**?** These jugglers toss the pins so that they rotate in midair. Each pin is of uniform composition, so its weight is concentrated toward its thick end. If we ignore air resistance but not the effects of gravity, will the angular speed of a pin in flight (i) increase continuously; (ii) decrease continuously; (iii) alternately increase and decrease; or (iv) remain the same?

# 10 Dynamics of Rotational Motion

**LEARNING OUTCOMES**

***In this chapter, you'll learn...***

**10.1** What is meant by the torque produced by a force.
**10.2** How the net torque on a rigid body affects the body's rotational motion.
**10.3** How to analyze the motion of a rigid body that both rotates and moves as a whole through space.
**10.4** How to solve problems that involve work and power for rotating rigid bodies.
**10.5** What is meant by the angular momentum of a particle or rigid body.
**10.6** How the angular momentum of an object can remain constant even if the object changes shape.
**10.7** Why a spinning gyroscope undergoes precession.

***You'll need to review...***

**1.10** Vector product of two vectors.
**5.2** Newton's second law.
**6.1–6.4** Work, the work–energy theorem, and power.
**8.2, 8.3, 8.5** External versus internal forces, inelastic collisions, and center-of-mass motion.
**9.1–9.5** Rotational motion and the parallel-axis theorem.

We learned in Chapters 4 and 5 that a net force applied to an object gives that object an acceleration. But what does it take to give an object an *angular* acceleration? That is, what does it take to start a stationary object rotating or to bring a spinning object to a halt? A force is required, but it must be applied in a way that gives a twisting or turning action.

In this chapter we'll define a new physical quantity, *torque,* that describes the twisting or turning effort of a force. We'll find that the net torque acting on a rigid body determines its angular acceleration, in the same way that the net force on an object determines its linear acceleration. We'll also look at work and power in rotational motion so as to understand, for example, how energy is transferred by an electric motor. Next we'll develop a new conservation principle, *conservation of angular momentum,* that is tremendously useful for understanding the rotational motion of both rigid and nonrigid bodies. We'll finish this chapter by studying *gyroscopes,* rotating devices that don't fall over when you might think they should—but that actually behave in accordance with the dynamics of rotational motion.

## 10.1 TORQUE

We know that forces acting on an object can affect its **translational motion**—that is, the motion of the object as a whole through space. Now we want to learn which aspects of a force determine how effective it is in causing or changing *rotational* motion. The magnitude and direction of the force are important, but so is the point on the object where the force is applied. In **Fig. 10.1** a wrench is being used to loosen a tight bolt. Force $\vec{F}_b$, applied near the end of the handle, is more effective than an equal force $\vec{F}_a$ applied near the bolt. Force $\vec{F}_c$ does no good; it's applied at the same point and has the same magnitude as $\vec{F}_b$, but it's directed along the length of the handle. The quantitative measure of the tendency of a force to cause or change an object's rotational motion is called *torque;* we say that $\vec{F}_a$ applies a torque about point $O$ to the wrench in Fig. 10.1, $\vec{F}_b$ applies a greater torque about $O$, and $\vec{F}_c$ applies zero torque about $O$.

**Figure 10.2** shows three examples of how to calculate torque. The object can rotate about an axis that is perpendicular to the plane of the figure and passes through point $O$. Three forces act on the object in the plane of the figure. The tendency of the first of these forces, $\vec{F}_1$, to cause a rotation about $O$ depends on its magnitude $F_1$. It also depends on the *perpendicular* distance $l_1$ between point $O$ and the **line of action** of the force (that is, the line along which the force vector lies). We call the distance $l_1$ the **lever arm** (or **moment arm**) of force $\vec{F}_1$ about $O$. The twisting effort is directly proportional to both $F_1$ and $l_1$, so we define the **torque** (or *moment*) of the force $\vec{F}_1$ with respect to $O$ as the product $F_1 l_1$. We use the Greek letter $\tau$ (tau) for torque. If a force of magnitude $F$ has a line of action that is a perpendicular distance $l$ from $O$, the torque is

$$\tau = Fl \qquad (10.1)$$

Physicists usually use the term "torque," while engineers usually use "moment" (unless they are talking about a rotating shaft).

The lever arm of $\vec{F}_1$ in Fig. 10.2 is the perpendicular distance $l_1$, and the lever arm of $\vec{F}_2$ is the perpendicular distance $l_2$. The line of action of $\vec{F}_3$ passes through point $O$, so the lever arm for $\vec{F}_3$ is zero and its torque with respect to $O$ is zero. In the same way, force $\vec{F}_c$ in Fig. 10.1 has zero torque with respect to point $O$; $\vec{F}_b$ has a greater torque than $\vec{F}_a$ because its lever arm is greater.

**CAUTION** **Torque is always measured about a point** Torque is *always* defined with reference to a specific point. If we shift the position of this point, the torque of each force may change. For example, the torque of force $\vec{F}_3$ in Fig. 10.2 is zero with respect to point $O$ but *not* with respect to point $A$. It's not enough to refer to "the torque of $\vec{F}$"; you must say "the torque of $\vec{F}$ with respect to point $X$" or "the torque of $\vec{F}$ about point $X$."

Force $\vec{F}_1$ in Fig. 10.2 tends to cause *counterclockwise* rotation about $O$, while $\vec{F}_2$ tends to cause *clockwise* rotation. To distinguish between these two possibilities, we need to choose a positive sense of rotation. With the choice that *counterclockwise torques are positive and clockwise torques are negative,* the torques of $\vec{F}_1$ and $\vec{F}_2$ about $O$ are

$$\tau_1 = +F_1 l_1 \qquad \tau_2 = -F_2 l_2$$

Figure 10.2 shows this choice for the sign of torque. We'll often use the symbol ⊕ to indicate our choice of the positive sense of rotation.

The SI unit of torque is the newton-meter. In our discussion of work and energy we called this combination the joule. But torque is *not* work or energy, and torque should be expressed in newton-meters, *not* joules.

**Figure 10.3** shows a force $\vec{F}$ applied at point $P$, located at position $\vec{r}$ with respect to point $O$. There are three ways to calculate the torque of $\vec{F}$:

1. Find the lever arm $l$ and use $\tau = Fl$.
2. Determine the angle $\phi$ between the vectors $\vec{r}$ and $\vec{F}$; the lever arm is $r\sin\phi$, so $\tau = rF\sin\phi$.
3. Represent $\vec{F}$ in terms of a radial component $F_{\text{rad}}$ along the direction of $\vec{r}$ and a tangential component $F_{\tan}$ at right angles, perpendicular to $\vec{r}$. (We call this component *tangential* because if the object rotates, the point where the force acts moves in a circle, and this component is tangent to that circle.) Then $F_{\tan} = F\sin\phi$ and $\tau = r(F\sin\phi) = F_{\tan}r$. The component $F_{\text{rad}}$ produces *no* torque with respect to $O$ because its lever arm with respect to that point is zero (compare to forces $\vec{F}_c$ in Fig. 10.1 and $\vec{F}_3$ in Fig. 10.2).

Summarizing these three expressions for torque, we have

**Magnitude of torque** due to force $\vec{F}$ relative to point $O$
Lever arm of $\vec{F}$
Magnitude of $\vec{r}$ (vector from $O$ to where $\vec{F}$ acts)

$$\tau = Fl = rF\sin\phi = F_{\tan}r \qquad (10.2)$$

Magnitude of $\vec{F}$
Angle between $\vec{r}$ and $\vec{F}$
Tangential component of $\vec{F}$

Figure **10.1** Which of these three equal-magnitude forces is most likely to loosen the tight bolt?

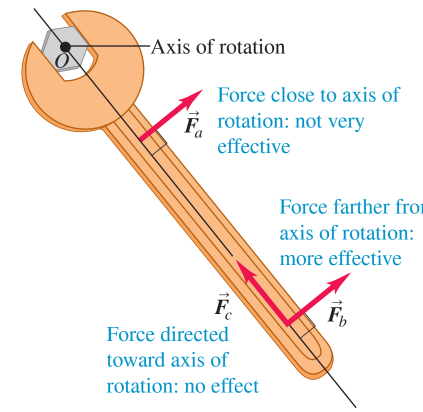



Figure **10.2** The torque of a force about a point is the product of the force magnitude and the lever arm of the force.

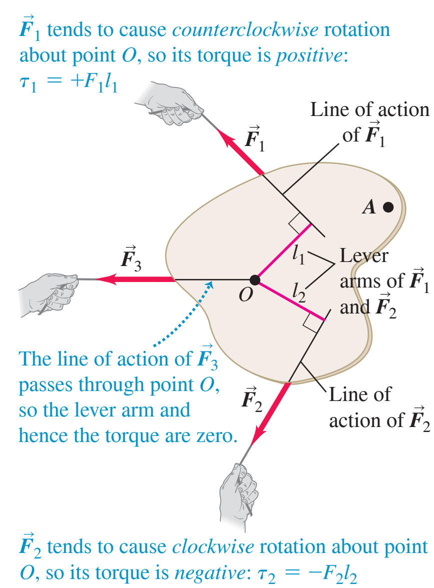



Figure **10.3** Three ways to calculate the torque of force $\vec{F}$ about point $O$. In this figure, $\vec{r}$ and $\vec{F}$ are in the plane of the page and the torque vector $\vec{\tau}$ points out of the page toward you.

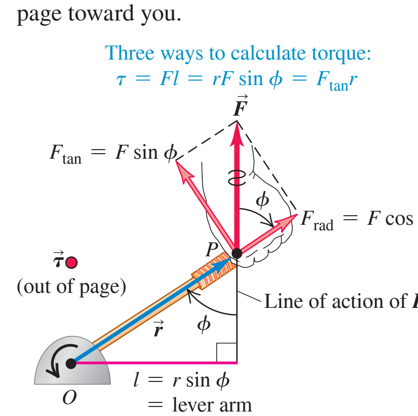
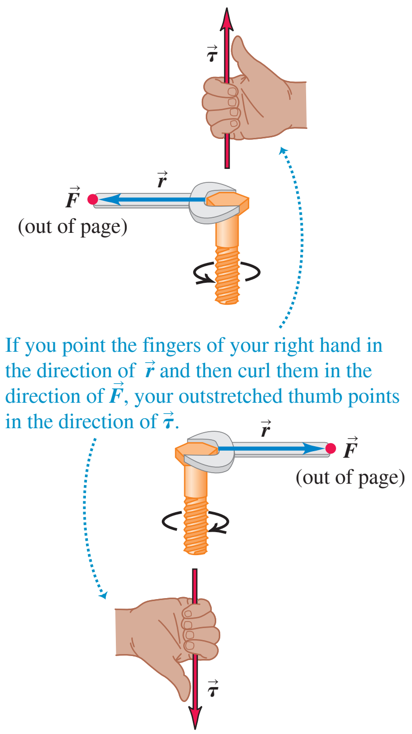


Figure **10.4** The torque vector $\vec{\tau} = \vec{r} \times \vec{F}$ is directed along the axis of the bolt, perpendicular to both $\vec{r}$ and $\vec{F}$. The fingers of the right hand curl in the direction of the rotation that the torque tends to cause.




## Torque as a Vector

We saw in Section 9.1 that angular velocity and angular acceleration can be represented as vectors; the same is true for torque. To see how to do this, note that the quantity $rF\sin\phi$ in Eq. (10.2) is the magnitude of the *vector product* $\vec{r} \times \vec{F}$ that we defined in Section 1.10. (Go back and review that definition.) We generalize the definition of torque as follows: When a force $\vec{F}$ acts at a point having a position vector $\vec{r}$ with respect to an origin $O$, as in Fig. 10.3, the torque $\vec{\tau}$ of the force with respect to $O$ is the *vector* quantity

Torque vector due to force $\vec{F}$ relative to point $O$ · Vector from $O$ to where $\vec{F}$ acts · Force $\vec{F}$

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{10.3}$$

The torque as defined in Eq. (10.2) is the magnitude of the torque vector $\vec{r} \times \vec{F}$. The direction of $\vec{\tau}$ is perpendicular to both $\vec{r}$ and $\vec{F}$. In particular, if both $\vec{r}$ and $\vec{F}$ lie in a plane perpendicular to the axis of rotation, as in Fig. 10.3, then the torque vector $\vec{\tau} = \vec{r} \times \vec{F}$ is directed along the axis of rotation, with a sense given by the right-hand rule (see Fig. 1.30 and **Fig. 10.4**).

Because $\vec{\tau} = \vec{r} \times \vec{F}$ is perpendicular to the plane of the vectors $\vec{r}$ and $\vec{F}$, it's common to have diagrams like Fig. 10.4, in which one of the vectors is perpendicular to the page. We use a dot (•) to represent a vector that points out of the page and a cross (×) to represent a vector that points into the page (see Figs. 10.3 and 10.4).

In the following sections we'll usually be concerned with rotation of an object about an axis oriented in a specified constant direction. In that case, only the component of torque along that axis will matter. We often call that component the torque with respect to the specified *axis*.

### EXAMPLE 10.1 Applying a torque

To loosen a pipe fitting, a plumber slips a piece of scrap pipe (a "cheater") over his wrench handle. He stands on the end of the cheater, applying his 900 N weight at a point 0.80 m from the center of the fitting (**Fig. 10.5a**). The wrench handle and cheater make an angle of 19° with the horizontal. Find the magnitude and direction of the torque he applies about the center of the fitting.

**IDENTIFY and SET UP** Figure 10.5b shows the vectors $\vec{r}$ and $\vec{F}$ and the angle between them ($\phi = 109°$). Equation (10.1) or (10.2) will tell us the magnitude of the torque. The right-hand rule with Eq. (10.3), $\vec{\tau} = \vec{r} \times \vec{F}$, will tell us the direction of the torque.

**EXECUTE** To use Eq. (10.1), we first calculate the lever arm $l$. As Fig. 10.5b shows,

$$l = r\sin\phi = (0.80\ \text{m})\sin 109° = 0.76\ \text{m}$$

Then Eq. (10.1) tells us that the magnitude of the torque is

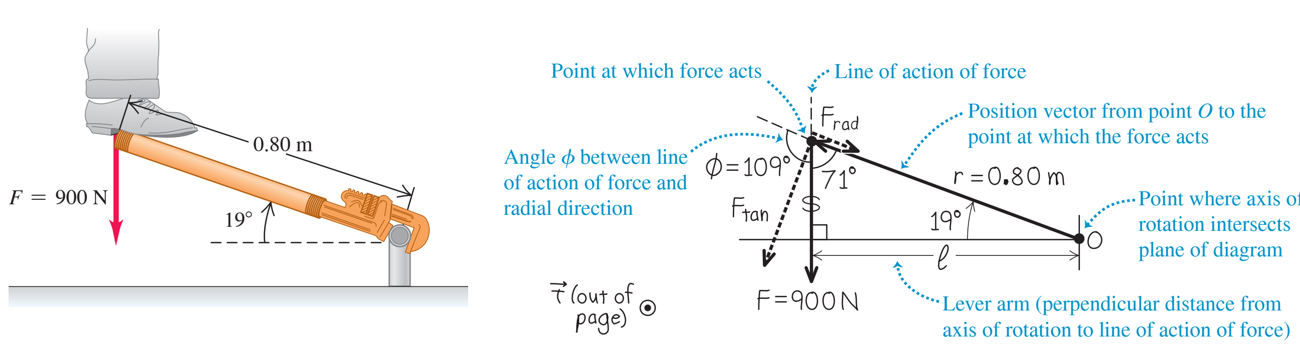
$$\tau = Fl = (900\ \text{N})(0.76\ \text{m}) = 680\ \text{N}\cdot\text{m}$$

We get the same result from Eq. (10.2):

$$\tau = rF\sin\phi = (0.80\ \text{m})(900\ \text{N})(\sin 109°) = 680\ \text{N}\cdot\text{m}$$

Figure **10.5** (a) Loosening a pipe fitting by standing on a "cheater." (b) Our vector diagram to find the torque about $O$.

(a) Diagram of situation (b) Free-body diagram



Alternatively, we can find $F_{\tan}$, the tangential component of $\vec{F}$ that acts perpendicular to $\vec{r}$. Figure 10.5b shows that this component is at an angle of $109° - 90° = 19°$ from $\vec{F}$, so $F_{\tan} = F(\cos 19°) = (900\ \text{N})(\cos 19°) = 851$ N. Then, from Eq. (10.2),

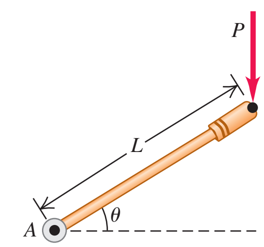
$$\tau = F_{\tan} r = (851\ \text{N})(0.80\ \text{m}) = 680\ \text{N} \cdot \text{m}$$

Curl the fingers of your right hand from the direction of $\vec{r}$ (in the plane of Fig. 10.5b, to the left and up) into the direction of $\vec{F}$ (straight down). Then your right thumb points out of the plane of the figure: This is the direction of $\vec{\tau}$.

**EVALUATE** To check the direction of $\vec{\tau}$, note that the force in Fig. 10.5 tends to produce a counterclockwise rotation about $O$. If you curl the fingers of your right hand in a counterclockwise direction, the thumb points out of the plane of Fig. 10.5, which is indeed the direction of the torque.

**KEYCONCEPT** You can determine the magnitude of the torque due to a force $\vec{F}$ in any of three ways: (i) from the magnitude of $\vec{F}$ and the lever arm; (ii) from the magnitude of $\vec{F}$, the magnitude of the vector $\vec{r}$ from the origin to where $\vec{F}$ acts, and the angle between $\vec{r}$ and $\vec{F}$; or (iii) from the magnitude of $\vec{r}$ and the tangential component of $\vec{F}$. Find the direction of the torque using the right-hand rule.

**TEST YOUR UNDERSTANDING OF SECTION 10.1** The accompanying figure shows a force of magnitude $P$ being applied to one end of a lever of length $L$. What is the magnitude of the torque of this force about point $A$? (i) $PL\sin\theta$; (ii) $PL\cos\theta$; (iii) $PL\tan\theta$; (iv) $PL/\sin\theta$; (v) $PL$.




**ANSWER**

**(ii)** The force of magnitude $P$ acts along a vertical line, so the lever arm is the horizontal distance from $A$ to the line of action. This is the horizontal component of distance $L$, which is $L\cos\theta$. Hence the magnitude of the torque is the product of the force magnitude $P$ and the lever arm $L\cos\theta$, or $\tau = PL\cos\theta$.

## 10.2 TORQUE AND ANGULAR ACCELERATION FOR A RIGID BODY

We're now ready to develop the fundamental relationship for the rotational dynamics of a rigid body (an object with a definite and unchanging shape and size). We'll show that the angular acceleration of a rotating rigid body is directly proportional to the sum of the torque components along the axis of rotation. The proportionality factor is the moment of inertia.

To develop this relationship, let's begin as we did in Section 9.4 by envisioning the rigid body as being made up of a large number of particles. We choose the axis of rotation to be the $z$-axis; the first particle has mass $m_1$ and distance $r_1$ from this axis (**Fig. 10.6**). The *net force* $\vec{F}_1$ acting on this particle has a component $F_{1,\text{rad}}$ along the radial direction, a component $F_{1,\tan}$ that is tangent to the circle of radius $r_1$ in which the particle moves as the body rotates, and a component $F_{1z}$ along the axis of rotation. Newton's second law for the tangential component is

$$F_{1,\tan} = m_1 a_{1,\tan} \qquad (10.4)$$

We can express the tangential acceleration of the first particle in terms of the angular acceleration $\alpha_z$ of the body by using Eq. (9.14): $a_{1,\tan} = r_1\alpha_z$. Using this relationship and multiplying both sides of Eq. (10.4) by $r_1$, we obtain

$$F_{1,\tan} r_1 = m_1 r_1^2 \alpha_z \qquad (10.5)$$

From Eq. (10.2), $F_{1,\tan} r_1$ is the *torque* of the net force with respect to the rotation axis, equal to the component $\tau_{1z}$ of the torque vector along the rotation axis. The subscript $z$ is a reminder that the torque affects rotation around the $z$-axis, in the same way that the subscript on $F_{1z}$ is a reminder that this force affects the motion of particle 1 along the $z$-axis.

Neither of the components $F_{1,\text{rad}}$ or $F_{1z}$ contributes to the torque about the $z$-axis, since neither tends to change the particle's rotation about that axis. So $\tau_{1z} = F_{1,\tan} r_1$ is the total torque acting on the particle with respect to the rotation axis. Also, $m_1 r_1^2$ is $I_1$, the moment of inertia of the particle about the rotation axis. Hence we can rewrite Eq. (10.5) as

$$\tau_{1z} = I_1\alpha_z = m_1 r_1^2 \alpha_z$$

Figure **10.6** As a rigid body rotates around the $z$-axis, a net force $\vec{F}_1$ acts on one particle of the body. Only the force component $F_{1,\tan}$ can affect the rotation, because only $F_{1,\tan}$ exerts a torque about $O$ with a $z$-component (along the rotation axis).

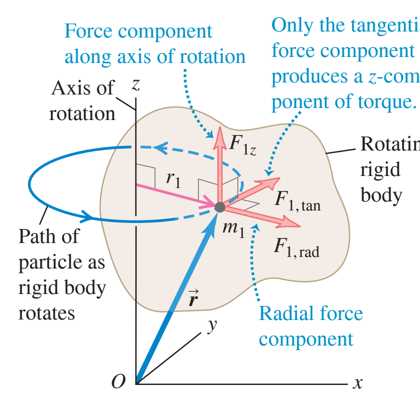
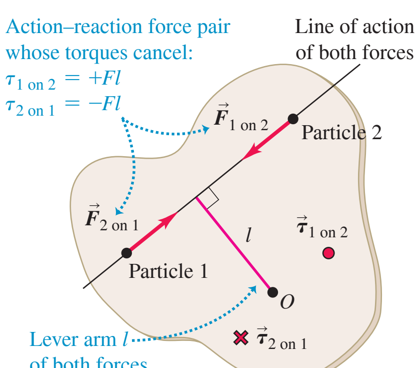


Figure **10.7** Loosening or tightening a screw requires giving it an angular acceleration and hence applying a torque. To make this easier, use a screwdriver with a large-radius handle. This provides a large lever arm for the force your hand applies.

Figure **10.8** Why only *external* torques affect a rigid body's rotation: Any two particles in the body exert equal and opposite forces on each other. If the forces act along the line joining the particles, the lever arms of the forces with respect to an axis through $O$ are the same and the torques due to the two forces are equal and opposite.




We write such an equation for every particle in the body, then add all these equations:

$$\tau_{1z} + \tau_{2z} + \cdots = I_1\alpha_z + I_2\alpha_z + \cdots = m_1 r_1^2 \alpha_z + m_2 r_2^2 \alpha_z + \cdots$$

or

$$\sum \tau_{iz} = \left(\sum m_i r_i^2\right)\alpha_z \tag{10.6}$$

The left side of Eq. (10.6) is the sum of all the torques about the rotation axis that act on all the particles. The right side is $I = \sum m_i r_i^2$, the total moment of inertia about the rotation axis, multiplied by the angular acceleration $\alpha_z$. Note that $\alpha_z$ is the same for every particle because this is a *rigid* body. Thus Eq. (10.6) says that for the rigid body as a whole,

**Rotational analog of Newton's second law for a rigid body:**

$$\sum \tau_z = I\alpha_z \tag{10.7}$$

Net torque on a rigid body about $z$-axis

Moment of inertia of rigid body about $z$-axis

Angular acceleration of rigid body about $z$-axis

Just as Newton's second law says that a net *force* on a particle causes an *acceleration* in the direction of the net force, Eq. (10.7) says that a net *torque* on a rigid body about an axis causes an *angular acceleration* about that axis (**Fig. 10.7**).

Our derivation assumed that the angular acceleration $\alpha_z$ is the same for all particles in the body. So Eq. (10.7) is valid *only* for *rigid* bodies. Hence this equation doesn't apply to a rotating tank of water or a swirling tornado of air, different parts of which have different angular accelerations. Note that since our derivation used Eq. (9.14), $a_{\tan} = r\alpha_z$, $\alpha_z$ must be measured in rad/s$^2$.

The torque on each particle is due to the net force on that particle, which is the vector sum of external and internal forces (see Section 8.2). According to Newton's third law, the *internal* forces that any pair of particles in the rigid body exert on each other are equal in magnitude and opposite in direction (**Fig. 10.8**). If these forces act along the line joining the two particles, their lever arms with respect to any axis are also equal. So the torques for each such pair are equal and opposite, and add to zero. Hence *all* the internal torques add to zero, so the sum $\sum \tau_z$ in Eq. (10.7) includes only the torques of the *external* forces.

Often, an important external force acting on a rigid body is its *weight*. This force is not concentrated at a single point; it acts on every particle in the entire body. Nevertheless, if $\vec{g}$ has the same value at all points, we always get the correct torque (about any specified axis) if we assume that all the weight is concentrated at the *center of mass* of the body. We'll prove this statement in Chapter 11, but meanwhile we'll use it for some of the problems in this chapter.

## PROBLEM-SOLVING STRATEGY 10.1 Rotational Dynamics for Rigid Bodies

Our strategy for solving problems in rotational dynamics is very similar to Problem-Solving Strategy 5.2 for solving problems involving Newton's second law.

**IDENTIFY** *the relevant concepts:* Equation (10.7), $\sum \tau_z = I\alpha_z$, is useful whenever torques act on a rigid body. Sometimes you can use an energy approach instead, as we did in Section 9.4. However, if the target variable is a force, a torque, an acceleration, an angular acceleration, or an elapsed time, using $\sum \tau_z = I\alpha_z$ is almost always best.

**SET UP** *the problem* using the following steps:

1. Sketch the situation and identify the body or bodies to be analyzed. Indicate the rotation axis.
2. For each body, draw a free-body diagram that shows the body's *shape,* including all dimensions and angles. Label pertinent quantities with algebraic symbols.
3. Choose coordinate axes for each body and indicate a positive sense of rotation (clockwise or counterclockwise) for each rotating body. If you know the sense of $\alpha_z$, pick that as the positive sense of rotation.

**EXECUTE** *the solution:*

1. For each body, decide whether it undergoes translational motion, rotational motion, or both. Then apply $\sum \vec{F} = m\vec{a}$ (as in Section 5.2), $\sum \tau_z = I\alpha_z$, or both to the body.
2. Express in algebraic form any *geometrical* relationships between the motions of two or more bodies. An example is a string that unwinds, without slipping, from a pulley or a wheel that rolls without slipping (discussed in Section 10.3). These relationships usually appear as relationships between linear and/or angular accelerations.
3. Ensure that you have as many independent equations as there are unknowns. Solve the equations to find the target variables.

**EVALUATE** *your answer:* Check that the algebraic signs of your results make sense. As an example, if you are unrolling thread from a spool, your answers should not tell you that the spool is turning in the direction that rolls the thread back onto the spool! Check that any algebraic results are correct for special cases or for extreme values of quantities.

## EXAMPLE 10.2 An unwinding cable I

WITH VARIATION PROBLEMS

**Figure 10.9a** shows the situation that we analyzed in Example 9.7 using energy methods. What is the cable's acceleration?

**IDENTIFY and SET UP** We can't use the energy method of Section 9.4, which doesn't involve acceleration. Instead we'll apply rotational dynamics to find the angular acceleration of the cylinder (Fig. 10.9b). We'll then find a relationship between the motion of the cable and the motion of the cylinder rim, and use this to find the acceleration of the cable. The cylinder rotates counterclockwise when the cable is pulled, so we take counterclockwise rotation to be positive. The net force on the cylinder must be zero because its center of mass remains at rest. The force $F$ exerted by the cable produces a torque about the rotation axis. The weight (magnitude $Mg$) and the normal force (magnitude $n$) exerted by the cylinder's bearings produce *no* torque about the rotation axis because both act along lines through that axis.

**EXECUTE** The lever arm of $F$ is equal to the radius $R = 0.060$ m of the cylinder, so the torque is $\tau_z = FR$. (This torque is positive, as it tends to cause a counterclockwise rotation.) From Table 9.2, case (f), the moment of inertia of the cylinder about the rotation axis is $I = \frac{1}{2}MR^2$. Then Eq. (10.7) tells us that

$$\alpha_z = \frac{\tau_z}{I} = \frac{FR}{MR^2/2} = \frac{2F}{MR} = \frac{2(9.0\text{ N})}{(50\text{ kg})(0.060\text{ m})} = 6.0\text{ rad/s}^2$$

(We can add "rad" to our result because radians are dimensionless.)

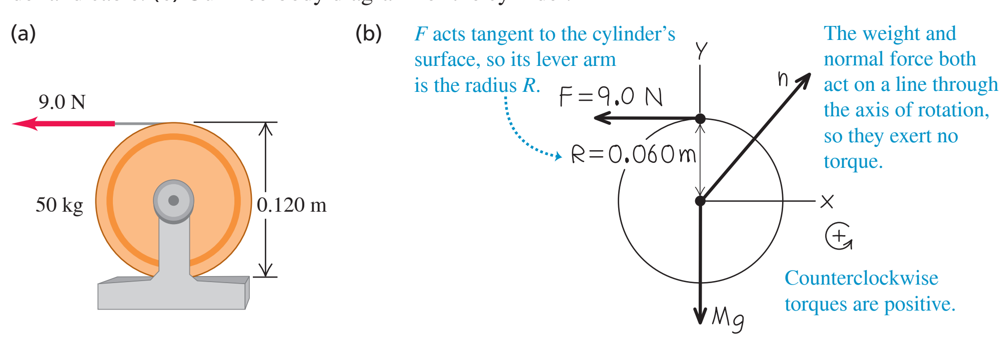
To get the linear acceleration of the cable, recall from Section 9.3 that the acceleration of a cable unwinding from a cylinder is the same as the tangential acceleration of a point on the surface of the cylinder where the cable is tangent to it. This tangential acceleration is given by Eq. (9.14):

$$a_{\tan} = R\alpha_z = (0.060\text{ m})(6.0\text{ rad/s}^2) = 0.36\text{ m/s}^2$$

**EVALUATE** Can you use this result, together with an equation from Chapter 2, to determine the speed of the cable after it has been pulled 2.0 m? Does your result agree with that of Example 9.7?

**KEYCONCEPT** For any problem involving torques on a rigid body, first draw a free-body diagram to identify where on the rigid body each external force acts with respect to the axis of rotation. Then apply the rotational analog of Newton's second law, $\sum\tau_z = I\alpha_z$.

Figure **10.9** (a) Cylinder and cable. (b) Our free-body diagram for the cylinder.




## EXAMPLE 10.3 An unwinding cable II

WITH VARIATION PROBLEMS

In Example 9.8 (Section 9.4), what are the acceleration of the falling block and the tension in the cable?

**IDENTIFY and SET UP** We'll apply translational dynamics to the block and rotational dynamics to the cylinder. As in Example 10.2, we'll relate the linear acceleration of the block (our target variable) to the angular acceleration of the cylinder. **Figure 10.10** (next page) shows our sketch of the situation and a free-body diagram for each object. We take the positive sense of rotation for the cylinder to be counterclockwise and the positive direction of the $y$-coordinate for the block to be downward.

**EXECUTE** For the block, Newton's second law gives

$$\sum F_y = mg + (-T) = ma_y$$

For the cylinder, the only torque about its axis is that due to the cable tension $T$. Hence Eq. (10.7) gives

$$\sum\tau_z = RT = I\alpha_z = \tfrac{1}{2}MR^2\alpha_z$$

As in Example 10.2, the acceleration of the cable is the same as the tangential acceleration of a point on the cylinder rim. From Eq. (9.14), this acceleration is $a_y = a_{\tan} = R\alpha_z$. We use this to replace $R\alpha_z$ with $a_y$ in the cylinder equation above, and divide by $R$. The result is $T = \frac{1}{2}Ma_y$. Now we substitute this expression for $T$ into Newton's second law for the block and solve for the acceleration $a_y$:

$$mg - \tfrac{1}{2}Ma_y = ma_y$$

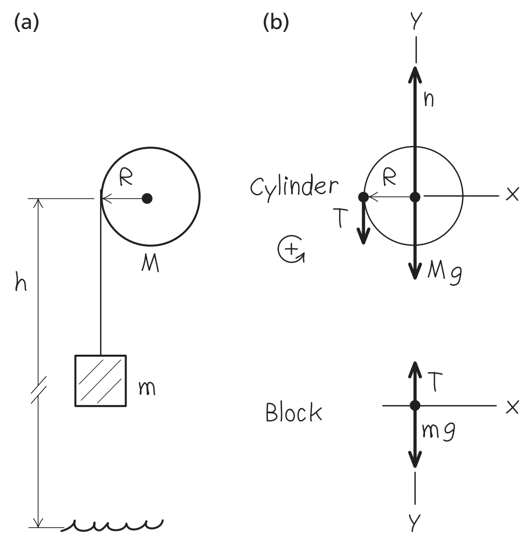
$$a_y = \frac{g}{1 + M/2m}$$

To find the cable tension $T$, we substitute our expression for $a_y$ into the block equation:

$$T = mg - ma_y = mg - m\left(\frac{g}{1 + M/2m}\right) = \frac{mg}{1 + 2m/M}$$

*Continued*

Figure **10.10** (a) Our diagram of the situation. (b) Our free-body diagrams for the cylinder and the block. We assume the cable has negligible mass.



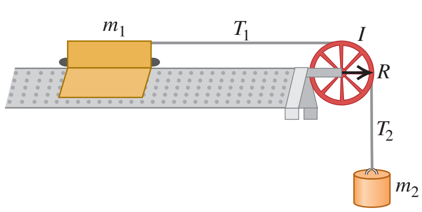

**EVALUATE** The acceleration is positive (in the downward direction) and less than $g$, as it should be, since the cable is holding back the block. The cable tension is *not* equal to the block's weight $mg$; if it were, the block could not accelerate.

Let's check some particular cases. When $M$ is much larger than $m$, the tension is nearly equal to $mg$ and the acceleration is correspondingly much less than $g$. When $M$ is zero, $T = 0$ and $a_y = g$; the object falls freely. If the object starts from rest ($v_{0y} = 0$) a height $h$ above the floor, its $y$-velocity when it strikes the floor is given by $v_y{}^2 = v_{0y}{}^2 + 2a_y h = 2a_y h$, so

$$v_y = \sqrt{2a_y h} = \sqrt{\frac{2gh}{1 + M/2m}}$$

We found this result from energy considerations in Example 9.8.

**KEYCONCEPT** When an object is connected to a string that wraps around a rotating pulley of radius $R$, the linear acceleration $a_y$ of the object is related to the angular acceleration $\alpha_z$ of the pulley by $a_y = R\alpha_z$.

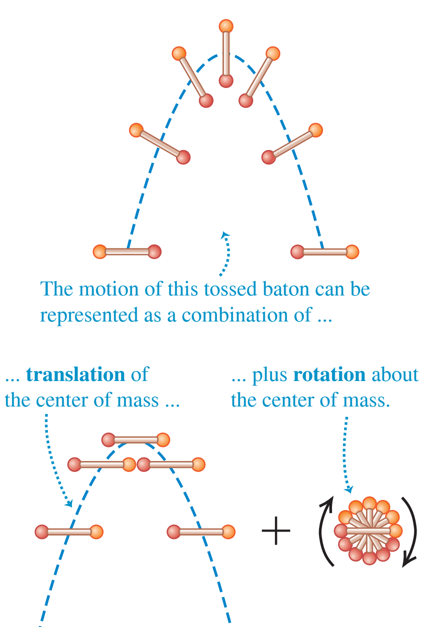



**TEST YOUR UNDERSTANDING OF SECTION 10.2** The figure shows a glider of mass $m_1$ that can slide without friction on a horizontal air track. It is attached to an object of mass $m_2$ by a massless string. The pulley has radius $R$ and moment of inertia $I$ about its axis of rotation. When released, the hanging object accelerates downward, the glider accelerates to the right, and the string turns the pulley without slipping or stretching. Rank the magnitudes of the following forces that act during the motion, in order from largest to smallest magnitude. (i) The tension force (magnitude $T_1$) in the horizontal part of the string; (ii) the tension force (magnitude $T_2$) in the vertical part of the string; (iii) the weight $m_2 g$ of the hanging object.

**ANSWER**

**(iii), (ii), (i)** For the hanging object of mass $m_2$ to accelerate downward, the net force on it must be downward. Hence the magnitude $m_2 g$ of the downward weight force must be greater than the magnitude $T_2$ of the upward tension force. For the pulley to have a clockwise angular acceleration, the net torque on the pulley must be clockwise. Tension $T_2$ tends to rotate the pulley clockwise, while tension $T_1$ tends to rotate the pulley counterclockwise. Both tension forces have the same lever arm $R$, so there is a clockwise torque $T_2 R$ and a counterclockwise torque $T_1 R$. For the net torque to be clockwise, $T_2$ must be greater than $T_1$. Hence $m_2 g > T_2 > T_1$.

Figure **10.11** The motion of a rigid body is a combination of translational motion of the center of mass and rotation around the center of mass.




## 10.3 RIGID-BODY ROTATION ABOUT A MOVING AXIS

We can extend our analysis of rigid-body rotational dynamics to some cases in which the axis of rotation moves. When that happens, the motion of the rigid body is **combined translation and rotation.** The key to understanding such situations is this: Every possible motion of a rigid body can be represented as a combination of *translational motion of the center of mass* and *rotation about an axis through the center of mass.* This is true even when the center of mass accelerates, so it is not at rest in any inertial frame. **Figure 10.11** illustrates this for the motion of a tossed baton: The center of mass of the baton follows a parabolic curve, as though the baton were a particle located at the center of mass. A rolling ball is another example of combined translational and rotational motions.

### Combined Translation and Rotation: Energy Relationships

It's beyond our scope to prove that rigid-body motion can always be divided into translation of the center of mass and rotation about the center of mass. But we *can* prove this for the kinetic energy $K$ of a rigid body that has both translational and rotational motions. For such a rigid body, $K$ is the sum of two parts:

**Kinetic energy of a rigid body with both translation and rotation**

$$K = \underbrace{\tfrac{1}{2}Mv_{\rm cm}^2}_{\text{Kinetic energy of } translation \text{ of center of mass (cm)}} + \underbrace{\tfrac{1}{2}I_{\rm cm}\omega^2}_{\text{Kinetic energy of } rotation \text{ around axis through cm}} \tag{10.8}$$

Mass of rigid body ($M$); Speed of cm ($v_{\rm cm}$); Moment of inertia of rigid body about axis through cm ($I_{\rm cm}$); Angular speed of rigid body ($\omega$)

To prove this relationship, we again imagine the rigid body to be made up of particles. For a typical particle with mass $m_i$ (**Fig. 10.12**), the velocity $\vec{v}_i$ of this particle relative to an inertial frame is the vector sum of the velocity $\vec{v}_{\rm cm}$ of the center of mass and the velocity $\vec{v}_i{}'$ of the particle *relative to* the center of mass:

$$\vec{v}_i = \vec{v}_{\rm cm} + \vec{v}_i{}' \tag{10.9}$$

The kinetic energy $K_i$ of this particle in the inertial frame is $\frac{1}{2}m_i v_i^2$, which we can also express as $\frac{1}{2}m_i(\vec{v}_i \cdot \vec{v}_i)$. Substituting Eq. (10.9) into this, we get

$$\begin{aligned} K_i &= \tfrac{1}{2}m_i(\vec{v}_{\rm cm} + \vec{v}_i{}') \cdot (\vec{v}_{\rm cm} + \vec{v}_i{}') \\ &= \tfrac{1}{2}m_i(\vec{v}_{\rm cm} \cdot \vec{v}_{\rm cm} + 2\vec{v}_{\rm cm} \cdot \vec{v}_i{}' + \vec{v}_i{}' \cdot \vec{v}_i{}') \\ &= \tfrac{1}{2}m_i(v_{\rm cm}^2 + 2\vec{v}_{\rm cm} \cdot \vec{v}_i{}' + v_i{}'^2) \end{aligned}$$

The total kinetic energy is the sum $\sum K_i$ for all the particles making up the rigid body. Expressing the three terms in this equation as separate sums, we get

$$K = \sum K_i = \sum\left(\tfrac{1}{2}m_i v_{\rm cm}^2\right) + \sum(m_i\vec{v}_{\rm cm} \cdot \vec{v}_i{}') + \sum\left(\tfrac{1}{2}m_i v_i{}'^2\right)$$

The first and second terms have common factors that we take outside the sum:

$$K = \tfrac{1}{2}\left(\sum m_i\right)v_{\rm cm}^2 + \vec{v}_{\rm cm} \cdot \left(\sum m_i\vec{v}_i{}'\right) + \sum\left(\tfrac{1}{2}m_i v_i{}'^2\right) \tag{10.10}$$

Now comes the reward for our effort. In the first term, $\sum m_i$ is the total mass $M$. The second term is zero because $\sum m_i\vec{v}_i{}'$ is $M$ times the velocity of the center of mass *relative to the center of mass,* and this is zero by definition. The last term is the sum of the kinetic energies of the particles computed by using their speeds with respect to the center of mass; this is just the kinetic energy of rotation around the center of mass. Using the same steps that led to Eq. (9.17) for the rotational kinetic energy of a rigid body, we can write this last term as $\frac{1}{2}I_{\rm cm}\omega^2$, where $I_{\rm cm}$ is the moment of inertia with respect to the axis through the center of mass and $\omega$ is the angular speed. So Eq. (10.10) becomes Eq. (10.8):

$$K = \tfrac{1}{2}Mv_{\rm cm}^2 + \tfrac{1}{2}I_{\rm cm}\omega^2$$

## Rolling Without Slipping

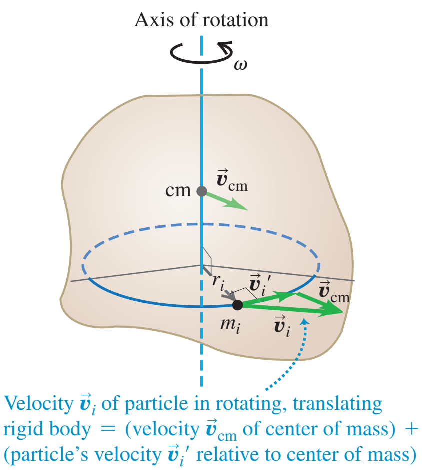
An important case of combined translation and rotation is **rolling without slipping.** The rolling wheel in **Fig. 10.13** (next page) is symmetrical, so its center of mass is at its geometric center. We view the motion in an inertial frame of reference in which the surface on which the wheel rolls is at rest. In this frame, the point on the wheel that contacts the surface must be instantaneously *at rest* so that it does not slip. Hence the velocity $\vec{v}_1{}'$ of the point of contact relative to the center of mass must have the same magnitude but opposite direction as the center-of-mass velocity $\vec{v}_{\rm cm}$. If the wheel's radius is $R$ and its angular speed about the center of mass is $\omega$, then the magnitude of $\vec{v}_1{}'$ is $R\omega$; hence

**Condition for rolling without slipping:**

$$v_{\rm cm} = R\omega \tag{10.11}$$

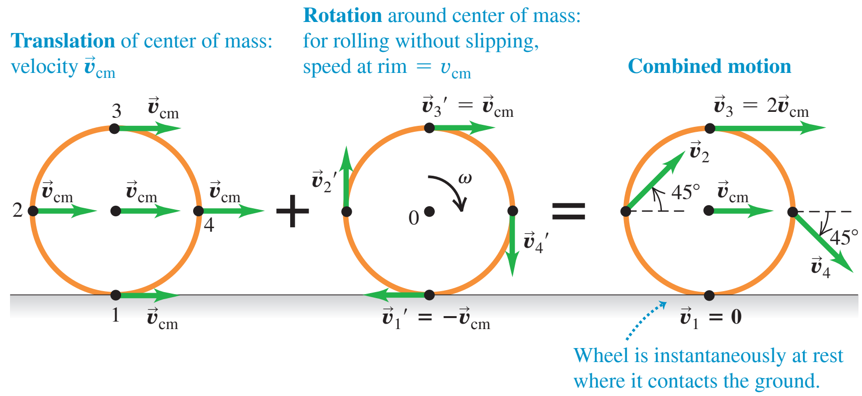
Speed of center of mass of rolling wheel ($v_{\rm cm}$); Radius of wheel ($R$); Angular speed of wheel ($\omega$)

Figure **10.12** A rigid body with both translational and rotational motions.




