? Each blade on a rotating airplane propeller is like a long, thin rod. If each blade were stretched to double its length (while the mass of each blade and the propeller's angular speed stay the same), by what factor would the kinetic energy of the rotating blades increase? (i) 2; (ii) 4; (iii) 8; (iv) the kinetic energy would not change; (v) the kinetic energy would decrease, not increase.

9 Rotation of Rigid Bodies

LEARNING OUTCOMES

In this chapter, you'll learn...

9.1 How to describe the rotation of a rigid body in terms of angular coordinate, angular velocity, and angular acceleration.
9.2 How to analyze rigid-body rotation when the angular acceleration is constant.
9.3 How to relate the rotation of a rigid body to the linear velocity and linear acceleration of a point on the body.
9.4 The meaning of a body's moment of inertia about a rotation axis, and how it relates to rotational kinetic energy.
9.5 How to relate the values of a body's moment of inertia for two different but parallel rotation axes.
9.6 How to calculate the moment of inertia of bodies with various shapes.

You'll need to review...

1.10 Vector product of two vectors.
2.2–2.4 Linear velocity, linear acceleration, and motion with constant acceleration.
3.4 Motion in a circle.
7.1 Using mechanical energy to solve problems.

What do the motions of an airplane propeller, a Blu-ray disc, a Ferris wheel, and a circular saw blade have in common? None of these can be represented adequately as a moving *point;* each involves an object that *rotates* about an axis that is stationary in some inertial frame of reference.

Rotation occurs at all scales, from the motions of electrons in atoms to the motions of entire galaxies. We need to develop some general methods for analyzing the motion of a rotating object. In this chapter and the next we consider objects that have definite size and definite shape, and that in general can have rotational as well as translational motion.

Real-world objects can be very complicated; the forces that act on them can deform them—stretching, twisting, and squeezing them. We'll ignore these deformations for now and assume that the object has a perfectly definite and unchanging shape and size. We call this idealized model a **rigid body.** This chapter and the next are mostly about rotational motion of a rigid body.

We begin with kinematic language for *describing* rotational motion. Next we look at the kinetic energy of rotation, the key to using energy methods for rotational motion. Then in Chapter 10 we'll develop dynamic principles that relate the forces on a body to its rotational motion.

9.1 ANGULAR VELOCITY AND ACCELERATION

In analyzing rotational motion, let's think first about a rigid body that rotates about a *fixed axis*—an axis that is at rest in some inertial frame of reference and does not change direction relative to that frame. The rotating rigid body might be a motor shaft, a chunk of beef on a barbecue skewer, or a merry-go-round.

Figure 9.1 shows a rigid body rotating about a fixed axis. The axis passes through point O and is perpendicular to the plane of the diagram, which we'll call the xy-plane. One way to describe the rotation of this body would be to choose a particular point P on the body and to keep track of the x- and y-coordinates of P. This isn't very convenient, since it takes two numbers (the two coordinates x and y) to specify the rotational position of the body. Instead, we notice that the line OP is fixed in the body and rotates with it. The angle θ that OP makes with the $+x$-axis is a single **angular cooordinate** that completely describes the body's rotational position.

The angular coordinate θ of a rigid body rotating around a fixed axis can be positive or negative. If we choose positive angles to be measured counterclockwise from the positive x-axis, then the angle θ in Fig. 9.1 is positive. If we instead choose the positive rotation direction to be clockwise, then θ in Fig. 9.1 is negative. When we considered the motion of a particle along a straight line, it was essential to specify the direction of positive displacement along that line; when we discuss rotation around a fixed axis, it's just as essential to specify the direction of positive rotation.

The most natural way to measure the angle θ is not in degrees but in **radians.** As **Fig. 9.2a** shows, one radian (1 rad) is the angle subtended at the center of a circle by an arc with a length equal to the radius of the circle. In Fig. 9.2b an angle θ is subtended by an arc of length s on a circle of radius r. The value of θ (in radians) is equal to s divided by r:

$$\theta = \frac{s}{r} \quad \text{or} \quad s = r\theta \quad (\theta \text{ in radians}) \tag{9.1}$$

An angle in radians is the ratio of two lengths, so it is a pure number, without dimensions. If $s = 3.0$ m and $r = 2.0$ m, then $\theta = 1.5$, but we'll often write this as 1.5 rad to distinguish it from an angle measured in degrees or revolutions.

The circumference of a circle (that is, the arc length all the way around the circle) is 2π times the radius, so there are 2π (about 6.283) radians in one complete revolution (360°). Therefore

$$1 \text{ rad} = \frac{360°}{2\pi} = 57.3°$$

Similarly, $180° = \pi$ rad, $90° = \pi/2$ rad, and so on. If we had measured angle θ in degrees, we would have needed an extra factor of $(2\pi/360)$ on the right-hand side of $s = r\theta$ in Eq. (9.1). By measuring angles in radians, we keep the relationship between angle and distance along an arc as simple as possible.

Angular Velocity

The coordinate θ shown in Fig. 9.1 specifies the rotational position of a rigid body at a given instant. We can describe the rotational *motion* of such a rigid body in terms of the rate of change of θ. We'll do this in an analogous way to our description of straight-line motion in Chapter 2. In **Fig. 9.3a**, a reference line OP in a rotating body makes an angle θ_1 with the $+x$-axis at time t_1. At a later time t_2 the angle has changed to θ_2. We define the **average angular velocity** $\omega_{\text{av-}z}$ (the Greek letter omega) of the body in the time interval $\Delta t = t_2 - t_1$ as the ratio of the **angular displacement** $\Delta\theta = \theta_2 - \theta_1$ to Δt:

$$\omega_{\text{av-}z} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t} \tag{9.2}$$

Figure 9.1 A speedometer needle (an example of a rigid body) rotating counterclockwise about a fixed axis.

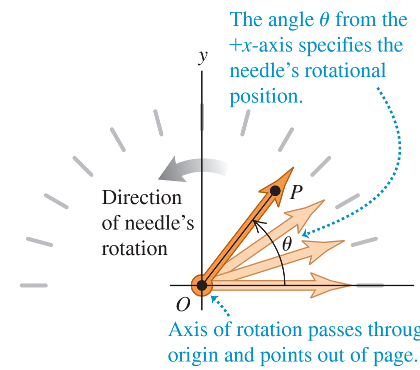
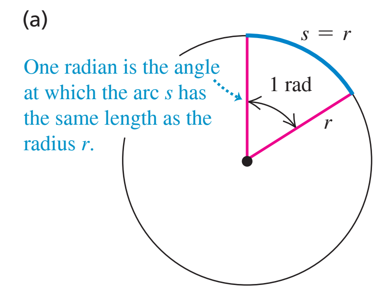
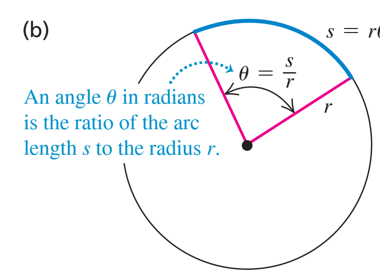


Figure 9.2 Measuring angles in radians.





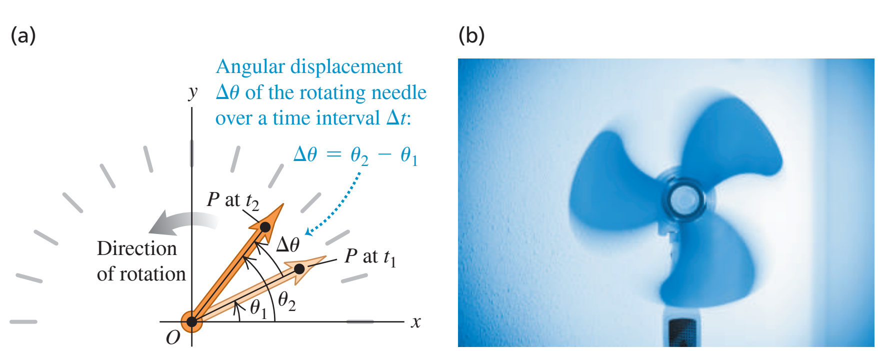


Figure 9.3 (a) Angular displacement $\Delta\theta$ of a rotating body. (b) Every part of a rotating rigid body has the same average angular velocity $\Delta\theta/\Delta t$.

The subscript z indicates that the body in Fig. 9.3a is rotating about the z-axis, which is perpendicular to the plane of the diagram. The **instantaneous angular velocity** ω_z is the limit of $\omega_{\text{av-}z}$ as Δt approaches zero:

The **instantaneous angular velocity** of a rigid body rotating around the z-axis ...

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \tag{9.3}$$

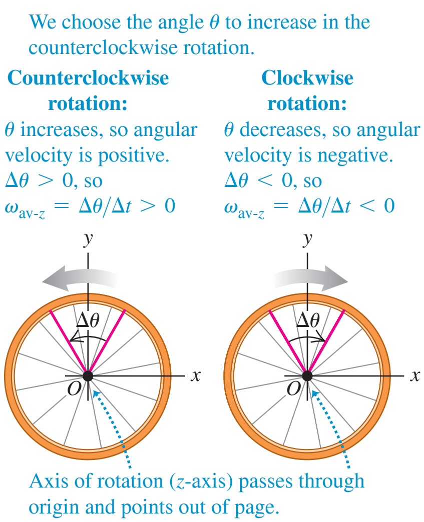
... equals the limit of the body's average angular velocity as the time interval approaches zero ...

... and equals the instantaneous rate of change of the body's angular coordinate.

When we refer simply to "angular velocity," we mean the instantaneous angular velocity, not the average angular velocity.

The angular velocity ω_z can be positive or negative, depending on the direction in which the rigid body is rotating (**Fig. 9.4**). The angular *speed* ω, which we'll use in Sections 9.3 and 9.4, is the magnitude of angular velocity. Like linear speed v, the angular speed is never negative.

Figure 9.4 A rigid body's average angular velocity (shown here) and instantaneous angular velocity can be positive or negative.



CAUTION **Angular velocity vs. linear velocity** Keep in mind the distinction between *angular* velocity ω_z and *linear* velocity v_x (see Section 2.2). If an object has a linear velocity v_x, the object as a whole is *moving* along the x-axis. By contrast, if an object has an angular velocity ω_z, then it is *rotating* around the z-axis. We do *not* mean that the object is moving along the z-axis.

Different points on a rotating rigid body move different distances in a given time interval, depending on how far each point lies from the rotation axis. But because the body is rigid, *all* points rotate through the same angle in the same time (Fig. 9.3b). Hence *at any instant, every part of a rotating rigid body has the same angular velocity.*

If angle θ is in radians, the unit of angular velocity is the radian per second (rad/s). Other units, such as the revolution per minute (rev/min or rpm), are often used. Since 1 rev $= 2\pi$ rad, two useful conversions are

$$1 \text{ rev/s} = 2\pi \text{ rad/s} \qquad \text{and} \qquad 1 \text{ rev/min} = 1 \text{ rpm} = \frac{2\pi}{60} \text{ rad/s}$$

That is, 1 rad/s is about 10 rpm.

EXAMPLE 9.1 Calculating angular velocity

The angular position θ of a 0.36-m-diameter flywheel is given by

$$\theta = (2.0 \text{ rad/s}^3)t^3$$

(a) Find θ, in radians and in degrees, at $t_1 = 2.0$ s and $t_2 = 5.0$ s. (b) Find the distance that a particle on the flywheel rim moves from $t_1 = 2.0$ s to $t_2 = 5.0$ s. (c) Find the average angular velocity, in rad/s and in rev/min, over that interval. (d) Find the instantaneous angular velocities at $t_1 = 2.0$ s and $t_2 = 5.0$ s.

IDENTIFY and SET UP Our target variables are θ_1 and θ_2 (the angular positions at times t_1 and t_2) and the angular displacement $\Delta\theta = \theta_2 - \theta_1$. We'll find these from the given expression for θ as a function of time. Knowing $\Delta\theta$, we'll find the distance traveled and the average angular velocity between t_1 and t_2 by using Eqs. (9.1) and (9.2), respectively. To find the instantaneous angular velocities ω_{1z} (at time t_1) and ω_{2z} (at time t_2), we'll take the derivative of the given equation for θ with respect to time, as in Eq. (9.3).

EXECUTE (a) We substitute the values of t into the equation for θ:

$$\theta_1 = (2.0 \text{ rad/s}^3)(2.0 \text{ s})^3 = 16 \text{ rad}$$
$$= (16 \text{ rad})\frac{360°}{2\pi \text{ rad}} = 920°$$
$$\theta_2 = (2.0 \text{ rad/s}^3)(5.0 \text{ s})^3 = 250 \text{ rad}$$
$$= (250 \text{ rad})\frac{360°}{2\pi \text{ rad}} = 14{,}000°$$

(b) During the interval from t_1 to t_2 the flywheel's angular displacement is $\Delta\theta = \theta_2 - \theta_1 = 250 \text{ rad} - 16 \text{ rad} = 234$ rad. The radius r is half the diameter, or 0.18 m. To use Eq. (9.1), the angles *must* be expressed in radians:

$$s = r\theta_2 - r\theta_1 = r\Delta\theta = (0.18 \text{ m})(234 \text{ rad}) = 42 \text{ m}$$

We can drop "radians" from the unit for s because θ is a dimensionless number; like r, s is measured in meters.

(c) From Eq. (9.2),

$$\omega_{\text{av-}z} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{250\text{ rad} - 16\text{ rad}}{5.0\text{ s} - 2.0\text{ s}} = 78\text{ rad/s}$$
$$= \left(78\,\frac{\text{rad}}{\text{s}}\right)\left(\frac{1\text{ rev}}{2\pi\text{ rad}}\right)\left(\frac{60\text{ s}}{1\text{ min}}\right) = 740\text{ rev/min}$$

(d) From Eq. (9.3),

$$\omega_z = \frac{d\theta}{dt} = \frac{d}{dt}[(2.0\text{ rad/s}^3)t^3] = (2.0\text{rad/s}^3)(3t^2) = (6.0\text{rad/s}^3)t^2$$

At times $t_1 = 2.0$ s and $t_2 = 5.0$ s we have

$$\omega_{1z} = (6.0\text{ rad/s}^3)(2.0\text{ s})^2 = 24\text{ rad/s}$$
$$\omega_{2z} = (6.0\text{ rad/s}^3)(5.0\text{ s})^2 = 150\text{ rad/s}$$

EVALUATE The angular velocity $\omega_z = (6.0\text{ rad/s}^3)t^2$ increases with time. Our results are consistent with this; the instantaneous angular velocity at the end of the interval ($\omega_{2z} = 150$ rad/s) is greater than at the beginning ($\omega_{1z} = 24$ rad/s), and the average angular velocity $\omega_{\text{av-}z} = 78$ rad/s over the interval is intermediate between these two values.

KEYCONCEPT To find the *average* angular velocity of a rotating rigid body, first find the body's angular displacement (final angular position minus initial angular position) during a time interval. Then divide the result by that time interval. To find the rigid body's *instantaneous* angular velocity, take the derivative of its angular position with respect to time.

Angular Velocity as a Vector

As we have seen, our notation for the angular velocity ω_z about the z-axis is reminiscent of the notation v_x for the ordinary velocity along the x-axis (see Section 2.2). Just as v_x is the x-component of the velocity vector $\vec{v}$, ω_z is the z-component of an angular velocity *vector* $\vec{\omega}$ directed along the axis of rotation. As **Fig. 9.5a** shows, the direction of $\vec{\omega}$ is given by the right-hand rule that we used to define the vector product in Section 1.10. If the rotation is about the z-axis, then $\vec{\omega}$ has only a z-component. This component is positive if $\vec{\omega}$ is along the positive z-axis and negative if $\vec{\omega}$ is along the negative z-axis (Fig. 9.5b).

The vector formulation is especially useful when the direction of the rotation axis *changes*. We'll examine such situations briefly at the end of Chapter 10. In this chapter, however, we'll consider only situations in which the rotation axis is fixed. Hence throughout this chapter we'll use "angular velocity" to refer to ω_z, the component of $\vec{\omega}$ along the axis.

CAUTION **The angular velocity vector is perpendicular to the plane of rotation, not in it** It's a common error to think that an object's angular velocity vector $\vec{\omega}$ points in the direction in which some particular part of the object is moving. Another error is to think that $\vec{\omega}$ is a "curved vector" that points around the rotation axis in the direction of rotation (like the curved arrows in Figs. 9.1, 9.3, and 9.4). Neither of these is true! Angular velocity is an attribute of the *entire* rotating rigid body, not any one part, and there's no such thing as a curved vector. We choose the direction of $\vec{\omega}$ to be along the rotation axis—*perpendicular* to the plane of rotation—because that axis is common to every part of a rotating rigid body.

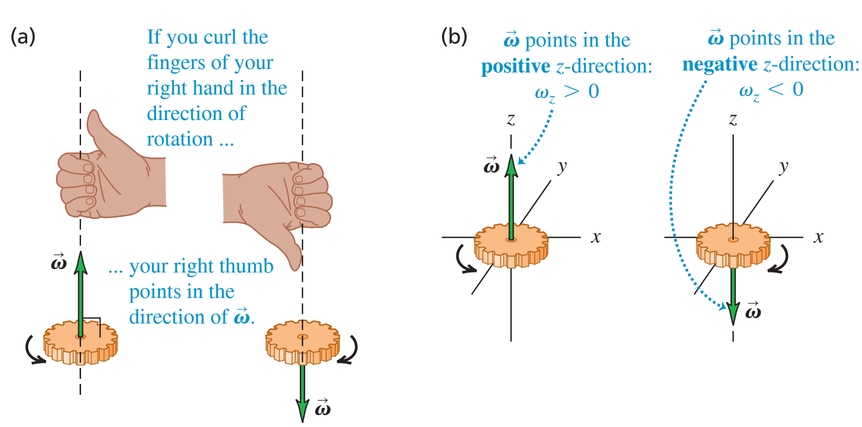


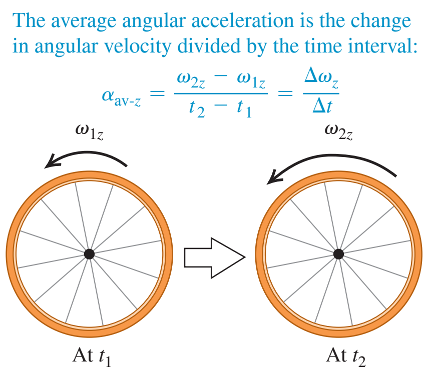
Figure 9.5 (a) The right-hand rule for the direction of the angular velocity vector $\vec{\omega}$. Reversing the direction of rotation reverses the direction of $\vec{\omega}$. (b) The sign of ω_z for rotation along the z-axis.

Angular Acceleration

A rigid body whose angular velocity changes has an *angular acceleration*. When you pedal your bicycle harder to make the wheels turn faster or apply the brakes to bring the wheels to a stop, you're giving the wheels an angular acceleration.

If ω_{1z} and ω_{2z} are the instantaneous angular velocities at times t_1 and t_2, we define the **average angular acceleration** $\alpha_{\text{av-}z}$ over the interval $\Delta t = t_2 - t_1$ as the change in angular velocity divided by Δt (**Fig. 9.6**):

Figure 9.6 Calculating the average angular acceleration of a rotating rigid body.



$$\alpha_{\text{av-}z} = \frac{\omega_{2z} - \omega_{1z}}{t_2 - t_1} = \frac{\Delta\omega_z}{\Delta t} \tag{9.4}$$

The **instantaneous angular acceleration** α_z is the limit of $\alpha_{\text{av-}z}$ as $\Delta t \to 0$:

The **instantaneous angular acceleration** of a rigid body rotating around the z-axis ...

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta\omega_z}{\Delta t} = \frac{d\omega_z}{dt} \tag{9.5}$$

... equals the limit of the body's average angular acceleration as the time interval approaches zero ...

... and equals the instantaneous rate of change of the body's angular velocity.

The usual unit of angular acceleration is the radian per second per second, or rad/s^2. From now on we'll use the term "angular acceleration" to mean the instantaneous angular acceleration rather than the average angular acceleration.

Because $\omega_z = d\theta/dt$, we can also express angular acceleration as the second derivative of the angular coordinate:

$$\alpha_z = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2} \tag{9.6}$$

You've probably noticed that we use Greek letters for angular kinematic quantities: θ for angular position, ω_z for angular velocity, and α_z for angular acceleration. These are analogous to x for position, v_x for velocity, and a_x for acceleration in straight-line motion. In each case, velocity is the rate of change of position with respect to time and acceleration is the rate of change of velocity with respect to time. We sometimes use the terms "*linear* velocity" for v_x and "*linear* acceleration" for a_x to distinguish clearly between these and the *angular* quantities introduced in this chapter.

If the angular acceleration α_z is positive, then the angular velocity ω_z is increasing; if α_z is negative, then ω_z is decreasing. The rotation is speeding up if α_z and ω_z have the same sign and slowing down if α_z and ω_z have opposite signs. (These are exactly the same relationships as those between *linear* acceleration a_x and *linear* velocity v_x for straight-line motion; see Section 2.3.)

EXAMPLE 9.2 Calculating angular acceleration

For the flywheel of Example 9.1, (a) find the average angular acceleration between $t_1 = 2.0$ s and $t_2 = 5.0$ s. (b) Find the instantaneous angular accelerations at $t_1 = 2.0$ s and $t_2 = 5.0$ s.

IDENTIFY and SET UP We use Eqs. (9.4) and (9.5) for the average and instantaneous angular accelerations.

EXECUTE (a) From Example 9.1, the values of ω_z at the two times are

$$\omega_{1z} = 24 \text{ rad/s} \qquad \omega_{2z} = 150 \text{ rad/s}$$

From Eq. (9.4), the average angular acceleration is

$$\alpha_{\text{av-}z} = \frac{150 \text{ rad/s} - 24 \text{ rad/s}}{5.0 \text{ s} - 2.0 \text{ s}} = 42 \text{ rad/s}^2$$

(b) We found in Example 9.1 that $\omega_z = (6.0 \text{ rad/s}^3)t^2$ for the flywheel. From Eq. (9.5), the value of α_z at any time t is

$$\alpha_z = \frac{d\omega_z}{dt} = \frac{d}{dt}[(6.0 \text{ rad/s}^3)(t^2)] = (6.0 \text{ rad/s}^3)(2t)$$
$$= (12 \text{ rad/s}^3)t$$

Hence

$$\alpha_{1z} = (12 \text{ rad/s}^3)(2.0 \text{ s}) = 24 \text{ rad/s}^2$$
$$\alpha_{2z} = (12 \text{ rad/s}^3)(5.0 \text{ s}) = 60 \text{ rad/s}^2$$

EVALUATE The angular acceleration is *not* constant in this situation. The angular velocity ω_z is always increasing because α_z is always positive. Furthermore, the rate at which angular velocity increases is itself increasing, since α_z increases with time.

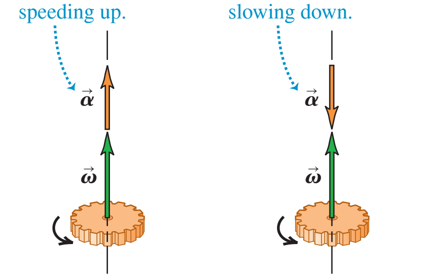
KEYCONCEPT To find the average angular acceleration of a rotating rigid body, first find the change in its angular velocity (final angular velocity minus initial angular velocity) during a time interval. Then divide the result by the time interval.

Angular Acceleration as a Vector

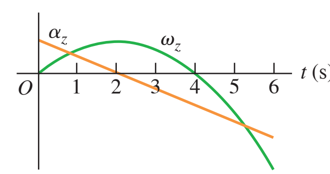
Just as we did for angular velocity, it's useful to define an angular acceleration *vector* $\vec{\boldsymbol{\alpha}}$. Mathematically, $\vec{\boldsymbol{\alpha}}$ is the time derivative of the angular velocity vector $\vec{\boldsymbol{\omega}}$. If the object rotates around the fixed z-axis, then $\vec{\boldsymbol{\alpha}}$ has only a z-component α_z. In this case, $\vec{\boldsymbol{\alpha}}$ is in the same direction as $\vec{\boldsymbol{\omega}}$ if the rotation is speeding up and opposite to $\vec{\boldsymbol{\omega}}$ if the rotation is slowing down (**Fig. 9.7**).

The vector $\vec{\boldsymbol{\alpha}}$ will be particularly useful in Chapter 10 when we discuss what happens when the rotation axis changes direction. In this chapter, however, the rotation axis will always be fixed and we need only the z-component α_z.

Figure **9.7** When the rotation axis is fixed, both the angular acceleration and angular velocity vectors lie along that axis.



TEST YOUR UNDERSTANDING OF SECTION 9.1 The figure shows a graph of ω_z and α_z versus time for a particular rotating body. (a) During which time intervals is the rotation speeding up? (i) $0 < t < 2$ s; (ii) $2\text{ s} < t < 4$ s; (iii) $4\text{ s} < t < 6$ s. (b) During which time intervals is the rotation slowing down? (i) $0 < t < 2$ s; (ii) $2\text{ s} < t < 4$ s; (iii) $4\text{ s} < t < 6$ s.



ANSWER

(a) (**i**) **and** (**iii**), (b) (**ii**) The rotation is speeding up when the angular velocity and angular acceleration have the same sign, and slowing down when they have opposite signs. Hence it is speeding up for $0 < t < 2$ s (both ω_z and α_z are positive) and for $4\text{ s} < t < 6$ s (both ω_z and α_z are negative) but is slowing down for $2\text{ s} < t < 4$ s (ω_z is positive and α_z is negative). Note that the body is rotating in one direction for $t < 4$ s (ω_z is positive) and in the opposite direction for $t > 4$ s (ω_z is negative).

9.2 ROTATION WITH CONSTANT ANGULAR ACCELERATION

In Chapter 2 we found that straight-line motion is particularly simple when the acceleration is constant. This is also true of rotational motion about a fixed axis. When the angular acceleration is constant, we can derive equations for angular velocity and angular position by using the same procedure that we used for straight-line motion in Section 2.4. In fact, the equations we are about to derive are identical to Eqs. (2.8), (2.12), (2.13), and (2.14) if we replace x with θ, v_x with ω_z, and a_x with α_z. We suggest that you review Section 2.4 before continuing.

Let ω_{0z} be the angular velocity of a rigid body at time $t = 0$ and ω_z be its angular velocity at a later time t. The angular acceleration α_z is constant and equal to the average value for any interval. From Eq. (9.4) with the interval from 0 to t,

$$\alpha_z = \frac{\omega_z - \omega_{0z}}{t - 0} \qquad \text{or}$$

Angular velocity at time t of a rigid body with **constant angular acceleration**

Angular velocity of body at time 0

$$\omega_z = \omega_{0z} + \alpha_z t \tag{9.7}$$

Time

Constant angular acceleration of body

The product $\alpha_z t$ is the total change in ω_z between $t = 0$ and the later time t; angular velocity ω_z at time t is the sum of the initial value ω_{0z} and this total change.

With constant angular acceleration, the angular velocity changes at a uniform rate, so its average value between 0 and t is the average of the initial and final values:

$$\omega_{\text{av-}z} = \frac{\omega_{0z} + \omega_z}{2} \tag{9.8}$$

We also know that $\omega_{\text{av-}z}$ is the total angular displacement $(\theta - \theta_0)$ divided by the time interval $(t - 0)$:

$$\omega_{\text{av-}z} = \frac{\theta - \theta_0}{t - 0} \tag{9.9}$$

BIO APPLICATION Rotational Motion in Bacteria *Escherichia coli* bacteria (about 2 μm by 0.5 μm) are found in the lower intestines of humans and other warm-blooded animals. The bacteria swim by rotating their long, corkscrew-shaped flagella, which act like the blades of a propeller. Each flagellum is powered by a remarkable motor (made of protein) located at the base of the bacterial cell. The motor can rotate the flagellum at angular speeds from 200 to 1000 rev/min (about 20 to 100 rad/s) and can vary its speed to give the flagellum an angular acceleration.

When we equate Eqs. (9.8) and (9.9) and multiply the result by t, we get

Angular position at time t of a rigid body with **constant angular acceleration**

Angular position of body at time 0

$$\theta - \theta_0 = \tfrac{1}{2}(\omega_{0z} + \omega_z)t \qquad (9.10)$$

Time

Angular velocity of body at time 0

Angular velocity of body at time t

To obtain a relationship between θ and t that doesn't contain ω_z, we substitute Eq. (9.7) into Eq. (9.10):

$$\theta - \theta_0 = \tfrac{1}{2}[\omega_{0z} + (\omega_{0z} + \alpha_z t)]t \quad \text{or}$$

Angular position of body at time 0

Time

Angular position at time t of a rigid body with **constant angular acceleration**

$$\theta = \theta_0 + \omega_{0z}t + \tfrac{1}{2}\alpha_z t^2 \qquad (9.11)$$

Angular velocity of body at time 0

Constant angular acceleration of body

That is, if at the initial time $t = 0$ the body is at angular position θ_0 and has angular velocity ω_{0z}, then its angular position θ at any later time t is θ_0, plus the rotation $\omega_{0z}t$ it would have if the angular velocity were constant, plus an additional rotation $\frac{1}{2}\alpha_z t^2$ caused by the changing angular velocity.

Following the same procedure as for straight-line motion in Section 2.4, we can combine Eqs. (9.7) and (9.11) to obtain a relationship between θ and ω_z that does not contain t. We invite you to work out the details, following the same procedure we used to get Eq. (2.13). We get

Angular velocity at time t of a rigid body with **constant angular acceleration**

Angular velocity of body at time 0

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) \qquad (9.12)$$

Constant angular acceleration of body

Angular position of body at time t

Angular position of body at time 0

CAUTION **Constant angular acceleration** Keep in mind that all of these results are valid *only* when the angular acceleration α_z is *constant;* do not try to apply them to problems in which α_z is *not* constant. **Table 9.1** shows the analogy between Eqs. (9.7), (9.10), (9.11), and (9.12) for fixed-axis rotation with constant angular acceleration and the corresponding equations for straight-line motion with constant linear acceleration.

TABLE 9.1 Comparison of Linear and Angular Motions with Constant Acceleration

Straight-Line Motion with Constant Linear Acceleration		Fixed-Axis Rotation with Constant Angular Acceleration	
$a_x = \text{constant}$		$\alpha_z = \text{constant}$	
$v_x = v_{0x} + a_x t$	(2.8)	$\omega_z = \omega_{0z} + \alpha_z t$	(9.7)
$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$	(2.12)	$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$	(9.11)
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	(2.13)	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$	(9.12)
$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$	(2.14)	$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$	(9.10)

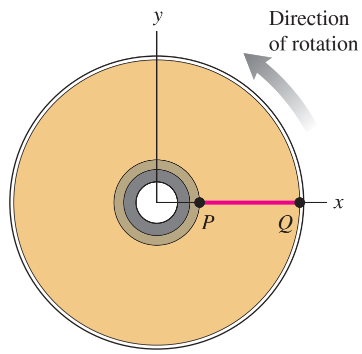
EXAMPLE 9.3 Rotation with constant angular acceleration

WITH VARIATION PROBLEMS

You have finished watching a movie on Blu-ray and the disc is slowing to a stop. The disc's angular velocity at $t = 0$ is 27.5 rad/s, and its angular acceleration is a constant $-10.0\ \text{rad/s}^2$. A line PQ on the disc's surface lies along the $+x$-axis at $t = 0$ (**Fig. 9.8**). (a) What is the disc's angular velocity at $t = 0.300$ s? (b) What angle does the line PQ make with the $+x$-axis at this time?

IDENTIFY and SET UP The angular acceleration of the disc is constant, so we can use any of the equations derived in this section (Table 9.1). Our target variables are the angular velocity ω_z and the angular displacement θ at $t = 0.300$ s. Given $\omega_{0z} = 27.5$ rad/s, $\theta_0 = 0$, and $\alpha_z = -10.0\ \text{rad/s}^2$, it's easiest to use Eqs. (9.7) and (9.11) to find the target variables.

Figure **9.8** A line PQ on a rotating Blu-ray disc at $t = 0$.



EXECUTE (a) From Eq. (9.7), at $t = 0.300$ s we have

$$\omega_z = \omega_{0z} + \alpha_z t = 27.5 \text{ rad/s} + (-10.0 \text{ rad/s}^2)(0.300 \text{ s})$$
$$= 24.5 \text{ rad/s}$$

(b) From Eq. (9.11),

$$\theta = \theta_0 + \omega_{0z}t + \tfrac{1}{2}\alpha_z t^2$$
$$= 0 + (27.5 \text{ rad/s})(0.300 \text{ s}) + \tfrac{1}{2}(-10.0 \text{ rad/s}^2)(0.300 \text{ s})^2$$
$$= 7.80 \text{ rad} = 7.80 \text{ rad}\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right) = 1.24 \text{ rev}$$

The disc has made one complete revolution plus an additional 0.24 revolution—that is, 360° plus $(0.24 \text{ rev})(360°/\text{rev}) = 87°$. Hence the line PQ makes an angle of 87° with the $+x$-axis.

EVALUATE Our answer to part (a) tells us that the disc's angular velocity has decreased, as it should since $\alpha_z < 0$. We can use our result for ω_z from part (a) with Eq. (9.12) to check our result for θ from part (b). To do so, we solve Eq. (9.12) for θ:

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$$
$$\theta = \theta_0 + \left(\frac{\omega_z^2 - \omega_{0z}^2}{2\alpha_z}\right)$$
$$= 0 + \frac{(24.5 \text{ rad/s})^2 - (27.5 \text{ rad/s})^2}{2(-10.0 \text{ rad/s}^2)} = 7.80 \text{ rad}$$

This agrees with our previous result from part (b).

KEYCONCEPT The relationships among angular position θ, angular velocity ω_z, and angular acceleration α_z for a rigid body rotating with constant angular acceleration are the same as the relationships among position x, velocity v_x, and acceleration a_x for an object moving in a straight line with constant linear acceleration.

TEST YOUR UNDERSTANDING OF SECTION 9.2 Suppose the disc in Example 9.3 was initially spinning at twice the rate (55.0 rad/s rather than 27.5 rad/s) and slowed down at twice the rate (-20.0 rad/s^2 rather than -10.0 rad/s^2). (a) Compared to the situation in Example 9.3, how long would it take the disc to come to a stop? (i) The same amount of time; (ii) twice as much time; (iii) 4 times as much time; (iv) $\frac{1}{2}$ as much time; (v) $\frac{1}{4}$ as much time. (b) Compared to the situation in Example 9.3, through how many revolutions would the disc rotate before coming to a stop? (i) The same number of revolutions; (ii) twice as many revolutions; (iii) 4 times as many revolutions; (iv) $\frac{1}{2}$ as many revolutions; (v) $\frac{1}{4}$ as many revolutions.

ANSWER

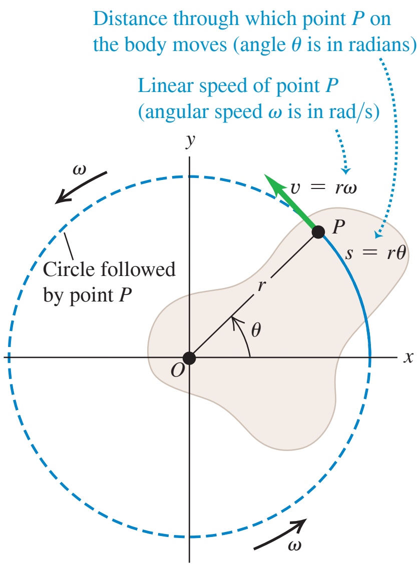
(a) **(i)**, (b) **(ii)** When the disc comes to rest, $\omega_z = 0$. From Eq. (9.7), the *time* when this occurs is $t = (\omega_z - \omega_{0z})/\alpha_z = -\omega_{0z}/\alpha_z$ (this is a positive time because α_z is negative). If we double the initial angular velocity ω_{0z} and also double the angular acceleration α_z, their ratio is unchanged and the rotation stops in the same amount of time. The *angle* through which the disc rotates is given by Eq. (9.10): $\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t = \frac{1}{2}\omega_{0z}t$ (since the final angular velocity is $\omega_z = 0$). The initial angular velocity ω_{0z} has been doubled but the time t is the same, so the angular displacement $\theta - \theta_0$ (and hence the number of revolutions) has doubled. You can also come to the same conclusion by using Eq. (9.12).

9.3 RELATING LINEAR AND ANGULAR KINEMATICS

How do we find the linear speed and acceleration of a particular point in a rotating rigid body? We need to answer this question to proceed with our study of rotation. For example, to find the kinetic energy of a rotating body, we have to start from $K = \frac{1}{2}mv^2$ for a particle, and this requires that we know the speed v for each particle in the body. So it's worthwhile to develop general relationships between the *angular* speed and acceleration of a rigid body rotating about a fixed axis and the *linear* speed and acceleration of a specific point or particle in the body.

Linear Speed in Rigid-Body Rotation

When a rigid body rotates about a fixed axis, every particle in the body moves in a circular path that lies in a plane perpendicular to the axis and is centered on the axis. A particle's speed is directly proportional to the body's angular velocity; the faster the rotation, Figure **9.9** A rigid body rotating about a fixed axis through point O.



the greater the speed of each particle. In **Fig. 9.9**, point P is a constant distance r from the axis, so it moves in a circle of radius r. At any time, Eq. (9.1) relates the angle θ (in radians) and the arc length s:

$$s = r\theta$$

We take the time derivative of this, noting that r is constant for any specific particle, and take the absolute value of both sides:

$$\left|\frac{ds}{dt}\right| = r\left|\frac{d\theta}{dt}\right|$$

Now $|ds/dt|$ is the absolute value of the rate of change of arc length, which is equal to the instantaneous *linear* speed v of the particle. The absolute value of the rate of change of the angle, $|d\theta/dt|$, is the instantaneous **angular speed** ω—that is, the magnitude of the instantaneous angular velocity in rad/s. Thus

Linear speed of a point on a rotating rigid body

$$v = r\omega \tag{9.13}$$

Angular speed of the rotating rigid body

Distance of that point from rotation axis

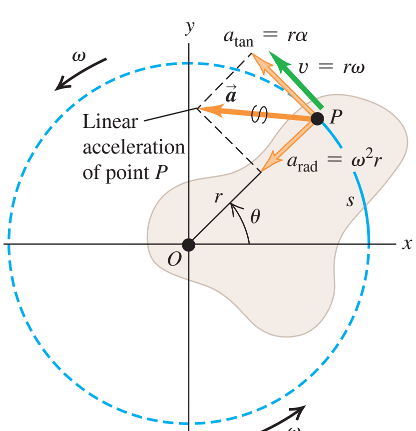
The farther a point is from the axis, the greater its linear speed. The *direction* of the linear velocity *vector* is tangent to its circular path at each point (Fig. 9.9).

CAUTION **Speed vs. velocity** Keep in mind the distinction between the linear and angular *speeds* v and ω, which appear in Eq. (9.13), and the linear and angular *velocities* v_x and ω_z. The quantities without subscripts, v and ω, are never negative; they are the magnitudes of the vectors $\vec{v}$ and $\vec{\omega}$, respectively, and their values tell you only how fast a particle is moving (v) or how fast a body is rotating (ω). The quantities with subscripts, v_x and ω_z, can be either positive or negative; their signs tell you the direction of the motion.

Linear Acceleration in Rigid-Body Rotation

Figure **9.10** A rigid body whose rotation is speeding up. The acceleration of point P has a component a_{rad} toward the rotation axis (perpendicular to $\vec{v}$) and a component $a_{\tan}$ along the circle that point P follows (parallel to $\vec{v}$).

Radial and tangential acceleration components:
- $a_{\text{rad}} = \omega^2 r$ is point P's centripetal acceleration.
- $a_{\tan} = r\alpha$ means that P's rotation is speeding up (the body has angular acceleration).



We can represent the acceleration $\vec{a}$ of a particle moving in a circle in terms of its centripetal and tangential components, a_{rad} and $a_{\tan}$ (**Fig. 9.10**), as we did in Section 3.4. (You should review that section now.) We found that the **tangential component of acceleration** $a_{\tan}$, the component parallel to the instantaneous velocity, acts to change the *magnitude* of the particle's velocity (i.e., the speed) and is equal to the rate of change of speed. Taking the derivative of Eq. (9.13), we find

Tangential acceleration of a point on a rotating rigid body

Distance of that point from rotation axis

$$a_{\tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \tag{9.14}$$

Rate of change of linear speed of that point

Rate of change of angular speed of body

This component of $\vec{a}$ is always tangent to the circular path of point P (Fig. 9.10).

The quantity $\alpha = d\omega/dt$ in Eq. (9.14) is the rate of change of the angular *speed.* It is not quite the same as $\alpha_z = d\omega_z/dt$, which is the rate of change of the angular *velocity.* For example, consider a object rotating so that its angular velocity vector points in the $-z$-direction (see Fig. 9.5b). If the body is gaining angular speed at a rate of 10 rad/s per second, then $\alpha = 10\ \text{rad/s}^2$. But ω_z is negative and becoming more negative as the rotation gains speed, so $\alpha_z = -10\ \text{rad/s}^2$. The rule for rotation about a fixed axis is that α is equal to α_z if ω_z is positive but equal to $-\alpha_z$ if ω_z is negative.

The component of $\vec{a}$ in Fig. 9.10 directed toward the rotation axis, the **centripetal component of acceleration** a_{rad}, is associated with the change of *direction* of the velocity of point P. In Section 3.4 we worked out the relationship $a_{\text{rad}} = v^2/r$. We can express this in terms of ω by using Eq. (9.13):

Centripetal acceleration of a point on a rotating rigid body

Linear speed of that point

Angular speed of body

$$a_{\text{rad}} = \frac{v^2}{r} = \omega^2 r \tag{9.15}$$

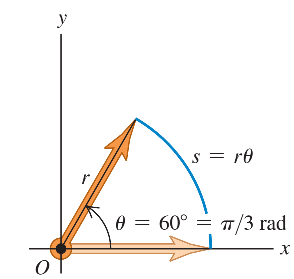
Distance of that point from rotation axis

This is true at each instant, *even when ω and v are not constant.* The centripetal component always points toward the axis of rotation.

CAUTION **Use angles in radians** Remember that Eq. (9.1), $s = r\theta$, is valid *only* when θ is measured in radians. The same is true of any equation derived from this, including Eqs. (9.13), (9.14), and (9.15). When you use these equations, you *must* express the angular quantities in radians, not revolutions or degrees (**Fig. 9.11**).

Equations (9.1), (9.13), and (9.14) also apply to any particle that has the same tangential velocity as a point in a rotating rigid body. For example, when a rope wound around a circular cylinder unwraps without stretching or slipping, its speed and acceleration at any instant are equal to the speed and tangential acceleration of the point at which it is tangent to the cylinder. The same principle holds for situations such as bicycle chains and sprockets, belts and pulleys that turn without slipping, and so on. We'll have several opportunities to use these relationships later in this chapter and in Chapter 10. Note that Eq. (9.15) for the centripetal component a_{rad} is applicable to the rope or chain *only* at points that are in contact with the cylinder or sprocket. Other points do not have the same acceleration toward the center of the circle that points on the cylinder or sprocket have.

