CHAPTER 2

Boolean Algebra

Contents

Objectives

Topics introduced in this chapter:

- Understand the basic operations and laws of Boolean algebra
- Relate these operations and laws to AND, OR, NOT gates and switches
- Prove these laws using a truth table
- Manipulation of algebraic expression using
  - Multiplying out
  - Factoring
  - Simplifying
  - Finding the complement of an expression

2.1 Introduction

- Basic mathematics for logic design: Boolean algebra
- Restrict to switching circuits(Two state values 0, 1) – Switching algebra
- Boolean Variable : X, Y, … can only have two state values (0, 1)

  \- representing True(1) False (0)

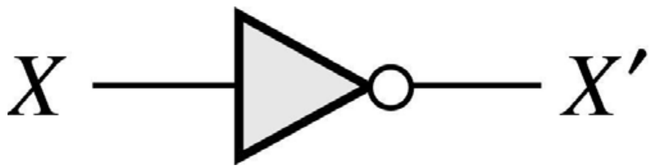
2.2 Basic Operations

NOT(Inverter)

$$0' = 1 \qquad \text{and} \qquad 1' = 0$$

$$X' = 1 \quad \text{if} \quad X = 0 \qquad \text{and} \qquad X' = 0 \quad \text{if} \quad X = 1$$

Gate Symbol



X\` = Not X

2.2 Basic Operations

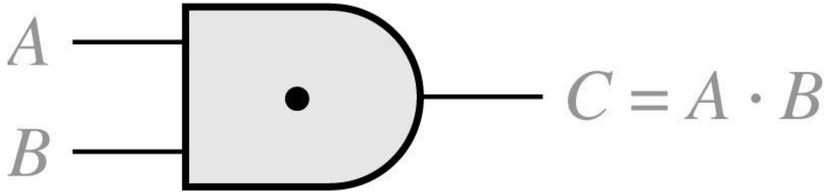
AND

$$0 \cdot 0 = 0 \qquad 0 \cdot 1 = 0 \qquad 1 \cdot 0 = 0 \qquad 1 \cdot 1 = 1$$

Truth Table

A B	C = A · B
0 0	0
0 1	0
1 0	0
1 1	1

Gate Symbol



2.2 Basic Operations

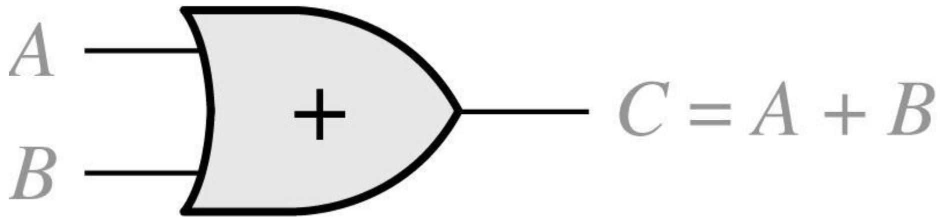
OR

$0+0=0$ $0+1=1$ $1+0=1$ $1+1=1$

Truth Table

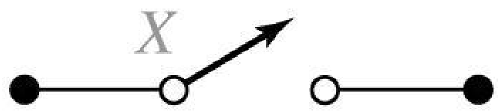
A B	C = A + B
0 0	0
0 1	1
1 0	1
1 1	1

Gate Symbol



2.2 Basic Operations

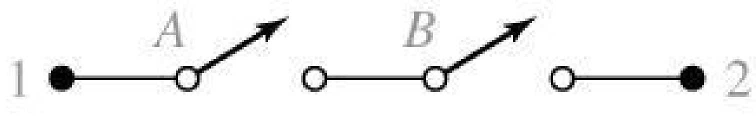
Apply to Switch



$X = 0 \rightarrow$ switch open

$X = 1 \rightarrow$ switch closed

AND $T = A \cdot B$



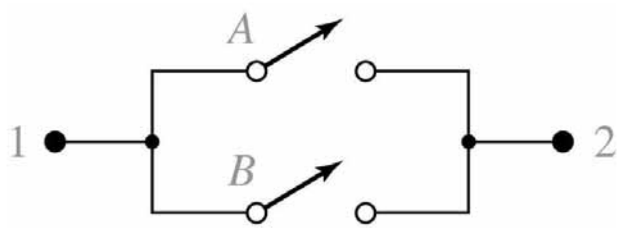
$T = 0 \rightarrow$ open circuit between terminals 1 and 2

