Chapter IV

Continuous Time Fourier Transform

- Representation of Aperiodic Signals: the Continuous-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of the Continuous-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Systems Characterized by Linear Constant-Coefficient Differential Equations

- CT Fourier Transform of an Aperiodic Signal:

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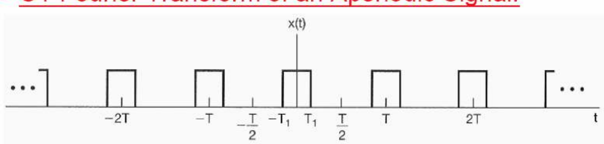


$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$$a_k = \frac{2\sin(kw_0 T_1)}{kw_0 T}$$ Fourier series coefficients

$$Ta_k = \left.\frac{2\sin(wT_1)}{w}\right|_{w=kw_0}$$ w as a continuous variable

$$Ta_k = \frac{2\sin(wT_1)}{w}$$

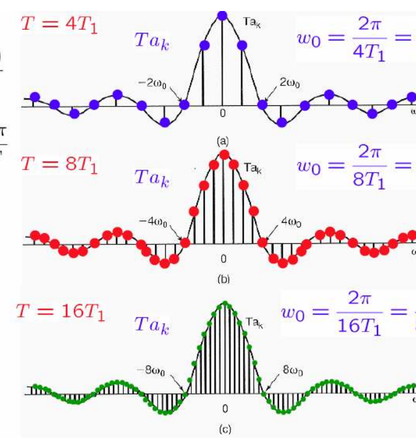
$$w = kw_0 = k\frac{2\pi}{T}$$

$$w_0 = \frac{2\pi}{T}$$

$T = 4T_1$ $\quad Ta_k \quad$ $w_0 = \frac{2\pi}{4T_1} = \frac{\pi}{2T_1}$

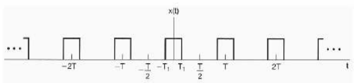
$T = 8T_1$ $\quad Ta_k \quad$ $w_0 = \frac{2\pi}{8T_1} = \frac{\pi}{4T_1}$

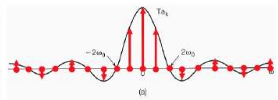
$T = 16T_1$ $\quad Ta_k \quad$ $w_0 = \frac{2\pi}{16T_1} = \frac{\pi}{8T_1}$

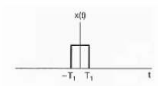


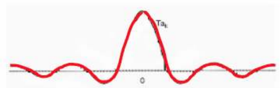
$$w = kw_0 = k\frac{2\pi}{T}$$

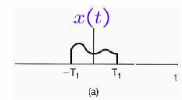
$$T \to \infty \;\Rightarrow\; \{Ta_k\} \to \left.\frac{2\sin(wT_1)}{w}\right|_{w=kw_0}$$



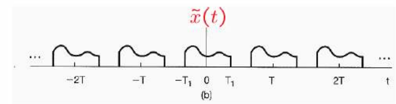








an aperiodic signal



a periodic signal

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jkw_0 t} dt$$

$$\Rightarrow a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jkw_0 t} dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jkw_0 t} dt$$

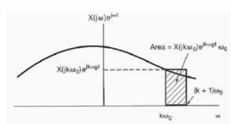
- Define the envelope $X(jw)$ of Ta_k as

$$Ta_k = \frac{2\sin(wT_1)}{w}$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

- Then,

$$a_k = \frac{1}{T} X(jkw_0)$$



- Hence,

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jkw_0)\, e^{jkw_0 t}$$

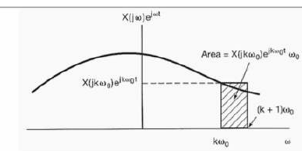
$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jkw_0)\, e^{jkw_0 t}\, w_0$$

$$w_0 = \frac{2\pi}{T}$$

$$\frac{1}{T} = \frac{1}{2\pi} w_0$$

- As $T \to \infty$, $\tilde{x}(t) \to x(t)$

also $w_0 \to 0$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

- inverse Fourier transform eqn
- synthesis eqn

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

- $X(jw)$: Fourier Transform of $x(t)$ spectrum
- analysis eqn

$$a_k = \left.\frac{1}{T} X(jw)\right|_{w=kw_0}$$

Sufficient conditions for the convergence of FT

$x(t) \xrightarrow{\mathcal{CTFT}} X(jw)$ $\qquad X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$

$\hat{x}(t) \xleftarrow{\mathcal{CTIFT}} X(jw)$ $\qquad \hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$

$$e(t) = \hat{x}(t) - x(t)$$

- If $x(t)$ has finite energy

  i.e., square integrable, $\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$

$\Rightarrow$ $X(jw)$ is finite

$\Rightarrow$ $\int_{-\infty}^{+\infty} |e(t)|^2 dt = 0$

Sufficient conditions for the convergence of FT

- Dirichlet conditions:

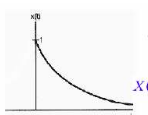
1. x(t) be absolutely integrable; that is, $\int_{-\infty}^{+\infty} |x(t)| dt < \infty$

