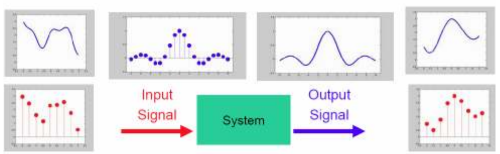
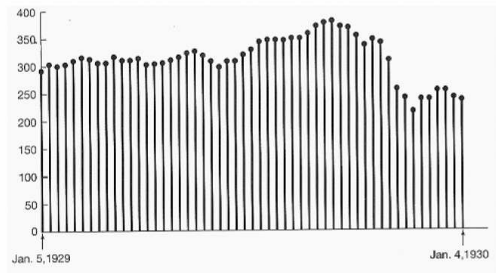
Chapter I
Signals And Systems

- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable
- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
- Continuous-Time & Discrete-Time Systems
- Basic System Properties

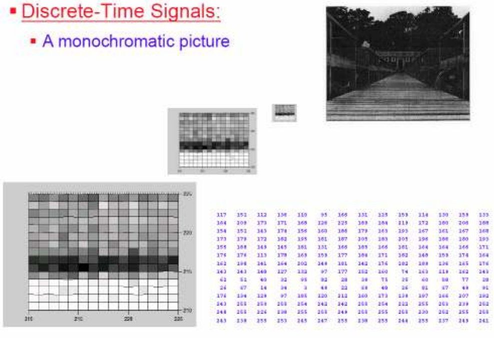
- Signals & Systems:
  - Is about using mathematical techniques to help describe and analyze systems which process signals
    - Signals are variables that carry information
    - Systems process input signals to produce output signals



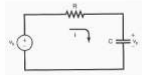
- Discrete-Time Signals:
  - The weekly Dow-Jones stock market index



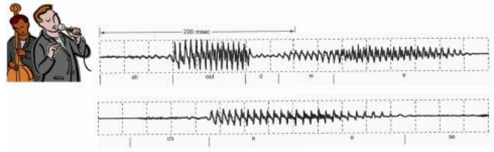
- Discrete-Time Signals:
  - A monochromatic picture



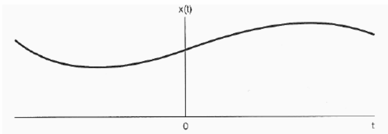
- Continuous-Time Signals:
  - Source voltage & capacity voltage in a simple RC circuit



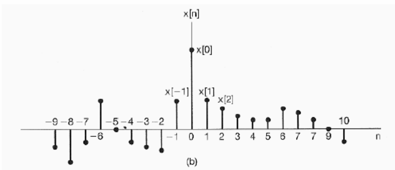
  - Recording of a speech signal



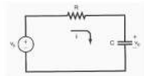
- Graphical Representations of Signals:
  - Continuous-time signals x(t) or $x_c(t)$



  - Discrete-time signals x[n] or $x_d[n]$



- Energy & Power of a resistor:



  - Instantaneous power

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$$

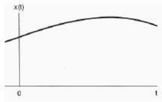
  - Total energy over a finite time interval

$$\int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

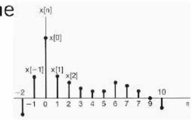
  - Average power over a finite time interval

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

- Signal Energy & Power:
  - Total energy over a finite time interval



$$E \triangleq \int_{t_1}^{t_2} |x(t)|^2 dt \qquad \text{continuous-time}$$



$$E \triangleq \sum_{n=n_1}^{n_2} |x[n]|^2 \qquad \text{discrete-time}$$

  - Time-averaged power over a finite time interval

$$P \triangleq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt \qquad \text{continuous-time}$$

$$P \triangleq \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2 \qquad \text{discrete-time}$$

- Signal Energy & Power:
  - Total energy over an infinite time interval

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[t]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

  - Time-averaged power over an infinite time interval

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[t]|^2$$

- Three Classes of Signals:

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

  - Finite total energy & zero average power

$$0 \le E_\infty < \infty \quad \Rightarrow \quad P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0$$

  - Finite average power & infinite total energy

$$0 \le P_\infty < \infty \quad \Rightarrow \quad E_\infty = \infty \ \ (\text{if } P_\infty > 0)$$

  - Infinite average power & infinite total energy

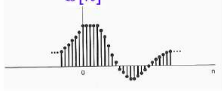
$$P_\infty = \infty \quad \& \quad E_\infty = \infty$$

- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable
  - Time Shift
  - Time Reversal
  - Time Scaling
  - Periodic Signals
  - Even & Odd Signals
- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
- Continuous-Time & Discrete-Time Systems
- Basic System Properties

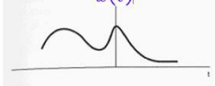
▪ Time Shift:

$$\begin{cases} n_0,\ t_0 > 0 : & \text{delay} \\ n_0,\ t_0 < 0 : & \text{advance} \end{cases}$$

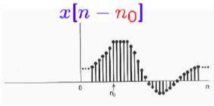
$x[n]$



$x(t)$



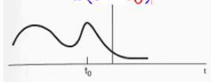
$x[n-n_0]$



$n_0 > 0$

$x[n-8]$

$x(t-t_0)$

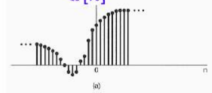


$t_0 < 0$

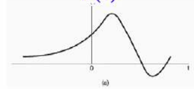
$x(t+5)$

▪ Time Reversal:

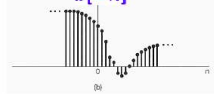
$x[n]$



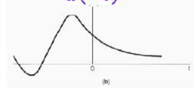
$x(t)$



$x[-n]$

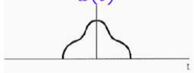


$x(-t)$



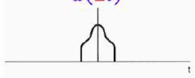
▪ Time Scaling:

$x(t)$



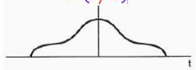
$t \rightarrow 2t$

$x(2t)$



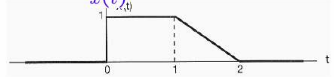
$t \rightarrow t/2$

$x(t/2)$



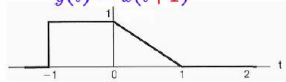
▪ x (t) → x (- t + 1)

$x(t)$



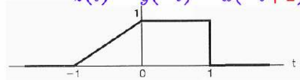
$t \rightarrow t+1$

$y(t) = x(t+1)$

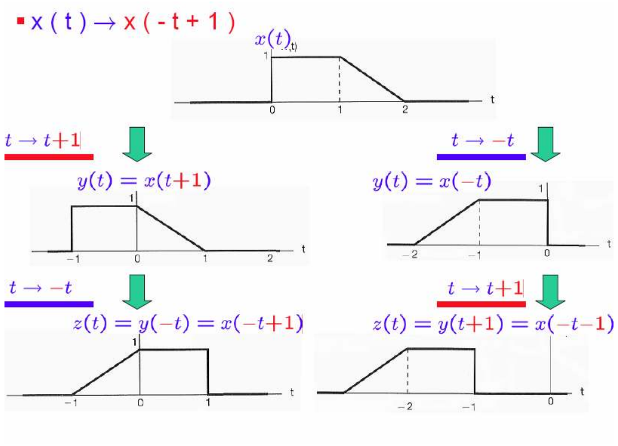


$t \rightarrow -t$

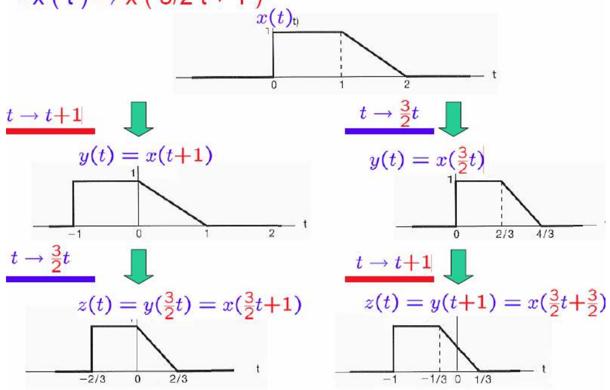
$z(t) = y(-t) = x(-t+1)$



- x (t) → x (- t + 1)

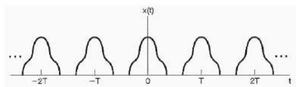


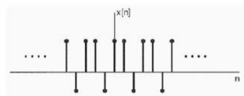
- x (t) → x (3/2 t + 1)



- x (t) → x (a t - b)
  - |a| < 1: linearly stretched
  - |a| > 1: linearly compressed
  - a < 0: time reversal
  - b > 0: delayed time shift
  - b < 0: advanced time shift

- CT & DT Periodic Signals:





$N = 3$

$x(t) = x(t+T)$ for $T > 0$ and all values of t

$x[n] = x[n+N]$ for $N > 0$ and all values of n

Periodic Signals:

$$x(t) = x(t+T) \quad \text{for } T > 0 \text{ and all values of } t$$

$$x[n] = x[n+N] \quad \text{for } N > 0 \text{ and all values of } n$$

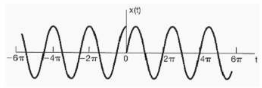
- A periodic signal is unchanged by a time shift of T or N
- They are also periodic with period
  - 2T, 3T, 4T, …
  - 2N, 3N, 4N, …
- T or N is called the fundamental period

  denoted as T_0 or N_0

Periodic signal ?

$$x(t) = x(t+T) \quad \forall t, \ T > 0$$

$$x(t) = \begin{cases} \cos(t), & \text{if } t < 0 \\ \sin(t), & \text{if } t \geq 0 \end{cases}$$

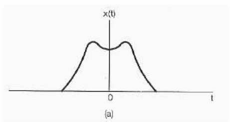


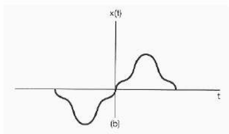
- Problems:
  - P1.25 for CT
  - P1.26 for DT

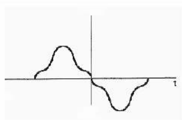
Even & odd signals:

A signal is even if $x(-t) = x(t)$ or $x[-n] = x[n]$

A signal is odd if $x(-t) = -x(t)$ or $x[-n] = -x[n]$







Even-odd decomposition of a signal:

- Any signal can be broken into a sum of one even signal and one odd signal

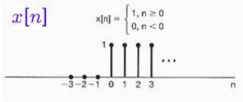
$$\mathcal{E}v\left\{x(t)\right\} = \frac{1}{2}\left[x(t) + x(-t)\right] = \frac{1}{2}\left[x(-t) + x(t)\right]$$

$$\mathcal{O}d\left\{x(t)\right\} = \frac{1}{2}\left[x(t) - x(-t)\right] = -\frac{1}{2}\left[x(-t) - x(t)\right]$$

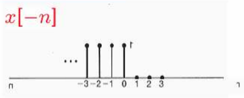
$$\Rightarrow x(t) = \mathcal{E}v\left\{x(t)\right\} + \mathcal{O}d\left\{x(t)\right\}$$

- Even-odd decomposition of a DT signal:

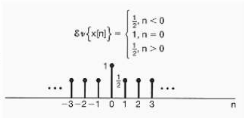
$x[n]$



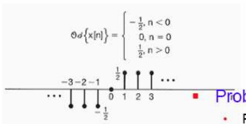
$x[-n]$



$$\mathcal{E}v\left\{x[n]\right\} = \frac{1}{2}\left[x[n] + x[-n]\right]$$



$$\mathcal{O}d\left\{x[n]\right\} = \frac{1}{2}\left[x[n] - x[-n]\right]$$



- Problems:
  - P1.23 for CT
  - P1.24 for DT

- Uniqueness of even-odd decomposition:

Assume that $x(t) = \mathcal{E}v_1(t) + \mathcal{O}d_1(t)$

and $x(t) = \mathcal{E}v_2(t) + \mathcal{O}d_2(t)$

So, $\mathcal{E}v_1(t) + \mathcal{O}d_1(t) = \mathcal{E}v_2(t) + \mathcal{O}d_2(t)$

and $\mathcal{E}v_1(-t) + \mathcal{O}d_1(-t) = \mathcal{E}v_2(-t) + \mathcal{O}d_2(-t)$

Because $\begin{cases} \mathcal{E}v_1(-t) = \mathcal{E}v_1(t) \\ \mathcal{E}v_2(-t) = \mathcal{E}v_2(t) \end{cases}$ and $\begin{cases} \mathcal{O}d_1(-t) = -\mathcal{O}d_1(t) \\ \mathcal{O}d_2(-t) = -\mathcal{O}d_2(t) \end{cases}$

Then, $\mathcal{E}v_1(t) - \mathcal{O}d_1(t) = \mathcal{E}v_2(t) - \mathcal{O}d_2(t)$

$\Rightarrow 2\mathcal{E}v_1(t) = 2\mathcal{E}v_2(t)$ or, $\mathcal{E}v_1(t) = \mathcal{E}v_2(t)$

$\Rightarrow 2\mathcal{O}d_1(t) = 2\mathcal{O}d_2(t)$ or, $\mathcal{O}d_1(t) = \mathcal{O}d_2(t)$

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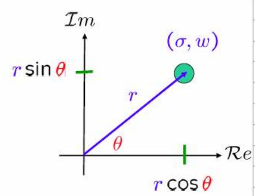
- Magnitude & Phase Representation:

$$\sigma + jw \;\Rightarrow\; \begin{cases} r = \sqrt{\sigma^2 + w^2} \\ \tan(\theta) = \frac{w}{\sigma} \end{cases}$$

$$\Rightarrow\; \sigma + jw \;=\; r\, e^{j\theta}$$



- Euler's relation:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\Rightarrow \sigma + jw = r\left(\cos\theta + j\sin\theta\right)$$

$$= (r\cos\theta) + j(r\sin\theta)$$

CT Complex Exponential Signals:

$$x(t) = Ce^{at}$$

- where C & a are, in general, complex numbers

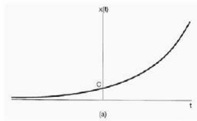
$$a = \sigma + jw$$

$$C = |C|\, e^{j\theta}$$

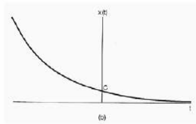
Real exponential signals:

- If C & a are real

$$x(t) = Ce^{at}$$



$a > 0$



$a < 0$

Periodic complex exponential signals:

$$a = \sigma + jw$$

- If a is purely imaginary

$$x(t) = e^{jw_0 t}$$

- It is periodic
  - Because let $T_0 = \dfrac{2\pi}{|w_0|}$
  - Then

$$e^{jw_0 T_0} = e^{jw_0 \frac{2\pi}{w_0}} = 1$$

  - Hence

$$e^{jw_0(t+T_0)} = e^{jw_0 t} e^{jw_0 T_0} = e^{jw_0 t}$$

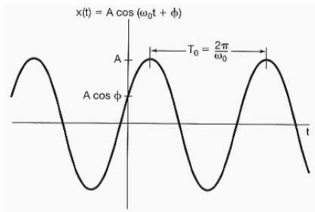
Periodic sinusoidal signals:

$$w_0 = 2\pi f_0$$

$$x(t) = A \cos(w_0 t + \phi)$$

$$T_0 = \frac{2\pi}{w_0}$$

$$T_0 = \frac{1}{f_0}$$

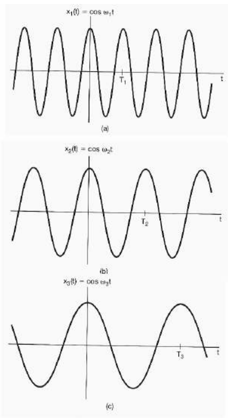


$T_0 : (sec)$

$w_0 : (rad/sec)$

$f_0 : (1/sec = Hz)$

- Period & Frequency:



$$T_0 = \frac{2\pi}{w_0}$$

$$w_0 = 2\pi f_0$$

$$T_0 = \frac{1}{f_0}$$

- Euler's relation:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos(\theta) = \mathcal{R}e\left\{e^{(j\theta)}\right\}$$

$$\sin(\theta) = \mathcal{I}m\left\{e^{(j\theta)}\right\}$$

$$e^{j(-\theta)} = \cos(-\theta) + j\sin(-\theta) \quad \Rightarrow \cos(\theta) = \frac{e^{(j\theta)} + e^{-(j\theta)}}{2}$$

$$= \cos(\theta) - j\sin(\theta) \quad \Rightarrow \sin(\theta) = \frac{e^{(j\theta)} - e^{-(j\theta)}}{2j}$$