$T = 1 \rightarrow$ closed circuit between terminals 1 and 2

T = 0 : switch A or switch B is open : A=0 or B=0

T = 1 : switch A and switch B is closed : A=1 and B=1

OR $T = A + B$



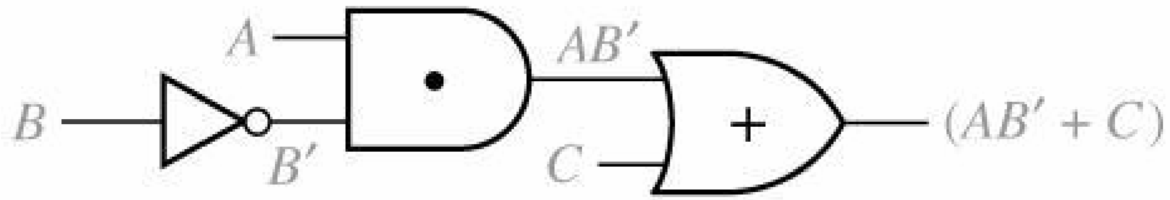
T = 0 : switch A and switch B is open : A=0 and B=0

T = 1 : switch A or switch B is closed : A=1 or B=1

2.3 Boolean Expressions and Truth Tables

Logic Expression : $[A(C+D)]'+BE$

Circuit of logic gates :



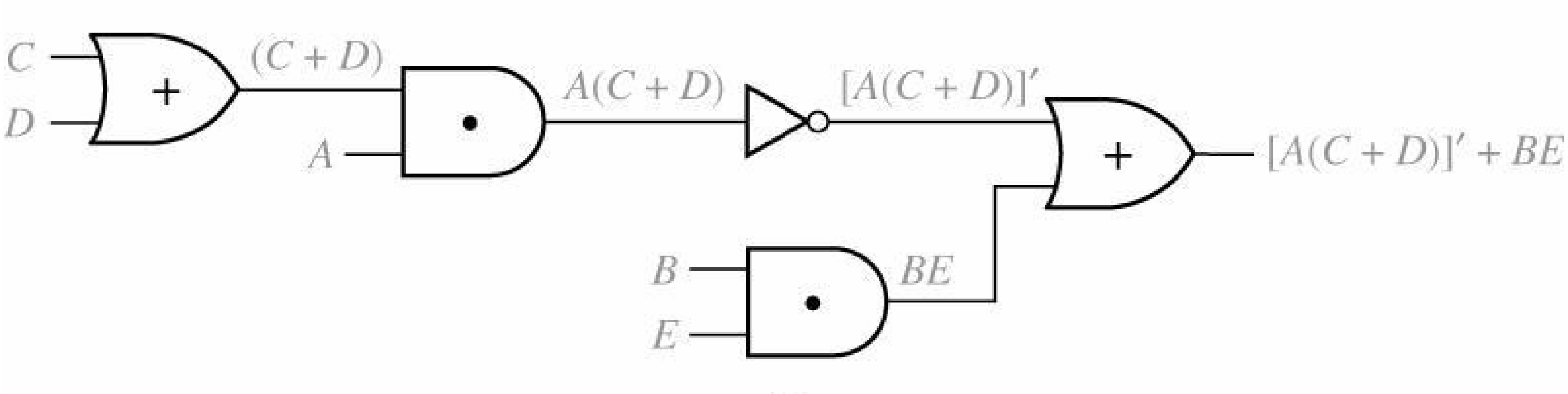
(a)