**BIO APPLICATION Combined Translation and Rotation** A maple seed consists of a pod attached to a much lighter, flattened wing. Airflow around the wing slows the falling seed to about 1 m/s and causes the seed to rotate about its center of mass. The seed's slow fall means that a breeze can carry the seed some distance from the parent tree. In the absence of wind, the seed's center of mass falls straight down.

Figure **10.13** The motion of a rolling wheel is the sum of the translational motion of the center of mass and the rotational motion of the wheel around the center of mass.




As Fig. 10.13 shows, the velocity of a point on the wheel is the vector sum of the velocity of the center of mass and the velocity of the point relative to the center of mass. Thus while point 1, the point of contact, is instantaneously at rest, point 3 at the top of the wheel is moving forward *twice as fast* as the center of mass, and points 2 and 4 at the sides have velocities at 45° to the horizontal.

At any instant we can think of the wheel as rotating about an "instantaneous axis" of rotation that passes through the point of contact with the ground. The angular velocity $\omega$ is the same for this axis as for an axis through the center of mass; an observer at the center of mass sees the rim make the same number of revolutions per second as does an observer at the rim watching the center of mass spin around him. If we think of the motion of the rolling wheel in Fig. 10.13 in this way, the kinetic energy of the wheel is $K = \frac{1}{2}I_1\omega^2$, where $I_1$ is the moment of inertia of the wheel about an axis through point 1. But by the parallel-axis theorem, Eq. (9.19), $I_1 = I_{cm} + MR^2$, where $M$ is the total mass of the wheel and $I_{cm}$ is the moment of inertia with respect to an axis through the center of mass. Using Eq. (10.11), we find that the wheel's kinetic energy is as given by Eq. (10.8):

$$K = \tfrac{1}{2}I_1\omega^2 = \tfrac{1}{2}I_{cm}\omega^2 + \tfrac{1}{2}MR^2\omega^2 = \tfrac{1}{2}I_{cm}\omega^2 + \tfrac{1}{2}Mv_{cm}{}^2$$

Figure **10.14** The smoke rising from this drag racer's rear tires shows that the tires are slipping on the road, so $v_{cm}$ is *not* equal to $R\omega$.

**CAUTION** **Rolling without slipping** The relationship $v_{cm} = R\omega$ holds *only* if there is rolling without slipping. When a drag racer first starts to move, the rear tires are spinning very fast even though the racer is hardly moving, so $R\omega$ is greater than $v_{cm}$ (**Fig. 10.14**). If a driver applies the brakes too heavily so that the car skids, the tires will spin hardly at all and $R\omega$ is less than $v_{cm}$.

If a rigid body changes height as it moves, we must also consider gravitational potential energy. We saw in Section 9.4 that for any extended object of mass $M$, rigid or not, the gravitational potential energy $U$ is the same as if we replaced the object by a particle of mass $M$ located at the object's center of mass, so

$$U = Mgy_{cm}$$

## EXAMPLE 10.4 Speed of a primitive yo-yo

A primitive yo-yo has a massless string wrapped around a solid cylinder with mass $M$ and radius $R$ (**Fig. 10.15**). You hold the free end of the string stationary and release the cylinder from rest. The string unwinds but does not slip or stretch as the cylinder descends and rotates. Using energy considerations, find the speed $v_{cm}$ of the cylinder's center of mass after it has descended a distance $h$.

**IDENTIFY and SET UP** Since you hold the upper end of the string fixed, your hand does no work on the string–cylinder system. There is friction between the string and the cylinder, but the string doesn't slip so no mechanical energy is lost. Hence we can use conservation of mechanical energy. The initial kinetic energy of the cylinder is $K_1 = 0$, and its final kinetic energy $K_2$ is given by Eq. (10.8); the massless string has no kinetic energy. The moment of inertia is $I_{cm} = \frac{1}{2}MR^2$, and by Eq. (9.13) $\omega = v_{cm}/R$ because the string doesn't slip. The potential energies are $U_1 = Mgh$ and $U_2 = 0$.

**EXECUTE** From Eq. (10.8), the kinetic energy at point 2 is

$$K_2 = \tfrac{1}{2}Mv_{cm}{}^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v_{cm}}{R}\right)^2 = \tfrac{3}{4}Mv_{cm}{}^2$$

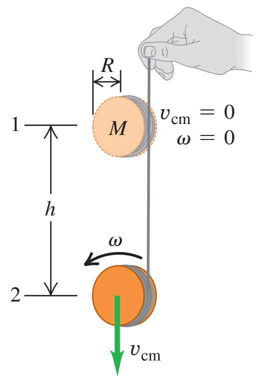



Figure **10.15** Calculating the speed of a primitive yo-yo.

The kinetic energy is $1\frac{1}{2}$ times what it would be if the yo-yo were falling at speed $v_{cm}$ without rotating. Two-thirds of the total kinetic energy $\left(\frac{1}{2}Mv_{cm}{}^2\right)$ is translational and one-third $\left(\frac{1}{4}Mv_{cm}{}^2\right)$ is rotational. Using conservation of energy,

$$K_1 + U_1 = K_2 + U_2$$
$$0 + Mgh = \tfrac{3}{4}Mv_{cm}{}^2 + 0$$
$$v_{cm} = \sqrt{\tfrac{4}{3}gh}$$

**EVALUATE** No mechanical energy was lost or gained, so from the energy standpoint the string is merely a way to convert some of the gravitational potential energy (which is released as the cylinder falls) into rotational kinetic energy rather than translational kinetic energy. Because not all of the released energy goes into translation, $v_{cm}$ is less than the speed $\sqrt{2gh}$ of an object dropped from height $h$ with no strings attached.

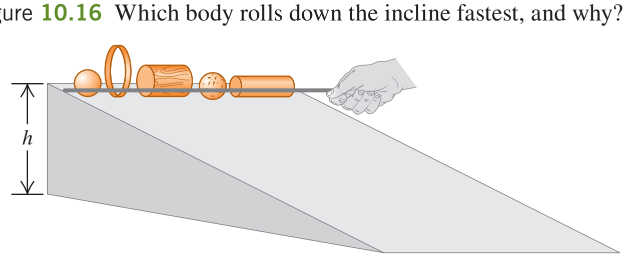
**KEYCONCEPT** If a rigid body is both translating (moving as a whole through space) and rotating, its total kinetic energy is the sum of the kinetic energy of translation of the center of mass and the kinetic energy of rotation around an axis through the center of mass.

### EXAMPLE 10.5 Race of the rolling bodies

In a physics demonstration, an instructor "races" various rigid bodies that roll without slipping from rest down an inclined plane (**Fig. 10.16**). What shape should a body have to reach the bottom of the incline first?

**IDENTIFY and SET UP** Kinetic friction does no work if the bodies roll without slipping. We can also ignore the effects of *rolling friction,* introduced in Section 5.3, if the bodies and the surface of the incline are rigid. (Later in this section we'll explain why this is so.) We can therefore use conservation of energy. Each body starts from rest at the top of an incline with height $h$, so $K_1 = 0$, $U_1 = Mgh$, and $U_2 = 0$. Equation (10.8) gives the kinetic energy at the bottom of the incline; since the bodies roll without slipping, $\omega = v_{cm}/R$. We can express the moments of inertia of the four round bodies in Table 9.2, cases (f)–(i), as $I_{cm} = cMR^2$, where $c$ is a number less than or equal to 1 that depends on the shape of the body, Our goal is to find the value of $c$ that gives the body the greatest speed $v_{cm}$ after its center of mass has descended a vertical distance $h$.

Figure **10.16** Which body rolls down the incline fastest, and why?




**EXECUTE** From conservation of energy,

$$K_1 + U_1 = K_2 + U_2$$
$$0 + Mgh = \tfrac{1}{2}Mv_{cm}{}^2 + \tfrac{1}{2}cMR^2\left(\frac{v_{cm}}{R}\right)^2 + 0$$
$$Mgh = \tfrac{1}{2}(1 + c)Mv_{cm}{}^2$$
$$v_{cm} = \sqrt{\frac{2gh}{1 + c}}$$

**EVALUATE** For a given value of $c$, the speed $v_{cm}$ after descending a distance $h$ is *independent* of the body's mass $M$ and radius $R$. Hence *all* uniform solid cylinders $\left(c = \frac{1}{2}\right)$ have the same speed at the bottom, regardless of their mass and radii. The values of $c$ tell us that the order of finish for uniform bodies will be as follows: (1) any solid sphere $\left(c = \frac{2}{5}\right)$, (2) any solid cylinder $\left(c = \frac{1}{2}\right)$, (3) any thin-walled, hollow sphere $\left(c = \frac{2}{3}\right)$, and (4) any thin-walled, hollow cylinder $(c = 1)$. Small-$c$ bodies always beat large-$c$ bodies because less of their kinetic energy is tied up in rotation, so more is available for translation.

**KEYCONCEPT** For a rigid body that rolls without slipping, has a given mass and radius, and moves with a given center-of-mass speed, the kinetic energy of rotation depends on the shape of the rigid body.

## Combined Translation and Rotation: Dynamics

We can also analyze the combined translational and rotational motions of a rigid body from the standpoint of dynamics. We showed in Section 8.5 that for an extended object, the acceleration of the center of mass is the same as that of a particle of the same mass acted on by all the external forces on the actual object:

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} \qquad (10.12)$$

Net external force on an object

Total mass of object

Acceleration of center of mass

The rotational motion about the center of mass is described by the rotational analog of Newton's second law, Eq. (10.7):

Net torque on a rigid body about $z$-axis through center of mass

Moment of inertia of rigid body about $z$-axis

$$\sum \tau_z = I_{cm}\alpha_z \qquad (10.13)$$

Angular acceleration of rigid body about $z$-axis

Figure **10.17** The axle of a bicycle wheel passes through the wheel's center of mass and is an axis of symmetry. Hence the rotation of the wheel is described by Eq. (10.13), provided the bicycle doesn't turn or tilt to one side (which would change the orientation of the axle).

It's not immediately obvious that Eq. (10.13) should apply to the motion of a translating rigid body; after all, our derivation of $\sum \tau_z = I\alpha_z$ in Section 10.2 assumed that the axis of rotation was stationary. But Eq. (10.13) is valid *even when the axis of rotation moves,* provided the following two conditions are met:

1. The axis through the center of mass must be an axis of symmetry.
2. The axis must not change direction.

These conditions are satisfied for many types of rotation (**Fig. 10.17**). Note that in general this moving axis of rotation is *not* at rest in an inertial frame of reference.

We can now solve dynamics problems involving a rigid body that undergoes translational and rotational motions at the same time, provided that the rotation axis satisfies the two conditions just mentioned. Problem-Solving Strategy 10.1 (Section 10.2) is equally useful here, and you should review it now. Keep in mind that when a rigid body undergoes translational and rotational motions at the same time, we need two separate equations of motion *for the same/body:* Eq. (10.12) for the translation of the center of mass and Eq. (10.13) for rotation about an axis through the center of mass.

## EXAMPLE 10.6 Acceleration of a primitive yo-yo

WITH VARIATION PROBLEMS

For the primitive yo-yo in Example 10.4 (**Fig. 10.18a**), find the downward acceleration of the cylinder and the tension in the string.

**IDENTIFY and SET UP** Figure 10.18b shows our free-body diagram for the yo-yo, including our choice of positive coordinate directions. Our target variables are $a_{cm\text{-}y}$ and $T$. We'll use Eq. (10.12) for the translational motion of the center of mass and Eq. (10.13) for the rotational motion around the center of mass. We'll also use Eq. (10.11), which says that the string unwinds without slipping. As in Example 10.4, the moment of inertia of the yo-yo for an axis through its center of mass is $I_{cm} = \frac{1}{2}MR^2$.

**EXECUTE** From Eq. (10.12),

$$\sum F_y = Mg + (-T) = Ma_{cm\text{-}y} \qquad (10.14)$$

From Eq. (10.13),

$$\sum \tau_z = TR = I_{cm}\alpha_z = \tfrac{1}{2}MR^2\alpha_z \qquad (10.15)$$

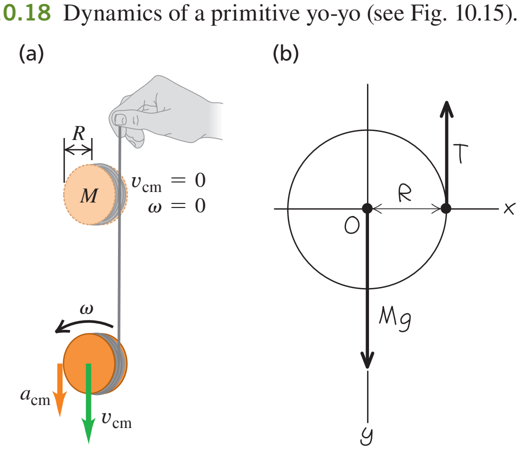
From Eq. (10.11), $v_{cm\text{-}z} = R\omega_z$; the derivative of this expression with respect to time gives us

$$a_{cm\text{-}y} = R\alpha_z \qquad (10.16)$$

We now use Eq. (10.16) to eliminate $\alpha_z$ from Eq. (10.15) and then solve Eqs. (10.14) and (10.15) simultaneously for $T$ and $a_{cm\text{-}y}$:

$$a_{cm\text{-}y} = \tfrac{2}{3}g \qquad T = \tfrac{1}{3}Mg$$

Figure **10.18** Dynamics of a primitive yo-yo (see Fig. 10.15).




**EVALUATE** The string slows the fall of the yo-yo, but not enough to stop it completely. Hence $a_{cm\text{-}y}$ is less than the free-fall value $g$ and $T$ is less than the yo-yo weight $Mg$.

**KEYCONCEPT** To analyze the motion of a rigid body that is both translating and rotating, use Newton's second law for the translational motion of the center of mass and the rotational analog of Newton's second law for the rotation around the center of mass.

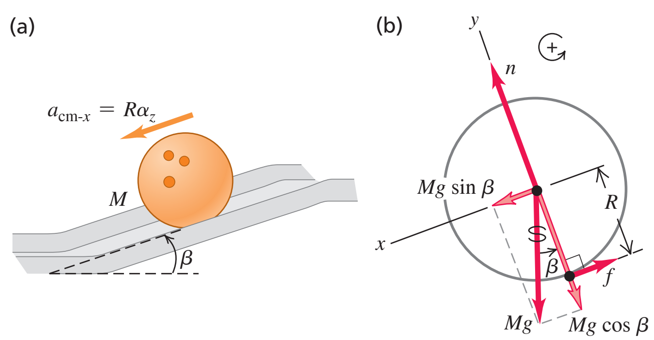
## EXAMPLE 10.7 Acceleration of a rolling sphere

WITH VARIATION PROBLEMS

A bowling ball of mass $M$ rolls without slipping down a ramp that is inclined at an angle $\beta$ to the horizontal (**Fig. 10.19a**). What are the ball's acceleration and the magnitude of the friction force on the ball? Treat the ball as a uniform solid sphere, ignoring the finger holes.

**IDENTIFY and SET UP** The free-body diagram (Fig. 10.19b) shows that only the friction force exerts a torque about the center of mass. Our target variables are the acceleration $a_{\text{cm-}x}$ of the ball's center of mass and the magnitude $f$ of the friction force. (Because the ball does not slip at the instantaneous point of contact with the ramp, this is a *static* friction force; it prevents slipping and gives the ball its angular acceleration.) We use Eqs. (10.12) and (10.13) as in Example 10.6.

Figure 10.19 A bowling ball rolling down a ramp.




**EXECUTE** The ball's moment of inertia is $I_{\text{cm}} = \frac{2}{5}MR^2$. The equations of motion are

$$\sum F_x = Mg\sin\beta + (-f) = Ma_{\text{cm-}x} \qquad (10.17)$$

$$\sum \tau_z = fR = I_{\text{cm}}\alpha_z = \left(\tfrac{2}{5}MR^2\right)\alpha_z \qquad (10.18)$$

The ball rolls without slipping, so as in Example 10.6 we use $a_{\text{cm-}x} = R\alpha_z$ to eliminate $\alpha_z$ from Eq. (10.18):

$$fR = \tfrac{2}{5}MRa_{\text{cm-}x}$$

This equation and Eq. (10.17) are two equations for the unknowns $a_{\text{cm-}x}$ and $f$. We solve Eq. (10.17) for $f$, substitute that expression into the above equation to eliminate $f$, and solve for $a_{\text{cm-}x}$:

$$a_{\text{cm-}x} = \tfrac{5}{7}g\sin\beta$$

Finally, we substitute this acceleration into Eq. (10.17) and solve for $f$:

$$f = \tfrac{2}{7}Mg\sin\beta$$

**EVALUATE** The ball's acceleration is just $\frac{5}{7}$ as large as that of an object *sliding* down the slope without friction. If the ball descends a vertical distance $h$ as it rolls down the ramp, its displacement along the ramp is $h/\sin\beta$. You can show that the speed of the ball at the bottom of the ramp is $v_{\text{cm}} = \sqrt{\frac{10}{7}gh}$, the same as our result from Example 10.5 with $c = \frac{2}{5}$.

If the ball were rolling *uphill* without slipping, the force of friction would still be directed uphill as in Fig. 10.19b. Can you see why?

**KEYCONCEPT** If an object is rolling without slipping on an incline, a friction force must act on it. The direction of this friction force is always such as to prevent slipping.

## Rolling Friction

In Example 10.5 we said that we can ignore rolling friction if both the rolling body and the surface over which it rolls are perfectly rigid. In **Fig. 10.20a** a perfectly rigid sphere is rolling down a perfectly rigid incline. The line of action of the normal force passes through the center of the sphere, so its torque is zero; there is no sliding at the point of contact, so the friction force does no work. Figure 10.20b shows a more realistic situation, in which the surface "piles up" in front of the sphere and the sphere rides in a shallow trench. Because of these deformations, the contact forces on the sphere no longer act along a single point but over an area; the forces are concentrated on the front of the sphere as shown. As a result, the normal force now exerts a torque that opposes the rotation. In addition, there is some sliding of the sphere over the surface due to the deformation, causing

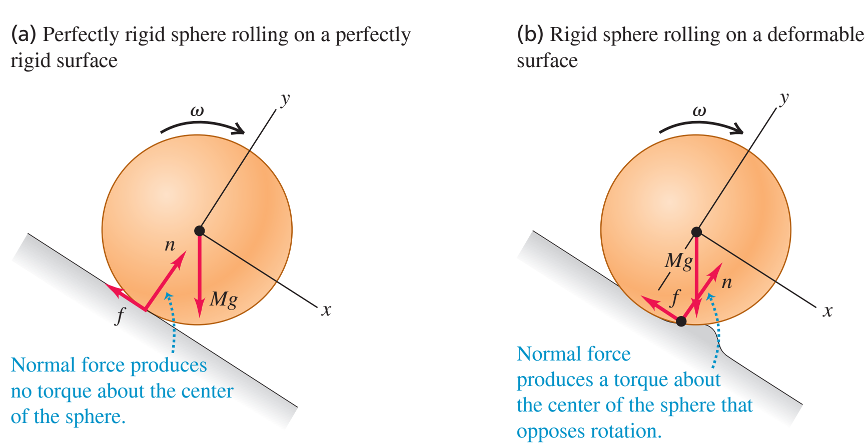



Figure 10.20 Rolling down (a) a perfectly rigid surface and (b) a deformable surface. In (b) the deformation is greatly exaggerated, and the force $n$ is the component of the contact force that points normal to the plane of the surface before it is deformed.

**BIO APPLICATION Rolling for Reproduction** One of the few organisms that uses rolling as a means of locomotion is the weed called Russian thistle (*Kali tragus*). The plant breaks off at its base, forming a rounded tumbleweed that disperses its seeds as it rolls. Because a tumbleweed deforms easily, it is subject to substantial rolling friction. Hence it quickly slows to a stop unless propelled by the wind.

mechanical energy to be lost. The combination of these two effects is the phenomenon of *rolling friction*. Rolling friction also occurs if the rolling body is deformable, such as an automobile tire. Often the rolling body and the surface are rigid enough that rolling friction can be ignored, as we have assumed in all the examples in this section.

**TEST YOUR UNDERSTANDING OF SECTION 10.3** Suppose the solid cylinder used as a yo-yo in Example 10.6 is replaced by a hollow cylinder of the same mass and radius. (a) Will the acceleration of the yo-yo (i) increase, (ii) decrease, or (iii) remain the same? (b) Will the string tension (i) increase, (ii) decrease, or (iii) remain the same?

**ANSWER**

(a) (**ii**), (b) (**i**) If you redo the calculation of Example 10.6 with a hollow cylinder (moment of inertia $I_{\mathrm{cm}} = MR^2$) instead of a solid cylinder (moment of inertia $I_{\mathrm{cm}} = \frac{1}{2}MR^2$), you'll find $a_{\mathrm{cm}\text{-}y} = \frac{1}{2}g$ and $T = \frac{1}{2}Mg$ (instead of $a_{\mathrm{cm}\text{-}y} = \frac{2}{3}g$ and $T = \frac{1}{3}Mg$ for a solid cylinder). Hence the acceleration is less but the tension is greater. You can come to the same conclusion without doing the calculation. The greater moment of inertia means that the hollow cylinder will rotate more slowly and hence will roll downward more slowly. To slow the downward motion, a greater upward tension force is needed to oppose the downward force of gravity.

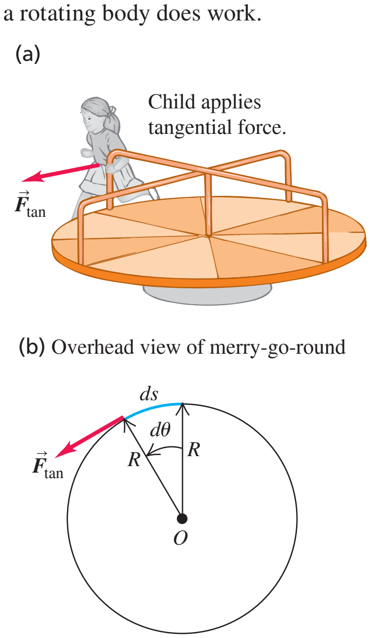
## 10.4 WORK AND POWER IN ROTATIONAL MOTION

When you pedal a bicycle, you apply forces to a rotating body and do work on it. Similar things happen in many other real-life situations, such as a rotating motor shaft driving a power tool or a car engine propelling the vehicle. Let's see how to apply our ideas about work from Chapter 6 to rotational motion.

Suppose a tangential force $\vec{F}_{\tan}$ acts at the rim of a pivoted disk—for example, a child running while pushing on a playground merry-go-round (**Fig. 10.21a**). The disk rotates through an infinitesimal angle $d\theta$ about a fixed axis during an infinitesimal time interval $dt$ (Fig. 10.21b). The work $dW$ done by the force $\vec{F}_{\tan}$ while a point on the rim moves a distance $ds$ is $dW = F_{\tan}\,ds$. If $d\theta$ is measured in radians, then $ds = R\,d\theta$ and

$$dW = F_{\tan}R\,d\theta$$

Figure **10.21** A tangential force applied to a rotating body does work.




Now $F_{\tan}R$ is the *torque* $\tau_z$ due to the force $\vec{F}_{\tan}$, so

$$dW = \tau_z\,d\theta \tag{10.19}$$

As the disk rotates from $\theta_1$ to $\theta_2$, the total work done by the torque is

Work done by a torque $\tau_z$

$$W = \int_{\theta_1}^{\theta_2} \tau_z\,d\theta \tag{10.20}$$

Upper limit = final angular position
Integral of the torque with respect to angle
Lower limit = initial angular position

If the torque remains *constant* while the angle changes, then the work is the product of torque and angular displacement:

Work done by a constant torque $\tau_z$

$$W = \overbrace{\tau_z}^{\text{Torque}}\underbrace{(\theta_2 - \theta_1)}_{\text{Final minus initial angular position}} = \tau_z\underbrace{\Delta\theta}_{\text{angular displacement}} \tag{10.21}$$

If torque is expressed in newton-meters $(\mathrm{N \cdot m})$ and angular displacement in radians, the work is in joules. Equation (10.21) is the rotational analog of Eq. (6.1), $W = Fs$, and Eq. (10.20) is the analog of Eq. (6.7), $W = \int F_x\,dx$, for the work done by a force in a straight-line displacement.

If the force in Fig. 10.21 had an axial component (parallel to the rotation axis) or a radial component (directed toward or away from the axis), that component would do no work because the displacement of the point of application has only a tangential component. An axial or radial component of force would also make no contribution to the torque about the axis of rotation. So Eqs. (10.20) and (10.21) are correct for *any* force, no matter what its components.

When a torque does work on a rotating rigid body, the kinetic energy changes by an amount equal to the work done. We can prove this by using exactly the same procedure that we used in Eqs. (6.11) through (6.13) for the translational kinetic energy of a particle. Let $\tau_z$ represent the *net* torque on the body so that $\tau_z = I\alpha_z$ from Eq. (10.7), and assume that the body is rigid so that the moment of inertia $I$ is constant. We then transform the integrand in Eq. (10.20) into an integrand with respect to $\omega_z$ as follows:

$$\tau_z\, d\theta = (I\alpha_z)\, d\theta = I\frac{d\omega_z}{dt}d\theta = I\frac{d\theta}{dt}d\omega_z = I\omega_z\, d\omega_z$$

Since $\tau_z$ is the net torque, the integral in Eq. (10.20) is the *total* work done on the rotating rigid body. This equation then becomes

Total work done on a rotating rigid body = work done by the net external torque

$$W_{\text{tot}} = \int_{\omega_1}^{\omega_2} I\omega_z\, d\omega_z = \overbrace{\tfrac{1}{2}I\omega_2^2}^{\text{Final rotational kinetic energy}} - \underbrace{\tfrac{1}{2}I\omega_1^2}_{\text{Initial rotational kinetic energy}} \qquad (10.22)$$

The change in the rotational kinetic energy of a *rigid* body equals the work done by forces exerted from outside the body (**Fig. 10.22**). This equation is analogous to Eq. (6.13), the work–energy theorem for a particle.

Figure **10.22** The rotational kinetic energy of a helicopter's main rotor is equal to the total work done to set it spinning. When it is spinning at a constant rate, positive work is done on the rotor by the engine and negative work is done on it by air resistance. Hence the net work being done is zero and the kinetic energy remains constant.

How does *power* relate to torque? When we divide both sides of Eq. (10.19) by the time interval $dt$ during which the angular displacement occurs, we find

$$\frac{dW}{dt} = \tau_z \frac{d\theta}{dt}$$

But $dW/dt$ is the rate of doing work, or *power P*, and $d\theta/dt$ is angular velocity $\omega_z$:

**Power due to a torque** acting on a rigid body

$$P = \tau_z\omega_z \qquad (10.23)$$

$\tau_z$: Torque with respect to rigid body's rotation axis
$\omega_z$: Angular velocity of rigid body about axis

This is the analog of the relationship $P = \vec{F} \cdot \vec{v}$ that we developed in Section 6.4 for particle motion.

## EXAMPLE 10.8 Calculating power from torque

An electric motor exerts a constant 10 N · m torque on a grindstone, which has a moment of inertia of 2.0 kg · m$^2$ about its shaft. The system starts from rest. Find the work $W$ done by the motor in 8.0 s and the grindstone's kinetic energy $K$ at this time. What average power $P_{\text{av}}$ is delivered by the motor?

**IDENTIFY and SET UP** The only torque acting is that due to the motor. Since this torque is constant, the grindstone's angular acceleration $\alpha_z$ is constant. We'll use Eq. (10.7) to find $\alpha_z$, and then use this in the kinematics equations from Section 9.2 to calculate the angle $\Delta\theta$ through which the grindstone rotates in 8.0 s and its final angular velocity $\omega_z$. From these we'll calculate $W$, $K$, and $P_{\text{av}}$.

**EXECUTE** We have $\sum\tau_z = 10\ \text{N}\cdot\text{m}$ and $I = 2.0\ \text{kg}\cdot\text{m}^2$, so $\sum\tau_z = I\alpha_z$ yields $\alpha_z = 5.0\ \text{rad/s}^2$. From Eqs. (9.11) and (10.21),

$$\Delta\theta = \tfrac{1}{2}\alpha_z t^2 = \tfrac{1}{2}(5.0\ \text{rad/s}^2)(8.0\ \text{s})^2 = 160\ \text{rad}$$
$$W = \tau_z\Delta\theta = (10\ \text{N}\cdot\text{m})(160\ \text{rad}) = 1600\ \text{J}$$

From Eqs. (9.7) and (9.17),

$$\omega_z = \alpha_z t = (5.0\ \text{rad/s}^2)(8.0\ \text{s}) = 40\ \text{rad/s}$$
$$K = \tfrac{1}{2}I\omega_z^2 = \tfrac{1}{2}(2.0\ \text{kg}\cdot\text{m}^2)(40\ \text{rad/s})^2 = 1600\ \text{J}$$

The average power is the work done divided by the time interval:

$$P_{\text{av}} = \frac{1600\ \text{J}}{8.0\ \text{s}} = 200\ \text{J/s} = 200\ \text{W}$$

*Continued*

**EVALUATE** The initial kinetic energy was zero, so the work done $W$ must equal the final kinetic energy $K$ [Eq. (10.22)]. This is just as we calculated. We can check our result $P_{av} = 200$ W by considering the *instantaneous* power $P = \tau_z \omega_z$. Because $\omega_z$ increases continuously, $P$ increases continuously as well; its value increases from zero at $t = 0$ to $(10\ \mathrm{N \cdot m})(40\ \mathrm{rad/s}) = 400$ W at $t = 8.0$ s. Both $\omega_z$ and $P$ increase *uniformly* with time, so the *average* power is just half this maximum value, or 200 W.

**KEYCONCEPT** If a torque acts on a rigid body, the work done equals torque times angular displacement and the power equals torque times angular velocity.

**TEST YOUR UNDERSTANDING OF SECTION 10.4** You apply equal torques to two different cylinders. Cylinder 1 has a moment of inertia twice as large as cylinder 2. Each cylinder is initially at rest. After one complete rotation, which cylinder has the greater kinetic energy? (i) Cylinder 1; (ii) cylinder 2; (iii) both cylinders have the same kinetic energy.

**ANSWER**

❙ **(iii)** You apply the same torque over the same angular displacement to both cylinders. Hence, by Eq. (10.21), you do the same amount of work to both cylinders and impart the same kinetic energy to both. (The one with the smaller moment of inertia ends up with a greater angular speed, but that isn't what we are asked. Compare Conceptual Example 6.5 in Section 6.2.)

## 10.5 ANGULAR MOMENTUM

Every rotational quantity that we have encountered in Chapters 9 and 10 is the analog of some quantity in the translational motion of a particle. The analog of *momentum* of a particle is **angular momentum,** a vector quantity denoted as $\vec{L}$. Its relationship to momentum $\vec{p}$ (which we'll often call *linear momentum* for clarity) is exactly the same as the relationship of torque to force, $\vec{\tau} = \vec{r} \times \vec{F}$. For a particle with constant mass $m$ and velocity $\vec{v}$, the angular momentum is

**Angular momentum of a particle** relative to origin $O$ of an inertial frame of reference

Position vector of particle relative to $O$

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \qquad (10.24)$$

Linear momentum of particle = mass times velocity

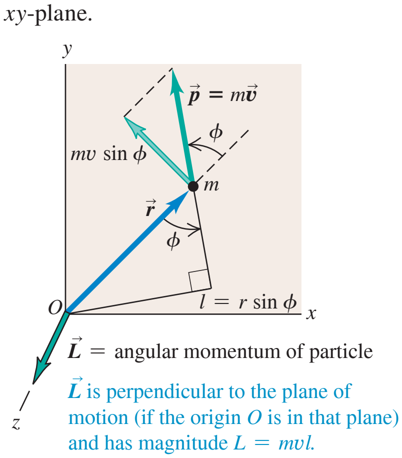
The value of $\vec{L}$ depends on the choice of origin $O$, since it involves the particle's position vector $\vec{r}$ relative to $O$. The units of angular momentum are $\mathrm{kg \cdot m^2/s}$.

In **Fig. 10.23** a particle moves in the $xy$-plane; its position vector $\vec{r}$ and momentum $\vec{p} = m\vec{v}$ are shown. The angular momentum vector $\vec{L}$ is perpendicular to the $xy$-plane. The right-hand rule for vector products shows that its direction is along the $+z$-axis, and its magnitude is

$$L = mvr\sin\phi = mvl \qquad (10.25)$$

where $l$ is the perpendicular distance from the line of $\vec{v}$ to $O$. This distance plays the role of "lever arm" for the momentum vector.

Figure 10.23 Calculating the angular momentum $\vec{L} = \vec{r} \times m\vec{v} = \vec{r} \times \vec{p}$ of a particle with mass $m$ moving in the $xy$-plane.




When a net force $\vec{F}$ acts on a particle, its velocity and momentum change, so its angular momentum may also change. We can show that the *rate of change* of angular momentum is equal to the torque of the net force. We take the time derivative of Eq. (10.24), using the rule for the derivative of a product:

$$\frac{d\vec{L}}{dt} = \left(\frac{d\vec{r}}{dt} \times m\vec{v}\right) + \left(\vec{r} \times m\frac{d\vec{v}}{dt}\right) = (\vec{v} \times m\vec{v}) + (\vec{r} \times m\vec{a})$$

The first term is zero because it contains the vector product of the vector $\vec{v} = d\vec{r}/dt$ with itself. In the second term we replace $m\vec{a}$ with the net force $\vec{F}$:

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \quad \text{(for a particle acted on by net force } \vec{F}\text{)} \qquad (10.26)$$

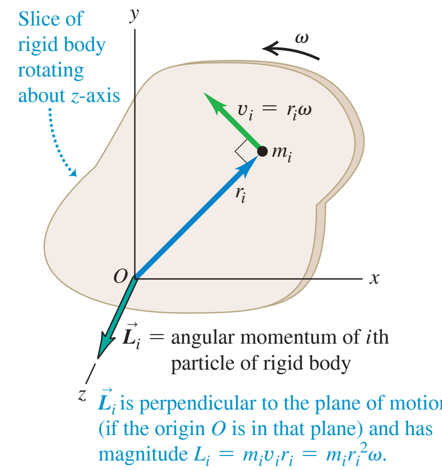
**The rate of change of angular momentum of a particle equals the torque of the net force acting on it.** Compare this result to Eq. (8.4): The rate of change $d\vec{p}/dt$ of the *linear* momentum of a particle equals the net force that acts on it.

## Angular Momentum of a Rigid Body

We can use Eq. (10.25) to find the total angular momentum of a *rigid body* rotating about the $z$-axis with angular speed $\omega$. First consider a thin slice of the body lying in the $xy$-plane (**Fig. 10.24**). Each particle in the slice moves in a circle centered at the origin, and at each instant its velocity $\vec{v}_i$ is perpendicular to its position vector $\vec{r}_i$, as shown. Hence in Eq. (10.25), $\phi = 90°$ for every particle. A particle with mass $m_i$ at a distance $r_i$ from $O$ has a speed $v_i$ equal to $r_i\omega$. From Eq. (10.25) the magnitude $L_i$ of its angular momentum is

$$L_i = m_i(r_i\omega)r_i = m_i r_i^2\omega \qquad (10.27)$$

Figure **10.24** Calculating the angular momentum of a particle of mass $m_i$ in a rigid body rotating at angular speed $\omega$. (Compare Fig. 10.23.)




The direction of each particle's angular momentum, as given by the right-hand rule for the vector product, is along the $+z$-axis.

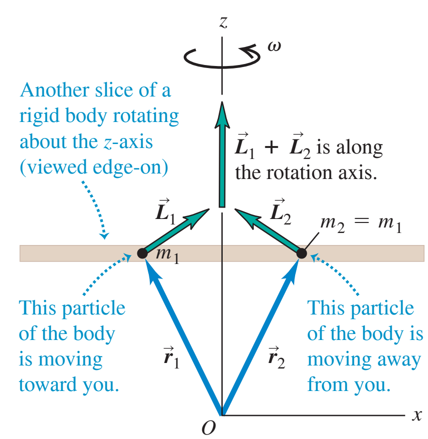
The *total* angular momentum of the slice of the rigid body that lies in the $xy$-plane is the sum $\sum L_i$ of the angular momenta $L_i$ of all of its particles. From Eq. (10.27),

$$L = \sum L_i = \left(\sum m_i r_i^2\right)\omega = I\omega$$

where $I$ is the moment of inertia of the slice about the $z$-axis.

We can do this same calculation for the other slices of the rigid body, all parallel to the $xy$-plane. For points that do not lie in the $xy$-plane, a complication arises because the $\vec{r}$ vectors have components in the $z$-direction as well as in the $x$- and $y$-directions; this gives the angular momentum of each particle a component perpendicular to the $z$-axis. But *if the z-axis is an axis of symmetry,* the perpendicular components for particles on opposite sides of this axis add up to zero (**Fig. 10.25**). So when a rigid body rotates about an axis of symmetry, its angular momentum vector $\vec{L}$ lies along the symmetry axis, and its magnitude is $L = I\omega$.

Figure **10.25** Two particles of the same mass located symmetrically on either side of the rotation axis of a rigid body. The angular momentum vectors $\vec{L}_1$ and $\vec{L}_2$ of the two particles do not lie along the rotation axis, but their vector sum $\vec{L}_1 + \vec{L}_2$ does.




The angular velocity vector $\vec{\omega}$ also lies along the rotation axis, as we saw in Section 9.1. Hence for a rigid body rotating around an axis of symmetry, $\vec{L}$ and $\vec{\omega}$ are in the same direction (**Fig. 10.26**). So we have the *vector* relationship

**Angular momentum of a rigid body rotating around a symmetry axis**

$$\vec{L} = I\vec{\omega} \qquad (10.28)$$

Moment of inertia of rigid body about symmetry axis

Angular velocity vector of rigid body

From Eq. (10.26) the rate of change of angular momentum of a particle equals the torque of the net force acting on the particle. For any system of particles (including both rigid and nonrigid bodies), the rate of change of the *total* angular momentum equals the sum of the torques of all forces acting on all the particles. The torques of the *internal* forces add to zero if these forces act along the line from one particle to another, as in Fig. 10.8, and so the sum of the torques includes only the torques of the *external* forces. (We saw a similar cancellation in our discussion of center-of-mass motion in Section 8.5.) So we conclude that

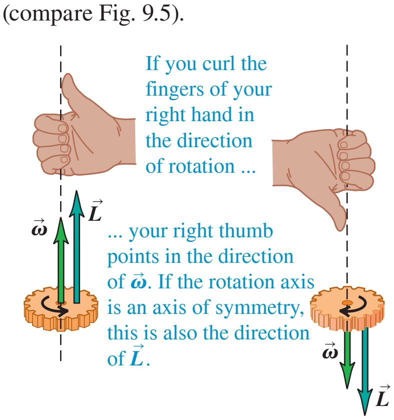
**For a system of particles:**

Sum of external torques on the system

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt} \qquad (10.29)$$

Rate of change of total angular momentum $\vec{L}$ of system

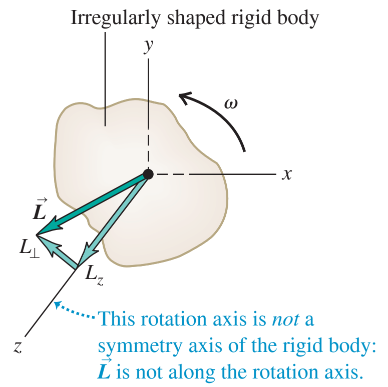
Figure **10.26** For rotation about an axis of symmetry, $\vec{\omega}$ and $\vec{L}$ are parallel and along the axis. The directions of both vectors are given by the right-hand rule (compare Fig. 9.5).




Finally, if the system of particles is a rigid body rotating about a symmetry axis (the $z$-axis), then $L_z = I\omega_z$ and $I$ is constant. If this axis has a fixed direction in space, then vectors $\vec{L}$ and $\vec{\omega}$ change only in magnitude, not in direction. In that case, $dL_z/dt = I\,d\omega_z/dt = I\alpha_z$, or

$$\sum \tau_z = I\alpha_z$$

Figure 10.27 If the rotation axis of a rigid body is not a symmetry axis, $\vec{L}$ does not in general lie along the rotation axis. Even if $\vec{\omega}$ is constant, the direction of $\vec{L}$ changes and a net torque is required to maintain rotation.




which is again our basic relationship for the dynamics of rigid-body rotation. If the body is *not* rigid, $I$ may change; in that case, $L$ changes even when $\omega$ is constant. For a nonrigid body, Eq. (10.29) is still valid, even though Eq. (10.7) is not.

When the axis of rotation is *not* a symmetry axis, the angular momentum is in general *not* parallel to the axis (**Fig. 10.27**). As the rigid body rotates, the angular momentum vector $\vec{L}$ traces out a cone around the rotation axis. Because $\vec{L}$ changes, there must be a net external torque acting on the body even though the angular velocity magnitude $\omega$ may be constant. If the body is an unbalanced wheel on a car, this torque is provided by friction in the bearings, which causes the bearings to wear out. "Balancing" a wheel means distributing the mass so that the rotation axis is an axis of symmetry; then $\vec{L}$ points along the rotation axis, and no net torque is required to keep the wheel turning.

In fixed-axis rotation we often use the term "angular momentum of the body" to refer to only the *component* of $\vec{L}$ along the rotation axis of the body (the $z$-axis in Fig. 10.27), with a positive or negative sign to indicate the sense of rotation just as with angular velocity.

## EXAMPLE 10.9 Angular momentum and torque

A turbine fan in a jet engine has a moment of inertia of $2.5\ \text{kg}\cdot\text{m}^2$ about its axis of rotation. As the turbine starts up, its angular velocity is given by $\omega_z = (40\ \text{rad/s}^3)t^2$. (a) Find the fan's angular momentum as a function of time, and find its value at $t = 3.0$ s. (b) Find the net torque on the fan as a function of time, and find its value at $t = 3.0$ s.

**IDENTIFY and SET UP** The fan rotates about its axis of symmetry (the $z$-axis). Hence the angular momentum vector has only a $z$-component $L_z$, which we can determine from the angular velocity $\omega_z$. Since the direction of angular momentum is constant, the net torque likewise has only a component $\tau_z$ along the rotation axis. We'll use Eq. (10.28) to find $L_z$ from $\omega_z$ and then Eq. (10.29) to find $\tau_z$.

**EXECUTE** (a) From Eq. (10.28),

$$L_z = I\omega_z = (2.5\ \text{kg}\cdot\text{m}^2)(40\ \text{rad/s}^3)t^2$$
$$= (100\ \text{kg}\cdot\text{m}^2/\text{s}^3)t^2$$

(We dropped the dimensionless quantity "rad" from the final expression.) At $t = 3.0$ s, $L_z = 900\ \text{kg}\cdot\text{m}^2/\text{s}$.

(b) From Eq. (10.29),

$$\tau_z = \frac{dL_z}{dt} = (100\ \text{kg}\cdot\text{m}^2/\text{s}^3)(2t) = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)t$$

At $t = 3.0$ s,

$$\tau_z = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)(3.0\ \text{s})$$
$$= 600\ \text{kg}\cdot\text{m}^2/\text{s}^2 = 600\ \text{N}\cdot\text{m}$$

**EVALUATE** As a check on our expression for $\tau_z$, note that the angular acceleration of the turbine is $\alpha_z = d\omega_z/dt = (40\ \text{rad/s}^3)(2t) = (80\ \text{rad/s}^3)t$. Hence from Eq. (10.7), the torque on the fan is $\tau_z = I\alpha_z = (2.5\ \text{kg}\cdot\text{m}^2)(80\ \text{rad/s}^3)t = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)t$, just as we calculated.

**KEYCONCEPT** The angular momentum vector of a rotating rigid body points along the rigid body's rotation axis. The rate of change of angular momentum equals the net external torque on the rigid body.

**TEST YOUR UNDERSTANDING OF SECTION 10.5** A ball is attached to one end of a piece of string. You hold the other end of the string and whirl the ball in a circle around your hand. (a) If the ball moves at a constant speed, is its linear momentum $\vec{p}$ constant? Why or why not? (b) Is its angular momentum $\vec{L}$ constant? Why or why not?