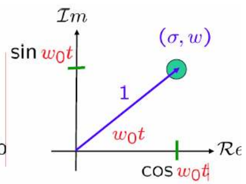
$$\Rightarrow A\cos(w_0 t + \phi) = \frac{A}{2}e^{j(\phi + w_0 t)} + \frac{A}{2}e^{-j(\phi + w_0 t)}$$

$$= \frac{A}{2}e^{j\phi}\,e^{jw_0 t} + \frac{A}{2}e^{-j\phi}\,e^{-jw_0 t}$$

- Total energy & average power:

$$E_{\text{period}} = \int_0^{T_0} \left|e^{jw_0 t}\right|^2 dt$$

$$= \int_0^{T_0} 1 \cdot dt = T_0$$



$$P_{\text{period}} = \frac{1}{T_0}E_{\text{period}} = 1$$

$$E_\infty = \infty$$

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T}\int_{-T}^{T} \left|e^{jw_0 t}\right|^2 dt = 1$$

- Problem:
  - P1.3

- Harmonically related periodic exponentials

$$e^{j0w_0 t},\ e^{j1w_0 t},\ e^{j2w_0 t},\ e^{j3w_0 t},\ \ldots,$$

$$e^{j(-1)w_0 t},\ e^{j(-2)w_0 t},\ \ldots$$

$$\phi_k(t) = e^{j\,k\,w_0\,t}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

- For $k = 0$, $\phi_k(t)$ is constant
- For $k \neq 0$, $\phi_k(t)$ is periodic with

  fundamental frequency $|k|w_0$ and

  fundamental period $\frac{T_0}{|k|}$

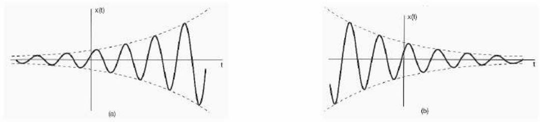
▪ General complex exponential signals:

$$Ce^{at} = (|C|e^{j\theta})(e^{(r+jw_0)t}) \qquad \sigma + jw = r\,e^{j\theta}$$

$$= (\,|C|\,e^{j\theta}\,)\,(\,e^{rt}\,e^{jw_0t}\,) \qquad e^{j\theta} = \cos\theta + j\sin\theta$$

$$= |C|\,e^{rt}\,e^{j(w_0t+\theta)}$$

$$= |C|e^{rt}\cos(w_0t+\theta) + j|C|e^{rt}\sin(w_0t+\theta)$$



▪ DT complex exponential signal or sequence:

$$x[n] = Ca^n$$

• where C & a are, in general, complex numbers

▪ Alternatively,

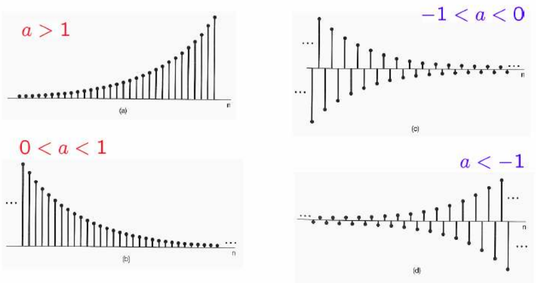
$$x[n] = Ce^{bn}$$

$$= C(e^b)^n \quad \text{with } a = e^b$$

▪ Real exponential signals:

• If C & a are real

$$x[n] = Ca^n$$



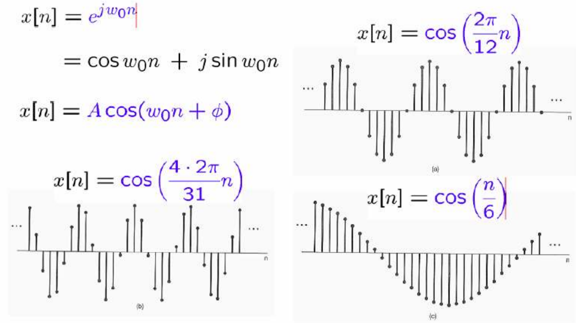
▪ DT Complex Exponential & Sinusoidal Signals

• If b is purely imaginary (or |a| = 1) $\qquad e^{j\theta} = \cos\theta + j\sin\theta$

$$x[n] = e^{jw_0n}$$

$$= \cos w_0n \; + \; j\sin w_0n$$

$$x[n] = A\cos(w_0n + \phi)$$



- **Euler's relation:**

$$e^{jw_0 n} = \cos w_0 n + j \sin w_0 n$$

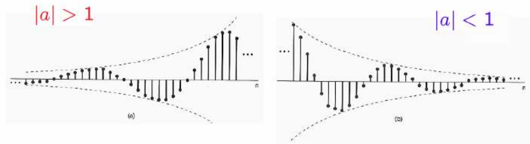
- And,

$$A\cos(w_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{jw_0 n} + \frac{A}{2} e^{-j\phi} e^{-jw_0 n}$$

- **General complex exponential signals:**

$$Ca^n = (|C|e^{j\theta})((|a|e^{jw_0})^n)$$

$$= |C||a|^n \cos(w_0 n + \theta) + j|C||a|^n \sin(w_0 n + \theta)$$



- **Periodicity properties of DT complex exponentials:**

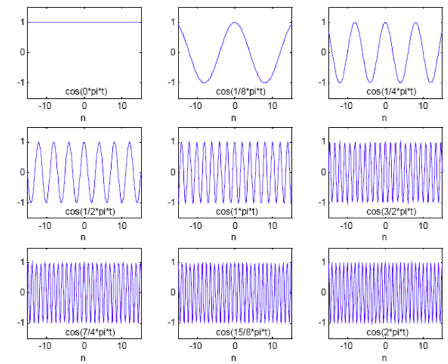
$$e^{j2\pi n} = \cos 2\pi n + j \sin 2\pi n$$