Figure **9.11** Always use radians when relating linear and angular quantities.



In any equation that relates linear quantities to angular quantities, the angles MUST be expressed in radians ...

RIGHT! ▶ $s = (\pi/3)r$

... never in degrees or revolutions.

WRONG ▶ $s = \cancel{60}r$

EXAMPLE 9.4 Throwing a discus

WITH VARIATION PROBLEMS

An athlete whirls a discus in a circle of radius 80.0 cm. At a certain instant, the athlete is rotating at 10.0 rad/s and the angular speed is increasing at 50.0 rad/s^2. For this instant, find the tangential and centripetal components of the acceleration of the discus and the magnitude of the acceleration.

IDENTIFY and SET UP We treat the discus as a particle traveling in a circular path (**Fig. 9.12a**), so we can use the ideas developed in this section. We are given $r = 0.800$ m, $\omega = 10.0$ rad/s, and $\alpha = 50.0$ rad/s^2 (Fig. 9.12b). We'll use Eqs. (9.14) and (9.15) to find the acceleration components a_{tan} and a_{rad}, respectively; we'll then find the magnitude a by using the Pythagorean theorem.

EXECUTE From Eqs. (9.14) and (9.15),

$$a_{\text{tan}} = r\alpha = (0.800\text{ m})(50.0\text{ rad/s}^2) = 40.0\text{ m/s}^2$$

$$a_{\text{rad}} = \omega^2 r = (10.0\text{ rad/s})^2(0.800\text{ m}) = 80.0\text{ m/s}^2$$

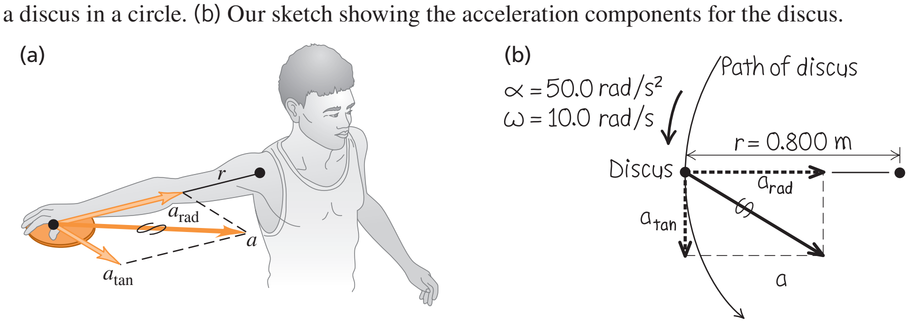
Then

$$a = \sqrt{a_{\text{tan}}{}^2 + a_{\text{rad}}{}^2} = 89.4\text{ m/s}^2$$

EVALUATE Note that we dropped the unit "radian" from our results for a_{tan}, a_{rad}, and a. We can do this because "radian" is a dimensionless quantity. Can you show that if the angular speed doubles to 20.0 rad/s while α remains the same, the acceleration magnitude a increases to 322 m/s^2?

KEYCONCEPT Points on a rigid body have a *centripetal* (radial) acceleration component $a_{\text{rad}} = \omega^2 r$ whenever the rigid body is rotating; they have a *tangential* acceleration component $a_{\text{tan}} = r\alpha$ *only* if the angular speed ω is changing. These two acceleration components are perpendicular, so you can use the Pythagorean theorem to relate them to the magnitude of the acceleration.

Figure **9.12** (a) Whirling a discus in a circle. (b) Our sketch showing the acceleration components for the discus.



EXAMPLE 9.5 Designing a propeller

WITH VARIATION PROBLEMS

You are designing an airplane propeller that is to turn at 2400 rpm (**Fig. 9.13a**). The forward airspeed of the plane is to be 75.0 m/s, and the speed of the propeller tips through the air must not exceed 270 m/s. (This is about 80% of the speed of sound in air. If the propeller tips moved faster, they would produce a lot of noise.) (a) What is the maximum possible propeller radius? (b) With this radius, what is the acceleration of the propeller tip?

IDENTIFY and SET UP We consider a particle at the tip of the propeller; our target variables are the particle's distance from the axis and its acceleration. The speed of this particle through the air, which cannot exceed 270 m/s, is due to both the propeller's rotation and the forward motion of the airplane. Figure 9.13b shows that the particle's velocity $\vec{v}_{tip}$ is the vector sum of its tangential velocity due to the propeller's rotation of magnitude $v_{tan} = \omega r$, given by Eq. (9.13), and the forward velocity of the airplane of magnitude $v_{plane} = 75.0$ m/s. The propeller rotates in a plane perpendicular to the direction of flight, so $\vec{v}_{tan}$ and $\vec{v}_{plane}$ are perpendicular to each other, and we can use the Pythagorean theorem to obtain an expression for v_{tip} from v_{tan} and v_{plane}. We'll then set $v_{tip} = 270$ m/s and solve for the radius r. The angular speed of the propeller is constant, so the acceleration of the propeller tip has only a radial component; we'll find it by using Eq. (9.15).

EXECUTE We first convert ω to rad/s (see Fig. 9.11):

$$\omega = 2400 \text{ rpm} = \left(2400 \frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 251 \text{ rad/s}$$

(a) From Fig. 9.13b and Eq. (9.13),

$$v_{tip}{}^2 = v_{plane}{}^2 + v_{tan}{}^2 = v_{plane}{}^2 + r^2\omega^2 \quad \text{so}$$

$$r^2 = \frac{v_{tip}{}^2 - v_{plane}{}^2}{\omega^2} \quad \text{and} \quad r = \frac{\sqrt{v_{tip}{}^2 - v_{plane}{}^2}}{\omega}$$

If $v_{tip} = 270$ m/s, the maximum propeller radius is

$$r = \frac{\sqrt{(270 \text{ m/s})^2 - (75.0 \text{ m/s})^2}}{251 \text{ rad/s}} = 1.03 \text{ m}$$

(b) The centripetal acceleration of the particle is, from Eq. (9.15),

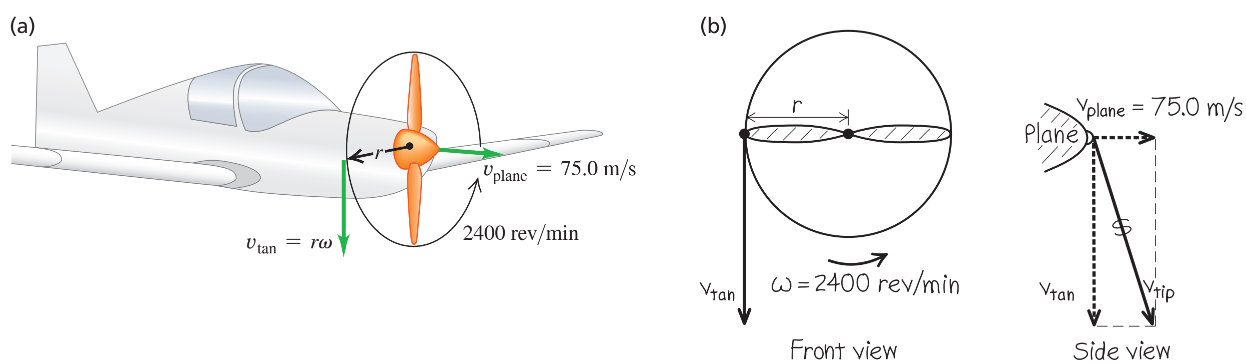
$$a_{rad} = \omega^2 r = (251 \text{ rad/s})^2(1.03 \text{ m})$$
$$= 6.5 \times 10^4 \text{ m/s}^2$$

The tangential acceleration a_{tan} is zero because ω is constant.

EVALUATE From $\sum \vec{F} = m\vec{a}$, the propeller must exert a force of 6.5×10^4 N on each kilogram of material at its tip! This is why propellers are made out of tough material such as aluminum alloy.

KEYCONCEPT If a rotating rigid body is also moving as a whole through space, use vector addition to find the velocity of a point on the rigid body.

Figure 9.13 (a) A propeller-driven airplane in flight. (b) Our sketch showing the velocity components for the propeller tip.



TEST YOUR UNDERSTANDING OF SECTION 9.3 Information is stored on a Blu-ray disc (see Fig. 9.8) in a coded pattern of tiny pits. The pits are arranged in a track that spirals outward toward the rim of the disc. As the disc spins inside a player, the track is scanned at a constant *linear* speed. How must the rotation speed ω of the disc change as the player's scanning head moves outward over the track? (i) ω must increase; (ii) ω must decrease; (iii) ω must stay the same.

ANSWER

(ii) From Eq. (9.13), $v = r\omega$. To maintain a constant linear speed v, the angular speed ω must decrease as the scanning head moves outward (greater r).

9.4 ENERGY IN ROTATIONAL MOTION

A rotating rigid body consists of mass in motion, so it has kinetic energy. As we'll see, we can express this kinetic energy in terms of the body's angular speed and a new quantity, called *moment of inertia,* that depends on the body's mass and how the mass is distributed.

To begin, we think of a body as being made up of a large number of particles, with masses $m_1, m_2, \ldots$ at distances $r_1, r_2, \ldots$ from the axis of rotation. We label the particles with the index i: The mass of the ith particle is m_i and r_i is the *perpendicular* distance from the axis to the ith particle. (The particles need not all lie in the same plane.)

When a rigid body rotates about a fixed axis, the speed v_i of the ith particle is given by Eq. (9.13), $v_i = r_i\omega$, where ω is the body's angular speed. Different particles have different values of r_i, but ω is the same for all (otherwise, the body wouldn't be rigid). The kinetic energy of the ith particle can be expressed as

$$\tfrac{1}{2}m_i v_i^{\,2} = \tfrac{1}{2}m_i r_i^{\,2}\omega^2$$

The body's *total* kinetic energy is the sum of the kinetic energies of all its particles:

$$K = \tfrac{1}{2}m_1 r_1^2\omega^2 + \tfrac{1}{2}m_2 r_2^2\omega^2 + \cdots = \sum_i \tfrac{1}{2}m_i r_i^{\,2}\omega^2$$

Taking the common factor $\frac{1}{2}\,\omega^2$ out of this expression, we get

$$K = \tfrac{1}{2}(m_1 r_1^{\,2} + m_2 r_2^{\,2} + \cdots)\omega^2 = \tfrac{1}{2}\Big(\sum_i m_i r_i^{\,2}\Big)\omega^2$$

The quantity in parentheses, obtained by multiplying the mass of each particle by the square of its distance from the axis of rotation and adding these products, is called the **moment of inertia** I of the body for this rotation axis:

Moment of inertia of a body for a given rotation axis

Masses of the particles that make up the body

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots = \sum_i m_i r_i^{\,2} \tag{9.16}$$

Perpendicular distances of the particles from rotation axis

"Moment" means that I depends on how the body's mass is distributed in space; it has nothing to do with a "moment" of time. For a body with a given rotation axis and a given total mass, the greater the distances from the axis to the particles that make up the body, the greater the moment of inertia I. In a rigid body, all distances r_i are constant and I is independent of how the body rotates around the given axis. The SI unit of I is the kilogram-meter^2 ($\mathrm{kg \cdot m^2}$).

Using Eq. (9.16), we see that the **rotational kinetic energy** K of a rigid body is

Rotational kinetic energy of a rigid body rotating around an axis

Moment of inertia of body for given rotation axis

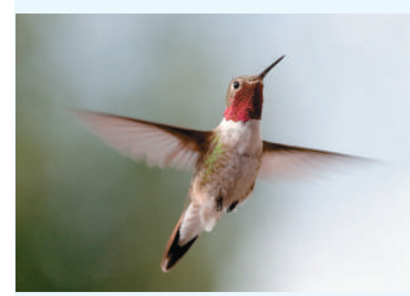
$$K = \tfrac{1}{2}I\omega^2 \tag{9.17}$$

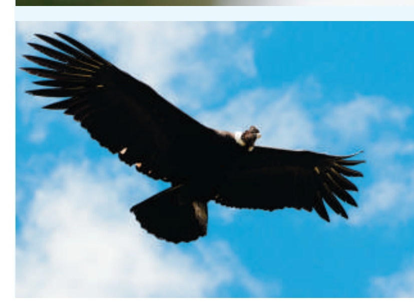
Angular speed of body

The kinetic energy given by Eq. (9.17) is *not* a new form of energy; it's simply the sum of the kinetic energies of the individual particles that make up the rotating rigid body. To use Eq. (9.17), ω *must* be measured in radians per second, not revolutions or degrees per second, to give K in joules. That's because we used $v_i = r_i\omega$ in our derivation.

Equation (9.17) gives a simple physical interpretation of moment of inertia: *The greater the moment of inertia, the greater the kinetic energy of a rigid body rotating with a given angular speed* ω. We learned in Chapter 6 that the kinetic energy of an object equals the amount of work done to accelerate that object from rest. So the greater a body's moment of inertia, the harder it is to start the body rotating if it's at rest and the harder it is to stop its rotation if it's already rotating (**Fig. 9.14**). For this reason, I is also called the *rotational inertia*.

BIO APPLICATION Moment of Inertia of a Bird's Wing When a bird flaps its wings, it rotates the wings up and down around the shoulder. A hummingbird has small wings with a small moment of inertia, so the bird can move its wings rapidly (up to 70 beats per second). By contrast, the Andean condor (*Vultur gryphus*) has immense wings that are hard to move due to their large moment of inertia. Condors flap their wings at about one beat per second on takeoff, but at most times prefer to soar while holding their wings steady.





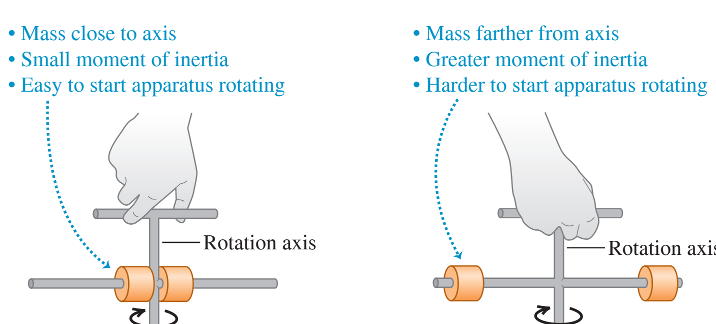


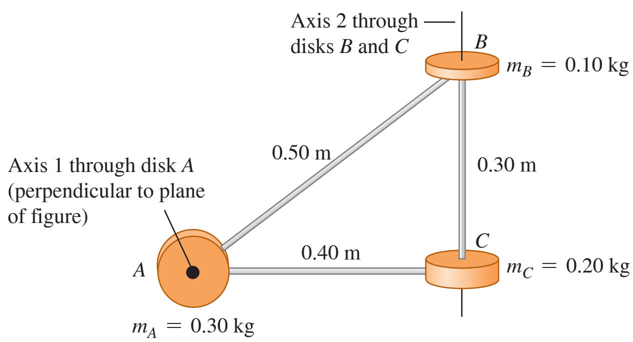
Figure **9.14** An apparatus free to rotate around a vertical axis. To vary the moment of inertia, the two equal-mass cylinders can be locked into different positions on the horizontal shaft.

EXAMPLE 9.6 Moments of inertia for different rotation axes

A machine part (**Fig. 9.15**) consists of three small disks linked by lightweight struts. (a) What is this body's moment of inertia about axis 1 through the center of disk A, perpendicular to the plane of the diagram? (b) What is its moment of inertia about axis 2 through the centers of disks B and C? (c) What is the body's kinetic energy if it rotates about axis 1 with angular speed $\omega = 4.0$ rad/s?

IDENTIFY and SET UP We'll consider the disks as massive particles located at the centers of the disks, and consider the struts as massless. In parts (a) and (b), we'll use Eq. (9.16) to find the moments of inertia. Given the moment of inertia about axis 1, we'll use Eq. (9.17) in part (c) to find the rotational kinetic energy.

Figure **9.15** An oddly shaped machine part.



EXECUTE (a) The particle at point A lies *on* axis 1 through A, so its distance r from the axis is zero and it contributes nothing to the moment of inertia. Hence only B and C contribute in Eq. (9.16):

$$I_1 = \sum m_i r_i^2 = (0.10\text{ kg})(0.50\text{ m})^2 + (0.20\text{ kg})(0.40\text{ m})^2$$
$$= 0.057\text{ kg}\cdot\text{m}^2$$

(b) The particles at B and C both lie on axis 2, so neither particle contributes to the moment of inertia. Hence only A contributes:

$$I_2 = \sum m_i r_i^2 = (0.30\text{ kg})(0.40\text{ m})^2 = 0.048\text{ kg}\cdot\text{m}^2$$

(c) From Eq. (9.17),

$$K_1 = \tfrac{1}{2} I_1 \omega^2 = \tfrac{1}{2}(0.057\text{ kg}\cdot\text{m}^2)(4.0\text{ rad/s})^2 = 0.46\text{ J}$$

EVALUATE The moment of inertia about axis 2 is smaller than that about axis 1. Hence, of the two axes, it's easier to make the machine part rotate about axis 2.

KEYCONCEPT The moment of inertia I of a rigid body about an axis depends on the position and orientation of the axis. The value of I for a given rigid body can be very different for different axes.

CAUTION Moment of inertia depends on the choice of axis Example 9.6 shows that the moment of inertia of a body depends on the location and orientation of the axis. It's not enough to say, "The moment of inertia is $0.048\text{ kg}\cdot\text{m}^2$." We have to be specific and say, "The moment of inertia *about the axis through B and C* is $0.048\text{ kg}\cdot\text{m}^2$."

In Example 9.6 we represented the body as several point masses, and we evaluated the sum in Eq. (9.16) directly. When the body is a *continuous* distribution of matter, such as a solid cylinder or plate, the sum becomes an integral, and we need to use calculus to calculate the moment of inertia. We'll give several examples of such calculations in Section 9.6; meanwhile, **Table 9.2** gives moments of inertia for several familiar shapes in terms of their masses and dimensions. Each body shown in Table 9.2 is *uniform;* that is, the density has the same value at all points within the solid parts of the body. ?

CAUTION Computing moments of inertia You may be tempted to try to compute the moment of inertia of a body by assuming that all the mass is concentrated at the center of mass and multiplying the total mass by the square of the distance from the center of mass to the axis. That doesn't work! For example, when a uniform thin rod of length L and mass M is pivoted about an axis through one end, perpendicular to the rod, the moment of inertia is $I = ML^2/3$ [case (b) in Table 9.2]. If we took the mass as concentrated at the center, a distance $L/2$ from the axis, we would obtain the *incorrect* result $I = M(L/2)^2 = ML^2/4$.

Now that we know how to calculate the kinetic energy of a rotating rigid body, we can apply the energy principles of Chapter 7 to rotational motion. The Problem-Solving Strategy on the next page, along with the examples that follow, shows how this is done.

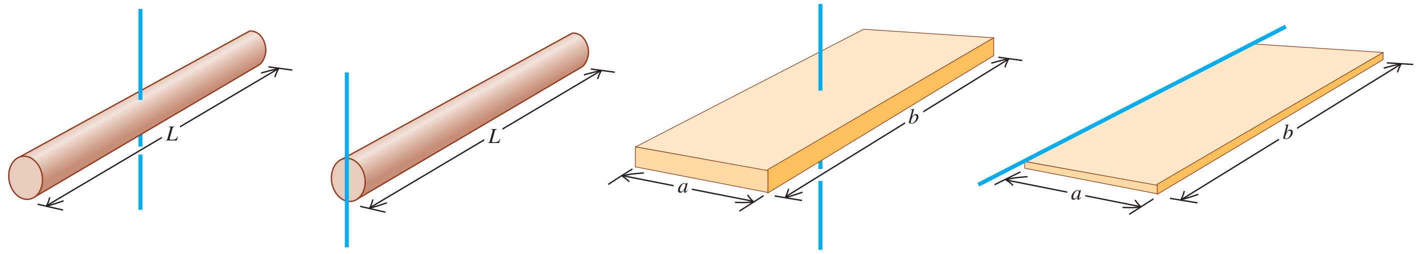
TABLE 9.2 Moments of Inertia of Various Bodies

(a) Slender rod, axis through center $I = \frac{1}{12}ML^2$

(b) Slender rod, axis through one end $I = \frac{1}{3}ML^2$

(c) Rectangular plate, axis through center $I = \frac{1}{12}M(a^2 + b^2)$

(d) Thin rectangular plate, axis along edge $I = \frac{1}{3}Ma^2$



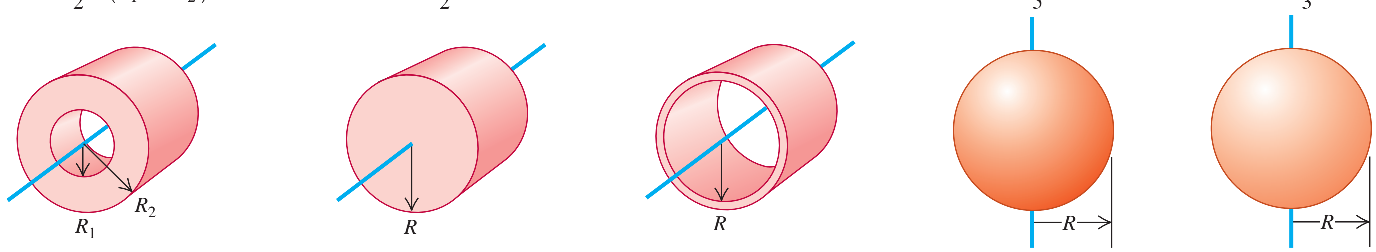
(e) Hollow cylinder $I = \frac{1}{2}M(R_1^2 + R_2^2)$

(f) Solid cylinder $I = \frac{1}{2}MR^2$

(g) Thin-walled hollow cylinder $I = MR^2$

(h) Solid sphere $I = \frac{2}{5}MR^2$

(i) Thin-walled hollow sphere $I = \frac{2}{3}MR^2$



PROBLEM-SOLVING STRATEGY 9.1 Rotational Energy

IDENTIFY *the relevant concepts:* You can use work–energy relationships and conservation of energy to find relationships involving the position and motion of a rigid body rotating around a fixed axis. The energy method is usually not helpful for problems that involve elapsed time. In Chapter 10 we'll see how to approach rotational problems of this kind.

SET UP *the problem* using Problem-Solving Strategy 7.1 (Section 7.1) with the following additional steps:

5. You can use Eqs. (9.13) and (9.14) in problems involving a rope (or the like) wrapped around a rotating rigid body, if the rope doesn't slip. These equations relate the linear speed and tangential acceleration of a point on the body to the body's angular velocity and angular acceleration. (See Examples 9.7 and 9.8.)
6. Use Table 9.2 to find moments of inertia. Use the parallel-axis theorem, Eq. (9.19) (to be derived in Section 9.5), to find moments of inertia for rotation about axes parallel to those shown in the table.

EXECUTE *the solution:* Write expressions for the initial and final kinetic and potential energies K_1, K_2, U_1, and U_2 and for the nonconservative work W_{other} (if any), where K_1 and K_2 must now include any rotational kinetic energy $K = \frac{1}{2}I\omega^2$. Substitute these expressions into Eq. (7.14), $K_1 + U_1 + W_{\text{other}} = K_2 + U_2$ (if nonconservative work is done), or Eq. (7.12), $K_1 + U_1 = K_2 + U_2$ (if only conservative work is done), and solve for the target variables. It's helpful to draw bar graphs showing the initial and final values of K, U, and $E = K + U$.

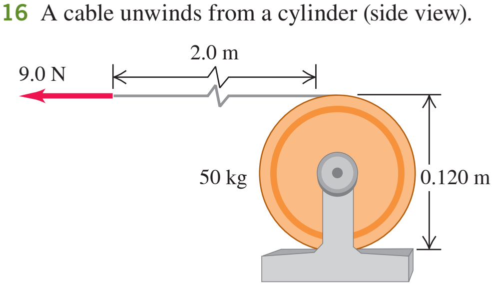
EVALUATE *your answer:* Check whether your answer makes physical sense.

EXAMPLE 9.7 An unwinding cable I

WITH VARIATION PROBLEMS

We wrap a light, nonstretching cable around a solid cylinder, of mass 50 kg and diameter 0.120 m, that rotates in frictionless bearings about a stationary horizontal axis (**Fig. 9.16**). We pull the free end of the cable with a constant 9.0 N force for a distance of 2.0 m; it turns the cylinder as it unwinds without slipping. The cylinder is initially at rest. Find its final angular speed and the final speed of the cable.

Figure **9.16** A cable unwinds from a cylinder (side view).



IDENTIFY We'll solve this problem by using energy methods. We'll assume that the cable is massless, so only the cylinder has kinetic energy. There are no changes in gravitational potential energy. There is friction

Continued

between the cable and the cylinder, but because the cable doesn't slip, there is no motion of the cable relative to the cylinder and no mechanical energy is lost in frictional work. Because the cable is massless, the force that the cable exerts on the cylinder rim is equal to the applied force F.

SET UP Point 1 is when the cable begins to move. The cylinder starts at rest, so $K_1 = 0$. Point 2 is when the cable has moved a distance $s = 2.0\text{ m}$ and the cylinder has kinetic energy $K_2 = \frac{1}{2}I\omega^2$. One of our target variables is ω; the other is the speed of the cable at point 2, which is equal to the tangential speed v of the cylinder at that point. We'll use Eq. (9.13) to find v from ω.

EXECUTE The work done on the cylinder is $W_{\text{other}} = Fs = (9.0\text{ N})(2.0\text{ m}) = 18\text{ J}$. From Table 9.2 the moment of inertia is

$$I = \tfrac{1}{2}mR^2 = \tfrac{1}{2}(50\text{ kg})(0.060\text{ m})^2 = 0.090\text{ kg}\cdot\text{m}^2$$

(The radius R is half the diameter.) From Eq. (7.14), $K_1 + U_1 + W_{\text{other}} = K_2 + U_2$, so

$$0 + 0 + W_{\text{other}} = \tfrac{1}{2}I\omega^2 + 0$$

$$\omega = \sqrt{\frac{2W_{\text{other}}}{I}} = \sqrt{\frac{2(18\text{ J})}{0.090\text{ kg}\cdot\text{m}^2}} = 20\text{ rad/s}$$

From Eq. (9.13), the final tangential speed of the cylinder, and hence the final speed of the cable, is

$$v = R\omega = (0.060\text{ m})(20\text{ rad/s}) = 1.2\text{ m/s}$$

EVALUATE If the cable mass is not negligible, some of the 18 J of work would go into the kinetic energy of the cable. Then the cylinder would have less kinetic energy and a lower angular speed than we calculated here.

KEYCONCEPT When using energy methods to solve problems about rotating rigid bodies, follow the same general steps as in Chapter 7, but include any rotational kinetic energy, $K = \frac{1}{2}I\omega^2$.

EXAMPLE 9.8 An unwinding cable II

WITH VARIATION PROBLEMS

We wrap a light, nonstretching cable around a solid cylinder with mass M and radius R. The cylinder rotates with negligible friction about a stationary horizontal axis. We tie the free end of the cable to a block of mass m and release the block from rest at a distance h above the floor. As the block falls, the cable unwinds without stretching or slipping. Find the speed of the falling block and the angular speed of the cylinder as the block strikes the floor.

IDENTIFY As in Example 9.7, the cable doesn't slip and so friction does no work. We assume that the cable is massless, so that the forces it exerts on the cylinder and the block have equal magnitudes. At its upper end the force and displacement are in the same direction, and at its lower end they are in opposite directions, so the cable does no *net* work and $W_{\text{other}} = 0$. Only gravity does work, and mechanical energy is conserved.

SET UP **Figure 9.17a** shows the situation before the block begins to fall (point 1). The initial kinetic energy is $K_1 = 0$. We take the gravitational potential energy to be zero when the block is at floor level (point 2), so $U_1 = mgh$ and $U_2 = 0$. (We ignore the gravitational potential energy for the rotating cylinder, since its height doesn't change.) Just before the block hits the floor (Fig. 9.17b), the block has kinetic energy due to its translational motion and the cylinder has kinetic energy due to its rotation. The total kinetic energy is the sum of these:

$$K_2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

The moment of inertia of the cylinder is $I = \frac{1}{2}MR^2$. Also, $v = R\omega$ since the speed of the falling block must be equal to the tangential speed at the outer surface of the cylinder.

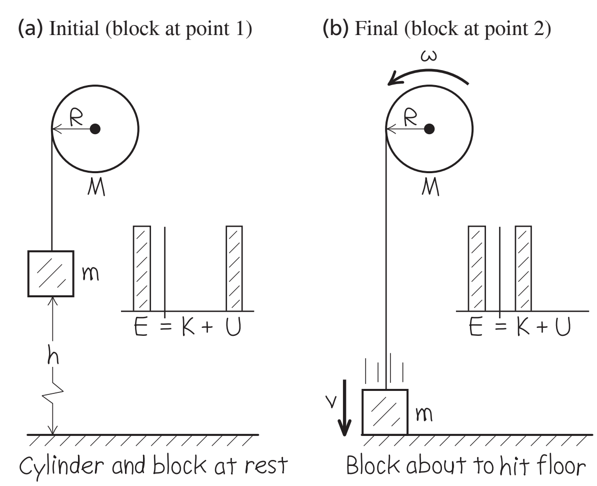
EXECUTE We use our expressions for K_1, U_1, K_2, and U_2 and the relationship $\omega = v/R$ in Eq. (7.4), $K_1 + U_1 = K_2 + U_2$, and solve for v:

$$0 + mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 + 0 = \tfrac{1}{2}\left(m + \tfrac{1}{2}M\right)v^2$$

$$v = \sqrt{\frac{2gh}{1 + M/2m}}$$

The final angular speed of the cylinder is $\omega = v/R$.

Figure **9.17** Our sketches for this problem.



EVALUATE When M is much larger than m, v is very small; when M is much smaller than m, v is nearly equal to $\sqrt{2gh}$, the speed of a body that falls freely from height h. Both of these results are as we would expect.

KEYCONCEPT When a block is attached to a string that wraps around a cylinder or pulley of radius R, the speed v of the block is related to the angular speed ω of the cylinder or pulley by $v = R\omega$. You can use this to find the combined kinetic energy of the two objects.

Gravitational Potential Energy for an Extended Body

In Example 9.8 the cable was of negligible mass, so we could ignore its kinetic energy as well as the gravitational potential energy associated with it. If the mass is *not* negligible, we need to know how to calculate the *gravitational potential energy* associated with such an extended body. If the acceleration of gravity g is the same at all points on the body, the gravitational potential energy is the same as though all the mass were concentrated at the center of mass of the body. Suppose we take the y-axis vertically upward. Then for a body with total mass M, the gravitational potential energy U is simply

$$U = Mgy_{\mathrm{cm}} \quad \text{(gravitational potential energy for an extended body)} \qquad (9.18)$$

where y_{cm} is the y-coordinate of the center of mass. This expression applies to any extended body, whether it is rigid or not (**Fig. 9.18**).

To prove Eq. (9.18), we again represent the body as a collection of mass elements m_i. The potential energy for element m_i is $m_i g y_i$, so the total potential energy is

$$U = m_1 g y_1 + m_2 g y_2 + \cdots = (m_1 y_1 + m_2 y_2 + \cdots)g$$

But from Eq. (8.28), which defines the coordinates of the center of mass,

$$m_1 y_1 + m_2 y_2 + \cdots = (m_1 + m_2 + \cdots)y_{\mathrm{cm}} = My_{\mathrm{cm}}$$

where $M = m_1 + m_2 + \cdots$ is the total mass. Combining this with the above expression for U, we find $U = Mgy_{\mathrm{cm}}$ in agreement with Eq. (9.18).

We leave the application of Eq. (9.18) to the problems. In Chapter 10 we'll use this equation to help us analyze rigid-body problems in which the axis of rotation moves.

Figure **9.18** In a technique called the "Fosbury flop" after its innovator, this athlete arches her body as she passes over the bar in the high jump. As a result, her center of mass actually passes *under* the bar. This technique requires a smaller increase in gravitational potential energy [Eq. (9.18)] than the older method of straddling the bar.

TEST YOUR UNDERSTANDING OF SECTION 9.4 Suppose the cylinder and block in Example 9.8 have the same mass, so $m = M$. Just before the block strikes the floor, which statement is correct about the relationship between the kinetic energy K_{block} of the falling block and the rotational kinetic energy K_{cylinder} of the cylinder? (i) $K_{\mathrm{block}} > K_{\mathrm{cylinder}}$; (ii) $K_{\mathrm{block}} < K_{\mathrm{cylinder}}$; (iii) $K_{\mathrm{block}} = K_{\mathrm{cylinder}}$.

ANSWER

(i) The kinetic energy in the falling block is $\frac{1}{2}mv^2$, and the kinetic energy in the rotating cylinder is $\frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mR^2\right)(v/R)^2 = \frac{1}{4}mv^2$. Hence the total kinetic energy of the system is $\frac{3}{4}mv^2$, of which two-thirds is in the block and one-third is in the cylinder.

9.5 PARALLEL-AXIS THEOREM

We pointed out in Section 9.4 that a body doesn't have just one moment of inertia. In fact, it has infinitely many, because there are infinitely many axes about which it might rotate. But there is a simple relationship, called the **parallel-axis theorem,** between the moment of inertia of a body about an axis through its center of mass and the moment of inertia about any other axis parallel to the original axis (**Fig. 9.19**):

Parallel-axis theorem:

Moment of inertia of a body for a rotation axis through point P

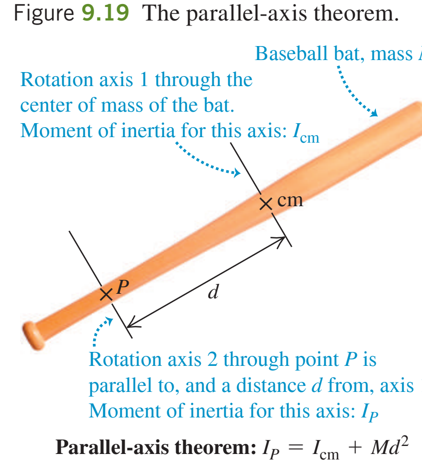
$$I_P = I_{\mathrm{cm}} + Md^2 \qquad (9.19)$$

Moment of inertia of body for a parallel axis through center of mass

Mass of body

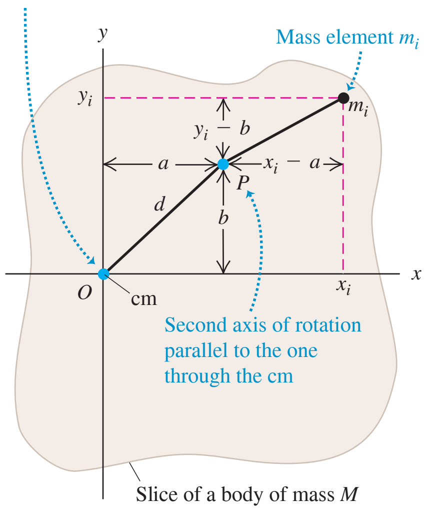
Distance between two parallel axes

Figure **9.19** The parallel-axis theorem.



To prove this theorem, we consider two axes, both parallel to the z-axis: one through the center of mass and the other through a point P (**Fig. 9.20**, next page). First we take a very thin slice of the body, parallel to the xy-plane and perpendicular to the z-axis. We take the origin of our coordinate system to be at the center of mass of the body; the coordinates of the center of mass are then $x_{\mathrm{cm}} = y_{\mathrm{cm}} = z_{\mathrm{cm}} = 0$. The axis through the center of mass passes through this thin slice at point O, and the parallel axis passes through point P, whose x- and y-coordinates are (a, b). The distance of this axis from the axis through the center of mass is d, where $d^2 = a^2 + b^2$.

Figure 9.20 The mass element m_i has coordinates (x_i, y_i) with respect to an axis of rotation through the center of mass (cm) and coordinates $(x_i - a, y_i - b)$ with respect to the parallel axis through point P.



We can write an expression for the moment of inertia I_P about the axis through point P. Let m_i be a mass element in our slice, with coordinates (x_i, y_i, z_i). Then the moment of inertia I_{cm} of the slice about the axis through the center of mass (at O) is

$$I_{cm} = \sum_i m_i(x_i^2 + y_i^2)$$

The moment of inertia of the slice about the axis through P is

$$I_P = \sum_i m_i[(x_i - a)^2 + (y_i - b)^2]$$

These expressions don't involve the coordinates z_i measured perpendicular to the slices, so we can extend the sums to include *all* particles in *all* slices. Then I_P becomes the moment of inertia of the *entire* body for an axis through P. We then expand the squared terms and regroup, and obtain

$$I_P = \sum_i m_i(x_i^2 + y_i^2) - 2a\sum_i m_i x_i - 2b\sum_i m_i y_i + (a^2 + b^2)\sum_i m_i$$

The first sum is I_{cm}. From Eq. (8.28), the definition of the center of mass, the second and third sums are proportional to x_{cm} and y_{cm}; these are zero because we have taken our origin to be the center of mass. The final term is d^2 multiplied by the total mass, or Md^2. This completes our proof that $I_P = I_{cm} + Md^2$.

As Eq. (9.19) shows, a rigid body has a lower moment of inertia about an axis through its center of mass than about any other parallel axis. Thus it's easier to start a body rotating if the rotation axis passes through the center of mass. This suggests that it's somehow most natural for a rotating body to rotate about an axis through its center of mass; we'll make this idea more quantitative in Chapter 10.

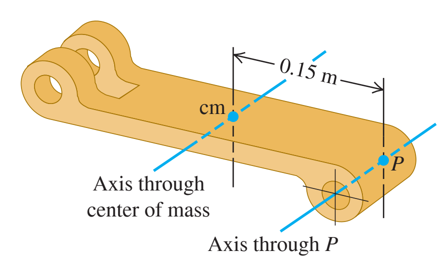
EXAMPLE 9.9 Using the parallel-axis theorem

A part of a mechanical linkage (**Fig. 9.21**) has a mass of 3.6 kg. Its moment of inertia I_P about an axis 0.15 m from its center of mass is $I_P = 0.132\ \mathrm{kg \cdot m^2}$. What is the moment of inertia I_{cm} about a parallel axis through the center of mass?

IDENTIFY, SET UP, and EXECUTE We'll determine the target variable I_{cm} by using the parallel-axis theorem, Eq. (9.19). Rearranging the equation, we obtain

$$\begin{aligned} I_{cm} &= I_P - Md^2 = 0.132\ \mathrm{kg \cdot m^2} - (3.6\ \mathrm{kg})(0.15\ \mathrm{m})^2 \\ &= 0.051\ \mathrm{kg \cdot m^2} \end{aligned}$$

Figure 9.21 Calculating I_{cm} from a measurement of I_P.



EVALUATE As we expect, I_{cm} is less than I_P; the moment of inertia for an axis through the center of mass has a lower value than for any other parallel axis.

KEYCONCEPT You can use the parallel-axis theorem to relate the moment of inertia of a rigid body about any axis to the moment of inertia of the same rigid body through a parallel axis through its center of mass.

TEST YOUR UNDERSTANDING OF SECTION 9.5 A pool cue is a wooden rod of uniform composition and is tapered with a larger diameter at one end than at the other end. Use the parallel-axis theorem to decide whether a pool cue has a larger moment of inertia for an axis perpendicular to the length of the rod that is (i) through the thicker end or (ii) through the thinner end.

ANSWER

(ii) More of the mass of the pool cue is concentrated at the thicker end, so the center of mass is closer to that end. The moment of inertia through a point P at either end is $I_P = I_{cm} + Md^2$; the thinner end is farther from the center of mass, so the distance d and the moment of inertia I_P are greater for the thinner end.

9.6 MOMENT-OF-INERTIA CALCULATIONS

If a rigid body is a continuous distribution of mass—like a solid cylinder or a solid sphere—it cannot be represented by a few point masses. In this case the *sum* of masses and distances that defines the moment of inertia [Eq. (9.16)] becomes an *integral*. Imagine dividing the body into elements of mass dm that are very small, so that all points in a particular element are at essentially the same perpendicular distance from the axis of rotation. We call this distance r, as before. Then the moment of inertia is

$$I = \int r^2\,dm \tag{9.20}$$

To evaluate the integral, we have to represent r and dm in terms of the same integration variable. When the body is effectively one-dimensional, such as the slender rods (a) and (b) in Table 9.2, we can use a coordinate x along the length and relate dm to an increment dx. For a three-dimensional body it is usually easiest to express dm in terms of an element of volume dV and the *density* ρ of the body. Density is mass per unit volume, $\rho = dm/dV$, so we may write Eq. (9.20) as

$$I = \int r^2 \rho\,dV$$

This expression tells us that a body's moment of inertia depends on how its density varies within its volume (**Fig. 9.22**). If the body is uniform in density, then we may take ρ outside the integral:

$$I = \rho \int r^2\,dV \tag{9.21}$$

To use this equation, we have to express the volume element dV in terms of the differentials of the integration variables, such as $dV = dx\,dy\,dz$. The element dV must always be chosen so that all points within it are at very nearly the same distance from the axis of rotation. The limits on the integral are determined by the shape and dimensions of the body. For regularly shaped bodies, this integration is often easy to do.

Figure 9.22 By measuring small variations in the orbits of satellites, geophysicists can measure the earth's moment of inertia. This tells us how our planet's mass is distributed within its interior. The data show that the earth is far denser at the core than in its outer layers.

EXAMPLE 9.10 Hollow or solid cylinder, rotating about axis of symmetry

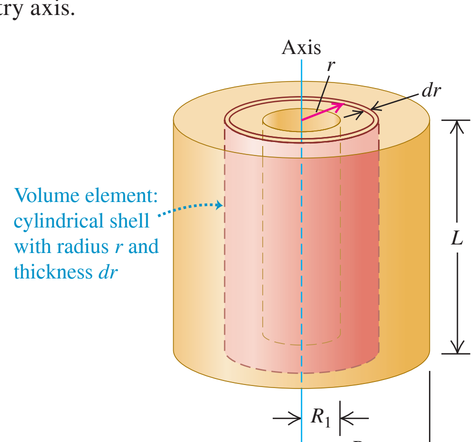
Figure 9.23 shows a hollow cylinder of uniform mass density ρ with length L, inner radius R_1, and outer radius R_2. (It might be a steel cylinder in a printing press.) Using integration, find its moment of inertia about its axis of symmetry.

IDENTIFY and SET UP We choose as a volume element a thin cylindrical shell of radius r, thickness dr, and length L. All parts of this shell are at very nearly the same distance r from the axis. The volume of the shell is very nearly that of a flat sheet with thickness dr, length L, and width $2\pi r$ (the circumference of the shell). The mass of the shell is

$$dm = \rho\,dV = \rho(2\pi r L\,dr)$$

We'll use this expression in Eq. (9.20), integrating from $r = R_1$ to $r = R_2$.

Figure 9.23 Finding the moment of inertia of a hollow cylinder about its symmetry axis.



