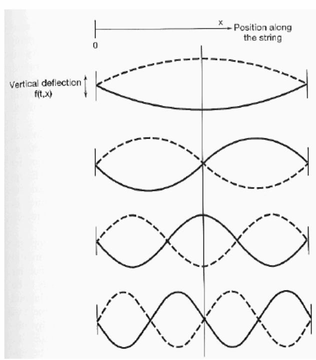
Chapter III

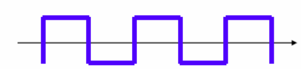
Fourier Series Representation of Periodic Signals

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

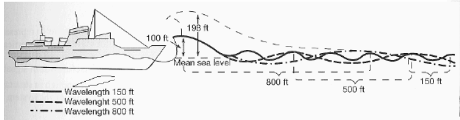
- L. Euler's study on the motion of a vibrating string in 1748



- L. Euler showed (in 1748)
  - The configuration of a vibrating string at some point in time is a linear combination of these normal modes
- D. Bernoulli argued (in 1753)
  - All physical motions of a string could be represented by linear combinations of normal modes
  - But, he did not pursue this mathematically
- J.L. Lagrange strongly criticized (in 1759)
  - The use of trigonometric series in examination of vibrating strings
  - Impossible to represent signals with corners using trigonometric series



- In 1807, Jean Baptiste Joseph Fourier
  - Submitted a paper of using trigonometric series to represent "any" periodic signal
  - It is examined by S.F. Lacroix, G. Monge, P.S. de Laplace, and J.L. Lagrange,
  - But Lagrange rejected it!
- In 1822, Fourier published a book "Theorie analytique de la chaleur"
  - "The Analytical Theory of Heat"



- Fourier's main contributions:
  - Studied vibration, heat diffusion, etc.
  - Found series of harmonically related sinusoids to be useful in representing the temperature distribution through a body
  - Claimed that "any" periodic signal could be represented by such a series (i.e., Fourier series discussed in Chap 3)
  - Obtained a representation for aperiodic signals (i.e., Fourier integral or transform discussed in Chap 4 & 5)
  - (Fourier did not actually contribute to the mathematical theory of Fourier series)

- Impact from Fourier's work:
  - Theory of integration, point-set topology, eigenfunction expansions, etc.
  - Motion of planets, periodic behavior of the earth's climate, wave in the ocean, radio & television stations
  - Harmonic time series in the 18th & 19th centuries
    - Gauss etc. on discrete-time signals and systems
  - Faster Fourier transform (FFT) in the mid-1960s
    - Cooley & Tukey in 1965 discovered independently
    - Can be found in Gauss's notebooks

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

- **Basic Idea:**
  - To represent signals as linear combinations of basic signals
- **Key Properties:**
  1. The set of basic signals can be used to construct a broad and useful class of signals
  2. The response of an LTI system to each signal should be simple enough in structure to provide us with a convenient representation for the response of the system to any signals constructed as linear combination of basic signals

---

- **One of Choices:**
  - The set of complex exponential signals

$$\begin{cases} \text{signals of form } e^{st} \text{ in CT} \\ \text{signals of form } z^{n} \text{ in DT} \end{cases}$$

- **The Response of an LTI System:**

input $\rightarrow$ LTI $\rightarrow$ output

$x(t)$ $h(t)$ $y(t)$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$\begin{cases} \text{CT:} & e^{st} \longrightarrow H(s)e^{st} \\ \text{DT:} & z^{n} \longrightarrow H(z)z^{n} \end{cases}$$

eigenfunction
eigenvalue

Let $x(t) = e^{st}$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$= e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

$$\Rightarrow y(t) = H(s)x(t) = H(s)e^{st}$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

Let $x[n] = z^{n}$

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} h[k]z^{n-k}$$

$$= z^{n}\sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

$$\Rightarrow y[n] = H(z)x[n] = H(z)z^{n}$$

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

- **Eigenfunctions and Superposition Properties:**

$e^{s_k t} \rightarrow$ LTI $\rightarrow H(s_k)\, e^{s_k t}$

$$a_1 e^{s_1 t} \longrightarrow a_1 H(s_1)e^{s_1 t}$$
$$a_2 e^{s_2 t} \longrightarrow a_2 H(s_2)e^{s_2 t}$$
$$a_3 e^{s_3 t} \longrightarrow a_3 H(s_3)e^{s_3 t}$$

$$x(t) = a_1\, e^{s_1 t} + a_2\, e^{s_2 t} + a_3\, e^{s_3 t}$$

$$y(t) = a_1\, H(s_1)\, e^{s_1 t} + a_2\, H(s_2)\, e^{s_2 t} + a_3\, H(s_3)\, e^{s_3 t}$$

$$\Rightarrow x(t) = \sum_k a_k e^{s_k t} \longrightarrow y(t) = \sum_k a_k H(s_k)e^{s_k t}$$

$$\Rightarrow x[n] = \sum_k a_k {z_k}^{n} \longrightarrow y[n] = \sum_k a_k H(z_k){z_k}^{n}$$

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

- <u>Harmonically related complex exponentials</u>

$$w_0 = \frac{2\pi}{T}$$

$$\phi_k(t) = e^{jkw_0 t} = e^{jk\left(\frac{2\pi}{T}\right)t}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

- <u>The Fourier Series Representation:</u>

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k\, \phi_k(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k\, e^{jk\left(\frac{2\pi}{T}\right)t}$$

$k = +1, -1$: the first harmonic components
or, the fundamental components

$k = +2, -2$: the second harmonic components

$\cdots$ etc.

- <u>Example 3.2:</u>

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk(2\pi)t}$$

$a_0 = 1$

$a_1 = a_{-1} = \frac{1}{4}$

$a_2 = a_{-2} = \frac{1}{2}$

$a_3 = a_{-3} = \frac{1}{3}$

$$\Rightarrow x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) + \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t})$$

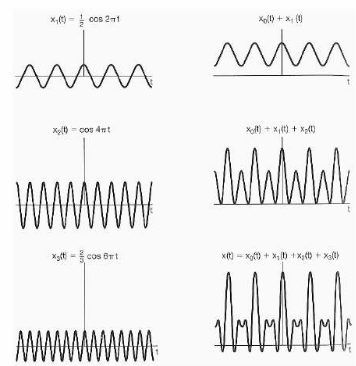
$$\Rightarrow x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$

$\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$

$\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$



- Procedure of Determining the Coefficients: $w_0 = \frac{2\pi}{T}$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$x(t)e^{-jnw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} e^{-jnw_0 t}$$

$$\int_0^T x(t)e^{-jnw_0 t}dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} e^{-jnw_0 t}dt$$

$$= \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)w_0 t}dt\right]$$

$$\int_0^T e^{j(k-n)w_0 t}dt = \int_0^T \cos\left((k-n)w_0 t\right)\, dt + j\int_0^T \sin\left((k-n)w_0 t\right)\, dt$$