**ANSWER**

(a) **no,** (b) **yes** As the ball goes around the circle, the magnitude of $\vec{p} = m\vec{v}$ remains the same (the speed is constant) but its direction changes, so the linear momentum vector isn't constant. But $\vec{L} = \vec{r} \times \vec{p}$ *is* constant: It has a constant magnitude (both the speed and the perpendicular distance from your hand to the ball are constant) and a constant direction (along the rotation axis, perpendicular to the plane of the ball's motion). The linear momentum changes because there is a net *force* $\vec{F}$ on the ball (toward the center of the circle). The angular momentum remains constant because there is no net *torque*; the vector $\vec{r}$ points from your hand to the ball and the force $\vec{F}$ on the ball is directed toward your hand, so the vector product $\vec{\tau} = \vec{r} \times \vec{F}$ is zero.

## 10.6 CONSERVATION OF ANGULAR MOMENTUM

We have just seen that angular momentum can be used for an alternative statement of the basic dynamic principle for rotational motion. It also forms the basis for the principle of **conservation of angular momentum.** Like conservation of energy and of linear momentum, this principle is a universal conservation law, valid at all scales from atomic and nuclear systems to the motions of galaxies. This principle follows directly from Eq. (10.29): $\sum \vec{\tau} = d\vec{L}/dt$. If $\sum \vec{\tau} = \mathbf{0}$, then $d\vec{L}/dt = \mathbf{0}$, and $\vec{L}$ is constant.

**CONSERVATION OF ANGULAR MOMENTUM When the net external torque acting on a system is zero, the total angular momentum of the system is constant (conserved).**

A circus acrobat, a diver, and an ice skater pirouetting on one skate all take advantage of this principle. Suppose an acrobat has just left a swing; she has her arms and legs extended and is rotating counterclockwise about her center of mass. When she pulls her arms and legs in, her moment of inertia $I_{cm}$ with respect to her center of mass changes from a large value $I_1$ to a much smaller value $I_2$. The only external force acting on her is her weight, which has no torque with respect to an axis through her center of mass. So her angular momentum $L_z = I_{cm}\omega_z$ remains constant, and her angular velocity $\omega_z$ increases as $I_{cm}$ decreases. That is,

$$I_1\omega_{1z} = I_2\omega_{2z} \quad \text{(zero net external torque)} \tag{10.30}$$

When a skater or ballerina spins with arms outstretched and then pulls her arms in, her angular velocity increases as her moment of inertia decreases. In each case there is conservation of angular momentum in a system in which the net external torque is zero.

When a system has several parts, the internal forces that the parts exert on one another cause changes in the angular momenta of the parts, but the *total* angular momentum doesn't change. Here's an example. Consider two objects $A$ and $B$ that interact with each other but not with anything else, such as the astronauts we discussed in Section 8.2 (see Fig. 8.9). Suppose object $A$ exerts a force $\vec{F}_{A \text{ on } B}$ on object $B$; the corresponding torque (with respect to whatever point we choose) is $\vec{\tau}_{A \text{ on } B}$. According to Eq. (10.29), this torque is equal to the rate of change of angular momentum of $B$:

$$\vec{\tau}_{A \text{ on } B} = \frac{d\vec{L}_B}{dt}$$

At the same time, object $B$ exerts a force $\vec{F}_{B \text{ on } A}$ on object $A$, with a corresponding torque $\vec{\tau}_{B \text{ on } A}$, and

$$\vec{\tau}_{B \text{ on } A} = \frac{d\vec{L}_A}{dt}$$

From Newton's third law, $\vec{F}_{B \text{ on } A} = -\vec{F}_{A \text{ on } B}$. Furthermore, if the forces act along the same line, as in Fig. 10.8, their lever arms with respect to the chosen axis are equal. Thus the *torques* of these two forces are equal and opposite, and $\vec{\tau}_{B \text{ on } A} = -\vec{\tau}_{A \text{ on } B}$. So if we add the two preceding equations, we find

$$\frac{d\vec{L}_A}{dt} + \frac{d\vec{L}_B}{dt} = \mathbf{0}$$

or, because $\vec{L}_A + \vec{L}_B$ is the *total* angular momentum $\vec{L}$ of the system,

$$\frac{d\vec{L}}{dt} = \mathbf{0} \quad \text{(zero net external torque)} \tag{10.31}$$

That is, the total angular momentum of the system is constant. The torques of the internal forces can transfer angular momentum from one object to the other, but they can't change the *total* angular momentum of the system (**Fig. 10.28**).

Figure 10.28 A falling cat twists different parts of its body in different directions so that it lands feet first. At all times during this process the angular momentum of the cat as a whole remains zero.

## EXAMPLE 10.10 Anyone can be a ballerina

A physics professor stands at the center of a frictionless turntable with arms outstretched and a 5.0 kg dumbbell in each hand (**Fig. 10.29**). He is set rotating about the vertical axis, making one revolution in 2.0 s. Find his final angular velocity if he pulls the dumbbells inward to his stomach. His moment of inertia (without the dumbbells) is $3.0\ \text{kg} \cdot \text{m}^2$ with arms outstretched and $2.2\ \text{kg} \cdot \text{m}^2$ with his hands at his stomach. The dumbbells are 1.0 m from the axis initially and 0.20 m at the end.

**IDENTIFY, SET UP, and EXECUTE** No external torques act about the $z$-axis, so $L_z$ is constant. We'll use Eq. (10.30) to find the final angular velocity $\omega_{2z}$. The moment of inertia of the system is $I = I_{\text{prof}} + I_{\text{dumbbells}}$. We treat each dumbbell as a particle of mass $m$ that contributes $mr^2$ to $I_{\text{dumbbells}}$, where $r$ is the perpendicular distance from the axis to the dumbbell. Initially we have

$$I_1 = 3.0\ \text{kg} \cdot \text{m}^2 + 2(5.0\ \text{kg})(1.0\ \text{m})^2 = 13\ \text{kg} \cdot \text{m}^2$$

$$\omega_{1z} = \frac{1\ \text{rev}}{2.0\ \text{s}} = 0.50\ \text{rev/s}$$

The final moment of inertia is

$$I_2 = 2.2\ \text{kg} \cdot \text{m}^2 + 2(5.0\ \text{kg})(0.20\ \text{m})^2 = 2.6\ \text{kg} \cdot \text{m}^2$$

From Eq. (10.30), the final angular velocity is

$$\omega_{2z} = \frac{I_1}{I_2}\omega_{1z} = \frac{13\ \text{kg} \cdot \text{m}^2}{2.6\ \text{kg} \cdot \text{m}^2}(0.50\ \text{rev/s}) = 2.5\ \text{rev/s} = 5\omega_{1z}$$

Can you see why we didn't have to change "revolutions" to "radians" in this calculation?

**EVALUATE** The angular momentum remained constant, but the angular velocity increased by a factor of 5, from $\omega_{1z} = (0.50\ \text{rev/s}) \times (2\pi\ \text{rad/rev}) = 3.14\ \text{rad/s}$ to $\omega_{2z} = (2.5\ \text{rev/s})(2\pi\ \text{rad/rev}) = 15.7\ \text{rad/s}$. The initial and final kinetic energies are then

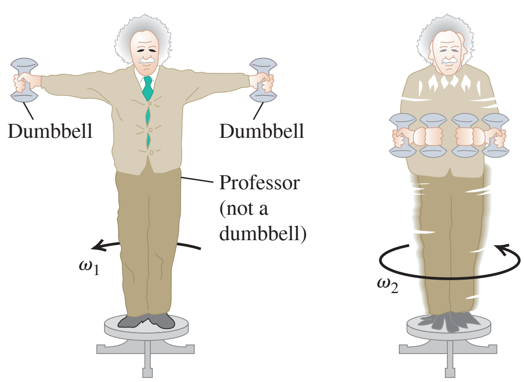
$$K_1 = \tfrac{1}{2}I_1\omega_{1z}{}^2 = \tfrac{1}{2}(13\ \text{kg} \cdot \text{m}^2)(3.14\ \text{rad/s})^2 = 64\ \text{J}$$

$$K_2 = \tfrac{1}{2}I_2\omega_{2z}{}^2 = \tfrac{1}{2}(2.6\ \text{kg} \cdot \text{m}^2)(15.7\ \text{rad/s})^2 = 320\ \text{J}$$

The fivefold increase in kinetic energy came from the work that the professor did in pulling his arms and the dumbbells inward.

**KEYCONCEPT** If there is zero net external torque on a rigid body, its angular momentum is conserved. If the body changes shape so that its moment of inertia changes, its angular velocity changes to keep the angular momentum constant.

Figure **10.29** Fun with conservation of angular momentum.




## EXAMPLE 10.11 A rotational "collision"

WITH VARIATION PROBLEMS

**Figure 10.30** shows two disks: an engine flywheel ($A$) and a clutch plate ($B$) attached to a transmission shaft. Their moments of inertia are $I_A$ and $I_B$; initially, they are rotating in the same direction with constant angular speeds $\omega_A$ and $\omega_B$, respectively. We push the disks together with forces acting along the axis, so as not to apply any torque on either disk. The disks rub against each other and eventually reach a common angular speed $\omega$. Derive an expression for $\omega$.

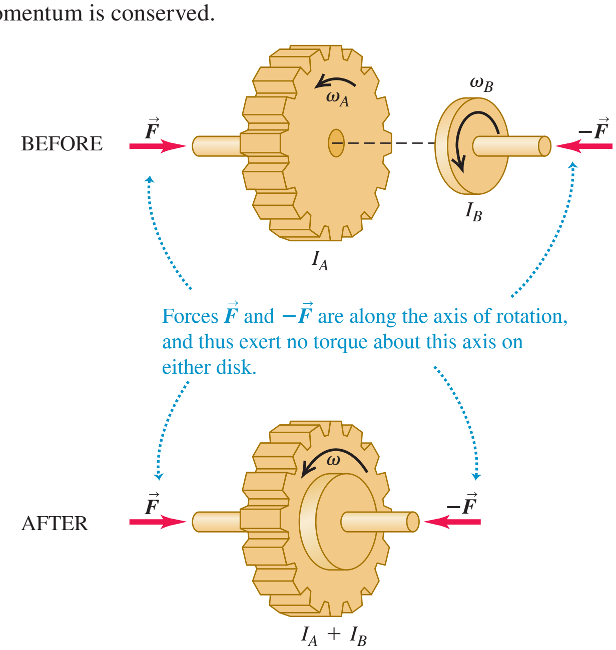
**IDENTIFY, SET UP, and EXECUTE** There are no external torques, so the only torque acting on either disk is the torque applied by the other disk. Hence the total angular momentum of the system of two disks is conserved. At the end they rotate together as one object with total moment of inertia $I = I_A + I_B$ and angular speed $\omega$. Figure 10.30 shows that all angular velocities are in the same direction, so we can regard $\omega_A$, $\omega_B$, and $\omega$ as components of angular velocity along the rotation axis. Conservation of angular momentum gives

$$I_A\omega_A + I_B\omega_B = (I_A + I_B)\omega$$

$$\omega = \frac{I_A\omega_A + I_B\omega_B}{I_A + I_B}$$

**EVALUATE** This "collision" is analogous to a completely inelastic collision (see Section 8.3). When two objects in translational motion along the

Figure **10.30** When the net external torque is zero, angular momentum is conserved.



same axis collide and stick, the linear momentum of the system is conserved. Here two objects in *rotational* motion around the same axis "collide" and stick, and the *angular* momentum of the system is conserved.

The kinetic energy of a system decreases in a completely inelastic collision. Here kinetic energy is lost because nonconservative (friction) internal forces act while the two disks rub together. Suppose flywheel $A$ has a mass of 2.0 kg, a radius of 0.20 m, and an initial angular speed of 50 rad/s (about 500 rpm), and clutch plate $B$ has a mass of 4.0 kg, a radius of 0.10 m, and an initial angular speed of 200 rad/s. Can you show that the final kinetic energy is only two-thirds of the initial kinetic energy?

**KEYCONCEPT** In processes that conserve angular momentum, the kinetic energy can change if nonconservative forces act.

## EXAMPLE 10.12 Angular momentum in a crime bust

WITH VARIATION PROBLEMS

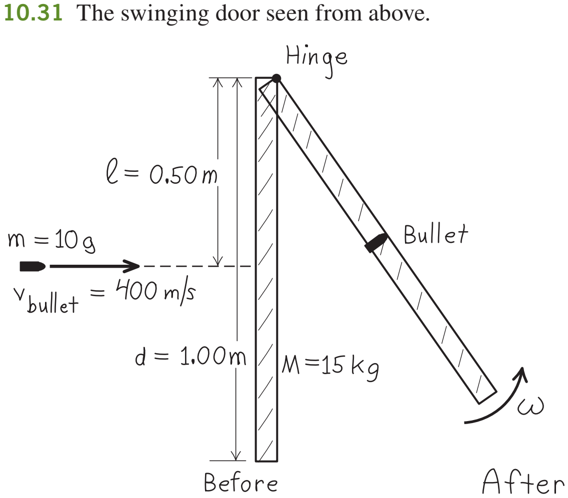
A door 1.00 m wide, of mass 15 kg, can rotate freely about a vertical axis through its hinges. A bullet with a mass of 10 g and a speed of 400 m/s strikes the center of the door, in a direction perpendicular to the plane of the door, and embeds itself there. Find the door's angular speed. Is kinetic energy conserved?

**IDENTIFY and SET UP** We consider the door and bullet as a system. There is no external torque about the hinge axis, so angular momentum about this axis is conserved. **Figure 10.31** shows our sketch. The initial angular momentum is that of the bullet, as given by Eq. (10.25). The final angular momentum is that of a rigid body composed of the door and the embedded bullet. We'll equate these quantities and solve for the resulting angular speed $\omega$ of the door and bullet.

**EXECUTE** From Eq. (10.25), the initial angular momentum of the bullet is

$$L = mvl = (0.010\text{ kg})(400\text{ m/s})(0.50\text{ m}) = 2.0\text{ kg}\cdot\text{m}^2/\text{s}$$

Figure **10.31** The swinging door seen from above.




The final angular momentum is $I\omega$, where $I = I_{door} + I_{bullet}$. From Table 9.2, case (d), for a door of width $d = 1.00$ m,

$$I_{door} = \frac{Md^2}{3} = \frac{(15\text{ kg})(1.00\text{ m})^2}{3} = 5.0\text{ kg}\cdot\text{m}^2$$

The moment of inertia of the bullet (with respect to the axis along the hinges) is

$$I_{bullet} = ml^2 = (0.010\text{ kg})(0.50\text{ m})^2 = 0.0025\text{ kg}\cdot\text{m}^2$$

Conservation of angular momentum requires that $mvl = I\omega$, or

$$\omega = \frac{mvl}{I} = \frac{2.0\text{ kg}\cdot\text{m}^2/\text{s}}{5.0\text{ kg}\cdot\text{m}^2 + 0.0025\text{ kg}\cdot\text{m}^2} = 0.40\text{ rad/s}$$

The initial and final kinetic energies are

$$K_1 = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(0.010\text{ kg})(400\text{ m/s})^2 = 800\text{ J}$$

$$K_2 = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}(5.0025\text{ kg}\cdot\text{m}^2)(0.40\text{ rad/s})^2 = 0.40\text{ J}$$

**EVALUATE** The final kinetic energy is only $\frac{1}{2000}$ of the initial value! We did not expect kinetic energy to be conserved: The collision is inelastic because nonconservative friction forces act during the impact. The door's final angular speed is quite slow: At 0.40 rad/s, it takes 3.9 s to swing through 90° ($\pi/2$ radians).

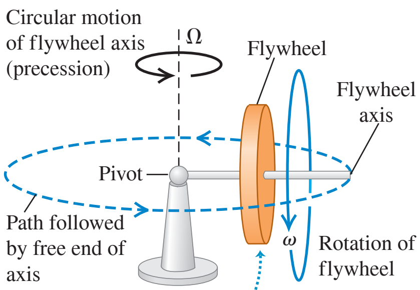
**KEYCONCEPT** The total angular momentum of a system that includes a rigid body and a particle is the sum of the angular momenta for the rigid body and for the particle. You can find the magnitude of the angular momentum of a particle about a rotation axis by multiplying the magnitude of its linear momentum by the perpendicular distance from the axis to the line of the particle's velocity.

**TEST YOUR UNDERSTANDING OF SECTION 10.6** If the polar ice caps were to melt completely due to global warming, the melted ice would redistribute itself over the earth. This change would cause the length of the day (the time needed for the earth to rotate once on its axis) to (i) increase; (ii) decrease; (iii) remain the same. (*Hint:* Use angular momentum ideas. Assume that the sun, moon, and planets exert negligibly small torques on the earth.)

**ANSWER**

**(i)** In the absence of external torques, the earth's angular momentum $L_z = I\omega_z$ would remain constant. The melted ice would move from the poles toward the equator—that is, away from our planet's rotation axis—and the earth's moment of inertia $I$ would increase slightly. Hence the angular velocity $\omega_z$ would decrease slightly and the day would be slightly longer.

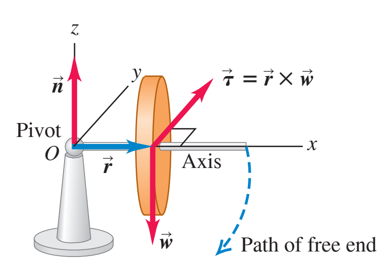
Figure 10.32 A gyroscope supported at one end. The horizontal circular motion of the flywheel and axis is called precession. The angular speed of precession is $\Omega$.




When the flywheel and its axis are stationary, they will fall to the table surface. When the flywheel spins, it and its axis "float" in the air while moving in a circle about the pivot.

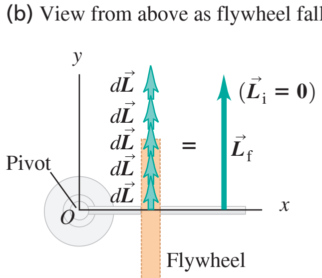
Figure 10.33 (a) If the flywheel in Fig. 10.32 is initially not spinning, its initial angular momentum is zero. (b) In each successive time interval $dt$, the torque produces a change $d\vec{L} = \vec{\tau}\,dt$ in the angular momentum. The flywheel acquires an angular momentum $\vec{L}$ in the same direction as $\vec{\tau}$, and the flywheel axis falls.

(a) Nonrotating flywheel falls




When the flywheel is not rotating, its weight creates a torque around the pivot, causing it to fall along a circular path until its axis rests on the table surface.

(b) View from above as flywheel falls




In falling, the flywheel rotates about the pivot and thus acquires an angular momentum $\vec{L}$. The *direction* of $\vec{L}$ stays constant.

## 10.7 GYROSCOPES AND PRECESSION

In all the situations we've looked at so far in this chapter, the axis of rotation either has stayed fixed or has moved and kept the same direction (such as rolling without slipping). But a variety of new physical phenomena, some quite unexpected, can occur when the axis of rotation changes direction. For example, consider a toy gyroscope that's supported at one end (**Fig. 10.32**). If we hold it with the flywheel axis horizontal and let go, the free end of the axis simply drops owing to gravity—*if* the flywheel isn't spinning. But if the flywheel *is* spinning, what happens is quite different. One possible motion is a steady circular motion of the axis in a horizontal plane, combined with the spin motion of the flywheel about the axis. This surprising, nonintuitive motion of the axis is called **precession.** Precession is found in nature as well as in rotating machines such as gyroscopes. As you read these words, the earth itself is precessing; its spin axis (through the north and south poles) slowly changes direction, going through a complete cycle of precession every 26,000 years.

To study this strange phenomenon of precession, we must remember that angular velocity, angular momentum, and torque are all *vector* quantities. In particular, we need the general relationship between the net torque $\sum\vec{\tau}$ that acts on an object and the rate of change of the object's angular momentum $\vec{L}$, given by Eq. (10.29), $\sum\vec{\tau} = d\vec{L}/dt$. Let's first apply this equation to the case in which the flywheel is *not* spinning (**Fig. 10.33a**). We take the origin $O$ at the pivot and assume that the flywheel is symmetrical, with mass $M$ and moment of inertia $I$ about the flywheel axis. The flywheel axis is initially along the $x$-axis. The only external forces on the gyroscope are the normal force $\vec{n}$ acting at the pivot (assumed to be frictionless) and the weight $\vec{w}$ of the flywheel that acts at its center of mass, a distance $r$ from the pivot. The normal force has zero torque with respect to the pivot, and the weight has a torque $\vec{\tau}$ in the $y$-direction, as shown in Fig. 10.33a. Initially, there is no rotation, and the initial angular momentum $\vec{L}_i$ is zero. From Eq. (10.29) the *change* $d\vec{L}$ in angular momentum in a short time interval $dt$ following this is

$$d\vec{L} = \vec{\tau}\,dt \tag{10.32}$$

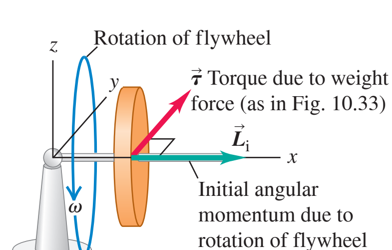
This change is in the $y$-direction because $\vec{\tau}$ is. As each additional time interval $dt$ elapses, the angular momentum changes by additional increments $d\vec{L}$ in the $y$-direction because the direction of the torque is constant (Fig. 10.33b). The steadily increasing horizontal angular momentum means that the gyroscope rotates downward faster and faster around the $y$-axis until it hits either the stand or the table on which it sits.

Now let's see what happens if the flywheel *is* spinning initially, so the initial angular momentum $\vec{L}_i$ is not zero (**Fig. 10.34a**). Since the flywheel rotates around its symmetry axis, $\vec{L}_i$ lies along this axis. But each change in angular momentum $d\vec{L}$ is perpendicular to the flywheel axis because the torque $\vec{\tau} = \vec{r} \times \vec{w}$ is perpendicular to that axis (Fig. 10.34b).

Figure 10.34 (a) The flywheel is spinning initially with angular momentum $\vec{L}_i$. The forces (not shown) are the same as those in Fig. 10.33a. (b) Because the initial angular momentum is not zero, each change $d\vec{L} = \vec{\tau}\,dt$ in angular momentum is perpendicular to $\vec{L}$. As a result, the magnitude of $\vec{L}$ remains the same but its direction changes continuously.

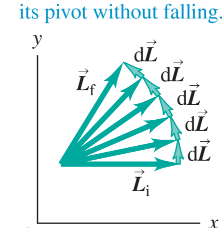
(a) Rotating flywheel

When the flywheel is rotating, the system starts with an angular momentum $\vec{L}_i$ parallel to the flywheel's axis of rotation.




(b) View from above

Now the effect of the torque is to cause the angular momentum to precess around the pivot. The gyroscope circles around its pivot without falling.



This causes the *direction* of $\vec{L}$ to change, but not its magnitude. The changes $d\vec{L}$ are always in the horizontal $xy$-plane, so the angular momentum vector and the flywheel axis with which it moves are always horizontal. That is, the axis doesn't fall—it precesses.

If this still seems mystifying to you, think about a ball attached to a string. If the ball is initially at rest and you pull the string toward you, the ball moves toward you also. But if the ball is initially moving and you continuously pull the string in a direction perpendicular to the ball's motion, the ball moves in a circle around your hand; it does not approach your hand at all. In the first case the ball has zero linear momentum $\vec{p}$ to start with; when you apply a force $\vec{F}$ toward you for a time $dt$, the ball acquires a momentum $d\vec{p} = \vec{F}\,dt$, which is also toward you. But if the ball already has linear momentum $\vec{p}$, a change in momentum $d\vec{p}$ that's perpendicular to $\vec{p}$ changes the direction of motion, not the speed. Replace $\vec{p}$ with $\vec{L}$ and $\vec{F}$ with $\vec{\tau}$ in this argument, and you'll see that precession is simply the rotational analog of uniform circular motion.

At the instant shown in Fig. 10.34a, the gyroscope has angular momentum $\vec{L}$. A short time interval $dt$ later, the angular momentum is $\vec{L} + d\vec{L}$; the infinitesimal change in angular momentum is $d\vec{L} = \vec{\tau}\,dt$, which is perpendicular to $\vec{L}$. As the vector diagram in **Fig. 10.35** shows, this means that the flywheel axis of the gyroscope has turned through a small angle $d\phi$ given by $d\phi = |d\vec{L}|/|\vec{L}|$. The rate at which the axis moves, $d\phi/dt$, is called the **precession angular speed;** denoting this quantity by $\Omega$, we find

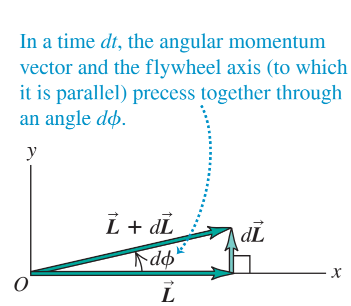
$$\Omega = \frac{d\phi}{dt} = \frac{|d\vec{L}|/|\vec{L}|}{dt} = \frac{\tau_z}{L_z} = \frac{wr}{I\omega} \tag{10.33}$$

Thus the precession angular speed is *inversely* proportional to the angular speed of spin about the axis. A rapidly spinning gyroscope precesses slowly; if friction in its bearings causes the flywheel to slow down, the precession angular speed *increases*! The precession angular speed of the earth is very slow (1 rev/26,000 yr) because its spin angular momentum $L_z$ is large and the torque $\tau_z$, due to the gravitational influences of the moon and sun, is relatively small.

As a gyroscope precesses, its center of mass moves in a circle with radius $r$ in a horizontal plane. Its vertical component of acceleration is zero, so the upward normal force $\vec{n}$ exerted by the pivot must be just equal in magnitude to the weight. The circular motion of the center of mass with angular speed $\Omega$ requires a force $\vec{F}$ directed toward the center of the circle, with magnitude $F = M\Omega^2 r$. This force must also be supplied by the pivot.

One key assumption that we made in our analysis of the gyroscope was that the angular momentum vector $\vec{L}$ is associated with only the spin of the flywheel and is purely horizontal. But there will also be a vertical component of angular momentum associated with the precessional motion of the gyroscope. By ignoring this, we've tacitly assumed that the precession is *slow*—that is, that the precession angular speed $\Omega$ is very much less than the spin angular speed $\omega$. As Eq. (10.33) shows, a large value of $\omega$ automatically gives a small value of $\Omega$, so this approximation is reasonable. When the precession is not slow, additional effects show up, including an up-and-down wobble or *nutation* of the flywheel axis that's superimposed on the precessional motion. You can see nutation occurring in a gyroscope as its spin slows down, so that $\Omega$ increases and the vertical component of $\vec{L}$ can no longer be ignored.

Figure **10.35** Detailed view of part of Fig. 10.34b.

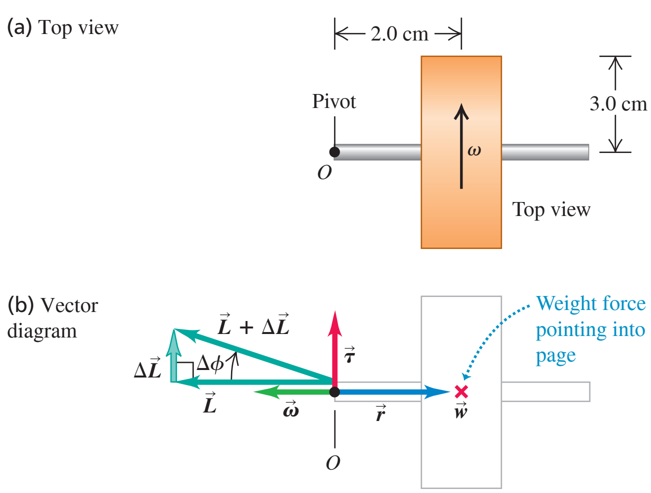


## EXAMPLE 10.13 A precessing gyroscope

**Figure 10.36a** shows a top view of a spinning, cylindrical gyroscope wheel. The pivot is at $O$, and the mass of the axle is negligible. (a) As seen from above, is the precession clockwise or counterclockwise? (b) If the gyroscope takes 4.0 s for one revolution of precession, what is the angular speed of the wheel?

**IDENTIFY and SET UP** We'll determine the direction of precession by using the right-hand rule as in Fig. 10.34, which shows the same kind of gyroscope as Fig. 10.36. We'll use the relationship between precession angular speed $\Omega$ and spin angular speed $\omega$, Eq. (10.33), to find $\omega$.

Figure 10.36 In which direction and at what speed does this gyroscope process?




**EXECUTE** (a) The right-hand rule shows that $\vec{\omega}$ and $\vec{L}$ are to the left in Fig. 10.36b. The weight $\vec{w}$ points into the page in this top view and acts at the center of mass (denoted by $\times$ in the figure). The torque $\vec{\tau} = \vec{r} \times \vec{w}$ is toward the top of the page, so $d\vec{L}/dt$ is also toward the top of the page. Adding a small $d\vec{L}$ to the initial vector $\vec{L}$ changes the direction of $\vec{L}$ as shown, so the precession is clockwise as seen from above.

(b) Be careful not to confuse $\omega$ and $\Omega$! The precession angular speed is $\Omega = (1\text{ rev})/(4.0\text{ s}) = (2\pi\text{ rad})/(4.0\text{ s}) = 1.57\text{ rad/s}$. The weight is $mg$, and if the wheel is a solid, uniform cylinder, its moment of inertia about its symmetry axis is $I = \frac{1}{2}mR^2$. From Eq. (10.33),

$$\begin{aligned}\omega &= \frac{wr}{I\Omega} = \frac{mgr}{(mR^2/2)\Omega} = \frac{2gr}{R^2\Omega}\\ &= \frac{2(9.8\text{ m/s}^2)(2.0 \times 10^{-2}\text{ m})}{(3.0 \times 10^{-2}\text{ m})^2(1.57\text{ rad/s})}\\ &= 280\text{ rad/s} = 2600\text{ rev/min}\end{aligned}$$

**EVALUATE** The precession angular speed $\Omega$ is only about 0.6% of the spin angular speed $\omega$, so this is an example of slow precession.

**KEYCONCEPT** A spinning rigid body will precess if the net external torque on the rigid body is perpendicular to the body's angular momentum vector.

**TEST YOUR UNDERSTANDING OF SECTION 10.7** Suppose the mass of the flywheel in Fig. 10.34 is doubled but all other dimensions and the spin angular speed remain the same. What effect would this change have on the precession angular speed $\Omega$? (i) $\Omega$ would increase by a factor of 4; (ii) $\Omega$ would double; (iii) $\Omega$ would be unaffected; (iv) $\Omega$ would be one-half as much; (v) $\Omega$ would be one-quarter as much.

**ANSWER**

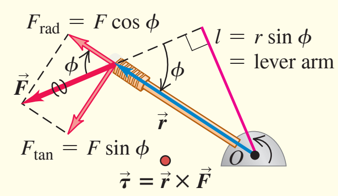
**(iii)** Doubling the flywheel mass would double both its moment of inertia $I$ and its weight $w$, so the ratio $I/w$ would be unchanged. Equation (10.33) shows that the precession angular speed depends on this ratio, so there would be *no* effect on the value of $\Omega$.

# CHAPTER 10 SUMMARY

**Torque:** When a force $\vec{F}$ acts on an object, the torque of that force with respect to a point $O$ has a magnitude given by the product of the force magnitude $F$ and the lever arm $l$. More generally, torque is a vector $\vec{\tau}$ equal to the vector product of $\vec{r}$ (the position vector of the point at which the force acts) and $\vec{F}$. (See Example 10.1.)

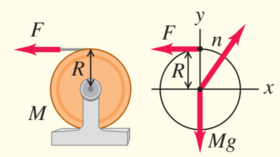
$$\tau = Fl = rF\sin\phi = F_{\tan}r \quad (10.2)$$

$$\vec{\tau} = \vec{r} \times \vec{F} \quad (10.3)$$




**Rotational dynamics:** The rotational analog of Newton's second law says that the net torque acting on an object equals the product of the object's moment of inertia and its angular acceleration. (See Examples 10.2 and 10.3.)

$$\sum \tau_z = I\alpha_z \quad (10.7)$$




**Combined translation and rotation:** If a rigid body is both moving through space and rotating, its motion can be regarded as translational motion of the center of mass plus rotational motion about an axis through the center of mass. Thus the kinetic energy is a sum of translational and rotational kinetic energies. For dynamics, Newton's second law describes the motion of the center of mass, and the rotational equivalent of Newton's second law describes rotation about the center of mass. In the case of rolling without slipping, there is a special relationship between the motion of the center of mass and the rotational motion. (See Examples 10.4–10.7.)

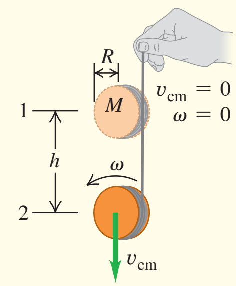
$$K = \tfrac{1}{2}Mv_{cm}^2 + \tfrac{1}{2}I_{cm}\omega^2 \quad (10.8)$$

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} \quad (10.12)$$

$$\sum \tau_z = I_{cm}\alpha_z \quad (10.13)$$

$$v_{cm} = R\omega \quad (10.11)$$

(rolling without slipping)




**Work done by a torque:** A torque that acts on a rigid body as it rotates does work on that body. The work can be expressed as an integral of the torque. The work–energy theorem says that the total rotational work done on a rigid body is equal to the change in rotational kinetic energy. The power, or rate at which the torque does work, is the product of the torque and the angular velocity (See Example 10.8.)

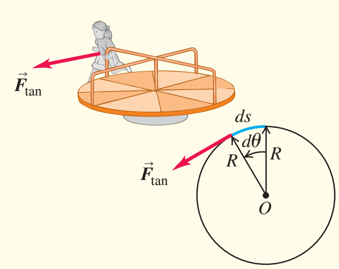
$$W = \int_{\theta_1}^{\theta_2} \tau_z \, d\theta \quad (10.20)$$

$$W = \tau_z(\theta_2 - \theta_1) = \tau_z \Delta\theta \quad (10.21)$$

(constant torque only)

$$W_{tot} = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2 \quad (10.22)$$

$$P = \tau_z\omega_z \quad (10.23)$$



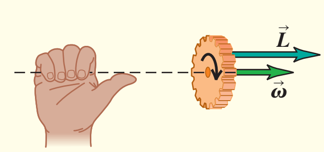

**Angular momentum:** The angular momentum of a particle with respect to point $O$ is the vector product of the particle's position vector $\vec{r}$ relative to $O$ and its momentum $\vec{p} = m\vec{v}$. When a symmetrical object rotates about a stationary axis of symmetry, its angular momentum is the product of its moment of inertia and its angular velocity vector $\vec{\omega}$. If the object is not symmetrical or the rotation $(z)$ axis is not an axis of symmetry, the component of angular momentum along the rotation axis is $I\omega_z$. (See Example 10.9.)

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \quad (10.24)$$

(particle)

$$\vec{L} = I\vec{\omega} \quad (10.28)$$

(rigid body rotating about axis of symmetry)

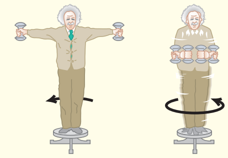



**Rotational dynamics and angular momentum:** The net external torque on a system is equal to the rate of change of its angular momentum. If the net external torque on a system is zero, the total angular momentum of the system is constant (conserved). (See Examples 10.10–10.13.)

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt} \quad (10.29)$$

$$\frac{d\vec{L}}{dt} = \mathbf{0} \quad (10.31)$$

(zero net external torque)



Chapter 10 Media Assets

## GUIDED PRACTICE

For assigned homework and other learning materials, go to Mastering Physics.

### KEY EXAMPLE VARIATION PROBLEMS

**Be sure to review EXAMPLES 10.2 and 10.3 (Section 10.2) before attempting these problems.**

**VP10.3.1** In the cylinder and cable apparatus of Example 10.2, you apply a force to the cable so that a point on the horizontal part of the cable accelerates to the left at $0.60\ \mathrm{m/s^2}$. What are the magnitudes of (a) the angular acceleration of the cylinder, (b) the torque that the cable exerts on the cylinder, and (c) the force that you exert on the cable?

**VP10.3.2** In the cylinder, cable, and block apparatus of Example 10.3, you replace the solid cylinder with a thin-walled, hollow cylinder of mass $M$ and radius $R$. Find (a) the acceleration of the falling block and (b) the tension in the cable as the block falls.

**VP10.3.3** A bucket of mass $m$ is hanging from the free end of a rope whose other end is wrapped around a drum (radius $R$, mass $M$) that can rotate with negligible friction about a stationary horizontal axis. The drum is not a uniform cylinder and has unknown moment of inertia. When you release the bucket from rest, you find that it has a downward acceleration of magnitude $a$. What are (a) the tension in the cable between the drum and the bucket and (b) the moment of inertia of the drum about its rotation axis?

**VP10.3.4** In the cylinder, cable, and block apparatus of Example 10.3, you attach an electric motor to the axis of the cylinder of mass $M$ and radius $R$ and turn the motor on. As a result the block of mass $m$ moves upward with an upward acceleration of magnitude $a$. What are (a) the tension in the cable between the cylinder and the block, (b) the magnitude of the torque that the cable exerts on the cylinder, and (c) the magnitude of the torque that the motor exerts on the cylinder?

**Be sure to review EXAMPLES 10.6 and 10.7 (Section 10.3) before attempting these problems.**

**VP10.7.1** In the primitive yo-yo apparatus of Example 10.6, you replace the solid cylinder with a hollow cylinder of mass $M$, outer radius $R$, and inner radius $R/2$. Find (a) the downward acceleration of the hollow cylinder and (b) the tension in the string.

**VP10.7.2** A thin-walled, hollow sphere of mass $M$ rolls without slipping down a ramp that is inclined at an angle $\beta$ to the horizontal. Find (a) the acceleration of the sphere, (b) the magnitude of the friction force that the ramp exerts on the sphere, and (c) the magnitude of the torque that this force exerts on the sphere.

**VP10.7.3** You redo the primitive yo-yo experiment of Example 10.6, but instead of holding the free end of the string stationary, you move your hand vertically so that the tension in the string equals $2Mg/3$. (a) What is the magnitude of the vertical acceleration of the yo-yo's center of mass? Does it accelerate upward or downward? (b) What is the angular acceleration of the yo-yo around its axis?

**VP10.7.4** You place a solid cylinder of mass $M$ on a ramp that is inclined at an angle $\beta$ to the horizontal. The coefficient of static friction for the cylinder on the ramp is $\mu_s$. (a) If the cylinder rolls downhill without slipping, what is the magnitude of the friction force that the ramp exerts on the cylinder? (b) You find by varying the angle of the ramp that the cylinder rolls without slipping if $\beta$ is less than a certain critical value but the cylinder slips if $\beta$ is greater than this critical value. What is this critical value of $\beta$?

**Be sure to review EXAMPLES 10.11 and 10.12 (Section 10.6) before attempting these problems.**

**VP10.12.1** In the situation shown in Example 10.11, suppose disk $A$ has moment of inertia $I_A$ and initial angular speed $\omega_A$, while disk $B$ has moment of inertia $I_A/4$ and initial angular speed $\omega_A/2$. Initially disks $A$ and $B$ are rotating in the *same* direction. (a) What is the final common angular speed of the two disks? (b) What fraction of the initial rotational kinetic energy remains as rotational kinetic energy after the disks have come to their final common angular speed?

**VP10.12.2** In the situation shown in Example 10.11, suppose disk $A$ has moment of inertia $I_A$ and initial angular speed $\omega_A$, while disk $B$ has moment of inertia $I_A/4$ and initial angular speed $\omega_A/2$. Initially disks $A$ and $B$ are rotating in *opposite* directions. (a) What is the final common angular speed of the two disks? (b) What fraction of the initial rotational kinetic energy remains as rotational kinetic energy after the disks have come to their final common angular speed?

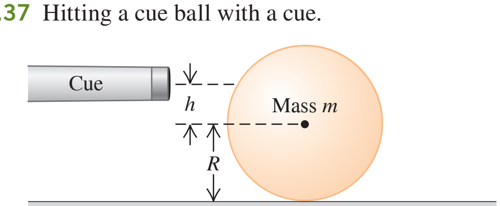
**VP10.12.3** Suppose that instead of hitting the center of the door, the bullet in Example 10.12 strikes the door at the edge farthest away from the hinge and embeds itself there. (a) What is the angular speed of the door just after the bullet embeds itself? (b) What fraction of the initial kinetic energy of the bullet remains as kinetic energy after the collision?

**VP10.12.4** A thin-walled, hollow sphere of mass $M$ and radius $R$ is free to rotate around a vertical shaft that passes through the center of the sphere. Initially the sphere is at rest. A small ball of clay of the same mass $M$ moving horizontally at speed $v$ grazes the surface of the sphere at its equator. After grazing the surface, the ball of clay is moving at speed $v/2$. (a) What is the angular momentum of the ball of clay about the shaft before it grazes the surface? After it grazes the surface? (b) What is the angular speed of the sphere after being grazed by the ball of clay? (c) What fraction of the ball of clay's initial kinetic energy remains as the combined kinetic energy of the sphere and the ball of clay?

### BRIDGING PROBLEM Billiard Physics

A cue ball (a uniform solid sphere of mass $m$ and radius $R$) is at rest on a level pool table. Using a pool cue, you give the ball a sharp, horizontal hit of magnitude $F$ at a height $h$ above the center of the ball (**Fig. 10.37**). The force of the hit is much greater than the friction force $f$ that the table surface exerts on the ball. The hit lasts for a short time $\Delta t$. (a) For what value of $h$ will the ball roll without slipping? (b) If you hit the ball dead center ($h = 0$), the ball will slide across the table for a while, but eventually it will roll without slipping. What will the speed of its center of mass be then?

Figure 10.37 Hitting a cue ball with a cue.



**SOLUTION GUIDE**

### IDENTIFY and SET UP

1. Draw a free-body diagram for the ball for the situation in part (a), including your choice of coordinate axes. Note that the cue exerts both an impulsive force on the ball and an impulsive torque around the center of mass.
2. The cue force applied for a time $\Delta t$ gives the ball's center of mass a speed $v_{cm}$, and the cue torque applied for that same time gives the ball an angular speed $\omega$. How must $v_{cm}$ and $\omega$ be related for the ball to roll without slipping?
3. Draw two free-body diagrams for the ball in part (b): one showing the forces during the hit and the other showing the forces after the hit but before the ball is rolling without slipping.
4. What is the angular speed of the ball in part (b) just after the hit? While the ball is sliding, does $v_{cm}$ increase or decrease? Does $\omega$ increase or decrease? What is the relationship between $v_{cm}$ and $\omega$ when the ball is finally rolling without slipping?

### EXECUTE

5. In part (a), use the impulse–momentum theorem to find the speed of the ball's center of mass immediately after the hit. Then use the rotational version of the impulse–momentum theorem to find the angular speed immediately after the hit. (*Hint:* To write the rotational version of the impulse–momentum theorem, remember that the relationship between torque and angular momentum is the same as that between force and linear momentum.)
6. Use your results from step 5 to find the value of $h$ that will cause the ball to roll without slipping immediately after the hit.
7. In part (b), again find the ball's center-of-mass speed and angular speed immediately after the hit. Then write Newton's second law for the translational motion and rotational motion of the ball as it slides. Use these equations to write expressions for $v_{cm}$ and $\omega$ as functions of the elapsed time $t$ since the hit.
8. Using your results from step 7, find the time $t$ when $v_{cm}$ and $\omega$ have the correct relationship for rolling without slipping. Then find the value of $v_{cm}$ at this time.

### EVALUATE

9. If you have access to a pool table, test the results of parts (a) and (b) for yourself!
10. Can you show that if you used a hollow cylinder rather than a solid ball, you would have to hit the top of the cylinder to cause rolling without slipping as in part (a)?

## PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

### DISCUSSION QUESTIONS

**Q10.1** Can a single force applied to an object change both its translational and rotational motions? Explain.

**Q10.2** Suppose you could use wheels of any type in the design of a soapbox-derby racer (an unpowered, four-wheel vehicle that coasts from rest down a hill). To conform to the rules on the total weight of the vehicle and rider, should you design with large massive wheels or small light wheels? Should you use solid wheels or wheels with most of the mass at the rim? Explain.

**Q10.3** Serious bicyclists say that if you reduce the weight of a bike, it is more effective if you do so in the wheels rather than in the frame. Why would reducing weight in the wheels make it easier on the bicyclist than reducing the same amount in the frame?