$$e^{j(w_0 + 2\pi)n} = e^{j2\pi n} e^{jw_0 n} = e^{jw_0 n}$$

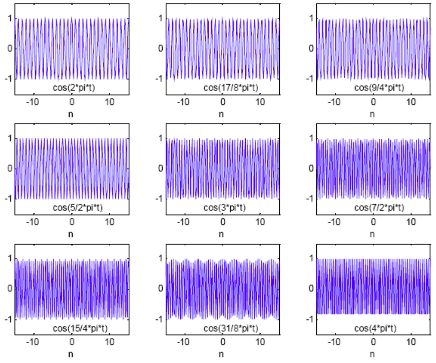
- The signal with frequency ω_0 is identical to the signals with frequencies $w_0 \pm 2\pi,\ w_0 \pm 4\pi,\ w_0 \pm 6\pi,\ \cdots$
- Only need to consider a frequency interval of length 2π
  - Usually use $0 \le w_0 < 2\pi$ or $-\pi \le w_0 < \pi,$
- The low frequencies are located at $w_0 = 0,\ \pm 2\pi,\ \cdots$
  The high frequencies are located at $w_0 = \pm\pi, \pm 3\pi,\ \cdots$

$$e^{j(0)n} = 1 \quad \text{and} \quad e^{j(\pi)n} = (e^{j(\pi)})^n = (-1)^n$$

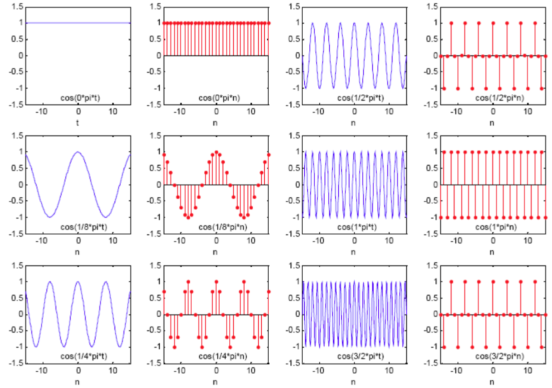
- **CT exponential signals**



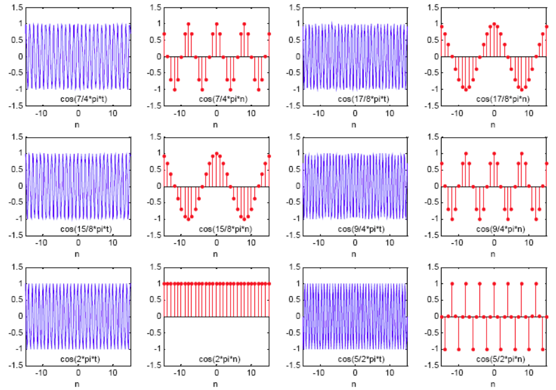
- CT exponential signals



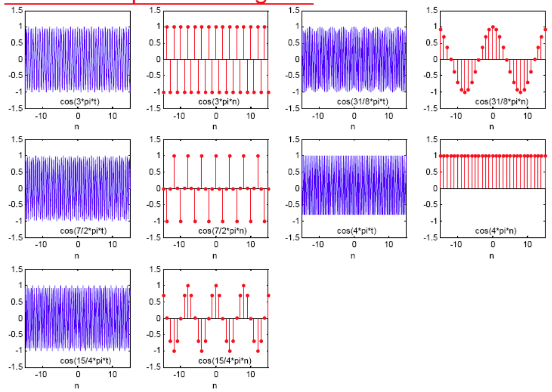
- CT & DT exponential signals



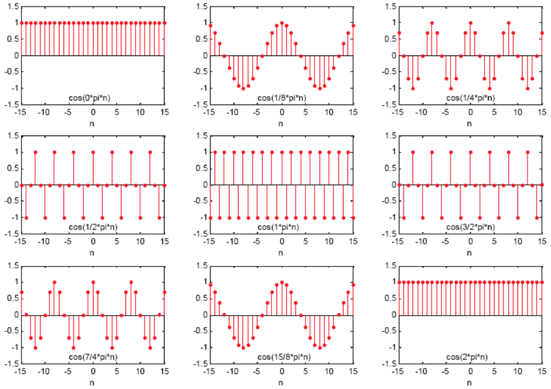
- CT & DT exponential signals



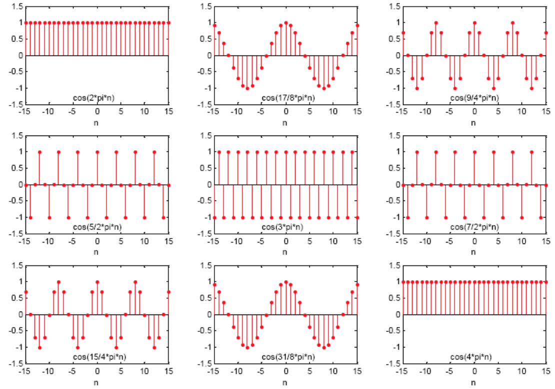
- CT & DT exponential signals



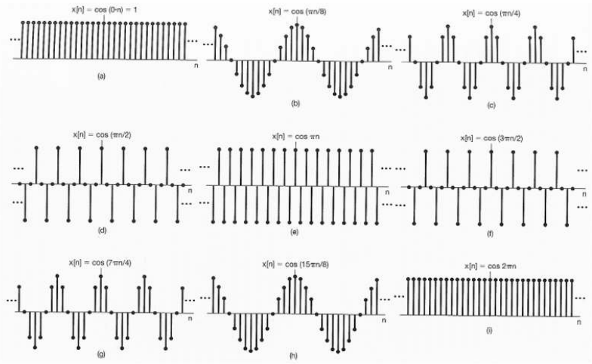
Periodicity properties of DT exponential signals



