{1}{2}\,\frac{1}{4}\,\frac{h^2}{4}$$

* $f(x)=x^{1/2}$; $f'(x)=\frac{1}{2}x^{-1/2}$; $f''(x)=-\frac{1}{4}x^{-3/2}$ $\Rightarrow$ $|f''(x)| = \left|\frac{1}{4\sqrt{x^3}}\right| \leq \frac{1}{4}$ $\forall x\in[1,4]$

* $|z-x_i||z-x_{i+1}| \leq \left(\frac{h}{2}\right)^2$ $\forall z\in[x_i,x_{i+1}]$

Per tant: És suficient imposar $\frac{1}{32}h^2 \leq \frac{1}{2}10^{-8}$ $\underset{h=\frac{3}{n}}{\Leftrightarrow}$ $\boxed{n\geq 7500}$

b) De manera semblant:

$$|f(z)-P_3(z)| = \left|\frac{f^{IV}(\eta)}{4!}(z-x_i)^2(z-x_{i+1})^2\right| \leq \frac{1}{24}\left[\max_{x\in[1,4]}|f^{IV}(x)|\right]\cdot \max_{z\in[x_i,x_{i+1}]}|(z-x_i)(z-x_{i+1})|^2 = \frac{1}{24}\,\frac{15}{16}\,\frac{h^4}{16}$$

* $f'''(x)=\frac{3}{8}x^{-5/2}$; $f^{IV}(x) = -\frac{15}{16}x^{-7/2}$ $\Rightarrow$ $|f^{IV}(x)| = \left|\frac{15}{16\sqrt{x^7}}\right| \leq \frac{15}{16}$ $\forall x\in[1,4]$

Ara, és suficient imposar $\frac{15}{6144}h^4 \leq \frac{1}{2}10^{-8}$ $\underset{h=\frac{3}{n}}{\Leftrightarrow}$ $n \geq \sqrt[4]{\frac{5}{4}}\cdot 75 \approx 79.3$ $\Leftrightarrow$ $\boxed{n\geq 80}$

2) ⓐ Per Taylor (amb abús de notació: no posem (a))

$$f(a) = f$$
$$f(a+h) = f + f' \cdot h + \frac{1}{2} f'' \cdot h^2 + \frac{1}{6} f''' \cdot h^3 + \frac{1}{24} f^{IV} \cdot h^4 + \dots$$
$$f(a+3h) = f + 3f' \cdot h + \frac{9}{2} f'' \cdot h^2 + \frac{27}{6} f''' \cdot h^3 + \frac{81}{24} f^{IV} \cdot h^4 + \dots$$

Llavors:

$$\frac{Af(a) + Bf(a+h) + Cf(a+3h)}{h} = \frac{1}{h}(A+B+C) \cdot f(a) + (B+3C) f'(a) + (B+9C)\frac{1}{2} f''(a) h + (B+27C)\frac{1}{6} f'''(a) h^2 + \dots$$

Imposem, doncs:

$$\left.\begin{aligned} A+B+C &= 0 \\ B+3C &= 1 \\ B+9C &= 0 \end{aligned}\right\} \Rightarrow \boxed{C = -\frac{1}{6},\ B = \frac{9}{6},\ A = -\frac{8}{6}}$$

Llavors: $B + 27C = \frac{9}{6} - \frac{27}{6} \neq 0$. Per tant,

$$\underbrace{\frac{-8f(a) + 9f(a+h) - f(a+3h)}{6h}}_{\substack{|||\\ F(h)}} = f'(a) + Kh^2 + O(h^3)$$

$(K \neq 0$ si $f'''(a) \neq 0)$

ⓑ

$$f'(0) \simeq F(0.1) = \frac{-8(1) + 9(1.05) - (1.14)}{6 \cdot (0.1)} = \frac{0.31}{0.6} = \boxed{0.51\hat{6}}$$

$$f'(0) \simeq F(0.3) = \frac{-8(1) + 9(1.14) - (1.38)}{6 \cdot (0.3)} = \frac{0.88}{1.8} = \boxed{0.4\hat{8}}$$

Extrapolació:

$$\left.\begin{aligned} F(h) &= f'(a) + Kh^2 + O(h^3) \\ F(3h) &= f'(a) + 9Kh^2 + O(h^3) \end{aligned}\right\} \Rightarrow 9F(h) - F(3h) = 8 \cdot f'(a) + O(h^3)$$

$$\Leftrightarrow \underbrace{\frac{9F(h) - F(3h)}{8}}_{\substack{|||\\ G(h)}} = f'(a) + O(h^3)$$