EXECUTE From Eq. (9.20), the moment of inertia is

$$\begin{aligned} I &= \int r^2\,dm = \int_{R_1}^{R_2} r^2 \rho(2\pi r L\,dr) \\ &= 2\pi\rho L \int_{R_1}^{R_2} r^3\,dr = \frac{2\pi\rho L}{4}(R_2^4 - R_1^4) \\ &= \frac{\pi\rho L}{2}(R_2^2 - R_1^2)(R_2^2 + R_1^2) \end{aligned}$$

[In the last step we used the identity $a^2 - b^2 = (a-b)(a+b)$.] Let's express this result in terms of the total mass M of the body, which is its density ρ multiplied by the total volume V. The cylinder's volume is

$$V = \pi L(R_2^2 - R_1^2)$$

so its total mass M is

$$M = \rho V = \pi L \rho(R_2^2 - R_1^2)$$

Comparing with the above expression for I, we see that

$$I = \tfrac{1}{2}M(R_1^2 + R_2^2)$$

Continued

EVALUATE Our result agrees with Table 9.2, case (e). If the cylinder is solid, with outer radius $R_2 = R$ and inner radius $R_1 = 0$, its moment of inertia is

$$I = \tfrac{1}{2}MR^2$$

in agreement with case (f). If the cylinder wall is very thin, we have $R_1 \approx R_2 = R$ and the moment of inertia is

$$I = MR^2$$

in agreement with case (g). We could have predicted this last result without calculation; in a thin-walled cylinder, all the mass is at the same distance $r = R$ from the axis, so $I = \int r^2\,dm = R^2 \int dm = MR^2$.

KEYCONCEPT Use integration to calculate the moment of inertia of a rigid body that is a continuous distribution of mass. If the body is symmetrical, divide it into volume elements that make use of its symmetry.

EXAMPLE 9.11 Uniform sphere with radius *R*, axis through center

Find the moment of inertia of a solid sphere of uniform mass density ρ (like a billiard ball) about an axis through its center.

IDENTIFY and SET UP We divide the sphere into thin, solid disks of thickness dx (**Fig. 9.24**), whose moment of inertia we know from Table 9.2, case (f). We'll integrate over these to find the total moment of inertia.

EXECUTE The radius and hence the volume and mass of a disk depend on its distance x from the center of the sphere. The radius r of the disk shown in Fig. 9.24 is

$$r = \sqrt{R^2 - x^2}$$

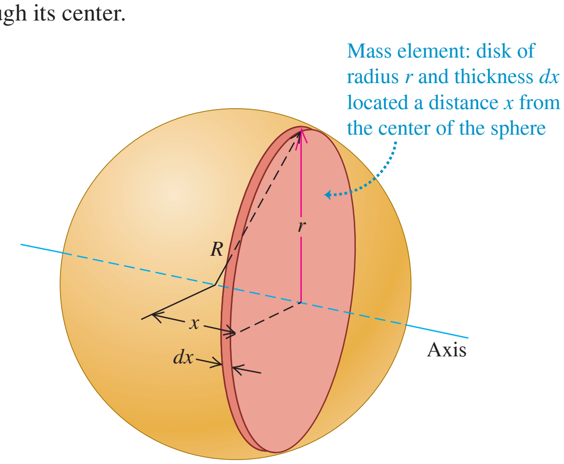
Its volume is

$$dV = \pi r^2\,dx = \pi(R^2 - x^2)\,dx$$

and so its mass is

$$dm = \rho\,dV = \pi\rho(R^2 - x^2)\,dx$$

Figure **9.24** Finding the moment of inertia of a sphere about an axis through its center.



From Table 9.2, case (f), the moment of inertia of a disk of radius r and mass dm is

$$dI = \tfrac{1}{2}r^2\,dm = \tfrac{1}{2}(R^2 - x^2)\left[\pi\rho(R^2 - x^2)dx\right]$$
$$= \frac{\pi\rho}{2}(R^2 - x^2)^2\,dx$$

Integrating this expression from $x = 0$ to $x = R$ gives the moment of inertia of the right hemisphere. The total I for the entire sphere, including both hemispheres, is just twice this:

$$I = (2)\frac{\pi\rho}{2}\int_0^R (R^2 - x^2)^2\,dx$$

Carrying out the integration, we find

$$I = \frac{8\pi\rho R^5}{15}$$

The volume of the sphere is $V = 4\pi R^3/3$, so in terms of its mass M its density is

$$\rho = \frac{M}{V} = \frac{3M}{4\pi R^3}$$

Hence our expression for I becomes

$$I = \left(\frac{8\pi R^5}{15}\right)\left(\frac{3M}{4\pi R^3}\right) = \tfrac{2}{5}MR^2$$

EVALUATE This is just as in Table 9.2, case (h). Note that the moment of inertia $I = \tfrac{2}{5}MR^2$ of a solid sphere of mass M and radius R is less than the moment of inertia $I = \tfrac{1}{2}MR^2$ of a solid *cylinder* of the same mass and radius, because more of the sphere's mass is located close to the axis.

KEYCONCEPT When calculating a moment of inertia by integration, in general use geometry to determine the size and moment of inertia of each volume element.

TEST YOUR UNDERSTANDING OF SECTION 9.6 Two hollow cylinders have the same inner and outer radii and the same mass, but they have different lengths. One is made of low-density wood and the other of high-density lead. Which cylinder has the greater moment of inertia around its axis of symmetry? (i) The wood cylinder; (ii) the lead cylinder; (iii) the two moments of inertia are equal.

ANSWER | **(iii)** Our result from Example 9.10 does *not* depend on the cylinder length L. The moment of inertia depends on only the *radial* distribution of mass, not on its distribution along the axis.

CHAPTER 9 SUMMARY

Rotational kinematics: When a rigid body rotates about a stationary axis (usually called the z-axis), the body's position is described by an angular coordinate θ. The angular velocity ω_z is the time derivative of θ, and the angular acceleration α_z is the time derivative of ω_z or the second derivative of θ. (See Examples 9.1 and 9.2.) If the angular acceleration is constant, then θ, ω_z, and α_z are related by simple kinematic equations analogous to those for straight-line motion with constant linear acceleration. (See Example 9.3.)

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \quad (9.3)$$

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta\omega_z}{\Delta t} = \frac{d\omega_z}{dt} \quad (9.5)$$

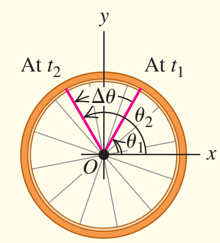
Constant α_z only:

$$\theta = \theta_0 + \omega_{0z}t + \tfrac{1}{2}\alpha_z t^2 \quad (9.11)$$

$$\theta - \theta_0 = \tfrac{1}{2}(\omega_{0z} + \omega_z)t \quad (9.10)$$

$$\omega_z = \omega_{0z} + \alpha_z t \quad (9.7)$$

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) \quad (9.12)$$

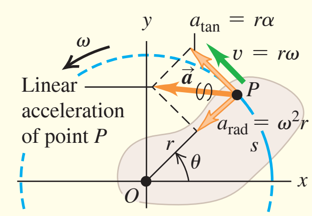


Relating linear and angular kinematics: The angular speed ω of a rigid body is the magnitude of the body's angular velocity. The rate of change of ω is $\alpha = d\omega/dt$. For a particle in the body a distance r from the rotation axis, the speed v and the components of the acceleration $\vec{a}$ are related to ω and α. (See Examples 9.4 and 9.5.)

$$v = r\omega \quad (9.13)$$

$$a_{tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \quad (9.14)$$

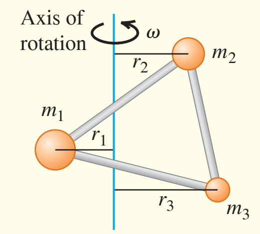
$$a_{rad} = \frac{v^2}{r} = \omega^2 r \quad (9.15)$$



Moment of inertia and rotational kinetic energy: The moment of inertia I of a body about a given axis is a measure of its rotational inertia: The greater the value of I, the more difficult it is to change the state of the body rotation. The moment of inertia can be expressed as a sum over the particles m_i that make up the body, each of which is at its own perpendicular distance r_i from the axis. The rotational kinetic energy of a rigid body rotating about a fixed axis depends on the angular speed ω and the moment of inertia I for that rotation axis. (See Examples 9.6–9.8.)

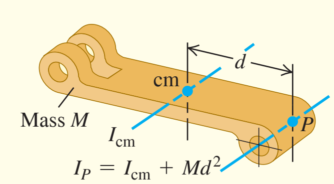
$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots = \sum_i m_i r_i^2 \quad (9.16)$$

$$K = \tfrac{1}{2}I\omega^2 \quad (9.17)$$



Calculating the moment of inertia: The parallel-axis theorem relates the moments of inertia of a rigid body of mass M about two parallel axes: an axis through the center of mass (moment of inertia I_{cm}) and a parallel axis a distance d from the first axis (moment of inertia I_P). (See Example 9.9.) If the body has a continuous mass distribution, the moment of inertia can be calculated by integration. (See Examples 9.10 and 9.11.)

$$I_P = I_{cm} + Md^2 \quad (9.19)$$



Chapter 9 Media Assets

GUIDED PRACTICE

For assigned homework and other learning materials, go to Mastering Physics.

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLE 9.3 (Section 9.2) before attempting these problems.

VP9.3.1 A machine part is initially rotating at 0.500 rad/s. Its rotation speeds up with constant angular acceleration 2.50 rad/s^2. Through what angle has the machine part rotated when its angular speed equals 3.25 rad/s? Give your answer in (a) radians, (b) degrees, and (c) revolutions.

VP9.3.2 The rotor of a helicopter is gaining angular speed with constant angular acceleration. At $t = 0$ it is rotating at 1.25 rad/s. From $t = 0$ to $t = 2.00$ s, the rotor rotates through 8.00 rad. (a) What is the angular acceleration of the rotor? (b) Through what angle (in radians) does the rotor rotate from $t = 0$ to $t = 4.00$ s?

VP9.3.3 A jeweler's grinding wheel slows down at a constant rate from 185 rad/s to 105 rad/s while it rotates through 16.0 revolutions. How much time does this take?

VP9.3.4 A disk rotates around an axis through its center that is perpendicular to the plane of the disk. The disk has a line drawn on it that extends from the axis of the disk to the rim. At $t = 0$ this line lies along the x-axis and the disk is rotating with positive angular velocity ω_{0z}. The disk has constant positive angular acceleration α_z. At what time after $t = 0$ has the line on the disk rotated through an angle θ?

Be sure to review EXAMPLES 9.4 and 9.5 (Section 9.3) before attempting these problems.

VP9.5.1 Shortly after a vinyl record (radius 0.152 m) starts rotating on a turntable, its angular velocity is 1.60 rad/s and increasing at a rate of 8.00 rad/s^2. At this instant, for a point at the rim of the record, what are (a) the tangential component of acceleration, (b) the centripetal component of acceleration, and (c) the magnitude of acceleration?

VP9.5.2 A superhero swings a magic hammer over her head in a horizontal plane. The end of the hammer moves around a circular path of radius 1.50 m at an angular speed of 6.00 rad/s. As the superhero swings the hammer, she then ascends vertically at a constant 2.00 m/s. (a) What is the speed of the end of the hammer relative to the ground? (b) What is the acceleration (magnitude and direction) of the end of the hammer?

VP9.5.3 If the magnitude of the acceleration of a propeller blade's tip exceeds a certain value a_{max}, the blade tip will fracture. If the propeller has radius r, is initially at rest, and has angular acceleration of magnitude α, at what angular speed ω will the blade tip fracture?

VP9.5.4 At a certain instant, a rotating turbine wheel of radius R has angular speed ω (measured in rad/s). (a) What must be the magnitude α of its angular acceleration (measured in rad/s^2) at this instant if the acceleration vector $\vec{a}$ of a point on the rim of the wheel makes an angle of exactly 30° with the velocity vector $\vec{v}$ of that point? (b) At this same instant, what is the angle between $\vec{a}$ and $\vec{v}$ for a point on the wheel halfway between the axis of rotation and the rim?

Be sure to review EXAMPLES 9.7 and 9.8 (Section 9.4) before attempting these problems.

VP9.8.1 A solid cylinder of mass 12.0 kg and radius 0.250 m is free to rotate without friction around its central axis. If you do 75.0 J of work on the cylinder to increase its angular speed, what will be its final angular speed if the cylinder (a) starts from rest; (b) is initially rotating at 12.0 rad/s?

VP9.8.2 A square plate has mass 0.600 kg and sides of length 0.150 m. It is free to rotate without friction around an axis through its center and perpendicular to the plane of the plate. How much work must you do on the plate to change its angular speed (a) from 0 to 40.0 rad/s and (b) from 40.0 rad/s to 80.0 rad/s?

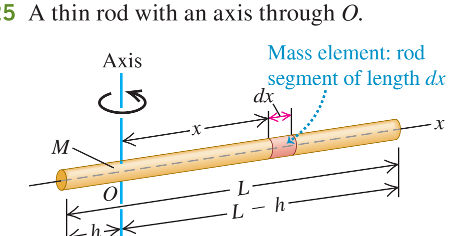
VP9.8.3 A hollow cylinder of mass 2.00 kg, inner radius 0.100 m, and outer radius 0.200 m is free to rotate without friction around a horizontal shaft of radius 0.100 m along the axis of the cylinder. You wrap a light, nonstretching cable around the cylinder and tie the free end to a 0.500 kg block of cheese. You release the cheese from rest a distance h above the floor. (a) If the cheese is moving downward at 4.00 m/s just before it hits the ground, what is the value of h? (b) What is the angular speed of the cylinder just before the cheese hits the ground?

VP9.8.4 A pulley in the shape of a solid cylinder of mass 1.50 kg and radius 0.240 m is free to rotate around a horizontal shaft along the axis of the pulley. There is friction between the pulley and this shaft. A light, nonstretching cable is wrapped around the pulley, and the free end is tied to a 2.00 kg textbook. You release the textbook from rest a distance 0.900 m above the floor. Just before the textbook hits the floor, the angular speed of the pulley is 10.0 rad/s. (a) What is the speed of the textbook just before it hits the floor? (b) How much work was done on the pulley by the force of friction while the textbook was falling to the floor?

BRIDGING PROBLEM A Rotating, Uniform Thin Rod

Figure 9.25 shows a slender uniform rod with mass M and length L. It might be a baton held by a twirler in a marching band (without the rubber end caps). (a) Use integration to compute its moment of inertia about an axis through O, at an arbitrary distance h from one end. (b) Initially the rod is at rest. It is given a constant angular acceleration of magnitude α around the axis through O. Find how much work is done on the rod in a time t. (c) At time t, what is the *linear* acceleration of the point on the rod farthest from the axis?

Figure 9.25 A thin rod with an axis through O.



SOLUTION GUIDE

IDENTIFY and SET UP

1. Make a list of the target variables for this problem.
2. To calculate the moment of inertia of the rod, you'll have to divide the rod into infinitesimal elements of mass. If an element has length dx, what is the mass of the element? What are the limits of integration?
3. What is the angular speed of the rod at time t? How does the work required to accelerate the rod from rest to this angular speed compare to the rod's kinetic energy at time t?
4. At time t, does the point on the rod farthest from the axis have a centripetal acceleration? A tangential acceleration? Why or why not?

EXECUTE

5. Do the integration required to find the moment of inertia.
6. Use your result from step 5 to calculate the work done in time t to accelerate the rod from rest.
7. Find the linear acceleration components for the point in question at time t. Use these to find the magnitude of the acceleration.

EVALUATE

8. Check your results for the special cases $h = 0$ (the axis passes through one end of the rod) and $h = L/2$ (the axis passes through the middle of the rod). Are these limits consistent with Table 9.2? With the parallel-axis theorem?
9. Is the acceleration magnitude from step 7 constant? Would you expect it to be?

PROBLEMS

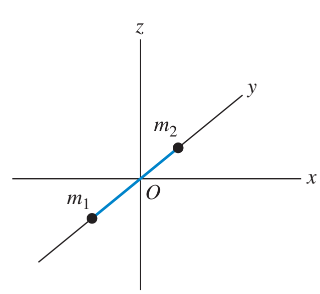
•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q9.1 Which of the following formulas is valid if the angular acceleration of an object is *not* constant? Explain your reasoning in each case. (a) $v = r\omega$; (b) $a_{\tan} = r\alpha$; (c) $\omega = \omega_0 + \alpha t$; (d) $a_{\tan} = r\omega^2$; (e) $K = \frac{1}{2}I\omega^2$.

Q9.2 A diatomic molecule can be modeled as two point masses, m_1 and m_2, slightly separated (**Fig. Q9.2**). If the molecule is oriented along the y-axis, it has kinetic energy K when it spins about the x-axis. What will its kinetic energy (in terms of K) be if it spins at the same angular speed about (a) the z-axis and (b) the y-axis?

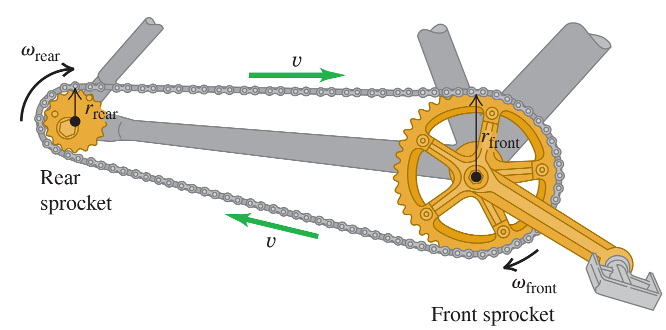
Figure **Q9.2**



Q9.3 What is the difference between tangential and radial acceleration for a point on a rotating body?

Q9.4 In **Fig. Q9.4,** all points on the chain have the same linear speed. Is the magnitude of the linear acceleration also the same for all points on the chain? How are the angular accelerations of the two sprockets related? Explain.

Figure **Q9.4**



Q9.5 In Fig. Q9.4, how are the radial accelerations of points at the teeth of the two sprockets related? Explain.

Q9.6 A flywheel rotates with constant angular velocity. Does a point on its rim have a tangential acceleration? A radial acceleration? Are these accelerations constant in magnitude? In direction? In each case give your reasoning.

Q9.7 What is the purpose of the spin cycle of a washing machine? Explain in terms of acceleration components.

Q9.8 You are designing a flywheel to store kinetic energy. If all of the following uniform objects have the same mass and same angular velocity, which one will store the greatest amount of kinetic energy? Which will store the least? Explain. (a) A solid sphere of diameter D rotating about a diameter; (b) a solid cylinder of diameter D rotating about an axis perpendicular to each face through its center; (c) a thin-walled hollow cylinder of diameter D rotating about an axis perpendicular to the plane of the circular face at its center; (d) a solid, thin bar of length D rotating about an axis perpendicular to it at its center.

Q9.9 Can you think of a body that has the same moment of inertia for all possible axes? If so, give an example, and if not, explain why this is not possible. Can you think of a body that has the same moment of inertia for all axes passing through a certain point? If so, give an example and indicate where the point is located.

Q9.10 To maximize the moment of inertia of a flywheel while minimizing its weight, what shape and distribution of mass should it have? Explain.

Q9.11 How might you determine experimentally the moment of inertia of an irregularly shaped body about a given axis?

Q9.12 A cylindrical body has mass M and radius R. Can the mass be distributed within the body in such a way that its moment of inertia about its axis of symmetry is greater than MR^2? Explain.

Q9.13 Describe how you could use part (b) of Table 9.2 to derive the result in part (d).

Q9.14 A hollow spherical shell of radius R that is rotating about an axis through its center has rotational kinetic energy K. If you want to modify this sphere so that it has three times as much kinetic energy at the same angular speed while keeping the same mass, what should be its radius in terms of R?

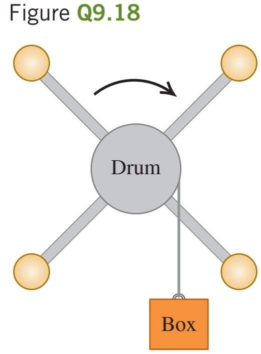
Q9.15 For the equations for I given in parts (a) and (b) of Table 9.2 to be valid, must the rod have a circular cross section? Is there any restriction on the size of the cross section for these equations to apply? Explain.

Q9.16 In part (d) of Table 9.2, the thickness of the plate must be much less than a for the expression given for I to apply. But in part (c), the expression given for I applies no matter how thick the plate is. Explain.

Q9.17 Two identical balls, A and B, are each attached to very light string, and each string is wrapped around the rim of a pulley of mass M on a frictionless axle. The only difference is that the pulley for ball A is a solid disk, while the one for ball B is a hollow disk, like part (e) in Table 9.2. If both balls are released from rest and fall the same distance, which one will have more kinetic energy, or will they have the same kinetic energy? Explain your reasoning.

Q9.18 An elaborate pulley consists of four identical balls at the ends of spokes extending out from a rotating drum (**Fig. Q9.18**). A box is connected to a light, thin rope wound around the rim of the drum. When it is released from rest, the box acquires a speed V after having fallen a distance d. Now the four balls are moved inward closer to the drum, and the box is again released from rest. After it has fallen a distance d, will its speed be equal to V, greater than V, or less than V? Show or explain why.

Figure **Q9.18**



Q9.19 You can use any angular measure—radians, degrees, or revolutions—in some of the equations in Chapter 9, but you can use only radian measure in others. Identify those for which using radians is necessary and those for which it is not, and in each case give your reasoning.

Q9.20 When calculating the moment of inertia of an object, can we treat all its mass as if it were concentrated at the center of mass of the object? Justify your answer.

Q9.21 A wheel is rotating about an axis perpendicular to the plane of the wheel and passing through the center of the wheel. The angular speed of the wheel is increasing at a constant rate. Point A is on the rim of the wheel and point B is midway between the rim and center of the wheel. For each of the following quantities, is its magnitude larger at point A or at point B, or is it the same at both points? (a) Angular speed; (b) tangential speed; (c) angular acceleration; (d) tangential acceleration; (e) radial acceleration. Justify each answer.

Q9.22 Estimate your own moment of inertia about a vertical axis through the center of the top of your head when you are standing up straight with your arms outstretched. Make reasonable approximations and measure or estimate necessary quantities.

EXERCISES

Section 9.1 Angular Velocity and Acceleration

9.1 • (a) What angle in radians is subtended by an arc 1.50 m long on the circumference of a circle of radius 2.50 m? What is this angle in degrees? (b) An arc 14.0 cm long on the circumference of a circle subtends an angle of 128°. What is the radius of the circle? (c) The angle between two radii of a circle with radius 1.50 m is 0.700 rad. What length of arc is intercepted on the circumference of the circle by the two radii?

9.2 • An airplane propeller is rotating at 1900 rpm (rev/min). (a) Compute the propeller's angular velocity in rad/s. (b) How many seconds does it take for the propeller to turn through 35°?

9.3 • **CP CALC** The angular velocity of a flywheel obeys the equation $\omega_z(t) = A + Bt^2$, where t is in seconds and A and B are constants having numerical values 2.75 (for A) and 1.50 (for B). (a) What are the units of A and B if ω_z is in rad/s? (b) What is the angular acceleration of the wheel at (i) $t = 0$ and (ii) $t = 5.00$ s? (c) Through what angle does the flywheel turn during the first 2.00 s? (*Hint:* See Section 2.6.)

9.4 •• **CALC** A fan blade rotates with angular velocity given by $\omega_z(t) = \gamma - \beta t^2$, where $\gamma = 5.00$ rad/s and $\beta = 0.800$ rad/s^3. (a) Calculate the angular acceleration as a function of time. (b) Calculate the instantaneous angular acceleration α_z at $t = 3.00$ s and the average angular acceleration $\alpha_{\text{av-}z}$ for the time interval $t = 0$ to $t = 3.00$ s. How do these two quantities compare? If they are different, why?

9.5 •• **CALC** A child is pushing a merry-go-round. The angle through which the merry-go-round has turned varies with time according to $\theta(t) = \gamma t + \beta t^3$, where $\gamma = 0.400$ rad/s and $\beta = 0.0120$ rad/s^3. (a) Calculate the angular velocity of the merry-go-round as a function of time. (b) What is the initial value of the angular velocity? (c) Calculate the instantaneous value of the angular velocity ω_z at $t = 5.00$ s and the average angular velocity $\omega_{\text{av-}z}$ for the time interval $t = 0$ to $t = 5.00$ s. Show that $\omega_{\text{av-}z}$ is *not* equal to the average of the instantaneous angular velocities at $t = 0$ and $t = 5.00$ s, and explain.

9.6 • **CALC** At $t = 0$ the current to a dc electric motor is reversed, resulting in an angular displacement of the motor shaft given by $\theta(t) = (250\text{ rad/s})t - (20.0\text{ rad/s}^2)t^2 - (1.50\text{ rad/s}^3)t^3$. (a) At what time is the angular velocity of the motor shaft zero? (b) Calculate the angular acceleration at the instant that the motor shaft has zero angular velocity. (c) How many revolutions does the motor shaft turn through between the time when the current is reversed and the instant when the angular velocity is zero? (d) How fast was the motor shaft rotating at $t = 0$, when the current was reversed? (e) Calculate the average angular velocity for the time period from $t = 0$ to the time calculated in part (a).

9.7 • **CALC** The angle θ through which a disk drive turns is given by $\theta(t) = a + bt - ct^3$, where a, b, and c are constants, t is in seconds, and θ is in radians. When $t = 0$, $\theta = \pi/4$ rad and the angular velocity is 2.00 rad/s. When $t = 1.50$ s, the angular acceleration is 1.25 rad/s^2. (a) Find a, b, and c, including their units. (b) What is the angular acceleration when $\theta = \pi/4$ rad? (c) What are θ and the angular velocity when the angular acceleration is 3.50 rad/s^2?

9.8 • A wheel is rotating about an axis that is in the z-direction. The angular velocity ω_z is -6.00 rad/s at $t = 0$, increases linearly with time, and is $+4.00$ rad/s at $t = 7.00$ s. We have taken counterclockwise rotation to be positive. (a) Is the angular acceleration during this time interval positive or negative? (b) During what time interval is the speed of the wheel increasing? Decreasing? (c) What is the angular displacement of the wheel at $t = 7.00$ s?

Section 9.2 Rotation with Constant Angular Acceleration

9.9 • A bicycle wheel has an initial angular velocity of 1.50 rad/s. (a) If its angular acceleration is constant and equal to 0.200 rad/s^2, what is its angular velocity at $t = 2.50$ s? (b) Through what angle has the wheel turned between $t = 0$ and $t = 2.50$ s?

9.10 •• An electric fan is turned off, and its angular velocity decreases uniformly from 500 rev/min to 200 rev/min in 4.00 s. (a) Find the angular acceleration in rev/s^2 and the number of revolutions made by the motor in the 4.00 s interval. (b) How many more seconds are required for the fan to come to rest if the angular acceleration remains constant at the value calculated in part (a)?

9.11 •• The rotating blade of a blender turns with constant angular acceleration 1.50 rad/s^2. (a) How much time does it take to reach an angular velocity of 36.0 rad/s, starting from rest? (b) Through how many revolutions does the blade turn in this time interval?

9.12 •• A wheel rotates from rest with constant angular acceleration. If it rotates through 8.00 revolutions in the first 2.50 s, how many more revolutions will it rotate through in the next 5.00 s?

9.13 •• A turntable rotates with a constant 2.25 rad/s^2 clockwise angular acceleration. After 4.00 s it has rotated through a clockwise angle of 30.0 rad. What was the angular velocity of the wheel at the beginning of the 4.00 s interval?

9.14 • A circular saw blade 0.200 m in diameter starts from rest. In 6.00 s it accelerates with constant angular acceleration to an angular velocity of 140 rad/s. Find the angular acceleration and the angle through which the blade has turned.

9.15 •• A high-speed flywheel in a motor is spinning at 500 rpm when a power failure suddenly occurs. The flywheel has mass 40.0 kg and diameter 75.0 cm. The power is off for 30.0 s, and during this time the flywheel slows due to friction in its axle bearings. During the time the power is off, the flywheel makes 200 complete revolutions. (a) At what rate is the flywheel spinning when the power comes back on? (b) How long after the beginning of the power failure would it have taken the flywheel to stop if the power had not come back on, and how many revolutions would the wheel have made during this time?

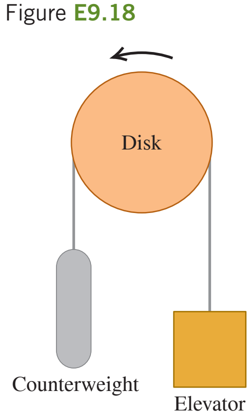
9.16 •• At $t = 0$ a grinding wheel has an angular velocity of 24.0 rad/s. It has a constant angular acceleration of 30.0 rad/s^2 until a circuit breaker trips at $t = 2.00$ s. From then on, it turns through 432 rad as it coasts to a stop at constant angular acceleration. (a) Through what total angle did the wheel turn between $t = 0$ and the time it stopped? (b) At what time did it stop? (c) What was its acceleration as it slowed down?

9.17 •• A safety device brings the blade of a power mower from an initial angular speed of ω_1 to rest in 1.00 revolution. At the same constant acceleration, how many revolutions would it take the blade to come to rest from an initial angular speed ω_3 that was three times as great, $\omega_3 = 3\omega_1$?

Section 9.3 Relating Linear and Angular Kinematics

9.18 • In a charming 19th-century hotel, an old-style elevator is connected to a counterweight by a cable that passes over a rotating disk 2.50 m in diameter (**Fig. E9.18**). The elevator is raised and lowered by turning the disk, and the cable does not slip on the rim of the disk but turns with it. (a) At how many rpm must the disk turn to raise the elevator at 25.0 cm/s? (b) To start the elevator moving, it must be accelerated at $\frac{1}{8}g$. What must be the angular acceleration of the disk, in rad/s^2? (c) Through what angle (in radians and degrees) has the disk turned when it has raised the elevator 3.25 m between floors?

Figure **E9.18**



9.19 • Spin cycles of washing machines remove water from clothes by producing a large radial acceleration at the rim of the cylindrical tub that holds the water and clothes. Estimate the diameter of the tub in a typical home washing machine. (a) What is the rotation rate, in rev/min, of the tub during the spin cycle if the radial acceleration of points on the tub wall is $3g$? (b) At this rotation rate, what is the tangential speed in m/s of a point on the tub wall?

9.20 • **Compact Disc.** A compact disc (CD) stores music in a coded pattern of tiny pits 10^{-7} m deep. The pits are arranged in a track that spirals outward toward the rim of the disc; the inner and outer radii of this spiral are 25.0 mm and 58.0 mm, respectively. As the disc spins inside a CD player, the track is scanned at a constant *linear* speed of 1.25 m/s. (a) What is the angular speed of the CD when the innermost part of the track is scanned? The outermost part of the track? (b) The maximum playing time of a CD is 74.0 min. What would be the length of the track on such a maximum-duration CD if it were stretched out in a straight line? (c) What is the average angular acceleration of a maximum-duration CD during its 74.0 min playing time? Take the direction of rotation of the disc to be positive.

9.21 •• A wheel of diameter 40.0 cm starts from rest and rotates with a constant angular acceleration of 3.00 rad/s^2. Compute the radial acceleration of a point on the rim for the instant the wheel completes its second revolution from the relationship (a) $a_{rad} = \omega^2 r$ and (b) $a_{rad} = v^2/r$.

9.22 •• You are to design a rotating cylindrical axle to lift 800 N buckets of cement from the ground to a rooftop 78.0 m above the ground. The buckets will be attached to a hook on the free end of a cable that wraps around the rim of the axle; as the axle turns, the buckets will rise. (a) What should the diameter of the axle be in order to raise the buckets at a steady 2.00 cm/s when it is turning at 7.5 rpm? (b) If instead the axle must give the buckets an upward acceleration of 0.400 m/s^2, what should the angular acceleration of the axle be?

9.23 • The blade of an electric saw rotates at 2600 rev/min. Estimate the diameter of a typical saw that is used to saw boards in home construction and renovation. What is the linear speed in m/s of a point on the rim of the circular saw blade?

9.24 •• An electric turntable 0.750 m in diameter is rotating about a fixed axis with an initial angular velocity of 0.250 rev/s and a constant angular acceleration of 0.900 rev/s^2. (a) Compute the angular velocity of the turntable after 0.200 s. (b) Through how many revolutions has the turntable spun in this time interval? (c) What is the tangential speed of a point on the rim of the turntable at $t = 0.200$ s? (d) What is the magnitude of the *resultant* acceleration of a point on the rim at $t = 0.200$ s?

9.25 •• **Centrifuge.** An advertisement claims that a centrifuge takes up only 0.127 m of bench space but can produce a radial acceleration of $3000g$ at 5000 rev/min. Calculate the required radius of the centrifuge. Is the claim realistic?

9.26 • At $t = 3.00$ s a point on the rim of a 0.200-m-radius wheel has a tangential speed of 50.0 m/s as the wheel slows down with a tangential acceleration of constant magnitude 10.0 m/s^2. (a) Calculate the wheel's constant angular acceleration. (b) Calculate the angular velocities at $t = 3.00$ s and $t = 0$. (c) Through what angle did the wheel turn between $t = 0$ and $t = 3.00$ s? (d) At what time will the radial acceleration equal g?

9.27 • A rotating wheel with diameter 0.600 m is speeding up with constant angular acceleration. The speed of a point on the rim of the wheel increases from 3.00 m/s to 6.00 m/s while the wheel turns through 4.00 revolutions. What is the angular acceleration of the wheel?

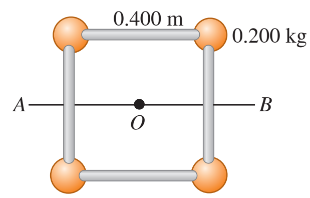
9.28 • The earth is approximately spherical, with a diameter of 1.27×10^7 m. It takes 24.0 hours for the earth to complete one revolution. What are the tangential speed and radial acceleration of a point on the surface of the earth, at the equator?

9.29 • A flywheel with radius 0.300 m starts from rest and accelerates with a constant angular acceleration of 0.600 rad/s^2. For a point on the rim of the flywheel, what are the magnitudes of the tangential, radial, and resultant accelerations after 2.00 s of acceleration?

Section 9.4 Energy in Rotational Motion

9.30 • Four small spheres, each of which you can regard as a point of mass 0.200 kg, are arranged in a square 0.400 m on a side and connected by extremely light rods (**Fig. E9.30**). Find the moment of inertia of the system about an axis (a) through the center of the square, perpendicular to its plane (an axis through point O in the figure); (b) bisecting two opposite sides of the square (an axis along the line AB in the figure); (c) that passes through the centers of the upper left and lower right spheres and through point O.

Figure **E9.30**



9.31 • Calculate the moment of inertia of each of the following uniform objects about the axes indicated. Consult Table 9.2 as needed. (a) A thin 2.50 kg rod of length 75.0 cm, about an axis perpendicular to it and passing through (i) one end and (ii) its center, and (iii) about an axis parallel to the rod and passing through it. (b) A 3.00 kg sphere 38.0 cm in diameter, about an axis through its center, if the sphere is (i) solid and (ii) a thin-walled hollow shell. (c) An 8.00 kg cylinder, of length 19.5 cm and diameter 12.0 cm, about the central axis of the cylinder, if the cylinder is (i) thin-walled and hollow, and (ii) solid.

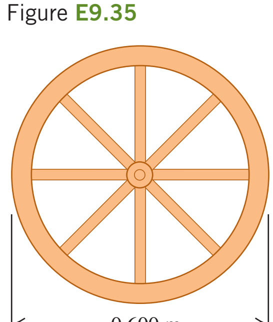
9.32 •• Three small blocks, each with mass m, are clamped at the ends and at the center of a rod of length L and negligible mass. Compute the moment of inertia of the system about an axis perpendicular to the rod and passing through (a) the center of the rod and (b) a point one-fourth of the length from one end.

9.33 • A uniform bar has two small balls glued to its ends. The bar is 2.00 m long and has mass 4.00 kg, while the balls each have mass 0.300 kg and can be treated as point masses. Find the moment of inertia of this combination about an axis (a) perpendicular to the bar through its center; (b) perpendicular to the bar through one of the balls; (c) parallel to the bar through both balls; and (d) parallel to the bar and 0.500 m from it.

9.34 •• You are a project manager for a manufacturing company. One of the machine parts on the assembly line is a thin, uniform rod that is 60.0 cm long and has mass 0.400 kg. (a) What is the moment of inertia of this rod for an axis at its center, perpendicular to the rod? (b) One of your engineers has proposed to reduce the moment of inertia by bending the rod at its center into a V-shape, with a 60.0° angle at its vertex. What would be the moment of inertia of this bent rod about an axis perpendicular to the plane of the V at its vertex?

9.35 •• A wagon wheel is constructed as shown in **Fig. E9.35**. The radius of the wheel is 0.300 m, and the rim has mass 1.40 kg. Each of the eight spokes that lie along a diameter and are 0.300 m long has mass 0.280 kg. What is the moment of inertia of the wheel about an axis through its center and perpendicular to the plane of the wheel? (Use Table 9.2.)

Figure **E9.35**



9.36 •• A uniform sphere made of modeling clay has radius R and moment of inertia I_1 for rotation about a diameter. It is flattened to a disk with the same radius R. In terms of I_1, what is the moment of inertia of the disk for rotation about an axis that is at the center of the disk and perpendicular to its flat surface?

9.37 •• A rotating flywheel has moment of inertia 12.0 kg • m^2 for an axis along the axle about which the wheel is rotating. Initially the flywheel has 30.0 J of kinetic energy. It is slowing down with an angular acceleration of magnitude 0.500 rev/s^2. How long does it take for the rotational kinetic energy to become half its initial value, so it is 15.0 J?

9.38 •• An airplane propeller is 2.08 m in length (from tip to tip) with mass 117 kg and is rotating at 2400 rpm (rev/min) about an axis through its center. You can model the propeller as a slender rod. (a) What is its rotational kinetic energy? (b) Suppose that, due to weight constraints, you had to reduce the propeller's mass to 75.0% of its original mass, but you still needed to keep the same size and kinetic energy. What would its angular speed have to be, in rpm?

9.39 •• A uniform sphere with mass M and radius R is rotating with angular speed ω_1 about a frictionless axle along a diameter of the sphere. The sphere has rotational kinetic energyK_1. A thin-walled hollow sphere has the same mass and radius as the uniform sphere. It is also rotating about a fixed axis along its diameter. In terms of ω_1, what angular speed must the hollow sphere have if its kinetic energy is also K_1, the same as for the uniform sphere?

9.40 • A wheel is turning about an axis through its center with constant angular acceleration. Starting from rest, at $t = 0$, the wheel turns through 8.20 revolutions in 12.0 s. At $t = 12.0$ s the kinetic energy of the wheel is 36.0 J. For an axis through its center, what is the moment of inertia of the wheel?

9.41 • A uniform sphere with mass 28.0 kg and radius 0.380 m is rotating at constant angular velocity about a stationary axis that lies along a diameter of the sphere. If the kinetic energy of the sphere is 236 J, what is the tangential velocity of a point on the rim of the sphere?

9.42 •• A hollow spherical shell has mass 8.20 kg and radius 0.220 m. It is initially at rest and then rotates about a stationary axis that lies along a diameter with a constant acceleration of 0.890 rad/s^2. What is the kinetic energy of the shell after it has turned through 6.00 rev?

9.43 • Wheel A has three times the moment of inertia about its axis of rotation as wheel B. Wheel B's angular speed is four times that of wheel A. (a) Which wheel has the greater rotational kinetic energy? (b) If K_A and K_B are the rotational kinetic energies of the wheels, what is K_A/K_B?

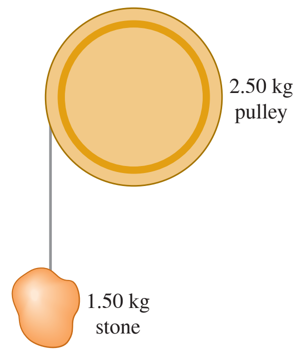
9.44 •• You need to design an industrial turntable that is 60.0 cm in diameter and has a kinetic energy of 0.250 J when turning at 45.0 rpm (rev/min). (a) What must be the moment of inertia of the turntable about the rotation axis? (b) If your workshop makes this turntable in the shape of a uniform solid disk, what must be its mass?

9.45 •• Energy is to be stored in a 70.0 kg flywheel in the shape of a uniform solid disk with radius $R = 1.20$ m. To prevent structural failure of the flywheel, the maximum allowed radial acceleration of a point on its rim is 3500 m/s^2. What is the maximum kinetic energy that can be stored in the flywheel?

9.46 •• You are designing a flywheel. It is to start from rest and then rotate with a constant angular acceleration of 0.200 rev/s^2. The design specifications call for it to have a rotational kinetic energy of 240 J after it has turned through 30.0 revolutions. What should be the moment of inertia of the flywheel about its rotation axis?

9.47 •• A pulley on a frictionless axle has the shape of a uniform solid disk of mass 2.50 kg and radius 20.0 cm. A 1.50 kg stone is attached to a very light wire that is wrapped around the rim of the pulley (**Fig. E9.47**), and the system is released from rest. (a) How far must the stone fall so that the pulley has 4.50 J of kinetic energy? (b) What percent of the total kinetic energy does the pulley have?

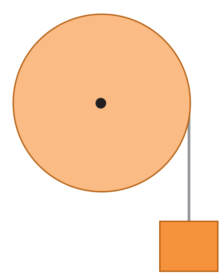
Figure **E9.47**



9.48 •• A bucket of mass m is tied to a massless cable that is wrapped around the outer rim of a uniform pulley of radius R, on a frictionless axle, similar to the system shown in Fig. E9.47. In terms of the stated variables, what must be the moment of inertia of the pulley so that it always has half as much kinetic energy as the bucket?

9.49 •• **CP** A thin, light wire is wrapped around the rim of a wheel (**Fig. E9.49**). The wheel rotates without friction about a stationary horizontal axis that passes through the center of the wheel. The wheel is a uniform disk with radius $R = 0.280$ m. An object of mass $m = 4.20$ kg is suspended from the free end of the wire. The system is released from rest and the suspended object descends with constant acceleration. If the suspended object moves downward a distance of 3.00 m in 2.00 s, what is the mass of the wheel?

Figure **E9.49**



Section 9.5 Parallel-Axis Theorem

9.50 •• Find the moment of inertia of a hoop (a thin-walled, hollow ring) with mass M and radius R about an axis perpendicular to the hoop's plane at an edge.

9.51 •• About what axis will a uniform, balsa-wood sphere have the same moment of inertia as does a thin-walled, hollow, lead sphere of the same mass and radius, with the axis along a diameter?

9.52 • (a) For the thin rectangular plate shown in part (d) of Table 9.2, find the moment of inertia about an axis that lies in the plane of the plate, passes through the center of the plate, and is parallel to the axis shown. (b) Find the moment of inertia of the plate for an axis that lies in the plane of the plate, passes through the center of the plate, and is perpendicular to the axis in part (a).

9.53 •• A thin, rectangular sheet of metal has mass M and sides of length a and b. Use the parallel-axis theorem to calculate the moment of inertia of the sheet for an axis that is perpendicular to the plane of the sheet and that passes through one corner of the sheet.

9.54 •• A thin uniform rod of mass M and length L is bent at its center so that the two segments are now perpendicular to each other. Find its moment of inertia about an axis perpendicular to its plane and passing through (a) the point where the two segments meet and (b) the midpoint of the line connecting its two ends.