Periodicity properties of DT exponential signals



Periodicity properties of DT exponential signals



Periodicity properties of DT exponential signals

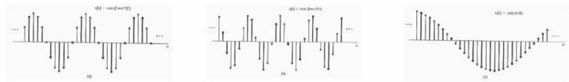
- Periodicity of N > 0

$$e^{jw_0(n+N)} = e^{jw_0 n} \quad \text{or} \quad e^{jw_0 N} = 1$$

- That is, $w_0 N = 2\pi m$ or $\dfrac{w_0}{2\pi} = \dfrac{m}{N}$
- Hence, $e^{jw_0 n}$ is periodic if $\frac{w_0}{2\pi}$ is a rational number

- Problem:
  - P1.35

$x[n] = \cos\left(\dfrac{2\pi}{12} n\right)$ $x[n] = \cos\left(\dfrac{4 \cdot 2\pi}{31} n\right)$ $x[n] = \cos\left(\dfrac{n}{6}\right)$



Comparison of CT & DT signals:

TABLE 1.1 Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

CT e^{jw_0t}	DT e^{jw_0n}
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

*Assumes that m and N do not have any factors in common.

- Problem:
  - P1.36

Harmonically related periodic exponentials

$$\phi_k[n] = e^{jk(2\pi/N)n}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$\phi_{k+N}[n] = e^{j(k+N)(2\pi/N)n}$$

$$= e^{jk(2\pi/N)n}\, e^{j2\pi n} = \phi_k[n]$$

- Only N distinct periodic exponentials in the set

$$\phi_0[n] = 1, \quad \phi_1[n] = e^{j(2\pi n/N)}, \quad \phi_2[n] = e^{j(4\pi n/N)},$$

$$\ldots, \quad \phi_{N-1}[n] = e^{j2\pi(N-1)n/N}$$

$$\phi_N[n] = e^{j2\pi(N)n/N} = e^{j2\pi n} = 1 = \phi_0[n], \; ; \quad \phi_{N+1}[n] = \phi_1[n], ..$$

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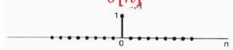
DT Unit Impulse & Unit Step Sequences

- Unit impulse (or unit sample)

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

$\delta[n]$

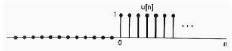


$\delta[n - n_0]$

n_0

- Unit step

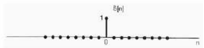
$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



Relationship Between Impulse & Step

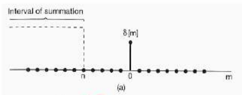
- First difference

$$\delta[n] = u[n] - u[n-1]$$

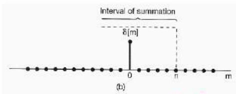


- Running sum

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] \qquad = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



$n < 0$



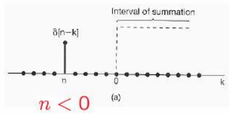
$n > 0$

Relationship Between Impulse & Step

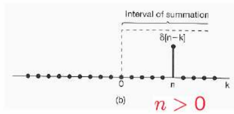
- Alternatively,

$$u[n] = \sum_{k=\infty}^{0} \delta[n-k], \quad \text{with } m = n - k$$

$$\text{or,} \quad u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$



$n < 0$



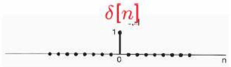
$n > 0$

Sample by Unit Impulse

- For x[n]

$$x[n]\delta[n] = x[0]\delta[n]$$

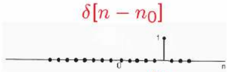
$\delta[n]$



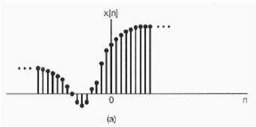
- More generally,

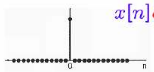
$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

$\delta[n-n_0]$

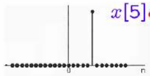


n_0





$x[n]\delta[n]$

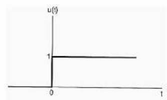


$x[5]\delta[n-5]$

CT Unit Impulse & Unit Step Functions

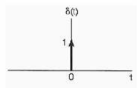
- Unit step function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



- Unit impulse function

$\delta(t)$



- Relationship Between Impulse & Step
  - Running integral

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \qquad = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

  - First derivative

$$\delta(t) = \frac{du(t)}{dt}$$

    - But, u(t) is discontinuous at t = 0, hence, not differentiable
    - Use approximation

- Relationship Between Impulse & Step
  - Use approximation

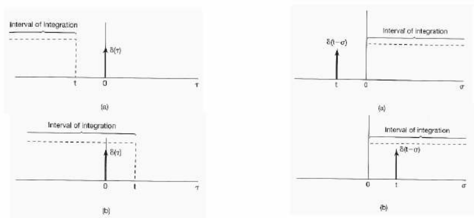
$$\delta_{\triangle}(t) = \frac{du_{\triangle}(t)}{dt}$$

$$\delta(t) = \lim_{\triangle \to 0} \delta_{\triangle}(t)$$

- Relationship Between Impulse & Step

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \quad = \int_{\infty}^{0} \delta(t-\sigma)(-d\sigma) \quad = \int_{0}^{\infty} \delta(t-\sigma)(d\sigma)$$

$$\tau = t - \sigma$$
$$d\tau = -d\sigma \qquad = \int_{0}^{\infty} \delta(t-\tau)(d\tau)$$

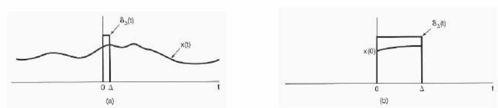


- Sample by Unit Impulse Function
  - For x(t)

$$x(t)\delta(t) = x(0)\delta(t)$$

  - More generally,

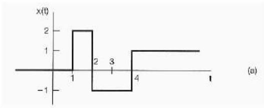
$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$