- Procedure of Determining the Coefficients:

$$\int_0^T e^{j(k-n)w_0 t}dt = \int_0^T \cos\left((k-n)w_0 t\right)\, dt + j\int_0^T \sin\left((k-n)w_0 t\right)\, dt$$

$$= \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

$$\Rightarrow \int_0^T x(t)e^{-jnw_0 t}dt = a_n T \quad \Rightarrow a_n = \frac{1}{T}\int_0^T x(t)e^{-jnw_0 t}dt$$

$$\Rightarrow a_k = \frac{1}{T}\int_0^T x(t)e^{-jkw_0 t}dt$$

- Furthermore,

$$\int_T e^{j(k-n)w_0 t}dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases} \quad \Rightarrow a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt$$

- In Summary:
  - The synthesis equation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

  - The analysis equation:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

  - $x(t) \xleftrightarrow{\mathcal{FS}} a_k$: CT Fouries series pair
  - $\{a_k\}$: the Fourier series coefficients or the spectral coefficients of $x(t)$
  - $a_0 = \frac{1}{T}\int_T x(t)dt$, the dc or constant component of $x(t)$

- Fourier Series of Real Periodic Signals:
  - If x(t) is real, then $x^*(t) = x(t)$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$\Rightarrow x(t) = x(t)^* = \left(\sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}\right)^*$$

$$= \sum_{k=-\infty}^{+\infty} a_k^* e^{-jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jkw_0 t}$$

$$\Rightarrow a_{-k}^* = a_k \quad \text{or,} \quad a_k^* = a_{-k}$$

- Alternative Forms of the Fourier Series:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$\Rightarrow x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jkw_0 t} + a_{-k} e^{-jkw_0 t}\right]$$

$$= a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jkw_0 t} + a_k^* e^{-jkw_0 t}\right]$$

$$= a_0 + \sum_{k=1}^{\infty} 2\, \mathcal{R}e\left\{a_k e^{jkw_0 t}\right\}$$

- Alternative Forms of the Fourier Series:
  - If $a_k = A_k\, e^{j\theta_k}$

$$\Rightarrow x(t) = a_0 + \sum_{k=1}^{\infty} 2\, \mathcal{R}e\left\{A_k\, e^{j\theta_k}\, e^{jkw_0 t}\right\}$$

$$= a_0 + \sum_{k=1}^{\infty} 2\, \mathcal{R}e\left\{A_k\, e^{j(kw_0 t + \theta_k)}\right\}$$

$$= a_0 + 2\sum_{k=1}^{\infty} A_k \cos(kw_0 t + \theta_k)$$

  - If $a_k = B_k + j\, C_k$

$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$

$\boxed{(a+jb)(c+jd) = (ac-bd) + j(ad+bc)}$

$$\Rightarrow x(t) = a_0 + \sum_{k=1}^{\infty} 2\, \mathcal{R}e\left\{\left(B_k + j\, C_k\right)\, e^{jkw_0 t}\right\}$$

$$= a_0 + 2\sum_{k=1}^{\infty} \left[B_k \cos(kw_0 t) - C_k \sin(kw_0 t)\right]$$

- Example 3.4: $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$

$$x(t) = 1 + \sin w_0 t + 2\cos w_0 t + \cos\left(2w_0 t + \frac{\pi}{4}\right)$$

$$\Rightarrow x(t) = 1 + \frac{1}{2j}\left[e^{jw_0 t} - e^{-jw_0 t}\right] + \left[e^{jw_0 t} + e^{-jw_0 t}\right]$$

$$+ \frac{1}{2}\left[e^{j(2w_0 t + \pi/4)} + e^{-j(2w_0 t + \pi/4)}\right]$$

$$\Rightarrow x(t) = 1 + \left(1 + \frac{1}{2j}\right) e^{jw_0 t} + \left(1 - \frac{1}{2j}\right) e^{-jw_0 t}$$

$$+ \left(\frac{1}{2} e^{j(\pi/4)}\right) e^{j2w_0 t} + \left(\frac{1}{2} e^{-j(\pi/4)}\right) e^{-j2w_0 t}$$

$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta); \quad \cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta}); \quad \sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})}$

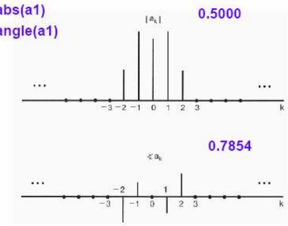
- Example 3.4:

$$\Rightarrow \begin{cases} a_0 = 1, \\ a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j, \\ a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j, \\ a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1 + j), \\ a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1 - j), \\ a_k = 0, \quad |k| > 2. \end{cases}$$

```
>> a1 = 1-0.5j
>> abs(a1)
>> angle(a1)
```

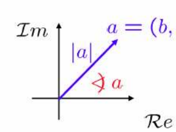
1.1180

0.5000



0.7854

-0.4636

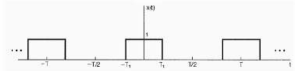


$\boxed{a = |a| e^{j\measuredangle a}}$

$\boxed{a = |a| \left[\cos(\measuredangle a) + j\sin(\measuredangle a)\right]}$

$\boxed{a = b + jc = \sqrt{b^2+c^2}\left[\frac{b}{\sqrt{b^2+c^2}} + j\frac{c}{\sqrt{b^2+c^2}}\right]}$

- Example 3.5: $a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$



$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

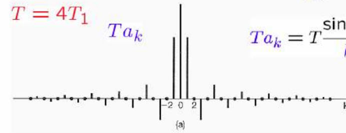
$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jkw_0 t}dt = -\frac{1}{jkw_0 T}e^{-jkw_0 t}\Big|_{-T_1}^{T_1}$$

$$= \frac{2}{kw_0 T}\left[\frac{e^{jkw_0 T_1} - e^{-jkw_0 T_1}}{2j}\right] \qquad w_0 = \frac{2\pi}{T}$$

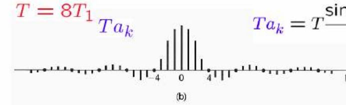
$$= \frac{2\sin(kw_0 T_1)}{kw_0 T} = \frac{\sin(kw_0 T_1)}{k\pi} = \frac{\sin(k(2\pi/T)T_1)}{k\pi}, \quad k \neq 0$$

- Example 3.5: $Ta_k = T\frac{\sin(k2\pi\frac{T_1}{T})}{k\pi}$

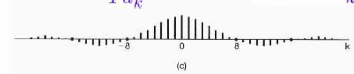
$T = 4T_1$ $\quad Ta_k$ $\quad Ta_k = T\frac{\sin(k\frac{\pi}{2})}{k\pi}$



$T = 8T_1$ $\quad Ta_k$ $\quad Ta_k = T\frac{\sin(k\frac{\pi}{4})}{k\pi}$



$T = 16T_1$ $\quad Ta_k$ $\quad Ta_k = T\frac{\sin(k\frac{\pi}{8})}{k\pi}$



$$w_0 = \frac{2\pi}{T}$$

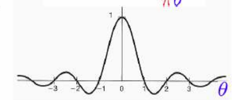
$$w = kw_0$$

$$Ta_k = \frac{2\sin(wT_1)}{w}$$

$$wT_1 = k\left(\frac{2\pi}{T}\right)\cdot T_1$$

$$= \frac{2k\pi}{A}$$

$$\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$



- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

- Fourier maintained that
  "any" periodic signal could be represented
  by a Fourier series
- The truth is that
  Fourier series can be used to represent
  an extremely large class of periodic signals
- The question is that
  when a periodic signal x(t) does in fact have a
  Fourier series representation?