Section 9.6 Moment-of-Inertia Calculations

9.55 •• **CALC** Use Eq. (9.20) to calculate the moment of inertia of a uniform, solid disk with mass M and radius R for an axis perpendicular to the plane of the disk and passing through its center.

9.56 • **CALC** Use Eq. (9.20) to calculate the moment of inertia of a slender, uniform rod with mass M and length L about an axis at one end, perpendicular to the rod.

9.57 •• **CALC** A slender rod with length L has a mass per unit length that varies with distance from the left end, where $x = 0$, according to $dm/dx = \gamma x$, where γ has units of kg/m^2. (a) Calculate the total mass of the rod in terms of γ and L. (b) Use Eq. (9.20) to calculate the moment of inertia of the rod for an axis at the left end, perpendicular to the rod. Use the expression you derived in part (a) to express I in terms of M and L. How does your result compare to that for a uniform rod? Explain. (c) Repeat part (b) for an axis at the right end of the rod. How do the results for parts (b) and (c) compare? Explain.

PROBLEMS

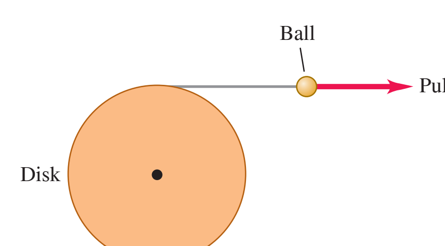
9.58 •• **CALC** A uniform disk with radius $R = 0.400$ m and mass 30.0 kg rotates in a horizontal plane on a frictionless vertical axle that passes through the center of the disk. The angle through which the disk has turned varies with time according to $\theta(t) = (1.10 \text{ rad/s})t + (6.30 \text{ rad/s}^2)t^2$. What is the resultant linear acceleration of a point on the rim of the disk at the instant when the disk has turned through 0.100 rev?

9.59 •• **CP** A circular saw blade with radius 0.120 m starts from rest and turns in a vertical plane with a constant angular acceleration of 2.00 rev/s^2. After the blade has turned through 155 rev, a small piece of the blade breaks loose from the top of the blade. After the piece breaks loose, it travels with a velocity that is initially horizontal and equal to the tangential velocity of the rim of the blade. The piece travels a vertical distance of 0.820 m to the floor. How far does the piece travel horizontally, from where it broke off the blade until it strikes the floor?

9.60 • **CALC** A roller in a printing press turns through an angle $\theta(t)$ given by $\theta(t) = \gamma t^2 - \beta t^3$, where $\gamma = 3.20 \text{ rad/s}^2$ and $\beta = 0.500 \text{ rad/s}^3$. (a) Calculate the angular velocity of the roller as a function of time. (b) Calculate the angular acceleration of the roller as a function of time. (c) What is the maximum positive angular velocity, and at what value of t does it occur?

9.61 •• **CP CALC** A disk of radius 25.0 cm is free to turn about an axle perpendicular to it through its center. It has very thin but strong string wrapped around its rim, and the string is attached to a ball that is pulled tangentially away from the rim of the disk (**Fig. P9.61**). The pull increases in magnitude and produces an acceleration of the ball that obeys the equation $a(t) = At$, where t is in seconds and A is a constant. The cylinder starts from rest, and at the end of the third second, the ball's acceleration is 1.80 m/s^2. (a) Find A. (b) Express the angular acceleration of the disk as a function of time. (c) How much time after the disk has begun to turn does it reach an angular speed of 15.0 rad/s? (d) Through what angle has the disk turned just as it reaches 15.0 rad/s? (*Hint:* See Section 2.6.)

Figure **P9.61**

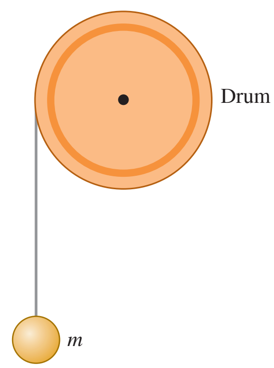


9.62 •• You are designing a rotating metal flywheel that will be used to store energy. The flywheel is to be a uniform disk with radius 25.0 cm. Starting from rest at $t = 0$, the flywheel rotates with constant angular acceleration 3.00 rad/s^2 about an axis perpendicular to the flywheel at its center. If the flywheel has a density (mass per unit volume) of 8600 kg/m^3, what thickness must it have to store 800 J of kinetic energy at $t = 8.00$ s?

9.63 •• A uniform wheel in the shape of a solid disk is mounted on a frictionless axle at its center. The wheel has mass 5.00 kg and radius 0.800 m. A thin rope is wrapped around the wheel, and a block is suspended from the free end of the rope. The system is released from rest and the block moves downward. What is the mass of the block if the wheel turns through 8.00 revolutions in the first 5.00 s after the block is released?

9.64 •• Engineers are designing a system by which a falling mass m imparts kinetic energy to a rotating uniform drum to which it is attached by thin, very light wire wrapped around the rim of the drum (**Fig. P9.64**). There is no appreciable friction in the axle of the drum, and everything starts from rest. This system is being tested on earth, but it is to be used on Mars, where the acceleration due to gravity is 3.71 m/s^2. In the earth tests, when m is set to 15.0 kg and allowed to fall through 5.00 m, it gives 250.0 J of kinetic energy to the drum. (a) If the system is operated on Mars, through what distance would the 15.0 kg mass have to fall to give the same amount of kinetic energy to the drum? (b) How fast would the 15.0 kg mass be moving on Mars just as the drum gained 250.0 J of kinetic energy?

Figure P9.64



9.65 •• Consider the system of two blocks shown in Fig. P9.77. There is no friction between block A and the tabletop. The mass of block B is 5.00 kg. The pulley rotates about a frictionless axle, and the light rope doesn't slip on the pulley surface. The pulley has radius 0.200 m and moment of inertia 1.30 kg • m^2. If the pulley is rotating with an angular speed of 8.00 rad/s after the block has descended 1.20 m, what is the mass of block A?

9.66 •• The motor of a table saw is rotating at 3450 rev/min. A pulley attached to the motor shaft drives a second pulley of half the diameter by means of a V-belt. A circular saw blade of diameter 0.208 m is mounted on the same rotating shaft as the second pulley. (a) The operator is careless and the blade catches and throws back a small piece of wood. This piece of wood moves with linear speed equal to the tangential speed of the rim of the blade. What is this speed? (b) Calculate the radial acceleration of points on the outer edge of the blade to see why sawdust doesn't stick to its teeth.

9.67 ••• While riding a multispeed bicycle, the rider can select the radius of the rear sprocket that is fixed to the rear axle. The front sprocket of a bicycle has radius 12.0 cm. If the angular speed of the front sprocket is 0.600 rev/s, what is the radius of the rear sprocket for which the tangential speed of a point on the rim of the rear wheel will be 5.00 m/s? The rear wheel has radius 0.330 m.

9.68 ••• A computer disk drive is turned on starting from rest and has constant angular acceleration. If it took 0.0865 s for the drive to make its *second* complete revolution, (a) how long did it take to make the first complete revolution, and (b) what is its angular acceleration, in rad/s^2?

9.69 •• **CP** Consider the system shown in Fig. E9.49. The suspended block has mass 1.50 kg. The system is released from rest and the block descends as the wheel rotates on a frictionless axle. As the wheel is rotating, the tension in the light wire is 9.00 N. What is the kinetic energy of the wheel 2.00 s after the system is released?

9.70 •• A uniform disk has radius R_0 and mass M_0. Its moment of inertia for an axis perpendicular to the plane of the disk at the disk's center is $\frac{1}{2}M_0R_0{}^2$. You have been asked to halve the disk's moment of inertia by cutting out a circular piece at the center of the disk. In terms of R_0, what should be the radius of the circular piece that you remove?

9.71 •• **Measuring *I*.** As an intern at an engineering firm, you are asked to measure the moment of inertia of a large wheel for rotation about an axis perpendicular to the wheel at its center. You measure the diameter of the wheel to be 0.640 m. Then you mount the wheel on frictionless bearings on a horizontal frictionless axle at the center of the wheel. You wrap a light rope around the wheel and hang an 8.20 kg block of wood from the free end of the rope, as in Fig. E9.49. You release the system from rest and find that the block descends 12.0 m in 4.00 s. What is the moment of inertia of the wheel for this axis?

9.72 ••• A uniform, solid disk with mass m and radius R is pivoted about a horizontal axis through its center. A small object of the same mass m is glued to the rim of the disk. If the disk is released from rest with the small object at the end of a horizontal radius, find the angular speed when the small object is directly below the axis.

9.73 •• **CP** A meter stick with a mass of 0.180 kg is pivoted about one end so it can rotate without friction about a horizontal axis. The meter stick is held in a horizontal position and released. As it swings through the vertical, calculate (a) the change in gravitational potential energy that has occurred; (b) the angular speed of the stick; (c) the linear speed of the end of the stick opposite the axis. (d) Compare the answer in part (c) to the speed of a particle that has fallen 1.00 m, starting from rest.

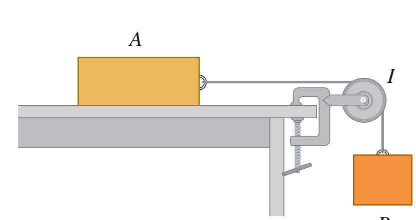
9.74 •• A physics student of mass 43.0 kg is standing at the edge of the flat roof of a building, 12.0 m above the sidewalk. An unfriendly dog is running across the roof toward her. Next to her is a large wheel mounted on a horizontal axle at its center. The wheel, used to lift objects from the ground to the roof, has a light crank attached to it and a light rope wrapped around it; the free end of the rope hangs over the edge of the roof. The student grabs the end of the rope and steps off the roof. If the wheel has radius 0.300 m and a moment of inertia of 9.60 kg • m^2 for rotation about the axle, how long does it take her to reach the sidewalk, and how fast will she be moving just before she lands? Ignore friction in the axle.

9.75 ••• A slender rod is 80.0 cm long and has mass 0.120 kg. A small 0.0200 kg sphere is welded to one end of the rod, and a small 0.0500 kg sphere is welded to the other end. The rod, pivoting about a stationary, frictionless axis at its center, is held horizontal and released from rest. What is the linear speed of the 0.0500 kg sphere as it passes through its lowest point?

9.76 •• Exactly one turn of a flexible rope with mass m is wrapped around a uniform cylinder with mass M and radius R. The cylinder rotates without friction about a horizontal axle along the cylinder axis. One end of the rope is attached to the cylinder. The cylinder starts with angular speed ω_0. After one revolution of the cylinder the rope has unwrapped and, at this instant, hangs vertically down, tangent to the cylinder. Find the angular speed of the cylinder and the linear speed of the lower end of the rope at this time. Ignore the thickness of the rope. [*Hint:* Use Eq. (9.18).]

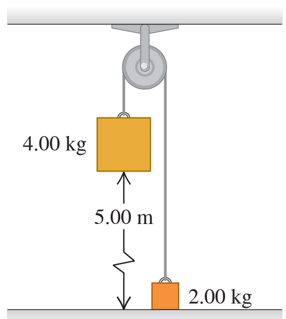
9.77 • The pulley in **Fig. P9.77** has radius R and a moment of inertia I. The rope does not slip over the pulley, and the pulley spins on a frictionless axle. The coefficient of kinetic friction between block A and the tabletop is μ_k. The system is released from rest, and block B descends. Block A has mass m_A and block B has mass m_B. Use energy methods to calculate the speed of block B as a function of the distance d that it has descended.

Figure P9.77



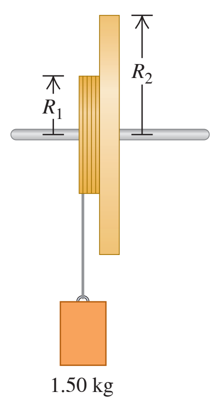
9.78 •• The pulley in **Fig. P9.78** has radius 0.160 m and moment of inertia 0.380 kg · m^2. The rope does not slip on the pulley rim. Use energy methods to calculate the speed of the 4.00 kg block just before it strikes the floor.

Figure **P9.78**



9.79 •• Two metal disks, one with radius $R_1 = 2.50$ cm and mass $M_1 = 0.80$ kg and the other with radius $R_2 = 5.00$ cm and mass $M_2 = 1.60$ kg, are welded together and mounted on a frictionless axis through their common center (**Fig. P9.79**). (a) What is the total moment of inertia of the two disks? (b) A light string is wrapped around the edge of the smaller disk, and a 1.50 kg block is suspended from the free end of the string. If the block is released from rest at a distance of 2.00 m above the floor, what is its speed just before it strikes the floor? (c) Repeat part (b), this time with the string wrapped around the edge of the larger disk. In which case is the final speed of the block greater? Explain.

Figure **P9.79**

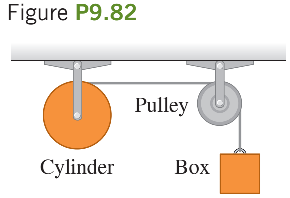


9.80 •• A thin, light wire is wrapped around the rim of a wheel as shown in Fig. E9.49. The wheel rotates about a stationary horizontal axle that passes through the center of the wheel. The wheel has radius 0.180 m and moment of inertia for rotation about the axle of $I = 0.480$ kg · m^2. A small block with mass 0.340 kg is suspended from the free end of the wire. When the system is released from rest, the block descends with constant acceleration. The bearings in the wheel at the axle are rusty, so friction there does −9.00 J of work as the block descends 3.00 m. What is the magnitude of the angular velocity of the wheel after the block has descended 3.00 m?

9.81 ••• In the system shown in Fig. 9.17, a 12.0 kg mass is released from rest and falls, causing the uniform 10.0 kg cylinder of diameter 30.0 cm to turn about a frictionless axle through its center. How far will the mass have to descend to give the cylinder 480 J of kinetic energy?

9.82 • In **Fig. P9.82**, the cylinder and pulley turn without friction about stationary horizontal axles that pass through their centers. A light rope is wrapped around the cylinder, passes over the pulley, and has a 3.00 kg box suspended from its free end. There is no slipping between the rope and the pulley surface. The uniform cylinder has mass 5.00 kg and radius 40.0 cm. The pulley is a uniform disk with mass 2.00 kg and radius 20.0 cm. The box is released from rest and descends as the rope unwraps from the cylinder. Find the speed of the box when it has fallen 2.50 m.

Figure **P9.82**



9.83 •• **BIO** **Human Rotational Energy.** A dancer is spinning at 72 rpm about an axis through her center with her arms outstretched (**Fig. P9.83**). From biomedical measurements, the typical distribution of mass in a human body is as follows:

Head: 7.0%
Arms: 13% (for both)
Trunk and legs: 80.0%

Suppose you are this dancer. Using this information plus length measurements on your own body, calculate (a) your moment of inertia about your spin axis and (b) your rotational kinetic energy. Use Table 9.2 to model reasonable approximations for the pertinent parts of your body.

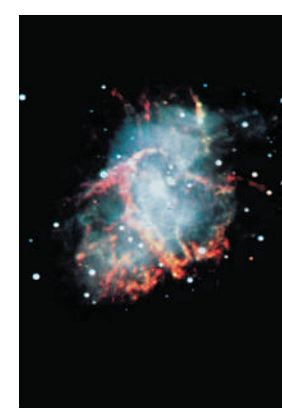
Figure **P9.83**

9.84 ••• A thin, uniform rod is bent into a square of side length a. If the total mass is M, find the moment of inertia about an axis through the center and perpendicular to the plane of the square. (*Hint:* Use the parallel-axis theorem.)

9.85 •• **CALC** A sphere with radius $R = 0.200$ m has density ρ that decreases with distance r from the center of the sphere according to $\rho = 3.00 \times 10^3\ \text{kg/m}^3 - (9.00 \times 10^3\ \text{kg/m}^4)r$. (a) Calculate the total mass of the sphere. (b) Calculate the moment of inertia of the sphere for an axis along a diameter.

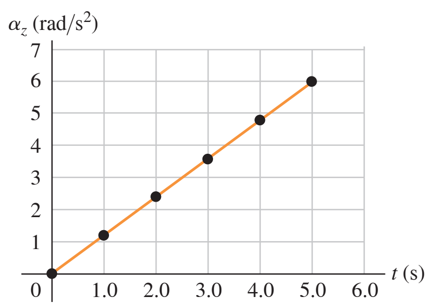
9.86 •• **CALC** **Neutron Stars and Supernova Remnants.** The Crab Nebula is a cloud of glowing gas about 10 light-years across, located about 6500 light-years from the earth (**Fig. P9.86**). It is the remnant of a star that underwent a *supernova explosion,* seen on earth in 1054 A.D. Energy is released by the Crab Nebula at a rate of about 5×10^{31} W, about 10^5 times the rate at which the sun radiates energy. The Crab Nebula obtains its energy from the rotational kinetic energy of a rapidly spinning *neutron star* at its center. This object rotates once every 0.0331 s, and this period is increasing by 4.22×10^{-13} s for each second of time that elapses. (a) If the rate at which energy is lost by the neutron star is equal to the rate at which energy is released by the nebula, find the moment of inertia of the neutron star. (b) Theories of supernovae predict that the neutron star in the Crab Nebula has a mass about 1.4 times that of the sun. Modeling the neutron star as a solid uniform sphere, calculate its radius in kilometers. (c) What is the linear speed of a point on the equator of the neutron star? Compare to the speed of light. (d) Assume that the neutron star is uniform and calculate its density. Compare to the density of ordinary rock (3000 kg/m^3) and to the density of an atomic nucleus (about 10^{17} kg/m^3). Justify the statement that a neutron star is essentially a large atomic nucleus.

Figure **P9.86**



9.87 •• DATA A technician is testing a computer-controlled, variable-speed motor. She attaches a thin disk to the motor shaft, with the shaft at the center of the disk. The disk starts from rest, and sensors attached to the motor shaft measure the angular acceleration α_z of the shaft as a function of time. The results from one test run are shown in **Fig. P9.87**. (a) Through how many revolutions has the disk turned in the first 5.0 s? Can you use Eq. (9.11)? Explain. What is the angular velocity, in rad/s, of the disk (b) at $t = 5.0$ s; (c) when it has turned through 2.00 rev?

Figure **P9.87**



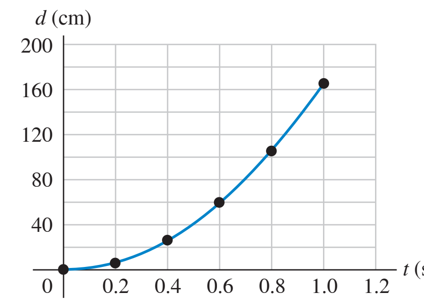
9.88 •• DATA You are analyzing the motion of a large flywheel that has radius 0.800 m. In one test run, the wheel starts from rest and turns in a horizontal plane with constant angular acceleration. An accelerometer on the rim of the flywheel measures the magnitude of the resultant acceleration a of a point on the rim of the flywheel as a function of the angle $\theta - \theta_0$ through which the wheel has turned. You collect these results:

$\boldsymbol{\theta - \theta_0}$ **(rad)**	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00
$\boldsymbol{a}$ **(m/s^2)**	0.678	1.07	1.52	1.98	2.45	2.92	3.39	3.87

Construct a graph of a^2 (in m^2/s^4) versus $(\theta - \theta_0)^2$ (in rad^2). (a) What are the slope and y-intercept of the straight line that gives the best fit to the data? (b) Use the slope from part (a) to find the angular acceleration of the flywheel. (c) What is the linear speed of a point on the rim of the flywheel when the wheel has turned through an angle of 135°? (d) When the flywheel has turned through an angle of 90.0°, what is the angle between the linear velocity of a point on its rim and the resultant acceleration of that point?

9.89 •• DATA You are rebuilding a 1965 Chevrolet. To decide whether to replace the flywheel with a newer, lighter-weight one, you want to determine the moment of inertia of the original, 35.6-cm-diameter flywheel. It is not a uniform disk, so you can't use $I = \frac{1}{2}MR^2$ to calculate the moment of inertia. You remove the flywheel from the car and use low-friction bearings to mount it on a horizontal, stationary rod that passes through the center of the flywheel, which can then rotate freely (about 2 m above the ground). After gluing one end of a long piece of flexible fishing line to the rim of the flywheel, you wrap the line a number of turns around the rim and suspend a 5.60 kg metal block from the free end of the line. When you release the block from rest, it descends as the flywheel rotates. With high-speed photography you measure the distance d the block has moved downward as a function of the time since it was released. The equation for the graph shown in **Fig. P9.89** that gives a good fit to the data points is $d = (165\ \text{cm/s}^2)t^2$. (a) Based on the graph, does the block fall with constant acceleration? Explain. (b) Use the graph to calculate the speed of the block when it has descended 1.50 m. (c) Apply conservation of mechanical energy to the system of flywheel and block to calculate the moment of inertia of the flywheel. (d) You are relieved that the fishing line doesn't break. Apply Newton's second law to the block to find the tension in the line as the block descended.

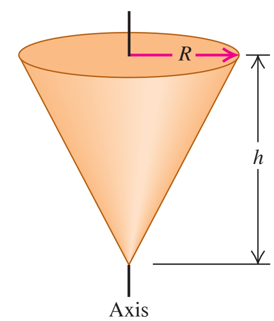
Figure **P9.89**



CHALLENGE PROBLEMS

9.90 ••• CALC Calculate the moment of inertia of a uniform solid cone about an axis through its center (**Fig. P9.90**). The cone has mass M and altitude h. The radius of its circular base is R.

Figure **P9.90**



9.91 ••• CALC On a compact disc (CD), music is coded in a pattern of tiny pits arranged in a track that spirals outward toward the rim of the disc. As the disc spins inside a CD player, the track is scanned at a constant *linear* speed of $v = 1.25$ m/s. Because the radius of the track varies as it spirals outward, the *angular* speed of the disc must change as the CD is played. (See Exercise 9.20.) Let's see what angular acceleration is required to keep v constant. The equation of a spiral is $r(\theta) = r_0 + \beta\theta$, where r_0 is the radius of the spiral at $\theta = 0$ and β is a constant. On a CD, r_0 is the inner radius of the spiral track. If we take the rotation direction of the CD to be positive, β must be positive so that r increases as the disc turns and θ increases. (a) When the disc rotates through a small angle $d\theta$, the distance scanned along the track is $ds = r\,d\theta$. Using the above expression for $r(\theta)$, integrate ds to find the total distance s scanned along the track as a function of the total angle θ through which the disc has rotated. (b) Since the track is scanned at a constant linear speed v, the distance s found in part (a) is equal to vt. Use this to find θ as a function of time. There will be two solutions for θ; choose the positive one, and explain why this is the solution to choose. (c) Use your expression for $\theta(t)$ to find the angular velocity ω_z and the angular acceleration α_z as functions of time. Is α_z constant? (d) On a CD, the inner radius of the track is 25.0 mm, the track radius increases by 1.55 μm per revolution, and the playing time is 74.0 min. Find r_0, β, and the total number of revolutions made during the playing time. (e) Using your results from parts (c) and (d), make graphs of ω_z (in rad/s) versus t and α_z (in rad/s^2) versus t between $t = 0$ and $t = 74.0$ min.

MCAT-STYLE PASSAGE PROBLEMS

BIO The Spinning Eel. American eels (*Anguilla rostrata*) are freshwater fish with long, slender bodies that we can treat as uniform cylinders 1.0 m long and 10 cm in diameter. An eel compensates for its small jaw and teeth by holding onto prey with its mouth and then rapidly spinning its body around its long axis to tear off a piece of flesh. Eels have been recorded to spin at up to 14 revolutions per second when feeding in this way. Although this feeding method is costly in terms of energy, it allows the eel to feed on larger prey than it otherwise could.

9.92 A field researcher uses the slow-motion feature on her phone's camera to shoot a video of an eel spinning at its maximum rate. The camera records at 120 frames per second. Through what angle does the eel rotate from one frame to the next? (a) 1°; (b) 10°; (c) 22°; (d) 42°.

9.93 The eel is observed to spin at 14 spins per second clockwise, and 10 seconds later it is observed to spin at 8 spins per second counterclockwise. What is the magnitude of the eel's average angular acceleration during this time? (a) $6/10\ \text{rad/s}^2$; (b) $6\pi/10\ \text{rad/s}^2$; (c) $12\pi/10\ \text{rad/s}^2$; (d) $44\pi/10\ \text{rad/s}^2$.

9.94 The eel has a certain amount of rotational kinetic energy when spinning at 14 spins per second. If it swam in a straight line instead, about how fast would the eel have to swim to have the same amount of kinetic energy as when it is spinning? (a) 0.5 m/s; (b) 0.7 m/s; (c) 3 m/s; (d) 5 m/s.

9.95 A new species of eel is found to have the same mass but one-quarter the length and twice the diameter of the American eel. How does its moment of inertia for spinning around its long axis compare to that of the American eel? The new species has (a) half the moment of inertia as the American eel; (b) the same moment of inertia as the American eel; (c) twice the moment of inertia as the American eel; (d) four times the moment of inertia as the American eel.

ANSWERS

Chapter Opening Question ?

(ii) The rotational kinetic energy of a rigid body rotating around an axis is $K = \frac{1}{2}I\omega^2$, where I is the body's moment of inertia for that axis and ω is the rotational speed. Table 9.2 shows that the moment of inertia for a slender rod of mass M and length L with an axis through one end (like a wind turbine blade) is $I = \frac{1}{3}ML^2$. If we double L while M and ω stay the same, both the moment of inertia I and the kinetic energy K increase by a factor of $2^2 = 4$.

Key Example VARIATION Problems

VP9.3.1 (a) 2.06 rad (b) 118° (c) 0.328 rev

VP9.3.2 (a) $2.75\ \text{rad/s}^2$ (b) 27.0 rad

VP9.3.3 0.693 s

VP9.3.4 $t = (-\omega_{0z} + \sqrt{\omega_{0z}{}^2 + 2\alpha_z\theta})/\alpha_z$

VP9.5.1 (a) $1.22\ \text{m/s}^2$ (b) $0.389\ \text{m/s}^2$ (c) $1.28\ \text{m/s}^2$

VP9.5.2 (a) 9.22 m/s (b) $54.0\ \text{m/s}^2$, toward the axis of rotation

VP9.5.3 $\left(\dfrac{a_{\text{max}}{}^2}{r^2} - \alpha^2\right)^{1/4}$

VP9.5.4 (a) $\sqrt{3}\omega^2$ (b) 30°

VP9.8.1 (a) 20.0 rad/s (b) 23.3 rad/s

VP9.8.2 (a) 1.80 J (b) 5.4 J

VP9.8.3 (a) 2.86 m (b) 20.0 rad/s

VP9.8.4 (a) 2.40 m/s (b) –9.72 J

Bridging Problem

(a) $I = \left[\dfrac{M}{L}\left(\dfrac{x^3}{3}\right)\right]_{-h}^{L-h} = \frac{1}{3}M(L^2 - 3Lh + 3h^2)$

(b) $W = \frac{1}{6}M(L^2 - 3Lh + 3h^2)\alpha^2t^2$

(c) $a = (L - h)\alpha\sqrt{1 + \alpha^2t^4}$

? These jugglers toss the pins so that they rotate in midair. Each pin is of uniform composition, so its weight is concentrated toward its thick end. If we ignore air resistance but not the effects of gravity, will the angular speed of a pin in flight (i) increase continuously; (ii) decrease continuously; (iii) alternately increase and decrease; or (iv) remain the same?

10 Dynamics of Rotational Motion

LEARNING OUTCOMES

In this chapter, you'll learn...

10.1 What is meant by the torque produced by a force.
10.2 How the net torque on a rigid body affects the body's rotational motion.
10.3 How to analyze the motion of a rigid body that both rotates and moves as a whole through space.
10.4 How to solve problems that involve work and power for rotating rigid bodies.
10.5 What is meant by the angular momentum of a particle or rigid body.
10.6 How the angular momentum of an object can remain constant even if the object changes shape.
10.7 Why a spinning gyroscope undergoes precession.

You'll need to review...

1.10 Vector product of two vectors.
5.2 Newton's second law.
6.1–6.4 Work, the work–energy theorem, and power.
8.2, 8.3, 8.5 External versus internal forces, inelastic collisions, and center-of-mass motion.
9.1–9.5 Rotational motion and the parallel-axis theorem.

We learned in Chapters 4 and 5 that a net force applied to an object gives that object an acceleration. But what does it take to give an object an *angular* acceleration? That is, what does it take to start a stationary object rotating or to bring a spinning object to a halt? A force is required, but it must be applied in a way that gives a twisting or turning action.

In this chapter we'll define a new physical quantity, *torque,* that describes the twisting or turning effort of a force. We'll find that the net torque acting on a rigid body determines its angular acceleration, in the same way that the net force on an object determines its linear acceleration. We'll also look at work and power in rotational motion so as to understand, for example, how energy is transferred by an electric motor. Next we'll develop a new conservation principle, *conservation of angular momentum,* that is tremendously useful for understanding the rotational motion of both rigid and nonrigid bodies. We'll finish this chapter by studying *gyroscopes,* rotating devices that don't fall over when you might think they should—but that actually behave in accordance with the dynamics of rotational motion.

10.1 TORQUE

We know that forces acting on an object can affect its **translational motion**—that is, the motion of the object as a whole through space. Now we want to learn which aspects of a force determine how effective it is in causing or changing *rotational* motion. The magnitude and direction of the force are important, but so is the point on the object where the force is applied. In **Fig. 10.1** a wrench is being used to loosen a tight bolt. Force $\vec{F}_b$, applied near the end of the handle, is more effective than an equal force $\vec{F}_a$ applied near the bolt. Force $\vec{F}_c$ does no good; it's applied at the same point and has the same magnitude as $\vec{F}_b$, but it's directed along the length of the handle. The quantitative measure of the tendency of a force to cause or change an object's rotational motion is called *torque;* we say that $\vec{F}_a$ applies a torque about point O to the wrench in Fig. 10.1, $\vec{F}_b$ applies a greater torque about O, and $\vec{F}_c$ applies zero torque about O.

Figure 10.2 shows three examples of how to calculate torque. The object can rotate about an axis that is perpendicular to the plane of the figure and passes through point O. Three forces act on the object in the plane of the figure. The tendency of the first of these forces, $\vec{F}_1$, to cause a rotation about O depends on its magnitude F_1. It also depends on the *perpendicular* distance l_1 between point O and the **line of action** of the force (that is, the line along which the force vector lies). We call the distance l_1 the **lever arm** (or **moment arm**) of force $\vec{F}_1$ about O. The twisting effort is directly proportional to both F_1 and l_1, so we define the **torque** (or *moment*) of the force $\vec{F}_1$ with respect to O as the product $F_1 l_1$. We use the Greek letter τ (tau) for torque. If a force of magnitude F has a line of action that is a perpendicular distance l from O, the torque is

$$\tau = Fl \qquad (10.1)$$

Physicists usually use the term "torque," while engineers usually use "moment" (unless they are talking about a rotating shaft).

The lever arm of $\vec{F}_1$ in Fig. 10.2 is the perpendicular distance l_1, and the lever arm of $\vec{F}_2$ is the perpendicular distance l_2. The line of action of $\vec{F}_3$ passes through point O, so the lever arm for $\vec{F}_3$ is zero and its torque with respect to O is zero. In the same way, force $\vec{F}_c$ in Fig. 10.1 has zero torque with respect to point O; $\vec{F}_b$ has a greater torque than $\vec{F}_a$ because its lever arm is greater.

CAUTION **Torque is always measured about a point** Torque is *always* defined with reference to a specific point. If we shift the position of this point, the torque of each force may change. For example, the torque of force $\vec{F}_3$ in Fig. 10.2 is zero with respect to point O but *not* with respect to point A. It's not enough to refer to "the torque of $\vec{F}$"; you must say "the torque of $\vec{F}$ with respect to point X" or "the torque of $\vec{F}$ about point X."

Force $\vec{F}_1$ in Fig. 10.2 tends to cause *counterclockwise* rotation about O, while $\vec{F}_2$ tends to cause *clockwise* rotation. To distinguish between these two possibilities, we need to choose a positive sense of rotation. With the choice that *counterclockwise torques are positive and clockwise torques are negative,* the torques of $\vec{F}_1$ and $\vec{F}_2$ about O are

$$\tau_1 = +F_1 l_1 \qquad \tau_2 = -F_2 l_2$$

Figure 10.2 shows this choice for the sign of torque. We'll often use the symbol ⊕ to indicate our choice of the positive sense of rotation.

The SI unit of torque is the newton-meter. In our discussion of work and energy we called this combination the joule. But torque is *not* work or energy, and torque should be expressed in newton-meters, *not* joules.

Figure 10.3 shows a force $\vec{F}$ applied at point P, located at position $\vec{r}$ with respect to point O. There are three ways to calculate the torque of $\vec{F}$:

1. Find the lever arm l and use $\tau = Fl$.
2. Determine the angle ϕ between the vectors $\vec{r}$ and $\vec{F}$; the lever arm is $r\sin\phi$, so $\tau = rF\sin\phi$.
3. Represent $\vec{F}$ in terms of a radial component F_{rad} along the direction of $\vec{r}$ and a tangential component $F_{\tan}$ at right angles, perpendicular to $\vec{r}$. (We call this component *tangential* because if the object rotates, the point where the force acts moves in a circle, and this component is tangent to that circle.) Then $F_{\tan} = F\sin\phi$ and $\tau = r(F\sin\phi) = F_{\tan}r$. The component F_{rad} produces *no* torque with respect to O because its lever arm with respect to that point is zero (compare to forces $\vec{F}_c$ in Fig. 10.1 and $\vec{F}_3$ in Fig. 10.2).

Summarizing these three expressions for torque, we have

Magnitude of torque due to force $\vec{F}$ relative to point O
Lever arm of $\vec{F}$
Magnitude of $\vec{r}$ (vector from O to where $\vec{F}$ acts)

$$\tau = Fl = rF\sin\phi = F_{\tan}r \qquad (10.2)$$

Magnitude of $\vec{F}$
Angle between $\vec{r}$ and $\vec{F}$
Tangential component of $\vec{F}$

Figure **10.1** Which of these three equal-magnitude forces is most likely to loosen the tight bolt?

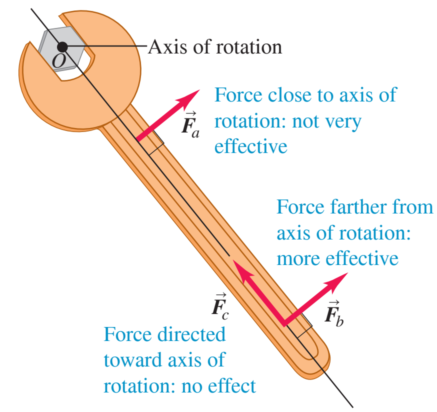


Figure **10.2** The torque of a force about a point is the product of the force magnitude and the lever arm of the force.

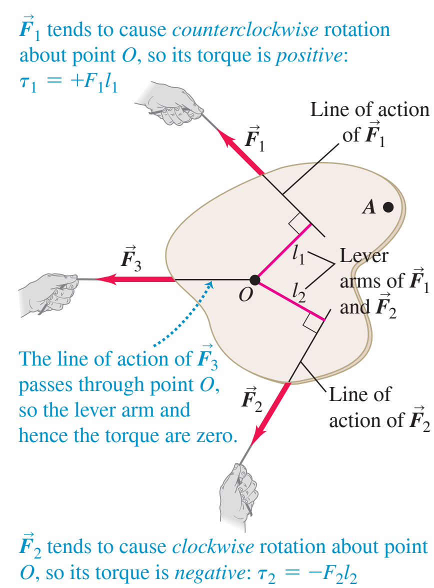


Figure **10.3** Three ways to calculate the torque of force $\vec{F}$ about point O. In this figure, $\vec{r}$ and $\vec{F}$ are in the plane of the page and the torque vector $\vec{\tau}$ points out of the page toward you.

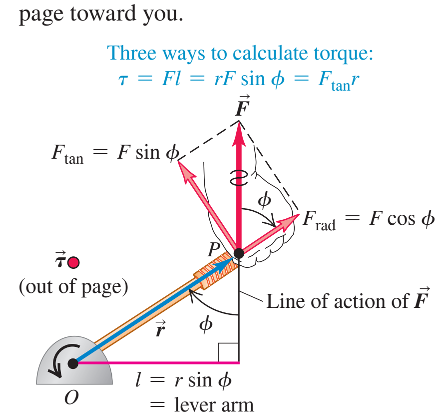
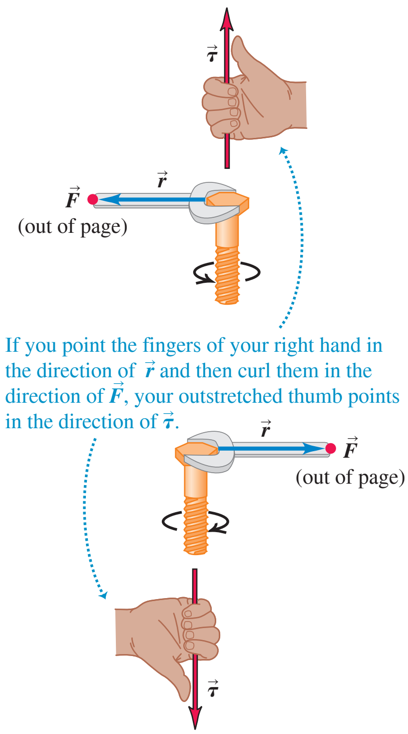


Figure **10.4** The torque vector $\vec{\tau} = \vec{r} \times \vec{F}$ is directed along the axis of the bolt, perpendicular to both $\vec{r}$ and $\vec{F}$. The fingers of the right hand curl in the direction of the rotation that the torque tends to cause.



Torque as a Vector

We saw in Section 9.1 that angular velocity and angular acceleration can be represented as vectors; the same is true for torque. To see how to do this, note that the quantity $rF\sin\phi$ in Eq. (10.2) is the magnitude of the *vector product* $\vec{r} \times \vec{F}$ that we defined in Section 1.10. (Go back and review that definition.) We generalize the definition of torque as follows: When a force $\vec{F}$ acts at a point having a position vector $\vec{r}$ with respect to an origin O, as in Fig. 10.3, the torque $\vec{\tau}$ of the force with respect to O is the *vector* quantity

Torque vector due to force $\vec{F}$ relative to point O · Vector from O to where $\vec{F}$ acts · Force $\vec{F}$

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{10.3}$$

The torque as defined in Eq. (10.2) is the magnitude of the torque vector $\vec{r} \times \vec{F}$. The direction of $\vec{\tau}$ is perpendicular to both $\vec{r}$ and $\vec{F}$. In particular, if both $\vec{r}$ and $\vec{F}$ lie in a plane perpendicular to the axis of rotation, as in Fig. 10.3, then the torque vector $\vec{\tau} = \vec{r} \times \vec{F}$ is directed along the axis of rotation, with a sense given by the right-hand rule (see Fig. 1.30 and **Fig. 10.4**).

Because $\vec{\tau} = \vec{r} \times \vec{F}$ is perpendicular to the plane of the vectors $\vec{r}$ and $\vec{F}$, it's common to have diagrams like Fig. 10.4, in which one of the vectors is perpendicular to the page. We use a dot (•) to represent a vector that points out of the page and a cross (×) to represent a vector that points into the page (see Figs. 10.3 and 10.4).

In the following sections we'll usually be concerned with rotation of an object about an axis oriented in a specified constant direction. In that case, only the component of torque along that axis will matter. We often call that component the torque with respect to the specified *axis*.

EXAMPLE 10.1 Applying a torque

To loosen a pipe fitting, a plumber slips a piece of scrap pipe (a "cheater") over his wrench handle. He stands on the end of the cheater, applying his 900 N weight at a point 0.80 m from the center of the fitting (**Fig. 10.5a**). The wrench handle and cheater make an angle of 19° with the horizontal. Find the magnitude and direction of the torque he applies about the center of the fitting.

IDENTIFY and SET UP Figure 10.5b shows the vectors $\vec{r}$ and $\vec{F}$ and the angle between them ($\phi = 109°$). Equation (10.1) or (10.2) will tell us the magnitude of the torque. The right-hand rule with Eq. (10.3), $\vec{\tau} = \vec{r} \times \vec{F}$, will tell us the direction of the torque.

EXECUTE To use Eq. (10.1), we first calculate the lever arm l. As Fig. 10.5b shows,

$$l = r\sin\phi = (0.80\ \text{m})\sin 109° = 0.76\ \text{m}$$

Then Eq. (10.1) tells us that the magnitude of the torque is

$$\tau = Fl = (900\ \text{N})(0.76\ \text{m}) = 680\ \text{N}\cdot\text{m}$$

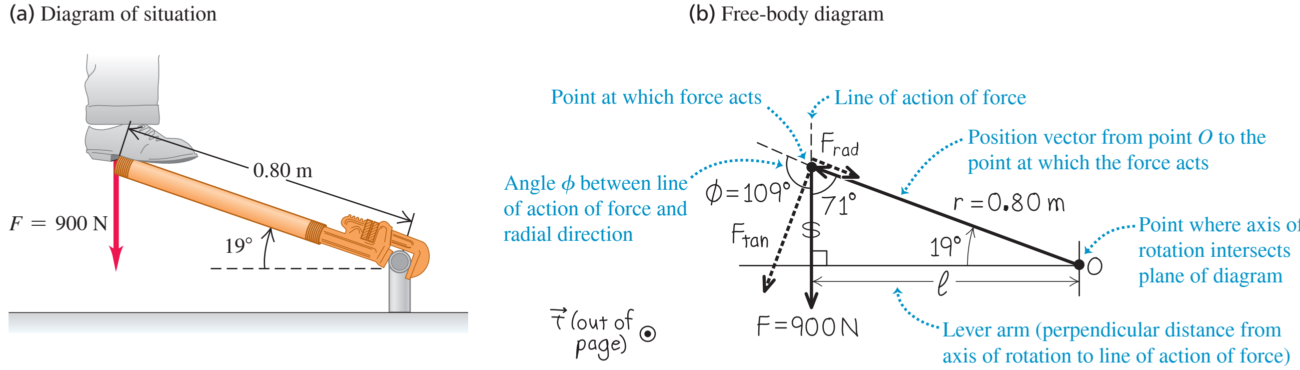
We get the same result from Eq. (10.2):

$$\tau = rF\sin\phi = (0.80\ \text{m})(900\ \text{N})(\sin 109°) = 680\ \text{N}\cdot\text{m}$$

Figure **10.5** (a) Loosening a pipe fitting by standing on a "cheater." (b) Our vector diagram to find the torque about O.

(a) Diagram of situation

(b) Free-body diagram



Alternatively, we can find F_{tan}, the tangential component of $\vec{F}$ that acts perpendicular to $\vec{r}$. Figure 10.5b shows that this component is at an angle of $109° - 90° = 19°$ from $\vec{F}$, so $F_{tan} = F(\cos 19°) = (900\text{ N})(\cos 19°) = 851\text{ N}$. Then, from Eq. (10.2),

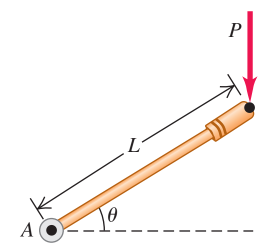
$$\tau = F_{tan} r = (851\text{ N})(0.80\text{ m}) = 680\text{ N}\cdot\text{m}$$

Curl the fingers of your right hand from the direction of $\vec{r}$ (in the plane of Fig. 10.5b, to the left and up) into the direction of $\vec{F}$ (straight down). Then your right thumb points out of the plane of the figure: This is the direction of $\vec{\tau}$.