Obtenim:

$$G(0.1) = \frac{9 \cdot (0.51\hat{6}) - (0.4\hat{8})}{8} = \frac{4.16\hat{1}}{8} = \boxed{0.52013\hat{8}}$$

3) ⓐ $f(x) = x - 3\sin(x) - \frac{1}{2}$

$\sin(x) \in [-1,+1]$. Per tant, $\begin{cases} \text{si } x > \frac{7}{2} \text{ llavors } f(x) > \frac{7}{2} - 3 - \frac{1}{2} = 0 \\ \text{si } x < -\frac{5}{2} \text{ llavors } f(x) < -\frac{5}{2} + 3 - \frac{1}{2} = 0 \end{cases}$

$\Rightarrow$ $\boxed{\text{Totes les arrels de } f(x)=0 \text{ són a l'interval } I = \left[-\frac{5}{2}, +\frac{7}{2}\right]}$

A més, $f(-\frac{5}{2}) \approx -1.2 < 0$; $f(+\frac{7}{2}) \approx 4.05 > 0$ $\Rightarrow$ Com a mínim hi ha una arrel a I

Estudiem els extrems relatius:

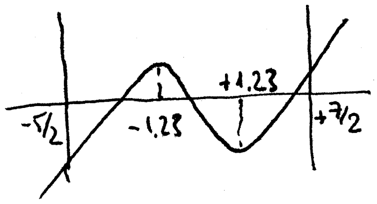
$0 = f'(x) = 1 - 3\cos(x) \Leftrightarrow \cos(x) = \frac{1}{3} \Leftrightarrow x \approx \pm 1.23 + 2k\pi \quad \forall k \in \mathbb{Z}$ (possibles extrems)

$f''(x) = 3\sin(x) \underset{\uparrow}{\neq} 0$ $\Rightarrow$ Són efectivament extrems relatius.

si $\cos(x) = 1/3$

Els únics que hi ha a I són $\begin{cases} x \approx -1.230959 \text{ ; màxim i } f(x) \approx +1.1 > 0 \\ x \approx +1.230959 \text{ ; mínim i } f(x) \approx -2.1 < 0 \end{cases}$

Per tant, la gràfica de $f(x)$ és:



$\Rightarrow$ $\boxed{\exists \text{ exactament 3 arrels: } \begin{cases} \alpha_1 \in (-2{,}5, -1.23\ldots) \\ \alpha_2 \in (-1.23\ldots, +1.23\ldots) \\ \alpha_3 \in (+1.23, +3{,}5) \end{cases}}$

ⓑ Fem N-R des de $x_0 = +3{,}5$

$$x_{k+1} = x_k - \frac{x_k - 3\sin(x_k) - 0.5}{1 - 3\cos(x_k)} \quad \Rightarrow \quad \begin{aligned} x_1 &= 2{,}4362\ldots \\ x_2 &= 2{,}438879\ldots \\ \boxed{x_3} &\boxed{= 2{,}43887702084\ldots} \end{aligned}$$

ⓒ

1. $x = g_1(x) \Leftrightarrow x = 3\sin(x) + \frac{1}{2} \Leftrightarrow f(x) = 0$ ok!

2. $x = g_2(x) \Leftrightarrow x = \frac{3 + 14x + 18\sin(x)}{20} \Leftrightarrow 0 = 3 - 6x + 18\sin(x) \Leftrightarrow 0 = 1 - 2x + 6\sin(x)$

$\Leftrightarrow f(x) = 0$ ok!

2. Tenim dues possibles iteracions: $x_{n+1} = g_1(x_n)$ o $x_{n+1} = g_2(x_n)$ $\forall n \geq 0$

Prop de l'arrel α, hi ha convergència si $|g_i'(\alpha)| < 1$

$\alpha \approx 2{,}438877$

$g_1'(x) = 3\cos(x) \Rightarrow |g_1'(\alpha)| \approx |-2.29| > 1 \Rightarrow$ divergent

$g_2'(x) = \frac{1}{20}(14 + 18\cos(x)) \Rightarrow |g_2'(\alpha)| \approx |+0.01322| < 1 \Rightarrow$ convergent

$\Rightarrow$ $\boxed{\text{és millor usar } x_{n+1} = g_2(x_n)}$