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} \qquad a_k = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

- One class of periodic signals:
  - Which have finite energy over a single period:

$$\int_T |x(t)|^2\, dt < \infty \quad \Rightarrow \quad a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt < \infty$$

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jkw_0 t}$$

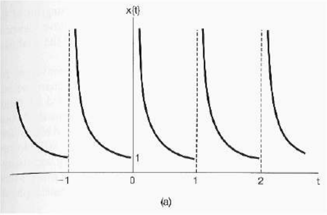
$$e_N(t) = x(t) - x_N(t) \qquad e(t) = x(t) - \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$E_N(t) = \int_T |e_N(t)|^2\, dt \qquad E(t) = \int_T |e(t)|^2\, dt \; = \; 0$$

$$\rightarrow\ 0 \quad \text{as } N \rightarrow \infty \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}, \quad \forall t\ ?$$

- The other class of periodic signals:
  - Which satisfy Dirichlet conditions:
  - Condition 1:
    - Over any period, x(t) must be absolutely integrable, i.e.,

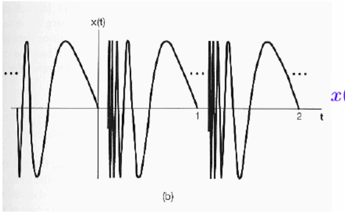
$$\int_T |x(t)|\, dt < \infty \quad \Rightarrow \quad |a_k| \;\le\; \frac{1}{T}\int_T \left|x(t)e^{-jkw_0 t}\right| dt$$



$$= \frac{1}{T}\int_T \left|x(t)\right| dt \;<\; \infty$$

$$x(t) = \frac{1}{t}, \quad 0 < t \le 1$$

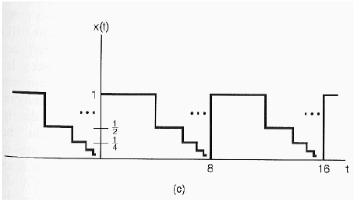
- The other class of periodic signals:
  - Which satisfy Dirichlet conditions:
  - Condition 2:
    - In any finite interval, x(t) is of bounded variation; i.e.,
    - There are no more than a finite number of maxima and minima during any single period of the signal



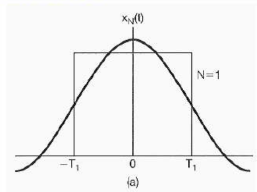
$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \le 1$$

$$\int_0^1 |x(t)|\, dt < 1$$

- The other class of periodic signals:
  - Which satisfy Dirichlet conditions:
  - Condition 3:
    - In any finite interval, x(t) has only finite number of discontinuities.
    - Furthermore, each of these discontinuities is finite

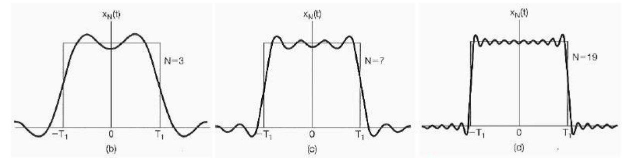


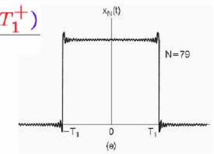
- How the Fourier series converges for a periodic signal with discontinuities
  - In 1898, Albert Michelson (an American physicist) used his harmonic analyzer to compute the truncated Fourier series approximation for the square wave



$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jkw_0 t}$$

$$x_1(t) = a_{-1}e^{-j\cdot 1\cdot w_0 t} + a_0 + a_1 e^{j\cdot 1\cdot w_0 t}$$



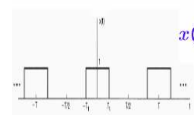


$$x_N(T_1) = \frac{x(T_1^-) + x(T_1^+)}{2}$$

- Michelson wrote to Josiah Gibbs
- In 1899, Gibbs showed that
  - the partial sum near discontinuity exhibits ripples &
  - the peak amplitude remains constant with increasing N
- The Gibbs phenomenon

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

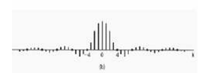
- CT Fourier Series Representation:
  - The synthesis equation:



$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

  - The analysis equation:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$



  - $x(t) \xleftrightarrow{\mathcal{FS}} a_k$: Fouries series pair

Section	Property
3.5.1	Linearity
3.5.2	Time Shifting
	Frequency Shifting
3.5.6	Conjugation
3.5.3	Time Reversal
3.5.4	Time Scaling
	Periodic Convolution
3.5.5	Multiplication
	Differentiation
	Integration
3.5.6	Conjugate Symmetry for Real Signals
3.5.6	Symmetry for Real and Even Signals
3.5.6	Symmetry for Real and Odd Signals
	Even-Odd Decomposition for Real Signals
3.5.7	Parseval's Relation for Periodic Signals

- Linearity: $a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0t}dt$
  - $x(t), y(t)$: periodic signals with period T

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t}$$

$$y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k \qquad y(t) = \sum_{m=-\infty}^{+\infty} b_m e^{jmw_0t}$$

$$\Rightarrow z(t) = Ax(t)+By(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k+Bb_k$$

$$z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jkw_0t}$$

- Time Shifting:
  - $x(t)$: periodic signal with period T

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\Rightarrow x(t-t_0) \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jkw_0t_0}a_k = e^{-jk\left(\frac{2\pi}{T}\right)t_0}a_k$$

b/c $$b_k = \frac{1}{T}\int_T x(t-t_0)e^{-jkw_0t}dt$$

$$= \frac{1}{T}\int_T x(\tau)e^{-jkw_0(\tau+t_0)}d\tau$$

$$= e^{-jkw_0t_0}\,\frac{1}{T}\int_T x(\tau)e^{-jkw_0\tau}d\tau$$

- Time Reversal:

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

$$\Rightarrow x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k} \qquad x(-t) = \sum_{k=-\infty}^{+\infty} a_k e^{-jk\left(\frac{2\pi}{T}\right)t}$$

$$= \sum_{m=-\infty}^{+\infty} a_{-m} e^{jm\left(\frac{2\pi}{T}\right)t}$$

- If $x(t)$ is even, i.e., $x(-t) = x(t)$
  - $\Rightarrow a_k$ is even, i.e., $a_{-k} = a_k$
- If $x(t)$ is odd, i.e., $x(-t) = -x(t)$
  - $\Rightarrow a_k$ is odd, i.e., $a_{-k} = -a_k$

- Time Scaling:
  - $x(t)$: periodic signals with period T and fundamental frequency w_0
  - $x(\alpha t)$: periodic signals with period $\frac{T}{\alpha}$ and fundamental frequency αw_0

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0(\alpha t)} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\alpha\left(\frac{2\pi}{T}\right)t}$$

$$= \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha w_0)t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{\left(\frac{T}{\alpha}\right)}\right)t}$$