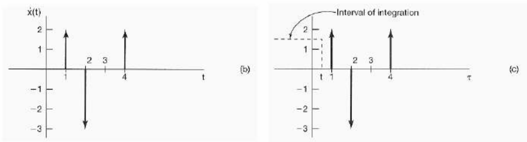
- Example 1.7:

$$\delta(t) = \frac{du(t)}{dt}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$



$$x(t) = \int_{0}^{t} \dot{x}(\tau)d\tau$$



- Introduction
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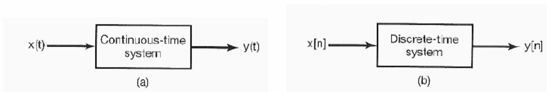
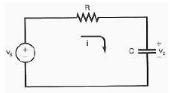
- Basic System Properties

- Physical Systems & Mathematical Descriptions
  - Examples of physical systems are signal processing, communications, electromechanical motors, automotive vehicles, chemical-processing plants
  - A system can be viewed as a process in which input signals are transformed by the system or cause the system to respond in some way, resulting in other signals or outputs



- Simple examples of CT systems
  - RC circuit

Input signal: $v_s(t)$



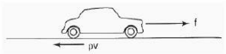
Output signal: $v_c(t)$

$$i(t) = \frac{v_s(t) - v_c(t)}{R} \qquad i(t) = C\frac{dv_c(t)}{dt}$$

$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

$$\Rightarrow \frac{dy(t)}{dt} + ay(t) = bx(t)$$

- Simple examples of CT systems
  - Automobile



Input signal: $f(t)$

Output signal: $v(t)$

$$f(t) - \rho v(t) = m\frac{dv(t)}{dt}$$

$$\frac{dv(t)}{dt} = \frac{1}{m}[f(t) - \rho v(t)]$$

$$\Rightarrow \frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

$$\Rightarrow \frac{dy(t)}{dt} + ay(t) = bx(t)$$

- Simple examples of DT systems
  - Balance in a band account

$$y[n] = 1.01y[n-1] + x[n]$$

or, $y[n] - 1.01y[n-1] = x[n]$

$$\Rightarrow y[n] + ay[n-1] = bx[n]$$

- Simple examples of DT systems
  - Digital simulation of differential equation

$$\frac{dv(t)}{dt} \approx \frac{v(n\Delta) - v((n-1)\Delta)}{\Delta} = \frac{v[n] - v[n-1]}{\Delta},$$

$t = n\Delta$

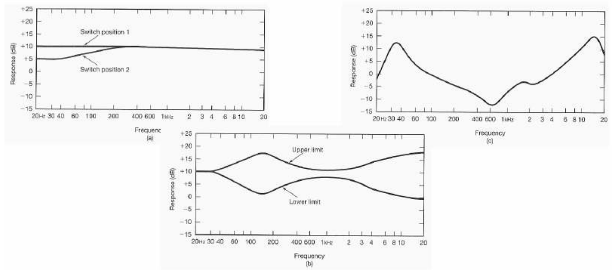
$$\frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

$$\Rightarrow v[n] - \frac{m}{m+\rho\Delta}v[n-1] = \frac{\Delta}{m+\rho\Delta}f[n]$$

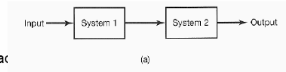
$$\Rightarrow y[n] + ay[n-1] = bx[n]$$

- Interconnections of Systems:
  - Audio System:

Microphone or Tape → Bass → Treble → Equalizer → Speaker

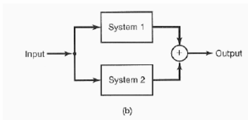


- Interconnections of Systems
  - Series or cascade interconnection of 2 systems



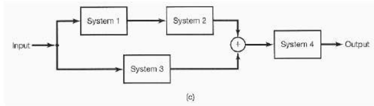
    > e.x. rac

  - Parallel interconnection of 2 systems

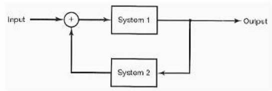


    > e.x. audio akers

- Interconnections of Systems
  - Series-parallel interconnection



  - Feedback interconnection



    > e.x. cruise control, electrical circuit

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  - Systems with & without memory
  - Invertibility & Inverse Systems
  - Causality
  - Stability
  - Time Invariance
  - Linearity

- Systems with & without memory
  - Memoryless systems
    - Output depends only on the input at that same time

$$y[n] = (2x[n] - x[n]^2)^2$$

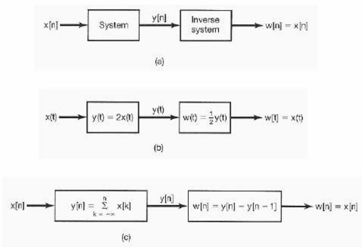
$$y(t) = Rx(t) \quad \text{(resistor)}$$

  - Systems with memory

$$y[n] = \sum_{k=-\infty}^{n} x[k] \quad \text{(accumulator)} \qquad y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

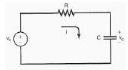
$$y[n] = x[n-1] \quad \text{(delay)}$$

- Invertibility & Inverse Systems
  - Invertible systems
    - Distinct inputs lead to distinct outputs

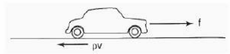


$y(t) = x(t)^2$ is not invertible

- Causality
  - Causal systems
    - Output depends only on input at present time & in the past