Logic Expression: AB’ + C

2.3 Boolean Expressions and Truth Tables

Logic Expression : $[A(C+D)]'+BE$

Circuit of logic gates :



Logic Evaluation : A=B=C=1, D=E=0

$$[A(C+D)]'+BE = [1(1+0)]+1\cdot 0 = [1(1)]'+0 = 0+0 = 0$$

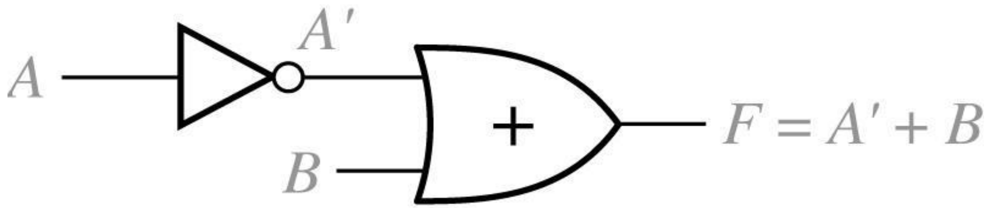
Literal : a variable or its complement in a logic expression

$$ab'c+a'b+a'bc'+b'c'$$

10 literals

2.3 Boolean Expressions and Truth Tables

2-Input Circuit and Truth Table



(a)

A B	A'	F = A' + B
0 0	1	1
0 1	1	1
1 0	0	0
1 1	0	1

Truth table of F

Truth table is specifies the values of a Boolean expression for every possible combination of values in the expression.

2.3 Boolean Expressions and Truth Tables

Proof using Truth Table $AB'+C=(A+C)(B'+C)$

n variable needs $\underbrace{2\times 2\times 2\times\ldots}_{\text{n times}}=2^n$ rows

TABLE 2.1

A B C		B'	AB'	AB' + C		A + C	B' + C	(A + C) (B' + C)
0 0 0		1	0	0		0	1	0
0 0 1		1	0	1		1	1	1
0 1 0		0	0	0		0	0	0
0 1 1		0	0	1		1	1	1
1 0 0		1	1	1		1	1	1
1 0 1		1	1	1		1	1	1
1 1 0		0	0	0		1	0	0
1 1 1		0	0	1		1	1	1

2.4 Basic Theorems

Operations with 0, 1	$X+0=X$	$X\cdot 1=X$
	$X+1=1$	$X\cdot 0=0$
Idempotent Laws	$X+X=X$	$X\cdot X=X$
Involution Laws	$(X')'=X$	
Complementary Laws	$X+X'=1$	$X\cdot X'=0$

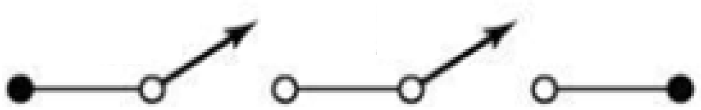
Proof $X=0,\quad 0+0'=0+1,\quad$ and if $X=1,\quad 1+1'=1+0=1$