- Multiplication: $z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jkw_0 t}$
  - $x(t), y(t)$: periodic signals with period T

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \qquad x(t) = \sum_{l=-\infty}^{+\infty} a_l e^{jlw_0 t}$$

$$y(t) \xleftrightarrow{\mathcal{FS}} b_k \qquad y(t) = \sum_{m=-\infty}^{+\infty} b_m e^{jmw_0 t}$$

$\Rightarrow x(t)y(t)$: also periodic with T

$$z(t) = x(t)y(t) \xleftrightarrow{\mathcal{FS}} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

- Differentiation: $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$
  - $x(t)$: periodic signals with period T

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{FS}} jkw_0\, a_k$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \quad e^{jkw_0 t}$$

- Integration: $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$
  - $x(t)$: periodic signals with period T

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{FS}} \frac{1}{jkw_0} a_k$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \quad e^{jkw_0 t}$$

- Conjugation & Conjugate Symmetry:

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k \ e^{jkw_0 t}$$

$$x(t)^* \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}^* \qquad = \sum_{k=-\infty}^{+\infty} a_k \ e^{jkw_0 t}$$

- $x(t) = x(t)^* \Rightarrow a_{-k} = a_k^*$

  $x(t)$ is real $\Rightarrow \{a_k\}$ are conjugate symmetric $\qquad = \sum_{k=-\infty}^{+\infty} a_k \ e^{j \ \ k \ w_0 t}$

- $x(t) = x(t)^*$ & $x(-t) = x(t) \Rightarrow a_{-k} = a_k^*$ & $a_{-k} = a_k$

  $\Rightarrow a_k = a_k^*$

  $x(t)$ is real & even $\Rightarrow \{a_k\}$ are real & even

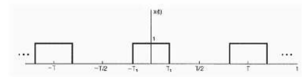
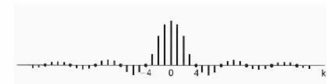
- $x(t)$ is real & odd $\Rightarrow \{a_k\}$ are purely imaginary & odd

- Parseval's relation for CT periodic signals:
  - As shown in Problem 3.46:

$$\frac{1}{T}\int_T \left|x(t)\right|^2 dt = \sum_{k=-\infty}^{+\infty} \left|a_k\right|^2$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt$$

(b)

  - Parseval's relation states that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components

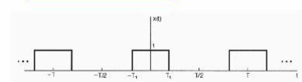
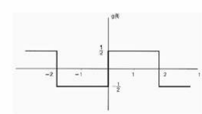
TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$, $y(t)$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k, b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real]	$\Re e\{a_k\}$
		$x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

- Example 3.6:

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$$a_0 = \frac{2T_1}{T}$$

$$a_k = \frac{\sin(k(2\pi/T)T_1)}{k\pi}, \quad k \neq 0$$

$$g(t) = x(t-1) - 1/2$$

with $T = 4, T_1 = 1$

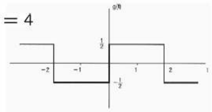
$$x(t-1) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = a_k e^{-jk\pi/2}$$

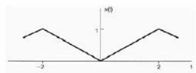
$$g(t) = x(t-1) - 1/2 \overset{\mathcal{FS}}{\longleftrightarrow} \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0 \\ a_0 - 1/2, & \text{for } k = 0 \end{cases}$$

$$g(t) \overset{\mathcal{FS}}{\longleftrightarrow} \begin{cases} \frac{\sin(k\pi/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}$$

- Example 3.7:

$T = 4$





$$g(t) \xleftrightarrow{\mathcal{FS}} d_k$$

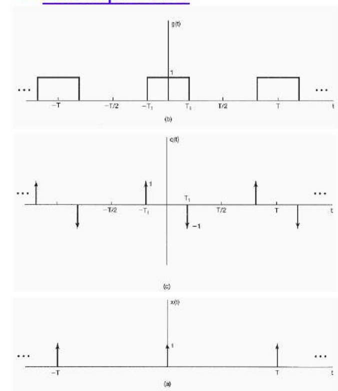
$$x(t) \xleftrightarrow{\mathcal{FS}} e_k$$

$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{FS}} jkw_0\, e_k$$

$$g(t) = \frac{d}{dt}x(t) \iff d_k = jk(\pi/2)e_k$$

$$e_k = \begin{cases} \frac{2}{jk\pi}\, d_k = \frac{2\sin(\pi k/2)}{j(k\pi)^2}e^{-jk\pi/2}, & \text{for } k \neq 0 \\ \frac{1}{2}, & \text{for } k = 0 \end{cases}$$

- Example 3.8:



$$g(t) \xleftrightarrow{\mathcal{FS}} c_k$$

$$q(t) = \frac{d}{dt}g(t) \iff b_k = jkw_0 c_k$$

$$q(t) \xleftrightarrow{\mathcal{FS}} b_k$$

$$q(t) = x(t+T_1) - x(t-T_1)$$

$$\iff b_k = e^{jkw_0T_1}a_k - e^{-jkw_0T_1}a_k$$

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k = \frac{1}{T}$$

- Example 3.8:

$$b_k = e^{jkw_0T_1}a_k - e^{-jkw_0T_1}a_k$$

$$= \frac{1}{T}\left[e^{jkw_0T_1} - e^{-jkw_0T_1}\right]$$

$$= \frac{2j\sin(kw_0T_1)}{T}$$

$$b_k = jkw_0 c_k$$

$$k \neq 0 \qquad c_k = \frac{b_k}{jkw_0} = \frac{2j\sin(kw_0T_1)}{jkw_0T} = \frac{\sin(kw_0T_1)}{k\pi}$$

$$k = 0 \qquad c_0 = \frac{2T_1}{T}$$

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

- Harmonically related complex exponentials

$$\phi_k[n] = e^{jkw_0 n} = e^{jk\left(\frac{2\pi}{N}\right)n}, \qquad k = 0, \pm 1, \pm 2, ...$$

$$\phi_{k+N}[n] = e^{j(k+N)\left(\frac{2\pi}{N}\right)n} = e^{jk\left(\frac{2\pi}{N}\right)n} e^{jN\left(\frac{2\pi}{N}\right)n}$$

$$\Rightarrow \phi_k[n] = \phi_{k+N}[n] = \cdots = \phi_{k+\,r\,N}[n]$$

- The Fourier Series Representation:

$$x[n] = \sum_{k=<N>} a_k \phi_k[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

- Procedure of Determining the Coefficients:

$$x[0] = \sum_{k=<N>} a_k$$

$$x[1] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)}$$

$$x[2] = \sum_{k=<N>} a_k e^{jk2\left(\frac{2\pi}{N}\right)}$$

$$\vdots$$

$$x[N-1] = \sum_{k=<N>} a_k e^{jk(N-1)\left(\frac{2\pi}{N}\right)}$$

and $$\sum_{n=<N>} e^{jm\left(\frac{2\pi}{N}\right)n} = \begin{cases} N, & m = 0, \pm N, \pm 2N, ... \\ 0, & \text{otherwise} \end{cases}$$