EVALUATE To check the direction of $\vec{\tau}$, note that the force in Fig. 10.5 tends to produce a counterclockwise rotation about O. If you curl the fingers of your right hand in a counterclockwise direction, the thumb points out of the plane of Fig. 10.5, which is indeed the direction of the torque.

KEYCONCEPT You can determine the magnitude of the torque due to a force $\vec{F}$ in any of three ways: (i) from the magnitude of $\vec{F}$ and the lever arm; (ii) from the magnitude of $\vec{F}$, the magnitude of the vector $\vec{r}$ from the origin to where $\vec{F}$ acts, and the angle between $\vec{r}$ and $\vec{F}$; or (iii) from the magnitude of $\vec{r}$ and the tangential component of $\vec{F}$. Find the direction of the torque using the right-hand rule.

TEST YOUR UNDERSTANDING OF SECTION 10.1 The accompanying figure shows a force of magnitude P being applied to one end of a lever of length L. What is the magnitude of the torque of this force about point A? (i) $PL\sin\theta$; (ii) $PL\cos\theta$; (iii) $PL\tan\theta$; (iv) $PL/\sin\theta$; (v) PL.



ANSWER

(ii) The force of magnitude P acts along a vertical line, so the lever arm is the horizontal distance from A to the line of action. This is the horizontal component of distance L, which is $L\cos\theta$. Hence the magnitude of the torque is the product of the force magnitude P and the lever arm $L\cos\theta$, or $\tau = PL\cos\theta$.

10.2 TORQUE AND ANGULAR ACCELERATION FOR A RIGID BODY

We're now ready to develop the fundamental relationship for the rotational dynamics of a rigid body (an object with a definite and unchanging shape and size). We'll show that the angular acceleration of a rotating rigid body is directly proportional to the sum of the torque components along the axis of rotation. The proportionality factor is the moment of inertia.

To develop this relationship, let's begin as we did in Section 9.4 by envisioning the rigid body as being made up of a large number of particles. We choose the axis of rotation to be the z-axis; the first particle has mass m_1 and distance r_1 from this axis (**Fig. 10.6**). The *net force* $\vec{F}_1$ acting on this particle has a component $F_{1,\text{rad}}$ along the radial direction, a component $F_{1,\text{tan}}$ that is tangent to the circle of radius r_1 in which the particle moves as the body rotates, and a component F_{1z} along the axis of rotation. Newton's second law for the tangential component is

$$F_{1,\text{tan}} = m_1 a_{1,\text{tan}} \tag{10.4}$$

We can express the tangential acceleration of the first particle in terms of the angular acceleration α_z of the body by using Eq. (9.14): $a_{1,\text{tan}} = r_1\alpha_z$. Using this relationship and multiplying both sides of Eq. (10.4) by r_1, we obtain

$$F_{1,\text{tan}} r_1 = m_1 r_1^2 \alpha_z \tag{10.5}$$

From Eq. (10.2), $F_{1,\text{tan}} r_1$ is the *torque* of the net force with respect to the rotation axis, equal to the component τ_{1z} of the torque vector along the rotation axis. The subscript z is a reminder that the torque affects rotation around the z-axis, in the same way that the subscript on F_{1z} is a reminder that this force affects the motion of particle 1 along the z-axis.

Neither of the components $F_{1,\text{rad}}$ or F_{1z} contributes to the torque about the z-axis, since neither tends to change the particle's rotation about that axis. So $\tau_{1z} = F_{1,\text{tan}} r_1$ is the total torque acting on the particle with respect to the rotation axis. Also, $m_1 r_1^2$ is I_1, the moment of inertia of the particle about the rotation axis. Hence we can rewrite Eq. (10.5) as

$$\tau_{1z} = I_1\alpha_z = m_1 r_1^2 \alpha_z$$

Figure **10.6** As a rigid body rotates around the z-axis, a net force $\vec{F}_1$ acts on one particle of the body. Only the force component $F_{1,\text{tan}}$ can affect the rotation, because only $F_{1,\text{tan}}$ exerts a torque about O with a z-component (along the rotation axis).

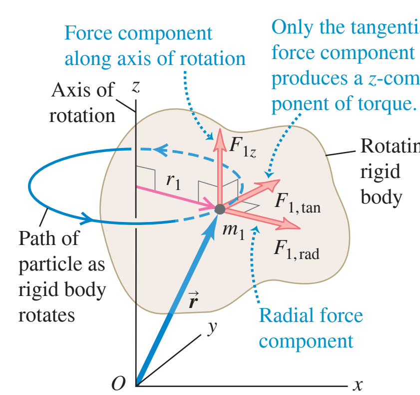
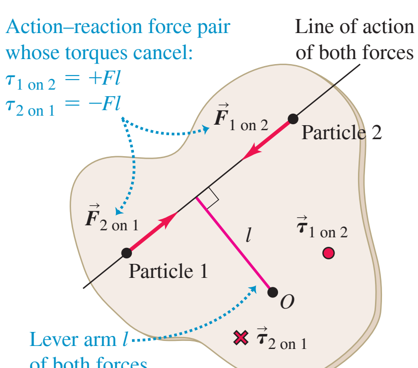


Figure 10.7 Loosening or tightening a screw requires giving it an angular acceleration and hence applying a torque. To make this easier, use a screwdriver with a large-radius handle. This provides a large lever arm for the force your hand applies.

Figure 10.8 Why only *external* torques affect a rigid body's rotation: Any two particles in the body exert equal and opposite forces on each other. If the forces act along the line joining the particles, the lever arms of the forces with respect to an axis through O are the same and the torques due to the two forces are equal and opposite.



We write such an equation for every particle in the body, then add all these equations:

$$\tau_{1z} + \tau_{2z} + \cdots = I_1\alpha_z + I_2\alpha_z + \cdots = m_1 r_1^{\,2}\alpha_z + m_2 r_2^{\,2}\alpha_z + \cdots$$

or

$$\sum \tau_{iz} = \left(\sum m_i r_i^{\,2}\right)\alpha_z \tag{10.6}$$

The left side of Eq. (10.6) is the sum of all the torques about the rotation axis that act on all the particles. The right side is $I = \sum m_i r_i^{\,2}$, the total moment of inertia about the rotation axis, multiplied by the angular acceleration α_z. Note that α_z is the same for every particle because this is a *rigid* body. Thus Eq. (10.6) says that for the rigid body as a whole,

Rotational analog of Newton's second law for a rigid body:

$$\underbrace{\sum \tau_z}_{\text{Net torque on a rigid body about } z\text{-axis}} = \overbrace{I}^{\text{Moment of inertia of rigid body about } z\text{-axis}}\underbrace{\alpha_z}_{\text{Angular acceleration of rigid body about } z\text{-axis}} \tag{10.7}$$

Just as Newton's second law says that a net *force* on a particle causes an *acceleration* in the direction of the net force, Eq. (10.7) says that a net *torque* on a rigid body about an axis causes an *angular acceleration* about that axis (**Fig. 10.7**).

Our derivation assumed that the angular acceleration α_z is the same for all particles in the body. So Eq. (10.7) is valid *only* for *rigid* bodies. Hence this equation doesn't apply to a rotating tank of water or a swirling tornado of air, different parts of which have different angular accelerations. Note that since our derivation used Eq. (9.14), $a_{\tan} = r\alpha_z$, α_z must be measured in rad/s^2.

The torque on each particle is due to the net force on that particle, which is the vector sum of external and internal forces (see Section 8.2). According to Newton's third law, the *internal* forces that any pair of particles in the rigid body exert on each other are equal in magnitude and opposite in direction (**Fig. 10.8**). If these forces act along the line joining the two particles, their lever arms with respect to any axis are also equal. So the torques for each such pair are equal and opposite, and add to zero. Hence *all* the internal torques add to zero, so the sum $\sum \tau_z$ in Eq. (10.7) includes only the torques of the *external* forces.

Often, an important external force acting on a rigid body is its *weight.* This force is not concentrated at a single point; it acts on every particle in the entire body. Nevertheless, if $\vec{g}$ has the same value at all points, we always get the correct torque (about any specified axis) if we assume that all the weight is concentrated at the *center of mass* of the body. We'll prove this statement in Chapter 11, but meanwhile we'll use it for some of the problems in this chapter.

PROBLEM-SOLVING STRATEGY 10.1 Rotational Dynamics for Rigid Bodies

Our strategy for solving problems in rotational dynamics is very similar to Problem-Solving Strategy 5.2 for solving problems involving Newton's second law.

IDENTIFY *the relevant concepts:* Equation (10.7), $\sum \tau_z = I\alpha_z$, is useful whenever torques act on a rigid body. Sometimes you can use an energy approach instead, as we did in Section 9.4. However, if the target variable is a force, a torque, an acceleration, an angular acceleration, or an elapsed time, using $\sum \tau_z = I\alpha_z$ is almost always best.

SET UP *the problem* using the following steps:

1. Sketch the situation and identify the body or bodies to be analyzed. Indicate the rotation axis.
2. For each body, draw a free-body diagram that shows the body's *shape,* including all dimensions and angles. Label pertinent quantities with algebraic symbols.
3. Choose coordinate axes for each body and indicate a positive sense of rotation (clockwise or counterclockwise) for each rotating body. If you know the sense of α_z, pick that as the positive sense of rotation.

EXECUTE *the solution:*

1. For each body, decide whether it undergoes translational motion, rotational motion, or both. Then apply $\sum \vec{F} = m\vec{a}$ (as in Section 5.2), $\sum \tau_z = I\alpha_z$, or both to the body.
2. Express in algebraic form any *geometrical* relationships between the motions of two or more bodies. An example is a string that unwinds, without slipping, from a pulley or a wheel that rolls without slipping (discussed in Section 10.3). These relationships usually appear as relationships between linear and/or angular accelerations.
3. Ensure that you have as many independent equations as there are unknowns. Solve the equations to find the target variables.

EVALUATE *your answer:* Check that the algebraic signs of your results make sense. As an example, if you are unrolling thread from a spool, your answers should not tell you that the spool is turning in the direction that rolls the thread back onto the spool! Check that any algebraic results are correct for special cases or for extreme values of quantities.

EXAMPLE 10.2 An unwinding cable I

WITH VARIATION PROBLEMS

Figure 10.9a shows the situation that we analyzed in Example 9.7 using energy methods. What is the cable's acceleration?

IDENTIFY and SET UP We can't use the energy method of Section 9.4, which doesn't involve acceleration. Instead we'll apply rotational dynamics to find the angular acceleration of the cylinder (Fig. 10.9b). We'll then find a relationship between the motion of the cable and the motion of the cylinder rim, and use this to find the acceleration of the cable. The cylinder rotates counterclockwise when the cable is pulled, so we take counterclockwise rotation to be positive. The net force on the cylinder must be zero because its center of mass remains at rest. The force F exerted by the cable produces a torque about the rotation axis. The weight (magnitude Mg) and the normal force (magnitude n) exerted by the cylinder's bearings produce *no* torque about the rotation axis because both act along lines through that axis.

EXECUTE The lever arm of F is equal to the radius $R = 0.060$ m of the cylinder, so the torque is $\tau_z = FR$. (This torque is positive, as it tends to cause a counterclockwise rotation.) From Table 9.2, case (f), the moment of inertia of the cylinder about the rotation axis is $I = \frac{1}{2}MR^2$. Then Eq. (10.7) tells us that

$$\alpha_z = \frac{\tau_z}{I} = \frac{FR}{MR^2/2} = \frac{2F}{MR} = \frac{2(9.0\ \text{N})}{(50\ \text{kg})(0.060\ \text{m})} = 6.0\ \text{rad/s}^2$$

(We can add "rad" to our result because radians are dimensionless.)

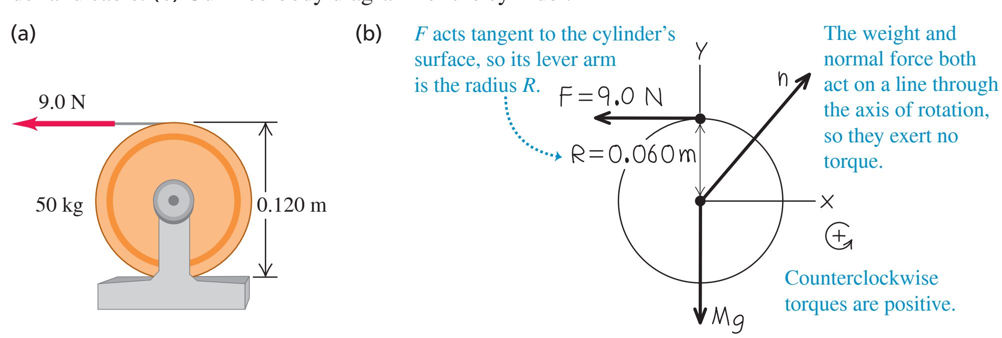
To get the linear acceleration of the cable, recall from Section 9.3 that the acceleration of a cable unwinding from a cylinder is the same as the tangential acceleration of a point on the surface of the cylinder where the cable is tangent to it. This tangential acceleration is given by Eq. (9.14):

$$a_{\tan} = R\alpha_z = (0.060\ \text{m})(6.0\ \text{rad/s}^2) = 0.36\ \text{m/s}^2$$

EVALUATE Can you use this result, together with an equation from Chapter 2, to determine the speed of the cable after it has been pulled 2.0 m? Does your result agree with that of Example 9.7?

KEYCONCEPT For any problem involving torques on a rigid body, first draw a free-body diagram to identify where on the rigid body each external force acts with respect to the axis of rotation. Then apply the rotational analog of Newton's second law, $\sum\tau_z = I\alpha_z$.

Figure **10.9** (a) Cylinder and cable. (b) Our free-body diagram for the cylinder.



EXAMPLE 10.3 An unwinding cable II

WITH VARIATION PROBLEMS

In Example 9.8 (Section 9.4), what are the acceleration of the falling block and the tension in the cable?

IDENTIFY and SET UP We'll apply translational dynamics to the block and rotational dynamics to the cylinder. As in Example 10.2, we'll relate the linear acceleration of the block (our target variable) to the angular acceleration of the cylinder. **Figure 10.10** (next page) shows our sketch of the situation and a free-body diagram for each object. We take the positive sense of rotation for the cylinder to be counterclockwise and the positive direction of the y-coordinate for the block to be downward.

EXECUTE For the block, Newton's second law gives

$$\sum F_y = mg + (-T) = ma_y$$

For the cylinder, the only torque about its axis is that due to the cable tension T. Hence Eq. (10.7) gives

$$\sum\tau_z = RT = I\alpha_z = \tfrac{1}{2}MR^2\alpha_z$$

As in Example 10.2, the acceleration of the cable is the same as the tangential acceleration of a point on the cylinder rim. From Eq. (9.14), this acceleration is $a_y = a_{\tan} = R\alpha_z$. We use this to replace $R\alpha_z$ with a_y in the cylinder equation above, and divide by R. The result is $T = \frac{1}{2}Ma_y$. Now we substitute this expression for T into Newton's second law for the block and solve for the acceleration a_y:

$$mg - \tfrac{1}{2}Ma_y = ma_y$$

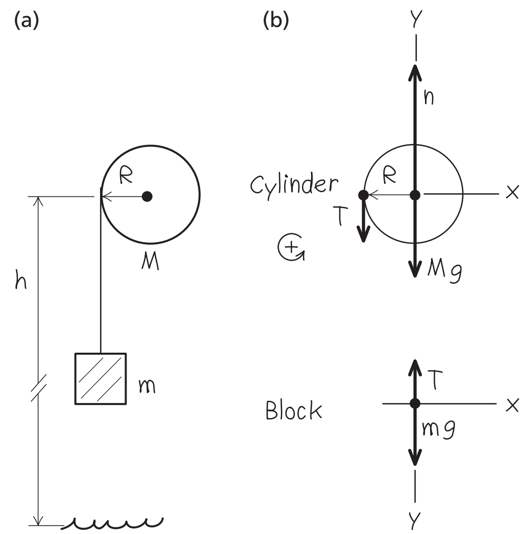
$$a_y = \frac{g}{1 + M/2m}$$

To find the cable tension T, we substitute our expression for a_y into the block equation:

$$T = mg - ma_y = mg - m\left(\frac{g}{1 + M/2m}\right) = \frac{mg}{1 + 2m/M}$$

Continued

Figure **10.10** (a) Our diagram of the situation. (b) Our free-body diagrams for the cylinder and the block. We assume the cable has negligible mass.



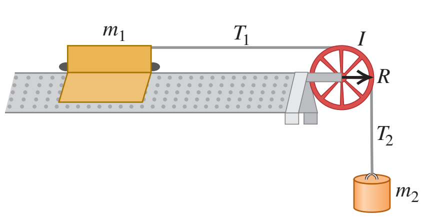
EVALUATE The acceleration is positive (in the downward direction) and less than g, as it should be, since the cable is holding back the block. The cable tension is *not* equal to the block's weight mg; if it were, the block could not accelerate.

Let's check some particular cases. When M is much larger than m, the tension is nearly equal to mg and the acceleration is correspondingly much less than g. When M is zero, $T = 0$ and $a_y = g$; the object falls freely. If the object starts from rest ($v_{0y} = 0$) a height h above the floor, its y-velocity when it strikes the floor is given by $v_y{}^2 = v_{0y}{}^2 + 2a_y h = 2a_y h$, so

$$v_y = \sqrt{2a_y h} = \sqrt{\frac{2gh}{1 + M/2m}}$$

We found this result from energy considerations in Example 9.8.

KEYCONCEPT When an object is connected to a string that wraps around a rotating pulley of radius R, the linear acceleration a_y of the object is related to the angular acceleration α_z of the pulley by $a_y = R\alpha_z$.

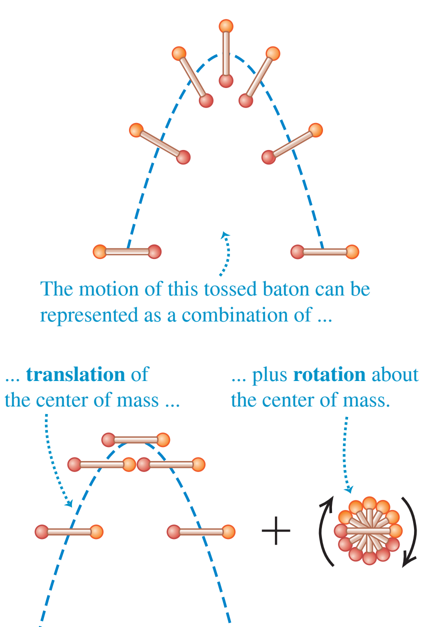


TEST YOUR UNDERSTANDING OF SECTION 10.2 The figure shows a glider of mass m_1 that can slide without friction on a horizontal air track. It is attached to an object of mass m_2 by a massless string. The pulley has radius R and moment of inertia I about its axis of rotation. When released, the hanging object accelerates downward, the glider accelerates to the right, and the string turns the pulley without slipping or stretching. Rank the magnitudes of the following forces that act during the motion, in order from largest to smallest magnitude. (i) The tension force (magnitude T_1) in the horizontal part of the string; (ii) the tension force (magnitude T_2) in the vertical part of the string; (iii) the weight $m_2 g$ of the hanging object.

ANSWER

(iii), (ii), (i) For the hanging object of mass m_2 to accelerate downward, the net force on it must be downward. Hence the magnitude m_2g of the downward weight force must be greater than the magnitude T_2 of the upward tension force. For the pulley to have a clockwise angular acceleration, the net torque on the pulley must be clockwise. Tension T_2 tends to rotate the pulley clockwise, while tension T_1 tends to rotate the pulley counterclockwise. Both tension forces have the same lever arm R, so there is a clockwise torque T_2R and a counterclockwise torque T_1R. For the net torque to be clockwise, T_2 must be greater than T_1. Hence $m_2g > T_2 > T_1$.

Figure **10.11** The motion of a rigid body is a combination of translational motion of the center of mass and rotation around the center of mass.



10.3 RIGID-BODY ROTATION ABOUT A MOVING AXIS

We can extend our analysis of rigid-body rotational dynamics to some cases in which the axis of rotation moves. When that happens, the motion of the rigid body is **combined translation and rotation.** The key to understanding such situations is this: Every possible motion of a rigid body can be represented as a combination of *translational motion of the center of mass* and *rotation about an axis through the center of mass.* This is true even when the center of mass accelerates, so it is not at rest in any inertial frame. **Figure 10.11** illustrates this for the motion of a tossed baton: The center of mass of the baton follows a parabolic curve, as though the baton were a particle located at the center of mass. A rolling ball is another example of combined translational and rotational motions.

Combined Translation and Rotation: Energy Relationships

It's beyond our scope to prove that rigid-body motion can always be divided into translation of the center of mass and rotation about the center of mass. But we *can* prove this for the kinetic energy K of a rigid body that has both translational and rotational motions. For such a rigid body, K is the sum of two parts:

Kinetic energy of a rigid body with both translation and rotation

$$K = \tfrac{1}{2}Mv_{cm}{}^2 + \tfrac{1}{2}I_{cm}\omega^2 \tag{10.8}$$

Kinetic energy of *translation* of center of mass (cm)
Kinetic energy of *rotation* around axis through cm
Mass of rigid body
Speed of cm
Moment of inertia of rigid body about axis through cm
Angular speed of rigid body

To prove this relationship, we again imagine the rigid body to be made up of particles. For a typical particle with mass m_i (**Fig. 10.12**), the velocity $\vec{v}_i$ of this particle relative to an inertial frame is the vector sum of the velocity $\vec{v}_{cm}$ of the center of mass and the velocity $\vec{v}_i{}'$ of the particle *relative to* the center of mass:

$$\vec{v}_i = \vec{v}_{cm} + \vec{v}_i{}' \tag{10.9}$$

The kinetic energy K_i of this particle in the inertial frame is $\frac{1}{2}m_iv_i{}^2$, which we can also express as $\frac{1}{2}m_i(\vec{v}_i \cdot \vec{v}_i)$. Substituting Eq. (10.9) into this, we get

$$\begin{aligned} K_i &= \tfrac{1}{2}m_i(\vec{v}_{cm} + \vec{v}_i{}') \cdot (\vec{v}_{cm} + \vec{v}_i{}') \\ &= \tfrac{1}{2}m_i(\vec{v}_{cm} \cdot \vec{v}_{cm} + 2\vec{v}_{cm} \cdot \vec{v}_i{}' + \vec{v}_i{}' \cdot \vec{v}_i{}') \\ &= \tfrac{1}{2}m_i(v_{cm}{}^2 + 2\vec{v}_{cm} \cdot \vec{v}_i{}' + v_i{}'^2) \end{aligned}$$

The total kinetic energy is the sum $\sum K_i$ for all the particles making up the rigid body. Expressing the three terms in this equation as separate sums, we get

$$K = \sum K_i = \sum\left(\tfrac{1}{2}m_iv_{cm}{}^2\right) + \sum(m_i\vec{v}_{cm} \cdot \vec{v}_i{}') + \sum\left(\tfrac{1}{2}m_iv_i{}'^2\right)$$

The first and second terms have common factors that we take outside the sum:

$$K = \tfrac{1}{2}\left(\sum m_i\right)v_{cm}{}^2 + \vec{v}_{cm} \cdot \left(\sum m_i\vec{v}_i{}'\right) + \sum\left(\tfrac{1}{2}m_iv_i{}'^2\right) \tag{10.10}$$

Now comes the reward for our effort. In the first term, $\sum m_i$ is the total mass M. The second term is zero because $\sum m_i\vec{v}_i{}'$ is M times the velocity of the center of mass *relative to the center of mass,* and this is zero by definition. The last term is the sum of the kinetic energies of the particles computed by using their speeds with respect to the center of mass; this is just the kinetic energy of rotation around the center of mass. Using the same steps that led to Eq. (9.17) for the rotational kinetic energy of a rigid body, we can write this last term as $\frac{1}{2}I_{cm}\omega^2$, where I_{cm} is the moment of inertia with respect to the axis through the center of mass and ω is the angular speed. So Eq. (10.10) becomes Eq. (10.8):

$$K = \tfrac{1}{2}Mv_{cm}{}^2 + \tfrac{1}{2}I_{cm}\omega^2$$

Rolling Without Slipping

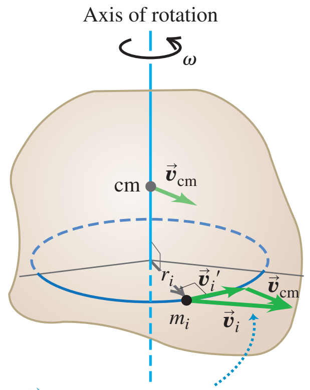
An important case of combined translation and rotation is **rolling without slipping.** The rolling wheel in **Fig. 10.13** (next page) is symmetrical, so its center of mass is at its geometric center. We view the motion in an inertial frame of reference in which the surface on which the wheel rolls is at rest. In this frame, the point on the wheel that contacts the surface must be instantaneously *at rest* so that it does not slip. Hence the velocity $\vec{v}_1{}'$ of the point of contact relative to the center of mass must have the same magnitude but opposite direction as the center-of-mass velocity $\vec{v}_{cm}$. If the wheel's radius is R and its angular speed about the center of mass is ω, then the magnitude of $\vec{v}_1{}'$ is $R\omega$; hence

Condition for rolling without slipping:

$$v_{cm} = R\omega \tag{10.11}$$

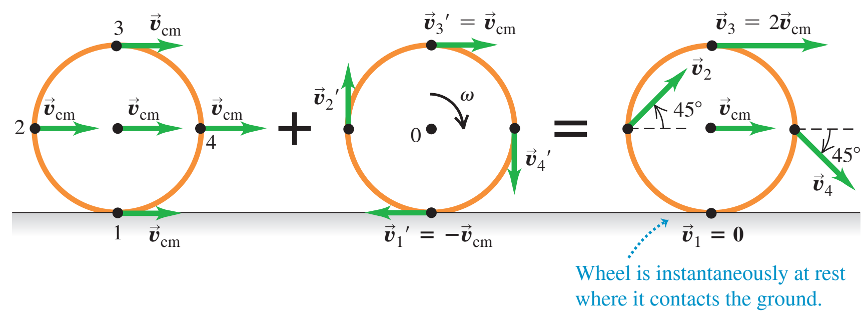
Speed of center of mass of rolling wheel
Radius of wheel
Angular speed of wheel

Figure **10.12** A rigid body with both translational and rotational motions.



BIO APPLICATION Combined Translation and Rotation A maple seed consists of a pod attached to a much lighter, flattened wing. Airflow around the wing slows the falling seed to about 1 m/s and causes the seed to rotate about its center of mass. The seed's slow fall means that a breeze can carry the seed some distance from the parent tree. In the absence of wind, the seed's center of mass falls straight down.

Figure **10.13** The motion of a rolling wheel is the sum of the translational motion of the center of mass and the rotational motion of the wheel around the center of mass.



As Fig. 10.13 shows, the velocity of a point on the wheel is the vector sum of the velocity of the center of mass and the velocity of the point relative to the center of mass. Thus while point 1, the point of contact, is instantaneously at rest, point 3 at the top of the wheel is moving forward *twice as fast* as the center of mass, and points 2 and 4 at the sides have velocities at 45° to the horizontal.

At any instant we can think of the wheel as rotating about an "instantaneous axis" of rotation that passes through the point of contact with the ground. The angular velocity ω is the same for this axis as for an axis through the center of mass; an observer at the center of mass sees the rim make the same number of revolutions per second as does an observer at the rim watching the center of mass spin around him. If we think of the motion of the rolling wheel in Fig. 10.13 in this way, the kinetic energy of the wheel is $K = \frac{1}{2}I_1\omega^2$, where I_1 is the moment of inertia of the wheel about an axis through point 1. But by the parallel-axis theorem, Eq. (9.19), $I_1 = I_{cm} + MR^2$, where M is the total mass of the wheel and I_{cm} is the moment of inertia with respect to an axis through the center of mass. Using Eq. (10.11), we find that the wheel's kinetic energy is as given by Eq. (10.8):

$$K = \tfrac{1}{2}I_1\omega^2 = \tfrac{1}{2}I_{cm}\omega^2 + \tfrac{1}{2}MR^2\omega^2 = \tfrac{1}{2}I_{cm}\omega^2 + \tfrac{1}{2}Mv_{cm}{}^2$$

CAUTION **Rolling without slipping** The relationship $v_{cm} = R\omega$ holds *only* if there is rolling without slipping. When a drag racer first starts to move, the rear tires are spinning very fast even though the racer is hardly moving, so $R\omega$ is greater than v_{cm} (**Fig. 10.14**). If a driver applies the brakes too heavily so that the car skids, the tires will spin hardly at all and $R\omega$ is less than v_{cm}.

Figure **10.14** The smoke rising from this drag racer's rear tires shows that the tires are slipping on the road, so v_{cm} is *not* equal to $R\omega$.

If a rigid body changes height as it moves, we must also consider gravitational potential energy. We saw in Section 9.4 that for any extended object of mass M, rigid or not, the gravitational potential energy U is the same as if we replaced the object by a particle of mass M located at the object's center of mass, so

$$U = Mgy_{cm}$$

EXAMPLE 10.4 Speed of a primitive yo-yo

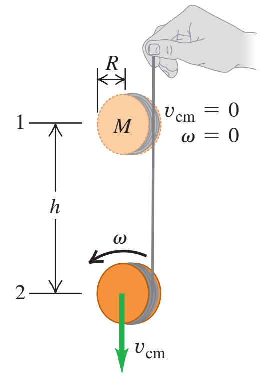
A primitive yo-yo has a massless string wrapped around a solid cylinder with mass M and radius R (**Fig. 10.15**). You hold the free end of the string stationary and release the cylinder from rest. The string unwinds but does not slip or stretch as the cylinder descends and rotates. Using energy considerations, find the speed v_{cm} of the cylinder's center of mass after it has descended a distance h.

IDENTIFY and SET UP Since you hold the upper end of the string fixed, your hand does no work on the string–cylinder system. There is friction between the string and the cylinder, but the string doesn't slip so no mechanical energy is lost. Hence we can use conservation of mechanical energy. The initial kinetic energy of the cylinder is $K_1 = 0$, and its final kinetic energy K_2 is given by Eq. (10.8); the massless string has no kinetic energy. The moment of inertia is $I_{cm} = \frac{1}{2}MR^2$, and by Eq. (9.13) $\omega = v_{cm}/R$ because the string doesn't slip. The potential energies are $U_1 = Mgh$ and $U_2 = 0$.

EXECUTE From Eq. (10.8), the kinetic energy at point 2 is

$$K_2 = \tfrac{1}{2}Mv_{cm}{}^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v_{cm}}{R}\right)^2 = \tfrac{3}{4}Mv_{cm}{}^2$$

Figure **10.15** Calculating the speed of a primitive yo-yo.



The kinetic energy is $1\frac{1}{2}$ times what it would be if the yo-yo were falling at speed v_{cm} without rotating. Two-thirds of the total kinetic energy $\left(\frac{1}{2}Mv_{cm}^2\right)$ is translational and one-third $\left(\frac{1}{4}Mv_{cm}^2\right)$ is rotational. Using conservation of energy,

$$K_1 + U_1 = K_2 + U_2$$
$$0 + Mgh = \tfrac{3}{4}Mv_{cm}^2 + 0$$
$$v_{cm} = \sqrt{\tfrac{4}{3}gh}$$

EVALUATE No mechanical energy was lost or gained, so from the energy standpoint the string is merely a way to convert some of the gravitational potential energy (which is released as the cylinder falls) into rotational kinetic energy rather than translational kinetic energy. Because not all of the released energy goes into translation, v_{cm} is less than the speed $\sqrt{2gh}$ of an object dropped from height h with no strings attached.

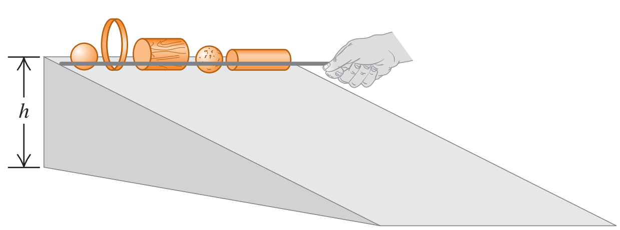
KEYCONCEPT If a rigid body is both translating (moving as a whole through space) and rotating, its total kinetic energy is the sum of the kinetic energy of translation of the center of mass and the kinetic energy of rotation around an axis through the center of mass.

EXAMPLE 10.5 Race of the rolling bodies

In a physics demonstration, an instructor "races" various rigid bodies that roll without slipping from rest down an inclined plane (**Fig. 10.16**). What shape should a body have to reach the bottom of the incline first?

IDENTIFY and SET UP Kinetic friction does no work if the bodies roll without slipping. We can also ignore the effects of *rolling friction,* introduced in Section 5.3, if the bodies and the surface of the incline are rigid. (Later in this section we'll explain why this is so.) We can therefore use conservation of energy. Each body starts from rest at the top of an incline with height h, so $K_1 = 0$, $U_1 = Mgh$, and $U_2 = 0$. Equation (10.8) gives the kinetic energy at the bottom of the incline; since the bodies roll without slipping, $\omega = v_{cm}/R$. We can express the moments of inertia of the four round bodies in Table 9.2, cases (f)–(i), as $I_{cm} = cMR^2$, where c is a number less than or equal to 1 that depends on the shape of the body, Our goal is to find the value of c that gives the body the greatest speed v_{cm} after its center of mass has descended a vertical distance h.

Figure **10.16** Which body rolls down the incline fastest, and why?



EXECUTE From conservation of energy,

$$K_1 + U_1 = K_2 + U_2$$
$$0 + Mgh = \tfrac{1}{2}Mv_{cm}^2 + \tfrac{1}{2}cMR^2\left(\frac{v_{cm}}{R}\right)^2 + 0$$
$$Mgh = \tfrac{1}{2}(1 + c)Mv_{cm}^2$$
$$v_{cm} = \sqrt{\frac{2gh}{1 + c}}$$

EVALUATE For a given value of c, the speed v_{cm} after descending a distance h is *independent* of the body's mass M and radius R. Hence *all* uniform solid cylinders $\left(c = \frac{1}{2}\right)$ have the same speed at the bottom, regardless of their mass and radii. The values of c tell us that the order of finish for uniform bodies will be as follows: (1) any solid sphere $\left(c = \frac{2}{5}\right)$, (2) any solid cylinder $\left(c = \frac{1}{2}\right)$, (3) any thin-walled, hollow sphere $\left(c = \frac{2}{3}\right)$, and (4) any thin-walled, hollow cylinder $(c = 1)$. Small-c bodies always beat large-c bodies because less of their kinetic energy is tied up in rotation, so more is available for translation.

KEYCONCEPT For a rigid body that rolls without slipping, has a given mass and radius, and moves with a given center-of-mass speed, the kinetic energy of rotation depends on the shape of the rigid body.

Combined Translation and Rotation: Dynamics

We can also analyze the combined translational and rotational motions of a rigid body from the standpoint of dynamics. We showed in Section 8.5 that for an extended object, the acceleration of the center of mass is the same as that of a particle of the same mass acted on by all the external forces on the actual object:

Net external force on an object · Total mass of object · Acceleration of center of mass

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} \qquad (10.12)$$

The rotational motion about the center of mass is described by the rotational analog of Newton's second law, Eq. (10.7):

Net torque on a rigid body about z-axis through center of mass

Moment of inertia of rigid body about z-axis

$$\sum \tau_z = I_{\mathrm{cm}} \alpha_z \qquad (10.13)$$

Angular acceleration of rigid body about z-axis

Figure **10.17** The axle of a bicycle wheel passes through the wheel's center of mass and is an axis of symmetry. Hence the rotation of the wheel is described by Eq. (10.13), provided the bicycle doesn't turn or tilt to one side (which would change the orientation of the axle).

It's not immediately obvious that Eq. (10.13) should apply to the motion of a translating rigid body; after all, our derivation of $\sum \tau_z = I\alpha_z$ in Section 10.2 assumed that the axis of rotation was stationary. But Eq. (10.13) is valid *even when the axis of rotation moves,* provided the following two conditions are met:

1. The axis through the center of mass must be an axis of symmetry.
2. The axis must not change direction.

These conditions are satisfied for many types of rotation (**Fig. 10.17**). Note that in general this moving axis of rotation is *not* at rest in an inertial frame of reference.

We can now solve dynamics problems involving a rigid body that undergoes translational and rotational motions at the same time, provided that the rotation axis satisfies the two conditions just mentioned. Problem-Solving Strategy 10.1 (Section 10.2) is equally useful here, and you should review it now. Keep in mind that when a rigid body undergoes translational and rotational motions at the same time, we need two separate equations of motion *for the same body:* Eq. (10.12) for the translation of the center of mass and Eq. (10.13) for rotation about an axis through the center of mass.

EXAMPLE 10.6 Acceleration of a primitive yo-yo

WITH VARIATION PROBLEMS

For the primitive yo-yo in Example 10.4 (**Fig. 10.18a**), find the downward acceleration of the cylinder and the tension in the string.

IDENTIFY and SET UP Figure 10.18b shows our free-body diagram for the yo-yo, including our choice of positive coordinate directions. Our target variables are $a_{\mathrm{cm}\text{-}y}$ and T. We'll use Eq. (10.12) for the translational motion of the center of mass and Eq. (10.13) for the rotational motion around the center of mass. We'll also use Eq. (10.11), which says that the string unwinds without slipping. As in Example 10.4, the moment of inertia of the yo-yo for an axis through its center of mass is $I_{\mathrm{cm}} = \frac{1}{2}MR^2$.

EXECUTE From Eq. (10.12),

$$\sum F_y = Mg + (-T) = Ma_{\mathrm{cm}\text{-}y} \qquad (10.14)$$

From Eq. (10.13),

$$\sum \tau_z = TR = I_{\mathrm{cm}}\alpha_z = \tfrac{1}{2}MR^2\alpha_z \qquad (10.15)$$

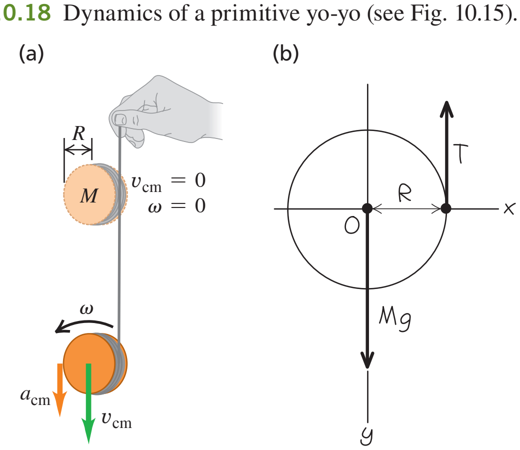
From Eq. (10.11), $v_{\mathrm{cm}\text{-}z} = R\omega_z$; the derivative of this expression with respect to time gives us

$$a_{\mathrm{cm}\text{-}y} = R\alpha_z \qquad (10.16)$$

We now use Eq. (10.16) to eliminate α_z from Eq. (10.15) and then solve Eqs. (10.14) and (10.15) simultaneously for T and $a_{\mathrm{cm}\text{-}y}$:

$$a_{\mathrm{cm}\text{-}y} = \tfrac{2}{3}g \qquad T = \tfrac{1}{3}Mg$$

Figure **10.18** Dynamics of a primitive yo-yo (see Fig. 10.15).



EVALUATE The string slows the fall of the yo-yo, but not enough to stop it completely. Hence $a_{\mathrm{cm}\text{-}y}$ is less than the free-fall value g and T is less than the yo-yo weight Mg.

KEYCONCEPT To analyze the motion of a rigid body that is both translating and rotating, use Newton's second law for the translational motion of the center of mass and the rotational analog of Newton's second law for the rotation around the center of mass.

EXAMPLE 10.7 Acceleration of a rolling sphere

WITH VARIATION PROBLEMS

A bowling ball of mass M rolls without slipping down a ramp that is inclined at an angle β to the horizontal (**Fig. 10.19a**). What are the ball's acceleration and the magnitude of the friction force on the ball? Treat the ball as a uniform solid sphere, ignoring the finger holes.

IDENTIFY and SET UP The free-body diagram (Fig. 10.19b) shows that only the friction force exerts a torque about the center of mass. Our target variables are the acceleration $a_{\text{cm-}x}$ of the ball's center of mass and the magnitude f of the friction force. (Because the ball does not slip at the instantaneous point of contact with the ramp, this is a *static* friction force; it prevents slipping and gives the ball its angular acceleration.) We use Eqs. (10.12) and (10.13) as in Example 10.6.

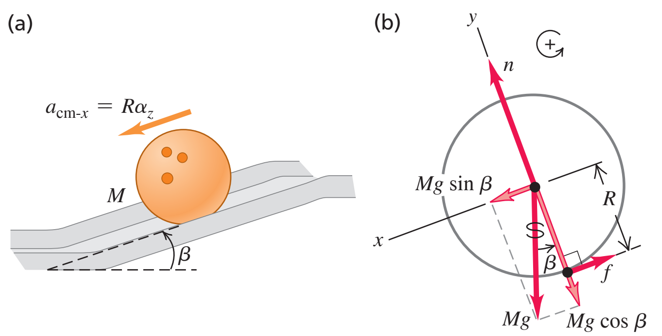


Figure 10.19 A bowling ball rolling down a ramp.

EXECUTE The ball's moment of inertia is $I_{\text{cm}} = \frac{2}{5}MR^2$. The equations of motion are

$$\sum F_x = Mg\sin\beta + (-f) = Ma_{\text{cm-}x} \qquad (10.17)$$

$$\sum \tau_z = fR = I_{\text{cm}}\alpha_z = \left(\tfrac{2}{5}MR^2\right)\alpha_z \qquad (10.18)$$

The ball rolls without slipping, so as in Example 10.6 we use $a_{\text{cm-}x} = R\alpha_z$ to eliminate α_z from Eq. (10.18):

$$fR = \tfrac{2}{5}MRa_{\text{cm-}x}$$

This equation and Eq. (10.17) are two equations for the unknowns $a_{\text{cm-}x}$ and f. We solve Eq. (10.17) for f, substitute that expression into the above equation to eliminate f, and solve for $a_{\text{cm-}x}$:

$$a_{\text{cm-}x} = \tfrac{5}{7}g\sin\beta$$

Finally, we substitute this acceleration into Eq. (10.17) and solve for f:

$$f = \tfrac{2}{7}Mg\sin\beta$$

EVALUATE The ball's acceleration is just $\frac{5}{7}$ as large as that of an object *sliding* down the slope without friction. If the ball descends a vertical distance h as it rolls down the ramp, its displacement along the ramp is $h/\sin\beta$. You can show that the speed of the ball at the bottom of the ramp is $v_{\text{cm}} = \sqrt{\frac{10}{7}gh}$, the same as our result from Example 10.5 with $c = \frac{2}{5}$.