**Q10.4** The harder you hit the brakes while driving forward, the more the front end of your car will move down (and the rear end move up). Why? What happens when cars accelerate forward? Why do drag racers not use front-wheel drive only?

**Q10.5** When an acrobat walks on a tightrope, she extends her arms straight out from her sides. She does this to make it easier for her to catch herself if she should tip to one side or the other. Explain how this works. [*Hint:* Think about Eq. (10.7).]

**Q10.6** When you turn on an electric motor, it takes longer to come up to final speed if a grinding wheel is attached to the shaft. Why?

**Q10.7** The work done by a force is the product of force and distance. The torque due to a force is the product of force and distance. Does this mean that torque and work are equivalent? Explain.

**Q10.8** A valued client brings a treasured ball to your engineering firm, wanting to know whether the ball is solid or hollow. He has tried tapping on it, but that has given insufficient information. Design a simple, inexpensive experiment that you could perform quickly, without injuring the precious ball, to find out whether it is solid or hollow.

**Q10.9** You make two versions of the same object out of the same material having uniform density. For one version, all the dimensions are exactly twice as great as for the other one. If the same torque acts on both versions, giving the smaller version angular acceleration $\alpha$, what will be the angular acceleration of the larger version in terms of $\alpha$?

**Q10.10** Two identical masses are attached to frictionless pulleys by very light strings wrapped around the rim of the pulley and are released from rest. Both pulleys have the same mass and same diameter, but one is solid and the other is a hoop. As the masses fall, in which case is the tension in the string greater, or is it the same in both cases? Justify your answer.

**Q10.11** The force of gravity acts on the baton in Fig. 10.11, and forces produce torques that cause a body's angular velocity to change. Why, then, is the angular velocity of the baton in the figure constant?

**Q10.12** Without slipping, a certain solid uniform ball rolls at speed $v$ on a horizontal surface and then up a hill to a maximum height $h_0$. How does the maximum height change (in terms of $h_0$) if you make only the following changes: (a) double the ball's diameter, (b) double its mass, (c) double both its diameter and mass, (d) double its angular speed at the bottom of the hill?

**Q10.13** A wheel is rolling without slipping on a horizontal surface. In an inertial frame of reference in which the surface is at rest, is there any point on the wheel that has a velocity that is purely vertical? Is there any point that has a horizontal velocity component opposite to the velocity of the center of mass? Explain. Do your answers change if the wheel is slipping as it rolls? Why or why not?

**Q10.14** A hoop, a uniform solid cylinder, a spherical shell, and a uniform solid sphere are released from rest at the top of an incline. What is the order in which they arrive at the bottom of the incline? Does it matter whether or not the masses and radii of the objects are all the same? Explain.

**Q10.15** A ball is rolling along at speed $v$ without slipping on a horizontal surface when it comes to a hill that rises at a constant angle above the horizontal. In which case will it go higher up the hill: if the hill has enough friction to prevent slipping or if the hill is perfectly smooth? Justify your answer in both cases in terms of energy conservation and in terms of Newton's second law.

**Q10.16** You are standing at the center of a large horizontal turntable in a carnival funhouse. The turntable is set rotating on frictionless bearings, and it rotates freely (that is, there is no motor driving the turntable). As you walk toward the edge of the turntable, what happens to the combined angular momentum of you and the turntable? What happens to the rotation speed of the turntable? Explain.

**Q10.17** **Global Warming.** If the earth's climate continues to warm, ice near the poles will melt, and the water will be added to the oceans. What effect will this have on the length of the day? Justify your answer.

**Q10.18** If two spinning objects have the same angular momentum, do they necessarily have the same rotational kinetic energy? If they have the same rotational kinetic energy, do they necessarily have the same angular momentum? Explain.

**Q10.19** A student is sitting on a frictionless rotating stool with her arms outstretched as she holds equal heavy weights in each hand. If she suddenly lets go of the weights, will her angular speed increase, stay the same, or decrease? Explain.

**Q10.20** A point particle travels in a straight line at constant speed, and the closest distance it comes to the origin of coordinates is a distance $l$. With respect to this origin, does the particle have nonzero angular momentum? As the particle moves along its straight-line path, does its angular momentum with respect to the origin change?

**Q10.21** In Example 10.10 (Section 10.6) the angular speed $\omega$ changes, and this must mean that there is nonzero angular acceleration. But there is no torque about the rotation axis if the forces the professor applies to the weights are directly, radially inward. Then, by Eq. (10.7), $\alpha_z$ must be zero. Explain what is wrong with this reasoning that leads to this apparent contradiction.

**Q10.22** In Example 10.10 (Section 10.6) the rotational kinetic energy of the professor and dumbbells increases. But since there are no external torques, no work is being done to change the rotational kinetic energy. Then, by Eq. (10.22), the kinetic energy must remain the same! Explain what is wrong with this reasoning, which leads to an apparent contradiction. Where *does* the extra kinetic energy come from?

**Q10.23** As discussed in Section 10.6, the angular momentum of a circus acrobat is conserved as she tumbles through the air. Is her *linear* momentum conserved? Why or why not?

**Q10.24** If you stop a spinning raw egg for the shortest possible instant and then release it, the egg will start spinning again. If you do the same to a hard-boiled egg, it will remain stopped. Try it. Explain it.

**Q10.25** A helicopter has a large main rotor that rotates in a horizontal plane and provides lift. There is also a small rotor on the tail that rotates in a vertical plane. What is the purpose of the tail rotor? (*Hint:* If there were no tail rotor, what would happen when the pilot changed the angular speed of the main rotor?) Some helicopters have no tail rotor, but instead have two large main rotors that rotate in a horizontal plane. Why is it important that the two main rotors rotate in opposite directions?

**Q10.26** In a common design for a gyroscope, the flywheel and flywheel axis are enclosed in a light, spherical frame with the flywheel at the center of the frame. The gyroscope is then balanced on top of a pivot so that the flywheel is directly above the pivot. Does the gyroscope precess if it is released while the flywheel is spinning? Explain.

**Q10.27** A gyroscope is precessing about a vertical axis. What happens to the precession angular speed if the following changes are made, with all other variables remaining the same? (a) The angular speed of the spinning flywheel is doubled; (b) the total weight is doubled; (c) the moment of inertia about the axis of the spinning flywheel is doubled; (d) the distance from the pivot to the center of gravity is doubled. (e) What happens if all of the variables in parts (a) through (d) are doubled? In each case justify your answer.

**Q10.28** A gyroscope takes 3.8 s to precess 1.0 revolution about a vertical axis. Two minutes later, it takes only 1.9 s to precess 1.0 revolution. No one has touched the gyroscope. Explain.

**Q10.29** A gyroscope is precessing as in Fig. 10.32. What happens if you gently add some weight to the end of the flywheel axis farthest from the pivot?

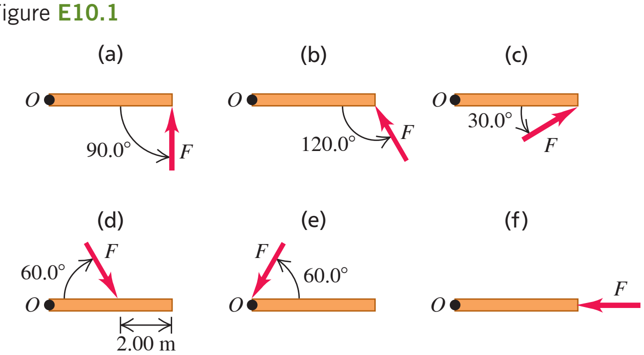
**Q10.30** A bullet spins on its axis as it emerges from a rifle. Explain how this prevents the bullet from tumbling and keeps the streamlined end pointed forward.

## EXERCISES

### Section 10.1 Torque

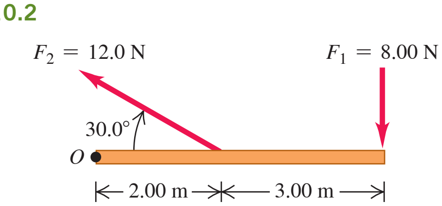
**10.1** • Calculate the torque (magnitude and direction) about point $O$ due to the force $\vec{F}$ in each of the cases sketched in **Fig. E10.1**. In each case, both the force $\vec{F}$ and the rod lie in the plane of the page, the rod has length 4.00 m, and the force has magnitude $F = 10.0$ N.

Figure **E10.1**



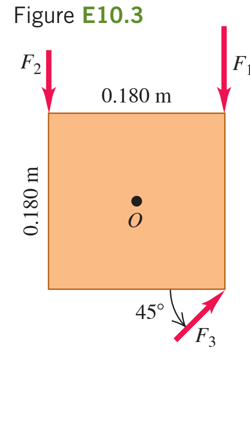

**10.2** • Calculate the net torque about point $O$ for the two forces applied as in **Fig. E10.2**. The rod and both forces are in the plane of the page.

Figure **E10.2**



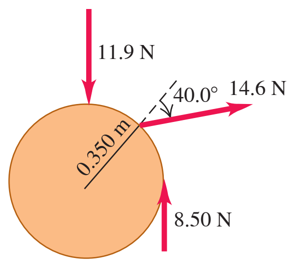

**10.3** •• A square metal plate 0.180 m on each side is pivoted about an axis through point $O$ at its center and perpendicular to the plate (**Fig. E10.3**). Calculate the net torque about this axis due to the three forces shown in the figure if the magnitudes of the forces are $F_1 = 18.0$ N, $F_2 = 26.0$ N, and $F_3 = 14.0$ N. The plate and all forces are in the plane of the page.

Figure **E10.3**



**10.4** • Three forces are applied to a wheel of radius 0.350 m, as shown in **Fig. E10.4**. One force is perpendicular to the rim, one is tangent to it, and the other makes a 40.0° angle with the radius. What is the net torque on the wheel due to these three forces for an axis perpendicular to the wheel and passing through its center?

Figure **E10.4**

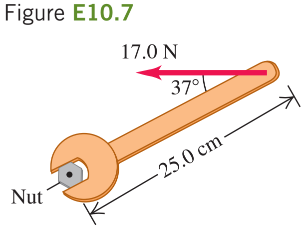



**10.5** • One force acting on a machine part is $\vec{F} = (-5.00\text{ N})\hat{\imath} + (4.00\text{ N})\hat{\jmath}$. The vector from the origin to the point where the force is applied is $\vec{r} = (-0.450\text{ m})\hat{\imath} + (0.150\text{ m})\hat{\jmath}$. (a) In a sketch, show $\vec{r}$, $\vec{F}$, and the origin. (b) Use the right-hand rule to determine the direction of the torque. (c) Calculate the vector torque for an axis at the origin produced by this force. Verify that the direction of the torque is the same as you obtained in part (b).

**10.6** • A metal bar is in the $xy$-plane with one end of the bar at the origin. A force $\vec{F} = (7.00\text{ N})\hat{\imath} + (-3.00\text{ N})\hat{\jmath}$ is applied to the bar at the point $x = 3.00$ m, $y = 4.00$ m. (a) In terms of unit vectors $\hat{\imath}$ and $\hat{\jmath}$, what is the position vector $\vec{r}$ for the point where the force is applied? (b) What are the magnitude and direction of the torque with respect to the origin produced by $\vec{F}$?

**10.7** • A machinist is using a wrench to loosen a nut. The wrench is 25.0 cm long, and he exerts a 17.0 N force at the end of the handle at 37° with the handle (**Fig. E10.7**). (a) What torque does the machinist exert about the center of the nut? (b) What is the maximum torque he could exert with a force of this magnitude, and how should the force be oriented?

Figure **E10.7**




## Section 10.2 Torque and Angular Acceleration for a Rigid Body

**10.8** •• A uniform disk with mass 40.0 kg and radius 0.200 m is pivoted at its center about a horizontal, frictionless axle that is stationary. The disk is initially at rest, and then a constant force $F = 30.0$ N is applied tangent to the rim of the disk. (a) What is the magnitude $v$ of the tangential velocity of a point on the rim of the disk after the disk has turned through 0.200 revolution? (b) What is the magnitude $a$ of the resultant acceleration of a point on the rim of the disk after the disk has turned through 0.200 revolution?

**10.9** •• The flywheel of an engine has moment of inertia 1.60 $\text{kg}\cdot\text{m}^2$ about its rotation axis. What constant torque is required to bring it up to an angular speed of 400 rev/min in 8.00 s, starting from rest?

**10.10** • A cord is wrapped around the rim of a solid uniform wheel 0.250 m in radius and of mass 9.20 kg. A steady horizontal pull of 40.0 N to the right is exerted on the cord, pulling it off tangentially from the wheel. The wheel is mounted on frictionless bearings on a horizontal axle through its center. (a) Compute the angular acceleration of the wheel and the acceleration of the part of the cord that has already been pulled off the wheel. (b) Find the magnitude and direction of the force that the axle exerts on the wheel. (c) Which of the answers in parts (a) and (b) would change if the pull were upward instead of horizontal?

**10.11** •• A machine part has the shape of a solid uniform sphere of mass 225 g and diameter 3.00 cm. It is spinning about a frictionless axle through its center, but at one point on its equator it is scraping against metal, resulting in a friction force of 0.0200 N at that point. (a) Find its angular acceleration. (b) How long will it take to decrease its rotational speed by 22.5 rad/s?

**10.12** •• **CP** A stone is suspended from the free end of a wire that is wrapped around the outer rim of a pulley, similar to what is shown in Fig. 10.10. The pulley is a uniform disk with mass 10.0 kg and radius 30.0 cm and turns on frictionless bearings. You measure that the stone travels 12.6 m in the first 3.00 s starting from rest. Find (a) the mass of the stone and (b) the tension in the wire.

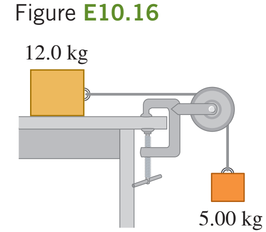
**10.13** •• **CP** A 2.00 kg textbook rests on a frictionless, horizontal surface. A cord attached to the book passes over a pulley whose diameter is 0.150 m, to a hanging book with mass 3.00 kg. The system is released from rest, and the books are observed to move 1.20 m in 0.800 s. (a) What is the tension in each part of the cord? (b) What is the moment of inertia of the pulley about its rotation axis?

**10.14** •• **CP** A 15.0 kg bucket of water is suspended by a very light rope wrapped around a solid uniform cylinder 0.300 m in diameter with mass 12.0 kg. The cylinder pivots on a frictionless axle through its center. The bucket is released from rest at the top of a well and falls 10.0 m to the water. (a) What is the tension in the rope while the bucket is falling? (b) With what speed does the bucket strike the water? (c) What is the time of fall? (d) While the bucket is falling, what is the force exerted on the cylinder by the axle?

**10.15** • A wheel rotates without friction about a stationary horizontal axis at the center of the wheel. A constant tangential force equal to 80.0 N is applied to the rim of the wheel. The wheel has radius 0.120 m. Starting from rest, the wheel has an angular speed of 12.0 rev/s after 2.00 s. What is the moment of inertia of the wheel?

**10.16** •• A 12.0 kg box resting on a horizontal, frictionless surface is attached to a 5.00 kg weight by a thin, light wire that passes over a frictionless pulley (**Fig. E10.16**). The pulley has the shape of a uniform solid disk of mass 2.00 kg and diameter 0.500 m. After the system is released, find (a) the tension in the wire on both sides of the pulley, (b) the acceleration of the box, and (c) the horizontal and vertical components of the force that the axle exerts on the pulley.

Figure **E10.16**




**10.17** •• **CP** A solid cylinder with radius 0.140 m is mounted on a frictionless, stationary axle that lies along the cylinder axis. The cylinder is initially at rest. Then starting at $t = 0$ a constant horizontal force of 3.00 N is applied tangential to the surface of the cylinder. You measure the angular displacement $\theta - \theta_0$ of the cylinder as a function of the time $t$ since the force was first applied. When you plot $\theta - \theta_0$ (in radians) as a function of $t^2$ (in $\text{s}^2$), your data lie close to a straight line. If the slope of this line is 16.0 rad/$\text{s}^2$, what is the moment of inertia of the cylinder for rotation about the axle?

**10.18** •• **CP** Two spheres are rolling without slipping on a horizontal floor. They are made of different materials, but each has mass 5.00 kg and radius 0.120 m. For each the translational speed of the center of mass is 4.00 m/s. Sphere $A$ is a uniform solid sphere and sphere $B$ is a thin-walled, hollow sphere. How much work, in joules, must be done on each sphere to bring it to rest? For which sphere is a greater magnitude of work required? Explain. (The spheres continue to roll without slipping as they slow down.)

## Section 10.3 Rigid-Body Rotation About a Moving Axis

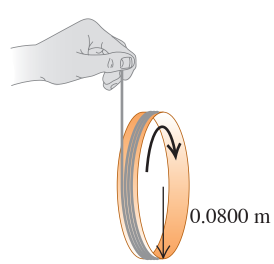
**10.19** • A 2.20 kg hoop 1.20 m in diameter is rolling to the right without slipping on a horizontal floor at a steady 2.60 rad/s. (a) How fast is its center moving? (b) What is the total kinetic energy of the hoop? (c) Find the velocity vector of each of the following points, as viewed by a person at rest on the ground: (i) the highest point on the hoop; (ii) the lowest point on the hoop; (iii) a point on the right side of the hoop, midway between the top and the bottom. (d) Find the velocity vector for each of the points in part (c), but this time as viewed by someone moving along with the same velocity as the hoop.

**10.20** •• Example 10.7 calculates the friction force needed for a uniform sphere to roll down an incline without slipping. The incline is at an angle $\beta$ above the horizontal. And the example discusses that the friction is static. (a) If the maximum friction force is given by $f = \mu_s n$, where $n$ is the normal force that the ramp exerts on the sphere, in terms of $\beta$ what is the minimum coefficient of static friction needed if the sphere is to roll without slipping? (b) Based on your result in part (a), what does the minimum required $\mu_s$ become in the limits $\beta \rightarrow 90^\circ$ and $\beta \rightarrow 0^\circ$?

**10.21** • What fraction of the total kinetic energy is rotational for the following objects rolling without slipping on a horizontal surface? (a) A uniform solid cylinder; (b) a uniform sphere; (c) a thin-walled, hollow sphere; (d) a hollow cylinder with outer radius $R$ and inner radius $R/2$.

**10.22** •• A string is wrapped several times around the rim of a small hoop with radius 8.00 cm and mass 0.180 kg. The free end of the string is held in place and the hoop is released from rest (**Fig. E10.22**). After the hoop has descended 75.0 cm, calculate (a) the angular speed of the rotating hoop and (b) the speed of its center.

Figure E10.22




**10.23** •• A solid ball is released from rest and slides down a hillside that slopes downward at 65.0° from the horizontal. (a) What minimum value must the coefficient of static friction between the hill and ball surfaces have for no slipping to occur? (b) Would the coefficient of friction calculated in part (a) be sufficient to prevent a hollow ball (such as a soccer ball) from slipping? Justify your answer. (c) In part (a), why did we use the coefficient of static friction and not the coefficient of kinetic friction?

**10.24** •• A hollow, spherical shell with mass 2.00 kg rolls without slipping down a 38.0° slope. (a) Find the acceleration, the friction force, and the minimum coefficient of static friction needed to prevent slipping. (b) How would your answers to part (a) change if the mass were doubled to 4.00 kg?

**10.25** •• A 392 N wheel comes off a moving truck and rolls without slipping along a highway. At the bottom of a hill it is rotating at 25.0 rad/s. The radius of the wheel is 0.600 m, and its moment of inertia about its rotation axis is $0.800MR^2$. Friction does work on the wheel as it rolls up the hill to a stop, a height $h$ above the bottom of the hill; this work has absolute value 2600 J. Calculate $h$.

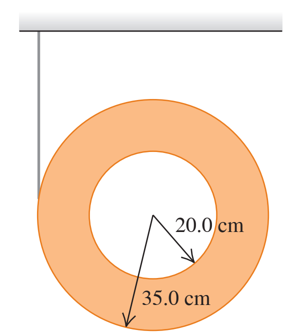
**10.26** •• A uniform marble rolls down a symmetrical bowl, starting from rest at the top of the left side. The top of each side is a distance $h$ above the bottom of the bowl. The left half of the bowl is rough enough to cause the marble to roll without slipping, but the right half has no friction because it is coated with oil. (a) How far up the smooth side will the marble go, measured vertically from the bottom? (b) How high would the marble go if both sides were as rough as the left side? (c) How do you account for the fact that the marble goes *higher* with friction on the right side than without friction?

**10.27** •• At a typical bowling alley the distance from the line where the ball is released (foul line) to the first pin is 60 ft. Estimate the time it takes the ball to reach the pins after you release it, if it rolls without slipping and has a constant translational speed. Assume the ball weighs 12 lb and has a diameter of 8.5 in. (a) Use your estimate to calculate the rotation rate of the ball, in rev/s. (b) What is its total kinetic energy in joules and what fraction of the total is its rotational kinetic energy? Ignore the finger holes and treat the bowling ball as a uniform sphere.

**10.28** • Two uniform solid balls are rolling without slipping at a constant speed. Ball 1 has twice the diameter, half the mass, and one-third the speed of ball 2. The kinetic energy of ball 2 is 27.0 J. What is the kinetic energy of ball 1?

**10.29** •• A thin, light string is wrapped around the outer rim of a uniform hollow cylinder of mass 4.75 kg having inner and outer radii as shown in **Fig. E10.29**. The cylinder is then released from rest. (a) How far must the cylinder fall before its center is moving at 6.66 m/s? (b) If you just dropped this cylinder without any string, how fast would its center be moving when it had fallen the distance in part (a)? (c) Why do you get two different answers when the cylinder falls the same distance in both cases?

Figure E10.29




**10.30** •• **A Ball Rolling Uphill.** A bowling ball rolls without slipping up a ramp that slopes upward at an angle $\beta$ to the horizontal (see Example 10.7 in Section 10.3). Treat the ball as a uniform solid sphere, ignoring the finger holes. (a) Draw the free-body diagram for the ball. Explain why the friction force must be directed *uphill.* (b) What is the acceleration of the center of mass of the ball? (c) What minimum coefficient of static friction is needed to prevent slipping?

**10.31** •• A size-5 soccer ball of diameter 22.6 cm and mass 426 g rolls up a hill without slipping, reaching a maximum height of 5.00 m above the base of the hill. We can model this ball as a thin-walled hollow sphere. (a) At what rate was it rotating at the base of the hill? (b) How much rotational kinetic energy did it have then? Neglect rolling friction and assume the system's total mechanical energy is conserved.

## Section 10.4 Work and Power in Rotational Motion

**10.32** • An engine delivers 175 hp to an aircraft propeller at 2400 rev/min. (a) How much torque does the aircraft engine provide? (b) How much work does the engine do in one revolution of the propeller?

**10.33** • A playground merry-go-round has radius 2.40 m and moment of inertia $2100\ \text{kg}\cdot\text{m}^2$ about a vertical axle through its center, and it turns with negligible friction. (a) A child applies an 18.0 N force tangentially to the edge of the merry-go-round for 15.0 s. If the merry-go-round is initially at rest, what is its angular speed after this 15.0 s interval? (b) How much work did the child do on the merry-go-round? (c) What is the average power supplied by the child?

**10.34** •• An electric motor consumes 9.00 kJ of electrical energy in 1.00 min. If one-third of this energy goes into heat and other forms of internal energy of the motor, with the rest going to the motor output, how much torque will this engine develop if you run it at 2500 rpm?

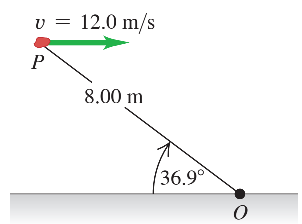
**10.35** • A 2.80 kg grinding wheel is in the form of a solid cylinder of radius 0.100 m. (a) What constant torque will bring it from rest to an angular speed of 1200 rev/min in 2.5 s? (b) Through what angle has it turned during that time? (c) Use Eq. (10.21) to calculate the work done by the torque. (d) What is the grinding wheel's kinetic energy when it is rotating at 1200 rev/min? Compare your answer to the result in part (c).

**10.36** •• An airplane propeller is 2.08 m in length (from tip to tip) and has a mass of 117 kg. When the airplane's engine is first started, it applies a constant net torque of 1950 N·m to the propeller, which starts from rest. (a) What is the angular acceleration of the propeller? Model the propeller as a slender rod and see Table 9.2. (b) What is the propeller's angular speed after making 5.00 revolutions? (c) How much work is done by the engine during the first 5.00 revolutions? (d) What is the average power output of the engine during the first 5.00 revolutions? (e) What is the instantaneous power output of the motor at the instant that the propeller has turned through 5.00 revolutions?

## Section 10.5 Angular Momentum

**10.37** • A 2.00 kg rock has a horizontal velocity of magnitude 12.0 m/s when it is at point $P$ in **Fig. E10.37**. (a) At this instant, what are the magnitude and direction of its angular momentum relative to point $O$? (b) If the only force acting on the rock is its weight, what is the rate of change (magnitude and direction) of its angular momentum at this instant?

Figure **E10.37**




**10.38** •• A woman with mass 50 kg is standing on the rim of a large horizontal disk that is rotating at 0.80 rev/s about an axis through its center. The disk has mass 110 kg and radius 4.0 m. Calculate the magnitude of the total angular momentum of the woman–disk system. (Assume that you can treat the woman as a point.)

**10.39** •• Find the magnitude of the angular momentum of the second hand on a clock about an axis through the center of the clock face. The clock hand has a length of 15.0 cm and a mass of 6.00 g. Take the second hand to be a slender rod rotating with constant angular velocity about one end.

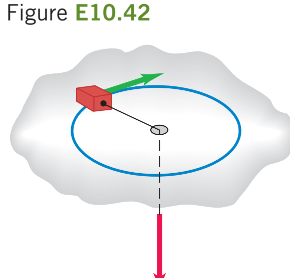
**10.40** •• (a) Calculate the magnitude of the angular momentum of the earth in a circular orbit around the sun. Is it reasonable to model it as a particle? (b) Calculate the magnitude of the angular momentum of the earth due to its rotation around an axis through the north and south poles, modeling it as a uniform sphere. Consult Appendix E and the astronomical data in Appendix F.

**10.41** •• CALC A hollow, thin-walled sphere of mass 12.0 kg and diameter 48.0 cm is rotating about an axle through its center. The angle (in radians) through which it turns as a function of time (in seconds) is given by $\theta(t) = At^2 + Bt^4$, where $A$ has numerical value 1.50 and $B$ has numerical value 1.10. (a) What are the units of the constants $A$ and $B$? (b) At the time 3.00 s, find (i) the angular momentum of the sphere and (ii) the net torque on the sphere.

## Section 10.6 Conservation of Angular Momentum

**10.42** • CP A small block on a frictionless, horizontal surface has a mass of 0.0250 kg. It is attached to a massless cord passing through a hole in the surface (**Fig. E10.42**). The block is originally revolving at a distance of 0.300 m from the hole with an angular speed of 2.85 rad/s. The cord is then pulled from below, shortening the radius of the circle in which the block revolves to 0.150 m. Model the block as a particle. (a) Is the angular momentum of the block conserved? Why or why not? (b) What is the new angular speed? (c) Find the change in kinetic energy of the block. (d) How much work was done in pulling the cord?

Figure **E10.42**

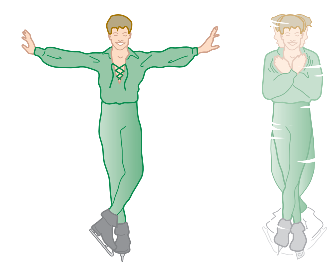


**10.43** •• Under some circumstances, a star can collapse into an extremely dense object made mostly of neutrons and called a *neutron star*. The density of a neutron star is roughly $10^{14}$ times as great as that of ordinary solid matter. Suppose we represent the star as a uniform, solid, rigid sphere, both before and after the collapse. The star's initial radius was $7.0 \times 10^5$ km (comparable to our sun); its final radius is 16 km. If the original star rotated once in 30 days, find the angular speed of the neutron star.

**10.44** •• A diver comes off a board with arms straight up and legs straight down, giving her a moment of inertia about her rotation axis of 18 kg·m$^2$. She then tucks into a small ball, decreasing this moment of inertia to 3.6 kg·m$^2$. While tucked, she makes two complete revolutions in 1.0 s. If she hadn't tucked at all, how many revolutions would she have made in the 1.5 s from board to water?

**10.45** •• **The Spinning Figure Skater.** The outstretched hands and arms of a figure skater preparing for a spin can be considered a slender rod pivoting about an axis through its center (**Fig. E10.45**). When the skater's hands and arms are brought in and wrapped around his body to execute the spin, the hands and arms can be considered a thin-walled, hollow cylinder. His hands and arms have a combined mass of 8.0 kg. When outstretched, they span 1.8 m; when wrapped, they form a cylinder of radius 25 cm. The moment of inertia about the rotation axis of the remainder of his body is constant and equal to 0.40 kg·m$^2$. If his original angular speed is 0.40 rev/s, what is his final angular speed?

Figure **E10.45**



**10.46** •• A solid wood door 1.00 m wide and 2.00 m high is hinged along one side and has a total mass of 40.0 kg. Initially open and at rest, the door is struck at its center by a handful of sticky mud with mass 0.500 kg, traveling perpendicular to the door at 12.0 m/s just before impact. Find the final angular speed of the door. Does the mud make a significant contribution to the moment of inertia?

**10.47** •• A large wooden turntable in the shape of a flat uniform disk has a radius of 2.00 m and a total mass of 120 kg. The turntable is initially rotating at 3.00 rad/s about a vertical axis through its center. Suddenly, a 70.0 kg parachutist makes a soft landing on the turntable at a point near the outer edge. (a) Find the angular speed of the turntable after the parachutist lands. (Assume that you can treat the parachutist as a particle.) (b) Compute the kinetic energy of the system before and after the parachutist lands. Why are these kinetic energies not equal?

**10.48** •• **Asteroid Collision!** Suppose that an asteroid traveling straight toward the center of the earth were to collide with our planet at the equator and bury itself just below the surface. What would have to be the mass of this asteroid, in terms of the earth's mass $M$, for the day to become 25.0% longer than it presently is as a result of the collision? Assume that the asteroid is very small compared to the earth and that the earth is uniform throughout.

**10.49** •• A small 10.0 g bug stands at one end of a thin uniform bar that is initially at rest on a smooth horizontal table. The other end of the bar pivots about a nail driven into the table and can rotate freely, without friction. The bar has mass 50.0 g and is 100 cm in length. The bug jumps off in the horizontal direction, perpendicular to the bar, with a speed of 20.0 cm/s relative to the table. (a) What is the angular speed of the bar just after the frisky insect leaps? (b) What is the total kinetic energy of the system just after the bug leaps? (c) Where does this energy come from?

**10.50** •• A thin uniform rod has a length of 0.500 m and is rotating in a circle on a frictionless table. The axis of rotation is perpendicular to the length of the rod at one end and is stationary. The rod has an angular velocity of 0.400 rad/s and a moment of inertia about the axis of $3.00 \times 10^{-3}$ kg·m$^2$. A bug initially standing on the rod at the axis of rotation decides to crawl out to the other end of the rod. When the bug has reached the end of the rod and sits there, its tangential speed is 0.160 m/s. The bug can be treated as a point mass. What is the mass of (a) the rod; (b) the bug?

**10.51** •• You live on a planet far from ours. Based on extensive communication with a physicist on earth, you have determined that all laws of physics on your planet are the same as ours and you have adopted the same units of seconds and meters as on earth. But you suspect that the value of $g$, the acceleration of an object in free fall near the surface of your planet, is different from what it is on earth. To test this, you take a solid uniform cylinder and let it roll down an incline. The vertical height $h$ of the top of the incline above the lower end of the incline can be varied. You measure the speed $v_{cm}$ of the center of mass of the cylinder when it reaches the bottom for various values of $h$. You plot $v_{cm}{}^2$ (in m$^2$/s$^2$) versus $h$ (in m) and find that your data lie close to a straight line with a slope of 6.42 m/s$^2$. What is the value of $g$ on your planet?

**10.52** •• A uniform, 4.5 kg, square, solid wooden gate 1.5 m on each side hangs vertically from a frictionless pivot at the center of its upper edge. A 1.1 kg raven flying horizontally at 5.0 m/s flies into this door at its center and bounces back at 2.0 m/s in the opposite direction. (a) What is the angular speed of the gate just after it is struck by the unfortunate raven? (b) During the collision, why is the angular momentum conserved but not the linear momentum?

**10.53** •• A teenager is standing at the rim of a large horizontal uniform wooden disk that can rotate freely about a vertical axis at its center. The mass of the disk (in kg) is $M$ and its radius (in m) is $R$. The mass of the teenager (in kg) is $m$. The disk and teenager are initially at rest. The teenager then throws a large rock that has a mass (in kg) of $m_{rock}$. As it leaves the thrower's hands, the rock is traveling horizontally with speed $v$ (in m/s) relative to the earth in a direction tangent to the rim of the disk. The teenager remains at rest relative to the disk and so rotates with it after throwing the rock. In terms of $M$, $R$, $m$, $m_{rock}$, and $v$, what is the angular speed of the disk? Treat the teenager as a point mass.

**10.54** •• A uniform solid disk made of wood is horizontal and rotates freely about a vertical axle at its center. The disk has radius 0.600 m and mass 1.60 kg and is initially at rest. A bullet with mass 0.0200 kg is fired horizontally at the disk, strikes the rim of the disk at a point perpendicular to the radius of the disk, and becomes embedded in its rim, a distance of 0.600 m from the axle. After being struck by the bullet, the disk rotates at 4.00 rad/s. What is the horizontal velocity of the bullet just before it strikes the disk?

### Section 10.7 Gyroscopes and Precession

**10.55** • **Stabilization of the Hubble Space Telescope.** The Hubble Space Telescope is stabilized to within an angle of about 2-millionths of a degree by means of a series of gyroscopes that spin at 19,200 rpm. Although the structure of these gyroscopes is actually quite complex, we can model each of the gyroscopes as a thin-walled cylinder of mass 2.0 kg and diameter 5.0 cm, spinning about its central axis. How large a torque would it take to cause these gyroscopes to precess through an angle of $1.0 \times 10^{-6}$ degree during a 5.0 hour exposure of a galaxy?

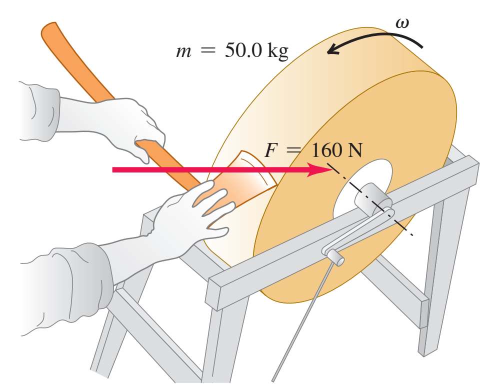
**10.56** • **A Gyroscope on the Moon.** A certain gyroscope precesses at a rate of 0.50 rad/s when used on earth. If it were taken to a lunar base, where the acceleration due to gravity is $0.165g$, what would be its precession rate?

## PROBLEMS

**10.57** •• You are riding your bicycle on a city street, and you are staying a constant distance behind a car that is traveling at the speed limit of 30 mph. Estimate the diameters of the bicycle wheels and sprockets and use these estimated quantities to calculate the number of revolutions per minute made by the large sprocket to which the pedals are attached. Do a Web search if you aren't familiar with the parts of a bicycle.

**10.58** •• A 50.0 kg grindstone is a solid disk 0.520 m in diameter. You press an ax down on the rim with a normal force of 160 N (**Fig. P10.58**). The coefficient of kinetic friction between the blade and the stone is 0.60, and there is a constant friction torque of 6.50 N·m between the axle of the stone and its bearings. (a) How much force must be applied tangentially at the end of a crank handle 0.500 m long to bring the stone from rest to 120 rev/min in 9.00 s? (b) After the grindstone attains an angular speed of 120 rev/min, what tangential force at the end of the handle is needed to maintain a constant angular speed of 120 rev/min? (c) How much time does it take the grindstone to come from 120 rev/min to rest if it is acted on by the axle friction alone?

Figure **P10.58**

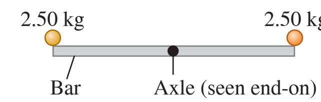



**10.59** ••• A grindstone in the shape of a solid disk with diameter 0.520 m and a mass of 50.0 kg is rotating at 850 rev/min. You press an ax against the rim with a normal force of 160 N (Fig. P10.58), and the grindstone comes to rest in 7.50 s. Find the coefficient of friction between the ax and the grindstone. You can ignore friction in the bearings.

**10.60** •• **CP** Block $A$ rests on a horizontal tabletop. A light horizontal rope is attached to it and passes over a pulley, and block $B$ is suspended from the free end of the rope. The light rope that connects the two blocks does not slip over the surface of the pulley (radius 0.080 m) because the pulley rotates on a frictionless axle. The horizontal surface on which block $A$ (mass 2.50 kg) moves is frictionless. The system is released from rest, and block $B$ (mass 6.00 kg) moves downward 1.80 m in 2.00 s. (a) What is the tension force that the rope exerts on block $B$? (b) What is the tension force on block $A$? (c) What is the moment of inertia of the pulley for rotation about the axle on which it is mounted?

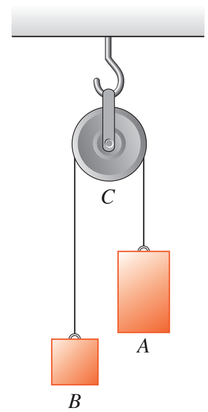
**10.61** ••• A thin, uniform, 3.80 kg bar, 80.0 cm long, has very small 2.50 kg balls glued on at either end (**Fig. P10.61**). It is supported horizontally by a thin, horizontal, frictionless axle passing through its center and perpendicular to the bar. Suddenly the right-hand ball becomes detached and falls off, but the other ball remains glued to the bar. (a) Find the angular acceleration of the bar just after the ball falls off. (b) Will the angular acceleration remain constant as the bar continues to swing? If not, will it increase or decrease? (c) Find the angular velocity of the bar just as it swings through its vertical position.

Figure **P10.61**



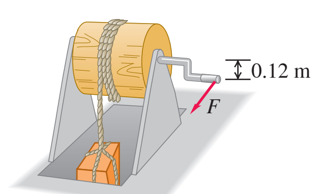
**10.62** •• Example 10.7 discusses a uniform solid sphere rolling without slipping down a ramp that is at an angle $\beta$ above the horizontal. Now consider the same sphere rolling without slipping up the ramp. (a) In terms of $g$ and $\beta$, calculate the acceleration of the center of mass of the sphere. Is your result larger or smaller than the acceleration when the sphere rolls down the ramp, or is it the same? (b) Calculate the friction force (in terms of $M$, $g$, and $\beta$) for the sphere to roll without slipping as it moves up the incline. Is the result larger, smaller, or the same as the friction force required to prevent slipping as the sphere rolls down the incline?

Figure **P10.63**




**10.63** •• **The Atwood's Machine. Figure P10.63** illustrates an Atwood's machine. Find the linear accelerations of blocks $A$ and $B$, the angular acceleration of the wheel $C$, and the tension in each side of the cord if there is no slipping between the cord and the surface of the wheel. Let the masses of blocks $A$ and $B$ be 4.00 kg and 2.00 kg, respectively, the moment of inertia of the wheel about its axis be $0.220\ \text{kg} \cdot \text{m}^2$, and the radius of the wheel be 0.120 m.

Figure **P10.64**

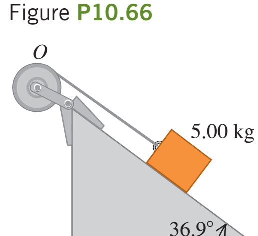



**10.64** ••• The mechanism shown in **Fig. P10.64** is used to raise a crate of supplies from a ship's hold. The crate has total mass 50 kg. A rope is wrapped around a wooden cylinder that turns on a metal axle. The cylinder has radius 0.25 m and moment of inertia $I = 2.9\ \text{kg} \cdot \text{m}^2$ about the axle. The crate is suspended from the free end of the rope. One end of the axle pivots on frictionless bearings; a crank handle is attached to the other end. When the crank is turned, the end of the handle rotates about the axle in a vertical circle of radius 0.12 m, the cylinder turns, and the crate is raised. What magnitude of the force $\vec{F}$ applied tangentially to the rotating crank is required to raise the crate with an acceleration of $1.40\ \text{m/s}^2$? (You can ignore the mass of the rope as well as the moments of inertia of the axle and the crank.)

**10.65** •• A solid uniform sphere and a thin-walled, hollow sphere have the same mass $M$ and radius $R$. If they roll without slipping up a ramp that is inclined at an angle $\beta$ above the horizontal and if both have the same $v_{\text{cm}}$ before they start up the incline, calculate the maximum height above their starting point reached by each object. Which object reaches the greater height, or do both of them reach the same height?

**10.66** •• A block with mass $m = 5.00$ kg slides down a surface inclined 36.9° to the horizontal (**Fig. P10.66**). The coefficient of kinetic friction is 0.25. A string attached to the block is wrapped around a flywheel on a fixed axis at $O$. The flywheel has mass 25.0 kg and moment of inertia $0.500\ \text{kg} \cdot \text{m}^2$ with respect to the axis of rotation. The string pulls without slipping at a perpendicular distance of 0.200 m from that axis. (a) What is the acceleration of the block down the plane? (b) What is the tension in the string?

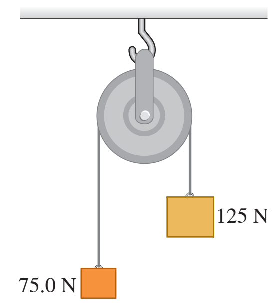
Figure **P10.66**




**10.67** •• **CP** A wheel with radius 0.0600 m rotates about a horizontal frictionless axle at its center. The moment of inertia of the wheel about the axle is $2.50\ \text{kg} \cdot \text{m}^2$. The wheel is initially at rest. Then at $t = 0$ a force $F = (5.00\ \text{N/s})t$ is applied tangentially to the wheel and the wheel starts to rotate. What is the magnitude of the force at the instant when the wheel has turned through 8.00 revolutions?

**10.68** •• A lawn roller in the form of a thin-walled, hollow cylinder with mass $M$ is pulled horizontally with a constant horizontal force $F$ applied by a handle attached to the axle. If it rolls without slipping, find the acceleration and the friction force.

Figure **P10.69**



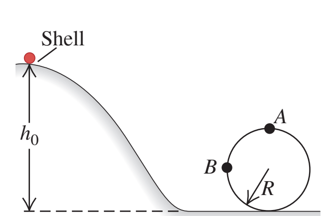

**10.69** • Two weights are connected by a very light, flexible cord that passes over an 80.0 N frictionless pulley of radius 0.300 m. The pulley is a solid uniform disk and is supported by a hook connected to the ceiling (**Fig. P10.69**). What force does the ceiling exert on the hook?

**10.70** •• A large uniform horizontal turntable rotates freely about a vertical axle at its center. You measure the radius of the turntable to be 3.00 m. To determine the moment of inertia $I$ of the turntable about the axle, you start the turntable rotating with angular speed $\omega$, which you measure. You then drop a small object of mass $m$ onto the rim of the turntable. After the object has come to rest relative to the turntable, you measure the angular speed $\omega_{\text{f}}$ of the rotating turntable. You plot the quantity $(\omega - \omega_{\text{f}})/\omega_{\text{f}}$ (with both $\omega$ and $\omega_{\text{f}}$ in rad/s) as a function of $m$ (in kg). You find that your data lie close to a straight line that has slope $0.250\ \text{kg}^{-1}$. What is the moment of inertia $I$ of the turntable?

**10.71** • **The Yo-yo.** A yo-yo is made from two uniform disks, each with mass $m$ and radius $R$, connected by a light axle of radius $b$. A light, thin string is wound several times around the axle and then held stationary while the yo-yo is released from rest, dropping as the string unwinds. Find the linear acceleration and angular acceleration of the yo-yo and the tension in the string.