- Procedure of Determining the Coefficients:

$$x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \qquad \sum_{n=<N>} e^{-jr\left(\frac{2\pi}{N}\right)n}$$

$$\sum_{n=<N>} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{n=<N>}\sum_{k=<N>} a_k e^{j(k-r)\left(\frac{2\pi}{N}\right)n}$$

$$\sum_{n=<N>} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{k=<N>} a_k \sum_{n=<N>} e^{j(k-r)\left(\frac{2\pi}{N}\right)n}$$

$$= a_r N$$

$$\Rightarrow a_r = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n}$$

- In Summary:
  - The synthesis equation:

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

  - The analysis equation:

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jkw_0 n} = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$a_k = a_{k+N}$$

  - $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$: DT Fouries series pair
  - $\{a_k\}$: the Fourier series coefficients or the spectral coefficients of $x[n]$

- Example 3.11:

$$x[n] \;=\; 1 \;+\; \sin\left(\frac{2\pi}{N}\right)n \;+\; 3\cos\left(\frac{2\pi}{N}\right)n \;+\; \cos\left(\frac{4\pi}{N}n+\frac{\pi}{2}\right)$$

$$\Rightarrow\; x[n] = 1 + \frac{1}{2j}\left[e^{j\left(\frac{2\pi}{N}\right)n} - e^{-j\left(\frac{2\pi}{N}\right)n}\right] + \frac{3}{2}\left[e^{j\left(\frac{2\pi}{N}\right)n} + e^{-j\left(\frac{2\pi}{N}\right)n}\right]$$

$$+\;\frac{1}{2}\left[e^{j\left(\frac{4\pi}{N}n+\frac{\pi}{2}\right)} \;+\; e^{-j\left(\frac{4\pi}{N}n+\frac{\pi}{2}\right)}\right]$$

$$\Rightarrow\; x[n] = 1 \;+\; \left(\frac{3}{2}+\frac{1}{2j}\right)e^{j\left(\frac{2\pi}{N}\right)n} \;+\; \left(\frac{3}{2}-\frac{1}{2j}\right)e^{-j\left(\frac{2\pi}{N}\right)n}$$

$$+\;\frac{1}{2}\,e^{j\left(\frac{\pi}{2}\right)}\,e^{j2\left(\frac{2\pi}{N}\right)n} \;+\; \frac{1}{2}\,e^{-j\left(\frac{\pi}{2}\right)}\,e^{-j2\left(\frac{2\pi}{N}\right)n}$$

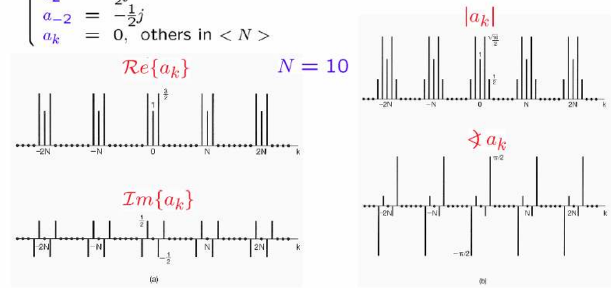
- Example 3.11:

$$\boxed{a = |a|e^{j\measuredangle a}}$$

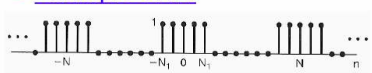
$$\boxed{a = |a|\left[\cos(\measuredangle a) + j\sin(\measuredangle a)\right]}$$

$$\boxed{a = b + jc = \sqrt{b^2+c^2}\left[\frac{b}{\sqrt{b^2+c^2}} + j\frac{c}{\sqrt{b^2+c^2}}\right]}$$

$$\Rightarrow \begin{cases} a_0 &= 1 \\ a_1 &= \left(\frac{3}{2}+\frac{1}{2j}\right) = \frac{3}{2}-\frac{1}{2}j \\ a_{-1} &= \left(\frac{3}{2}-\frac{1}{2j}\right) = \frac{3}{2}+\frac{1}{2}j \\ a_2 &= \frac{1}{2}j \\ a_{-2} &= -\frac{1}{2}j \\ a_k &= 0,\ \text{others in } < N > \end{cases}$$



- Example 3.12:



$$x[n] = \begin{cases} 1, & -N_1 \le n \le N_1 \\ 0, & \text{others in } < N > \end{cases}$$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} 1\cdot e^{-jk\left(\frac{2\pi}{N}\right)n} = \frac{1}{N}\sum_{n=-N_1}^{N_1}\left(e^{-jk\left(\frac{2\pi}{N}\right)}\right)^n$$

$$= \frac{1}{N}\left[(\cdot)^{-N_1} + (\cdot)^{-N_1+1} + \cdots + (\cdot)^{N_1}\right]$$

$$= \frac{1}{N}\left[\frac{1-(\cdot)^{(2N_1+1)}}{1-(\cdot)}\right] \qquad (\cdot) \neq 1$$

- Let $m = n + N_1$ or $n = m - N_1$

$$a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)(m-N_1)} \qquad = \frac{1}{N}e^{jk\left(\frac{2\pi}{N}\right)N_1}\sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)m}$$

- Example 3.12:

$$\boxed{1 - e^{-j\theta} = e^{-j\theta/2}\left(e^{j\theta/2} - e^{-j\theta/2}\right)}$$

- $k \neq 0, \pm N, \pm 2N, \ldots$

$$a_k = \frac{1}{N}\,e^{jk\left(\frac{2\pi}{N}\right)N_1}\left(\frac{1-e^{-jk\left(\frac{2\pi}{N}\right)(2N_1+1)}}{1-e^{-jk\left(\frac{2\pi}{N}\right)}}\right) \qquad (N_1 + \frac{1}{2})$$

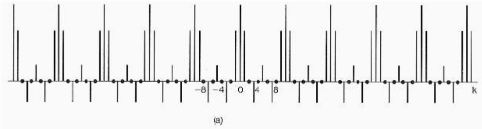
$$= \frac{1}{N}\,\frac{e^{-jk\left(\frac{2\pi}{2N}\right)}\left[e^{jk\left(\frac{2\pi}{2N}\right)(2N_1+1)} - e^{-jk\left(\frac{2\pi}{2N}\right)(2N_1+1)}\right]}{e^{-jk\left(\frac{2\pi}{2N}\right)}\left[e^{jk\left(\frac{2\pi}{2N}\right)} - e^{-jk\left(\frac{2\pi}{2N}\right)}\right]}$$

$$= \frac{1}{N}\,\frac{\sin\left[\left(\frac{2\pi}{N}\right)k(N_1+\frac{1}{2})\right]}{\sin\left[\left(\frac{\pi}{N}\right)k\right]}$$

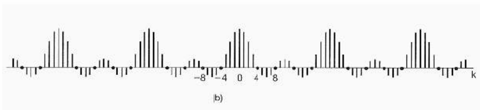
- $k = 0, \pm N, \pm 2N, \ldots$

$$a_k = \frac{2N_1+1}{N}$$

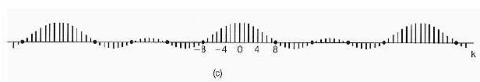
- Example 3.12:
  - $2N_1 + 1 = 5$
  - $N = 10$