2. x(t) have a finite number of maxima and minima within any finite interval

3. x(t) have a finite number of discontinuities within any finite interval
   Furthermore, each of these discontinuities must be finite

Example 4.1:

$x(t) = e^{-at}u(t), \quad a > 0$



$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$

$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$

$$\Rightarrow X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} e^{-at}\,u(t)\,e^{-jwt}dt$$

$$= \int_{0}^{\infty} e^{-at}\,e^{-jwt}dt$$

$$= \int_{0}^{\infty} e^{-(a+w)t}\,dt$$

$$= -\frac{1}{a+jw}e^{-(a+jw)t}\Big|_0^{\infty}$$

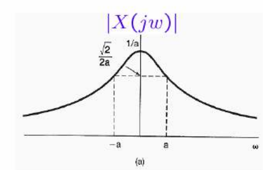
$$= 0 - \left(-\frac{1}{a+jw}e^{-(a+jw)0}\right)$$

$$= \frac{1}{a+jw}, \quad a > 0$$

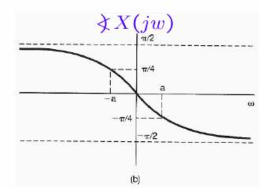
Example 4.1:

$$\Rightarrow X(jw) = \frac{1}{a+jw}, \quad a > 0$$

$$\Rightarrow |X(jw)| = \frac{1}{\sqrt{a^2+w^2}}$$



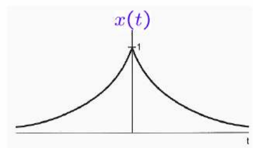
$$\Rightarrow \measuredangle X(jw) = -\tan^{-1}\left(\frac{w}{a}\right)$$



- Example 4.2:

$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$

$x(t) = e^{-a|t|}, \quad a > 0$

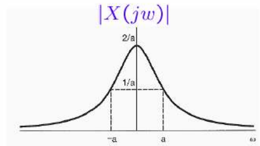


$$\Rightarrow X(jw) = \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-jwt} dt$$

$$= \int_{-\infty}^{0} e^{at}\, e^{-jwt} dt \quad + \int_{0}^{\infty} e^{-at}\, e^{-jwt} dt$$

$$= \frac{1}{a - jw} + \frac{1}{a + jw}$$

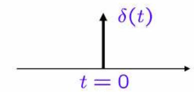
$$= \frac{2a}{a^2 + w^2}$$



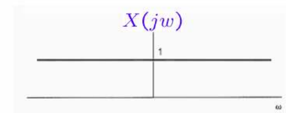
- Example 4.3:

$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$

$x(t) = \delta(t)$, i.e., unit impules



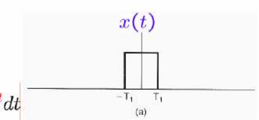
$$\Rightarrow X(jw) = \int_{-\infty}^{\infty} \delta(t)\, e^{-jwt} dt = 1$$



- Example 4.4:

$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



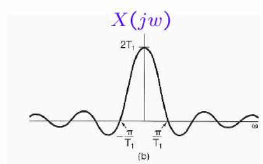
$$\Rightarrow X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt = \int_{-T_1}^{T_1} e^{-jwt} dt$$

$$= \frac{1}{-jw} e^{-jwt} \Big|_{-T_1}^{T_1}$$

$$= \frac{1}{-jw} \left(e^{-jwT_1} - e^{jwT_1} \right)$$

$$= \frac{1}{jw} \left(e^{jwT_1} - e^{-jwT_1} \right)$$

$$= 2 \frac{\sin(wT_1)}{w}$$



- Example 4.5:

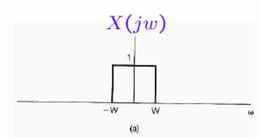
$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$

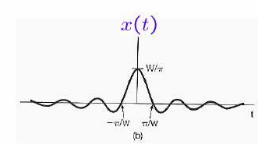
$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$

$$X(jw) = \begin{cases} 1, & |w| < W \\ 0, & |w| > W \end{cases}$$

$$\Rightarrow x(t) = \frac{1}{2\pi} \int_{-W}^{W} e^{jwt} dw$$

$$= \frac{\sin(Wt)}{\pi t}$$

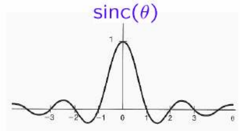


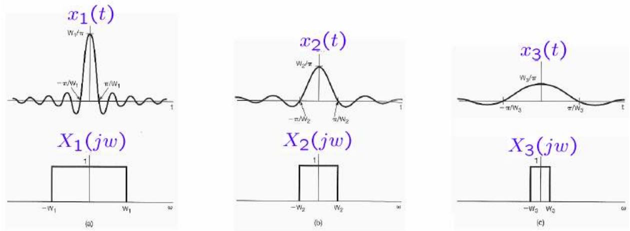


- sinc functions:

$$\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

$$\frac{\sin(Wt)}{\pi t} = \frac{W}{\pi}\text{sinc}\left(\frac{Wt}{\pi}\right)$$





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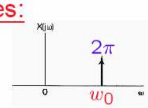
- Fourier Transform from Fourier Series:

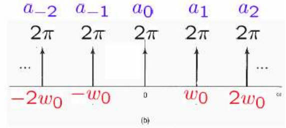
$$X(jw) = 2\pi \delta(w - w_0)$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi \delta(w - w_0)e^{jwt}dw$$

$$= e^{jw_0t}$$



- more generally,

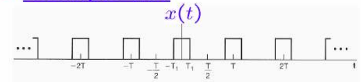


$$X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - kw_0)$$

$$\Rightarrow x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t}$$

Fourier series represntation of a periodic signal

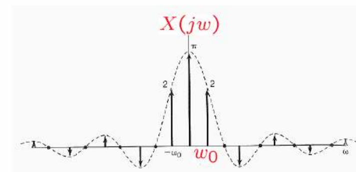
- Example 4.6:



$$\Rightarrow a_k = \frac{\sin(kw_0T_1)}{\pi k}$$

$$\Rightarrow X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - kw_0)$$

$$= \sum_{k=-\infty}^{+\infty} \frac{2\sin(kw_0T_1)}{k} \delta(w - kw_0)$$



- Example 4.7:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$x(t) = \sin(w_0 t) = \frac{e^{jw_0 t} - e^{-jw_0 t}}{2j}$$

$$\Rightarrow a_1 = \frac{1}{2j} \quad a_{-1} = -\frac{1}{2j} \quad a_k = 0, \quad k \neq 1, -1$$

$$x(t) = \cos(w_0 t) = \frac{e^{jw_0 t} + e^{-jw_0 t}}{2}$$

$$\Rightarrow a_1 = \frac{1}{2} \quad a_{-1} = \frac{1}{2} \quad a_k = 0, \quad k \neq 1, -1$$

- Example 4.8:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

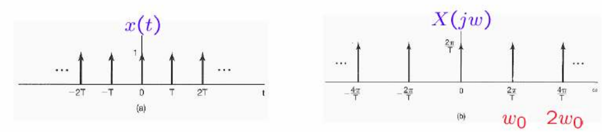
$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \tilde{x}(t) e^{-jkw_0 t} dt$$

$$X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - kw_0)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

$$\Rightarrow a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t) e^{-jkw_0 t} dt = \frac{1}{T}$$

$$\Rightarrow X(jw) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(w - \frac{2\pi}{T}k)$$



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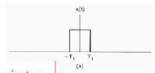
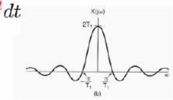
Section	Property
4.3.1	Linearity
4.3.2	Time Shifting
4.3.6	Frequency Shifting
4.3.3	Conjugation
4.3.5	Time Reversal
4.3.5	Time and Frequency Scaling
4.4	Convolution
4.5	Multiplication
4.3.4	Differentiation in Time
4.3.4	Integration
4.3.6	Differentiation in Frequency
4.3.3	Conjugate Symmetry for Real Signals
4.3.3	Symmetry for Real and Even Signals
4.3.3	Symmetry for Real and Odd Signals
4.3.3	Even-Odd Decomposition for Real Signals
4.3.7	Parseval's Relation for Aperiodic Signals

- Fourier Transform Pair:



  - Synthesis equation: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$
  - Analysis equation: $X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$



  - Notations:

$X(jw) = \mathcal{F}\{x(t)\}$ $\qquad \frac{1}{a+jw} = \mathcal{F}\{e^{-at}u(t)\}$

$x(t) = \mathcal{F}^{-1}\{X(jw)\}$ $\qquad e^{-at}u(t) = \mathcal{F}^{-1}\{\frac{1}{a+jw}\}$

$x(t) \xleftrightarrow{\mathcal{CTFT}} X(jw)$ $\qquad e^{-at}u(t) \xleftrightarrow{\mathcal{CTFT}} \frac{1}{a+jw}$

- Linearity:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$
$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$x(t) \xleftrightarrow{\mathcal{F}} X(jw)$$
$$y(t) \xleftrightarrow{\mathcal{F}} Y(jw)$$
$$\Rightarrow \quad a\,x(t)+b\,y(t) \xleftrightarrow{\mathcal{F}} a\,X(jw)+b\,Y(jw)$$

- Time Shifting:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$
$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$x(t) \xleftrightarrow{\mathcal{F}} X(jw)$$
$$\Rightarrow x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-jwt_0}X(jw)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$
$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jw(t-t_0)}dw$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(e^{-jwt_0}X(jw)\right)e^{jwt}dw$$

$$Y(jw) = \int_{-\infty}^{+\infty} x(t-t_0)e^{-jwt}dt$$
$$= \int_{-\infty}^{+\infty} x(\tau)e^{-jw(\tau+t_0)}d\tau$$
$$= e^{-jwt_0}\int_{-\infty}^{+\infty} x(\tau)e^{-jw\tau}d\tau$$

- Time Shift $\rightarrow$ Phase Shift:

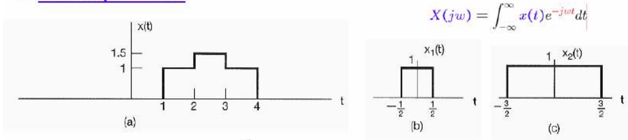
$$\mathcal{F}\{x(t)\} = X(jw) = |X(jw)|e^{j\sphericalangle X(jw)}$$

$$\mathcal{F}\{x(t-t_0)\} = e^{-jwt_0}X(jw) = |X(jw)|e^{j[\sphericalangle X(jw)-wt_0]}$$

Example 4.9:

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$

$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$



$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$$

$$X_1(jw) = \frac{2\sin(w/2)}{w}$$

$$X_2(jw) = \frac{2\sin(3w/2)}{w}$$

$$\Rightarrow X(jw) = e^{-j5w/2}\left\{\frac{\sin(w/2) + 2\sin(3w/2)}{w}\right\}$$

Conjugation & Conjugate Symmetry:

$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$x(t)^* \overset{\mathcal{F}}{\longleftrightarrow} X^*(-jw)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$= \frac{1}{2\pi}\int_{\infty}^{\infty} X(j \quad w)e^{jwt} \quad dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j \quad w)e^{jwt}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{j \quad wt}dw$$

Conjugation & Conjugate Symmetry:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$x(t)^* \overset{\mathcal{F}}{\longleftrightarrow} X^*(-jw)$$

- $x(t) = x^*(t) \Rightarrow X(-jw) = X^*(jw)$

  $x(t)$ is real $\Rightarrow$ $X(jw)$ is conjugate symmetric

- $x(t) = x^*(t)$ & $x(-t) = x(t)$

  $\Rightarrow X(-jw) = X^*(jw)$ & $X(-jw) = X(jw)$

  $\Rightarrow X(jw) = X^*(jw)$

  $x(t)$ is real & even $\Rightarrow$ $X(jw)$ are real & even

- $x(t)$ is real & odd $\Rightarrow$ $X(jw)$ are purely imaginary & odd

Conjugation & Conjugate Symmetry:

If $x(t)$ is a real function

$$x(t) = \mathcal{E}v\{x(t)\} + \mathcal{O}d\{x(t)\} = x_e(t) + x_o(t)$$

$$\Rightarrow \mathcal{F}\{x(t)\} = \mathcal{F}\{x_e(t)\} + \mathcal{F}\{x_o(t)\}$$

$\Rightarrow \mathcal{F}\{x_e(t)\}$: a real function

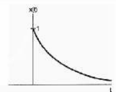
$\Rightarrow \mathcal{F}\{x_o(t)\}$: a purely imaginary function

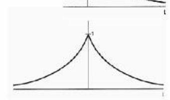
$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$\mathcal{E}v\{x(t)\} \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e\{X(jw)\}$$

$$\mathcal{O}d\{x(t)\} \overset{\mathcal{F}}{\longleftrightarrow} j\,\mathcal{I}m\{X(jw)\}$$

- **Example 4.10:**





$$e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a+jw}$$

$$e^{-a|t|} \xleftrightarrow{\mathcal{F}} ?$$

$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

$$= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] \qquad = 2\mathcal{E}v\left\{e^{-at}u(t)\right\}$$

$$\mathcal{E}v\left\{e^{-at}u(t)\right\} \xleftrightarrow{\mathcal{F}} \mathcal{R}e\left\{\frac{1}{a+jw}\right\}$$

$$\mathcal{O}d\left\{e^{-at}u(t)\right\} \xleftrightarrow{\mathcal{F}} j\,\mathcal{I}m\left\{\frac{1}{a+jw}\right\}$$

$$X(jw) = 2\mathcal{R}e\left\{\frac{1}{a+jw}\right\} = \frac{2a}{a^2+w^2}$$

- **Differentiation & Integration:**

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$x(t) \xleftrightarrow{\mathcal{F}} X(jw) \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

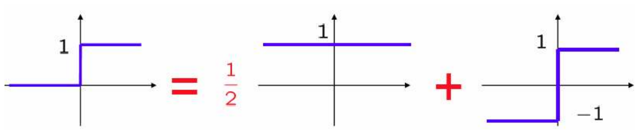
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw) \quad e^{jwt} \quad dw$$

$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{F}} jwX(jw)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw) \quad e^{jwt} \quad dw$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{jw}X(jw) + \pi X(0)\delta(w)$$

dc or average value



$u(t)$ $\qquad \frac{1}{2}$ $\qquad 1$ $\qquad$ sgn(t)

$$1 \xleftrightarrow{\mathcal{FT}} 2\pi\delta(jw) \qquad \text{sgn(t)} \xleftrightarrow{\mathcal{FT}} S(jw)$$

$$\frac{d}{dt}\text{sgn(t)} \xleftrightarrow{\mathcal{FT}} jw\,S(jw)$$

$$2\,\delta(t) \xleftrightarrow{\mathcal{FT}} jw\,S(jw) \qquad \Rightarrow U(jw) =$$

$$\Rightarrow S(jw) =$$

- **Example 4.11:**

$$x(t) = u(t) \xleftrightarrow{\mathcal{F}} X(jw) = ?$$

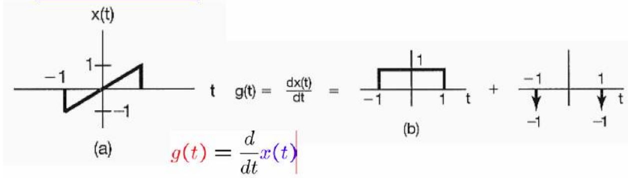
$$g(t) = \delta(t) \xleftrightarrow{\mathcal{F}} G(jw) = 1$$