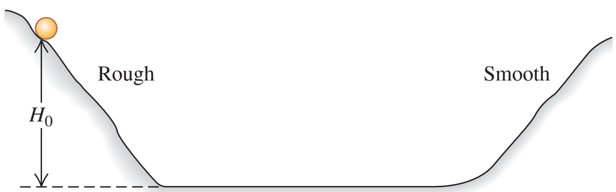
**10.72** •• **CP** A thin-walled, hollow spherical shell of mass $m$ and radius $r$ starts from rest and rolls without slipping down a track (**Fig. P10.72**). Points $A$ and $B$ are on a circular part of the track having radius $R$. The diameter of the shell is very small compared to $h_0$ and $R$, and the work done by rolling friction is negligible. (a) What is the minimum height $h_0$ for which this shell will make a complete loop-the-loop on the circular part of the track? (b) How hard does the track push on the shell at point $B$, which is at the same level as the center of the circle? (c) Suppose that the track had no friction and the shell was released from the same height $h_0$ you found in part (a). Would it make a complete loop-the-loop? How do you know? (d) In part (c), how hard does the track push on the shell at point $A$, the top of the circle? How hard did it push on the shell in part (a)?

Figure **P10.72**



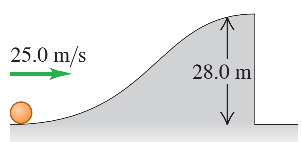
**10.73** •• A basketball (which can be closely modeled as a hollow spherical shell) rolls down a mountainside into a valley and then up the opposite side, starting from rest at a height $H_0$ above the bottom. In **Fig. P10.73**, the rough part of the terrain prevents slipping while the smooth part has no friction. Neglect rolling friction and assume the system's total mechanical energy is conserved. (a) How high, in terms of $H_0$, will the ball go up the other side? (b) Why doesn't the ball return to height $H_0$? Has it lost any of its original potential energy?

Figure **P10.73**



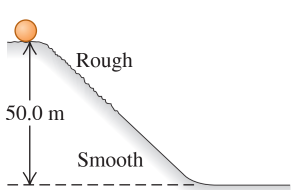

**10.74** •• **CP** A solid uniform ball rolls without slipping up a hill (**Fig. P10.74**). At the top of the hill, it is moving horizontally, and then it goes over the vertical cliff. Neglect rolling friction and assume the system's total mechanical energy is conserved. (a) How far from the foot of the cliff does the ball land, and how fast is it moving just before it lands? (b) Notice that when the balls lands, it has a greater translational speed than when it was at the bottom of the hill. Does this mean that the ball somehow gained energy? Explain!

Figure **P10.74**




**10.75** •• **Rolling Stones.** A solid, uniform, spherical boulder starts from rest and rolls down a 50.0-m-high hill, as shown in **Fig. P10.75**. The top half of the hill is rough enough to cause the boulder to roll without slipping, but the lower half is covered with ice and there is no friction. What is the translational speed of the boulder when it reaches the bottom of the hill? Neglect rolling friction and assume the system's total mechanical energy is conserved.

Figure **P10.75**



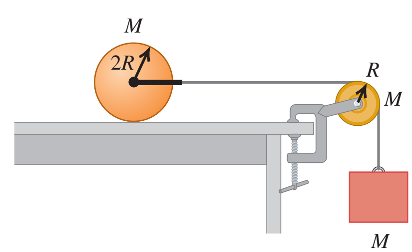

**10.76** ••• You are designing a system for moving aluminum cylinders from the ground to a loading dock. You use a sturdy wooden ramp that is 6.00 m long and inclined at 37.0° above the horizontal. Each cylinder is fitted with a light, frictionless yoke through its center, and a light (but strong) rope is attached to the yoke. Each cylinder is uniform and has mass 460 kg and radius 0.300 m. The cylinders are pulled up the ramp by applying a constant force $\vec{F}$ to the free end of the rope. $\vec{F}$ is parallel to the surface of the ramp and exerts no torque on the cylinder. The coefficient of static friction between the ramp surface and the cylinder is 0.120. (a) What is the largest magnitude $\vec{F}$ can have so that the cylinder still rolls without slipping as it moves up the ramp? (b) If the cylinder starts from rest at the bottom of the ramp and rolls without slipping as it moves up the ramp, what is the shortest time it can take the cylinder to reach the top of the ramp?

**10.77** •• A 42.0-cm-diameter wheel, consisting of a rim and six spokes, is constructed from a thin, rigid plastic material having a linear mass density of 25.0 g/cm. This wheel is released from rest at the top of a hill 58.0 m high. (a) How fast is it rolling when it reaches the bottom of the hill? (b) How would your answer change if the linear mass density and the diameter of the wheel were each doubled?

**10.78** ••• A uniform, 0.0300 kg rod of length 0.400 m rotates in a horizontal plane about a fixed axis through its center and perpendicular to the rod. Two small rings, each with mass 0.0200 kg, are mounted so that they can slide without friction along the rod. They are initially held by catches at positions 0.0500 m on each side of the center of the rod, and the system is rotating at 48.0 rev/min. With no other changes in the system, the catches are released, and the rings slide outward along the rod and fly off at the ends. What is the angular speed (a) of the system at the instant when the rings reach the ends of the rod; (b) of the rod after the rings leave it?

**10.79** • A uniform solid cylinder with mass $M$ and radius $2R$ rests on a horizontal tabletop. A string is attached by a yoke to a frictionless axle through the center of the cylinder so that the cylinder can rotate about the axle. The string runs over a disk-shaped pulley with mass $M$ and radius $R$ that is mounted on a frictionless axle through its center. A block of mass $M$ is suspended from the free end of the string (**Fig. P10.79**). The string doesn't slip over the pulley surface, and the cylinder rolls without slipping on the tabletop. Find the magnitude of the acceleration of the block after the system is released from rest.

Figure **P10.79**




**10.80** ••• A 5.00 kg ball is dropped from a height of 12.0 m above one end of a uniform bar that pivots at its center. The bar has mass 8.00 kg and is 4.00 m in length. At the other end of the bar sits another 5.00 kg ball, unattached to the bar. The dropped ball sticks to the bar after the collision. How high will the other ball go after the collision?

**10.81** •• A uniform rod of length $L$ rests on a frictionless horizontal surface. The rod pivots about a fixed frictionless axis at one end. The rod is initially at rest. A bullet traveling parallel to the horizontal surface and perpendicular to the rod with speed $v$ strikes the rod at its center and becomes embedded in it. The mass of the bullet is one-fourth the mass of the rod. (a) What is the final angular speed of the rod? (b) What is the ratio of the kinetic energy of the system after the collision to the kinetic energy of the bullet before the collision?

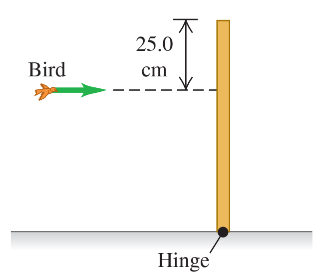
**10.82** •• **CP** A large turntable with radius 6.00 m rotates about a fixed vertical axis, making one revolution in 8.00 s. The moment of inertia of the turntable about this axis is 1200 kg · m$^2$. You stand, barefooted, at the rim of the turntable and very slowly walk toward the center, along a radial line painted on the surface of the turntable. Your mass is 70.0 kg. Since the radius of the turntable is large, it is a good approximation to treat yourself as a point mass. Assume that you can maintain your balance by adjusting the positions of your feet. You find that you can reach a point 3.00 m from the center of the turntable before your feet begin to slip. What is the coefficient of static friction between the bottoms of your feet and the surface of the turntable?

**10.83** •• In your job as a mechanical engineer you are designing a flywheel and clutch-plate system like the one in Example 10.11. Disk $A$ is made of a lighter material than disk $B$, and the moment of inertia of disk $A$ about the shaft is one-third that of disk $B$. The moment of inertia of the shaft is negligible. With the clutch disconnected, $A$ is brought up to an angular speed $\omega_0$; $B$ is initially at rest. The accelerating torque is then removed from $A$, and $A$ is coupled to $B$. (Ignore bearing friction.) The design specifications allow for a maximum of 2400 J of thermal energy to be developed when the connection is made. What can be the maximum value of the original kinetic energy of disk $A$ so as not to exceed the maximum allowed value of the thermal energy?

**10.84** •• A local ice hockey team has asked you to design an apparatus for measuring the speed of the hockey puck after a slap shot. Your design is a 2.00-m-long, uniform rod pivoted about one end so that it is free to rotate horizontally on the ice without friction. The 0.800 kg rod has a light basket at the other end to catch the 0.163 kg puck. The puck slides across the ice with velocity $\vec{v}$ (perpendicular to the rod), hits the basket, and is caught. After the collision, the rod rotates. If the rod makes one revolution every 0.736 s after the puck is caught, what was the puck's speed just before it hit the rod?

**10.85** ••• A 500.0 g bird is flying horizontally at 2.25 m/s, not paying much attention, when it suddenly flies into a stationary vertical bar, hitting it 25.0 cm below the top (**Fig. P10.85**). The bar is uniform, 0.750 m long, has a mass of 1.50 kg, and is hinged at its base. The collision stuns the bird so that it just drops to the ground afterward (but soon recovers to fly happily away). What is the angular velocity of the bar (a) just after it is hit by the bird and (b) just as it reaches the ground?

Figure **P10.85**



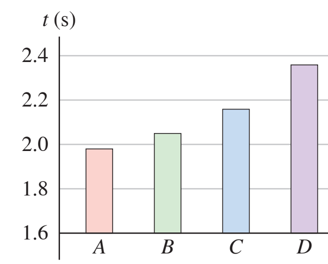

**10.86** ••• **CP** A small block with mass 0.130 kg is attached to a string passing through a hole in a frictionless, horizontal surface (see Fig. E10.42). The block is originally revolving in a circle with a radius of 0.800 m about the hole with a tangential speed of 4.00 m/s. The string is then pulled slowly from below, shortening the radius of the circle in which the block revolves. The breaking strength of the string is 30.0 N. What is the radius of the circle when the string breaks?

**10.87** • A 55 kg runner runs around the edge of a horizontal turntable mounted on a vertical, frictionless axis through its center. The runner's velocity relative to the earth has magnitude 2.8 m/s. The turntable is rotating in the opposite direction with an angular velocity of magnitude 0.20 rad/s relative to the earth. The radius of the turntable is 3.0 m, and its moment of inertia about the axis of rotation is 80 kg·m$^2$. Find the final angular velocity of the system if the runner comes to rest relative to the turntable. (You can model the runner as a particle.)

**10.88** •• **DATA** The V6 engine in a 2014 Chevrolet Silverado 1500 pickup truck is reported to produce a maximum power of 285 hp at 5300 rpm and a maximum torque of 305 ft·lb at 3900 rpm. (a) Calculate the torque, in both ft·lb and N·m, at 5300 rpm. Is your answer in ft·lb smaller than the specified maximum value? (b) Calculate the power, in both horsepower and watts, at 3900 rpm. Is your answer in hp smaller than the specified maximum value? (c) The relationship between power in hp and torque in ft·lb at a particular angular velocity in rpm is often written as hp = [torque (in ft·lb) × rpm]/$c$, where $c$ is a constant. What is the numerical value of $c$? (d) The engine of a 2012 Chevrolet Camaro ZL1 is reported to produce 580 hp at 6000 rpm. What is the torque (in ft·lb) at 6000 rpm?

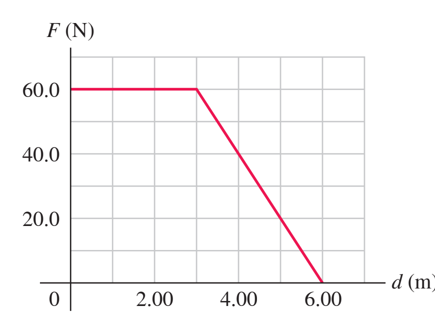
**10.89** •• **DATA** You have one object of each of these shapes, all with mass 0.840 kg: a uniform solid cylinder, a thin-walled hollow cylinder, a uniform solid sphere, and a thin-walled hollow sphere. You release each object from rest at the same vertical height $h$ above the bottom of a long wooden ramp that is inclined at 35.0° from the horizontal. Each object rolls without slipping down the ramp. You measure the time $t$ that it takes each one to reach the bottom of the ramp; **Fig. P10.89** shows the results. (a) From the bar graphs, identify objects $A$ through $D$ by shape. (b) Which of objects $A$ through $D$ has the greatest total kinetic energy at the bottom of the ramp, or do all have the same kinetic energy? (c) Which of objects $A$ through $D$ has the greatest rotational kinetic energy $\frac{1}{2}I\omega^2$ at the bottom of the ramp, or do all have the same rotational kinetic energy? (d) What minimum coefficient of static friction is required for all four objects to roll without slipping?

Figure **P10.89**




**10.90** ••• **DATA** You are testing a small flywheel (radius 0.166 m) that will be used to store a small amount of energy. The flywheel is pivoted with low-friction bearings about a horizontal shaft through the flywheel's center. A thin, light cord is wrapped multiple times around the rim of the flywheel. Your lab has a device that can apply a specified horizontal force $\vec{F}$ to the free end of the cord. The device records both the magnitude of that force as a function of the horizontal distance the end of the cord has traveled and the time elapsed since the force was first applied. The flywheel is initially at rest. (a) You start with a test run to determine the flywheel's moment of inertia $I$. The magnitude $F$ of the force is a constant 25.0 N, and the end of the rope moves 8.35 m in 2.00 s. What is $I$? (b) In a second test, the flywheel again starts from rest but the free end of the rope travels 6.00 m; **Fig. P10.90** shows the force magnitude $F$ as a function of the distance $d$ that the end of the rope has moved. What is the kinetic energy of the flywheel when $d$ = 6.00 m? (c) What is the angular speed of the flywheel, in rev/min, when $d$ = 6.00 m?

Figure **P10.90**




## CHALLENGE PROBLEMS

**10.91** ••• **CP CALC** A block with mass $m$ is revolving with linear speed $v_1$ in a circle of radius $r_1$ on a frictionless horizontal surface (see Fig. E10.42). The string is slowly pulled from below until the radius of the circle in which the block is revolving is reduced to $r_2$. (a) Calculate the tension $T$ in the string as a function of $r$, the distance of the block from the hole. Your answer will be in terms of the initial velocity $v_1$ and the radius $r_1$. (b) Use $W = \int_{r_1}^{r_2} \vec{T}(r) \cdot d\vec{r}$ to calculate the work done by $\vec{T}$ when $r$ changes from $r_1$ to $r_2$. (c) Compare the results of part (b) to the change in the kinetic energy of the block.

**10.92** ••• When an object is rolling without slipping, the rolling friction force is much less than the friction force when the object is sliding; a silver dollar will roll on its edge much farther than it will slide on its flat side (see Section 5.3). When an object is rolling without slipping on a horizontal surface, we can approximate the friction force to be zero, so that $a_x$ and $\alpha_z$ are approximately zero and $v_x$ and $\omega_z$ are approximately constant. Rolling without slipping means $v_x = r\omega_z$ and $a_x = r\alpha_z$. If an object is set in motion on a surface *without* these equalities, sliding (kinetic) friction will act on the object as it slips until rolling without slipping is established. A solid cylinder with mass $M$ and radius $R$, rotating with angular speed $\omega_0$ about an axis through its center, is set on a horizontal surface for which the kinetic friction coefficient is $\mu_k$. (a) Draw a free-body diagram for the cylinder on the surface. Think carefully about the direction of the kinetic friction force on the cylinder. Calculate the accelerations $a_x$ of the center of mass and $\alpha_z$ of rotation about the center of mass. (b) The cylinder is initially slipping completely, so initially $\omega_z = \omega_0$ but $v_x = 0$. Rolling without slipping sets in when $v_x = r\omega_z$. Calculate the *distance* the cylinder rolls before slipping stops. (c) Calculate the work done by the friction force on the cylinder as it moves from where it was set down to where it begins to roll without slipping.

**10.93** ••• A demonstration gyroscope wheel is constructed by removing the tire from a bicycle wheel 0.650 m in diameter, wrapping lead wire around the rim, and taping it in place. The shaft projects 0.200 m at each side of the wheel, and a woman holds the ends of the shaft in her hands. The mass of the system is 8.00 kg; its entire mass may be assumed to be located at its rim. The shaft is horizontal, and the wheel is spinning about the shaft at 5.00 rev/s. Find the magnitude and direction of the force each hand exerts on the shaft (a) when the shaft is at rest; (b) when the shaft is rotating in a horizontal plane about its center at 0.050 rev/s; (c) when the shaft is rotating in a horizontal plane about its center at 0.300 rev/s. (d) At what rate must the shaft rotate in order that it may be supported at one end only?

## MCAT-STYLE PASSAGE PROBLEMS

**BIO Human Moment of Inertia.** The moment of inertia of the human body about an axis through its center of mass is important in the application of biomechanics to sports such as diving and gymnastics. We can measure the body's moment of inertia in a particular position while a person remains in that position on a horizontal turntable, with the body's center of mass on the turntable's rotational axis. The turntable with the person on it is then accelerated from rest by a torque that is produced by using a rope wound around a pulley on the shaft of the turntable. From the measured tension in the rope and the angular acceleration, we can calculate the body's moment of inertia about an axis through its center of mass.

Overhead view of a female gymnast lying in somersault position atop a turntable

**10.94** The moment of inertia of the empty turntable is 1.5 kg·m$^2$. With a constant torque of 2.5 N·m, the turntable–person system takes 3.0 s to spin from rest to an angular speed of 1.0 rad/s. What is the person's moment of inertia about an axis through her center of mass? Ignore friction in the turntable axle. (a) 2.5 kg·m$^2$; (b) 6.0 kg·m$^2$; (c) 7.5 kg·m$^2$; (d) 9.0 kg·m$^2$.

**10.95** While the turntable is being accelerated, the person suddenly extends her legs. What happens to the turntable? (a) It suddenly speeds up; (b) it rotates with constant speed; (c) its acceleration decreases; (d) it suddenly stops rotating.

**10.96** A doubling of the torque produces a greater angular acceleration. Which of the following would do this, assuming that the tension in the rope doesn't change? (a) Increasing the pulley diameter by a factor of $\sqrt{2}$; (b) increasing the pulley diameter by a factor of 2; (c) increasing the pulley diameter by a factor of 4; (d) decreasing the pulley diameter by a factor of $\sqrt{2}$.

**10.97** If the body's center of mass were not placed on the rotational axis of the turntable, how would the person's measured moment of inertia compare to the moment of inertia for rotation about the center of mass? (a) The measured moment of inertia would be too large; (b) the measured moment of inertia would be too small; (c) the two moments of inertia would be the same; (d) it depends on where the body's center of mass is placed relative to the center of the turntable.

# ANSWERS

### Chapter Opening Question ?

**(iv)** A tossed pin rotates around its center of mass (which is located toward its thick end). This is also the point at which the gravitational force acts on the pin, so this force exerts no torque on the pin. Hence the pin rotates with constant angular momentum, and its angular speed remains the same.

### Key Example √ARIATION Problems

**VP10.3.1** (a) 10 rad/s$^2$ (b) 0.90 N·m (c) 15 N

**VP10.3.2** (a) $a_y = \dfrac{g}{1 + M/m}$ (b) $T = \dfrac{mg}{1 + m/M} = \dfrac{Mg}{1 + M/m}$

**VP10.3.3** (a) $T = m(g - a)$ (b) $I = mR^2\left(\dfrac{g}{a} - 1\right)$

**VP10.3.4** (a) $T = m(g + a)$ (b) $\tau_{\text{cable on cylinder}} = mR(g + a)$ (c) $\tau_{\text{motor on cylinder}} = mR(g + a) + \frac{1}{2}MRa$

**VP10.7.1** (a) $a_{\text{cm-}y} = \frac{8}{13}g$ (b) $T = \frac{5}{13}Mg$

**VP10.7.2** (a) $a_{\text{cm-}x} = \frac{3}{5}g\sin\beta$ (b) $f = \frac{2}{5}Mg\sin\beta$ (c) $\tau = \frac{2}{5}MgR\sin\beta$

**VP10.7.3** (a) $a_{\text{cm}} = \frac{1}{3}g$, downward (b) $\alpha_z = \frac{4g}{3R}$

**VP10.7.4** (a) $f = \frac{1}{3}Mg\sin\beta$ (b) $\beta_{\text{critical}} = \arctan 3\mu_s$

**VP10.12.1** (a) $\frac{9}{10}\omega_A$ (b) $\frac{81}{85} = 0.953$

**VP10.12.2** (a) $\frac{7}{10}\omega_A$ (b) $\frac{49}{85} = 0.576$

**VP10.12.3** (a) 0.80 rad/s (b) $0.0020 = 1/500$

**VP10.12.4** (a) $MRv$ before, $MRv/2$ after (b) $\omega = \dfrac{3v}{4R}$ (c) $\frac{5}{8}$

### Bridging Problem

(a) $h = \dfrac{2R}{5}$

(b) $\frac{5}{7}$ of the speed it had just after the hit

? This Roman aqueduct uses the principle of the arch to sustain the weight of the structure and the water it carries. Are the blocks that make up the arch being (i) compressed, (ii) stretched, (iii) a combination of these, or (iv) neither compressed nor stretched?

# 11 Equilibrium and Elasticity

We've devoted a good deal of effort to understanding why and how objects accelerate in response to the forces that act on them. But very often we're interested in making sure that objects *don't* accelerate. Any building, from a multistory skyscraper to the humblest shed, must be designed so that it won't topple over. Similar concerns arise with a suspension bridge, a ladder leaning against a wall, or a crane hoisting a bucket full of concrete.

An object that can be modeled as a *particle* is in equilibrium whenever the vector sum of the forces acting on it is zero. But for the situations we've just described, that condition isn't enough. If forces act at different points on an extended object, an additional requirement must be satisfied to ensure that the object has no tendency to *rotate:* The sum of the *torques* about any point must be zero. This requirement is based on the principles of rotational dynamics developed in Chapter 10. We can compute the torque due to the weight of an object by using the concept of center of gravity, which we introduce in this chapter.

Idealized rigid bodies don't bend, stretch, or squash when forces act on them. But all real materials are *elastic* and do deform to some extent. Elastic properties of materials are tremendously important. You want the wings of an airplane to be able to bend a little, but you'd rather not have them break off. Tendons in your limbs need to stretch when you exercise, but they must return to their relaxed lengths when you stop. Many of the necessities of everyday life, from rubber bands to suspension bridges, depend on the elastic properties of materials. In this chapter we'll introduce the concepts of *stress, strain,* and *elastic modulus* and a simple principle called *Hooke's law,* which helps us predict what deformations will occur when forces are applied to a real (not perfectly rigid) object.

### LEARNING OUTCOMES

***In this chapter, you'll learn...***

**11.1** The conditions that must be satisfied for an object or structure to be in equilibrium.
**11.2** What the center of gravity of an object is and how it relates to the object's stability.
**11.3** How to solve problems that involve rigid bodies in equilibrium.
**11.4** How to analyze situations in which an object is deformed by tension, compression, pressure, or shear.
**11.5** What happens when an object is stretched so much that it deforms or breaks.

***You'll need to review...***

**4.2, 5.1** Newton's first law.
**5.3** Static friction.
**6.3, 7.2** Hooke's law for an ideal spring.
**8.5** Center of mass.
**10.2, 10.5** Torque, rotational dynamics, and angular momentum.

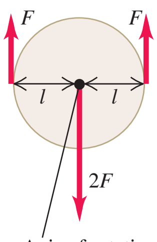
## 11.1 CONDITIONS FOR EQUILIBRIUM

We learned in Sections 4.2 and 5.1 that a particle is in *equilibrium*—that is, the particle does not accelerate—in an inertial frame of reference if the vector sum of all the forces acting on the particle is zero, $\sum\vec{F} = \mathbf{0}$. For an *extended* object, the equivalent statement is that the center of mass of the object has zero acceleration if the vector sum of all external

Figure **11.1** To be in static equilibrium, an object at rest must satisfy *both* conditions for equilibrium: It can have no tendency to accelerate as a whole or to start rotating.

(a) This object is in static equilibrium.

**Equilibrium conditions:**

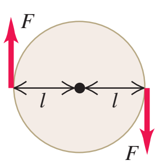



**First condition satisfied:** Net force = 0, so object at rest has no tendency to start moving as a whole.

**Second condition satisfied:** Net torque about the axis = 0, so object at rest has no tendency to start rotating.

Axis of rotation (perpendicular to figure)

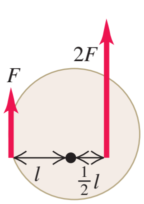
(b) This object has no tendency to accelerate as a whole, but it has a tendency to start rotating.




**First condition satisfied:** Net force = 0, so object at rest has no tendency to start moving as a whole.

**Second condition NOT satisfied:** There is a net clockwise torque about the axis, so object at rest will start rotating clockwise.

(c) This object has a tendency to accelerate as a whole but no tendency to start rotating.




**First condition NOT satisfied:** There is a net upward force, so object at rest will start moving upward.

**Second condition satisfied:** Net torque about the axis = 0, so object at rest has no tendency to start rotating.

forces acting on the object is zero, as discussed in Section 8.5. This is often called the **first condition for equilibrium:**

**First condition for equilibrium:** For the center of mass of an object at rest to remain at rest ... $$\sum \vec{F} = \mathbf{0} \tag{11.1}$$ ... the *net external force* on the object must be *zero.*

A second condition for an extended object to be in equilibrium is that the object must have no tendency to *rotate*. A rigid body that, in an inertial frame, is not rotating about a certain point has zero angular momentum about that point. If it is not to start rotating about that point, the rate of change of angular momentum must *also* be zero. From the discussion in Section 10.5, particularly Eq. (10.29), this means that the sum of torques due to all the external forces acting on the object must be zero. A rigid body in equilibrium can't have any tendency to start rotating about *any* point, so the sum of external torques must be zero about any point. This is the **second condition for equilibrium:**

**Second condition for equilibrium:** For a nonrotating object to remain nonrotating ... $$\sum \vec{\tau} = \mathbf{0} \tag{11.2}$$ ...the *net external torque around any point* on the object must be *zero.*

In this chapter we'll apply the first and second conditions for equilibrium to situations in which a rigid body is at rest (no translation or rotation). Such a rigid body is said to be in **static equilibrium** (**Fig. 11.1**). But the same conditions apply to a rigid body in uniform *translational* motion (without rotation), such as an airplane in flight with constant speed, direction, and altitude. Such a rigid body is in equilibrium but is not static.

**TEST YOUR UNDERSTANDING OF SECTION 11.1** Which situation satisfies both the first and second conditions for equilibrium? (i) A seagull gliding at a constant angle below the horizontal and at a constant speed; (ii) an automobile crankshaft turning at an increasing angular speed in the engine of a parked car; (iii) a thrown baseball that does not rotate as it sails through the air.

**ANSWER**

| (i) Situation (i) satisfies both equilibrium conditions because the seagull has zero acceleration (so $\sum \vec{F} = \mathbf{0}$) and no tendency to start rotating (so $\sum \vec{\tau} = \mathbf{0}$). Situation (ii) satisfies the first condition because the crankshaft as a whole does not accelerate through space, but it does not satisfy the second condition: the crankshaft has an angular acceleration, so $\sum \vec{\tau}$ is not zero. Situation (iii) satisfies the second condition (there is no tendency to rotate) but not the first one; the baseball accelerates in its flight (due to gravity and air resistance), so $\sum \vec{F}$ is not zero.

## 11.2 CENTER OF GRAVITY

In most equilibrium problems, one of the forces acting on the object is its weight. We need to be able to calculate the *torque* of this force. The weight doesn't act at a single point; it is distributed over the entire object. But we can always calculate the torque due to the object's weight by assuming that the entire force of gravity (weight) is concentrated at a point called the **center of gravity** (abbreviated "cg"). The acceleration due to gravity decreases with altitude; but if we can ignore this variation over the vertical dimension of the object, then the object's center of gravity is identical to its *center of mass* (abbreviated "cm"), which we defined in Section 8.5. We stated this result without proof in Section 10.2, and now we'll prove it.

First let's review the definition of the center of mass. For a collection of particles with masses $m_1, m_2, \ldots$ and coordinates $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, $\ldots$, the coordinates $x_{cm}$, $y_{cm}$, and $z_{cm}$ of the center of mass of the collection are

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_ix_i}{\sum_i m_i}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i y_i}{\sum_i m_i} \quad \text{(center of mass)} \tag{11.3}$$

$$z_{cm} = \frac{m_1 z_1 + m_2 z_2 + m_3 z_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i z_i}{\sum_i m_i}$$

Also, $x_{cm}$, $y_{cm}$, and $z_{cm}$ are the components of the position vector $\vec{r}_{cm}$ of the center of mass, so Eqs. (11.3) are equivalent to the vector equation

Position vectors of individual particles

Position vector of center of mass of a system of particles

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \tag{11.4}$$

Masses of individual particles

Now consider the gravitational torque on an object of arbitrary shape (**Fig. 11.2**). We assume that the acceleration due to gravity $\vec{g}$ is the same at every point in the object. Every particle in the object experiences a gravitational force, and the total weight of the object is the vector sum of a large number of parallel forces. A typical particle has mass $m_i$ and weight $\vec{w}_i = m_i\vec{g}$. If $\vec{r}_i$ is the position vector of this particle with respect to an arbitrary origin $O$, then the torque vector $\vec{\tau}_i$ of the weight $\vec{w}_i$ with respect to $O$ is, from Eq. (10.3),

$$\vec{\tau}_i = \vec{r}_i \times \vec{w}_i = \vec{r}_i \times m_i\vec{g}$$

The *total* torque due to the gravitational forces on all the particles is

$$\begin{aligned}\vec{\tau} &= \sum_i \vec{\tau}_i = \vec{r}_1 \times m_1\vec{g} + \vec{r}_2 \times m_2\vec{g} + \cdots \\ &= (m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots) \times \vec{g} \\ &= \left(\sum_i m_i\vec{r}_i\right) \times \vec{g}\end{aligned}$$

When we multiply and divide this result by the total mass of the object,

$$M = m_1 + m_2 + \cdots = \sum_i m_i$$

we get

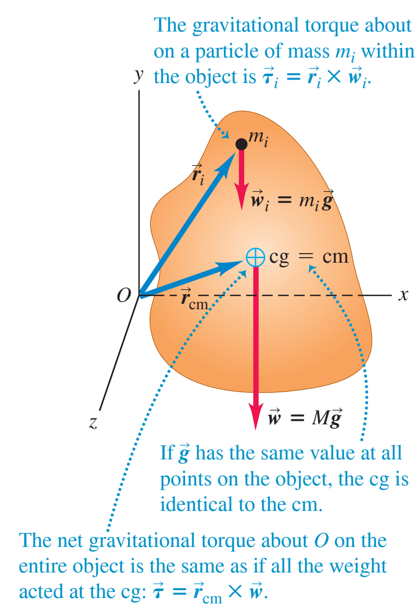
$$\vec{\tau} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots}{m_1 + m_2 + \cdots} \times M\vec{g} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} \times M\vec{g}$$

The fraction in this equation is just the position vector $\vec{r}_{cm}$ of the center of mass, with components $x_{cm}$, $y_{cm}$, and $z_{cm}$, as given by Eq. (11.4), and $M\vec{g}$ is equal to the total weight $\vec{w}$ of the object. Thus

$$\vec{\tau} = \vec{r}_{cm} \times M\vec{g} = \vec{r}_{cm} \times \vec{w} \tag{11.5}$$

The total gravitational torque, given by Eq. (11.5), is the same as though the total weight $\vec{w}$ were acting at the position $\vec{r}_{cm}$ of the center of mass, which we also call the *center of gravity*. **If $\vec{g}$ has the same value at all points on an object, its center of gravity**

Figure **11.2** The center of gravity (cg) and center of mass (cm) of an extended object.



**is identical to its center of mass.** Note, however, that the center of mass is defined independently of any gravitational effect.

While the value of $\vec{g}$ varies somewhat with elevation, the variation is extremely slight (**Fig. 11.3**). We'll assume throughout this chapter that the center of gravity and center of mass are identical unless explicitly stated otherwise.

Figure 11.3 The acceleration due to gravity at the bottom of the 452-m-tall Petronas Towers in Malaysia is only 0.014% greater than at the top. The center of gravity of the towers is only about 2 cm below the center of mass.

## Finding and Using the Center of Gravity

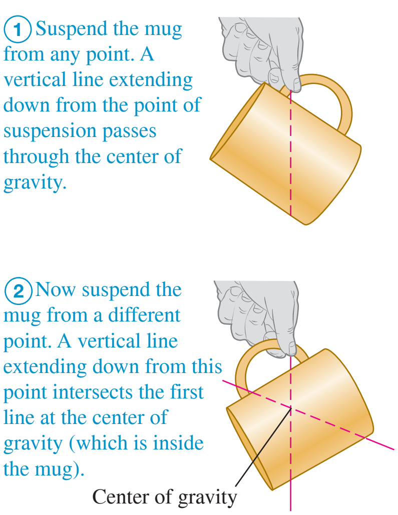
We can often use symmetry considerations to locate the center of gravity of an object, just as we did for the center of mass. The center of gravity of a homogeneous sphere, cube, or rectangular plate is at its geometric center. The center of gravity of a right circular cylinder or cone is on its axis of symmetry.

For an object with a more complex shape, we can sometimes locate the center of gravity by thinking of the object as being made of symmetrical pieces. For example, we could approximate the human body as a collection of solid cylinders, with a sphere for the head. Then we can locate the center of gravity of the combination with Eqs. (11.3), letting $m_1$, $m_2$, . . . be the masses of the individual pieces and $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, . . . be the coordinates of their centers of gravity.

When an object in rotational equilibrium and acted on by gravity is supported or suspended at a single point, the center of gravity is always at or directly above or below the point of suspension. If it were anywhere else, the weight would have a torque with respect to the point of suspension, and the object could not be in rotational equilibrium. **Figure 11.4** shows an application of this idea.

Figure 11.4 Finding the center of gravity of an irregularly shaped object—in this case, a coffee mug.

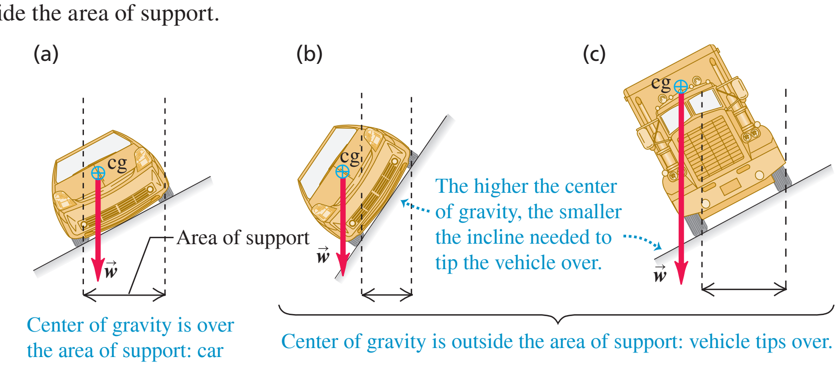
Where is the center of gravity of this mug?




Using the same reasoning, we can see that an object supported at several points must have its center of gravity somewhere within the area bounded by the supports. This explains why a car can drive on a straight but slanted road if the slant angle is relatively small (**Fig. 11.5a**) but will tip over if the angle is too steep (Fig. 11.5b). The truck in Fig. 11.5c has a higher center of gravity than the car and will tip over on a shallower incline.

The lower the center of gravity and the larger the area of support, the harder it is to overturn an object. Four-legged animals such as deer and horses have a large area of support bounded by their legs; hence they are naturally stable and need only small feet or hooves. Animals that walk on two legs, such as humans and birds, need relatively large feet to give them a reasonable area of support. If a two-legged animal holds its body approximately horizontal, like a chicken or the dinosaur *Tyrannosaurus rex,* it must perform a balancing act as it walks to keep its center of gravity over the foot that is on the ground. A chicken does this by moving its head; *T. rex* probably did it by moving its massive tail.

Figure 11.5 In (a) the center of gravity is within the area bounded by the supports, and the car is in equilibrium. The car in (b) and the truck in (c) will tip over because their centers of gravity lie outside the area of support.



## EXAMPLE 11.1 Walking the plank

WITH VARIATION PROBLEMS

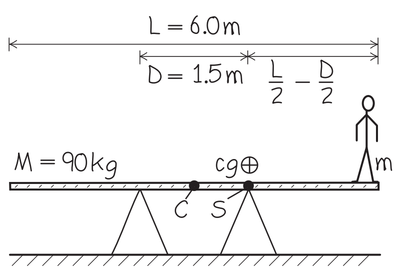
A uniform plank of length $L = 6.0$ m and mass $M = 90$ kg rests on sawhorses separated by $D = 1.5$ m and equidistant from the center of the plank. Cousin Throckmorton wants to stand on the right-hand end of the plank. If the plank is to remain at rest, how massive can Throckmorton be?

**IDENTIFY and SET UP** To just balance, Throckmorton's mass $m$ must be such that the center of gravity of the plank–Throcky system is directly over the right-hand sawhorse (**Fig. 11.6**). We take the origin at $C$, the geometric center and center of gravity of the plank, and take the positive $x$-axis horizontally to the right. Then the centers of gravity of the plank and Throcky are at $x_P = 0$ and $x_T = L/2 = 3.0$ m, respectively, and the right-hand sawhorse is at $x_S = D/2$. We'll use Eqs. (11.3) to locate the center of gravity $x_{cg}$ of the plank–Throcky system.

**EXECUTE** From the first of Eqs. (11.3),

$$x_{cg} = \frac{M(0) + m(L/2)}{M + m} = \frac{m}{M + m}\frac{L}{2}$$

Figure 11.6 Our sketch for this problem.




We set $x_{cg} = x_S$ and solve for $m$:

$$\frac{m}{M + m}\frac{L}{2} = \frac{D}{2}$$

$$mL = (M + m)D$$

$$m = M\frac{D}{L - D} = (90\text{ kg})\frac{1.5\text{ m}}{6.0\text{ m} - 1.5\text{ m}} = 30\text{ kg}$$

**EVALUATE** As a check, let's repeat the calculation with the origin at the right-hand sawhorse. Now $x_S = 0$, $x_P = -D/2$, and $x_T = (L/2) - (D/2)$, and we require $x_{cg} = x_S = 0$:

$$x_{cg} = \frac{M(-D/2) + m[(L/2) - (D/2)]}{M + m} = 0$$

$$m = \frac{MD/2}{(L/2) - (D/2)} = M\frac{D}{L - D} = 30\text{ kg}$$

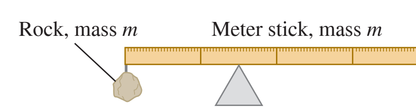
The result doesn't depend on our choice of origin.

A 60 kg adult could stand only halfway between the right-hand sawhorse and the end of the plank. Can you see why?

**KEYCONCEPT** If an extended object supported at two or more points is to be in equilibrium, its center of gravity must be somewhere within the area bounded by the supports. If the object is supported at only one point, its center of gravity must be above that point.

**TEST YOUR UNDERSTANDING OF SECTION 11.2** A rock is attached to the left end of a uniform meter stick that has the same mass as the rock. In order for the combination of rock and meter stick to balance atop the triangular object in **Fig. 11.7**, how far from the left end of the stick should the triangular object be placed? (i) Less than 0.25 m; (ii) 0.25 m; (iii) between 0.25 m and 0.50 m; (iv) 0.50 m; (v) more than 0.50 m.

Figure 11.7 At what point will the meter stick with rock attached be in balance?




**ANSWER**

**(ii)** In equilibrium, the center of gravity must be at the point of support. Since the rock and meter stick have the same mass and hence the same weight, the center of gravity of the system is midway between their respective centers. The center of gravity of the meter stick alone is 0.50 m from the left end (that is, at the middle of the meter stick), so the center of gravity of the combination of rock and meter stick is 0.25 m from the left end.

## 11.3 SOLVING RIGID-BODY EQUILIBRIUM PROBLEMS

There are just two key conditions for rigid-body equilibrium: The vector sum of the forces on the object must be zero, and the sum of the torques about any point must be zero. To keep things simple, we'll restrict our attention to situations in which we can treat all forces as acting in a single plane, which we'll call the $xy$-plane. Then we need consider only the $x$- and $y$-components of force in Eq. (11.1), and in Eq. (11.2) we need consider only the $z$-components of torque (perpendicular to the plane). The first and second conditions for equilibrium are then

$$\sum F_x = 0 \quad \text{and} \quad \sum F_y = 0 \quad \text{(first condition for equilibrium, forces in } xy\text{-plane)}$$
$$\sum \tau_z = 0 \quad \text{(second condition for equilibrium, forces in } xy\text{-plane)} \qquad (11.6)$$

**CAUTION Choosing the reference point for calculating torques** In equilibrium problems, the choice of reference point for calculating torques in $\sum \tau_z$ is completely arbitrary. But once you make your choice, you must use the *same* point to calculate *all* the torques on an object. Choose the point so as to simplify the calculations as much as possible.

The challenge is to apply these simple conditions to specific problems. Problem-Solving Strategy 11.1 is very similar to the suggestions given in Section 5.1 for the equilibrium of a particle. You should compare it with Problem-Solving Strategy 10.1 (Section 10.2) for rotational dynamics problems.

## PROBLEM-SOLVING STRATEGY 11.1 Equilibrium of a Rigid Body

**IDENTIFY** *the relevant concepts:* The first and second conditions for equilibrium $(\sum F_x = 0, \sum F_y = 0, \text{ and } \sum \tau_z = 0)$ are applicable to any rigid body that is not accelerating in space and not rotating.

**SET UP** *the problem* using the following steps:

1. Sketch the physical situation and identify the object in equilibrium to be analyzed. Sketch the object accurately; do *not* represent it as a point. Include dimensions.
2. Draw a free-body diagram showing all forces acting *on* the object. Show the point on the object at which each force acts.
3. Choose coordinate axes and specify their direction. Specify a positive direction of rotation for torques. Represent forces in terms of their components with respect to the chosen axes.
4. Choose a reference point about which to compute torques. Choose wisely; you can eliminate from your torque equation any force whose line of action goes through the point you choose. The object doesn't actually have to be pivoted about an axis through the reference point.

**EXECUTE** *the solution* as follows:

1. Write equations expressing the equilibrium conditions. Remember that $\sum F_x = 0$, $\sum F_y = 0$, and $\sum \tau_z = 0$ are *separate* equations. You can compute the torque of a force by finding the torque of each of its components separately, each with its appropriate lever arm and sign, and adding the results.
2. To obtain as many equations as you have unknowns, you may need to compute torques with respect to two or more reference points; choose them wisely, too.

**EVALUATE** *your answer:* Check your results by writing $\sum \tau_z = 0$ with respect to a different reference point. You should get the same answers.

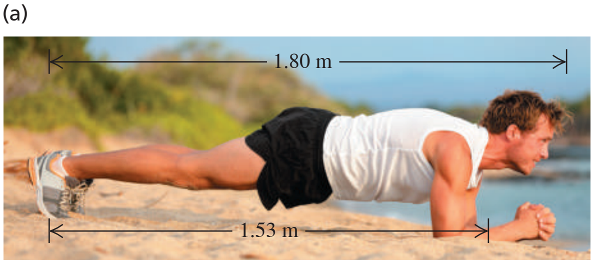
## EXAMPLE 11.2 Locating your center of gravity while you work out

WITH VARIATION PROBLEMS

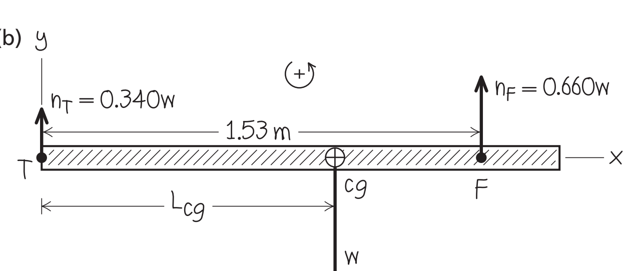
The *plank* (**Fig. 11.8a**) is a great way to strengthen abdominal, back, and shoulder muscles. You can also use this exercise position to locate your center of gravity. Holding plank position with a scale under his toes and another under his forearms, one athlete measured that 66.0% of his weight was supported by his forearms and 34.0% by his toes. (That is, the total normal forces on his forearms and toes were $0.660w$ and $0.340w$, respectively, where $w$ is the athlete's weight.) He is 1.80 m tall, and in plank position the distance from his toes to the middle of his forearms is 1.53 m. How far from his toes is his center of gravity?