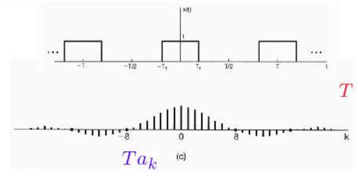
  - $N = 20$



  - $N = 40$



- Examples 3.5 (CT) & 3.12 (DT):



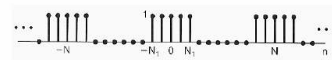
$$Ta_k = T\frac{\sin(k\frac{\pi}{8})}{k\pi}$$

$T = 16T_1$

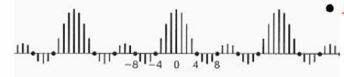
Ta_k

$$a_k = \frac{1}{N}\frac{\sin\left[\left(\frac{2\pi}{N}\right)k(N_1+\frac{1}{2})\right]}{\sin\left[\left(\frac{\pi}{N}\right)k\right]}$$

$$a_k = \frac{2N_1+1}{N}$$



- $N = 20$



- Partial Sum:

$N = 9$

$$x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

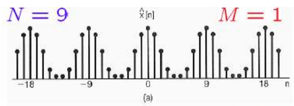
- If N is odd

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

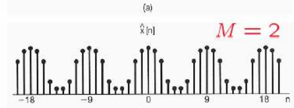
- If N is even

$$\hat{x}[n] = \sum_{k=-M+1}^{M} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

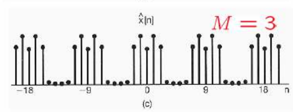
$M = 1$



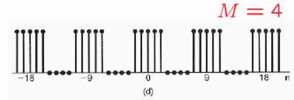
$M = 2$



$M = 3$



$M = 4$



- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

Section	Property
	Linearity
	Time Shifting
	Frequency Shifting
	Conjugation
	Time Reversal
	Time Scaling
	Periodic Convolution
3.7.1	Multiplication
3.7.2	First Difference
	Running Sum
	Conjugate Symmetry for Real Signals
	Symmetry for Real and Even Signals
	Symmetry for Real and Odd Signals
	Even-Odd Decomposition for Real Signals
3.7.3	Parseval's Relation for Periodic Signals

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

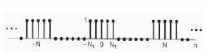
Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} \lvert x[n]\rvert^2 = \sum_{k=\langle N\rangle} \lvert a_k\rvert^2$$

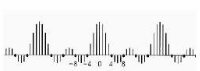
- In Summary:
  - The synthesis equation:

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} \qquad = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$



  - The analysis equation:

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jkw_0 n} = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n}$$



$$a_k = a_{k+N}$$

  - $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$: DT Fouries series pair

- Linearity:
  - $x[n], y[n]$: periodic signals with period N

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$y[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

$$\Rightarrow z[n] = Ax[n] + By[n] \overset{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k$$

- Time Shifting:

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\Rightarrow x[n - n_0] \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jkw_0 n_0}a_k = e^{-jk\left(\frac{2\pi}{N}\right)n_0}a_k$$

- Multiplication:

  - $x[n], y[n]$: periodic signals with period N

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x[n] = \sum_{l=<N>} a_l e^{jlw_0 n}$$

$$y[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k \qquad y[n] = \sum_{m=<N>} b_m e^{jmw_0 n}$$

$\Rightarrow x[n]y[n]$: also periodic with N

$$x[n]y[n] \overset{\mathcal{FS}}{\longleftrightarrow} d_k = \sum_{l=<N>} a_l\, b_{k-l}$$

$\Rightarrow$ a periodic convolution

- First Difference:

$$x[n] = \sum_{l=<N>} a_k e^{jkw_0 n}$$

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

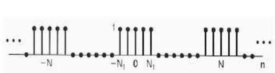
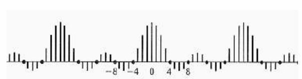
$$\Rightarrow x[n-n_0] \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jkw_0 n_0} a_k \quad = e^{-jk\left(\frac{2\pi}{N}\right)n_0} a_k$$

$$\Rightarrow x[n-1] \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jkw_0} a_k \quad = e^{-jk\left(\frac{2\pi}{N}\right)} a_k$$

$$x[n] - x[n-1] \overset{\mathcal{FS}}{\longleftrightarrow} \left(1 - e^{-jk\left(\frac{2\pi}{N}\right)}\right) a_k$$

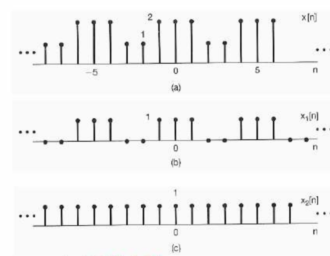
- Parseval's relation for DT periodic signals:
  - As shown in Problem 3.57:

$$\frac{1}{N}\sum_{k=<N>} \left|x[n]\right|^2 = \sum_{k=<N>} \left|a_k\right|^2 \qquad x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} \qquad a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jkw_0 n}$$

  - Parseval's relation states that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components (only N distinct harmonic components in DT)

- Example 3.13:



$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$x_1[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

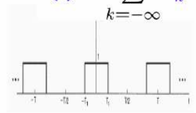
$$x_2[n] \overset{\mathcal{FS}}{\longleftrightarrow} c_k$$

$$\Rightarrow b_k = \begin{cases} \frac{1}{5}\frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\ \frac{3}{5}, & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases} \qquad x[n] = x_1[n] + x_2[n]$$

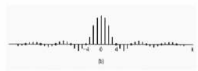
$$\Rightarrow c_k = \begin{cases} 0, & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\ 1, & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases} \quad \Rightarrow \quad a_k = b_k + c_k$$

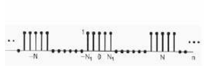
▪ CT & DT Fourier Series Representation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} \qquad a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt$$



$x(t) \overset{\mathcal{CTFS}}{\longleftrightarrow} a_k$





$x[n] \overset{\mathcal{DTFS}}{\longleftrightarrow} a_k$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jkw_0 n} \qquad a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jkw_0 n}$$

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

▪ The Response of an LTI System:

$in \rightarrow$ LTI $\rightarrow out$

$$\begin{cases} \text{CT:} & e^{st} \longrightarrow H(s)e^{st} \\ \text{DT:} & z^n \longrightarrow H(z)z^n \end{cases}$$

$$H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt \qquad \Rightarrow \text{the impulse response}$$

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k} \qquad \Rightarrow \text{the system functions}$$

- If $s = jw$ or $z = e^{jw}$:

$$H(jw) = \int_{-\infty}^{+\infty} h(t)e^{-jwt}dt$$

$$H(e^{jw}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-jwn}$$

$\Rightarrow$ the frequency response

▪ In Summary:

$$\boxed{a = |a|e^{j\measuredangle a}}$$

$$\boxed{H = |H|e^{j\measuredangle H}}$$

$in \rightarrow$ LTI H(s/z/w) $\rightarrow out$

($s_i = jw_i$ or $z_i = e^{jw_i}$)

$$\begin{cases} \text{CT:} & e^{s_i t} \longrightarrow H(s_i)e^{s_i t} \\ \text{DT:} & z_i^n \longrightarrow H(z_i)z_i^n \end{cases}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{jkw_0 t} \longrightarrow y(t) = \sum_{k=-\infty}^{+\infty} a_k\, H(jkw_0)\, e^{jkw_0 t}$$

$$x[n] = \sum_{k=\langle N\rangle} a_k\, e^{jk\left(\frac{2\pi}{N}\right)n} \longrightarrow y[n] = \sum_{k=\langle N\rangle} a_k\, H(e^{j\left(\frac{2\pi}{N}\right)k})\, e^{jk\left(\frac{2\pi}{N}\right)n}$$