If the ball were rolling *uphill* without slipping, the force of friction would still be directed uphill as in Fig. 10.19b. Can you see why?

KEYCONCEPT If an object is rolling without slipping on an incline, a friction force must act on it. The direction of this friction force is always such as to prevent slipping.

Rolling Friction

In Example 10.5 we said that we can ignore rolling friction if both the rolling body and the surface over which it rolls are perfectly rigid. In **Fig. 10.20a** a perfectly rigid sphere is rolling down a perfectly rigid incline. The line of action of the normal force passes through the center of the sphere, so its torque is zero; there is no sliding at the point of contact, so the friction force does no work. Figure 10.20b shows a more realistic situation, in which the surface "piles up" in front of the sphere and the sphere rides in a shallow trench. Because of these deformations, the contact forces on the sphere no longer act along a single point but over an area; the forces are concentrated on the front of the sphere as shown. As a result, the normal force now exerts a torque that opposes the rotation. In addition, there is some sliding of the sphere over the surface due to the deformation, causing

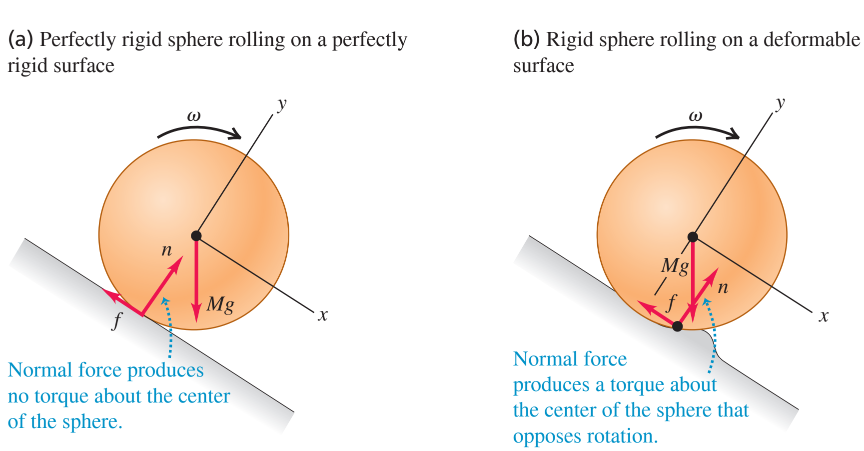


Figure 10.20 Rolling down (a) a perfectly rigid surface and (b) a deformable surface. In (b) the deformation is greatly exaggerated, and the force n is the component of the contact force that points normal to the plane of the surface before it is deformed.

BIO APPLICATION Rolling for Reproduction One of the few organisms that uses rolling as a means of locomotion is the weed called Russian thistle (*Kali tragus*). The plant breaks off at its base, forming a rounded tumbleweed that disperses its seeds as it rolls. Because a tumbleweed deforms easily, it is subject to substantial rolling friction. Hence it quickly slows to a stop unless propelled by the wind.

mechanical energy to be lost. The combination of these two effects is the phenomenon of *rolling friction*. Rolling friction also occurs if the rolling body is deformable, such as an automobile tire. Often the rolling body and the surface are rigid enough that rolling friction can be ignored, as we have assumed in all the examples in this section.

TEST YOUR UNDERSTANDING OF SECTION 10.3 Suppose the solid cylinder used as a yo-yo in Example 10.6 is replaced by a hollow cylinder of the same mass and radius. (a) Will the acceleration of the yo-yo (i) increase, (ii) decrease, or (iii) remain the same? (b) Will the string tension (i) increase, (ii) decrease, or (iii) remain the same?

ANSWER

(a) (**ii**), (b) (**i**) If you redo the calculation of Example 10.6 with a hollow cylinder (moment of inertia $I_{cm} = MR^2$) instead of a solid cylinder (moment of inertia $I_{cm} = \frac{1}{2}MR^2$), you'll find $a_{cm\text{-}y} = \frac{1}{2}g$ and $T = \frac{1}{2}Mg$ (instead of $a_{cm\text{-}y} = \frac{2}{3}g$ and $T = \frac{1}{3}Mg$ for a solid cylinder). Hence the acceleration is less but the tension is greater. You can come to the same conclusion without doing the calculation. The greater moment of inertia means that the hollow cylinder will rotate more slowly and hence will roll downward more slowly. To slow the downward motion, a greater upward tension force is needed to oppose the downward force of gravity.

10.4 WORK AND POWER IN ROTATIONAL MOTION

When you pedal a bicycle, you apply forces to a rotating body and do work on it. Similar things happen in many other real-life situations, such as a rotating motor shaft driving a power tool or a car engine propelling the vehicle. Let's see how to apply our ideas about work from Chapter 6 to rotational motion.

Suppose a tangential force $\vec{F}_{tan}$ acts at the rim of a pivoted disk—for example, a child running while pushing on a playground merry-go-round (**Fig. 10.21a**). The disk rotates through an infinitesimal angle $d\theta$ about a fixed axis during an infinitesimal time interval dt (Fig. 10.21b). The work dW done by the force $\vec{F}_{tan}$ while a point on the rim moves a distance ds is $dW = F_{tan}\,ds$. If $d\theta$ is measured in radians, then $ds = R\,d\theta$ and

$$dW = F_{tan} R\,d\theta$$

Now $F_{tan}R$ is the *torque* τ_z due to the force $\vec{F}_{tan}$, so

$$dW = \tau_z\,d\theta \tag{10.19}$$

As the disk rotates from θ_1 to θ_2, the total work done by the torque is

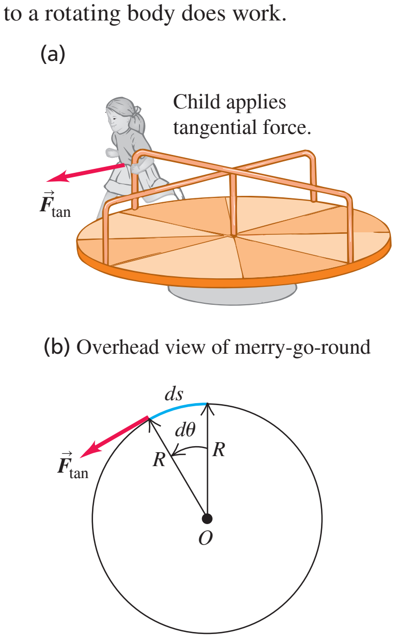
$$\underbrace{W}_{\text{Work done by a torque } \tau_z} = \int_{\theta_1}^{\theta_2} \underbrace{\tau_z\,d\theta}_{\text{Integral of the torque with respect to angle}} \tag{10.20}$$

Upper limit = final angular position (θ_2); Lower limit = initial angular position (θ_1)

If the torque remains *constant* while the angle changes, then the work is the product of torque and angular displacement:

$$\underbrace{W}_{\text{Work done by a constant torque } \tau_z} = \overbrace{\tau_z}^{\text{Torque}}\underbrace{(\theta_2 - \theta_1)}_{\text{Final minus initial angular position = angular displacement}} = \tau_z\,\Delta\theta \tag{10.21}$$

Figure **10.21** A tangential force applied to a rotating body does work.



If torque is expressed in newton-meters ($N \cdot m$) and angular displacement in radians, the work is in joules. Equation (10.21) is the rotational analog of Eq. (6.1), $W = Fs$, and Eq. (10.20) is the analog of Eq. (6.7), $W = \int F_x\,dx$, for the work done by a force in a straight-line displacement.

If the force in Fig. 10.21 had an axial component (parallel to the rotation axis) or a radial component (directed toward or away from the axis), that component would do no work because the displacement of the point of application has only a tangential component. An axial or radial component of force would also make no contribution to the torque about the axis of rotation. So Eqs. (10.20) and (10.21) are correct for *any* force, no matter what its components.

When a torque does work on a rotating rigid body, the kinetic energy changes by an amount equal to the work done. We can prove this by using exactly the same procedure that we used in Eqs. (6.11) through (6.13) for the translational kinetic energy of a particle. Let τ_z represent the *net* torque on the body so that $\tau_z = I\alpha_z$ from Eq. (10.7), and assume that the body is rigid so that the moment of inertia I is constant. We then transform the integrand in Eq. (10.20) into an integrand with respect to ω_z as follows:

$$\tau_z\, d\theta = (I\alpha_z)\, d\theta = I\frac{d\omega_z}{dt}d\theta = I\frac{d\theta}{dt}d\omega_z = I\omega_z\, d\omega_z$$

Since τ_z is the net torque, the integral in Eq. (10.20) is the *total* work done on the rotating rigid body. This equation then becomes

Total work done on a rotating rigid body = work done by the net external torque

$$W_{\text{tot}} = \int_{\omega_1}^{\omega_2} I\omega_z\, d\omega_z = \overbrace{\tfrac{1}{2}I\omega_2^2}^{\text{Final rotational kinetic energy}} - \underbrace{\tfrac{1}{2}I\omega_1^2}_{\text{Initial rotational kinetic energy}} \qquad (10.22)$$

The change in the rotational kinetic energy of a *rigid* body equals the work done by forces exerted from outside the body (**Fig. 10.22**). This equation is analogous to Eq. (6.13), the work–energy theorem for a particle.

How does *power* relate to torque? When we divide both sides of Eq. (10.19) by the time interval dt during which the angular displacement occurs, we find

$$\frac{dW}{dt} = \tau_z\frac{d\theta}{dt}$$

But dW/dt is the rate of doing work, or *power* P, and $d\theta/dt$ is angular velocity ω_z:

Power due to a torque acting on a rigid body

$$P = \overbrace{\tau_z}^{\text{Torque with respect to rigid body's rotation axis}}\underbrace{\omega_z}_{\text{Angular velocity of rigid body about axis}} \qquad (10.23)$$

This is the analog of the relationship $P = \vec{F} \cdot \vec{v}$ that we developed in Section 6.4 for particle motion.

Figure **10.22** The rotational kinetic energy of a helicopter's main rotor is equal to the total work done to set it spinning. When it is spinning at a constant rate, positive work is done on the rotor by the engine and negative work is done on it by air resistance. Hence the net work being done is zero and the kinetic energy remains constant.

EXAMPLE 10.8 Calculating power from torque

An electric motor exerts a constant $10\ \text{N}\cdot\text{m}$ torque on a grindstone, which has a moment of inertia of $2.0\ \text{kg}\cdot\text{m}^2$ about its shaft. The system starts from rest. Find the work W done by the motor in 8.0 s and the grindstone's kinetic energy K at this time. What average power P_{av} is delivered by the motor?

IDENTIFY and SET UP The only torque acting is that due to the motor. Since this torque is constant, the grindstone's angular acceleration α_z is constant. We'll use Eq. (10.7) to find α_z, and then use this in the kinematics equations from Section 9.2 to calculate the angle $\Delta\theta$ through which the grindstone rotates in 8.0 s and its final angular velocity ω_z. From these we'll calculate W, K, and P_{av}.

EXECUTE We have $\sum\tau_z = 10\ \text{N}\cdot\text{m}$ and $I = 2.0\ \text{kg}\cdot\text{m}^2$, so $\sum\tau_z = I\alpha_z$ yields $\alpha_z = 5.0\ \text{rad/s}^2$. From Eqs. (9.11) and (10.21),

$$\Delta\theta = \tfrac{1}{2}\alpha_z t^2 = \tfrac{1}{2}(5.0\ \text{rad/s}^2)(8.0\ \text{s})^2 = 160\ \text{rad}$$
$$W = \tau_z\Delta\theta = (10\ \text{N}\cdot\text{m})(160\ \text{rad}) = 1600\ \text{J}$$

From Eqs. (9.7) and (9.17),

$$\omega_z = \alpha_z t = (5.0\ \text{rad/s}^2)(8.0\ \text{s}) = 40\ \text{rad/s}$$
$$K = \tfrac{1}{2}I\omega_z^2 = \tfrac{1}{2}(2.0\ \text{kg}\cdot\text{m}^2)(40\ \text{rad/s})^2 = 1600\ \text{J}$$

The average power is the work done divided by the time interval:

$$P_{\text{av}} = \frac{1600\ \text{J}}{8.0\ \text{s}} = 200\ \text{J/s} = 200\ \text{W}$$

Continued

EVALUATE The initial kinetic energy was zero, so the work done W must equal the final kinetic energy K [Eq. (10.22)]. This is just as we calculated. We can check our result $P_{\text{av}} = 200\text{ W}$ by considering the *instantaneous* power $P = \tau_z \omega_z$. Because ω_z increases continuously, P increases continuously as well; its value increases from zero at $t = 0$ to $(10\text{ N}\cdot\text{m})(40\text{ rad/s}) = 400\text{ W}$ at $t = 8.0\text{ s}$. Both ω_z and P increase *uniformly* with time, so the *average* power is just half this maximum value, or 200 W.

KEYCONCEPT If a torque acts on a rigid body, the work done equals torque times angular displacement and the power equals torque times angular velocity.

TEST YOUR UNDERSTANDING OF SECTION 10.4 You apply equal torques to two different cylinders. Cylinder 1 has a moment of inertia twice as large as cylinder 2. Each cylinder is initially at rest. After one complete rotation, which cylinder has the greater kinetic energy? (i) Cylinder 1; (ii) cylinder 2; (iii) both cylinders have the same kinetic energy.

ANSWER

(iii) You apply the same torque over the same angular displacement to both cylinders. Hence, by Eq. (10.21), you do the same amount of work to both cylinders and impart the same kinetic energy to both. (The one with the smaller moment of inertia ends up with a greater angular speed, but that isn't what we are asked. Compare Conceptual Example 6.5 in Section 6.2.)

10.5 ANGULAR MOMENTUM

Every rotational quantity that we have encountered in Chapters 9 and 10 is the analog of some quantity in the translational motion of a particle. The analog of *momentum* of a particle is **angular momentum,** a vector quantity denoted as $\vec{L}$. Its relationship to momentum $\vec{p}$ (which we'll often call *linear momentum* for clarity) is exactly the same as the relationship of torque to force, $\vec{\tau} = \vec{r} \times \vec{F}$. For a particle with constant mass m and velocity $\vec{v}$, the angular momentum is

Angular momentum of a particle relative to origin O of an inertial frame of reference

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \tag{10.24}$$

Position vector of particle relative to O

Linear momentum of particle = mass times velocity

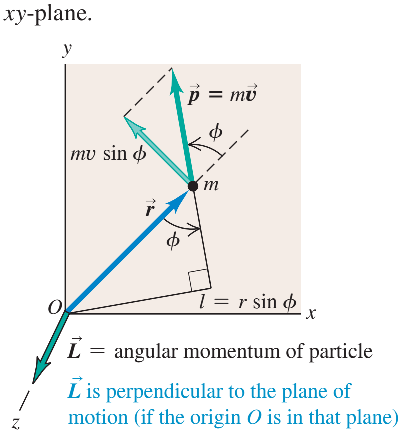
The value of $\vec{L}$ depends on the choice of origin O, since it involves the particle's position vector $\vec{r}$ relative to O. The units of angular momentum are $\text{kg}\cdot\text{m}^2/\text{s}$.

In **Fig. 10.23** a particle moves in the xy-plane; its position vector $\vec{r}$ and momentum $\vec{p} = m\vec{v}$ are shown. The angular momentum vector $\vec{L}$ is perpendicular to the xy-plane. The right-hand rule for vector products shows that its direction is along the $+z$-axis, and its magnitude is

$$L = mvr\sin\phi = mvl \tag{10.25}$$

where l is the perpendicular distance from the line of $\vec{v}$ to O. This distance plays the role of "lever arm" for the momentum vector.

Figure 10.23 Calculating the angular momentum $\vec{L} = \vec{r} \times m\vec{v} = \vec{r} \times \vec{p}$ of a particle with mass m moving in the xy-plane.



When a net force $\vec{F}$ acts on a particle, its velocity and momentum change, so its angular momentum may also change. We can show that the *rate of change* of angular momentum is equal to the torque of the net force. We take the time derivative of Eq. (10.24), using the rule for the derivative of a product:

$$\frac{d\vec{L}}{dt} = \left(\frac{d\vec{r}}{dt} \times m\vec{v}\right) + \left(\vec{r} \times m\frac{d\vec{v}}{dt}\right) = (\vec{v} \times m\vec{v}) + (\vec{r} \times m\vec{a})$$

The first term is zero because it contains the vector product of the vector $\vec{v} = d\vec{r}/dt$ with itself. In the second term we replace $m\vec{a}$ with the net force $\vec{F}$:

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \quad \text{(for a particle acted on by net force } \vec{F}\text{)} \tag{10.26}$$

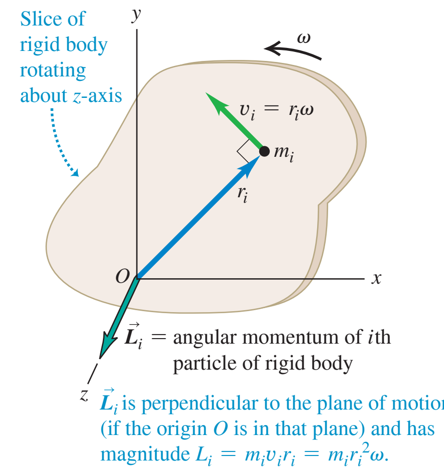
The rate of change of angular momentum of a particle equals the torque of the net force acting on it. Compare this result to Eq. (8.4): The rate of change $d\vec{p}/dt$ of the *linear* momentum of a particle equals the net force that acts on it.

Angular Momentum of a Rigid Body

We can use Eq. (10.25) to find the total angular momentum of a *rigid body* rotating about the z-axis with angular speed ω. First consider a thin slice of the body lying in the xy-plane (**Fig. 10.24**). Each particle in the slice moves in a circle centered at the origin, and at each instant its velocity $\vec{v}_i$ is perpendicular to its position vector $\vec{r}_i$, as shown. Hence in Eq. (10.25), $\phi = 90°$ for every particle. A particle with mass m_i at a distance r_i from O has a speed v_i equal to $r_i\omega$. From Eq. (10.25) the magnitude L_i of its angular momentum is

$$L_i = m_i(r_i\omega)\,r_i = m_i r_i^2\omega \qquad (10.27)$$

Figure **10.24** Calculating the angular momentum of a particle of mass m_i in a rigid body rotating at angular speed ω. (Compare Fig. 10.23.)



The direction of each particle's angular momentum, as given by the right-hand rule for the vector product, is along the $+z$-axis.

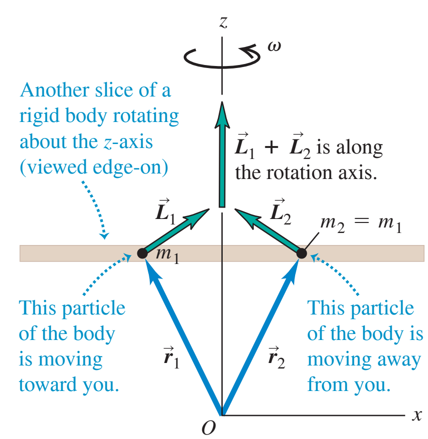
The *total* angular momentum of the slice of the rigid body that lies in the xy-plane is the sum $\sum L_i$ of the angular momenta L_i of all of its particles. From Eq. (10.27),

$$L = \sum L_i = \left(\sum m_i r_i^2\right)\omega = I\omega$$

where I is the moment of inertia of the slice about the z-axis.

We can do this same calculation for the other slices of the rigid body, all parallel to the xy-plane. For points that do not lie in the xy-plane, a complication arises because the $\vec{r}$ vectors have components in the z-direction as well as in the x- and y-directions; this gives the angular momentum of each particle a component perpendicular to the z-axis. But *if the z-axis is an axis of symmetry,* the perpendicular components for particles on opposite sides of this axis add up to zero (**Fig. 10.25**). So when a rigid body rotates about an axis of symmetry, its angular momentum vector $\vec{L}$ lies along the symmetry axis, and its magnitude is $L = I\omega$.

Figure **10.25** Two particles of the same mass located symmetrically on either side of the rotation axis of a rigid body. The angular momentum vectors $\vec{L}_1$ and $\vec{L}_2$ of the two particles do not lie along the rotation axis, but their vector sum $\vec{L}_1 + \vec{L}_2$ does.



The angular velocity vector $\vec{\omega}$ also lies along the rotation axis, as we saw in Section 9.1. Hence for a rigid body rotating around an axis of symmetry, $\vec{L}$ and $\vec{\omega}$ are in the same direction (**Fig. 10.26**). So we have the *vector* relationship

Angular momentum of a rigid body rotating around a symmetry axis $\vec{L} = I\vec{\omega}$ (10.28)

Moment of inertia of rigid body about symmetry axis

Angular velocity vector of rigid body

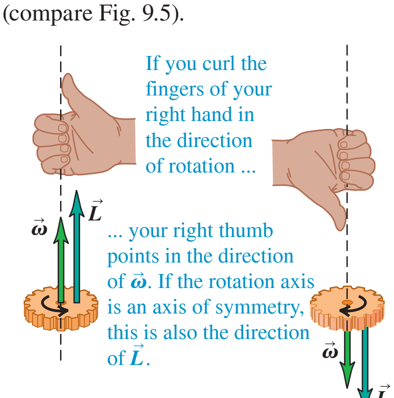
From Eq. (10.26) the rate of change of angular momentum of a particle equals the torque of the net force acting on the particle. For any system of particles (including both rigid and nonrigid bodies), the rate of change of the *total* angular momentum equals the sum of the torques of all forces acting on all the particles. The torques of the *internal* forces add to zero if these forces act along the line from one particle to another, as in Fig. 10.8, and so the sum of the torques includes only the torques of the *external* forces. (We saw a similar cancellation in our discussion of center-of-mass motion in Section 8.5.) So we conclude that

For a system of particles: $\sum \vec{\tau} = \dfrac{d\vec{L}}{dt}$ (10.29)

Sum of external torques on the system

Rate of change of total angular momentum $\vec{L}$ of system

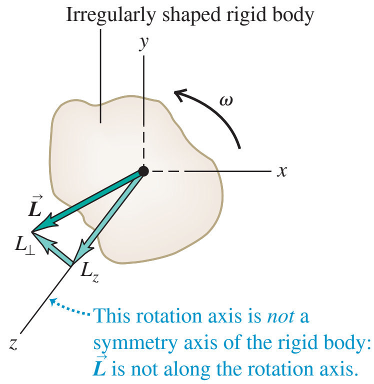
Figure **10.26** For rotation about an axis of symmetry, $\vec{\omega}$ and $\vec{L}$ are parallel and along the axis. The directions of both vectors are given by the right-hand rule (compare Fig. 9.5).



Finally, if the system of particles is a rigid body rotating about a symmetry axis (the z-axis), then $L_z = I\omega_z$ and I is constant. If this axis has a fixed direction in space, then vectors $\vec{L}$ and $\vec{\omega}$ change only in magnitude, not in direction. In that case, $dL_z/dt = I\,d\omega_z/dt = I\alpha_z$, or

$$\sum \tau_z = I\alpha_z$$

Figure **10.27** If the rotation axis of a rigid body is not a symmetry axis, $\vec{L}$ does not in general lie along the rotation axis. Even if $\vec{\omega}$ is constant, the direction of $\vec{L}$ changes and a net torque is required to maintain rotation.



which is again our basic relationship for the dynamics of rigid-body rotation. If the body is *not* rigid, I may change; in that case, L changes even when ω is constant. For a nonrigid body, Eq. (10.29) is still valid, even though Eq. (10.7) is not.

When the axis of rotation is *not* a symmetry axis, the angular momentum is in general *not* parallel to the axis (**Fig. 10.27**). As the rigid body rotates, the angular momentum vector $\vec{L}$ traces out a cone around the rotation axis. Because $\vec{L}$ changes, there must be a net external torque acting on the body even though the angular velocity magnitude ω may be constant. If the body is an unbalanced wheel on a car, this torque is provided by friction in the bearings, which causes the bearings to wear out. "Balancing" a wheel means distributing the mass so that the rotation axis is an axis of symmetry; then $\vec{L}$ points along the rotation axis, and no net torque is required to keep the wheel turning.

In fixed-axis rotation we often use the term "angular momentum of the body" to refer to only the *component* of $\vec{L}$ along the rotation axis of the body (the z-axis in Fig. 10.27), with a positive or negative sign to indicate the sense of rotation just as with angular velocity.

EXAMPLE 10.9 Angular momentum and torque

A turbine fan in a jet engine has a moment of inertia of $2.5\ \text{kg}\cdot\text{m}^2$ about its axis of rotation. As the turbine starts up, its angular velocity is given by $\omega_z = (40\ \text{rad/s}^3)t^2$. (a) Find the fan's angular momentum as a function of time, and find its value at $t = 3.0$ s. (b) Find the net torque on the fan as a function of time, and find its value at $t = 3.0$ s.

IDENTIFY and SET UP The fan rotates about its axis of symmetry (the z-axis). Hence the angular momentum vector has only a z-component L_z, which we can determine from the angular velocity ω_z. Since the direction of angular momentum is constant, the net torque likewise has only a component τ_z along the rotation axis. We'll use Eq. (10.28) to find L_z from ω_z and then Eq. (10.29) to find τ_z.

EXECUTE (a) From Eq. (10.28),

$$L_z = I\omega_z = (2.5\ \text{kg}\cdot\text{m}^2)(40\ \text{rad/s}^3)t^2$$
$$= (100\ \text{kg}\cdot\text{m}^2/\text{s}^3)t^2$$

(We dropped the dimensionless quantity "rad" from the final expression.) At $t = 3.0$ s, $L_z = 900\ \text{kg}\cdot\text{m}^2/\text{s}$.

(b) From Eq. (10.29),

$$\tau_z = \frac{dL_z}{dt} = (100\ \text{kg}\cdot\text{m}^2/\text{s}^3)(2t) = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)t$$

At $t = 3.0$ s,

$$\tau_z = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)(3.0\ \text{s})$$
$$= 600\ \text{kg}\cdot\text{m}^2/\text{s}^2 = 600\ \text{N}\cdot\text{m}$$

EVALUATE As a check on our expression for τ_z, note that the angular acceleration of the turbine is $\alpha_z = d\omega_z/dt = (40\ \text{rad/s}^3)(2t) = (80\ \text{rad/s}^3)t$. Hence from Eq. (10.7), the torque on the fan is $\tau_z = I\alpha_z = (2.5\ \text{kg}\cdot\text{m}^2)(80\ \text{rad/s}^3)t = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)t$, just as we calculated.

KEYCONCEPT The angular momentum vector of a rotating rigid body points along the rigid body's rotation axis. The rate of change of angular momentum equals the net external torque on the rigid body.

TEST YOUR UNDERSTANDING OF SECTION 10.5 A ball is attached to one end of a piece of string. You hold the other end of the string and whirl the ball in a circle around your hand. (a) If the ball moves at a constant speed, is its linear momentum $\vec{p}$ constant? Why or why not? (b) Is its angular momentum $\vec{L}$ constant? Why or why not?

ANSWER (a) **no,** (b) **yes** As the ball goes around the circle, the magnitude of $\vec{p} = m\vec{v}$ remains the same (the speed is constant) but its direction changes, so the linear momentum vector isn't constant. But $\vec{L} = \vec{r} \times \vec{p}$ *is* constant: It has a constant magnitude (both the speed and the perpendicular distance from your hand to the ball are constant) and a constant direction (along the rotation axis, perpendicular to the plane of the ball's motion). The linear momentum changes because there is a net *force* $\vec{F}$ on the ball (toward the center of the circle). The angular momentum remains constant because there is no net *torque*; the vector $\vec{r}$ points from your hand to the ball and the force $\vec{F}$ on the ball is directed toward your hand, so the vector product $\vec{\tau} = \vec{r} \times \vec{F}$ is zero.

10.6 CONSERVATION OF ANGULAR MOMENTUM

We have just seen that angular momentum can be used for an alternative statement of the basic dynamic principle for rotational motion. It also forms the basis for the principle of **conservation of angular momentum.** Like conservation of energy and of linear momentum, this principle is a universal conservation law, valid at all scales from atomic and nuclear systems to the motions of galaxies. This principle follows directly from Eq. (10.29): $\sum \vec{\boldsymbol{\tau}} = d\vec{\boldsymbol{L}}/dt$. If $\sum \vec{\boldsymbol{\tau}} = \mathbf{0}$, then $d\vec{\boldsymbol{L}}/dt = \mathbf{0}$, and $\vec{\boldsymbol{L}}$ is constant.

CONSERVATION OF ANGULAR MOMENTUM **When the net external torque acting on a system is zero, the total angular momentum of the system is constant (conserved).**

A circus acrobat, a diver, and an ice skater pirouetting on one skate all take advantage of this principle. Suppose an acrobat has just left a swing; she has her arms and legs extended and is rotating counterclockwise about her center of mass. When she pulls her arms and legs in, her moment of inertia I_{cm} with respect to her center of mass changes from a large value I_1 to a much smaller value I_2. The only external force acting on her is her weight, which has no torque with respect to an axis through her center of mass. So her angular momentum $L_z = I_{cm}\omega_z$ remains constant, and her angular velocity ω_z increases as I_{cm} decreases. That is,

$$I_1\omega_{1z} = I_2\omega_{2z} \quad \text{(zero net external torque)} \tag{10.30}$$

When a skater or ballerina spins with arms outstretched and then pulls her arms in, her angular velocity increases as her moment of inertia decreases. In each case there is conservation of angular momentum in a system in which the net external torque is zero.

When a system has several parts, the internal forces that the parts exert on one another cause changes in the angular momenta of the parts, but the *total* angular momentum doesn't change. Here's an example. Consider two objects A and B that interact with each other but not with anything else, such as the astronauts we discussed in Section 8.2 (see Fig. 8.9). Suppose object A exerts a force $\vec{\boldsymbol{F}}_{A \text{ on } B}$ on object B; the corresponding torque (with respect to whatever point we choose) is $\vec{\boldsymbol{\tau}}_{A \text{ on } B}$. According to Eq. (10.29), this torque is equal to the rate of change of angular momentum of B:

$$\vec{\boldsymbol{\tau}}_{A \text{ on } B} = \frac{d\vec{\boldsymbol{L}}_B}{dt}$$

At the same time, object B exerts a force $\vec{\boldsymbol{F}}_{B \text{ on } A}$ on object A, with a corresponding torque $\vec{\boldsymbol{\tau}}_{B \text{ on } A}$, and

$$\vec{\boldsymbol{\tau}}_{B \text{ on } A} = \frac{d\vec{\boldsymbol{L}}_A}{dt}$$

From Newton's third law, $\vec{\boldsymbol{F}}_{B \text{ on } A} = -\vec{\boldsymbol{F}}_{A \text{ on } B}$. Furthermore, if the forces act along the same line, as in Fig. 10.8, their lever arms with respect to the chosen axis are equal. Thus the *torques* of these two forces are equal and opposite, and $\vec{\boldsymbol{\tau}}_{B \text{ on } A} = -\vec{\boldsymbol{\tau}}_{A \text{ on } B}$. So if we add the two preceding equations, we find

$$\frac{d\vec{\boldsymbol{L}}_A}{dt} + \frac{d\vec{\boldsymbol{L}}_B}{dt} = \mathbf{0}$$

or, because $\vec{\boldsymbol{L}}_A + \vec{\boldsymbol{L}}_B$ is the *total* angular momentum $\vec{\boldsymbol{L}}$ of the system,

$$\frac{d\vec{\boldsymbol{L}}}{dt} = \mathbf{0} \quad \text{(zero net external torque)} \tag{10.31}$$

That is, the total angular momentum of the system is constant. The torques of the internal forces can transfer angular momentum from one object to the other, but they can't change the *total* angular momentum of the system (**Fig. 10.28**).

Figure 10.28 A falling cat twists different parts of its body in different directions so that it lands feet first. At all times during this process the angular momentum of the cat as a whole remains zero.

EXAMPLE 10.10 Anyone can be a ballerina

A physics professor stands at the center of a frictionless turntable with arms outstretched and a 5.0 kg dumbbell in each hand (**Fig. 10.29**). He is set rotating about the vertical axis, making one revolution in 2.0 s. Find his final angular velocity if he pulls the dumbbells inward to his stomach. His moment of inertia (without the dumbbells) is $3.0\ \text{kg} \cdot \text{m}^2$ with arms outstretched and $2.2\ \text{kg} \cdot \text{m}^2$ with his hands at his stomach. The dumbbells are 1.0 m from the axis initially and 0.20 m at the end.

IDENTIFY, SET UP, and EXECUTE No external torques act about the z-axis, so L_z is constant. We'll use Eq. (10.30) to find the final angular velocity ω_{2z}. The moment of inertia of the system is $I = I_{\text{prof}} + I_{\text{dumbbells}}$. We treat each dumbbell as a particle of mass m that contributes mr^2 to $I_{\text{dumbbells}}$, where r is the perpendicular distance from the axis to the dumbbell. Initially we have

$$I_1 = 3.0\ \text{kg} \cdot \text{m}^2 + 2(5.0\ \text{kg})(1.0\ \text{m})^2 = 13\ \text{kg} \cdot \text{m}^2$$

$$\omega_{1z} = \frac{1\ \text{rev}}{2.0\ \text{s}} = 0.50\ \text{rev/s}$$

The final moment of inertia is

$$I_2 = 2.2\ \text{kg} \cdot \text{m}^2 + 2(5.0\ \text{kg})(0.20\ \text{m})^2 = 2.6\ \text{kg} \cdot \text{m}^2$$

From Eq. (10.30), the final angular velocity is

$$\omega_{2z} = \frac{I_1}{I_2}\omega_{1z} = \frac{13\ \text{kg} \cdot \text{m}^2}{2.6\ \text{kg} \cdot \text{m}^2}(0.50\ \text{rev/s}) = 2.5\ \text{rev/s} = 5\omega_{1z}$$

Can you see why we didn't have to change "revolutions" to "radians" in this calculation?

EVALUATE The angular momentum remained constant, but the angular velocity increased by a factor of 5, from $\omega_{1z} = (0.50\ \text{rev/s}) \times (2\pi\ \text{rad/rev}) = 3.14\ \text{rad/s}$ to $\omega_{2z} = (2.5\ \text{rev/s})(2\pi\ \text{rad/rev}) = 15.7\ \text{rad/s}$. The initial and final kinetic energies are then

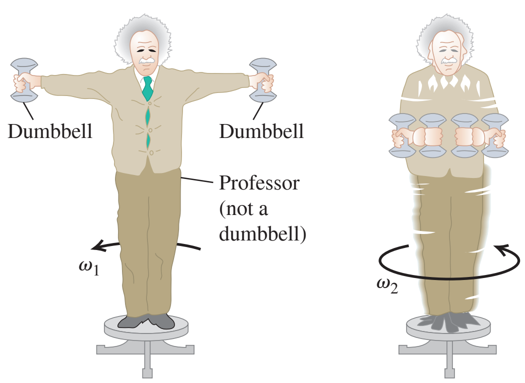
$$K_1 = \tfrac{1}{2}I_1\omega_{1z}{}^2 = \tfrac{1}{2}(13\ \text{kg} \cdot \text{m}^2)(3.14\ \text{rad/s})^2 = 64\ \text{J}$$

$$K_2 = \tfrac{1}{2}I_2\omega_{2z}{}^2 = \tfrac{1}{2}(2.6\ \text{kg} \cdot \text{m}^2)(15.7\ \text{rad/s})^2 = 320\ \text{J}$$

The fivefold increase in kinetic energy came from the work that the professor did in pulling his arms and the dumbbells inward.

KEYCONCEPT If there is zero net external torque on a rigid body, its angular momentum is conserved. If the body changes shape so that its moment of inertia changes, its angular velocity changes to keep the angular momentum constant.

Figure 10.29 Fun with conservation of angular momentum.



EXAMPLE 10.11 A rotational "collision"

WITH VARIATION PROBLEMS

Figure 10.30 shows two disks: an engine flywheel (A) and a clutch plate (B) attached to a transmission shaft. Their moments of inertia are I_A and I_B; initially, they are rotating in the same direction with constant angular speeds ω_A and ω_B, respectively. We push the disks together with forces acting along the axis, so as not to apply any torque on either disk. The disks rub against each other and eventually reach a common angular speed ω. Derive an expression for ω.

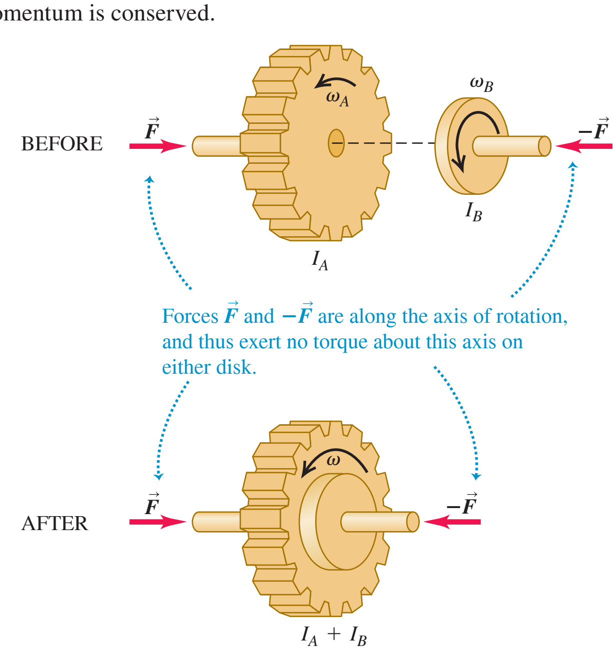
IDENTIFY, SET UP, and EXECUTE There are no external torques, so the only torque acting on either disk is the torque applied by the other disk. Hence the total angular momentum of the system of two disks is conserved. At the end they rotate together as one object with total moment of inertia $I = I_A + I_B$ and angular speed ω. Figure 10.30 shows that all angular velocities are in the same direction, so we can regard ω_A, ω_B, and ω as components of angular velocity along the rotation axis. Conservation of angular momentum gives

$$I_A\omega_A + I_B\omega_B = (I_A + I_B)\omega$$

$$\omega = \frac{I_A\omega_A + I_B\omega_B}{I_A + I_B}$$

EVALUATE This "collision" is analogous to a completely inelastic collision (see Section 8.3). When two objects in translational motion along the

Figure 10.30 When the net external torque is zero, angular momentum is conserved.



same axis collide and stick, the linear momentum of the system is conserved. Here two objects in *rotational* motion around the same axis "collide" and stick, and the *angular* momentum of the system is conserved.

The kinetic energy of a system decreases in a completely inelastic collision. Here kinetic energy is lost because nonconservative (friction) internal forces act while the two disks rub together. Suppose flywheel A has a mass of 2.0 kg, a radius of 0.20 m, and an initial angular speed of 50 rad/s (about 500 rpm), and clutch plate B has a mass of 4.0 kg, a radius of 0.10 m, and an initial angular speed of 200 rad/s. Can you show that the final kinetic energy is only two-thirds of the initial kinetic energy?

KEYCONCEPT In processes that conserve angular momentum, the kinetic energy can change if nonconservative forces act.

EXAMPLE 10.12 Angular momentum in a crime bust

WITH VARIATION PROBLEMS

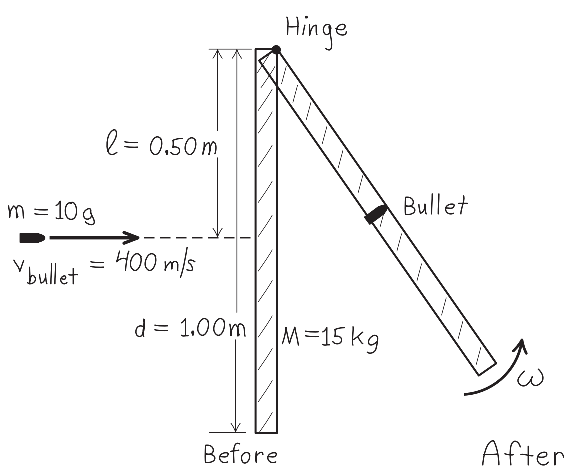
A door 1.00 m wide, of mass 15 kg, can rotate freely about a vertical axis through its hinges. A bullet with a mass of 10 g and a speed of 400 m/s strikes the center of the door, in a direction perpendicular to the plane of the door, and embeds itself there. Find the door's angular speed. Is kinetic energy conserved?

IDENTIFY and SET UP We consider the door and bullet as a system. There is no external torque about the hinge axis, so angular momentum about this axis is conserved. **Figure 10.31** shows our sketch. The initial angular momentum is that of the bullet, as given by Eq. (10.25). The final angular momentum is that of a rigid body composed of the door and the embedded bullet. We'll equate these quantities and solve for the resulting angular speed ω of the door and bullet.

EXECUTE From Eq. (10.25), the initial angular momentum of the bullet is

$$L = mvl = (0.010\text{ kg})(400\text{ m/s})(0.50\text{ m}) = 2.0\text{ kg}\cdot\text{m}^2/\text{s}$$

Figure 10.31 The swinging door seen from above.



The final angular momentum is $I\omega$, where $I = I_{\text{door}} + I_{\text{bullet}}$. From Table 9.2, case (d), for a door of width $d = 1.00$ m,

$$I_{\text{door}} = \frac{Md^2}{3} = \frac{(15\text{ kg})(1.00\text{ m})^2}{3} = 5.0\text{ kg}\cdot\text{m}^2$$

The moment of inertia of the bullet (with respect to the axis along the hinges) is

$$I_{\text{bullet}} = ml^2 = (0.010\text{ kg})(0.50\text{ m})^2 = 0.0025\text{ kg}\cdot\text{m}^2$$

Conservation of angular momentum requires that $mvl = I\omega$, or

$$\omega = \frac{mvl}{I} = \frac{2.0\text{ kg}\cdot\text{m}^2/\text{s}}{5.0\text{ kg}\cdot\text{m}^2 + 0.0025\text{ kg}\cdot\text{m}^2} = 0.40\text{ rad/s}$$

The initial and final kinetic energies are

$$K_1 = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(0.010\text{ kg})(400\text{ m/s})^2 = 800\text{ J}$$

$$K_2 = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}(5.0025\text{ kg}\cdot\text{m}^2)(0.40\text{ rad/s})^2 = 0.40\text{ J}$$

EVALUATE The final kinetic energy is only $\frac{1}{2000}$ of the initial value! We did not expect kinetic energy to be conserved: The collision is inelastic because nonconservative friction forces act during the impact. The door's final angular speed is quite slow: At 0.40 rad/s, it takes 3.9 s to swing through 90° ($\pi/2$ radians).

KEYCONCEPT The total angular momentum of a system that includes a rigid body and a particle is the sum of the angular momenta for the rigid body and for the particle. You can find the magnitude of the angular momentum of a particle about a rotation axis by multiplying the magnitude of its linear momentum by the perpendicular distance from the axis to the line of the particle's velocity.

TEST YOUR UNDERSTANDING OF SECTION 10.6 If the polar ice caps were to melt completely due to global warming, the melted ice would redistribute itself over the earth. This change would cause the length of the day (the time needed for the earth to rotate once on its axis) to (i) increase; (ii) decrease; (iii) remain the same. (*Hint:* Use angular momentum ideas. Assume that the sun, moon, and planets exert negligibly small torques on the earth.)