$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau \qquad X(jw) = \frac{1}{jw}G(jw) + \pi G(0)\delta(w)$$

$$= \frac{1}{jw} + \pi\delta(w)$$

$$\delta(t) = \frac{d}{dt}u(t) \xleftrightarrow{\mathcal{F}} jw\left[\frac{1}{jw} + \pi\delta(w)\right] = 1$$

- Example 4.12:



$$g(t) = \frac{d}{dt}x(t)$$

$$G(jw) = \frac{2\sin(w)}{w} - e^{jw} - e^{-jw}$$

$$\Rightarrow X(jw) = \frac{G(jw)}{jw} + \pi G(0)\delta(w)$$

$$= \frac{2\sin(w)}{jw^2} - \frac{2\cos(w)}{jw}$$

- Time & Frequency Scaling:

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{jw}{a}\right)$$

$$\frac{1}{|b|}x\left(\frac{t}{b}\right) \overset{\mathcal{F}}{\longleftrightarrow} X(jbw)$$

$$x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X(-jw)$$

$$x(\ \ t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jw\ \ t}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\ \ \bar{w})e^{j\ \bar{w}\ t}\ \ d\bar{w}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\ \ X(j\ \ \bar{w})e^{j\ \bar{w}\ t}d\bar{w}$$

$$= \frac{1}{2\pi}\int_{+\infty}^{-\infty} X(j\ \ \bar{w})e^{j\ \bar{w}\ t}\ \ d\bar{w}$$

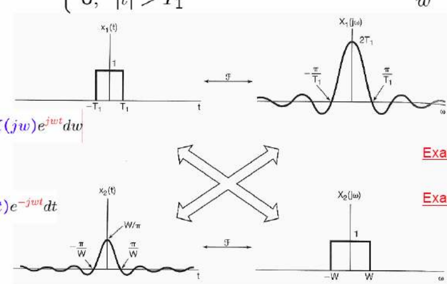
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\ \ X(j\ \ \bar{w})e^{j\ \bar{w}\ t}d\bar{w}$$

- Duality:

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \overset{\mathcal{F}}{\longleftrightarrow} X_1(jw) = \frac{2\sin(wT_1)}{w}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$



Example 4.4

Example 4.5

$$x_2(t) = \frac{\sin(Wt)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} X_2(jw) = \begin{cases} 1, & |w| < W \\ 0, & |w| > W \end{cases}$$

- Duality:

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$B(s) = \int_{-\infty}^{+\infty} A(\tau)e^{-js\tau}d\tau$$

$$A(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} B(s)e^{js\tau}ds$$

$$B(-s) = \int_{-\infty}^{+\infty} A(\tau)e^{js\tau}d\tau$$

$$A(s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} B(\tau)e^{js\tau}d\tau$$

$$A(-s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} B(\tau)e^{-js\tau}d\tau$$

- Duality:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$

$$\frac{d}{dt}x(t) \overset{\mathcal{F}}{\longleftrightarrow} jwX(jw)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{jw}X(jw) + \pi X(0)\delta(w)$$

$$-jtx(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{d}{dw}X(jw)$$

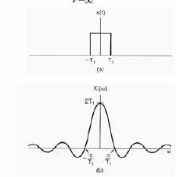
$$e^{jw_0t}x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j(w-w_0))$$

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{w} X(\eta)d\eta$$

- Parseval's relation:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$



$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$\Rightarrow \int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(jw)|^2 dw$$

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)x^*(t)dt$$

$$= \int_{-\infty}^{+\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(jw)e^{-jwt}dw\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(jw)\left[\int_{-\infty}^{+\infty} x(t)e^{-jwt}dt\right]dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(jw)|^2 dw$$

- Representation of Aperiodic Signals: the Continuous-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of the Continuous-Time Fourier Transform
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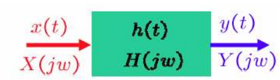
- Systems Characterized by Linear Constant-Coefficient Differential Equations

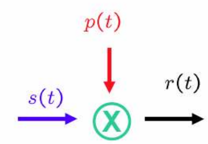
- Convolution Property:



$$y(t) = x(t) * h(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(jw) = X(jw)H(jw)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- Multiplication Property:

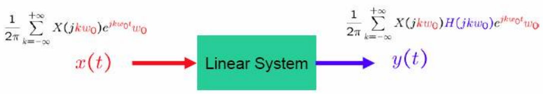


$$r(t) = s(t)p(t) \overset{\mathcal{F}}{\longleftrightarrow} R(jw) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(w-\theta))d\theta$$