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

- Filtering:

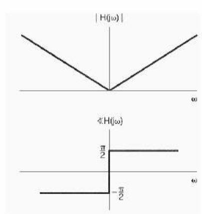
$in \rightarrow$ filter $\rightarrow out$

  - Change the relative amplitudes of the frequency components in a signal,
    - Frequency-shaping filters
  - OR, significantly attenuate or eliminate some frequency components entirely
    - Frequency-selective filters

- Frequency-Shaping Filters:
  - Differentiating filter:

$x(t) \rightarrow$ Differentiating Filter $\rightarrow y(t) = \frac{d}{dt}x(t)$

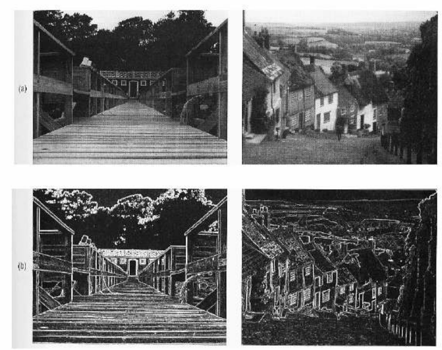
$H(jw) = jw$



- Frequency-Shaping Filters:

$x(t) \rightarrow$ Differentiating Filter $\rightarrow y(t) = \frac{d}{dt}x(t)$

  - Differentiating filter on enhancing edges: $H(jw) = jw$



- <u>Frequency-Shaping Filters:</u>
  - A simple DT filter: Two-point average

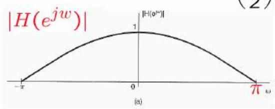
$$y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right)$$

$$y[n] = H(e^{jw})\, x[n]$$

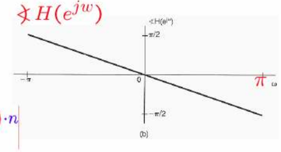
$$\Rightarrow H(e^{jw}) = \frac{1}{2}\left[1 + e^{-jw}\right] = \frac{1}{2}e^{-j\left(\frac{w}{2}\right)}\left[e^{j\left(\frac{w}{2}\right)} + e^{-j\left(\frac{w}{2}\right)}\right]$$

$$= e^{-j\left(\frac{w}{2}\right)} \cos\left(\frac{w}{2}\right)$$

$|H(e^{jw})|$

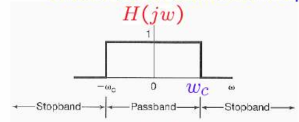


$\measuredangle H(e^{jw})$



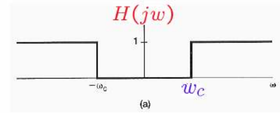
$$y[n] = H\left(e^{j\left(\frac{\pi}{2}\right)}\right) K e^{j\left(\frac{\pi}{2}\right)\cdot n}$$

- <u>Frequency-Selective Filters:</u>
  - Select some bands of frequencies and reject others



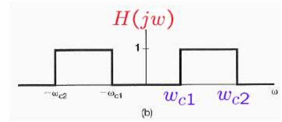
CT ideal lowpass filter

$$H(jw) = \begin{cases} 1, & |w| \le w_c \\ 0, & |w| > w_c \end{cases}$$



CT ideal highpass filter

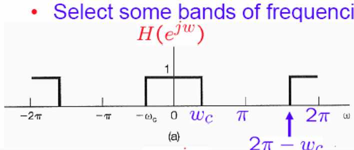
$$H(jw) = \begin{cases} 0, & |w| < w_c \\ 1, & |w| \ge w_c \end{cases}$$



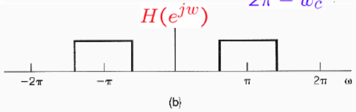
CT ideal bandpass filter

$$H(jw) = \begin{cases} 1, & w_{c1} \le |w| \le w_{c2} \\ 0, & \text{otherwise} \end{cases}$$

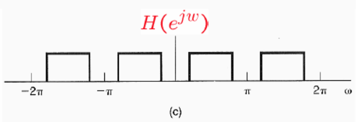
- <u>Frequency-Selective Filters:</u>
  - Select some bands of frequencies and reject others



DT ideal lowpass filter



DT ideal highpass filter

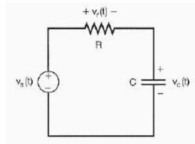


DT ideal bandpass filter

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

▪ A Simple RC Lowpass Filter:

Input signal: $v_s(t) = e^{jwt}$ → $\delta(t)$, $u(t)$



Output signal: $v_c(t) = H(jw)e^{jwt}$ → $h(t)$, $s(t)$

$$\Rightarrow RC\,\frac{d}{dt}v_c(t) + v_c(t) = v_s(t)$$

$$\Rightarrow RC\,\frac{d}{dt}\left[H(jw)e^{jwt}\right] + H(jw)e^{jwt} = e^{jwt}$$

$$\Rightarrow RC\;jw\;H(jw)e^{jwt} + H(jw)e^{jwt} = e^{jwt}$$

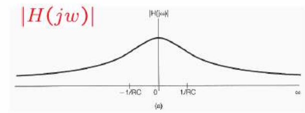
$$\Rightarrow H(jw)e^{jwt} = \frac{1}{1+RCjw}e^{jwt}$$

▪ A Simple RC Lowpass Filter:

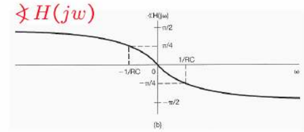
$$\Rightarrow H(jw) = \frac{1}{1+RCjw}$$

$$\boxed{H = |H|e^{j\measuredangle H}}$$

$|H(jw)|$



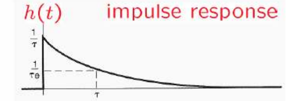
$\measuredangle H(jw)$



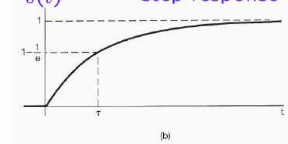
$$\Rightarrow h(t) = \frac{1}{RC}\,e^{-t/RC}\,u(t)$$

$$\Rightarrow s(t) = \left[1 - e^{-t/RC}\right]u(t)$$

$h(t)$ impulse response



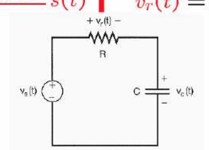
$s(t)$ step response



▪ A Simple RC Highpass Filter:

$h(t)$, $s(t)$ ↑ Output signal: $v_r(t) = G(jw)e^{jwt}$

Input signal: $v_s(t) = e^{jwt}$ → $\delta(t)$, $u(t)$



$$\Rightarrow RC\,\frac{d}{dt}v_r(t) + v_r(t) = RC\,\frac{d}{dt}v_s(t)$$

$$\Rightarrow RC\,\frac{d}{dt}\left[G(jw)e^{jwt}\right] + G(jw)e^{jwt} = RC\,\frac{d}{dt}e^{jwt}$$

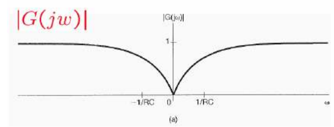
$$\Rightarrow RC\;jw\;G(jw)e^{jwt} + G(jw)e^{jwt} = RC\;jw\;e^{jwt}$$

$$\Rightarrow G(jw)e^{jwt} = \frac{jw\;RC}{1+jw\;RC}e^{jwt}$$

▪ A Simple RC Highpass Filter:

$$\Rightarrow G(jw) = \frac{jw\;RC}{1+jw\;RC}$$

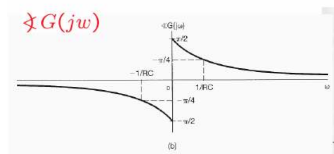
$|G(jw)|$



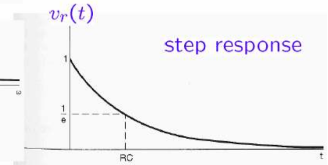
$v_r(t) = v_s(t) - v_c(t)$

$$\Rightarrow v_r(t) = e^{-t/RC}\,u(t)$$

$\measuredangle G(jw)$



$v_r(t)$ step response



First-Order Recursive DT Filters:

$$y[n] - ay[n-1] = x[n]$$

- If $x[n] = e^{jwn}$, then $y[n] = H(e^{jw})e^{jwn}$

  where $H(e^{jw})$: the frequency response

$$\Rightarrow \ H(e^{jw})\, e^{jwn} - a\, H(e^{jw})\, e^{jw(n-1)} = e^{jwn}$$

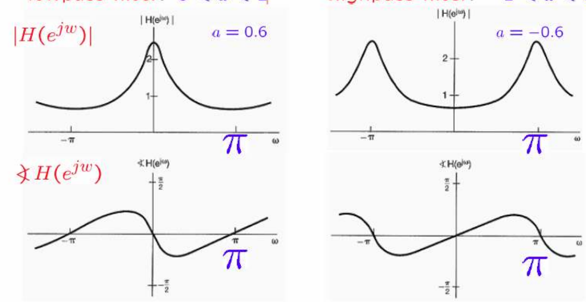
$$\Rightarrow \ \left[1 - a\, e^{-jw}\right] H(e^{jw})\, e^{jwn} = e^{jwn}$$

$$\Rightarrow \ H(e^{jw}) = \frac{1}{1 - a\, e^{-jw}}$$

First-Order Recursive DT Filters:

$$H(e^{jw}) = \frac{1}{1 - a\, e^{-jw}} \qquad y[n] = ay[n-1] + x[n]$$

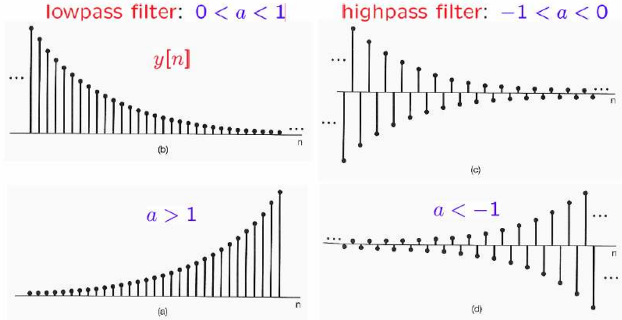
lowpass filter: $0 < a < 1$ highpass filter: $-1 < a < 0$



First-Order Recursive DT Filters:

$$y[n] = ay[n-1] + x[n]$$

lowpass filter: $0 < a < 1$ highpass filter: $-1 < a < 0$



Nonrecursive DT Filters:

- An FIR nonrecursive difference equation:

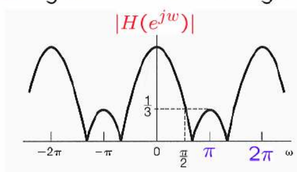
$$y[n] = \sum_{k=-N}^{M} b_k\, x[n-k]$$

- **Nonrecursive DT Filters:**
  - Three-point moving average (lowpass) filter:

$$y[n] = \frac{1}{3}\left(x[n+1] + x[n] + x[n-1]\right)$$

$$\Rightarrow h[n] = \frac{1}{3}\left(\delta[n+1] + \delta[n] + \delta[n-1]\right)$$

$$\Rightarrow H(e^{jw}) = \frac{1}{3}\left(e^{jw} + 1 + e^{-jw}\right) = \frac{1}{3}\left(1 + 2\cos w\right)$$



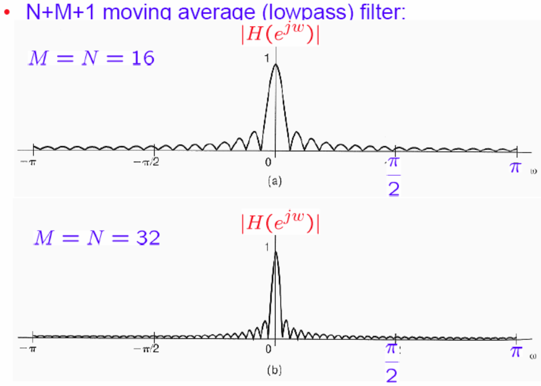
- **Nonrecursive DT Filters:**
  - N+M+1 moving average (lowpass) filter:

$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k]$$

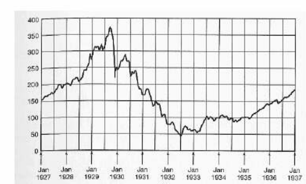
$$\Rightarrow H(e^{jw}) = \frac{1}{N+M+1}\sum_{k=-N}^{M} e^{-jwk}$$

$$\Rightarrow H(e^{jw}) = \frac{1}{N+M+1} e^{jw\left(\frac{N-M}{2}\right)} \frac{\sin w\left(\frac{M+N+1}{2}\right)}{\sin\left(\frac{w}{2}\right)}$$

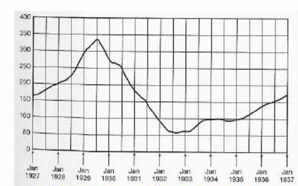
- **Nonrecursive DT Filters:**
  - N+M+1 moving average (lowpass) filter:



- **Lowpass Filtering on Dow Jones Weekly Stock Market Index:**



51-day moving average



201-day moving average

- Nonrecursive DT Filters:
  - Highpass filters:

$$y[n] = \frac{x[n] - x[n-1]}{2}$$

$$\Rightarrow h[n] = \frac{1}{2}\left\{\delta[n] - \delta[n-1]\right\}$$

$$\Rightarrow H(e^{jw}) = \frac{1}{2}\left[1 - e^{-jw}\right] = j\, e^{jw/2} \sin(w/2)$$