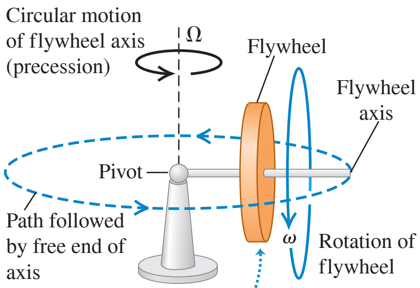
ANSWER

(i) In the absence of external torques, the earth's angular momentum $L_z = I\omega_z$ would remain constant. The melted ice would move from the poles toward the equator—that is, away from our planet's rotation axis—and the earth's moment of inertia I would increase slightly. Hence the angular velocity ω_z would decrease slightly and the day would be slightly longer.

10.7 GYROSCOPES AND PRECESSION

In all the situations we've looked at so far in this chapter, the axis of rotation either has stayed fixed or has moved and kept the same direction (such as rolling without slipping). But a variety of new physical phenomena, some quite unexpected, can occur when the axis of rotation changes direction. For example, consider a toy gyroscope that's supported at one end (**Fig. 10.32**). If we hold it with the flywheel axis horizontal and let go, the free end of the axis simply drops owing to gravity—*if* the flywheel isn't spinning. But if the flywheel *is* spinning, what happens is quite different. One possible motion is a steady circular motion of the axis in a horizontal plane, combined with the spin motion of the flywheel about the axis. This surprising, nonintuitive motion of the axis is called **precession.** Precession is found in nature as well as in rotating machines such as gyroscopes. As you read these words, the earth itself is precessing; its spin axis (through the north and south poles) slowly changes direction, going through a complete cycle of precession every 26,000 years.

Figure 10.32 A gyroscope supported at one end. The horizontal circular motion of the flywheel and axis is called precession. The angular speed of precession is Ω.



To study this strange phenomenon of precession, we must remember that angular velocity, angular momentum, and torque are all *vector* quantities. In particular, we need the general relationship between the net torque $\sum\vec{\tau}$ that acts on an object and the rate of change of the object's angular momentum $\vec{L}$, given by Eq. (10.29), $\sum\vec{\tau} = d\vec{L}/dt$. Let's first apply this equation to the case in which the flywheel is *not* spinning (**Fig. 10.33a**). We take the origin O at the pivot and assume that the flywheel is symmetrical, with mass M and moment of inertia I about the flywheel axis. The flywheel axis is initially along the x-axis. The only external forces on the gyroscope are the normal force $\vec{n}$ acting at the pivot (assumed to be frictionless) and the weight $\vec{w}$ of the flywheel that acts at its center of mass, a distance r from the pivot. The normal force has zero torque with respect to the pivot, and the weight has a torque $\vec{\tau}$ in the y-direction, as shown in Fig. 10.33a. Initially, there is no rotation, and the initial angular momentum $\vec{L}_i$ is zero. From Eq. (10.29) the *change* $d\vec{L}$ in angular momentum in a short time interval dt following this is

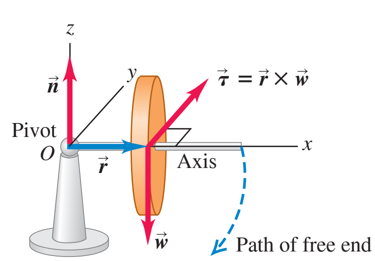
$$d\vec{L} = \vec{\tau}\,dt \qquad (10.32)$$

This change is in the y-direction because $\vec{\tau}$ is. As each additional time interval dt elapses, the angular momentum changes by additional increments $d\vec{L}$ in the y-direction because the direction of the torque is constant (Fig. 10.33b). The steadily increasing horizontal angular momentum means that the gyroscope rotates downward faster and faster around the y-axis until it hits either the stand or the table on which it sits.

Now let's see what happens if the flywheel *is* spinning initially, so the initial angular momentum $\vec{L}_i$ is not zero (**Fig. 10.34a**). Since the flywheel rotates around its symmetry axis, $\vec{L}_i$ lies along this axis. But each change in angular momentum $d\vec{L}$ is perpendicular to the flywheel axis because the torque $\vec{\tau} = \vec{r} \times \vec{w}$ is perpendicular to that axis (Fig. 10.34b).

Figure 10.33 (a) If the flywheel in Fig. 10.32 is initially not spinning, its initial angular momentum is zero. (b) In each successive time interval dt, the torque produces a change $d\vec{L} = \vec{\tau}\,dt$ in the angular momentum. The flywheel acquires an angular momentum $\vec{L}$ in the same direction as $\vec{\tau}$, and the flywheel axis falls.

(a) Nonrotating flywheel falls



(b) View from above as flywheel falls

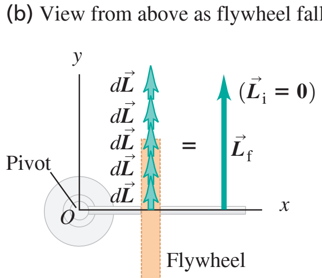
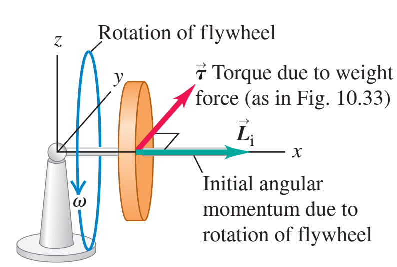
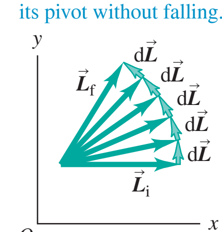


Figure 10.34 (a) The flywheel is spinning initially with angular momentum $\vec{L}_i$. The forces (not shown) are the same as those in Fig. 10.33a. (b) Because the initial angular momentum is not zero, each change $d\vec{L} = \vec{\tau}\,dt$ in angular momentum is perpendicular to $\vec{L}$. As a result, the magnitude of $\vec{L}$ remains the same but its direction changes continuously.

(a) Rotating flywheel



(b) View from above



This causes the *direction* of $\vec{L}$ to change, but not its magnitude. The changes $d\vec{L}$ are always in the horizontal xy-plane, so the angular momentum vector and the flywheel axis with which it moves are always horizontal. That is, the axis doesn't fall—it precesses.

If this still seems mystifying to you, think about a ball attached to a string. If the ball is initially at rest and you pull the string toward you, the ball moves toward you also. But if the ball is initially moving and you continuously pull the string in a direction perpendicular to the ball's motion, the ball moves in a circle around your hand; it does not approach your hand at all. In the first case the ball has zero linear momentum $\vec{p}$ to start with; when you apply a force $\vec{F}$ toward you for a time dt, the ball acquires a momentum $d\vec{p} = \vec{F}\,dt$, which is also toward you. But if the ball already has linear momentum $\vec{p}$, a change in momentum $d\vec{p}$ that's perpendicular to $\vec{p}$ changes the direction of motion, not the speed. Replace $\vec{p}$ with $\vec{L}$ and $\vec{F}$ with $\vec{\tau}$ in this argument, and you'll see that precession is simply the rotational analog of uniform circular motion.

At the instant shown in Fig. 10.34a, the gyroscope has angular momentum $\vec{L}$. A short time interval dt later, the angular momentum is $\vec{L} + d\vec{L}$; the infinitesimal change in angular momentum is $d\vec{L} = \vec{\tau}\,dt$, which is perpendicular to $\vec{L}$. As the vector diagram in **Fig. 10.35** shows, this means that the flywheel axis of the gyroscope has turned through a small angle $d\phi$ given by $d\phi = |d\vec{L}|/|\vec{L}|$. The rate at which the axis moves, $d\phi/dt$, is called the **precession angular speed;** denoting this quantity by Ω, we find

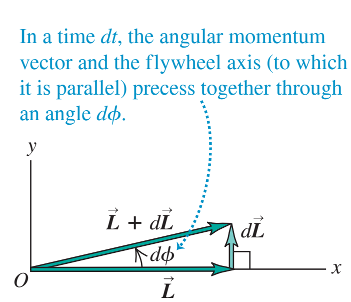
$$\Omega = \frac{d\phi}{dt} = \frac{|d\vec{L}|/|\vec{L}|}{dt} = \frac{\tau_z}{L_z} = \frac{wr}{I\omega} \tag{10.33}$$

Thus the precession angular speed is *inversely* proportional to the angular speed of spin about the axis. A rapidly spinning gyroscope precesses slowly; if friction in its bearings causes the flywheel to slow down, the precession angular speed *increases*! The precession angular speed of the earth is very slow (1 rev/26,000 yr) because its spin angular momentum L_z is large and the torque τ_z, due to the gravitational influences of the moon and sun, is relatively small.

As a gyroscope precesses, its center of mass moves in a circle with radius r in a horizontal plane. Its vertical component of acceleration is zero, so the upward normal force $\vec{n}$ exerted by the pivot must be just equal in magnitude to the weight. The circular motion of the center of mass with angular speed Ω requires a force $\vec{F}$ directed toward the center of the circle, with magnitude $F = M\Omega^2 r$. This force must also be supplied by the pivot.

One key assumption that we made in our analysis of the gyroscope was that the angular momentum vector $\vec{L}$ is associated with only the spin of the flywheel and is purely horizontal. But there will also be a vertical component of angular momentum associated with the precessional motion of the gyroscope. By ignoring this, we've tacitly assumed that the precession is *slow*—that is, that the precession angular speed Ω is very much less than the spin angular speed ω. As Eq. (10.33) shows, a large value of ω automatically gives a small value of Ω, so this approximation is reasonable. When the precession is not slow, additional effects show up, including an up-and-down wobble or *nutation* of the flywheel axis that's superimposed on the precessional motion. You can see nutation occurring in a gyroscope as its spin slows down, so that Ω increases and the vertical component of $\vec{L}$ can no longer be ignored.

Figure **10.35** Detailed view of part of Fig. 10.34b.

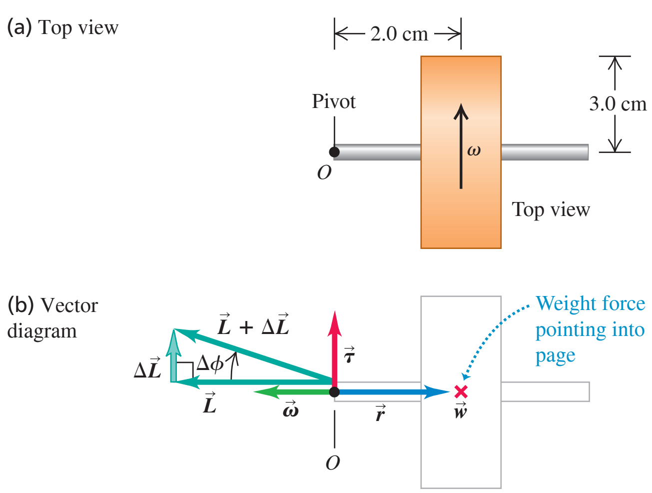


EXAMPLE 10.13 A precessing gyroscope

Figure 10.36a shows a top view of a spinning, cylindrical gyroscope wheel. The pivot is at O, and the mass of the axle is negligible. (a) As seen from above, is the precession clockwise or counterclockwise? (b) If the gyroscope takes 4.0 s for one revolution of precession, what is the angular speed of the wheel?

IDENTIFY and SET UP We'll determine the direction of precession by using the right-hand rule as in Fig. 10.34, which shows the same kind of gyroscope as Fig. 10.36. We'll use the relationship between precession angular speed Ω and spin angular speed ω, Eq. (10.33), to find ω.

Figure **10.36** In which direction and at what speed does this gyroscope process?



EXECUTE (a) The right-hand rule shows that $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{L}}$ are to the left in Fig. 10.36b. The weight $\vec{\boldsymbol{w}}$ points into the page in this top view and acts at the center of mass (denoted by $\times$ in the figure). The torque $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{w}}$ is toward the top of the page, so $d\vec{\boldsymbol{L}}/dt$ is also toward the top of the page. Adding a small $d\vec{\boldsymbol{L}}$ to the initial vector $\vec{\boldsymbol{L}}$ changes the direction of $\vec{\boldsymbol{L}}$ as shown, so the precession is clockwise as seen from above.

(b) Be careful not to confuse ω and Ω! The precession angular speed is $\Omega = (1\text{ rev})/(4.0\text{ s}) = (2\pi\text{ rad})/(4.0\text{ s}) = 1.57\text{ rad/s}$. The weight is mg, and if the wheel is a solid, uniform cylinder, its moment of inertia about its symmetry axis is $I = \frac{1}{2}mR^2$. From Eq. (10.33),

$$\begin{aligned}\omega &= \frac{wr}{I\Omega} = \frac{mgr}{(mR^2/2)\Omega} = \frac{2gr}{R^2\Omega}\\ &= \frac{2(9.8\text{ m/s}^2)(2.0\times10^{-2}\text{ m})}{(3.0\times10^{-2}\text{ m})^2(1.57\text{ rad/s})}\\ &= 280\text{ rad/s} = 2600\text{ rev/min}\end{aligned}$$

EVALUATE The precession angular speed Ω is only about 0.6% of the spin angular speed ω, so this is an example of slow precession.

KEYCONCEPT A spinning rigid body will precess if the net external torque on the rigid body is perpendicular to the body's angular momentum vector.

TEST YOUR UNDERSTANDING OF SECTION 10.7 Suppose the mass of the flywheel in Fig. 10.34 is doubled but all other dimensions and the spin angular speed remain the same. What effect would this change have on the precession angular speed Ω? (i) Ω would increase by a factor of 4; (ii) Ω would double; (iii) Ω would be unaffected; (iv) Ω would be one-half as much; (v) Ω would be one-quarter as much.

ANSWER

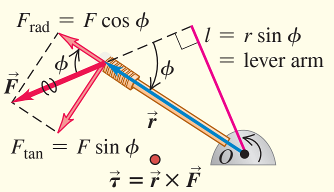
(iii) Doubling the flywheel mass would double both its moment of inertia I and its weight w, so the ratio I/w would be unchanged. Equation (10.33) shows that the precession angular speed depends on this ratio, so there would be *no* effect on the value of Ω.

CHAPTER 10 SUMMARY

Torque: When a force $\vec{F}$ acts on an object, the torque of that force with respect to a point O has a magnitude given by the product of the force magnitude F and the lever arm l. More generally, torque is a vector $\vec{\tau}$ equal to the vector product of $\vec{r}$ (the position vector of the point at which the force acts) and $\vec{F}$. (See Example 10.1.)

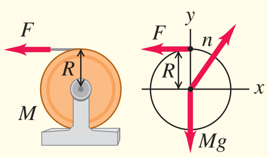
$$\tau = Fl = rF\sin\phi = F_{\tan}r \tag{10.2}$$

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{10.3}$$



Rotational dynamics: The rotational analog of Newton's second law says that the net torque acting on an object equals the product of the object's moment of inertia and its angular acceleration. (See Examples 10.2 and 10.3.)

$$\sum\tau_z = I\alpha_z \tag{10.7}$$



Combined translation and rotation: If a rigid body is both moving through space and rotating, its motion can be regarded as translational motion of the center of mass plus rotational motion about an axis through the center of mass. Thus the kinetic energy is a sum of translational and rotational kinetic energies. For dynamics, Newton's second law describes the motion of the center of mass, and the rotational equivalent of Newton's second law describes rotation about the center of mass. In the case of rolling without slipping, there is a special relationship between the motion of the center of mass and the rotational motion. (See Examples 10.4–10.7.)

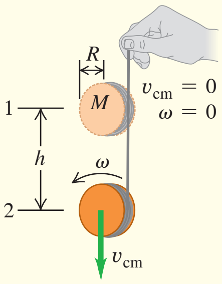
$$K = \tfrac{1}{2}Mv_{cm}^2 + \tfrac{1}{2}I_{cm}\omega^2 \tag{10.8}$$

$$\sum\vec{F}_{ext} = M\vec{a}_{cm} \tag{10.12}$$

$$\sum\tau_z = I_{cm}\alpha_z \tag{10.13}$$

$$v_{cm} = R\omega \tag{10.11}$$

(rolling without slipping)



Work done by a torque: A torque that acts on a rigid body as it rotates does work on that body. The work can be expressed as an integral of the torque. The work–energy theorem says that the total rotational work done on a rigid body is equal to the change in rotational kinetic energy. The power, or rate at which the torque does work, is the product of the torque and the angular velocity (See Example 10.8.)

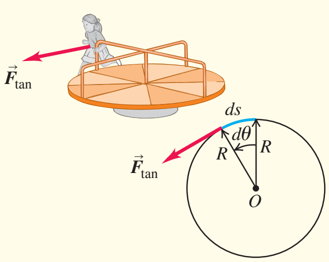
$$W = \int_{\theta_1}^{\theta_2} \tau_z\, d\theta \tag{10.20}$$

$$W = \tau_z(\theta_2 - \theta_1) = \tau_z\Delta\theta \tag{10.21}$$

(constant torque only)

$$W_{tot} = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2 \tag{10.22}$$

$$P = \tau_z\omega_z \tag{10.23}$$



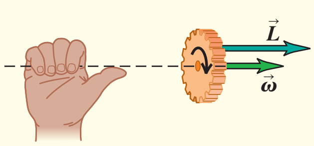
Angular momentum: The angular momentum of a particle with respect to point O is the vector product of the particle's position vector $\vec{r}$ relative to O and its momentum $\vec{p} = m\vec{v}$. When a symmetrical object rotates about a stationary axis of symmetry, its angular momentum is the product of its moment of inertia and its angular velocity vector $\vec{\omega}$. If the object is not symmetrical or the rotation (z) axis is not an axis of symmetry, the component of angular momentum along the rotation axis is $I\omega_z$. (See Example 10.9.)

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \tag{10.24}$$

(particle)

$$\vec{L} = I\vec{\omega} \tag{10.28}$$

(rigid body rotating about axis of symmetry)

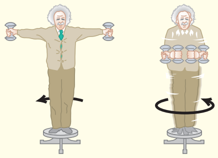


Rotational dynamics and angular momentum: The net external torque on a system is equal to the rate of change of its angular momentum. If the net external torque on a system is zero, the total angular momentum of the system is constant (conserved). (See Examples 10.10–10.13.)

$$\sum\vec{\tau} = \frac{d\vec{L}}{dt} \tag{10.29}$$

$$\frac{d\vec{L}}{dt} = \mathbf{0} \tag{10.31}$$

(zero net external torque)



Chapter 10 Media Assets

GUIDED PRACTICE

For assigned homework and other learning materials, go to Mastering Physics.

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 10.2 and 10.3 (Section 10.2) before attempting these problems.

VP10.3.1 In the cylinder and cable apparatus of Example 10.2, you apply a force to the cable so that a point on the horizontal part of the cable accelerates to the left at 0.60 m/s^2. What are the magnitudes of (a) the angular acceleration of the cylinder, (b) the torque that the cable exerts on the cylinder, and (c) the force that you exert on the cable?

VP10.3.2 In the cylinder, cable, and block apparatus of Example 10.3, you replace the solid cylinder with a thin-walled, hollow cylinder of mass M and radius R. Find (a) the acceleration of the falling block and (b) the tension in the cable as the block falls.

VP10.3.3 A bucket of mass m is hanging from the free end of a rope whose other end is wrapped around a drum (radius R, mass M) that can rotate with negligible friction about a stationary horizontal axis. The drum is not a uniform cylinder and has unknown moment of inertia. When you release the bucket from rest, you find that it has a downward acceleration of magnitude a. What are (a) the tension in the cable between the drum and the bucket and (b) the moment of inertia of the drum about its rotation axis?

VP10.3.4 In the cylinder, cable, and block apparatus of Example 10.3, you attach an electric motor to the axis of the cylinder of mass M and radius R and turn the motor on. As a result the block of mass m moves upward with an upward acceleration of magnitude a. What are (a) the tension in the cable between the cylinder and the block, (b) the magnitude of the torque that the cable exerts on the cylinder, and (c) the magnitude of the torque that the motor exerts on the cylinder?

Be sure to review EXAMPLES 10.6 and 10.7 (Section 10.3) before attempting these problems.

VP10.7.1 In the primitive yo-yo apparatus of Example 10.6, you replace the solid cylinder with a hollow cylinder of mass M, outer radius R, and inner radius $R/2$. Find (a) the downward acceleration of the hollow cylinder and (b) the tension in the string.

VP10.7.2 A thin-walled, hollow sphere of mass M rolls without slipping down a ramp that is inclined at an angle β to the horizontal. Find (a) the acceleration of the sphere, (b) the magnitude of the friction force that the ramp exerts on the sphere, and (c) the magnitude of the torque that this force exerts on the sphere.

VP10.7.3 You redo the primitive yo-yo experiment of Example 10.6, but instead of holding the free end of the string stationary, you move your hand vertically so that the tension in the string equals $2Mg/3$. (a) What is the magnitude of the vertical acceleration of the yo-yo's center of mass? Does it accelerate upward or downward? (b) What is the angular acceleration of the yo-yo around its axis?

VP10.7.4 You place a solid cylinder of mass M on a ramp that is inclined at an angle β to the horizontal. The coefficient of static friction for the cylinder on the ramp is μ_s. (a) If the cylinder rolls downhill without slipping, what is the magnitude of the friction force that the ramp exerts on the cylinder? (b) You find by varying the angle of the ramp that the cylinder rolls without slipping if β is less than a certain critical value but the cylinder slips if β is greater than this critical value. What is this critical value of β?

Be sure to review EXAMPLES 10.11 and 10.12 (Section 10.6) before attempting these problems.

VP10.12.1 In the situation shown in Example 10.11, suppose disk A has moment of inertia I_A and initial angular speed ω_A, while disk B has moment of inertia $I_A/4$ and initial angular speed $\omega_A/2$. Initially disks A and B are rotating in the *same* direction. (a) What is the final common angular speed of the two disks? (b) What fraction of the initial rotational kinetic energy remains as rotational kinetic energy after the disks have come to their final common angular speed?

VP10.12.2 In the situation shown in Example 10.11, suppose disk A has moment of inertia I_A and initial angular speed ω_A, while disk B has moment of inertia $I_A/4$ and initial angular speed $\omega_A/2$. Initially disks A and B are rotating in *opposite* directions. (a) What is the final common angular speed of the two disks? (b) What fraction of the initial rotational kinetic energy remains as rotational kinetic energy after the disks have come to their final common angular speed?

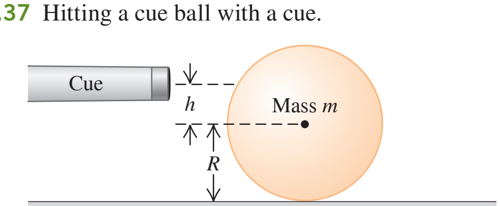
VP10.12.3 Suppose that instead of hitting the center of the door, the bullet in Example 10.12 strikes the door at the edge farthest away from the hinge and embeds itself there. (a) What is the angular speed of the door just after the bullet embeds itself? (b) What fraction of the initial kinetic energy of the bullet remains as kinetic energy after the collision?

VP10.12.4 A thin-walled, hollow sphere of mass M and radius R is free to rotate around a vertical shaft that passes through the center of the sphere. Initially the sphere is at rest. A small ball of clay of the same mass M moving horizontally at speed v grazes the surface of the sphere at its equator. After grazing the surface, the ball of clay is moving at speed $v/2$. (a) What is the angular momentum of the ball of clay about the shaft before it grazes the surface? After it grazes the surface? (b) What is the angular speed of the sphere after being grazed by the ball of clay? (c) What fraction of the ball of clay's initial kinetic energy remains as the combined kinetic energy of the sphere and the ball of clay?

BRIDGING PROBLEM Billiard Physics

A cue ball (a uniform solid sphere of mass m and radius R) is at rest on a level pool table. Using a pool cue, you give the ball a sharp, horizontal hit of magnitude F at a height h above the center of the ball (**Fig. 10.37**). The force of the hit is much greater than the friction force f that the table surface exerts on the ball. The hit lasts for a short time Δt. (a) For what value of h will the ball roll without slipping? (b) If you hit the ball dead center ($h = 0$), the ball will slide across the table for a while, but eventually it will roll without slipping. What will the speed of its center of mass be then?

Figure 10.37 Hitting a cue ball with a cue.



SOLUTION GUIDE

IDENTIFY and SET UP

1. Draw a free-body diagram for the ball for the situation in part (a), including your choice of coordinate axes. Note that the cue exerts both an impulsive force on the ball and an impulsive torque around the center of mass.
2. The cue force applied for a time Δt gives the ball's center of mass a speed v_{cm}, and the cue torque applied for that same time gives the ball an angular speed ω. How must v_{cm} and ω be related for the ball to roll without slipping?
3. Draw two free-body diagrams for the ball in part (b): one showing the forces during the hit and the other showing the forces after the hit but before the ball is rolling without slipping.
4. What is the angular speed of the ball in part (b) just after the hit? While the ball is sliding, does v_{cm} increase or decrease? Does ω increase or decrease? What is the relationship between v_{cm} and ω when the ball is finally rolling without slipping?

EXECUTE

5. In part (a), use the impulse–momentum theorem to find the speed of the ball's center of mass immediately after the hit. Then use the rotational version of the impulse–momentum theorem to find the angular speed immediately after the hit. (*Hint:* To write the rotational version of the impulse–momentum theorem, remember that the relationship between torque and angular momentum is the same as that between force and linear momentum.)
6. Use your results from step 5 to find the value of h that will cause the ball to roll without slipping immediately after the hit.
7. In part (b), again find the ball's center-of-mass speed and angular speed immediately after the hit. Then write Newton's second law for the translational motion and rotational motion of the ball as it slides. Use these equations to write expressions for v_{cm} and ω as functions of the elapsed time t since the hit.
8. Using your results from step 7, find the time t when v_{cm} and ω have the correct relationship for rolling without slipping. Then find the value of v_{cm} at this time.

EVALUATE

9. If you have access to a pool table, test the results of parts (a) and (b) for yourself!
10. Can you show that if you used a hollow cylinder rather than a solid ball, you would have to hit the top of the cylinder to cause rolling without slipping as in part (a)?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q10.1 Can a single force applied to an object change both its translational and rotational motions? Explain.

Q10.2 Suppose you could use wheels of any type in the design of a soapbox-derby racer (an unpowered, four-wheel vehicle that coasts from rest down a hill). To conform to the rules on the total weight of the vehicle and rider, should you design with large massive wheels or small light wheels? Should you use solid wheels or wheels with most of the mass at the rim? Explain.

Q10.3 Serious bicyclists say that if you reduce the weight of a bike, it is more effective if you do so in the wheels rather than in the frame. Why would reducing weight in the wheels make it easier on the bicyclist than reducing the same amount in the frame?

Q10.4 The harder you hit the brakes while driving forward, the more the front end of your car will move down (and the rear end move up). Why? What happens when cars accelerate forward? Why do drag racers not use front-wheel drive only?

Q10.5 When an acrobat walks on a tightrope, she extends her arms straight out from her sides. She does this to make it easier for her to catch herself if she should tip to one side or the other. Explain how this works. [*Hint:* Think about Eq. (10.7).]

Q10.6 When you turn on an electric motor, it takes longer to come up to final speed if a grinding wheel is attached to the shaft. Why?

Q10.7 The work done by a force is the product of force and distance. The torque due to a force is the product of force and distance. Does this mean that torque and work are equivalent? Explain.

Q10.8 A valued client brings a treasured ball to your engineering firm, wanting to know whether the ball is solid or hollow. He has tried tapping on it, but that has given insufficient information. Design a simple, inexpensive experiment that you could perform quickly, without injuring the precious ball, to find out whether it is solid or hollow.

Q10.9 You make two versions of the same object out of the same material having uniform density. For one version, all the dimensions are exactly twice as great as for the other one. If the same torque acts on both versions, giving the smaller version angular acceleration α, what will be the angular acceleration of the larger version in terms of α?

Q10.10 Two identical masses are attached to frictionless pulleys by very light strings wrapped around the rim of the pulley and are released from rest. Both pulleys have the same mass and same diameter, but one is solid and the other is a hoop. As the masses fall, in which case is the tension in the string greater, or is it the same in both cases? Justify your answer.

Q10.11 The force of gravity acts on the baton in Fig. 10.11, and forces produce torques that cause a body's angular velocity to change. Why, then, is the angular velocity of the baton in the figure constant?

Q10.12 Without slipping, a certain solid uniform ball rolls at speed v on a horizontal surface and then up a hill to a maximum height h_0. How does the maximum height change (in terms of h_0) if you make only the following changes: (a) double the ball's diameter, (b) double its mass, (c) double both its diameter and mass, (d) double its angular speed at the bottom of the hill?

Q10.13 A wheel is rolling without slipping on a horizontal surface. In an inertial frame of reference in which the surface is at rest, is there any point on the wheel that has a velocity that is purely vertical? Is there any point that has a horizontal velocity component opposite to the velocity of the center of mass? Explain. Do your answers change if the wheel is slipping as it rolls? Why or why not?

Q10.14 A hoop, a uniform solid cylinder, a spherical shell, and a uniform solid sphere are released from rest at the top of an incline. What is the order in which they arrive at the bottom of the incline? Does it matter whether or not the masses and radii of the objects are all the same? Explain.

Q10.15 A ball is rolling along at speed v without slipping on a horizontal surface when it comes to a hill that rises at a constant angle above the horizontal. In which case will it go higher up the hill: if the hill has enough friction to prevent slipping or if the hill is perfectly smooth? Justify your answer in both cases in terms of energy conservation and in terms of Newton's second law.

Q10.16 You are standing at the center of a large horizontal turntable in a carnival funhouse. The turntable is set rotating on frictionless bearings, and it rotates freely (that is, there is no motor driving the turntable). As you walk toward the edge of the turntable, what happens to the combined angular momentum of you and the turntable? What happens to the rotation speed of the turntable? Explain.

Q10.17 **Global Warming.** If the earth's climate continues to warm, ice near the poles will melt, and the water will be added to the oceans. What effect will this have on the length of the day? Justify your answer.

Q10.18 If two spinning objects have the same angular momentum, do they necessarily have the same rotational kinetic energy? If they have the same rotational kinetic energy, do they necessarily have the same angular momentum? Explain.

Q10.19 A student is sitting on a frictionless rotating stool with her arms outstretched as she holds equal heavy weights in each hand. If she suddenly lets go of the weights, will her angular speed increase, stay the same, or decrease? Explain.

Q10.20 A point particle travels in a straight line at constant speed, and the closest distance it comes to the origin of coordinates is a distance l. With respect to this origin, does the particle have nonzero angular momentum? As the particle moves along its straight-line path, does its angular momentum with respect to the origin change?

Q10.21 In Example 10.10 (Section 10.6) the angular speed ω changes, and this must mean that there is nonzero angular acceleration. But there is no torque about the rotation axis if the forces the professor applies to the weights are directly, radially inward. Then, by Eq. (10.7), α_z must be zero. Explain what is wrong with this reasoning that leads to this apparent contradiction.

Q10.22 In Example 10.10 (Section 10.6) the rotational kinetic energy of the professor and dumbbells increases. But since there are no external torques, no work is being done to change the rotational kinetic energy. Then, by Eq. (10.22), the kinetic energy must remain the same! Explain what is wrong with this reasoning, which leads to an apparent contradiction. Where *does* the extra kinetic energy come from?

Q10.23 As discussed in Section 10.6, the angular momentum of a circus acrobat is conserved as she tumbles through the air. Is her *linear* momentum conserved? Why or why not?

Q10.24 If you stop a spinning raw egg for the shortest possible instant and then release it, the egg will start spinning again. If you do the same to a hard-boiled egg, it will remain stopped. Try it. Explain it.

Q10.25 A helicopter has a large main rotor that rotates in a horizontal plane and provides lift. There is also a small rotor on the tail that rotates in a vertical plane. What is the purpose of the tail rotor? (*Hint:* If there were no tail rotor, what would happen when the pilot changed the angular speed of the main rotor?) Some helicopters have no tail rotor, but instead have two large main rotors that rotate in a horizontal plane. Why is it important that the two main rotors rotate in opposite directions?

Q10.26 In a common design for a gyroscope, the flywheel and flywheel axis are enclosed in a light, spherical frame with the flywheel at the center of the frame. The gyroscope is then balanced on top of a pivot so that the flywheel is directly above the pivot. Does the gyroscope precess if it is released while the flywheel is spinning? Explain.

Q10.27 A gyroscope is precessing about a vertical axis. What happens to the precession angular speed if the following changes are made, with all other variables remaining the same? (a) The angular speed of the spinning flywheel is doubled; (b) the total weight is doubled; (c) the moment of inertia about the axis of the spinning flywheel is doubled; (d) the distance from the pivot to the center of gravity is doubled. (e) What happens if all of the variables in parts (a) through (d) are doubled? In each case justify your answer.

Q10.28 A gyroscope takes 3.8 s to precess 1.0 revolution about a vertical axis. Two minutes later, it takes only 1.9 s to precess 1.0 revolution. No one has touched the gyroscope. Explain.

Q10.29 A gyroscope is precessing as in Fig. 10.32. What happens if you gently add some weight to the end of the flywheel axis farthest from the pivot?

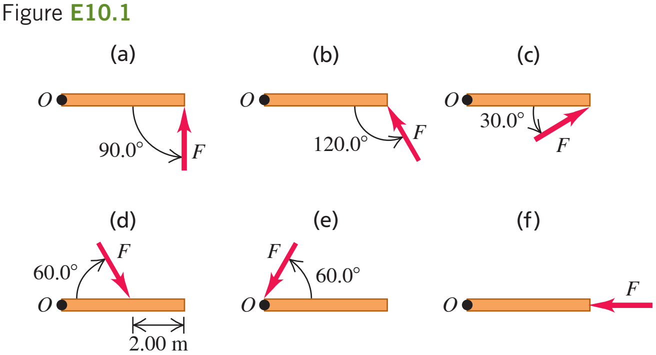
Q10.30 A bullet spins on its axis as it emerges from a rifle. Explain how this prevents the bullet from tumbling and keeps the streamlined end pointed forward.

EXERCISES

Section 10.1 Torque

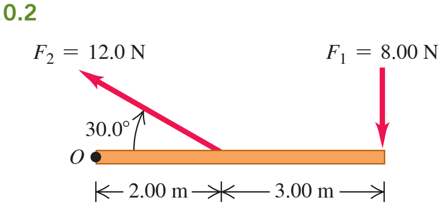
10.1 • Calculate the torque (magnitude and direction) about point O due to the force $\vec{F}$ in each of the cases sketched in **Fig. E10.1**. In each case, both the force $\vec{F}$ and the rod lie in the plane of the page, the rod has length 4.00 m, and the force has magnitude $F = 10.0$ N.

Figure **E10.1**



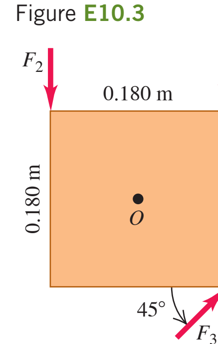
10.2 • Calculate the net torque about point O for the two forces applied as in **Fig. E10.2**. The rod and both forces are in the plane of the page.

Figure **E10.2**



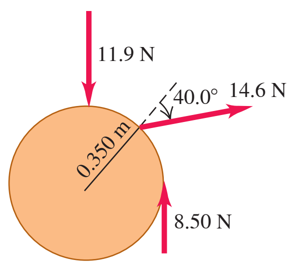
10.3 •• A square metal plate 0.180 m on each side is pivoted about an axis through point O at its center and perpendicular to the plate (**Fig. E10.3**). Calculate the net torque about this axis due to the three forces shown in the figure if the magnitudes of the forces are $F_1 = 18.0$ N, $F_2 = 26.0$ N, and $F_3 = 14.0$ N. The plate and all forces are in the plane of the page.

Figure **E10.3**



10.4 • Three forces are applied to a wheel of radius 0.350 m, as shown in **Fig. E10.4**. One force is perpendicular to the rim, one is tangent to it, and the other one makes a 40.0° angle with the radius. What is the net torque on the wheel due to these three forces for an axis perpendicular to the wheel and passing through its center?

Figure **E10.4**

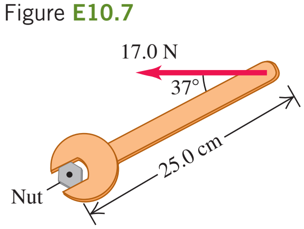


10.5 • One force acting on a machine part is $\vec{F} = (-5.00\text{ N})\hat{\imath} + (4.00\text{ N})\hat{\jmath}$. The vector from the origin to the point where the force is applied is $\vec{r} = (-0.450\text{ m})\hat{\imath} + (0.150\text{ m})\hat{\jmath}$. (a) In a sketch, show $\vec{r}$, $\vec{F}$, and the origin. (b) Use the right-hand rule to determine the direction of the torque. (c) Calculate the vector torque for an axis at the origin produced by this force. Verify that the direction of the torque is the same as you obtained in part (b).

10.6 • A metal bar is in the xy-plane with one end of the bar at the origin. A force $\vec{F} = (7.00\text{ N})\hat{\imath} + (-3.00\text{ N})\hat{\jmath}$ is applied to the bar at the point $x = 3.00$ m, $y = 4.00$ m. (a) In terms of unit vectors $\hat{\imath}$ and $\hat{\jmath}$, what is the position vector $\vec{r}$ for the point where the force is applied? (b) What are the magnitude and direction of the torque with respect to the origin produced by $\vec{F}$?

10.7 • A machinist is using a wrench to loosen a nut. The wrench is 25.0 cm long, and he exerts a 17.0 N force at the end of the handle at 37° with the handle (**Fig. E10.7**). (a) What torque does the machinist exert about the center of the nut? (b) What is the maximum torque he could exert with a force of this magnitude, and how should the force be oriented?

Figure **E10.7**



Section 10.2 Torque and Angular Acceleration for a Rigid Body

10.8 •• A uniform disk with mass 40.0 kg and radius 0.200 m is pivoted at its center about a horizontal, frictionless axle that is stationary. The disk is initially at rest, and then a constant force $F = 30.0$ N is applied tangent to the rim of the disk. (a) What is the magnitude v of the tangential velocity of a point on the rim of the disk after the disk has turned through 0.200 revolution? (b) What is the magnitude a of the resultant acceleration of a point on the rim of the disk after the disk has turned through 0.200 revolution?

10.9 •• The flywheel of an engine has moment of inertia $1.60\text{ kg}\cdot\text{m}^2$ about its rotation axis. What constant torque is required to bring it up to an angular speed of 400 rev/min in 8.00 s, starting from rest?

10.10 • A cord is wrapped around the rim of a solid uniform wheel 0.250 m in radius and of mass 9.20 kg. A steady horizontal pull of 40.0 N to the right is exerted on the cord, pulling it off tangentially from the wheel. The wheel is mounted on frictionless bearings on a horizontal axle through its center. (a) Compute the angular acceleration of the wheel and the acceleration of the part of the cord that has already been pulled off the wheel. (b) Find the magnitude and direction of the force that the axle exerts on the wheel. (c) Which of the answers in parts (a) and (b) would change if the pull were upward instead of horizontal?

10.11 •• A machine part has the shape of a solid uniform sphere of mass 225 g and diameter 3.00 cm. It is spinning about a frictionless axle through its center, but at one point on its equator it is scraping against metal, resulting in a friction force of 0.0200 N at that point. (a) Find its angular acceleration. (b) How long will it take to decrease its rotational speed by 22.5 rad/s?

10.12 •• **CP** A stone is suspended from the free end of a wire that is wrapped around the outer rim of a pulley, similar to what is shown in Fig. 10.10. The pulley is a uniform disk with mass 10.0 kg and radius 30.0 cm and turns on frictionless bearings. You measure that the stone travels 12.6 m in the first 3.00 s starting from rest. Find (a) the mass of the stone and (b) the tension in the wire.

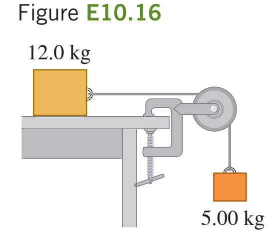
10.13 •• **CP** A 2.00 kg textbook rests on a frictionless, horizontal surface. A cord attached to the book passes over a pulley whose diameter is 0.150 m, to a hanging book with mass 3.00 kg. The system is released from rest, and the books are observed to move 1.20 m in 0.800 s. (a) What is the tension in each part of the cord? (b) What is the moment of inertia of the pulley about its rotation axis?

10.14 •• **CP** A 15.0 kg bucket of water is suspended by a very light rope wrapped around a solid uniform cylinder 0.300 m in diameter with mass 12.0 kg. The cylinder pivots on a frictionless axle through its center. The bucket is released from rest at the top of a well and falls 10.0 m to the water. (a) What is the tension in the rope while the bucket is falling? (b) With what speed does the bucket strike the water? (c) What is the time of fall? (d) While the bucket is falling, what is the force exerted on the cylinder by the axle?

10.15 • A wheel rotates without friction about a stationary horizontal axis at the center of the wheel. A constant tangential force equal to 80.0 N is applied to the rim of the wheel. The wheel has radius 0.120 m. Starting from rest, the wheel has an angular speed of 12.0 rev/s after 2.00 s. What is the moment of inertia of the wheel?

10.16 •• A 12.0 kg box resting on a horizontal, frictionless surface is attached to a 5.00 kg weight by a thin, light wire that passes over a frictionless pulley (**Fig. E10.16**). The pulley has the shape of a uniform solid disk of mass 2.00 kg and diameter 0.500 m. After the system is released, find (a) the tension in the wire on both sides of the pulley, (b) the acceleration of the box, and (c) the horizontal and vertical components of the force that the axle exerts on the pulley.

Figure **E10.16**



10.17 •• **CP** A solid cylinder with radius 0.140 m is mounted on a frictionless, stationary axle that lies along the cylinder axis. The cylinder is initially at rest. Then starting at $t = 0$ a constant horizontal force of 3.00 N is applied tangential to the surface of the cylinder. You measure the angular displacement $\theta - \theta_0$ of the cylinder as a function of the time t since the force was first applied. When you plot $\theta - \theta_0$ (in radians) as a function of t^2 (in s^2), your data lie close to a straight line. If the slope of this line is 16.0 rad/s^2, what is the moment of inertia of the cylinder for rotation about the axle?

10.18 •• **CP** Two spheres are rolling without slipping on a horizontal floor. They are made of different materials, but each has mass 5.00 kg and radius 0.120 m. For each the translational speed of the center of mass is 4.00 m/s. Sphere A is a uniform solid sphere and sphere B is a thin-walled, hollow sphere. How much work, in joules, must be done on each sphere to bring it to rest? For which sphere is a greater magnitude of work required? Explain. (The spheres continue to roll without slipping as they slow down.)

Section 10.3 Rigid-Body Rotation About a Moving Axis

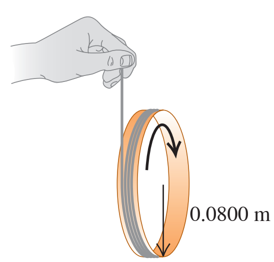
10.19 • A 2.20 kg hoop 1.20 m in diameter is rolling to the right without slipping on a horizontal floor at a steady 2.60 rad/s. (a) How fast is its center moving? (b) What is the total kinetic energy of the hoop? (c) Find the velocity vector of each of the following points, as viewed by a person at rest on the ground: (i) the highest point on the hoop; (ii) the lowest point on the hoop; (iii) a point on the right side of the hoop, midway between the top and the bottom. (d) Find the velocity vector for each of the points in part (c), but this time as viewed by someone moving along with the same velocity as the hoop.