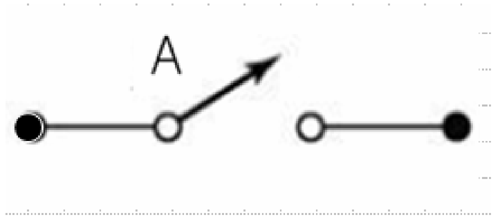
Example

$$(AB'+D)E+1=1$$

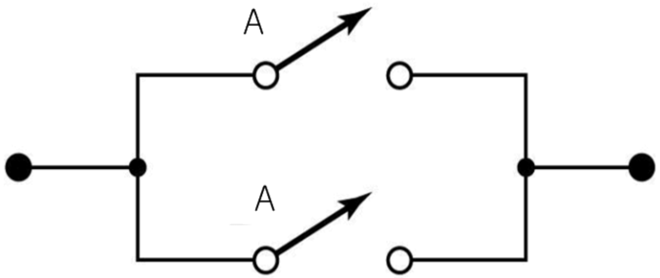
$$(AB'+D)(AB'+D)'=0$$

2.4 Basic Theorems with Switch Circuits

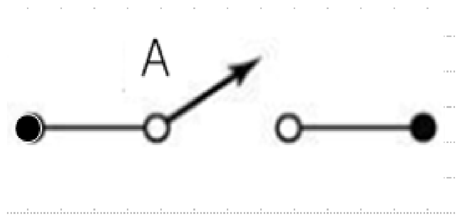




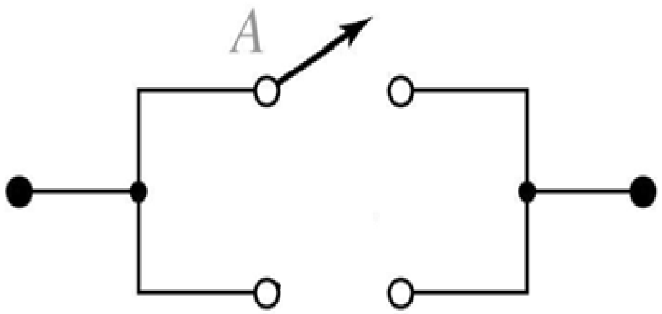
Logic expression: $A \cdot A = A$