Figure **11.8** An athlete in plank position.

(a)




(b)




**IDENTIFY and SET UP** We can use the two conditions for equilibrium, Eqs. (11.6), for an athlete at rest. So both the net force and net torque on the athlete are zero. Figure 11.8b shows a free-body diagram, including $x$- and $y$-axes and our convention that counterclockwise torques are positive. The weight $w$ acts at the center of gravity, which is between the two supports (as it must be; see Section 11.2). Our target variable is the distance $L_{cg}$, the lever arm of the weight with respect to the toes $T$, so it is wise to take torques with respect to $T$. The torque due to the weight is negative (it tends to cause a clockwise rotation around $T$), and the torque due to the upward normal force at the forearms $F$ is positive (it tends to cause a counterclockwise rotation around $T$).

**EXECUTE** The first condition for equilibrium is satisfied (Fig. 11.8b): $\sum F_x = 0$ because there are no $x$-components and $\sum F_y = 0$ because $0.340w + 0.660w + (-w) = 0$. We write the torque equation and solve for $L_{cg}$:

$$\sum \tau_z = 0.340w(0) - wL_{cg} + 0.660w(1.53\text{ m}) = 0$$
$$L_{cg} = 1.01\text{ m}$$

**EVALUATE** The center of gravity is slightly below our athlete's navel (as it is for most people), closer to his head than to his toes. It's also closer to his forearms than to his toes, which is why his forearms support most of his weight. You can check our result by writing the torque equation about the forearms $F$. You'll find that his center of gravity is 0.52 m from his forearms, or $(1.53\text{ m}) - (0.52\text{ m}) = 1.01\text{ m}$ from his toes.

**KEYCONCEPT** For an extended object to be in equilibrium, both the net external *force* and the net external *torque* on the object must be zero. The weight of the object acts at its center of gravity.

## EXAMPLE 11.3 Will the ladder slip?

WITH VARIATION PROBLEMS

Sir Lancelot, who weighs 800 N, is assaulting a castle by climbing a uniform ladder that is 5.0 m long and weighs 180 N (**Fig. 11.9a**). The bottom of the ladder rests on a ledge and leans across the moat in equilibrium against a frictionless, vertical castle wall. The ladder makes an angle of 53.1° with the horizontal. Lancelot pauses one-third of the way up the ladder. (a) Find the normal and friction forces on the base of the ladder. (b) Find the minimum coefficient of static friction needed to prevent slipping at the base. (c) Find the magnitude and direction of the contact force on the base of the ladder.

**IDENTIFY and SET UP** The ladder–Lancelot system is stationary, so we can use the two conditions for equilibrium to solve part (a). In part (b), we need the relationship among the static friction force, coefficient of static friction, and normal force (see Section 5.3). In part (c), the contact force is the vector sum of the normal and friction forces acting at the base of the ladder, found in part (a). Figure 11.9b shows the free-body diagram, with $x$- and $y$-directions as shown and with counterclockwise torques taken to be positive. The ladder's center of gravity is at its geometric center. Lancelot's 800 N weight acts at a point one-third of the way up the ladder.

The wall exerts only a normal force $n_1$ on the top of the ladder. The forces on the base are an upward normal force $n_2$ and a static friction force $f_s$, which must point to the right to prevent slipping. The magnitudes $n_2$ and $f_s$ are the target variables in part (a). From Eq. (5.4), these magnitudes are related by $f_s \le \mu_s n_2$; the coefficient of static friction $\mu_s$ is the target variable in part (b).

**EXECUTE** (a) From Eqs. (11.6), the first condition for equilibrium gives

$$\sum F_x = f_s + (-n_1) = 0$$
$$\sum F_y = n_2 + (-800\text{ N}) + (-180\text{ N}) = 0$$

These are two equations for the three unknowns $n_1$, $n_2$, and $f_s$. The second equation gives $n_2 = 980$ N. To obtain a third equation, we use the second condition for equilibrium. We take torques about point $B$, about which $n_2$ and $f_s$ have no torque. The 53.1° angle creates a 3-4-5 right triangle, so from Fig. 11.9b the lever arm for the ladder's weight is 1.5 m, the lever arm for Lancelot's weight is 1.0 m, and the lever arm for $n_1$ is 4.0 m. The torque equation for point $B$ is then

$$\sum \tau_B = n_1(4.0\text{ m}) - (180\text{ N})(1.5\text{ m}) - (800\text{ N})(1.0\text{ m}) + n_2(0) + f_s(0) = 0$$

Solving for $n_1$, we get $n_1 = 268$ N. We substitute this into the $\sum F_x = 0$ equation and get $f_s = 268$ N.

(b) The static friction force $f_s$ cannot exceed $\mu_s n_2$, so the *minimum* coefficient of static friction to prevent slipping is

$$(\mu_s)_{\min} = \frac{f_s}{n_2} = \frac{268\text{ N}}{980\text{ N}} = 0.27$$

(c) The components of the contact force $\vec{F}_B$ at the base are the static friction force $f_s$ and the normal force $n_2$, so

$$\vec{F}_B = f_s\hat{\imath} + n_2\hat{\jmath} = (268\text{ N})\hat{\imath} + (980\text{ N})\hat{\jmath}$$

The magnitude and direction of $\vec{F}_B$ (Fig. 11.9c) are

$$F_B = \sqrt{(268\text{ N})^2 + (980\text{ N})^2} = 1020\text{ N}$$
$$\theta = \arctan\frac{980\text{ N}}{268\text{ N}} = 75°$$

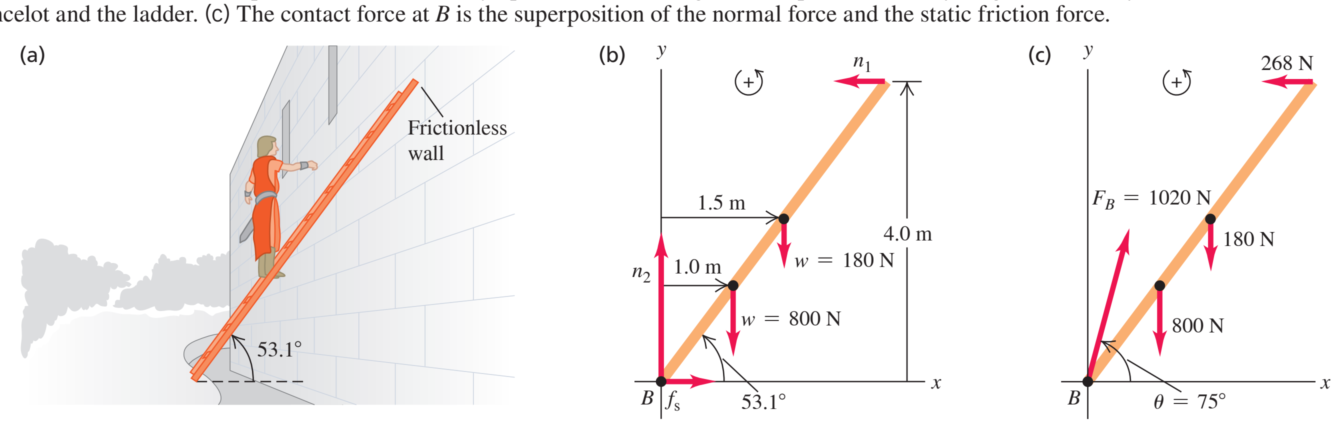
**EVALUATE** As Fig. 11.9c shows, the contact force $\vec{F}_B$ is *not* directed along the length of the ladder. Can you show that if $\vec{F}_B$ were directed along the ladder, there would be a net counterclockwise torque with respect to the top of the ladder, and equilibrium would be impossible?

As Lancelot climbs higher on the ladder, the lever arm and torque of his weight about $B$ increase. This increases the values of $n_1$, $f_s$, and the required friction coefficient $(\mu_s)_{\min}$, so the ladder is more and more likely to slip as he climbs (see Exercise 11.10). A simple way to make slipping less likely is to use a larger ladder angle (say, 75° rather than 53.1°). This decreases the lever arms with respect to $B$ of the weights of the ladder and Lancelot and increases the lever arm of $n_1$, all of which decrease the required friction force.

If we had assumed friction on the wall as well as on the floor, the problem would be impossible to solve by using the equilibrium conditions alone. (Try it!) The difficulty is that it's no longer adequate to treat the object as being perfectly rigid. Another problem of this kind is a four-legged table; there's no way to use the equilibrium conditions alone to find the force on each separate leg.

**KEYCONCEPT** In an equilibrium problem, you can calculate torques around any point you choose. The torque equation will not include any force whose line of action goes through your chosen point, so choosing the point wisely can simplify your calculations.

Figure 11.9 (a) Sir Lancelot pauses a third of the way up the ladder, fearing it will slip. (b) Free-body diagram for the system of Sir Lancelot and the ladder. (c) The contact force at $B$ is the superposition of the normal force and the static friction force.



## EXAMPLE 11.4 Equilibrium and pumping iron

WITH VARIATION PROBLEMS

**Figure 11.10a** shows a horizontal human arm lifting a dumbbell. The forearm is in equilibrium under the action of the weight $\vec{w}$ of the dumbbell, the tension $\vec{T}$ in the tendon connected to the biceps muscle, and the force $\vec{E}$ exerted on the forearm by the upper arm at the elbow joint. We ignore the weight of the forearm itself. (For clarity, in the drawing we've exaggerated the distance from the elbow to the point $A$ where the tendon is attached.) Given the weight $w$ and the angle $\theta$ between the tension force and the horizontal, find $T$ and the two components of $\vec{E}$ (three unknown scalar quantities in all).

**IDENTIFY and SET UP** The system is at rest, so we use the conditions for equilibrium. We represent $\vec{T}$ and $\vec{E}$ in terms of their components (Fig. 11.10b). We guess that the directions of $E_x$ and $E_y$ are as shown; the signs of $E_x$ and $E_y$ as given by our solution will tell us the actual directions. Our target variables are $T$, $E_x$, and $E_y$.

**EXECUTE** To find $T$, we take torques about the elbow joint so that the torque equation does not contain $E_x$, $E_y$, or $T_x$, then solve for $T_y$ and hence $T$:

$$\sum \tau_{\text{elbow}} = Lw - DT_y = 0$$

$$T_y = \frac{Lw}{D} = T\sin\theta \qquad \text{and} \qquad T = \frac{Lw}{D\sin\theta}$$

To find $E_x$ and $E_y$, we use the first conditions for equilibrium:

$$\sum F_x = T_x + (-E_x) = 0$$

$$E_x = T_x = T\cos\theta = \frac{Lw}{D\sin\theta}\cos\theta = \frac{Lw}{D}\cot\theta = \frac{Lw}{D}\frac{D}{h} = \frac{Lw}{h}$$

$$\sum F_y = T_y + E_y + (-w) = 0$$

$$E_y = w - \frac{Lw}{D} = -\frac{(L - D)w}{D}$$

The negative sign for $E_y$ tells us that it should actually point *down* in Fig. 11.10b.

**EVALUATE** We can check our results for $E_x$ and $E_y$ by taking torques about points $A$ and $B$, about both of which $T$ has zero torque:

$$\sum \tau_A = (L - D)w + DE_y = 0 \qquad \text{so} \qquad E_y = -\frac{(L - D)w}{D}$$

$$\sum \tau_B = Lw - hE_x = 0 \qquad \text{so} \qquad E_x = \frac{Lw}{h}$$

As a realistic example, take $w = 200$ N, $D = 0.050$ m, $L = 0.30$ m, and $\theta = 80°$, so that $h = D\tan\theta = (0.050\text{ m})(5.67) = 0.28$ m. Using our results for $T$, $E_x$, and $E_y$, we find

$$T = \frac{Lw}{D\sin\theta} = \frac{(0.30\text{ m})(200\text{ N})}{(0.050\text{ m})(0.98)} = 1220\text{ N}$$

$$E_y = -\frac{(L - D)w}{D} = -\frac{(0.30\text{ m} - 0.050\text{ m})(200\text{ N})}{0.050\text{ m}} = -1000\text{ N}$$

$$E_x = \frac{Lw}{h} = \frac{(0.30\text{ m})(200\text{ N})}{0.28\text{ m}} = 210\text{ N}$$

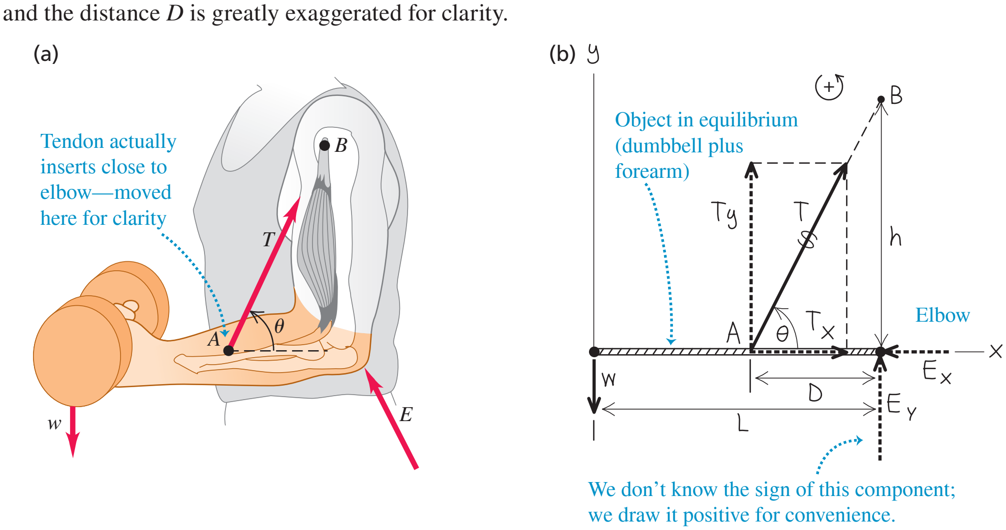
The magnitude of the force at the elbow is

$$E = \sqrt{E_x^2 + E_y^2} = 1020\text{ N}$$

Note that $T$ and $E$ are *much* larger than the 200 N weight of the dumbbell. A forearm weighs only about 20 N, so it was reasonable to ignore its weight.

**KEYCONCEPT** In a two-dimensional equilibrium problem, you have *three* equilibrium equations: two from the condition that the net external force is zero and one from the condition that the net external torque is zero. Use these equations to solve for the unknowns.

Figure 11.10 (a) The situation. (b) Our free-body diagram for the forearm. The weight of the forearm is ignored, and the distance $D$ is greatly exaggerated for clarity.



**TEST YOUR UNDERSTANDING OF SECTION 11.3** A metal advertising sign (weight $w$) for a specialty shop is suspended from the end of a horizontal rod of length $L$ and negligible mass (**Fig. 11.11**). The rod is supported by a cable at an angle $\theta$ from the horizontal and by a hinge at point $P$. Rank the following force magnitudes in order from greatest to smallest: (i) the weight $w$ of the sign; (ii) the tension in the cable; (iii) the vertical component of force exerted on the rod by the hinge at $P$.

**ANSWER**

❙ **(ii), (i), (iii)** This is the same situation described in Example 11.4, with the rod replacing the forearm, the hinge replacing the elbow, and the cable replacing the tendon. The only difference is that the cable attachment point is at the end of the rod, so the distances $D$ and $L$ are identical. From Example 11.4, the tension is

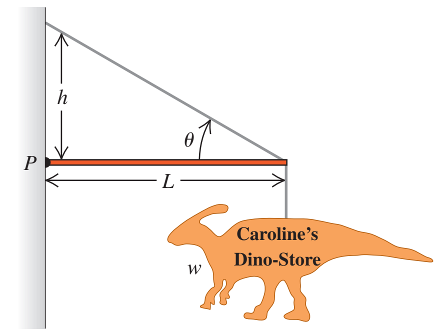
$$T = \frac{Lw}{L\sin\theta} = \frac{w}{\sin\theta}$$

Since $\sin\theta$ is less than 1, the tension $T$ is greater than the weight $w$. The vertical component of the force exerted by the hinge is

$$E_y = -\frac{(L - L)w}{L} = 0$$

In this situation, the hinge exerts *no* vertical force. To see this, calculate torques around the right end of the horizontal rod: The only force that exerts a torque around this point is the vertical component of the hinge force, so this force component must be zero.

Figure **11.11** What are the tension in the diagonal cable and the vertical component of force exerted by the hinge at $P$?




## 11.4 STRESS, STRAIN, AND ELASTIC MODULI

The rigid body is a useful idealized model, but the stretching, squeezing, and twisting of real objects when forces are applied are often too important to ignore. **Figure 11.12** shows three examples. We want to study the relationship between the forces and deformations for each case.

You don't have to look far to find a deformable object; it's as plain as the nose on your face (**Fig. 11.13**). If you grasp the tip of your nose between your index finger and thumb, you'll find that the harder you pull your nose outward or push it inward, the more it stretches or compresses. Likewise, the harder you squeeze your index finger and thumb together, the more the tip of your nose compresses. If you try to twist the tip of your nose, you'll get a greater amount of twist if you apply stronger forces.

These observations illustrate a general rule. In each case you apply a **stress** to your nose; the amount of stress is a measure of the forces causing the deformation, on a "force per unit area" basis. And in each case the stress causes a deformation, or **strain.** More careful versions of the experiments with your nose suggest that for relatively small stresses, the resulting strain is proportional to the stress: The greater the deforming forces, the greater the resulting deformation. This proportionality is called **Hooke's law,** and the ratio of stress to strain is called the **elastic modulus:**

Measure of forces applied to deform an object

**Hooke's law:** $$\frac{\text{Stress}}{\text{Strain}} = \text{Elastic modulus} \qquad (11.7)$$

Property of material of which object is made

Measure of how much deformation results from stress

Figure **11.12** Three types of stress. (a) Guitar strings under *tensile stress,* being stretched by forces acting at their ends. (b) A diver under *bulk stress,* being squeezed from all sides by forces due to water pressure. (c) A ribbon under *shear stress,* being deformed and eventually cut by forces exerted by the scissors.

(a)

(b)

(c)

Figure **11.13** When you pinch your nose, the force per area that you apply to your nose is called *stress*. The fractional change in the size of your nose (the change in size divided by the initial size) is called *strain*. The deformation is *elastic* because your nose springs back to its initial size when you stop pinching.

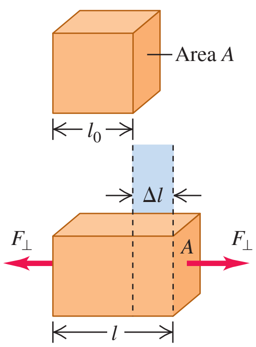
The value of the elastic modulus depends on what the object is made of but not its shape or size. If a material returns to its original state after the stress is removed, it is called **elastic;** Hooke's law is a special case of elastic behavior. If a material instead remains deformed after the stress is removed, it is called **plastic.** Here we'll consider elastic behavior only; we'll return to plastic behavior in Section 11.5.

We used one form of Hooke's law in Section 6.3: The elongation of an ideal spring is proportional to the stretching force. Remember that Hooke's "law" is not really a general law; it is valid over only a limited range of stresses. In Section 11.5 we'll see what happens beyond that limited range.

## Tensile and Compressive Stress and Strain

The simplest elastic behavior to understand is the stretching of a bar, rod, or wire when its ends are pulled (Fig. 11.12a). **Figure 11.14** shows an object that initially has uniform cross-sectional area $A$ and length $l_0$. We then apply forces of equal magnitude $F_\perp$ but opposite directions at the ends (this ensures that the object has no tendency to move left or right). We say that the object is in **tension.** We've already talked a lot about tension in ropes and strings; it's the same concept here. The subscript $\perp$ is a reminder that the forces act perpendicular to the cross section.

Figure 11.14 An object in tension. The net force on the object is zero, but the object deforms. The tensile stress (the ratio of the force to the cross-sectional area) produces a tensile strain (the elongation divided by the initial length). The elongation $\Delta l$ is exaggerated for clarity.




Tensile stress $= \dfrac{F_\perp}{A}$ Tensile strain $= \dfrac{\Delta l}{l_0}$

We define the **tensile stress** at the cross section as the ratio of the force $F_\perp$ to the cross-sectional area $A$:

$$\text{Tensile stress} = \frac{F_\perp}{A} \tag{11.8}$$

This is a *scalar* quantity because $F_\perp$ is the *magnitude* of the force. The SI unit of stress is the **pascal** (abbreviated Pa and named for the 17th-century French scientist and philosopher Blaise Pascal). Equation (11.8) shows that 1 pascal equals 1 newton per square meter ($\text{N/m}^2$):

$$1 \text{ pascal} = 1 \text{ Pa} = 1 \text{ N/m}^2$$

In the British system the most common unit of stress is the pound per square inch ($\text{lb/in.}^2$ or psi). The conversion factors are

$$1 \text{ psi} = 6895 \text{ Pa} \qquad \text{and} \qquad 1 \text{ Pa} = 1.450 \times 10^{-4} \text{ psi}$$

The units of stress are the same as those of *pressure,* which we'll encounter often in later chapters.

Under tension the object in Fig. 11.14 stretches to a length $l = l_0 + \Delta l$. The elongation $\Delta l$ does not occur only at the ends; every part of the object stretches in the same proportion. The **tensile strain** of the object equals the fractional change in length, which is the ratio of the elongation $\Delta l$ to the original length $l_0$:

$$\text{Tensile strain} = \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0} \tag{11.9}$$

Tensile strain is stretch per unit length. It is a ratio of two lengths, always measured in the same units, and so is a pure (dimensionless) number with no units.

Experiment shows that for a sufficiently small tensile stress, stress and strain are proportional, as in Eq. (11.7). The corresponding elastic modulus is called **Young's modulus,** denoted by $Y$:

Force applied perpendicular to cross section

**Young's modulus** for tension

Original length (see Fig. 11.14)

$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F_\perp/A}{\Delta l/l_0} = \frac{F_\perp}{A}\frac{l_0}{\Delta l} \tag{11.10}$$

Cross-sectional area of object

Elongation (see Fig. 11.14)

**BIO APPLICATION Young's Modulus of a Tendon** The anterior tibial tendon connects your foot to the large muscle that runs along the side of your shinbone. (You can feel this tendon at the front of your ankle.) Measurements show that this tendon has a Young's modulus of $1.2 \times 10^9$ Pa, much less than for the metals listed in Table 11.1. Hence this tendon stretches substantially (up to 2.5% of its length) in response to the stresses experienced in walking and running.

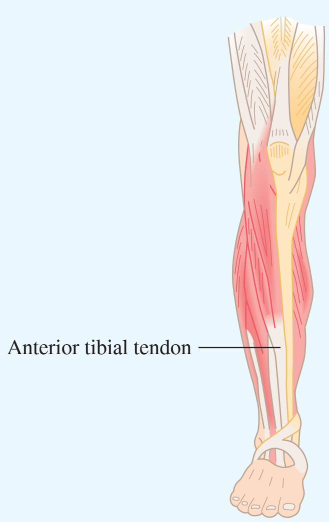


**TABLE 11.1** Approximate Elastic Moduli

| Material | Young's Modulus, $Y$ (Pa) | Bulk Modulus, $B$ (Pa) | Shear Modulus, $S$ (Pa) |
|---|---|---|---|
| Aluminum | $7.0 \times 10^{10}$ | $7.5 \times 10^{10}$ | $2.5 \times 10^{10}$ |
| Brass | $9.0 \times 10^{10}$ | $6.0 \times 10^{10}$ | $3.5 \times 10^{10}$ |
| Copper | $11 \times 10^{10}$ | $14 \times 10^{10}$ | $4.4 \times 10^{10}$ |
| Iron | $21 \times 10^{10}$ | $16 \times 10^{10}$ | $7.7 \times 10^{10}$ |
| Lead | $1.6 \times 10^{10}$ | $4.1 \times 10^{10}$ | $0.6 \times 10^{10}$ |
| Nickel | $21 \times 10^{10}$ | $17 \times 10^{10}$ | $7.8 \times 10^{10}$ |
| Silicone rubber | $0.001 \times 10^{10}$ | $0.2 \times 10^{10}$ | $0.0002 \times 10^{10}$ |
| Steel | $20 \times 10^{10}$ | $16 \times 10^{10}$ | $7.5 \times 10^{10}$ |
| Tendon (typical) | $0.12 \times 10^{10}$ | — | — |

Since strain is a pure number, the units of Young's modulus are the same as those of stress: force per unit area. **Table 11.1** lists some typical values. (This table also gives values of two other elastic moduli that we'll discuss later in this chapter.) A material with a large value of $Y$ is relatively unstretchable; a large stress is required for a given strain. For example, the value of $Y$ for cast steel $(2 \times 10^{11}\ \text{Pa})$ is much larger than that for a tendon $(1.2 \times 10^{9}\ \text{Pa})$.

When the forces on the ends of a bar are pushes rather than pulls (**Fig. 11.15**), the bar is in **compression** and the stress is a **compressive stress.** The **compressive strain** of an object in compression is defined in the same way as the tensile strain, but $\Delta l$ has the opposite direction. Hooke's law and Eq. (11.10) are valid for compression as well as tension if the compressive stress is not too great. For many materials, Young's modulus has the same value for both tensile and compressive stresses. Composite materials such as concrete and stone are an exception; they can withstand compressive stresses but fail under comparable tensile stresses. Stone was the primary building material used by ancient civilizations such as the Babylonians, Assyrians, and Romans, so their structures had to be designed to avoid tensile stresses. Hence they used arches in doorways and bridges, where the weight of the overlying material compresses the stones of the arch together and does not place them under tension.

In many situations, objects can experience both tensile and compressive stresses at the same time. For example, a horizontal beam supported at each end sags under its own weight. As a result, the top of the beam is under compression while the bottom of the beam is under tension (**Fig. 11.16a**). To minimize the stress and hence the bending strain, the top and bottom of the beam are given a large cross-sectional area. There is neither compression nor tension along the centerline of the beam, so this part can have a small cross section; this helps keep the weight of the beam to a minimum and further helps reduce the stress. The result is an I-beam of the familiar shape used in building construction (Fig. 11.16b).

Figure **11.15** An object in compression. The compressive stress and compressive strain are defined in the same way as tensile stress and strain (see Fig. 11.14), except that $\Delta l$ now denotes the distance that the object contracts.

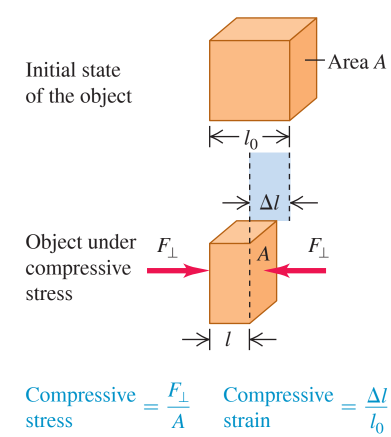
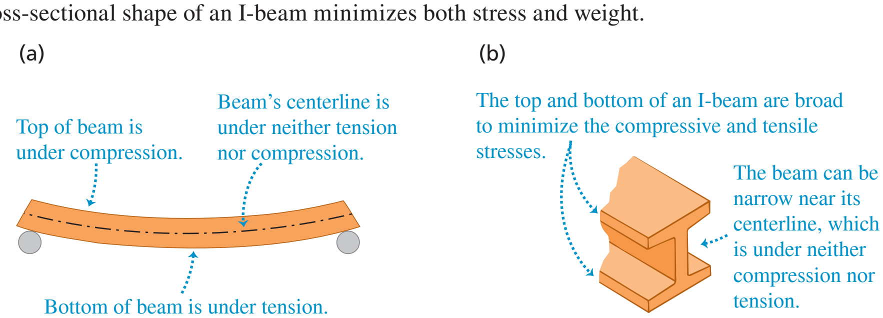



Figure **11.16** (a) A beam supported at both ends is under both compression and tension. (b) The cross-sectional shape of an I-beam minimizes both stress and weight.



### EXAMPLE 11.5 Tensile stress and strain

WITH VARIATION PROBLEMS

A steel rod 2.0 m long has a cross-sectional area of 0.30 $cm^2$. It is hung by one end from a support, and a 550 kg milling machine is hung from its other end. Determine the stress on the rod and the resulting strain and elongation.

**IDENTIFY, SET UP, and EXECUTE** The rod is under tension, so we can use Eq. (11.8) to find the tensile stress; Eq. (11.9), with the value of Young's modulus $Y$ for steel from Table 11.1, to find the corresponding strain; and Eq. (11.10) to find the elongation $\Delta l$:

$$\text{Tensile stress} = \frac{F_\perp}{A} = \frac{(550\text{ kg})(9.8\text{ m/s}^2)}{3.0 \times 10^{-5}\text{ m}^2} = 1.8 \times 10^8\text{ Pa}$$

$$\text{Strain} = \frac{\Delta l}{l_0} = \frac{\text{Stress}}{Y} = \frac{1.8 \times 10^8\text{ Pa}}{20 \times 10^{10}\text{ Pa}} = 9.0 \times 10^{-4}$$

$$\text{Elongation} = \Delta l = (\text{Strain}) \times l_0$$
$$= (9.0 \times 10^{-4})(2.0\text{ m}) = 0.0018\text{ m} = 1.8\text{ mm}$$

**EVALUATE** This small elongation, resulting from a load of over half a ton, is a testament to the stiffness of steel. (We've ignored the relatively small stress due to the weight of the rod itself.)

**KEYCONCEPT** For an object under tensile or compressive stress, the stress equals the force exerted on either end of the object divided by the cross-sectional area of either end. The strain equals the fractional change in length. The ratio of stress to strain equals Young's modulus for the material of which the object is made.

## Bulk Stress and Strain

When a scuba diver plunges deep into the ocean, the water exerts nearly uniform pressure everywhere on his surface and squeezes him to a slightly smaller volume (see Fig. 11.12b). This is a different situation from the tensile and compressive stresses and strains we have discussed. The uniform pressure on all sides of the diver is a **bulk stress** (or **volume stress**), and the resulting deformation—a **bulk strain** (or **volume strain**)—is a change in his volume.

If an object is immersed in a fluid (liquid or gas) at rest, the fluid exerts a force on any part of the object's surface; this force is *perpendicular* to the surface. (If we tried to make the fluid exert a force parallel to the surface, the fluid would slip sideways to counteract the effort.) The force $F_\perp$ per unit area that the fluid exerts on an immersed object is called the **pressure** $p$ in the fluid:

**CAUTION** **Pressure vs. force** Unlike force, pressure has no intrinsic direction: The pressure on the surface of an immersed object is the same no matter how the surface is oriented. Hence pressure is a *scalar* quantity, not a vector quantity.

$$\underbrace{p}_{\text{Pressure in a fluid}} = \frac{\overbrace{F_\perp}^{\text{Force that fluid applies to surface of an immersed object}}}{\underbrace{A}_{\text{Area over which force is exerted}}} \qquad (11.11)$$

Pressure has the same units as stress; commonly used units include 1 Pa ($= 1\text{ N/m}^2$), 1 lb/in.$^2$ (1 psi), and 1 **atmosphere** (1 atm). One atmosphere is the approximate average pressure of the earth's atmosphere at sea level:

$$1\text{ atmosphere} = 1\text{ atm} = 1.013 \times 10^5\text{ Pa} = 14.7\text{ lb/in.}^2$$

The pressure in a fluid increases with depth. For example, the pressure in the ocean increases by about 1 atm every 10 m. If an immersed object is relatively small, however, we can ignore these pressure differences for purposes of calculating bulk stress. We'll then treat the pressure as having the same value at all points on an immersed object's surface.

Pressure plays the role of stress in a volume deformation. The corresponding strain is the fractional change in volume (**Fig. 11.17**)—that is, the ratio of the volume change $\Delta V$ to the original volume $V_0$:

$$\text{Bulk (volume) strain} = \frac{\Delta V}{V_0} \qquad (11.12)$$

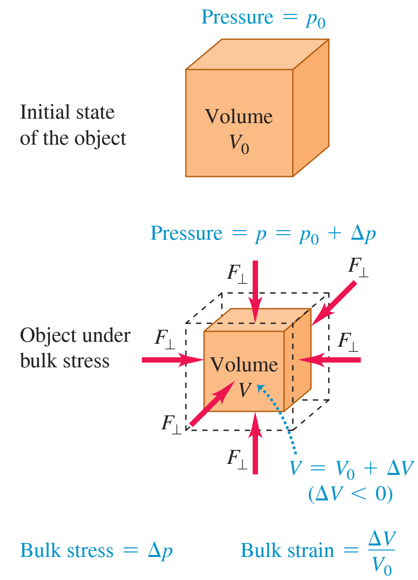
Volume strain is the change in volume per unit volume. Like tensile or compressive strain, it is a pure number, without units.

**BIO APPLICATION Bulk Stress on an Anglerfish** The anglerfish (*Melanocetus johnsonii*) is found in oceans throughout the world at depths as great as 1000 m, where the pressure (that is, the bulk stress) is about 100 atmospheres. Anglerfish are able to withstand such stress because they have no internal air spaces, unlike fish found in the upper ocean, where pressures are lower. The largest anglerfish are about 12 cm (5 in.) long.

When Hooke's law is obeyed, an increase in pressure (bulk stress) produces a *proportional* bulk strain (fractional change in volume). The corresponding elastic modulus (ratio of stress to strain) is called the **bulk modulus,** denoted by $B$. When the pressure on an object changes by a small amount $\Delta p$, from $p_0$ to $p_0 + \Delta p$, and the resulting bulk strain is $\Delta V/V_0$, Hooke's law takes the form

$$\underset{\text{Bulk modulus for compression}}{B} = \frac{\text{Bulk stress}}{\text{Bulk strain}} = -\frac{\overset{\text{Additional pressure on object}}{\Delta p}}{\underset{\text{Change in volume (see Fig. 11.17)}}{\Delta V}/\underset{\text{Original volume (see Fig. 11.17)}}{V_0}} \qquad (11.13)$$

Figure **11.17** An object under bulk stress. Without the stress, the cube has volume $V_0$; when the stress is applied, the cube has a smaller volume $V$. The volume change $\Delta V$ is exaggerated for clarity.




We include a minus sign in this equation because an *increase* of pressure always causes a *decrease* in volume. In other words, if $\Delta p$ is positive, $\Delta V$ is negative. The bulk modulus $B$ itself is a positive quantity.

For small pressure changes in a solid or a liquid, we consider $B$ to be constant. The bulk modulus of a *gas,* however, depends on the initial pressure $p_0$. Table 11.1 includes values of $B$ for several solid materials. Its units, force per unit area, are the same as those of pressure (and of tensile or compressive stress).

The reciprocal of the bulk modulus is called the **compressibility** and is denoted by $k$. From Eq. (11.13),

$$k = \frac{1}{B} = -\frac{\Delta V/V_0}{\Delta p} = -\frac{1}{V_0}\frac{\Delta V}{\Delta p} \qquad \text{(compressibility)} \qquad (11.14)$$

Compressibility is the fractional decrease in volume, $-\Delta V/V_0$, per unit increase $\Delta p$ in pressure. The units of compressibility are those of *reciprocal pressure,* $\text{Pa}^{-1}$ or $\text{atm}^{-1}$.

**Table 11.2** lists the values of compressibility $k$ for several liquids. For example, the compressibility of water is $46.4 \times 10^{-6}\ \text{atm}^{-1}$, which means that the volume of water decreases by 46.4 parts per million for each 1 atmosphere increase in pressure. Materials with small bulk modulus $B$ and large compressibility $k$ are easiest to compress.

**TABLE 11.2** Compressibilities of Liquids

| | **Compressibility, $k$** | |
|---|---|---|
| **Liquid** | **$\text{Pa}^{-1}$** | **$\text{atm}^{-1}$** |
| Carbon disulfide | $93 \times 10^{-11}$ | $94 \times 10^{-6}$ |
| Ethyl alcohol | $110 \times 10^{-11}$ | $111 \times 10^{-6}$ |
| Glycerin | $21 \times 10^{-11}$ | $21 \times 10^{-6}$ |
| Mercury | $3.7 \times 10^{-11}$ | $3.8 \times 10^{-6}$ |
| Water | $45.8 \times 10^{-11}$ | $46.4 \times 10^{-6}$ |

## EXAMPLE 11.6 Bulk stress and strain

WITH VARIATION PROBLEMS

A hydraulic press contains $0.25\ \text{m}^3$ (250 L) of oil. Find the decrease in the volume of the oil when it is subjected to a pressure increase $\Delta p = 1.6 \times 10^7$ Pa (about 160 atm or 2300 psi). The bulk modulus of the oil is $B = 5.0 \times 10^9$ Pa (about $5.0 \times 10^4$ atm), and its compressibility is $k = 1/B = 20 \times 10^{-6}\ \text{atm}^{-1}$.

**IDENTIFY, SET UP, and EXECUTE** This example uses the ideas of bulk stress and strain. We are given both the bulk modulus and the compressibility, and our target variable is $\Delta V$. Solving Eq. (11.13) for $\Delta V$, we find

$$\Delta V = -\frac{V_0 \Delta p}{B} = -\frac{(0.25\ \text{m}^3)(1.6 \times 10^7\ \text{Pa})}{5.0 \times 10^9\ \text{Pa}}$$
$$= -8.0 \times 10^{-4}\ \text{m}^3 = -0.80\ \text{L}$$

Alternatively, we can use Eq. (11.14) with the approximate unit conversions given above:

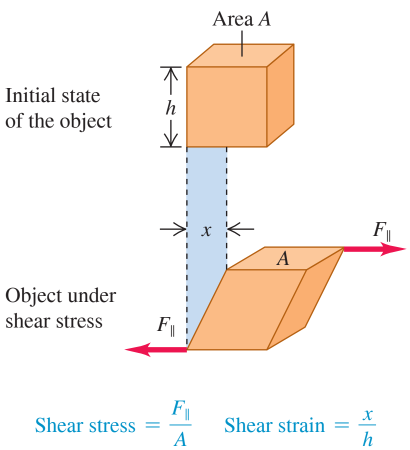
$$\Delta V = -kV_0\Delta p = -(20 \times 10^{-6}\ \text{atm}^{-1})(0.25\ \text{m}^3)(160\ \text{atm})$$
$$= -8.0 \times 10^{-4}\ \text{m}^3$$

**EVALUATE** The negative value of $\Delta V$ means that the volume decreases when the pressure increases. The 160 atm pressure increase is large, but the *fractional* volume change is very small:

$$\frac{\Delta V}{V_0} = \frac{-8.0 \times 10^{-4}\ \text{m}^3}{0.25\ \text{m}^3} = -0.0032 \qquad \text{or} \qquad -0.32\%$$

**KEYCONCEPT** For an object under bulk stress, the stress equals the additional pressure applied to all sides of the object. The strain equals the fractional change in volume. The ratio of stress to strain equals the bulk modulus for the material of which the object is made.

Figure **11.18** An object under shear stress. Forces are applied tangent to opposite surfaces of the object (in contrast to the situation in Fig. 11.14, in which the forces act perpendicular to the surfaces). The deformation $x$ is exaggerated for clarity.




## Shear Stress and Strain

The third kind of stress-strain situation is called *shear*. The ribbon in Fig. 11.12c is under **shear stress:** One part of the ribbon is being pushed up while an adjacent part is being pushed down, producing a deformation of the ribbon. **Figure 11.18** shows an object being deformed by a shear stress. In the figure, forces of equal magnitude but opposite direction act *tangent* to the surfaces of opposite ends of the object. We define the shear stress as the force $F_\parallel$ acting tangent to the surface divided by the area $A$ on which it acts:

$$\text{Shear stress} = \frac{F_\parallel}{A} \tag{11.15}$$

Shear stress, like the other two types of stress, is a force per unit area.

Figure 11.18 shows that one face of the object under shear stress is displaced by a distance $x$ relative to the opposite face. We define **shear strain** as the ratio of the displacement $x$ to the transverse dimension $h$:

$$\text{Shear strain} = \frac{x}{h} \tag{11.16}$$

In real-life situations, $x$ is typically much smaller than $h$. Like all strains, shear strain is a dimensionless number; it is a ratio of two lengths.

If the forces are small enough that Hooke's law is obeyed, the shear strain is *proportional* to the shear stress. The corresponding elastic modulus (ratio of shear stress to shear strain) is called the **shear modulus,** denoted by $S$:

Force applied tangent to surface of object

Transverse dimension (see Fig. 11.18)

**Shear modulus** for shear

$$S = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{F_\parallel/A}{x/h} = \frac{F_\parallel}{A}\frac{h}{x} \tag{11.17}$$

Area over which force is exerted

Deformation (see Fig. 11.18)

Table 11.1 gives several values of shear modulus. For a given material, $S$ is usually one-third to one-half as large as Young's modulus $Y$ for tensile stress. Keep in mind that the concepts of shear stress, shear strain, and shear modulus apply to *solid* materials only. The reason is that *shear* refers to deforming an object that has a definite shape (see Fig. 11.18). This concept doesn't apply to gases and liquids, which do not have definite shapes.

### EXAMPLE 11.7 Shear stress and strain

WITH VARIATION PROBLEMS

Suppose the object in Fig. 11.18 is the brass base plate of an outdoor sculpture that experiences shear forces in an earthquake. The plate is 0.80 m square and 0.50 cm thick. What is the force exerted on each of its edges if the resulting displacement $x$ is 0.16 mm?

**IDENTIFY and SET UP** This example uses the relationship among shear stress, shear strain, and shear modulus. Our target variable is the force $F_\parallel$ exerted parallel to each edge, as shown in Fig. 11.18. We'll find the shear strain from Eq. (11.16), the shear stress from Eq. (11.17), and $F_\parallel$ from Eq. (11.15). Table 11.1 gives the shear modulus of brass. In Fig. 11.18, $h$ represents the 0.80 m length of each side of the plate. The area $A$ in Eq. (11.15) is the product of the 0.80 m length and the 0.50 cm thickness.

**EXECUTE** From Eq. (11.16),

$$\text{Shear strain} = \frac{x}{h} = \frac{1.6 \times 10^{-4}\ \text{m}}{0.80\ \text{m}} = 2.0 \times 10^{-4}$$

From Eq. (11.17),

$$\begin{aligned}\text{Shear stress} &= (\text{Shear strain}) \times S \\ &= (2.0 \times 10^{-4})(3.5 \times 10^{10}\ \text{Pa}) = 7.0 \times 10^{6}\ \text{Pa}\end{aligned}$$

Finally, from Eq. (11.15),

$$\begin{aligned}F_\parallel &= (\text{Shear stress}) \times A \\ &= (7.0 \times 10^{6}\ \text{Pa})(0.80\ \text{m})(0.0050\ \text{m}) = 2.8 \times 10^{4}\ \text{N}\end{aligned}$$

**EVALUATE** The shear force supplied by the earthquake is more than 3 tons! The large shear modulus of brass makes it hard to deform. Further, the plate is relatively thick (0.50 cm), so the area $A$ is relatively large and a substantial force $F_\parallel$ is needed to provide the necessary stress $F_\parallel/A$.

**KEYCONCEPT** For an object under shear stress, the stress equals the force exerted tangent to either of two opposite surfaces of the object divided by the area of that surface. The strain equals the deformation divided by the distance between the two surfaces. The ratio of stress to strain equals the shear modulus for the material of which the object is made.

**TEST YOUR UNDERSTANDING OF SECTION 11.4** A copper rod of cross-sectional area $0.500\ \text{cm}^2$ and length 1.00 m is elongated by $2.00 \times 10^{-2}$ mm, and a steel rod of the same cross-sectional area but 0.100 m in length is elongated by $2.00 \times 10^{-3}$ mm. (a) Which rod has greater tensile *strain*? (i) The copper rod; (ii) the steel rod; (iii) the strain is the same for both. (b) Which rod is under greater tensile *stress*? (i) The copper rod; (ii) the steel rod; (iii) the stress is the same for both.

**ANSWER**

(a) **(iii)**, (b) **(ii)** In (a), the copper rod has 10 times the elongation $\Delta l$ of the steel rod, but it also has 10 times the original length $l_0$. Hence the tensile strain $\Delta l/l_0$ is the same for both rods. In (b), the stress is equal to Young's modulus $Y$ multiplied by the strain. From Table 11.1, steel has a larger value of $Y$, so a greater stress is required to produce the same strain.