- From Superposition (or Linearity):

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$= \lim_{w_0 \to 0} \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jkw_0)e^{jkw_0 t}w_0$$



$$H(jkw_0) = \int_{-\infty}^{\infty} h(t)e^{-jkw_0 t}dt$$

$$y(t) = \lim_{w_0 \to 0} \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jkw_0)H(jkw_0)e^{jkw_0 t}w_0$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)H(jw)e^{jwt}dw$$

- From Superposition (or Linearity):

$$\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jkw_0)e^{jkw_0 t}w_0 \longrightarrow \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jkw_0)H(jkw_0)e^{jkw_0 t}w_0$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)H(jw)e^{jwt}dw$$

Since $y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(jw)e^{jwt}dw$

$$\Rightarrow Y(jw) = X(jw)H(jw)$$

$$y(t) = x(t) * h(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(jw) = X(jw)H(jw)$$

- From Convolution Integral:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$x(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$\Rightarrow Y(jw) = \mathcal{F}\{y(t)\} = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right]e^{-jwt}dt$$

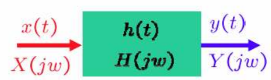
$$= \int_{-\infty}^{+\infty} x(\tau)\left[\int_{-\infty}^{+\infty} h(t-\tau)e^{-jwt}dt\right]d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)\left[e^{-jw\tau}H(jw)\right]d\tau$$

$$= H(jw)\int_{-\infty}^{+\infty} x(\tau)e^{-jw\tau}d\tau$$

$$\Rightarrow Y(jw) = H(jw)X(jw)$$

- Equivalent LTI Systems:

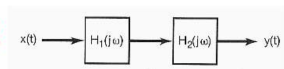


$$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(jw)$$

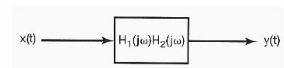
impulse response | frequency response

$$y(t) = x(t) * h(t)$$

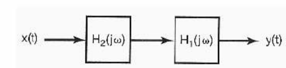
$$Y(jw) = X(jw)H(jw)$$



(a)



(b)



(c)

$$\Rightarrow Y(jw) = H_1(jw)H_2(jw)X(jw)$$

- Example 4.15: Time Shift

$x(t)$, $X(jw)$ → **$h(t)$** / **$H(jw)$** → $y(t)$, $Y(jw)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-jwt_0}X(jw)$$

$$h(t) = \delta(t-t_0)$$

$$\Rightarrow H(jw) = e^{-jwt_0}$$

$$\begin{aligned} Y(jw) &= H(jw)X(jw) \\ &= e^{-jwt_0}X(jw) \end{aligned}$$

$$\Rightarrow y(t) = x(t-t_0)$$

- Examples 4.16 & 17: Differentiator & Integrator

$$y(t) = \frac{d}{dt}x(t) \qquad \Rightarrow Y(jw) = jwX(jw)$$

$$\Rightarrow H(jw) = jw$$

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \qquad \Rightarrow h(t) = u(t)$$

impulse response

$$\Rightarrow H(jw) = \frac{1}{jw} + \pi\delta(w)$$

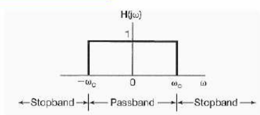
$$\begin{aligned} \Rightarrow Y(jw) &= H(jw)X(jw) \\ &= \frac{1}{jw}X(jw) + \pi\delta(w)X(jw) \\ &= \frac{1}{jw}X(jw) + \pi\delta(w)X(0) \end{aligned}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

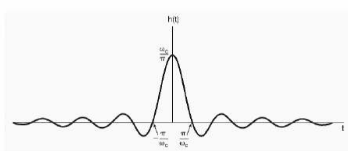
- Example 4.18: Ideal Lowpass Filter

$$H(jw) = \begin{cases} 1, & |w| < w_c \\ 0, & |w| > w_c \end{cases}$$



$$\Rightarrow h(t) = \frac{1}{2\pi}\int_{-w_c}^{+w_c} e^{jwt}dw$$

$$= \frac{\sin(w_c t)}{\pi t}$$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

- Filter Design:

$$H(jw) = \int_{-\infty}^{\infty} h(t)e^{-jwt}dt$$

$\delta(t)$, $x(t)$ → Filter LTI System → $h(t)$, $y(t) = h(t) * x(t)$

$$= \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

$$H(jw) = \int_{-\infty}^{\infty} h(t)e^{-jwt}dt$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$\Rightarrow Y(jw) = H(jw)X(jw)$$

$$\Rightarrow y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(jw)e^{jwt}dw$$