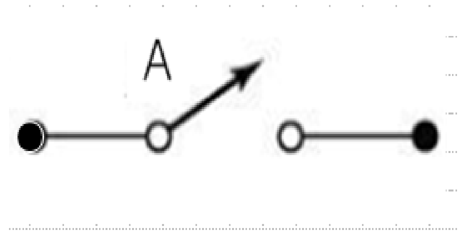
A



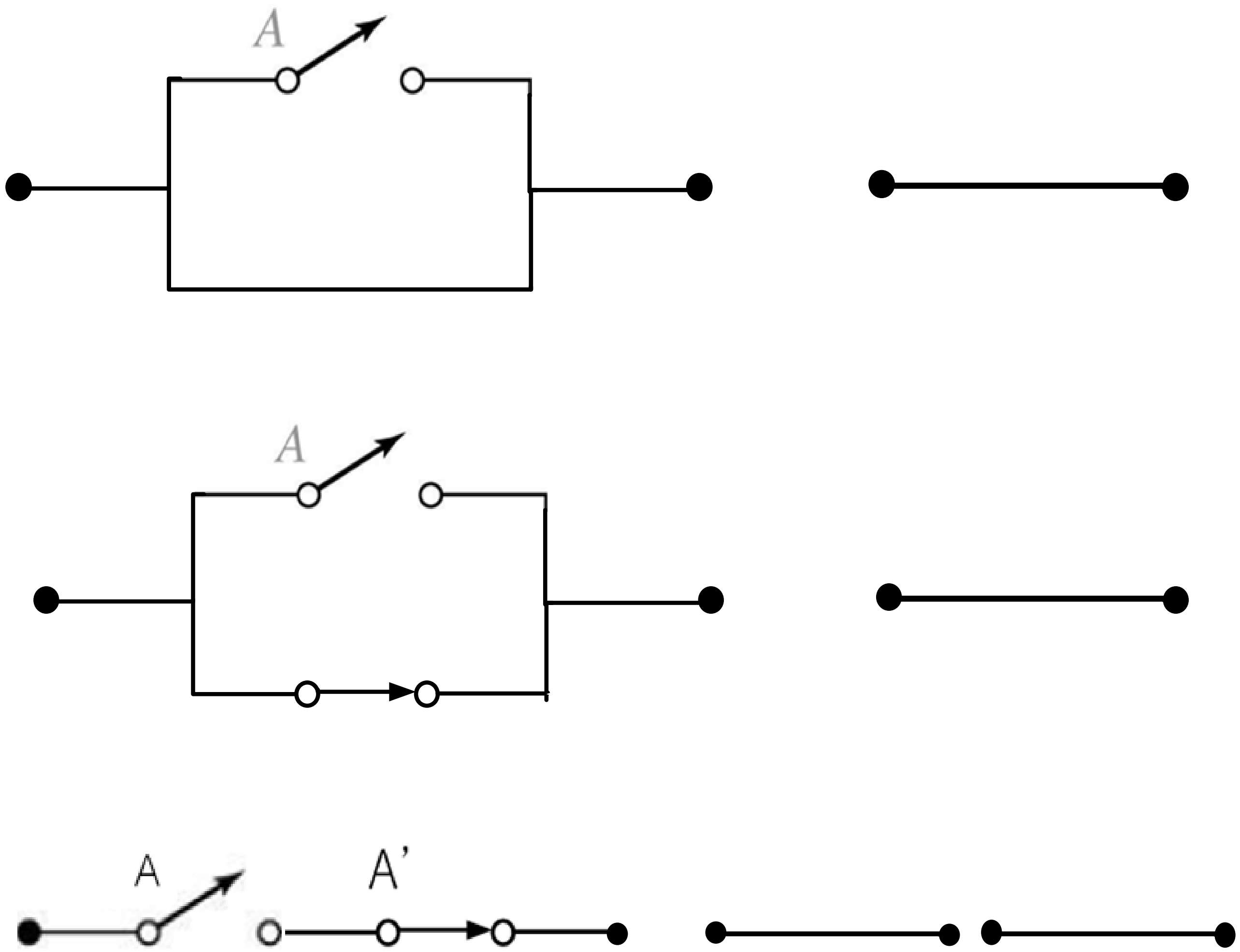
Logic expression: $A + A = A$



(A+0=A)



2.4 Basic Theorems with Switch Circuits



(A+1=1)

2.5 Commutative, Associative, and Distributive Laws

Commutative Laws:

$$XY = YX \qquad X + Y = Y + X$$

Associative Laws:

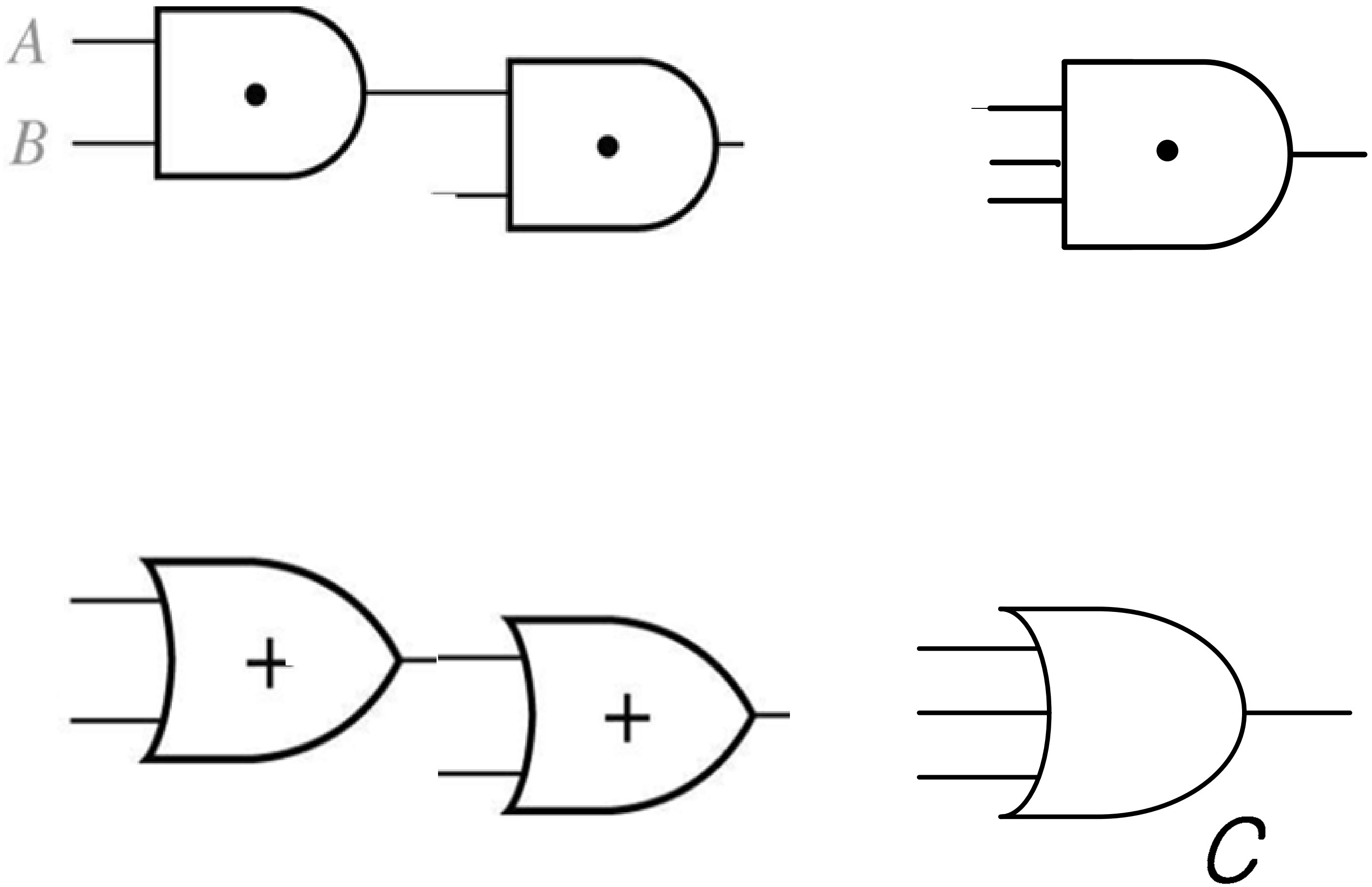
$$(XY)Z = X(YZ) = XYZ$$

$$(X + Y) + Z = X + (Y + Z) = X + Y + Z$$

Proof of Associate Law for AND

X Y Z	XY	YZ	(XY)Z	X(YZ)
0 0 0	0	0	0	0
0 0 1	0	0	0	0
0 1 0	0	0	0	0
0 1 1	0	1	0	0
1 0 0	0	0	0	0
1 0 1	0	0	0	0
1 1 0	1	0	0	0
1 1 1	1	1	1	1

Associative Laws for AND and OR



(AB)C

2.5 Commutative, Associative, and Distributive Laws

AND $XYZ = 1 \text{ iff } X = Y = Z = 1$

OR $X + Y + Z = 0 \text{ iff } X = Y = Z = 0$

Distributive Laws:

$$X(Y+Z) = XY + XZ$$

$$X + YZ = (X+Y)(X+Z)$$

Valid only Boolean algebra not for ordinary algebra

Proof

$$(X+Y)(X+Z) = X(X+Z) + Y(X+Z) = XX = XZ + YX + YZ$$

$$= X + XZ + XY + YZ = X \cdot 1 + XZ + XY + YZ$$

$$= X(1+Z+Y) + YZ = X \cdot 1 + YZ = X + YZ$$

2.6 Simplification Theorems

Useful Theorems for Simplification

$$XY + XY' = X \qquad (X+Y)(X+Y') = X$$

$$X + XY = X \qquad X(X+Y) = X$$

$$(X+Y')Y = XY \qquad XY' + Y = X + Y$$

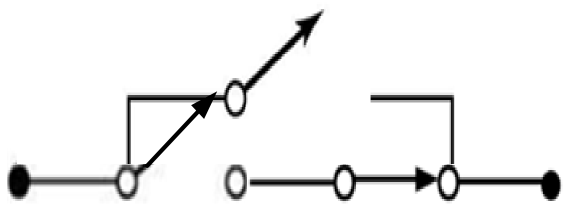
Proof

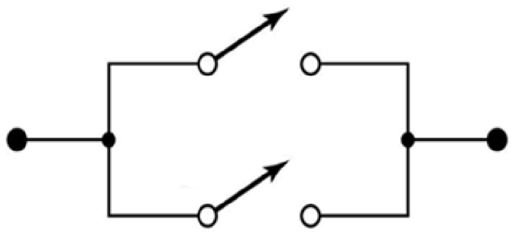
$$X + XY = X \cdot 1 + XY = X(1+Y) = X \cdot 1 = X$$