$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$



$$\Rightarrow \frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

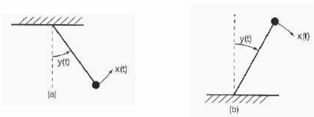
- Non-causal systems

$$y[n] = x[n] - x[n+1]$$

$$y(t) = x(t+1)$$

$$y(t) = x(t)\cos(t+1) \quad ???$$

- Stability
  - Stable systems
    - Small inputs lead to responses that do not diverge
    - Every bounded input excites a bounded output
      - Bounded-input bounded-output stable (BIBO stable)
      - For all $|x(t)| < a$, then $|y(t)| < b$, for all t



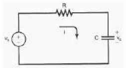
- Balance in a bank account? $y[n] = 1.01y[n-1] + x[n]$

- Example 1.13: Stability

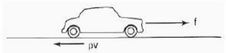
$$S_1: \quad y(t) = t\, x(t)$$

$$S_2: \quad y(t) = e^{x(t)}$$

- Time Invariance
  - Time-invariant systems
    - Behavior & characteristics of system are fixed over time



$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$



$$\Rightarrow \frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

    - A time shift in the input signal results in an identical time shift in the output signal

$$x[n] \rightarrow y[n] \iff x[n-n_0] \rightarrow y[n-n_0]$$

- Time Invariance
  - Example of time-invariant system (Example 1.14)

$$y(t) = \sin[x(t)]$$

$x_1(t)$ $\qquad y_1(t) = \sin[x_1(t)]$

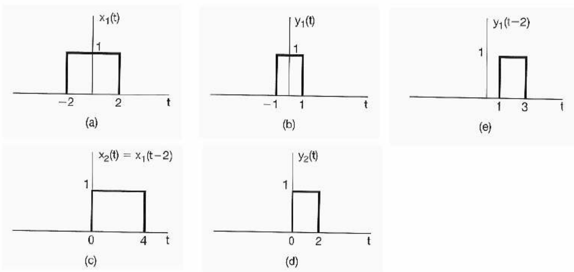
$x_2(t) = x_1(t-t_0)$ $\qquad y_2(t) = \sin[x_2(t)] = \sin[x_1(t-t_0)]$

$$y_1(t-t_0) = \sin[x_1(t-t_0)]$$

$$y_2(t) = y_1(t-t_0)$$

- Time Invariance
  - Example of time-varying system (Example 1.16)

$$y(t) = x(2t)$$



- Linearity
  - Linear systems
    - If an input consists of the weighted sum of several signals, then the output is the superposition of the responses of the system to each of those signals

$$x_1[n] \rightarrow y_1[n]$$

$$x_2[n] \rightarrow y_2[n]$$

IF (1) $x_1[n] + x_2[n] \rightarrow y_1[n] + y_2[n]$ (additivity)

(2) $a\,x_1[n] \rightarrow a\,y_1[n]$ (scaling or homogeneity)

a: any complex constant

THEN, the system is linear

- Linearity
  - Linear systems
    - In general, a, b: any complex constants

$$a x_1[n] + b x_2[n] \to a y_1[n] + b y_2[n] \quad \text{for DT}$$

$$a x_1(t) + b x_2(t) \to a y_1(t) + b y_2(t) \quad \text{for CT}$$

    - Or,

$$x[n] = \sum_k a_k x_k[n] = a_1 x_1[n] + a_2 x_2[n] + \ldots$$

$$\longrightarrow y[n] = \sum_k a_k y_k[n] = a_1 y_1[n] + a_2 y_2[n] + \ldots$$

This is known as the superposition property

- Linearity
  - Example 1.17: $S: \; y(t) = t x(t)$

$$x_1(t) \to y_1(t) = t x_1(t)$$

$$x_2(t) \to y_2(t) = t x_2(t)$$

$$x_3(t) = a x_1(t) + b x_2(t)$$

$$\to \; y_3(t) = t x_3(t)$$

$$= t(a x_1(t) + b x_2(t)) = a t x_1(t) + b t x_2(t)$$

$$= a y_1(t) + b y_2(t)$$

- Linearity
  - Example 1.18: $S: \; y(t) = (x(t))^2$

$$x_1(t) \; \to \; y_1(t) = (x_1(t))^2$$

$$x_2(t) \; \to \; y_2(t) = (x_2(t))^2$$

$$x_3(t) = a x_1(t) + b x_2(t)$$

$$\to \; y_3(t) = (x_3(t))^2 \qquad = (a x_1(t) + b x_2(t))^2$$

$$= a^2 (x_1(t))^2 + b^2 (x_2(t))^2 + 2ab x_1(t) x_2(t)$$

$$= a^2 y_1(t) + b^2 y_2(t) + 2ab x_1(t) x_2(t)$$

- Linearity
  - Example 1.20: $S: \; y[n] = 2x[n] + 3$

$$x_1[n] \; \to \; y_1[n] = 2x_1[n] + 3$$

$$x_2[n] \; \to \; y_2[n] = 2x_2[n] + 3$$

$$x_3[n] = a x_1[n] + b x_2[n]$$

$$\to \; y_3[n] = 2x_3[n] + 3$$

$$= 2(a x_1[n] + b x_2[n]) + 3$$

$$= a(2x_1[n] + 3) + b(2x_2[n] + 3) + 3 - 3a - 3b$$

$$= a y_1[n] + b y_2[n] + 3(1 - a - b)$$

- Linearity
  - Example 1.20: $S: \; y[n] = 2x[n] + 3$

$$x_1[n] \quad \rightarrow \quad y_1[n] = 2x_1[n] + 3$$

$$x_2[n] \quad \rightarrow \quad y_2[n] = 2x_2[n] + 3$$

  - However,

$$y_1[n] - y_2[n] = 2\left(x_1[n] + 3\right) - 2\left(x_2[n] + 3\right)$$

$$= 2\left[x_1[n] - x_2[n]\right]$$

It is a incrementally linear system