- **Filter Design:**

$$H(jw) = \int_{-\infty}^{\infty} h(t)e^{-jwt}dt$$

$\delta(t)$ → Filter LTI System → $h(t)$

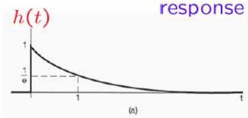
$x(t)$ → Filter LTI System → $y(t) = h(t) * x(t)$

$$Y(jw) = H(jw)X(jw)$$

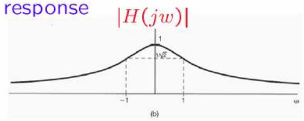
$$h(t) = e^{-t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} H(jw) = \frac{1}{jw+1}$$

impulse response — frequency response

$h(t)$



$|H(jw)|$



- **Example 4.19:**

$x(t)$ → Filter LTI System → $y(t)$

$$h(t) = e^{-at}u(t), \quad a > 0 \quad \Rightarrow H(jw) = \frac{1}{a+jw}$$

$$x(t) = e^{-bt}u(t), \quad b > 0 \quad \Rightarrow X(jw) = \frac{1}{b+jw}$$

$$\Rightarrow Y(jw) = H(jw)X(jw) = \frac{1}{a+jw}\,\frac{1}{b+jw}$$

if $a \neq b$ $$= \frac{1}{b-a}\left[\frac{1}{a+jw} - \frac{1}{b+jw}\right]$$

- **Example 4.19:**

if $a \neq b$ $$Y(jw) = \frac{1}{b-a}\left[\frac{1}{a+jw} - \frac{1}{b+jw}\right]$$

$$\Rightarrow y(t) = \frac{1}{b-a}\left[e^{-at}u(t) - e^{-bt}u(t)\right]$$

if $a = b$ $$Y(jw) = \frac{1}{(a+jw)^2}$$

since $$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+jw}$$

and $$t\,e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{dw}\left[\frac{1}{a+jw}\right] = \frac{1}{(a+jw)^2}$$

$$\Rightarrow y(t) = te^{-at}u(t)$$

- **Example 4.20:**

$$H(jw) = \int_{-\infty}^{\infty} h(t)e^{-jwt}dt$$

$x(t) = \frac{\sin(w_i t)}{\pi t}$ → Filter LTI System → $y(t) = ?$

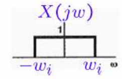
$$h(t) = \frac{\sin(w_c t)}{\pi t}$$

$$\Rightarrow Y(jw) = H(jw)X(jw)$$

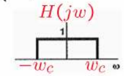
$$\Rightarrow X(jw) = \begin{cases} 1, & |w| \le w_i \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow H(jw) = \begin{cases} 1, & |w| \le w_c \\ 0, & \text{otherwise} \end{cases}$$

$X(jw)$



$H(jw)$



$$w_0 = \min(w_c, w_i)$$

$$\Rightarrow Y(jw) = \begin{cases} 1, & |w| \le w_0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow y(t) = \begin{cases} \frac{\sin(w_c t)}{\pi t}, & w_c \le w_i \\ \frac{\sin(w_i t)}{\pi t}, & w_c \ge w_i \end{cases}$$

- Representation of Aperiodic Signals:
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- The Multiplication Property
- Systems Characterized by Linear Constant-Coefficient Differential Equations

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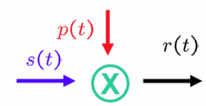
- Convolution & Multiplication:

$$y(t) = x(t) * h(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(jw) = X(jw)H(jw)$$

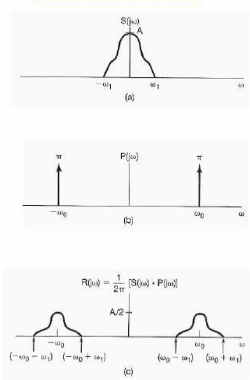
$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$r(t) = s(t)p(t) \overset{\mathcal{F}}{\longleftrightarrow} R(jw) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(w-\theta))d\theta$$

- Multiplication of One Signal by Another:
  - Scale or modulate the amplitude of the other signal
  - Modulation



- Example 4.21:



$$r(t) = s(t)p(t)$$

$$s(t) \overset{\mathcal{F}}{\longleftrightarrow} S(jw)$$

$$p(t) \overset{\mathcal{F}}{\longleftrightarrow} P(jw)$$

$$p(t) = \cos(w_0 t)$$

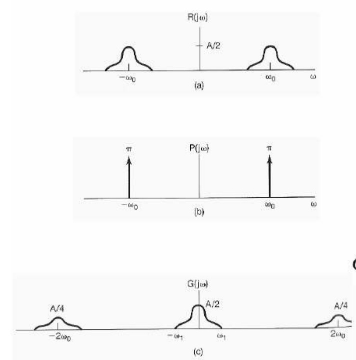
$$P(jw) = \pi\delta(w-w_0) + \pi\delta(w+w_0)$$

$$R(jw) = \frac{1}{2\pi}[S(jw) * P(jw)]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(w-\theta))d\theta$$

$$= \frac{1}{2}S\Big(j(w-w_0)\Big) + \frac{1}{2}S\Big(j(w+w_0)\Big)$$

- Example 4.22:



$$g(t) = r(t)p(t)$$

$$r(t) \overset{\mathcal{F}}{\longleftrightarrow} R(jw)$$

$$p(t) \overset{\mathcal{F}}{\longleftrightarrow} P(jw)$$

$$p(t) = \cos(w_0 t)$$

$$G(jw) = \frac{1}{2\pi}[R(jw) * P(jw)]$$

- **Example 4.23:**

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

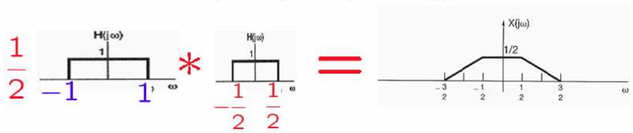
$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2}$$

$$X(jw) = \int_{-\infty}^{\infty} \frac{\sin(t)\sin(t/2)}{\pi t^2} e^{-jwt}dt$$

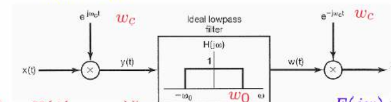
$$= \pi\left(\frac{\sin(t)}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right)$$

$$\Rightarrow X(jw) = \frac{1}{2}\mathcal{F}\left\{\frac{\sin(t)}{\pi t}\right\} * \mathcal{F}\left\{\frac{\sin(t/2)}{\pi t}\right\}$$



- **Bandpass Filter Using Amplitude Modulation:**

$$e^{jw_c t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(w - w_c)$$



$$Y(jw) = X(j(w - w_c))$$

$$F(jw) = W(j(w + w_c))$$

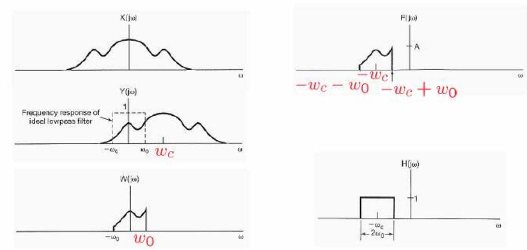


TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals $\int_{-\infty}^{+\infty}\lvert x(t)\rvert^2dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\lvert X(j\omega)\rvert^2d\omega$		

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi\sum_{k=-\infty}^{+\infty} a_k\delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1,\quad a_k = 0,\ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi}\operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$

$\sum_{n=-\infty}^{+\infty} \delta(t-nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \|t\| < T_1 \\ 0, & \|t\| > T_1 \end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \|\omega\| < W \\ 0, & \|\omega\| > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t-t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a+j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a+j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a+j\omega)^n}$	—

- Representation of Aperiodic Signals: the Continuous-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of the Continuous-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Systems Characterized by Linear Constant-Coefficient Differential Equations

- A useful class of CT LTI systems:

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1}\frac{d^{N-1}y(t)}{dt^{N-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_M \frac{d^M x(t)}{dt^M} + b_{M-1}\frac{d^{M-1}x(t)}{dt^{M-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t)$$

$$Y(jw) = X(jw)H(jw) \qquad H(jw) = \frac{Y(jw)}{X(jw)}$$

$$\mathcal{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k \mathcal{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathcal{F}\left\{\frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k (jw)^k Y(jw) = \sum_{k=0}^{M} b_k (jw)^k X(jw)$$

$$Y(jw)\left[\sum_{k=0}^{N} a_k (jw)^k\right] = X(jw)\left[\sum_{k=0}^{M} b_k (jw)^k\right]$$

$$\Rightarrow H(jw) = \frac{Y(jw)}{X(jw)} = \frac{\sum_{k=0}^{M} b_k (jw)^k}{\sum_{k=0}^{N} a_k (jw)^k} = \frac{b_M (jw)^M + \cdots + b_1 (jw) + b_0}{a_N (jw)^N + \cdots + a_1 (jw) + a_0}$$

- Examples 4.24 & 4.25:

$$H = \frac{Y}{X}$$

$$\frac{dy(t)}{dt} + ay(t) = x(t) \quad \Rightarrow \quad H(jw) = \frac{1}{jw + a}$$

$$(jw)Y(jw) + aY(jw) = X(jw) \quad \Rightarrow \quad h(t) = e^{-at}u(t)$$

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3\,y(t) = \frac{dx(t)}{dt} + 2\,x(t)$$

$$\Rightarrow \quad H(jw) = \frac{(jw)+2}{(jw)^2 + 4(jw) + 3} = \frac{(jw+2)}{(jw+1)(jw+3)}$$

$$= \frac{1/2}{jw+1} + \frac{1/2}{jw+3}$$

$$\Rightarrow \quad h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

- Example 4.26:

$$x(t) = e^{-t}u(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = ???$$

$$H(jw) = \frac{(jw+2)}{(jw+1)(jw+3)}$$

$$\Rightarrow \quad Y(jw) = X(jw)H(jw)$$

$$= \left[\frac{1}{jw+1}\right]\left[\frac{jw+2}{(jw+1)(jw+3)}\right]$$

$$= \frac{jw+2}{(jw+1)^2(jw+3)}$$

$$= \frac{\frac{1}{4}}{jw+1} + \frac{\frac{1}{2}}{(jw+1)^2} - \frac{\frac{1}{4}}{jw+3}$$

$$\Rightarrow \quad y(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}t\,e^{-t} - \frac{1}{4}e^{-3t}\right]u(t)$$