## 11.5 ELASTICITY AND PLASTICITY

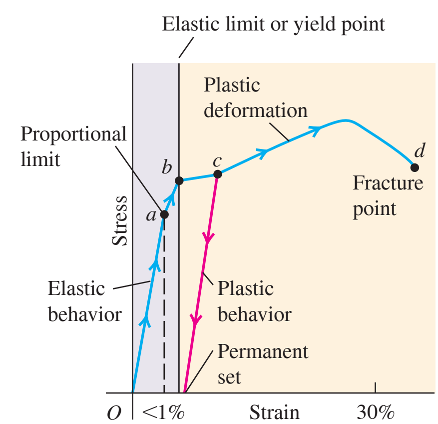
Hooke's law—the proportionality of stress and strain in elastic deformations—has a limited range of validity. In the preceding section we used phrases such as "if the forces are small enough that Hooke's law is obeyed." Just what *are* the limitations of Hooke's law? What's more, if you pull, squeeze, or twist *anything* hard enough, it will bend or break. Can we be more precise than that?

To address these questions, let's look at a graph of tensile stress as a function of tensile strain. **Figure 11.19** shows a typical graph of this kind for a metal such as copper or soft iron. The strain is shown as the *percent* elongation; the horizontal scale is not uniform beyond the first portion of the curve, up to a strain of less than 1%. The first portion is a straight line, indicating Hooke's law behavior with stress directly proportional to strain. This straight-line portion ends at point *a*; the stress at this point is called the *proportional limit.*

From *a* to *b*, stress and strain are no longer proportional, and Hooke's law is *not* obeyed. However, from *a* to *b* (and *O* to *a*), the behavior of the material is *elastic:* If the load is gradually removed starting at any point between *O* and *b*, the curve is retraced until the material returns to its original length. This elastic deformation is *reversible*.

Point *b*, the end of the elastic region, is called the *yield point;* the stress at the yield point is called the *elastic limit*. When we increase the stress beyond point *b*, the strain continues to increase. But if we remove the load at a point like *c* beyond the elastic limit, the material does *not* return to its original length. Instead, it follows the red line in Fig. 11.19. The material has deformed *irreversibly* and acquired a *permanent set*. This is the *plastic* behavior mentioned in Section 11.4.

Figure **11.19** Typical stress-strain diagram for a ductile metal under tension.

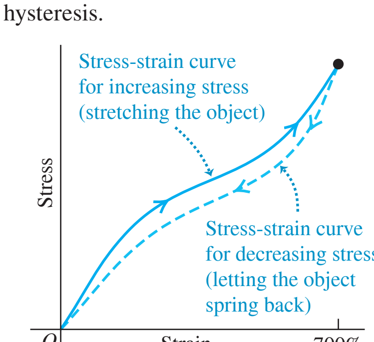



Once the material has become plastic, a small additional stress produces a relatively large increase in strain, until a point *d* is reached at which *fracture* takes place. That's what happens if a steel guitar string in Fig. 11.12a is tightened too much: The string breaks at the fracture point. Steel is *brittle* because it breaks soon after reaching its elastic limit; other materials, such as soft iron, are *ductile*—they can be given a large permanent stretch without breaking. (The material depicted in Fig. 11.19 is ductile, since it can stretch by more than 30% before breaking.)

Unlike uniform materials such as metals, stretchable biological materials such as tendons and ligaments have no true plastic region. That's because these materials are made of a collection of microscopic fibers; when stressed beyond the elastic limit, the fibers tear apart from each other. (A torn ligament or tendon is one that has fractured in this way.)

If a material is still within its elastic region, something very curious can happen when it is stretched and then allowed to relax. **Figure 11.20** is a stress-strain curve for vulcanized rubber that has been stretched by more than seven times its original length. The stress is not proportional to the strain, but the behavior is elastic because when the load is removed, the material returns to its original length. However, the material follows *different* curves for increasing and decreasing stress. This is called *elastic hysteresis*. The work done by the material when it returns to its original shape is less than the work required to deform it; that's due to internal friction. Rubber with large elastic hysteresis is very useful for absorbing vibrations, such as in engine mounts and shock-absorber bushings for cars. Tendons display similar behavior.

Figure **11.20** Typical stress-strain diagram for vulcanized rubber. The curves are different for increasing and decreasing stress, a phenomenon called elastic hysteresis.




The stress required to cause actual fracture of a material is called the *breaking stress,* the *ultimate strength,* or (for tensile stress) the *tensile strength*. Two materials, such as

**TABLE 11.3** Approximate Breaking Stresses

| Material | Breaking Stress (Pa or $N/m^2$) |
|---|---|
| Aluminum | $2.2 \times 10^8$ |
| Brass | $4.7 \times 10^8$ |
| Glass | $10 \times 10^8$ |
| Iron | $3.0 \times 10^8$ |
| Steel | $5\text{–}20 \times 10^8$ |
| Tendon (typical) | $1 \times 10^8$ |

two types of steel, may have very similar elastic constants but vastly different breaking stresses. **Table 11.3** gives typical values of breaking stress for several materials in tension. Comparing Tables 11.1 and 11.3 shows that iron and steel are comparably *stiff* (they have almost the same value of Young's modulus), but steel is *stronger* (it has a larger breaking stress than does iron).

**TEST YOUR UNDERSTANDING OF SECTION 11.5** While parking your car, you accidentally back into a steel post. You pull forward until the car no longer touches the post and then get out to inspect the damage. What does your rear bumper look like if the strain in the impact was (a) less than at the proportional limit; (b) greater than at the proportional limit but less than at the yield point; (c) greater than at the yield point but less than at the fracture point; and (d) greater than at the fracture point?

**ANSWER**

In (a) and (b), the bumper will have sprung back to its original shape (although the paint may be scratched). In (c), the bumper will have a permanent dent or deformation. In (d), the bumper will be torn or broken.

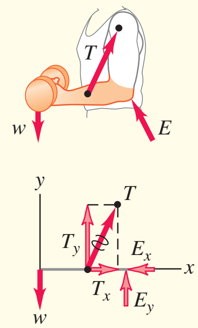
# CHAPTER 11 SUMMARY

**Conditions for equilibrium:** For a rigid body to be in equilibrium, two conditions must be satisfied. First, the vector sum of forces must be zero. Second, the sum of torques about any point must be zero. The torque due to the weight of an object can be found by assuming the entire weight is concentrated at the center of gravity, which is at the same point as the center of mass if $\vec{g}$ has the same value at all points. (See Examples 11.1–11.4.)

$$\sum \vec{F} = \mathbf{0} \tag{11.1}$$

$$\sum \vec{\tau} = \mathbf{0} \quad \text{about } any \text{ point} \tag{11.2}$$

$$\vec{r}_{\mathrm{cm}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \tag{11.4}$$

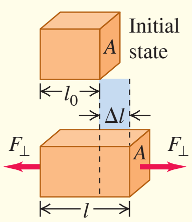



**Stress, strain, and Hooke's law:** Hooke's law states that in elastic deformations, stress (force per unit area) is proportional to strain (fractional deformation). The proportionality constant is called the elastic modulus.

$$\frac{\text{Stress}}{\text{Strain}} = \text{Elastic modulus} \tag{11.7}$$

**Tensile and compressive stress:** Tensile stress is tensile force per unit area, $F_\perp/A$. Tensile strain is fractional change in length, $\Delta l/l_0$. The elastic modulus for tension is called Young's modulus $Y$. Compressive stress and strain are defined in the same way. (See Example 11.5.)

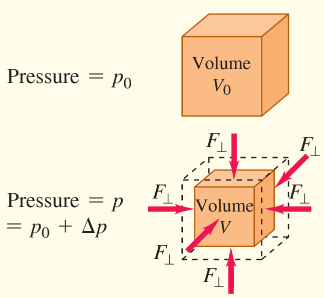
$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F_\perp/A}{\Delta l/l_0} = \frac{F_\perp}{A}\frac{l_0}{\Delta l} \tag{11.10}$$




**Bulk stress:** Pressure in a fluid is force per unit area. Bulk stress is pressure change, $\Delta p$, and bulk strain is fractional volume change, $\Delta V/V_0$. The elastic modulus for compression is called the bulk modulus, $B$. Compressibility, $k$, is the reciprocal of bulk modulus: $k = 1/B$. (See Example 11.6.)

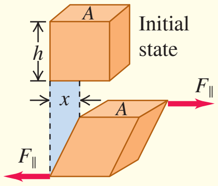
$$p = \frac{F_\perp}{A} \tag{11.11}$$

$$B = \frac{\text{Bulk stress}}{\text{Bulk strain}} = -\frac{\Delta p}{\Delta V/V_0} \tag{11.13}$$




**Shear stress:** Shear stress is force per unit area, $F_\parallel/A$, for a force applied tangent to a surface. Shear strain is the displacement $x$ of one side divided by the transverse dimension $h$. The elastic modulus for shear is called the shear modulus, $S$. (See Example 11.7.)

$$S = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{F_\parallel/A}{x/h} = \frac{F_\parallel}{A}\frac{h}{x} \tag{11.17}$$




**The limits of Hooke's law:** The proportional limit is the maximum stress for which stress and strain are proportional. Beyond the proportional limit, Hooke's law is not valid. The elastic limit is the stress beyond which irreversible deformation occurs. The breaking stress, or ultimate strength, is the stress at which the material breaks.

Chapter 11 Media Assets

## GUIDED PRACTICE

For assigned homework and other learning materials, go to Mastering Physics.

### KEY EXAMPLE VARIATION PROBLEMS

**Be sure to review EXAMPLE 11.1 (Section 11.2) before attempting these problems.**

**VP11.1.1** A uniform plank 8.00 m in length with mass 40.0 kg is supported at two points located 1.00 m and 5.00 m, respectively, from the left-hand end. What is the maximum additional mass you could place on the right-hand end of the plank and have the plank still be at rest?

**VP11.1.2** A bowling ball (which we can regard as a uniform sphere) has a mass of 7.26 kg and a radius of 0.216 m. A baseball has a mass of 0.145 kg. If you connect these two balls with a lightweight rod, what must be the distance between the center of the bowling ball and the center of the baseball so that the system of the two balls and the rod will balance at the point where the rod touches the surface of the bowling ball?

**VP11.1.3** Three small objects are arranged along a uniform rod of mass $m$ and length $L$: one of mass $m$ at the left end, one of mass $m$ at the center, and one of mass $2m$ at the right end. How far to the left or right of the rod's center should you place a support so that the rod with the attached objects will balance there?

**VP11.1.4** A small airplane with full fuel tanks, but no occupants or baggage, has a mass of $1.17 \times 10^3$ kg and a center of gravity that is 2.58 m behind the nose of the airplane. The pilot's seat is 2.67 m behind the nose, and the baggage compartment is 4.30 m behind the nose. A 75.0 kg pilot boards the plane and is the only occupant. If the center of gravity of the airplane with pilot can be no more than 2.76 m behind the nose for in-flight stability, what is the maximum mass that the baggage compartment can hold?

**Be sure to review EXAMPLES 11.2, 11.3, and 11.4 (Section 11.3) before attempting these problems.**

**VP11.4.1** The rear wheels of a truck support 57.0% of the weight of the truck, while the front wheels support 43.0% of the weight. The center of gravity of the truck is 2.52 m in front of the rear wheels. What is the wheelbase of the truck (the distance between the front and rear wheels)?

**VP11.4.2** A small airplane is sitting at rest on the ground. Its center of gravity is 2.58 m behind the nose of the airplane, the front wheel (nose wheel) is 0.800 m behind the nose, and the main wheels are 3.02 m behind the nose. What percentage of the airplane's weight is supported by the nose wheel, and what percentage is supported by the main wheels?

**VP11.4.3** Figure 11.11 shows a metal advertising sign of weight $w$ suspended from the end of a horizontal rod of negligible mass and length $L$. The end of the rod with the sign is supported by a cable at an angle $\theta$ from the horizontal, and the other end is supported by a hinge at point $P$. (a) Using the idea that there is zero net torque about the end of the rod with the attached sign, find the vertical component of the force $\vec{F}_{\text{hinge}}$ exerted by the hinge. (b) Using the idea that there is zero net vertical force on the rod with the attached sign, find the tension in the cable. (c) Using the idea that there is zero net horizontal force on the rod with the attached sign, find the horizontal component of the force exerted by the hinge.

**VP11.4.4** Suppose that in Figure 11.11 the horizontal rod has weight $w$ (the same as the hanging sign). (a) Using the idea that there is zero net torque about the end of the rod with the attached sign, find the vertical component of the force $\vec{F}_{\text{hinge}}$ exerted by the hinge. (b) Using the idea that there is zero net vertical force on the rod with the attached sign, find the tension in the cable. (c) Using the idea that there is zero net horizontal force on the rod with the attached sign, find the horizontal component of the force exerted by the hinge.

**Be sure to review EXAMPLES 11.5, 11.6, and 11.7 (Section 11.4) before attempting these problems.**

**VP11.7.1** A copper wire has a radius of 4.5 mm. When forces of a certain equal magnitude but opposite directions are applied to the ends of the wire, the wire stretches by $5.0 \times 10^{-3}$ of its original length. (a) What is the tensile stress on the wire? (b) What is the magnitude of the force on either end?

**VP11.7.2** An aluminum cylinder with a radius of 2.5 cm and a height of 82 cm is used as one leg of a workbench. The workbench pushes down on the cylinder with a force of $3.2 \times 10^4$ N. (a) What is the compressive strain of the cylinder? (b) By what distance does the cylinder's height decrease as a result of the forces on it?

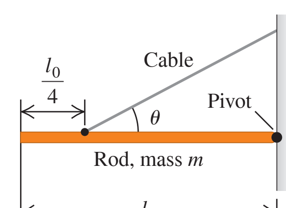
**VP11.7.3** The pressure on the surface of a sphere of radius 1.2 cm is increased by $2.5 \times 10^7$ Pa. Calculate the resulting decrease in volume of the sphere if it is made (a) of lead and (b) of mercury.

**VP11.7.4** You apply forces of magnitude $4.2 \times 10^4$ N to the top and bottom surfaces of a brass cube. The forces are tangent to each surface and parallel to the sides of each surface. If the cube is 2.5 cm on a side, what is the resulting shear displacement?

## BRIDGING PROBLEM In Equilibrium and Under Stress

A horizontal, uniform, solid copper rod has an original length $l_0$, cross-sectional area $A$, Young's modulus $Y$, bulk modulus $B$, shear modulus $S$, and mass $m$. It is supported by a frictionless pivot at its right end and by a cable a distance $l_0/4$ from its left end (**Fig. 11.21**). Both pivot and cable are attached so that they exert their forces uniformly over the rod's cross section. The cable makes an angle $\theta$ with the rod and compresses it. (a) Find the tension in the cable. (b) Find the magnitude and direction of the force exerted by the pivot on the right end of the rod. How does this magnitude compare to the cable tension? How does this angle compare to $\theta$? (c) Find the change in length of the rod due to the stresses exerted by the cable and pivot on the rod. (The length change is small compared to the original length $l_0$.) (d) By what factor would your answer in part (c) increase if the solid copper rod were twice as long but had the same cross-sectional area?

Figure 11.21 What are the forces on the rod? What are the stress and strain?



**SOLUTION GUIDE**

### IDENTIFY and SET UP

1. Draw a free-body diagram for the rod. Be careful to place each force in the correct location.
2. List the unknown quantities, and decide which are the target variables.
3. What conditions must be met so that the rod remains at rest? What kind of stress (and resulting strain) is involved? Use your answers to select the appropriate equations.

### EXECUTE

4. Use your equations to solve for the target variables. (*Hint:* You can make the solution easier by carefully choosing the point around which you calculate torques.)
5. Use trigonometry to decide whether the pivot force or the cable tension has the greater magnitude and whether the angle of the pivot force is greater than, less than, or equal to $\theta$.

### EVALUATE

6. Check whether your answers are reasonable. Which force, the cable tension or the pivot force, holds up more of the weight of the rod? Does this make sense?

## PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

### DISCUSSION QUESTIONS

**Q11.1** Does a rigid object in uniform rotation about a fixed axis satisfy the first and second conditions for equilibrium? Why? Does it then follow that every particle in this object is in equilibrium? Explain.

**Q11.2** (a) Is it possible for an object to be in translational equilibrium (the first condition) but *not* in rotational equilibrium (the second condition)? Illustrate your answer with a simple example. (b) Can an object be in rotational equilibrium yet *not* in translational equilibrium? Justify your answer with a simple example.

**Q11.3** Car tires are sometimes "balanced" on a machine that pivots the tire and wheel about the center. Weights are placed around the wheel rim until it does not tip from the horizontal plane. Discuss this procedure in terms of the center of gravity.

**Q11.4** Does the center of gravity of a solid object always lie within the material of the object? If not, give a counterexample.

**Q11.5** In Section 11.2 we always assumed that the value of $g$ was the same at all points on the object. This is *not* a good approximation if the dimensions of the object are great enough, because the value of $g$ decreases with altitude. If this is taken into account, will the center of gravity of a long, vertical rod be above, below, or at its center of mass? Explain how this can be used to keep the long axis of an orbiting spacecraft pointed toward the earth. (This would be useful for a weather satellite that must always keep its camera lens trained on the earth.) The moon is not exactly spherical but is somewhat elongated. Explain why this same effect is responsible for keeping the same face of the moon pointed toward the earth at all times.

**Q11.6** You are balancing a wrench by suspending it at a single point. Is the equilibrium stable, unstable, or neutral if the point is above, at, or below the wrench's center of gravity? In each case give the reasoning behind your answer. (For rotation, a rigid object is in *stable* equilibrium if a small rotation of the object produces a torque that tends to return the object to equilibrium; it is in *unstable* equilibrium if a small rotation produces a torque that tends to take the object farther from equilibrium; and it is in *neutral* equilibrium if a small rotation produces no torque.)

**Q11.7** You can probably stand flatfooted on the floor and then rise up and balance on your tiptoes. Why are you unable do it if your toes are touching the wall of your room? (Try it!)

**Q11.8** You freely pivot a horseshoe from a horizontal nail through one of its nail holes. You then hang a long string with a weight at its bottom from the same nail, so that the string hangs vertically in front of the horseshoe without touching it. How do you know that the horseshoe's center of gravity is along the line behind the string? How can you locate the center of gravity by repeating the process at another nail hole? Will the center of gravity be within the solid material of the horseshoe?

**Q11.9** An object consists of a ball of weight $W$ glued to the end of a uniform bar also of weight $W$. If you release it from rest, with the bar horizontal, what will its behavior be as it falls if air resistance is negligible? Will it (a) remain horizontal; (b) rotate about its center of gravity; (c) rotate about the ball; or (d) rotate so that the ball swings downward? Explain your reasoning.

**Q11.10** Suppose that the object in Question 11.9 is released from rest with the bar tilted at 60° above the horizontal with the ball at the upper end. As it is falling, will it (a) rotate about its center of gravity until it is horizontal; (b) rotate about its center of gravity until it is vertical with the ball at the bottom; (c) rotate about the ball until it is vertical with the ball at the bottom; or (d) remain at 60° above the horizontal?

**Q11.11** Why must a water skier moving with constant velocity lean backward? What determines how far back she must lean? Draw a free-body diagram for the water skier to justify your answers.

**Q11.12** In pioneer days, when a Conestoga wagon was stuck in the mud, people would grasp the wheel spokes and try to turn the wheels, rather than simply pushing the wagon. Why?

**Q11.13** The mighty Zimbo claims to have leg muscles so strong that he can stand flat on his feet and lean forward to pick up an apple on the floor with his teeth. Should you pay to see him perform, or do you have any suspicions about his claim? Why?

**Q11.14** Why is it easier to hold a 10 kg dumbbell in your hand at your side than it is to hold it with your arm extended horizontally?

**Q11.15** Certain features of a person, such as height and mass, are fixed (at least over relatively long periods of time). Are the following features also fixed? (a) location of the center of gravity of the body; (b) moment of inertia of the body about an axis through the person's center of mass. Explain your reasoning.

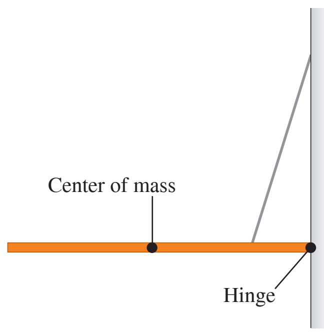
**Q11.16** During pregnancy, women often develop back pains from leaning backward while walking. Why do they have to walk this way?

**Q11.17** Why is a tapered water glass with a narrow base easier to tip over than a glass with straight sides? Does it matter whether the glass is full or empty?

**Q11.18** When a tall, heavy refrigerator is pushed across a rough floor, what factors determine whether it slides or tips?

**Q11.19** A uniform beam is suspended horizontally and attached to a wall by a small hinge (**Fig. Q11.19**). What are the directions (upward or downward, and to the left or the right) of the components of the force that the hinge exerts *on the beam*? Explain.

Figure **Q11.19**




**Q11.20** If a metal wire has its length doubled and its diameter tripled, by what factor does its Young's modulus change?

**Q11.21** A metal wire of diameter $D$ stretches by 0.100 mm when supporting a weight $W$. If the same-length wire is used to support a weight three times as heavy, what would its diameter have to be (in terms of $D$) so it still stretches only 0.100 mm?

**Q11.22** Compare the mechanical properties of a steel cable, made by twisting many thin wires together, with the properties of a solid steel rod of the same diameter. What advantages does each have?

**Q11.23** The material in human bones and elephant bones is essentially the same, but an elephant has much thicker legs. Explain why, in terms of breaking stress.

**Q11.24** There is a small but appreciable amount of elastic hysteresis in the large tendon at the back of a horse's leg. Explain how this can cause damage to the tendon if a horse runs too hard for too long a time.

**Q11.25** When rubber mounting blocks are used to absorb machine vibrations through elastic hysteresis, as mentioned in Section 11.5, what becomes of the energy associated with the vibrations?

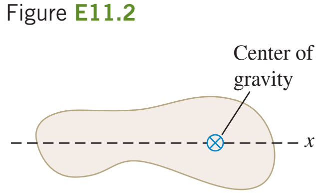
## EXERCISES

### Section 11.2 Center of Gravity

**11.1 ••** A 0.120 kg, 50.0-cm-long uniform bar has a small 0.055 kg mass glued to its left end and a small 0.110 kg mass glued to the other end. The two small masses can each be treated as point masses. You want to balance this system horizontally on a fulcrum placed just under its center of gravity. How far from the left end should the fulcrum be placed?

**11.2 ••** The center of gravity of a 5.00 kg irregular object is shown in **Fig. E11.2.** You need to move the center of gravity 2.20 cm to the left by gluing on a 1.50 kg mass, which will then be considered as part of the object. Where should the center of gravity of this additional mass be located?

Figure **E11.2**




**11.3 •** A uniform rod is 2.00 m long and has mass 1.80 kg. A 2.40 kg clamp is attached to the rod. How far should the center of gravity of the clamp be from the left-hand end of the rod in order for the center of gravity of the composite object to be 1.20 m from the left-hand end of the rod?

**11.4 •** Consider the free-body diagram shown in Fig. 11.9b. (a) What is the horizontal distance of the center of gravity of the person–ladder system from the point where the ladder touches the ground? (b) What is the torque about the rotation axis shown in the figure (point $B$) computed by taking the total weight of the person plus ladder acting at the center of gravity? (c) How does the result in part (b) compare to the sum of the individual torques computed in Example 11.3 for the weight of the person and the weight of the ladder?

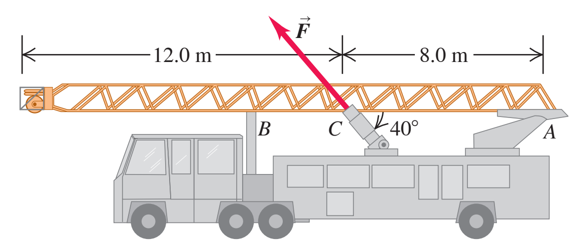
**11.5 ••** A uniform steel rod has mass 0.300 kg and length 40.0 cm and is horizontal. A uniform sphere with radius 8.00 cm and mass 0.900 kg is welded to one end of the bar, and a uniform sphere with radius 6.00 cm and mass 0.380 kg is welded to the other end of the bar. The centers of the rod and of each sphere all lie along a horizontal line. How far is the center of gravity of the combined object from the center of the rod?

### Section 11.3 Solving Rigid-Body Equilibrium Problems

**11.6 •** A uniform 300 N trapdoor in a floor is hinged at one side. Find the net upward force needed to begin to open it and the total force exerted on the door by the hinges (a) if the upward force is applied at the center and (b) if the upward force is applied at the center of the edge opposite the hinges.

**11.7 ••** **Raising a Ladder.** A ladder carried by a fire truck is 20.0 m long. The ladder weighs 3400 N and its center of gravity is at its center. The ladder is pivoted at one end ($A$) about a pin (**Fig. E11.7**); ignore the friction torque at the pin. The ladder is raised into position by a force applied by a hydraulic piston at $C$. Point $C$ is 8.0 m from $A$, and the force $\vec{F}$ exerted by the piston makes an angle of 40° with the ladder. What magnitude must $\vec{F}$ have to just lift the ladder off the support bracket at $B$? Start with a free-body diagram of the ladder.

Figure **E11.7**

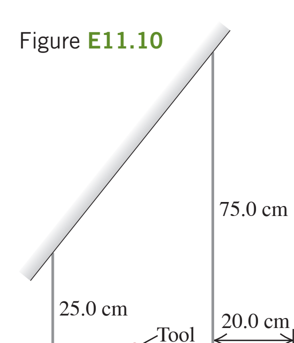



**11.8 ••** Two people are carrying a uniform wooden board that is 3.00 m long and weighs 160 N. If one person applies an upward force equal to 60 N at one end, at what point does the other person lift? Begin with a free-body diagram of the board.

**11.9 ••** Two people carry a heavy electric motor by placing it on a light board 2.00 m long. One person lifts at one end with a force of 400 N, and the other lifts the opposite end with a force of 600 N. (a) What is the weight of the motor, and where along the board is its center of gravity located? (b) Suppose the board is not light but weighs 200 N, with its center of gravity at its center, and the two people exert the same forces as before. What is the weight of the motor in this case, and where is its center of gravity located?

**11.10 ••** A 60.0 cm, uniform, 50.0 N shelf is supported horizontally by two vertical wires attached to the sloping ceiling (**Fig. E11.10**). A very small 25.0 N tool is placed on the shelf midway between the points where the wires are attached to it. Find the tension in each wire. Begin by making a free-body diagram of the shelf.

Figure **E11.10**

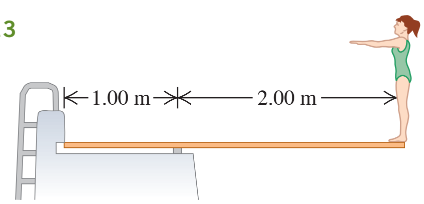


**11.11** •• A 350 N, uniform, 1.50 m bar is suspended horizontally by two vertical cables at each end. Cable *A* can support a maximum tension of 500.0 N without breaking, and cable *B* can support up to 400.0 N. You want to place a small weight on this bar. (a) What is the heaviest weight you can put on without breaking either cable, and (b) where should you put this weight?

**11.12** •• A uniform ladder 5.0 m long rests against a frictionless, vertical wall with its lower end 3.0 m from the wall. The ladder weighs 160 N. The coefficient of static friction between the foot of the ladder and the ground is 0.40. A man weighing 740 N climbs slowly up the ladder. Start by drawing a free-body diagram of the ladder. (a) What is the maximum friction force that the ground can exert on the ladder at its lower end? (b) What is the actual friction force when the man has climbed 1.0 m along the ladder? (c) How far along the ladder can the man climb before the ladder starts to slip?

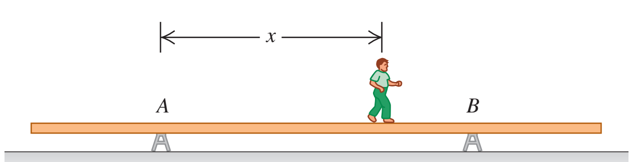
**11.13** • A diving board 3.00 m long is supported at a point 1.00 m from the end, and a diver weighing 500 N stands at the free end (**Fig. E11.13**). The diving board is of uniform cross section and weighs 280 N. Find (a) the force at the support point and (b) the force at the left-hand end.

Figure **E11.13**




**11.14** • A uniform aluminum beam 9.00 m long, weighing 300 N, rests symmetrically on two supports 5.00 m apart (**Fig. E11.14**). A boy weighing 600 N starts at point *A* and walks toward the right. (a) In the same diagram construct two graphs showing the upward forces $F_A$ and $F_B$ exerted on the beam at points *A* and *B*, as functions of the coordinate *x* of the boy. Let 1 cm = 100 N vertically and 1 cm = 1.00 m horizontally. (b) From your diagram, how far beyond point *B* can the boy walk before the beam tips? (c) How far from the right end of the beam should support *B* be placed so that the boy can walk just to the end of the beam without causing it to tip?

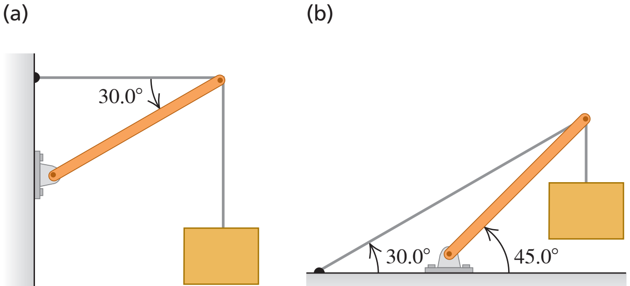
Figure **E11.14**




**11.15** • Find the tension *T* in each cable and the magnitude and direction of the force exerted on the strut by the pivot in each of the arrangements in **Fig. E11.15**. In each case let *w* be the weight of the suspended crate full of priceless art objects. The strut is uniform and also has weight *w*. Start each case with a free-body diagram of the strut.

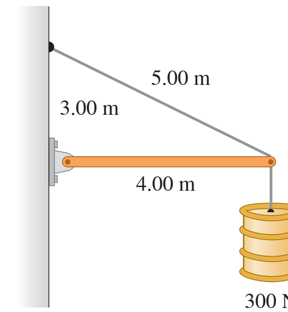
Figure **E11.15**

(a) (b)



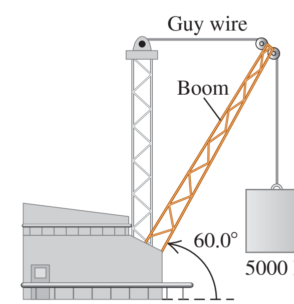

**11.16** • The horizontal beam in **Fig. E11.16** weighs 190 N, and its center of gravity is at its center. Find (a) the tension in the cable and (b) the horizontal and vertical components of the force exerted on the beam at the wall.

Figure **E11.16**



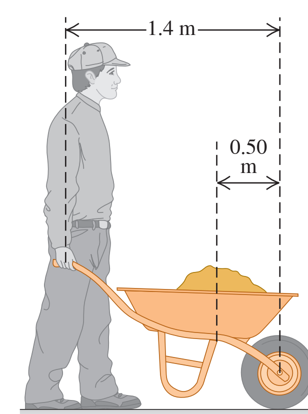

**11.17** •• The boom shown in **Fig. E11.17** weighs 2600 N and is attached to a frictionless pivot at its lower end. It is not uniform; the distance of its center of gravity from the pivot is 35% of its length. Find (a) the tension in the guy wire and (b) the horizontal and vertical components of the force exerted on the boom at its lower end. Start with a free-body diagram of the boom.

Figure **E11.17**



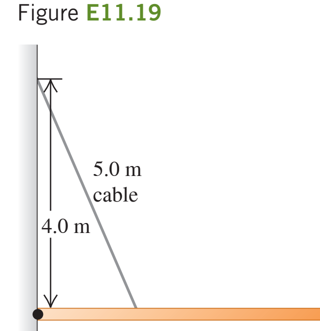

**11.18** •• You are pushing an 80.0 N wheelbarrow as shown in **Fig. E11.18**. You lift upward on the handle of the wheelbarrow so that the only point of contact between the wheelbarrow and the ground is at the front wheel. Assume the distances are as shown in the figure, where 0.50 m is the horizontal distance from the center of the wheel to the center of gravity of the wheelbarrow. The center of gravity of the dirt in the wheelbarrow is assumed to also be a horizontal distance of 0.50 m from the center of the wheel. Estimate the maximum total upward force that you can apply to the wheelbarrow handles. (a) If you apply this estimated force, what is the maximum weight of dirt that you can carry in the wheelbarrow? Express your answer in pounds. (b) If the dirt has the weight you calculated in part (a), what upward force does the ground apply to the wheel of the wheelbarrow?

Figure **E11.18**



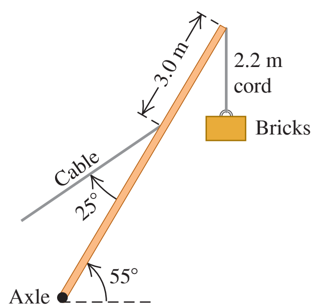

**11.19** •• A 9.00-m-long uniform beam is hinged to a vertical wall and held horizontally by a 5.00-m-long cable attached to the wall 4.00 m above the hinge (**Fig. E11.19**). The metal of this cable has a test strength of 1.00 kN, which means that it will break if the tension in it exceeds that amount. (a) Draw a free-body diagram of the beam. (b) What is the heaviest beam that the cable can support in this configuration? (c) Find the horizontal and vertical components of the force the hinge exerts on the beam. Is the vertical component upward or downward?

Figure **E11.19**



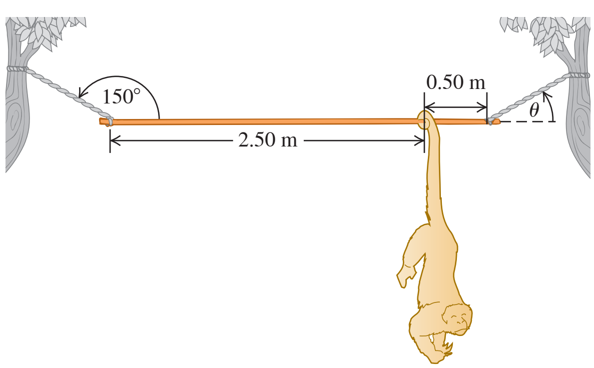
**11.20** •• A 15,000 N crane pivots around a friction-free axle at its base and is supported by a cable making a 25° angle with the crane (**Fig. E11.20**). The crane is 16 m long and is not uniform, its center of gravity being 7.0 m from the axle as measured along the crane. The cable is attached 3.0 m from the upper end of the crane. When the crane is raised to 55° above the horizontal holding an 11,000 N pallet of bricks by a 2.2 m, very light cord, find (a) the tension in the cable and (b) the horizontal and vertical components of the force that the axle exerts on the crane. Start with a free-body diagram of the crane.

Figure **E11.20**



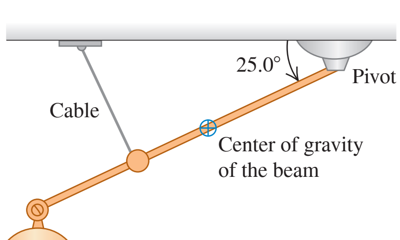

**11.21** •• A 3.00-m-long, 190 N, uniform rod at the zoo is held in a horizontal position by two ropes at its ends (**Fig. E11.21**). The left rope makes an angle of 150° with the rod, and the right rope makes an angle $\theta$ with the horizontal. A 90 N howler monkey (*Alouatta seniculus*) hangs motionless 0.50 m from the right end of the rod as he carefully studies you. Calculate the tensions in the two ropes and the angle $\theta$. First make a free-body diagram of the rod.

Figure **E11.21**



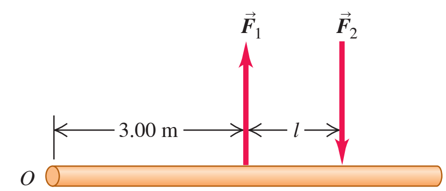

**11.22** •• A nonuniform beam 4.50 m long and weighing 1.40 kN makes an angle of 25.0° below the horizontal. It is held in position by a frictionless pivot at its upper right end and by a cable 3.00 m farther down the beam and perpendicular to it (**Fig. E11.22**). The center of gravity of the beam is 2.00 m down the beam from the pivot. Lighting equipment exerts a 5.00 kN downward force on the lower left end of the beam. Find the tension $T$ in the cable and the horizontal and vertical components of the force exerted on the beam by the pivot. Start by sketching a free-body diagram of the beam.

Figure **E11.22**



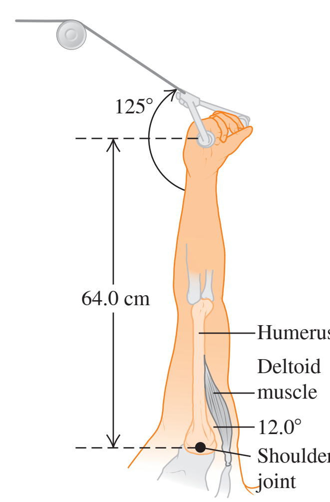

**11.23** • **A Couple.** Two forces equal in magnitude and opposite in direction, acting on an object at two different points, form what is called a *couple.* Two antiparallel forces with equal magnitudes $F_1 = F_2 = 8.00$ N are applied to a rod as shown in **Fig. E11.23.** (a) What should the distance $l$ between the forces be if they are to provide a net torque of 6.40 N·m about the left end of the rod? (b) Is the sense of this torque clockwise or counterclockwise? (c) Repeat parts (a) and (b) for a pivot at the point on the rod where $\vec{F}_2$ is applied.

Figure **E11.23**




**11.24** •• **BIO A Good Workout.** You are doing exercises on a Nautilus machine in a gym to strengthen your deltoid (shoulder) muscles. Your arms are raised vertically and can pivot around the shoulder joint, and you grasp the cable of the machine in your hand 64.0 cm from your shoulder joint. The deltoid muscle is attached to the humerus 15.0 cm from the shoulder joint and makes a 12.0° angle with that bone (**Fig. E11.24**). If you have set the tension in the cable of the machine to 36.0 N on each arm, what is the tension in each deltoid muscle if you simply hold your outstretched arms in place? (*Hint:* Start by making a clear free-body diagram of your arm.)

Figure **E11.24**




**11.25** •• A uniform rod has one end attached to a vertical wall by a frictionless hinge. A horizontal wire runs from the other end of the rod to a point on the wall above the hinge and holds the rod at an angle $\theta$ above the horizontal. You vary the angle $\theta$ by changing the length of the wire, and for each $\theta$ you measure the tension $T$ in the wire. You plot your data in the form of a $T$-versus-$\cot\theta$ graph. The data lie close to a straight line that has slope 30.0 N. What is the mass of the rod?

## Section 11.4 Stress, Strain, and Elastic Moduli

**11.26** • **BIO Biceps Muscle.** A relaxed biceps muscle requires a force of 25.0 N for an elongation of 3.0 cm; the same muscle under maximum tension requires a force of 500 N for the same elongation. Find Young's modulus for the muscle tissue under each of these conditions if the muscle is assumed to be a uniform cylinder with length 0.200 m and cross-sectional area 50.0 cm$^2$.

**11.27** •• A circular steel wire 2.00 m long must stretch no more than 0.25 cm when a tensile force of 700 N is applied to each end of the wire. What minimum diameter is required for the wire?

**11.28** •• Two cylindrical rods, one steel and the other copper, are joined end to end. Each rod is 0.750 m long and 1.50 cm in diameter. The combination is subjected to a tensile force with magnitude 4000 N. For each rod, what are (a) the strain and (b) the elongation?

**11.29** •• A metal rod that is 4.00 m long and 0.50 $\text{cm}^2$ in cross-sectional area is found to stretch 0.20 cm under a tension of 5000 N. What is Young's modulus for this metal?

**11.30** •• **Stress on a Mountaineer's Rope.** A nylon rope used by mountaineers elongates 1.10 m under the weight of a 65.0 kg climber. If the rope is 45.0 m in length and 7.0 mm in diameter, what is Young's modulus for nylon?

**11.31** • A lead sphere has volume 6.0 $\text{cm}^3$ when it is resting on a lab table, where the pressure applied to the sphere is atmospheric pressure. The sphere is then placed in the fluid of a hydraulic press. What increase in the pressure above atmospheric pressure produces a 0.50% decrease in the volume of the sphere?

**11.32** •• A vertical, solid steel post 25 cm in diameter and 2.50 m long is required to support a load of 8000 kg. You can ignore the weight of the post. What are (a) the stress in the post; (b) the strain in the post; and (c) the change in the post's length when the load is applied?

**11.33** •• **BIO Compression of Human Bone.** The bulk modulus for bone is 15 GPa. (a) If a diver-in-training is put into a pressurized suit, by how much would the pressure have to be raised (in atmospheres) above atmospheric pressure to compress her bones by 0.10% of their original volume? (b) Given that the pressure in the ocean increases by $1.0 \times 10^4$ Pa for every meter of depth below the surface, how deep would this diver have to go for her bones to compress by 0.10%? Does it seem that bone compression is a problem she needs to be concerned with when diving?

**11.34** • A solid gold bar is pulled up from the hold of the sunken RMS *Titanic*. (a) What happens to its volume as it goes from the pressure at the ship to the lower pressure at the ocean's surface? (b) The pressure difference is proportional to the depth. How many times greater would the volume change have been had the ship been twice as deep? (c) The bulk modulus of lead is one-fourth that of gold. Find the ratio of the volume change of a solid lead bar to that of a gold bar of equal volume for the same pressure change.

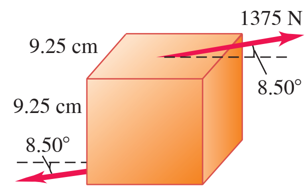
**11.35** • A specimen of oil having an initial volume of 600 $\text{cm}^3$ is subjected to a pressure increase of $3.6 \times 10^6$ Pa, and the volume is found to decrease by 0.45 $\text{cm}^3$. What is the bulk modulus of the material? The compressibility?

**11.36** •• In the Challenger Deep of the Marianas Trench, the depth of seawater is 10.9 km and the pressure is $1.16 \times 10^8$ Pa (about $1.15 \times 10^3$ atm). (a) If a cubic meter of water is taken from the surface to this depth, what is the change in its volume? (Normal atmospheric pressure is about $1.0 \times 10^5$ Pa. Assume that $k$ for seawater is the same as the freshwater value given in Table 11.2.) (b) What is the density of seawater at this depth? (At the surface, seawater has a density of $1.03 \times 10^3$ $\text{kg/m}^3$.)

**11.37** •• A square steel plate is 10.0 cm on a side and 0.500 cm thick. (a) Find the shear strain that results if two forces, each of magnitude $9.0 \times 10^5$ N and in opposite directions, act tangent to the surfaces of a pair of opposite sides of the object, as in Fig. E11.18. (b) Find the displacement $x$ in centimeters.

**11.38** • In lab tests on a 9.25 cm cube of a certain material, a force of 1375 N directed at 8.50° to the cube (**Fig. E11.38**) causes the cube to deform through an angle of 1.24°. What is the shear modulus of the material?

Figure **E11.38**




**11.39** •• A steel wire with radius $r_{\text{steel}}$ has a fractional increase in length of $(\Delta l/l_0)_{\text{steel}}$ when the tension in the wire is increased from zero to $T_{\text{steel}}$. An aluminum wire has radius $r_{\text{al}}$ that is twice the radius of the steel wire: $r_{\text{al}} = 2r_{\text{steel}}$. In terms of $T_{\text{steel}}$, what tension in the aluminum wire produces the same fractional change in length as in the steel wire?

**11.40** •• You apply a force of magnitude $F_\perp$ to one end of a wire and another force $F_\perp$ in the opposite direction to the other end of the wire. The cross-sectional area of the wire is 8.00 $\text{mm}^2$. You measure the fractional change in the length of the wire, $\Delta l/l_0$, for several values of $F_\perp$. When you plot your data with $\Delta l/l_0$ on the vertical axis and $F_\perp$ (in units of N) on the horizontal axis, the data lie close to a line that has slope $8.0 \times 10^{-7}\ \text{N}^{-1}$. What is the value of Young's modulus for this wire?