$$X(X+Y) = XX + XY = X + XY = X$$

$$Y + XY' = (Y+X)(Y+Y') = (Y+X)1 = Y + X$$

Proof with Switch

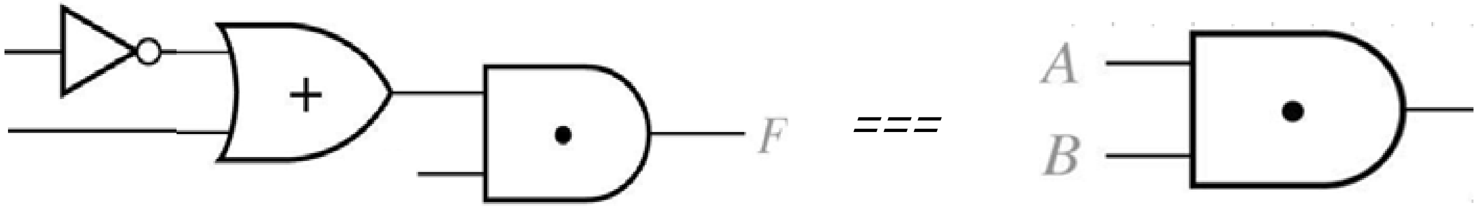




2.6 Simplification Theorems

Equivalent Gate Circuits

$$F = A(A'+B) = AB$$



Proof

A(A’ + B) = AA’ + AB : Distributive Law

= AB : AA’ = 0

2.7 Multiplying Out and Factoring

To obtain a sum-of-product form ➔ Multiplying out using distributive laws

Sum of product form:

$$AB'+CD'E+AC'E$$

Still considered to be in sum of product form:

$$ABC'+DEFG+H$$

$$A+B'+C+D'E$$

Not in Sum of product form:

$$(A+B)CD+EF$$

Multiplying out and eliminating redundant terms

$$\begin{aligned}(A+BC)(A+D+E) &= A+AD+AE+ABC+BCD+BCE \\ &= A(1+D+E+BC)+BCD+BCE \\ &= A+BCD+BCE\end{aligned}$$

2.7 Multiplying Out and Factoring

To obtain a product of sum form ➔ all sums are the sum of single variable

Product of sum form:

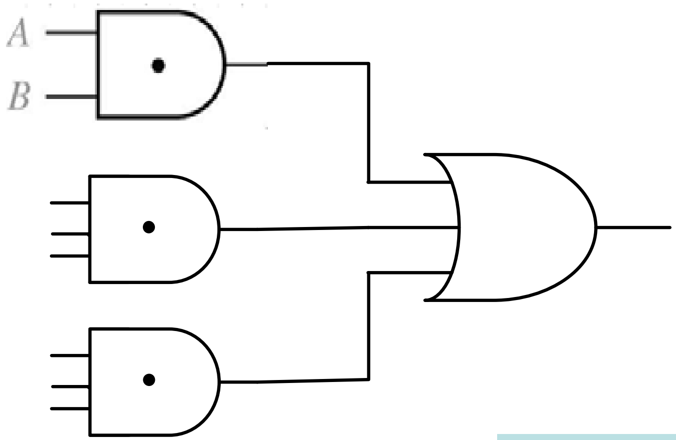
$$(A+B'(C+D'+E)(A+C'+E')$$

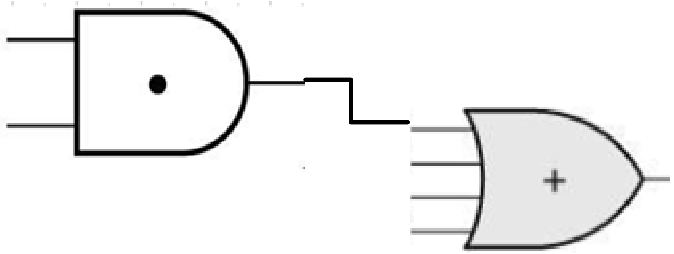
Still considered to be in product of sum form:

$$(A+B)(C+D+E)F$$

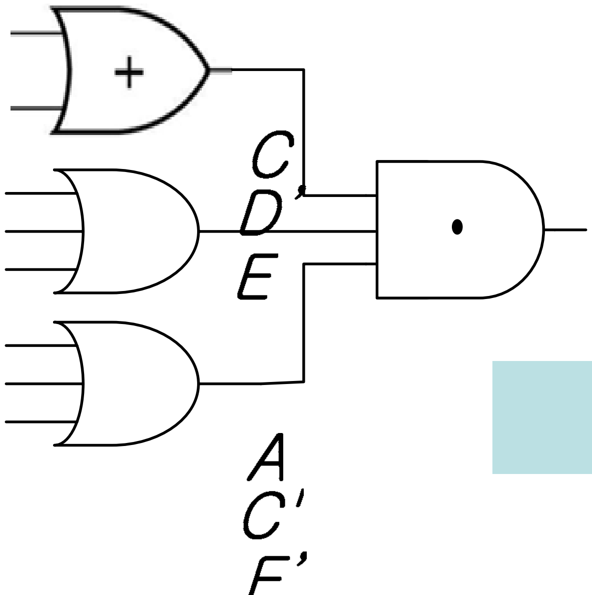
$$AB'C(D'+E)$$

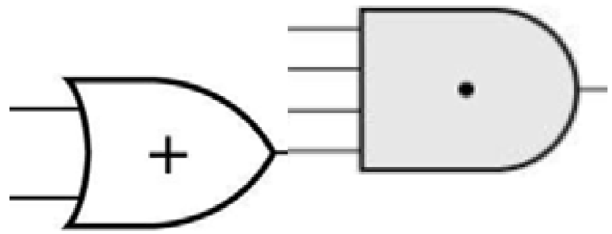
Circuits for SOP and POS form





Sum of product form:





Product of sum form:

2.8 DeMorgan's Laws

DeMorgan's Laws

$$(X+Y)' = X'Y'$$

$$(XY)' = X'+Y'$$

Proof

X Y	X' Y'	X + Y	(X + Y)'	X' Y'	XY	(XY)'	X' + Y'
0 0	1 1	0	1	1	0	1	1
0 1	1 0	1	0	0	0	1	1
1 0	0 1	1	0	0	0	1	1
1 1	0 0	1	0	0	1	0	0

DeMorgan's Laws for *n* variables

$$(X_1 + X_2 + X_3 + \ldots + X_n)' = X_1' X_2' X_3' \ldots X_n'$$

$$(X_1 X_2 X_3 \ldots X_n)' = X_1' + X_2' + X_3' + \ldots + X_n'$$

Example

$$(X_1 + X_2 + X_3)' = (X_1 + X_2)' X_3' = X_1' X_2' X_3'$$

2.8 DeMorgan's Laws

Inverse of *A'B=AB'*

$$F' = (A'B + AB')' = (A'B)'(AB')' = (A + B')(A' + B)$$
$$= AA' + AB + B'A' + BB' = A'B' + AB$$

A B		A' B	A B'	F = A'B+AB'		A' B'	A B	F' = A'B' + AB
0 0		0	0	0		1	0	1
0 1		1	0	1		0	0	0
1 0		0	1	1		0	0	0
1 1		0	0	0		0	1	1

Dual: 'dual' is formed by replacing AND with OR, OR with AND, 0 with 1, 1 with 0

$$(XYZ\ldots)^D = X + Y + Z + \ldots \qquad (X + Y + Z + \ldots)^D = XYZ\ldots$$

$$(AB' + C)' = (AB')'C' = (A'B)C', \quad \text{so} \quad (AB' + C)^D = (A + B')C$$