**11.41** • An increase in applied pressure $\Delta p_1$ produces a fractional volume change of $(\Delta V/V_0)_1$ for a sample of glycerin. In terms of $\Delta p_1$, what pressure increase above atmospheric pressure is required to produce the same volume change $(\Delta V/V_0)_1$ for a sample of ethyl alcohol?

### Section 11.5 Elasticity and Plasticity

**11.42** •• A brass wire is to withstand a tensile force of 350 N without breaking. What minimum diameter must the wire have?

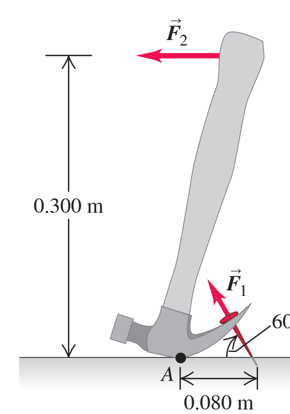
**11.43** •• In a materials testing laboratory, a metal wire made from a new alloy is found to break when a tensile force of 90.8 N is applied perpendicular to each end. If the diameter of the wire is 1.84 mm, what is the breaking stress of the alloy?

**11.44** •• **CP** A steel cable with cross-sectional area 3.00 $\text{cm}^2$ has an elastic limit of $2.40 \times 10^8$ Pa. Find the maximum upward acceleration that can be given a 1200 kg elevator supported by the cable if the stress is not to exceed one-third of the elastic limit.

## PROBLEMS

**11.45** •• You are using a hammer to pull a nail from the floor, as shown in **Fig. P11.45**. Force $\vec{F}_2$ is the force you apply to the handle, and force $\vec{F}_1$ is the force the hammer applies to the nail. Estimate the maximum magnitude of the force $\vec{F}_2$ that you could apply to the hammer. If you apply this force, what is the magnitude of the force $\vec{F}_1$ that the hammer applies to the nail?

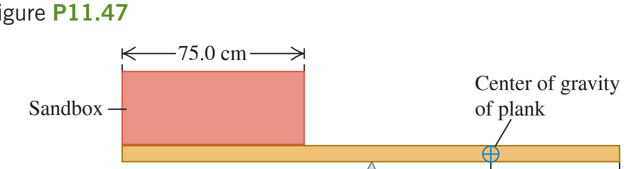
Figure **P11.45**




**11.46** ••• A door 1.00 m wide and 2.00 m high weighs 330 N and is supported by two hinges, one 0.50 m from the top and the other 0.50 m from the bottom. Each hinge supports half the total weight of the door. Assuming that the door's center of gravity is at its center, find the horizontal components of force exerted on the door by each hinge.

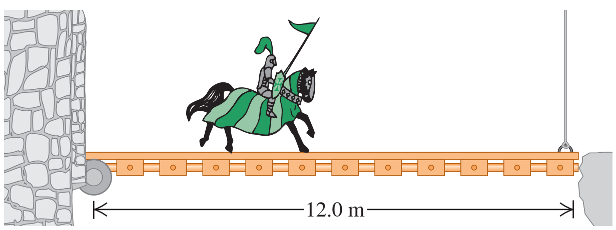
**11.47** ••• A box of negligible mass rests at the left end of a 2.00 m, 25.0 kg plank (**Fig. P11.47**). The width of the box is 75.0 cm, and sand is to be distributed uniformly throughout it. The center of gravity of the nonuniform plank is 50.0 cm from the right end. What mass of sand should be put into the box so that the plank balances horizontally on a fulcrum placed just below its midpoint?

Figure **P11.47**



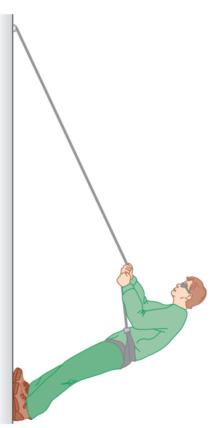
**11.48** • Sir Lancelot rides slowly out of the castle at Camelot and onto the 12.0-m-long drawbridge that passes over the moat (**Fig. P11.48**). Unbeknownst to him, his enemies have partially severed the vertical cable holding up the front end of the bridge so that it will break under a tension of $5.80 \times 10^3$ N. The bridge has mass 200 kg and its center of gravity is at its center. Lancelot, his lance, his armor, and his horse together have a combined mass of 600 kg. Will the cable break before Lancelot reaches the end of the drawbridge? If so, how far from the castle end of the bridge will the center of gravity of the horse plus rider be when the cable breaks?

Figure **P11.48**



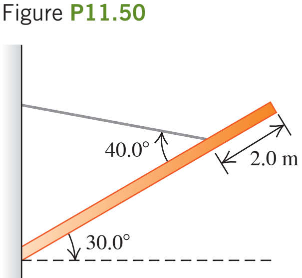

**11.49** ••• **Mountain Climbing.** Mountaineers often use a rope to lower themselves down the face of a cliff (this is called *rappelling*). They do this with their body nearly horizontal and their feet pushing against the cliff (**Fig. P11.49**). Suppose that an 82.0 kg climber, who is 1.90 m tall and has a center of gravity 1.1 m from his feet, rappels down a vertical cliff with his body raised 35.0° above the horizontal. He holds the rope 1.40 m from his feet, and it makes a 25.0° angle with the cliff face. (a) What tension does his rope need to support? (b) Find the horizontal and vertical components of the force that the cliff face exerts on the climber's feet. (c) What minimum coefficient of static friction is needed to prevent the climber's feet from slipping on the cliff face if he has one foot at a time against the cliff?

Figure **P11.49**



**11.50** •• A uniform, 8.0 m, 1150 kg beam is hinged to a wall and supported by a thin cable attached 2.0 m from the free end of the beam (**Fig. P11.50**). The beam is supported at an angle of 30.0° above the horizontal. (a) Draw a free-body diagram of the beam. (b) Find the tension in the cable. (c) How hard does the beam push inward on the wall?

Figure **P11.50**



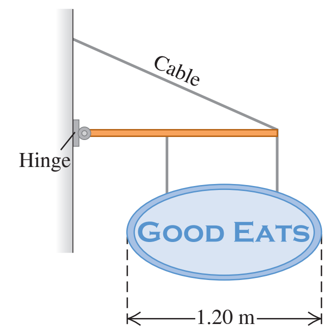

**11.51** •• A uniform, 255 N rod that is 2.00 m long carries a 225 N weight at its right end and an unknown weight $W$ toward the left end (**Fig. P11.51**). When $W$ is placed 50.0 cm from the left end of the rod, the system just balances horizontally when the fulcrum is located 75.0 cm from the right end. (a) Find $W$. (b) If $W$ is now moved 25.0 cm to the right, how far and in what direction must the fulcrum be moved to restore balance?

Figure **P11.51**


**11.52** ••• A claw hammer is used to pull a nail out of a board (see Fig. P11.45). The nail is at an angle of 60° to the board, and a force $\vec{F}_1$ of magnitude 400 N applied to the nail is required to pull it from the board. The hammer head contacts the board at point $A$, which is 0.080 m from where the nail enters the board. A horizontal force $\vec{F}_2$ is applied to the hammer handle at a distance of 0.300 m above the board. What magnitude of force $\vec{F}_2$ is required to apply the required 400 N force ($F_1$) to the nail? (Ignore the weight of the hammer.)

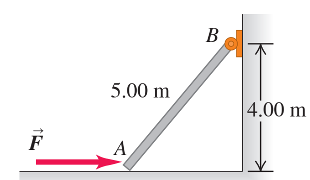
**11.53** •• You open a restaurant and hope to entice customers by hanging out a sign (**Fig. P11.53**). The uniform horizontal beam supporting the sign is 1.50 m long, has a mass of 16.0 kg, and is hinged to the wall. The sign itself is uniform with a mass of 28.0 kg and overall length of 1.20 m. The two wires supporting the sign are each 32.0 cm long, are 90.0 cm apart, and are equally spaced from the middle of the sign. The cable supporting the beam is 2.00 m long. (a) What minimum tension must your cable be able to support without having your sign come crashing down? (b) What minimum vertical force must the hinge be able to support without pulling out of the wall?

Figure **P11.53**



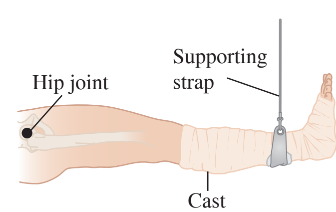

**11.54** • End $A$ of the bar $AB$ in **Fig. P11.54** rests on a frictionless horizontal surface, and end $B$ is hinged. A horizontal force $\vec{F}$ of magnitude 220 N is exerted on end $A$. Ignore the weight of the bar. What are the horizontal and vertical components of the force exerted by the bar on the hinge at $B$?

Figure **P11.54**




**11.55** •• BIO **Supporting a Broken Leg.** A therapist tells a 74 kg patient with a broken leg that he must have his leg in a cast suspended horizontally. For minimum discomfort, the leg should be supported by a vertical strap attached at the center of mass of the leg–cast system (**Fig. P11.55**). To comply with these instructions, the patient consults a table of typical mass distributions and finds that both upper legs (thighs) together typically account for 21.5% of body weight and the center of mass of each thigh is 18.0 cm from the hip joint. The patient also reads that the two lower legs (including the feet) are 14.0% of body weight, with a center of mass 69.0 cm from the hip joint. The cast has a mass of 5.50 kg, and its center of mass is 78.0 cm from the hip joint. How far from the hip joint should the supporting strap be attached to the cast?

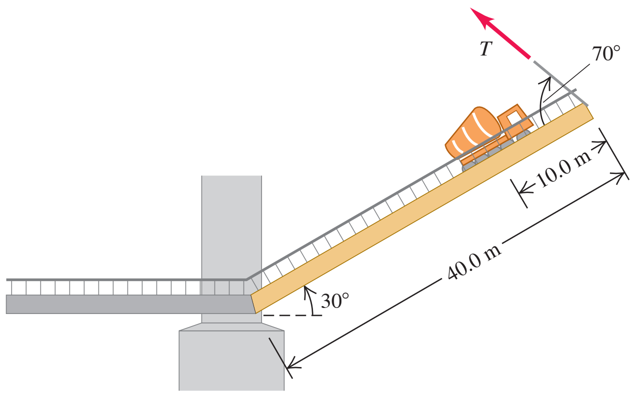
Figure **P11.55**




**11.56** • **A Truck on a Drawbridge.** A loaded cement mixer drives onto an old drawbridge, where it stalls with its center of gravity three-quarters of the way across the span. The truck driver radios for help, sets the handbrake, and waits. Meanwhile, a boat approaches, so the drawbridge is raised by means of a cable attached to the end opposite the hinge (**Fig. P11.56**). The drawbridge is 40.0 m long and has a mass of 18,000 kg; its center of gravity is at its midpoint. The cement mixer, with driver, has mass 30,000 kg. When the drawbridge has been raised to an angle of 30° above the horizontal, the cable makes an angle of

70° with the surface of the bridge. (a) What is the tension $T$ in the cable when the drawbridge is held in this position? (b) What are the horizontal and vertical components of the force the hinge exerts on the span?

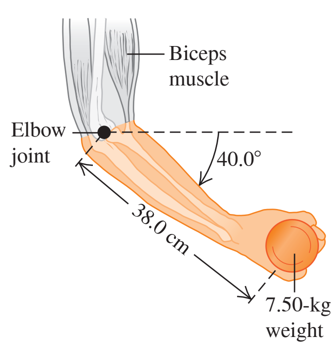
Figure **P11.56**




**11.57** •• **CP** The left-hand end of a uniform rod of length $L$ and mass $m$ is attached to a vertical wall by a frictionless hinge. The rod is held at an angle $\theta$ above the horizontal by a horizontal wire that runs between the wall and the right-hand end of the rod. (a) If the tension in the wire is $T$, what is the magnitude of the angle $\theta$ that the rod makes with the horizontal? (b) The wire breaks and the rod rotates about the hinge. What is the angular speed of the rod as the rod passes through a horizontal position?

**11.58** •• **BIO Pumping Iron.** A 72.0 kg weightlifter doing arm raises holds a 7.50 kg weight. Her arm pivots around the elbow joint, starting 40.0° below the horizontal (**Fig. P11.58**). Biometric measurements have shown that, together, the forearms and the hands account for 6.00% of a person's weight. Since the upper arm is held vertically, the biceps muscle always acts vertically and is attached to the bones of the forearm 5.50 cm from the elbow joint. The center of mass of this person's forearm–hand combination is 16.0 cm from the elbow joint, along the bones of the forearm, and she holds the weight 38.0 cm from her elbow joint. (a) Draw a free-body diagram of the forearm. (b) What force does the biceps muscle exert on the forearm? (c) Find the magnitude and direction of the force that the elbow joint exerts on the forearm. (d) As the weightlifter raises her arm toward a horizontal position, will the force in the biceps muscle increase, decrease, or stay the same? Why?

Figure **P11.58**




**11.59** •• The left-hand end of a light rod of length $L$ is attached to a vertical wall by a frictionless hinge. An object of mass $m$ is suspended from the rod at a point a distance $\alpha L$ from the hinge, where $0 < \alpha \leq 1.00$. The rod is held in a horizontal position by a light wire that runs from the right-hand end of the rod to the wall. The wire makes an angle $\theta$ with the rod. (a) What is the angle $\beta$ that the net force exerted by the hinge on the rod makes with the horizontal? (b) What is the value of $\alpha$ for which $\beta = \theta$? (c) What is $\beta$ when $\alpha = 1.00$?

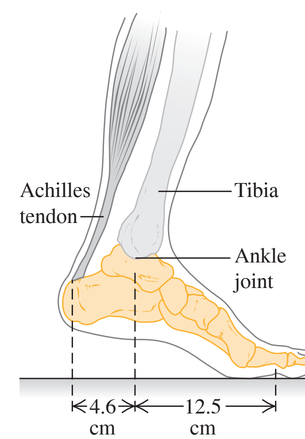
**11.60** •• The left-hand end of a slender uniform rod of mass $m$ is placed against a vertical wall. The rod is held in a horizontal position by friction at the wall and by a light wire that runs from the right-hand end of the rod to a point on the wall above the rod. The wire makes an angle $\theta$ with the rod. (a) What must the magnitude of the friction force be in order for the rod to remain at rest? (b) If the coefficient of static friction between the rod and the wall is $\mu_s$, what is the maximum angle between the wire and the rod at which the rod doesn't slip at the wall?

**11.61** •• A uniform, 7.5-m-long beam weighing 6490 N is hinged to a wall and supported by a thin cable attached 1.5 m from the free end of the beam. The cable runs between the beam and the wall and makes a 40° angle with the beam. What is the tension in the cable when the beam is at an angle of 30° above the horizontal?

**11.62** •• **CP** A uniform drawbridge must be held at a 37° angle above the horizontal to allow ships to pass underneath. The drawbridge weighs 45,000 N and is 14.0 m long. A cable is connected 3.5 m from the hinge where the bridge pivots (measured along the bridge) and pulls horizontally on the bridge to hold it in place. (a) What is the tension in the cable? (b) Find the magnitude and direction of the force the hinge exerts on the bridge. (c) If the cable suddenly breaks, what is the magnitude of the angular acceleration of the drawbridge just after the cable breaks? (d) What is the angular speed of the drawbridge as it becomes horizontal?

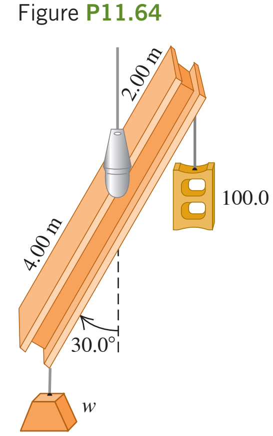
**11.63** •• **BIO Tendon-Stretching Exercises.** As part of an exercise program, a 75 kg person does toe raises in which he raises his entire body weight on the ball of one foot (**Fig. P11.63**). The Achilles tendon pulls straight upward on the heel bone of his foot. This tendon is 25 cm long and has a cross-sectional area of 78 $mm^2$ and a Young's modulus of 1470 MPa. (a) Draw a free-body diagram of the person's foot (everything below the ankle joint). Ignore the weight of the foot. (b) What force does the Achilles tendon exert on the heel during this exercise? Express your answer in newtons and in multiples of his weight. (c) By how many millimeters does the exercise stretch his Achilles tendon?

Figure **P11.63**




**11.64** •• (a) In **Fig. P11.64** a 6.00-m-long, uniform beam is hanging from a point 1.00 m to the right of its center. The beam weighs 140 N and makes an angle of 30.0° with the vertical. At the right-hand end of the beam a 100.0 N weight is hung; an unknown weight $w$ hangs at the left end. If the system is in equilibrium, what is $w$? You can ignore the thickness of the beam. (b) If the beam makes, instead, an angle of 45.0° with the vertical, what is $w$?

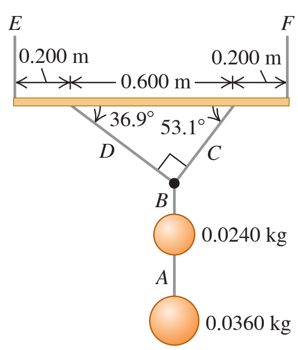
Figure **P11.64**



**11.65** •• The left-hand end of a uniform rod of mass 2.00 kg and length 1.20 m is attached to a vertical wall by a frictionless hinge. The rod is held in a horizontal position by an aluminum wire that runs between the right-hand end of the rod and a point on the wall that is above the hinge. The cross-sectional radius of the wire is 2.50 mm, and the wire makes an angle of 30.0° with the rod. (a) What is the length of the wire? (b) An object of mass 90.0 kg is suspended from the right-hand end of the rod. What is the increase in the length of the wire when this object is added? In your analysis do you need to be concerned that the lengthening of the wire means that the rod is no longer horizontal?

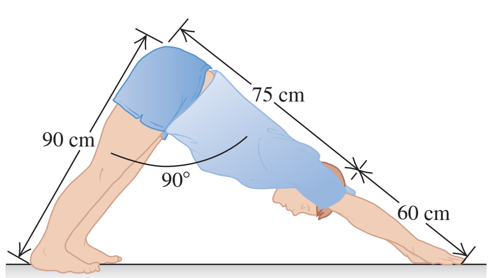
**11.66** • A holiday decoration consists of two shiny glass spheres with masses 0.0240 kg and 0.0360 kg suspended from a uniform rod with mass 0.120 kg and length 1.00 m (**Fig. P11.66**). The rod is suspended from the ceiling by a vertical cord at each end, so that it is horizontal. Calculate the tension in each of the cords *A* through *F*.

Figure **P11.66**




**11.67** •• BIO **Downward-Facing Dog.** The yoga exercise "Downward-Facing Dog" requires stretching your hands straight out above your head and bending down to lean against the floor. This exercise is performed by a 750 N person as shown in **Fig. P11.67**. When he bends his body at the hip to a 90° angle between his legs and trunk, his legs, trunk, head, and arms have the dimensions indicated. Furthermore, his legs and feet weigh a total of 277 N, and their center of mass is 41 cm from his hip, measured along his legs. The person's trunk, head, and arms weigh 473 N, and their center of gravity is 65 cm from his hip, measured along the upper body. (a) Find the normal force that the floor exerts on each foot and on each hand, assuming that the person does not favor either hand or either foot. (b) Find the friction force on each foot and on each hand, assuming that it is the same on both feet and on both hands (but not necessarily the same on the feet as on the hands). [*Hint:* First treat his entire body as a system; then isolate his legs (or his upper body).]

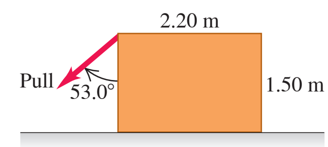
Figure **P11.67**




**11.68** •• CP A brass wire is 1.40 m long and has a cross-sectional area of 6.00 mm$^2$. A small steel ball with mass 0.0800 kg is attached to the end of the wire. You hold the other end of the wire and whirl the ball in a vertical circle of radius 1.40 m. What speed must the ball have at the lowest point of its path if its fractional change in length of the brass wire at this point from its unstretched length is $2.0 \times 10^{-5}$? Treat the ball as a point mass.

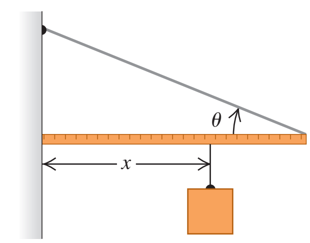
**11.69** • A worker wants to turn over a uniform, 1250 N, rectangular crate by pulling at 53.0° on one of its vertical sides (**Fig. P11.69**). The floor is rough enough to prevent the crate from slipping. (a) What pull is needed to just start the crate to tip? (b) How hard does the floor push upward on the crate? (c) Find the friction force on the crate. (d) What is the minimum coefficient of static friction needed to prevent the crate from slipping on the floor?

Figure **P11.69**



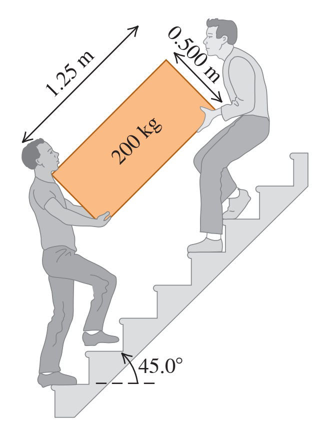

**11.70** ••• One end of a uniform meter stick is placed against a vertical wall (**Fig. P11.70**). The other end is held by a lightweight cord that makes an angle $\theta$ with the stick. The coefficient of static friction between the end of the meter stick and the wall is 0.40. (a) What is the maximum value the angle $\theta$ can have if the stick is to remain in equilibrium? (b) Let the angle $\theta$ be 15°. A block of the same weight as the meter stick is suspended from the stick, as shown, at a distance $x$ from the wall. What is the minimum value of $x$ for which the stick will remain in equilibrium? (c) When $\theta = 15°$, how large must the coefficient of static friction be so that the block can be attached 10 cm from the left end of the stick without causing it to slip?

Figure **P11.70**




**11.71** •• Two friends are carrying a 200 kg crate up a flight of stairs. The crate is 1.25 m long and 0.500 m high, and its center of gravity is at its center. The stairs make a 45.0° angle with respect to the floor. The crate also is carried at a 45.0° angle, so that its bottom side is parallel to the slope of the stairs (**Fig. P11.71**). If the force each person applies is vertical, what is the magnitude of each of these forces? Is it better to be the person above or below on the stairs?

Figure **P11.71**

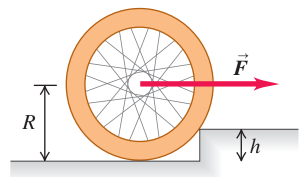



**11.72** ••• In a city park a nonuniform wooden beam 4.00 m long is suspended horizontally by a light steel cable at each end. The cable at the left-hand end makes an angle of 30.0° with the vertical and has tension 620 N. The cable at the right-hand end of the beam makes an angle of 50.0° with the vertical. As an employee of the Parks and Recreation Department, you are asked to find the weight of the beam and the location of its center of gravity.

**11.73** •• CALC BIO Refer to the discussion of holding a dumbbell in Example 11.4 (Section 11.3). The maximum weight that can be held in this way is limited by the maximum allowable tendon tension $T$ (determined by the strength of the tendons) and by the distance $D$ from the elbow to where the tendon attaches to the forearm. (a) Let $T_{max}$ represent the maximum value of the tendon tension. Use the results of Example 11.4 to express $w_{max}$ (the maximum weight that can be held) in terms of $T_{max}$, $L$, $D$, and $h$. Your expression should *not* include the angle $\theta$. (b) The tendons of different primates are attached to the forearm at different values of $D$. Calculate the derivative of $w_{max}$ with respect to $D$, and determine whether the derivative is positive or negative. (c) A chimpanzee tendon is attached to the forearm at a point farther from the elbow than for humans. Use this to explain why chimpanzees have stronger arms than humans. (The disadvantage is that chimpanzees have less flexible arms than do humans.)

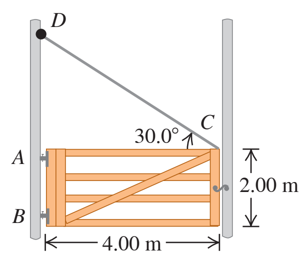
**11.74** •• You are trying to raise a bicycle wheel of mass $m$ and radius $R$ up over a curb of height $h$. To do this, you apply a horizontal force $\vec{F}$ (**Fig. P11.74**). What is the smallest magnitude of the force $\vec{F}$ that will succeed in raising the wheel onto the curb when the force is applied (a) at the center of the wheel and (b) at the top of the wheel? (c) In which case is less force required?

Figure P11.74



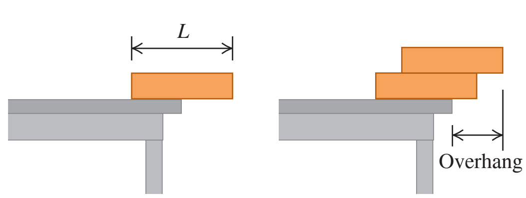

**11.75** • **The Farmyard Gate.** A gate 4.00 m wide and 2.00 m high weighs 700 N. Its center of gravity is at its center, and it is hinged at $A$ and $B$. To relieve the strain on the top hinge, a wire $CD$ is connected as shown in **Fig. P11.75**. The tension in $CD$ is increased until the horizontal force at hinge $A$ is zero. What are (a) the tension in the wire $CD$; (b) the magnitude of the horizontal component of the force at hinge $B$; (c) the combined vertical force exerted by hinges $A$ and $B$?

Figure P11.75



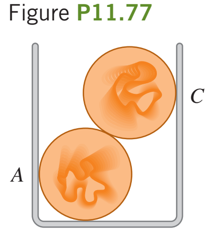

**11.76** • If you put a uniform block at the edge of a table, the center of the block must be over the table for the block not to fall off. (a) If you stack two identical blocks at the table edge, the center of the top block must be over the bottom block, and the center of gravity of the two blocks together must be over the table. In terms of the length $L$ of each block, what is the maximum overhang possible (**Fig. P11.76**)? (b) Repeat part (a) for three identical blocks and for four identical blocks. (c) Is it possible to make a stack of blocks such that the uppermost block is not directly over the table at all? How many blocks would it take to do this? (Try.)

Figure P11.76



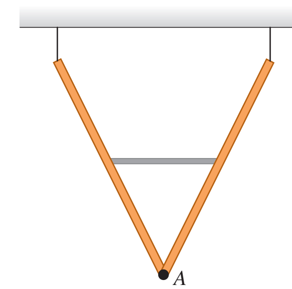

**11.77** ••• Two uniform, 75.0 g marbles 2.00 cm in diameter are stacked as shown in **Fig. P11.77** in a container that is 3.00 cm wide. (a) Find the force that the container exerts on the marbles at the points of contact $A$, $B$, and $C$. (b) What force does each marble exert on the other?

Figure P11.77



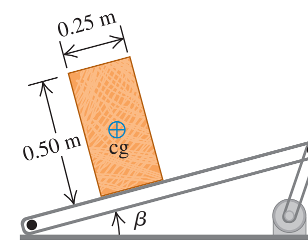

**11.78** •• Two identical, uniform beams weighing 260 N each are connected at one end by a frictionless hinge. A light horizontal crossbar attached at the midpoints of the beams maintains an angle of 53.0° between the beams. The beams are suspended from the ceiling by vertical wires such that they form a "V" (**Fig. P11.78**). (a) What force does the crossbar exert on each beam? (b) Is the crossbar under tension or compression? (c) What force (magnitude and direction) does the hinge at point $A$ exert on each beam?

Figure P11.78



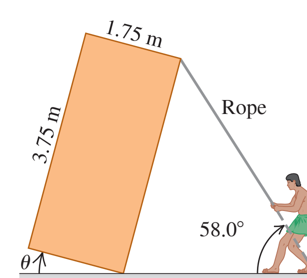

**11.79** • An engineer is designing a conveyor system for loading hay bales into a wagon (**Fig. P11.79**). Each bale is 0.25 m wide, 0.50 m high, and 0.80 m long (the dimension perpendicular to the plane of the figure), with mass 30.0 kg. The center of gravity of each bale is at its geometrical center. The coefficient of static friction between a bale and the conveyor belt is 0.60, and the belt moves with constant speed. (a) The angle $\beta$ of the conveyor is slowly increased. At some critical angle a bale will tip (if it doesn't slip first), and at some different critical angle it will slip (if it doesn't tip first). Find the two critical angles and determine which happens at the smaller angle. (b) Would the outcome of part (a) be different if the coefficient of friction were 0.40?

Figure P11.79



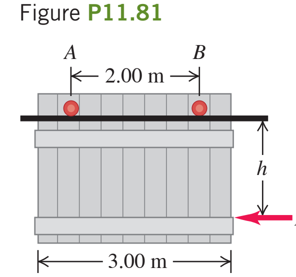

**11.80** ••• **Pyramid Builders.** Ancient pyramid builders are balancing a uniform rectangular stone slab of weight $w$, tipped at an angle $\theta$ above the horizontal, using a rope (**Fig. P11.80**). The rope is held by five workers who share the force equally. (a) If $\theta = 20.0°$, what force does each worker exert on the rope? (b) As $\theta$ increases, does each worker have to exert more or less force than in part (a), assuming they do not change the angle of the rope? Why? (c) At what angle do the workers need to exert *no force* to balance the slab? What happens if $\theta$ exceeds this value?

Figure P11.80




**11.81** •• A garage door is mounted on an overhead rail (**Fig. P11.81**). The wheels at $A$ and $B$ have rusted so that they do not roll, but rather slide along the track. The coefficient of kinetic friction is 0.52. The distance between the wheels is 2.00 m, and each is 0.50 m from the vertical sides of the door. The door is uniform and weighs 950 N. It is pushed to the left at constant speed by a horizontal force $\vec{F}$, that is applied as shown in the figure. (a) If the distance $h$ is 1.60 m, what is the vertical component of the force exerted on each wheel by the track? (b) Find the maximum value $h$ can have without causing one wheel to leave the track.

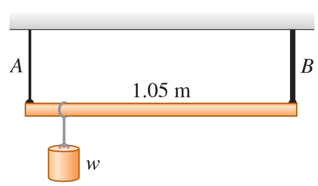
Figure P11.81



**11.82 ••• CP** A 12.0 kg mass, fastened to the end of an aluminum rod with an unstretched length of 0.70 m, is whirled in a vertical circle with a constant angular speed of 120 rev/min. The cross-sectional area of the rod is $0.014 \text{ cm}^2$. Calculate the elongation of the rod when the mass is (a) at the lowest point of the path and (b) at the highest point of its path.

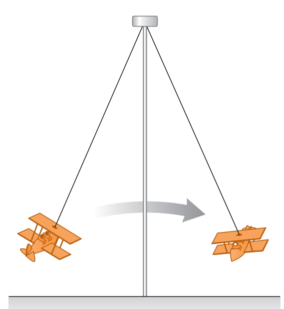
**11.83 •••** A 1.05-m-long rod of negligible weight is supported at its ends by wires $A$ and $B$ of equal length (**Fig. P11.83**). The cross-sectional area of $A$ is $2.00 \text{ mm}^2$ and that of $B$ is $4.00 \text{ mm}^2$. Young's modulus for wire $A$ is $1.80 \times 10^{11}$ Pa; that for $B$ is $1.20 \times 10^{11}$ Pa. At what point along the rod should a weight $w$ be suspended to produce (a) equal stresses in $A$ and $B$ and (b) equal strains in $A$ and $B$?

Figure **P11.83**




**11.84 ••• CP** An amusement park ride consists of airplane-shaped cars attached to steel rods (**Fig. P11.84**). Each rod has a length of 15.0 m and a cross-sectional area of $8.00 \text{ cm}^2$. The rods are attached to a frictionless hinge at the top, so that the cars can swing outward when the ride rotates. (a) How much is each rod stretched when it is vertical and the ride is at rest? (Assume that each car plus two people seated in it has a total weight of 1900 N.) (b) When operating, the ride has a maximum angular speed of 12.0 rev/min. How much is the rod stretched then?

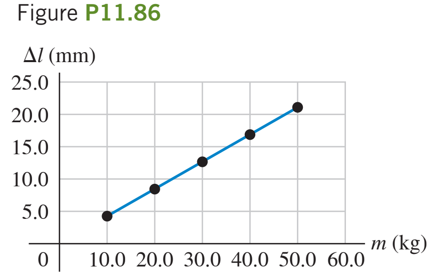
Figure **P11.84**



**11.85 ••• CP BIO Stress on the Shin Bone.** The compressive strength of our bones is important in everyday life. Young's modulus for bone is about $1.4 \times 10^{10}$ Pa. Bone can take only about a 1.0% change in its length before fracturing. (a) What is the maximum force that can be applied to a bone whose minimum cross-sectional area is $3.0 \text{ cm}^2$? (This is approximately the cross-sectional area of a tibia, or shin bone, at its narrowest point.) (b) Estimate the maximum height from which a 70 kg man could jump and not fracture his tibia. Take the time between when he first touches the floor and when he has stopped to be 0.030 s, and assume that the stress on his two legs is distributed equally.

**11.86 •• DATA** You are to use a long, thin wire to build a pendulum in a science museum. The wire has an unstretched length of 22.0 m and a circular cross section of diameter 0.860 mm; it is made of an alloy that has a large breaking stress. One end of the wire will be attached to the ceiling, and a 9.50 kg metal sphere will be attached to the other end. As the pendulum swings back and forth, the wire's maximum angular displacement from the vertical will be 36.0°. You must determine the maximum amount the wire will stretch during this motion. So, before you attach the metal sphere, you suspend a test mass (mass $m$) from the wire's lower end. You then measure the increase in length $\Delta l$ of the wire for several different test masses. **Figure P11.86**, a graph of $\Delta l$ versus $m$, shows the results and the straight line that gives the best fit to the data. The equation for this line is $\Delta l = (0.422 \text{ mm/kg})m$. (a) Assume that $g = 9.80 \text{ m/s}^2$, and use Fig. P11.86 to calculate Young's modulus $Y$ for this wire. (b) You remove the test masses, attach the 9.50 kg sphere, and release the sphere from rest, with the wire displaced by 36.0°. Calculate the amount the wire will stretch as it swings through the vertical. Ignore air resistance.

Figure **P11.86**




**11.87 •• DATA** You need to measure the mass $M$ of a 4.00-m-long bar. The bar has a square cross section but has some holes drilled along its length, so you suspect that its center of gravity isn't in the middle of the bar. The bar is too long for you to weigh on your scale. So, first you balance the bar on a knife-edge pivot and determine that the bar's center of gravity is 1.88 m from its left-hand end. You then place the bar on the pivot so that the point of support is 1.50 m from the left-hand end of the bar. Next you suspend a 2.00 kg mass $(m_1)$ from the bar at a point 0.200 m from the left-hand end. Finally, you suspend a mass $m_2 = 1.00$ kg from the bar at a distance $x$ from the left-hand end and adjust $x$ so that the bar is balanced. You repeat this step for other values of $m_2$ and record each corresponding value of $x$. The table gives your results.

| $m_2$ (kg) | 1.00 | 1.50 | 2.00 | 2.50 | 3.00 | 4.00 |
|---|---|---|---|---|---|---|
| $x$ (m) | 3.50 | 2.83 | 2.50 | 2.32 | 2.16 | 2.00 |

(a) Draw a free-body diagram for the bar when $m_1$ and $m_2$ are suspended from it. (b) Apply the static equilibrium equation $\sum \tau_z = 0$ with the axis at the location of the knife-edge pivot. Solve the equation for $x$ as a function of $m_2$. (c) Plot $x$ versus $1/m_2$. Use the slope of the best-fit straight line and the equation you derived in part (b) to calculate that bar's mass $M$. Use $g = 9.80 \text{ m/s}^2$. (d) What is the $y$-intercept of the straight line that fits the data? Explain why it has this value.

**11.88 ••• DATA** You are a construction engineer working on the interior design of a retail store in a mall. A 2.00-m-long uniform bar of mass 8.50 kg is to be attached at one end to a wall, by means of a hinge that allows the bar to rotate freely with very little friction. The bar will be held in a horizontal position by a light cable from a point on the bar (a distance $x$ from the hinge) to a point on the wall above the hinge. The cable makes an angle $\theta$ with the bar. The architect has proposed four possible ways to connect the cable and asked you to assess them:

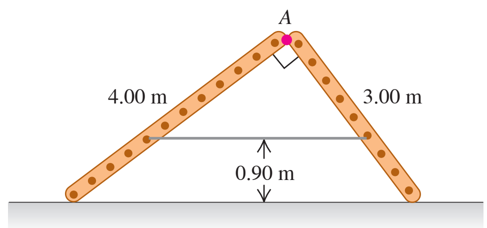
| Alternative | A | B | C | D |
|---|---|---|---|---|
| $x$ (m) | 2.00 | 1.50 | 0.75 | 0.50 |
| $\theta$ (degrees) | 30 | 60 | 37 | 75 |

(a) There is concern about the strength of the cable that will be required. Which set of $x$ and $\theta$ values in the table produces the smallest tension in the cable? The greatest? (b) There is concern about the breaking strength of the sheetrock wall where the hinge will be attached. Which set of $x$ and $\theta$ values produces the smallest horizontal component of the force the bar exerts on the hinge? The largest? (c) There is also concern about the required strength of the hinge and the strength of its attachment to the wall. Which set of $x$ and $\theta$ values produces the smallest magnitude of the vertical component of the force the bar exerts on the hinge? The largest? (*Hint:* Does the direction of the vertical component of the force the hinge exerts on the bar depend on where along the bar the cable is attached?) (d) Is one of the alternatives given in the table preferable? Should any of the alternatives be avoided? Discuss.

## CHALLENGE PROBLEMS

**11.89 •••** Two ladders, 4.00 m and 3.00 m long, are hinged at point $A$ and tied together by a horizontal rope 0.90 m above the floor (**Fig. P11.89**). The ladders weigh 480 N and 360 N, respectively, and the center of gravity of each is at its center. Assume that the floor is freshly waxed and frictionless. (a) Find the upward force at the bottom of each ladder. (b) Find the tension in the rope. (c) Find the magnitude

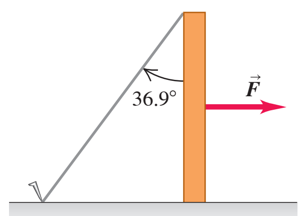
Figure **P11.89**




of the force one ladder exerts on the other at point $A$. (d) If an 800 N painter stands at point $A$, find the tension in the horizontal rope.

**11.90 ••• Knocking Over a Post.** One end of a post weighing 400 N and with height $h$ rests on a rough horizontal surface with $\mu_s = 0.30$. The upper end is held by a rope fastened to the surface and making an angle of 36.9° with the post (**Fig. P11.90**). A horizontal force $\vec{F}$ is exerted on the post as shown. (a) If the force $\vec{F}$ is applied at the midpoint of the post, what is the largest value it can have without causing the post to slip? (b) How large can the force be without causing the post to slip if its point of application is $\frac{6}{10}$ of the way from the ground to the top of the post? (c) Show that if the point of application of the force is too high, the post cannot be made to slip, no matter how great the force. Find the critical height for the point of application.

Figure **P11.90**

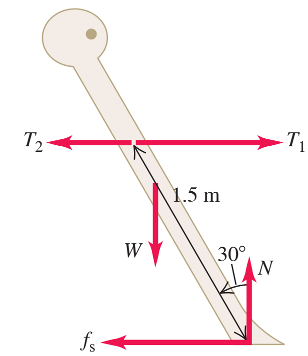



**11.91 ••• CP** An angler hangs a 4.50 kg fish from a vertical steel wire 1.50 m long and $5.00 \times 10^{-3}$ cm$^2$ in cross-sectional area. The upper end of the wire is securely fastened to a support. (a) Calculate the amount the wire is stretched by the hanging fish. The angler now applies a varying force $\vec{F}$ at the lower end of the wire, pulling it very slowly downward by 0.500 mm from its equilibrium position. For this downward motion, calculate (b) the work done by gravity; (c) the work done by the force $\vec{F}$; (d) the work done by the force the wire exerts on the fish; and (e) the change in the elastic potential energy (the potential energy associated with the tensile stress in the wire). Compare the answers in parts (d) and (e).

## MCAT-STYLE PASSAGE PROBLEMS

**BIO Torques and Tug-of-War.** In a study of the biomechanics of the tug-of-war, a 2.0-m-tall, 80.0 kg competitor in the middle of the line is considered to be a rigid body leaning back at an angle of 30.0° to the vertical. The competitor is pulling on a rope that is held horizontal a distance of 1.5 m from his feet (as measured along the line of the body). At the moment shown in the figure, the man is stationary and the tension in the rope in front of him is $T_1 = 1160$ N. Since there is friction between the rope and his hands, the tension in the rope behind him, $T_2$, is not equal to $T_1$. His center of mass is halfway between his feet and the top of his head. The coefficient of static friction between his feet and the ground is 0.65.

Figure **11.92** Side view.




**11.92** What is tension $T_2$ in the rope behind him? (a) 590 N; (b) 650 N; (c) 860 N; (d) 1100 N.

**11.93** If he leans slightly farther back (increasing the angle between his body and the vertical) but remains stationary in this new position, which of the following statements is true? Assume that the rope remains horizontal. (a) The difference between $T_1$ and $T_2$ will increase, balancing the increased torque about his feet that his weight produces when he leans farther back; (b) the difference between $T_1$ and $T_2$ will decrease, balancing the increased torque about his feet that his weight produces when he leans farther back; (c) neither $T_1$ nor $T_2$ will change, because no other forces are changing; (d) both $T_1$ and $T_2$ will change, but the difference between them will remain the same.

**11.94** His body is again leaning back at 30.0° to the vertical, but now the height at which the rope is held above—but still parallel to—the ground is varied. The tension in the rope in front of the competitor ($T_1$) is measured as a function of the shortest distance between the rope and the ground (the holding height). Tension $T_1$ is found to decrease as the holding height increases. What could explain this observation? As the holding height increases, (a) the moment arm of the rope about his feet decreases due to the angle that his body makes with the vertical; (b) the moment arm of the weight about his feet decreases due to the angle that his body makes with the vertical; (c) a smaller tension in the rope is needed to produce a torque sufficient to balance the torque of the weight about his feet; (d) his center of mass moves down to compensate, so less tension in the rope is required to maintain equilibrium.

**11.95** His body is leaning back at 30.0° to the vertical, but the coefficient of static friction between his feet and the ground is suddenly reduced to 0.50. What will happen? (a) His entire body will accelerate forward; (b) his feet will slip forward; (c) his feet will slip backward; (d) his feet will not slip.

# ANSWERS

## Chapter Opening Question ?

**(i)** Each stone in the arch is under compression, not tension. This is because the forces on the stones tend to push them inward toward the center of the arch and thus squeeze them together. Compared to a solid supporting wall, a wall with arches is just as strong yet much more economical to build.

## Key Example √ARIATION Problems

**VP11.1.1** 13.3 kg
**VP11.1.2** 11.0 m
**VP11.1.3** $L/10$ to the right of center
**VP11.1.4** 141 kg
**VP11.4.1** 5.86 m
**VP11.4.2** nose wheel: 19.8%, main wheels: 80.2%
**VP11.4.3** (a) $F_{\text{hinge-}y} = 0$ (b) $T = w/\sin\theta$ (c) $F_{\text{hinge-}x} = w/\tan\theta$
**VP11.4.4** (a) $F_{\text{hinge-}y} = w/2$ (b) $T = 3w/2\sin\theta$ (c) $F_{\text{hinge-}x} = 3w/2\tan\theta$
**VP11.7.1** (a) $5.5 \times 10^8$ Pa (b) $3.5 \times 10^4$ N
**VP11.7.2** (a) $2.3 \times 10^{-4}$ (b) 0.19 mm
**VP11.7.3** (a) $4.4 \times 10^{-9}\,\text{m}^3$ (b) $6.7 \times 10^{-9}\,\text{m}^3$
**VP11.7.4** $4.8 \times 10^{-5}$ m

## Bridging Problem

(a) $T = \dfrac{2mg}{3\sin\theta}$

(b) $F = \dfrac{2mg}{3\sin\theta}\sqrt{\cos^2\theta + \frac{1}{4}\sin^2\theta}, \quad \phi = \arctan\left(\frac{1}{2}\tan\theta\right)$

(c) $\Delta l = \dfrac{mgl_0}{2AY\tan\theta}$

(d) 4