10.20 •• Example 10.7 calculates the friction force needed for a uniform sphere to roll down an incline without slipping. The incline is at an angle β above the horizontal. And the example discusses that the friction is static. (a) If the maximum friction force is given by $f = \mu_s n$, where n is the normal force that the ramp exerts on the sphere, in terms of β what is the minimum coefficient of static friction needed if the sphere is to roll without slipping? (b) Based on your result in part (a), what does the minimum required μ_s become in the limits $\beta \rightarrow 90°$ and $\beta \rightarrow 0°$?

10.21 • What fraction of the total kinetic energy is rotational for the following objects rolling without slipping on a horizontal surface? (a) A uniform solid cylinder; (b) a uniform sphere; (c) a thin-walled, hollow sphere; (d) a hollow cylinder with outer radius R and inner radius $R/2$.

10.22 •• A string is wrapped several times around the rim of a small hoop with radius 8.00 cm and mass 0.180 kg. The free end of the string is held in place and the hoop is released from rest (**Fig. E10.22**). After the hoop has descended 75.0 cm, calculate (a) the angular speed of the rotating hoop and (b) the speed of its center.

Figure **E10.22**



10.23 •• A solid ball is released from rest and slides down a hillside that slopes downward at 65.0° from the horizontal. (a) What minimum value must the coefficient of static friction between the hill and ball surfaces have for no slipping to occur? (b) Would the coefficient of friction calculated in part (a) be sufficient to prevent a hollow ball (such as a soccer ball) from slipping? Justify your answer. (c) In part (a), why did we use the coefficient of static friction and not the coefficient of kinetic friction?

10.24 •• A hollow, spherical shell with mass 2.00 kg rolls without slipping down a 38.0° slope. (a) Find the acceleration, the friction force, and the minimum coefficient of static friction needed to prevent slipping. (b) How would your answers to part (a) change if the mass were doubled to 4.00 kg?

10.25 •• A 392 N wheel comes off a moving truck and rolls without slipping along a highway. At the bottom of a hill it is rotating at 25.0 rad/s. The radius of the wheel is 0.600 m, and its moment of inertia about its rotation axis is $0.800MR^2$. Friction does work on the wheel as it rolls up the hill to a stop, a height h above the bottom of the hill; this work has absolute value 2600 J. Calculate h.

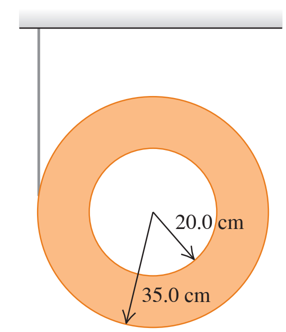
10.26 •• A uniform marble rolls down a symmetrical bowl, starting from rest at the top of the left side. The top of each side is a distance h above the bottom of the bowl. The left half of the bowl is rough enough to cause the marble to roll without slipping, but the right half has no friction because it is coated with oil. (a) How far up the smooth side will the marble go, measured vertically from the bottom? (b) How high would the marble go if both sides were as rough as the left side? (c) How do you account for the fact that the marble goes *higher* with friction on the right side than without friction?

10.27 •• At a typical bowling alley the distance from the line where the ball is released (foul line) to the first pin is 60 ft. Estimate the time it takes the ball to reach the pins after you release it, if it rolls without slipping and has a constant translational speed. Assume the ball weighs 12 lb and has a diameter of 8.5 in. (a) Use your estimate to calculate the rotation rate of the ball, in rev/s. (b) What is its total kinetic energy in joules and what fraction of the total is its rotational kinetic energy? Ignore the finger holes and treat the bowling ball as a uniform sphere.

10.28 • Two uniform solid balls are rolling without slipping at a constant speed. Ball 1 has twice the diameter, half the mass, and one-third the speed of ball 2. The kinetic energy of ball 2 is 27.0 J. What is the kinetic energy of ball 1?

10.29 •• A thin, light string is wrapped around the outer rim of a uniform hollow cylinder of mass 4.75 kg having inner and outer radii as shown in **Fig. E10.29**. The cylinder is then released from rest. (a) How far must the cylinder fall before its center is moving at 6.66 m/s? (b) If you just dropped this cylinder without any string, how fast would its center be moving when it had fallen the distance in part (a)? (c) Why do you get two different answers when the cylinder falls the same distance in both cases?

Figure **E10.29**



10.30 •• **A Ball Rolling Uphill.** A bowling ball rolls without slipping up a ramp that slopes upward at an angle β to the horizontal (see Example 10.7 in Section 10.3). Treat the ball as a uniform solid sphere, ignoring the finger holes. (a) Draw the free-body diagram for the ball. Explain why the friction force must be directed *uphill.* (b) What is the acceleration of the center of mass of the ball? (c) What minimum coefficient of static friction is needed to prevent slipping?

10.31 •• A size-5 soccer ball of diameter 22.6 cm and mass 426 g rolls up a hill without slipping, reaching a maximum height of 5.00 m above the base of the hill. We can model this ball as a thin-walled hollow sphere. (a) At what rate was it rotating at the base of the hill? (b) How much rotational kinetic energy did it have then? Neglect rolling friction and assume the system's total mechanical energy is conserved.

Section 10.4 Work and Power in Rotational Motion

10.32 • An engine delivers 175 hp to an aircraft propeller at 2400 rev/min. (a) How much torque does the aircraft engine provide? (b) How much work does the engine do in one revolution of the propeller?

10.33 • A playground merry-go-round has radius 2.40 m and moment of inertia $2100 \text{ kg} \cdot \text{m}^2$ about a vertical axle through its center, and it turns with negligible friction. (a) A child applies an 18.0 N force tangentially to the edge of the merry-go-round for 15.0 s. If the merry-go-round is initially at rest, what is its angular speed after this 15.0 s interval? (b) How much work did the child do on the merry-go-round? (c) What is the average power supplied by the child?

10.34 •• An electric motor consumes 9.00 kJ of electrical energy in 1.00 min. If one-third of this energy goes into heat and other forms of internal energy of the motor, with the rest going to the motor output, how much torque will this engine develop if you run it at 2500 rpm?

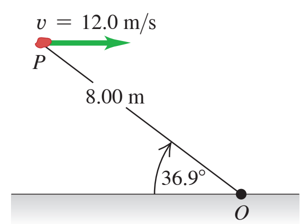
10.35 • A 2.80 kg grinding wheel is in the form of a solid cylinder of radius 0.100 m. (a) What constant torque will bring it from rest to an angular speed of 1200 rev/min in 2.5 s? (b) Through what angle has it turned during that time? (c) Use Eq. (10.21) to calculate the work done by the torque. (d) What is the grinding wheel's kinetic energy when it is rotating at 1200 rev/min? Compare your answer to the result in part (c).

10.36 •• An airplane propeller is 2.08 m in length (from tip to tip) and has a mass of 117 kg. When the airplane's engine is first started, it applies a constant net torque of 1950 N·m to the propeller, which starts from rest. (a) What is the angular acceleration of the propeller? Model the propeller as a slender rod and see Table 9.2. (b) What is the propeller's angular speed after making 5.00 revolutions? (c) How much work is done by the engine during the first 5.00 revolutions? (d) What is the average power output of the engine during the first 5.00 revolutions? (e) What is the instantaneous power output of the motor at the instant that the propeller has turned through 5.00 revolutions?

Section 10.5 Angular Momentum

10.37 • A 2.00 kg rock has a horizontal velocity of magnitude 12.0 m/s when it is at point P in **Fig. E10.37**. (a) At this instant, what are the magnitude and direction of its angular momentum relative to point O? (b) If the only force acting on the rock is its weight, what is the rate of change (magnitude and direction) of its angular momentum at this instant?

Figure **E10.37**



10.38 •• A woman with mass 50 kg is standing on the rim of a large horizontal disk that is rotating at 0.80 rev/s about an axis through its center. The disk has mass 110 kg and radius 4.0 m. Calculate the magnitude of the total angular momentum of the woman–disk system. (Assume that you can treat the woman as a point.)

10.39 •• Find the magnitude of the angular momentum of the second hand on a clock about an axis through the center of the clock face. The clock hand has a length of 15.0 cm and a mass of 6.00 g. Take the second hand to be a slender rod rotating with constant angular velocity about one end.

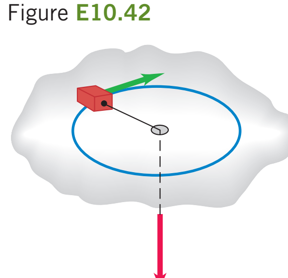
10.40 •• (a) Calculate the magnitude of the angular momentum of the earth in a circular orbit around the sun. Is it reasonable to model it as a particle? (b) Calculate the magnitude of the angular momentum of the earth due to its rotation around an axis through the north and south poles, modeling it as a uniform sphere. Consult Appendix E and the astronomical data in Appendix F.

10.41 •• CALC A hollow, thin-walled sphere of mass 12.0 kg and diameter 48.0 cm is rotating about an axle through its center. The angle (in radians) through which it turns as a function of time (in seconds) is given by $\theta(t) = At^2 + Bt^4$, where A has numerical value 1.50 and B has numerical value 1.10. (a) What are the units of the constants A and B? (b) At the time 3.00 s, find (i) the angular momentum of the sphere and (ii) the net torque on the sphere.

Section 10.6 Conservation of Angular Momentum

10.42 • CP A small block on a frictionless, horizontal surface has a mass of 0.0250 kg. It is attached to a massless cord passing through a hole in the surface (**Fig. E10.42**). The block is originally revolving at a distance of 0.300 m from the hole with an angular speed of 2.85 rad/s. The cord is then pulled from below, shortening the radius of the circle in which the block revolves to 0.150 m. Model the block as a particle. (a) Is the angular momentum of the block conserved? Why or why not? (b) What is the new angular speed? (c) Find the change in kinetic energy of the block. (d) How much work was done in pulling the cord?

Figure **E10.42**

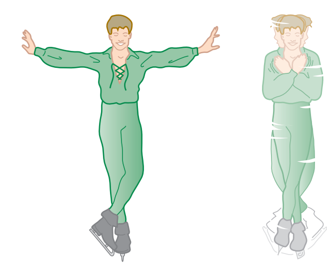


10.43 •• Under some circumstances, a star can collapse into an extremely dense object made mostly of neutrons and called a *neutron star*. The density of a neutron star is roughly 10^{14} times as great as that of ordinary solid matter. Suppose we represent the star as a uniform, solid, rigid sphere, both before and after the collapse. The star's initial radius was 7.0×10^5 km (comparable to our sun); its final radius is 16 km. If the original star rotated once in 30 days, find the angular speed of the neutron star.

10.44 •• A diver comes off a board with arms straight up and legs straight down, giving her a moment of inertia about her rotation axis of 18 kg·m^2. She then tucks into a small ball, decreasing this moment of inertia to 3.6 kg·m^2. While tucked, she makes two complete revolutions in 1.0 s. If she hadn't tucked at all, how many revolutions would she have made in the 1.5 s from board to water?

10.45 •• The Spinning Figure Skater. The outstretched hands and arms of a figure skater preparing for a spin can be considered a slender rod pivoting about an axis through its center (**Fig. E10.45**). When the skater's hands and arms are brought in and wrapped around his body to execute the spin, the hands and arms can be considered a thin-walled, hollow cylinder. His hands and arms have a combined mass of 8.0 kg. When outstretched, they span 1.8 m; when wrapped, they form a cylinder of radius 25 cm. The moment of inertia about the rotation axis of the remainder of his body is constant and equal to 0.40 kg·m^2. If his original angular speed is 0.40 rev/s, what is his final angular speed?

Figure **E10.45**



10.46 •• A solid wood door 1.00 m wide and 2.00 m high is hinged along one side and has a total mass of 40.0 kg. Initially open and at rest, the door is struck at its center by a handful of sticky mud with mass 0.500 kg, traveling perpendicular to the door at 12.0 m/s just before impact. Find the final angular speed of the door. Does the mud make a significant contribution to the moment of inertia?

10.47 •• A large wooden turntable in the shape of a flat uniform disk has a radius of 2.00 m and a total mass of 120 kg. The turntable is initially rotating at 3.00 rad/s about a vertical axis through its center. Suddenly, a 70.0 kg parachutist makes a soft landing on the turntable at a point near the outer edge. (a) Find the angular speed of the turntable after the parachutist lands. (Assume that you can treat the parachutist as a particle.) (b) Compute the kinetic energy of the system before and after the parachutist lands. Why are these kinetic energies not equal?

10.48 •• Asteroid Collision! Suppose that an asteroid traveling straight toward the center of the earth were to collide with our planet at the equator and bury itself just below the surface. What would have to be the mass of this asteroid, in terms of the earth's mass M, for the day to become 25.0% longer than it presently is as a result of the collision? Assume that the asteroid is very small compared to the earth and that the earth is uniform throughout.

10.49 •• A small 10.0 g bug stands at one end of a thin uniform bar that is initially at rest on a smooth horizontal table. The other end of the bar pivots about a nail driven into the table and can rotate freely, without friction. The bar has mass 50.0 g and is 100 cm in length. The bug jumps off in the horizontal direction, perpendicular to the bar, with a speed of 20.0 cm/s relative to the table. (a) What is the angular speed of the bar just after the frisky insect leaps? (b) What is the total kinetic energy of the system just after the bug leaps? (c) Where does this energy come from?

10.50 •• A thin uniform rod has a length of 0.500 m and is rotating in a circle on a frictionless table. The axis of rotation is perpendicular to the length of the rod at one end and is stationary. The rod has an angular velocity of 0.400 rad/s and a moment of inertia about the axis of 3.00×10^{-3} kg · m^2. A bug initially standing on the rod at the axis of rotation decides to crawl out to the other end of the rod. When the bug has reached the end of the rod and sits there, its tangential speed is 0.160 m/s. The bug can be treated as a point mass. What is the mass of (a) the rod; (b) the bug?

10.51 •• You live on a planet far from ours. Based on extensive communication with a physicist on earth, you have determined that all laws of physics on your planet are the same as ours and you have adopted the same units of seconds and meters as on earth. But you suspect that the value of g, the acceleration of an object in free fall near the surface of your planet, is different from what it is on earth. To test this, you take a solid uniform cylinder and let it roll down an incline. The vertical height h of the top of the incline above the lower end of the incline can be varied. You measure the speed v_{cm} of the center of mass of the cylinder when it reaches the bottom for various values of h. You plot v_{cm}^2 (in m^2/s^2) versus h (in m) and find that your data lie close to a straight line with a slope of 6.42 m/s^2. What is the value of g on your planet?

10.52 •• A uniform, 4.5 kg, square, solid wooden gate 1.5 m on each side hangs vertically from a frictionless pivot at the center of its upper edge. A 1.1 kg raven flying horizontally at 5.0 m/s flies into this door at its center and bounces back at 2.0 m/s in the opposite direction. (a) What is the angular speed of the gate just after it is struck by the unfortunate raven? (b) During the collision, why is the angular momentum conserved but not the linear momentum?

10.53 •• A teenager is standing at the rim of a large horizontal uniform wooden disk that can rotate freely about a vertical axis at its center. The mass of the disk (in kg) is M and its radius (in m) is R. The mass of the teenager (in kg) is m. The disk and teenager are initially at rest. The teenager then throws a large rock that has a mass (in kg) of m_{rock}. As it leaves the thrower's hands, the rock is traveling horizontally with speed v (in m/s) relative to the earth in a direction tangent to the rim of the disk. The teenager remains at rest relative to the disk and so rotates with it after throwing the rock. In terms of M, R, m, m_{rock}, and v, what is the angular speed of the disk? Treat the teenager as a point mass.

10.54 •• A uniform solid disk made of wood is horizontal and rotates freely about a vertical axle at its center. The disk has radius 0.600 m and mass 1.60 kg and is initially at rest. A bullet with mass 0.0200 kg is fired horizontally at the disk, strikes the rim of the disk at a point perpendicular to the radius of the disk, and becomes embedded in its rim, a distance of 0.600 m from the axle. After being struck by the bullet, the disk rotates at 4.00 rad/s. What is the horizontal velocity of the bullet just before it strikes the disk?

Section 10.7 Gyroscopes and Precession

10.55 • **Stabilization of the Hubble Space Telescope.** The Hubble Space Telescope is stabilized to within an angle of about 2-millionths of a degree by means of a series of gyroscopes that spin at 19,200 rpm. Although the structure of these gyroscopes is actually quite complex, we can model each of the gyroscopes as a thin-walled cylinder of mass 2.0 kg and diameter 5.0 cm, spinning about its central axis. How large a torque would it take to cause these gyroscopes to precess through an angle of 1.0×10^{-6} degree during a 5.0 hour exposure of a galaxy?

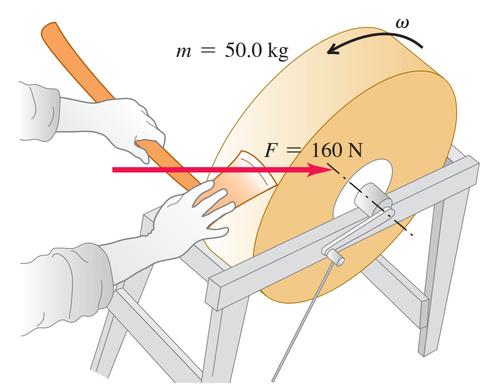
10.56 • **A Gyroscope on the Moon.** A certain gyroscope precesses at a rate of 0.50 rad/s when used on earth. If it were taken to a lunar base, where the acceleration due to gravity is 0.165g, what would be its precession rate?

PROBLEMS

10.57 •• You are riding your bicycle on a city street, and you are staying a constant distance behind a car that is traveling at the speed limit of 30 mph. Estimate the diameters of the bicycle wheels and sprockets and use these estimated quantities to calculate the number of revolutions per minute made by the large sprocket to which the pedals are attached. Do a Web search if you aren't familiar with the parts of a bicycle.

10.58 •• A 50.0 kg grindstone is a solid disk 0.520 m in diameter. You press an ax down on the rim with a normal force of 160 N (**Fig. P10.58**). The coefficient of kinetic friction between the blade and the stone is 0.60, and there is a constant friction torque of 6.50 N · m between the axle of the stone and its bearings. (a) How much force must be applied tangentially at the end of a crank handle 0.500 m long to bring the stone from rest to 120 rev/min in 9.00 s? (b) After the grindstone attains an angular speed of 120 rev/min, what tangential force at the end of the handle is needed to maintain a constant angular speed of 120 rev/min? (c) How much time does it take the grindstone to come from 120 rev/min to rest if it is acted on by the axle friction alone?

Figure **P10.58**

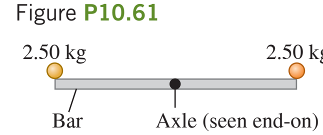


10.59 ••• A grindstone in the shape of a solid disk with diameter 0.520 m and a mass of 50.0 kg is rotating at 850 rev/min. You press an ax against the rim with a normal force of 160 N (Fig. P10.58), and the grindstone comes to rest in 7.50 s. Find the coefficient of friction between the ax and the grindstone. You can ignore friction in the bearings.

10.60 •• **CP** Block A rests on a horizontal tabletop. A light horizontal rope is attached to it and passes over a pulley, and block B is suspended from the free end of the rope. The light rope that connects the two blocks does not slip over the surface of the pulley (radius 0.080 m) because the pulley rotates on a frictionless axle. The horizontal surface on which block A (mass 2.50 kg) moves is frictionless. The system is released from rest, and block B (mass 6.00 kg) moves downward 1.80 m in 2.00 s. (a) What is the tension force that the rope exerts on block B? (b) What is the tension force on block A? (c) What is the moment of inertia of the pulley for rotation about the axle on which it is mounted?

10.61 ••• A thin, uniform, 3.80 kg bar, 80.0 cm long, has very small 2.50 kg balls glued on at either end (**Fig. P10.61**). It is supported horizontally by a thin, horizontal, frictionless axle passing through its center and perpendicular to the bar. Suddenly the right-hand ball becomes detached and falls off, but the other ball remains glued to the bar. (a) Find the angular acceleration of the bar just after the ball falls off. (b) Will the angular acceleration remain constant as the bar continues to swing? If not, will it increase or decrease? (c) Find the angular velocity of the bar just as it swings through its vertical position.

Figure **P10.61**



10.62 •• Example 10.7 discusses a uniform solid sphere rolling without slipping down a ramp that is at an angle β above the horizontal. Now consider the same sphere rolling without slipping up the ramp. (a) In terms of g and β, calculate the acceleration of the center of mass of the sphere. Is your result larger or smaller than the acceleration when the sphere rolls down the ramp, or is it the same? (b) Calculate the friction force (in terms of M, g, and β) for the sphere to roll without slipping as it moves up the incline. Is the result larger, smaller, or the same as the friction force required to prevent slipping as the sphere rolls down the incline?

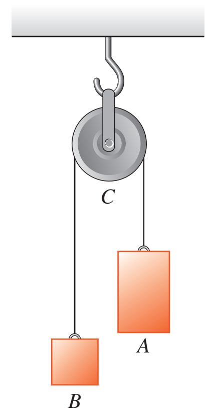


Figure **P10.63**

10.63 •• The Atwood's Machine. Figure P10.63 illustrates an Atwood's machine. Find the linear accelerations of blocks A and B, the angular acceleration of the wheel C, and the tension in each side of the cord if there is no slipping between the cord and the surface of the wheel. Let the masses of blocks A and B be 4.00 kg and 2.00 kg, respectively, the moment of inertia of the wheel about its axis be 0.220 $kg \cdot m^2$, and the radius of the wheel be 0.120 m.

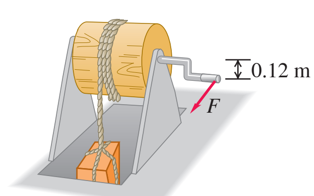


Figure **P10.64**

10.64 ••• The mechanism shown in **Fig. P10.64** is used to raise a crate of supplies from a ship's hold. The crate has total mass 50 kg. A rope is wrapped around a wooden cylinder that turns on a metal axle. The cylinder has radius 0.25 m and moment of inertia $I = 2.9\ kg \cdot m^2$ about the axle. The crate is suspended from the free end of the rope. One end of the axle pivots on frictionless bearings; a crank handle is attached to the other end. When the crank is turned, the end of the handle rotates about the axle in a vertical circle of radius 0.12 m, the cylinder turns, and the crate is raised. What magnitude of the force $\vec{F}$ applied tangentially to the rotating crank is required to raise the crate with an acceleration of 1.40 m/s^2? (You can ignore the mass of the rope as well as the moments of inertia of the axle and the crank.)

10.65 •• A solid uniform sphere and a thin-walled, hollow sphere have the same mass M and radius R. If they roll without slipping up a ramp that is inclined at an angle β above the horizontal and if both have the same v_{cm} before they start up the incline, calculate the maximum height above their starting point reached by each object. Which object reaches the greater height, or do both of them reach the same height?

10.66 •• A block with mass $m = 5.00$ kg slides down a surface inclined 36.9° to the horizontal (**Fig. P10.66**). The coefficient of kinetic friction is 0.25. A string attached to the block is wrapped around a flywheel on a fixed axis at O. The flywheel has mass 25.0 kg and moment of inertia 0.500 $kg \cdot m^2$ with respect to the axis of rotation. The string pulls without slipping at a perpendicular distance of 0.200 m from that axis. (a) What is the acceleration of the block down the plane? (b) What is the tension in the string?

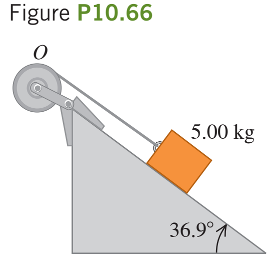


Figure **P10.66**

10.67 •• CP A wheel with radius 0.0600 m rotates about a horizontal frictionless axle at its center. The moment of inertia of the wheel about the axle is 2.50 $kg \cdot m^2$. The wheel is initially at rest. Then at $t = 0$ a force $F = (5.00\ N/s)t$ is applied tangentially to the wheel and the wheel starts to rotate. What is the magnitude of the force at the instant when the wheel has turned through 8.00 revolutions?

10.68 •• A lawn roller in the form of a thin-walled, hollow cylinder with mass M is pulled horizontally with a constant horizontal force F applied by a handle attached to the axle. If it rolls without slipping, find the acceleration and the friction force.

10.69 • Two weights are connected by a very light, flexible cord that passes over an 80.0 N frictionless pulley of radius 0.300 m. The pulley is a solid uniform disk and is supported by a hook connected to the ceiling (**Fig. P10.69**). What force does the ceiling exert on the hook?

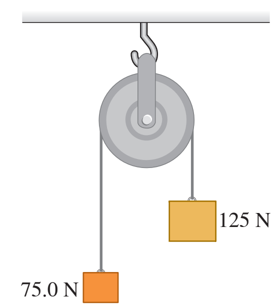


Figure **P10.69**

10.70 •• A large uniform horizontal turntable rotates freely about a vertical axle at its center. You measure the radius of the turntable to be 3.00 m. To determine the moment of inertia I of the turntable about the axle, you start the turntable rotating with angular speed ω, which you measure. You then drop a small object of mass m onto the rim of the turntable. After the object has come to rest relative to the turntable, you measure the angular speed ω_f of the rotating turntable. You plot the quantity $(\omega - \omega_f)/\omega_f$ (with both ω and ω_f in rad/s) as a function of m (in kg). You find that your data lie close to a straight line that has slope 0.250 kg^{-1}. What is the moment of inertia I of the turntable?

10.71 • The Yo-yo. A yo-yo is made from two uniform disks, each with mass m and radius R, connected by a light axle of radius b. A light, thin string is wound several times around the axle and then held stationary while the yo-yo is released from rest, dropping as the string unwinds. Find the linear acceleration and angular acceleration of the yo-yo and the tension in the string.

10.72 •• CP A thin-walled, hollow spherical shell of mass m and radius r starts from rest and rolls without slipping down a track (**Fig. P10.72**). Points A and B are on a circular part of the track having radius R. The diameter of the shell is very small compared to h_0 and R, and the work done by rolling friction is negligible. (a) What is the minimum height h_0 for which this shell will make a complete loop-the-loop on the circular part of the track? (b) How hard does the track push on the shell at point B, which is at the same level as the center of the circle? (c) Suppose that the track had no friction and the shell was released from the same height h_0 you found in part (a). Would it make a complete loop-the-loop? How do you know? (d) In part (c), how hard does the track push on the shell at point A, the top of the circle? How hard did it push on the shell in part (a)?

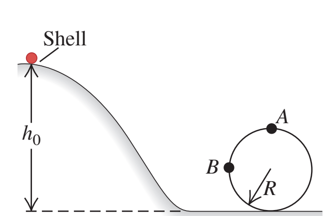
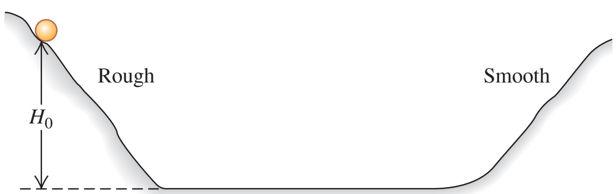


Figure **P10.72**

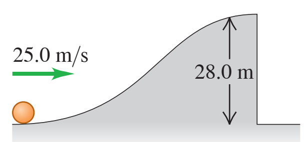
10.73 •• A basketball (which can be closely modeled as a hollow spherical shell) rolls down a mountainside into a valley and then up the opposite side, starting from rest at a height H_0 above the bottom. In **Fig. P10.73**, the rough part of the terrain prevents slipping while the smooth part has no friction. Neglect rolling friction and assume the system's total mechanical energy is conserved. (a) How high, in terms of H_0, will the ball go up the other side? (b) Why doesn't the ball return to height H_0? Has it lost any of its original potential energy?

Figure **P10.73**



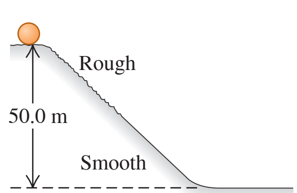
10.74 •• CP A solid uniform ball rolls without slipping up a hill (**Fig. P10.74**). At the top of the hill, it is moving horizontally, and then it goes over the vertical cliff. Neglect rolling friction and assume the system's total mechanical energy is conserved. (a) How far from the foot of the cliff does the ball land, and how fast is it moving just before it lands? (b) Notice that when the balls lands, it has a greater translational speed than when it was at the bottom of the hill. Does this mean that the ball somehow gained energy? Explain!

Figure **P10.74**



10.75 •• Rolling Stones. A solid, uniform, spherical boulder starts from rest and rolls down a 50.0-m-high hill, as shown in **Fig. P10.75**. The top half of the hill is rough enough to cause the boulder to roll without slipping, but the lower half is covered with ice and there is no friction. What is the translational speed of the boulder when it reaches the bottom of the hill? Neglect rolling friction and assume the system's total mechanical energy is conserved.

Figure **P10.75**



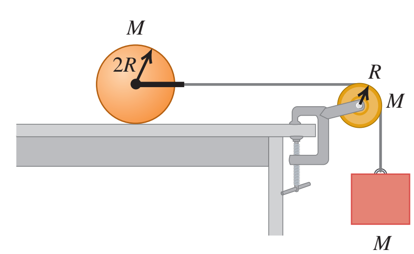
10.76 ••• You are designing a system for moving aluminum cylinders from the ground to a loading dock. You use a sturdy wooden ramp that is 6.00 m long and inclined at 37.0° above the horizontal. Each cylinder is fitted with a light, frictionless yoke through its center, and a light (but strong) rope is attached to the yoke. Each cylinder is uniform and has mass 460 kg and radius 0.300 m. The cylinders are pulled up the ramp by applying a constant force $\vec{F}$ to the free end of the rope. $\vec{F}$ is parallel to the surface of the ramp and exerts no torque on the cylinder. The coefficient of static friction between the ramp surface and the cylinder is 0.120. (a) What is the largest magnitude $\vec{F}$ can have so that the cylinder still rolls without slipping as it moves up the ramp? (b) If the cylinder starts from rest at the bottom of the ramp and rolls without slipping as it moves up the ramp, what is the shortest time it can take the cylinder to reach the top of the ramp?

10.77 •• A 42.0-cm-diameter wheel, consisting of a rim and six spokes, is constructed from a thin, rigid plastic material having a linear mass density of 25.0 g/cm. This wheel is released from rest at the top of a hill 58.0 m high. (a) How fast is it rolling when it reaches the bottom of the hill? (b) How would your answer change if the linear mass density and the diameter of the wheel were each doubled?

10.78 ••• A uniform, 0.0300 kg rod of length 0.400 m rotates in a horizontal plane about a fixed axis through its center and perpendicular to the rod. Two small rings, each with mass 0.0200 kg, are mounted so that they can slide without friction along the rod. They are initially held by catches at positions 0.0500 m on each side of the center of the rod, and the system is rotating at 48.0 rev/min. With no other changes in the system, the catches are released, and the rings slide outward along the rod and fly off at the ends. What is the angular speed (a) of the system at the instant when the rings reach the ends of the rod; (b) of the rod after the rings leave it?

10.79 • A uniform solid cylinder with mass M and radius $2R$ rests on a horizontal tabletop. A string is attached by a yoke to a frictionless axle through the center of the cylinder so that the cylinder can rotate about the axle. The string runs over a disk-shaped pulley with mass M and radius R that is mounted on a frictionless axle through its center. A block of mass M is suspended from the free end of the string (**Fig. P10.79**). The string doesn't slip over the pulley surface, and the cylinder rolls without slipping on the tabletop. Find the magnitude of the acceleration of the block after the system is released from rest.

Figure **P10.79**



10.80 ••• A 5.00 kg ball is dropped from a height of 12.0 m above one end of a uniform bar that pivots at its center. The bar has mass 8.00 kg and is 4.00 m in length. At the other end of the bar sits another 5.00 kg ball, unattached to the bar. The dropped ball sticks to the bar after the collision. How high will the other ball go after the collision?

10.81 •• A uniform rod of length L rests on a frictionless horizontal surface. The rod pivots about a fixed frictionless axis at one end. The rod is initially at rest. A bullet traveling parallel to the horizontal surface and perpendicular to the rod with speed v strikes the rod at its center and becomes embedded in it. The mass of the bullet is one-fourth the mass of the rod. (a) What is the final angular speed of the rod? (b) What is the ratio of the kinetic energy of the system after the collision to the kinetic energy of the bullet before the collision?

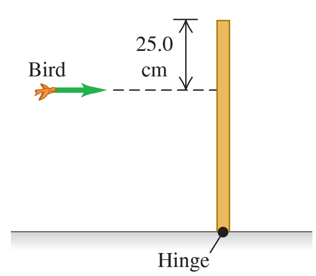
10.82 •• CP A large turntable with radius 6.00 m rotates about a fixed vertical axis, making one revolution in 8.00 s. The moment of inertia of the turntable about this axis is 1200 kg $\cdot$ m^2. You stand, barefooted, at the rim of the turntable and very slowly walk toward the center, along a radial line painted on the surface of the turntable. Your mass is 70.0 kg. Since the radius of the turntable is large, it is a good approximation to treat yourself as a point mass. Assume that you can maintain your balance by adjusting the positions of your feet. You find that you can reach a point 3.00 m from the center of the turntable before your feet begin to slip. What is the coefficient of static friction between the bottoms of your feet and the surface of the turntable?

10.83 •• In your job as a mechanical engineer you are designing a flywheel and clutch-plate system like the one in Example 10.11. Disk A is made of a lighter material than disk B, and the moment of inertia of disk A about the shaft is one-third that of disk B. The moment of inertia of the shaft is negligible. With the clutch disconnected, A is brought up to an angular speed ω_0; B is initially at rest. The accelerating torque is then removed from A, and A is coupled to B. (Ignore bearing friction.) The design specifications allow for a maximum of 2400 J of thermal energy to be developed when the connection is made. What can be the maximum value of the original kinetic energy of disk A so as not to exceed the maximum allowed value of the thermal energy?

10.84 •• A local ice hockey team has asked you to design an apparatus for measuring the speed of the hockey puck after a slap shot. Your design is a 2.00-m-long, uniform rod pivoted about one end so that it is free to rotate horizontally on the ice without friction. The 0.800 kg rod has a light basket at the other end to catch the 0.163 kg puck. The puck slides across the ice with velocity $\vec{v}$ (perpendicular to the rod), hits the basket, and is caught. After the collision, the rod rotates. If the rod makes one revolution every 0.736 s after the puck is caught, what was the puck's speed just before it hit the rod?

10.85 ••• A 500.0 g bird is flying horizontally at 2.25 m/s, not paying much attention, when it suddenly flies into a stationary vertical bar, hitting it 25.0 cm below the top (**Fig. P10.85**). The bar is uniform, 0.750 m long, has a mass of 1.50 kg, and is hinged at its base. The collision stuns the bird so that it just drops to the ground afterward (but soon recovers to fly happily away). What is the angular velocity of the bar (a) just after it is hit by the bird and (b) just as it reaches the ground?

Figure **P10.85**



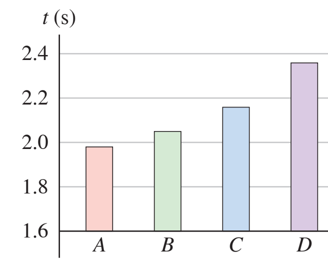
10.86 ••• CP A small block with mass 0.130 kg is attached to a string passing through a hole in a frictionless, horizontal surface (see Fig. E10.42). The block is originally revolving in a circle with a radius of 0.800 m about the hole with a tangential speed of 4.00 m/s. The string is then pulled slowly from below, shortening the radius of the circle in which the block revolves. The breaking strength of the string is 30.0 N. What is the radius of the circle when the string breaks?

10.87 • A 55 kg runner runs around the edge of a horizontal turntable mounted on a vertical, frictionless axis through its center. The runner's velocity relative to the earth has magnitude 2.8 m/s. The turntable is rotating in the opposite direction with an angular velocity of magnitude 0.20 rad/s relative to the earth. The radius of the turntable is 3.0 m, and its moment of inertia about the axis of rotation is 80 kg · m^2. Find the final angular velocity of the system if the runner comes to rest relative to the turntable. (You can model the runner as a particle.)

10.88 •• DATA The V6 engine in a 2014 Chevrolet Silverado 1500 pickup truck is reported to produce a maximum power of 285 hp at 5300 rpm and a maximum torque of 305 ft · lb at 3900 rpm. (a) Calculate the torque, in both ft · lb and N · m, at 5300 rpm. Is your answer in ft · lb smaller than the specified maximum value? (b) Calculate the power, in both horsepower and watts, at 3900 rpm. Is your answer in hp smaller than the specified maximum value? (c) The relationship between power in hp and torque in ft · lb at a particular angular velocity in rpm is often written as hp = [torque (in ft · lb) × rpm]/c, where c is a constant. What is the numerical value of c? (d) The engine of a 2012 Chevrolet Camaro ZL1 is reported to produce 580 hp at 6000 rpm. What is the torque (in ft · lb) at 6000 rpm?

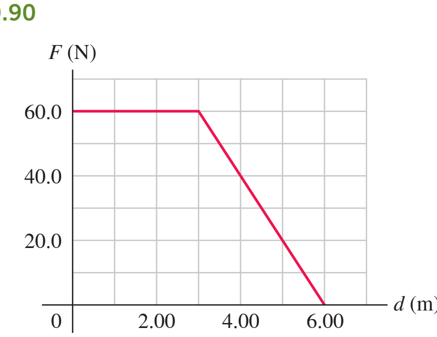
10.89 •• DATA You have one object of each of these shapes, all with mass 0.840 kg: a uniform solid cylinder, a thin-walled hollow cylinder, a uniform solid sphere, and a thin-walled hollow sphere. You release each object from rest at the same vertical height h above the bottom of a long wooden ramp that is inclined at 35.0° from the horizontal. Each object rolls without slipping down the ramp. You measure the time t that it takes each one to reach the bottom of the ramp; **Fig. P10.89** shows the results. (a) From the bar graphs, identify objects A through D by shape. (b) Which of objects A through D has the greatest total kinetic energy at the bottom of the ramp, or do all have the same kinetic energy? (c) Which of objects A through D has the greatest rotational kinetic energy $\frac{1}{2}I\omega^2$ at the bottom of the ramp, or do all have the same rotational kinetic energy? (d) What minimum coefficient of static friction is required for all four objects to roll without slipping?

Figure **P10.89**



10.90 ••• DATA You are testing a small flywheel (radius 0.166 m) that will be used to store a small amount of energy. The flywheel is pivoted with low-friction bearings about a horizontal shaft through the flywheel's center. A thin, light cord is wrapped multiple times around the rim of the flywheel. Your lab has a device that can apply a specified horizontal force $\vec{F}$ to the free end of the cord. The device records both the magnitude of that force as a function of the horizontal distance the end of the cord has traveled and the time elapsed since the force was first applied. The flywheel is initially at rest. (a) You start with a test run to determine the flywheel's moment of inertia I. The magnitude F of the force is a constant 25.0 N, and the end of the rope moves 8.35 m in 2.00 s. What is I? (b) In a second test, the flywheel again starts from rest but the free end of the rope travels 6.00 m; **Fig. P10.90** shows the force magnitude F as a function of the distance d that the end of the rope has moved. What is the kinetic energy of the flywheel when $d = 6.00$ m? (c) What is the angular speed of the flywheel, in rev/min, when $d = 6.00$ m?

Figure **P10.90**



CHALLENGE PROBLEMS

10.91 ••• CP CALC A block with mass m is revolving with linear speed v_1 in a circle of radius r_1 on a frictionless horizontal surface (see Fig. E10.42). The string is slowly pulled from below until the radius of the circle in which the block is revolving is reduced to r_2. (a) Calculate the tension T in the string as a function of r, the distance of the block from the hole. Your answer will be in terms of the initial velocity v_1 and the radius r_1. (b) Use $W = \int_{r_1}^{r_2} \vec{T}(r) \cdot d\vec{r}$ to calculate the work done by $\vec{T}$ when r changes from r_1 to r_2. (c) Compare the results of part (b) to the change in the kinetic energy of the block.

10.92 ••• When an object is rolling without slipping, the rolling friction force is much less than the friction force when the object is sliding; a silver dollar will roll on its edge much farther than it will slide on its flat side (see Section 5.3). When an object is rolling without slipping on a horizontal surface, we can approximate the friction force to be zero, so that a_x and α_z are approximately zero and v_x and ω_z are approximately constant. Rolling without slipping means $v_x = r\omega_z$ and $a_x = r\alpha_z$. If an object is set in motion on a surface *without* these equalities, sliding (kinetic) friction will act on the object as it slips until rolling without slipping is established. A solid cylinder with mass M and radius R, rotating with angular speed ω_0 about an axis through its center, is set on a horizontal surface for which the kinetic friction coefficient is μ_k. (a) Draw a free-body diagram for the cylinder on the surface. Think carefully about the direction of the kinetic friction force on the cylinder. Calculate the accelerations a_x of the center of mass and α_z of rotation about the center of mass. (b) The cylinder is initially slipping completely, so initially $\omega_z = \omega_0$ but $v_x = 0$. Rolling without slipping sets in when $v_x = r\omega_z$. Calculate the *distance* the cylinder rolls before slipping stops. (c) Calculate the work done by the friction force on the cylinder as it moves from where it was set down to where it begins to roll without slipping.

10.93 ••• A demonstration gyroscope wheel is constructed by removing the tire from a bicycle wheel 0.650 m in diameter, wrapping lead wire around the rim, and taping it in place. The shaft projects 0.200 m at each side of the wheel, and a woman holds the ends of the shaft in her hands. The mass of the system is 8.00 kg; its entire mass may be assumed to be located at its rim. The shaft is horizontal, and the wheel is spinning about the shaft at 5.00 rev/s. Find the magnitude and direction of the force each hand exerts on the shaft (a) when the shaft is at rest; (b) when the shaft is rotating in a horizontal plane about its center at 0.050 rev/s; (c) when the shaft is rotating in a horizontal plane about its center at 0.300 rev/s. (d) At what rate must the shaft rotate in order that it may be supported at one end only?

MCAT-STYLE PASSAGE PROBLEMS

BIO Human Moment of Inertia. The moment of inertia of the human body about an axis through its center of mass is important in the application of biomechanics to sports such as diving and gymnastics. We can measure the body's moment of inertia in a particular position while a person remains in that position on horizontal turntable, with the body's center of mass on the turntable's rotational axis. The turntable with the person on it is then accelerated from rest by a torque that is produced by using a rope wound around a pulley on the shaft of the turntable. From the measured tension in the rope and the angular acceleration, we can calculate the body's moment of inertia about an axis through its center of mass.

10.94 The moment of inertia of the empty turntable is 1.5 kg · m². With a constant torque of 2.5 N · m, the turntable–person system takes 3.0 s to spin from rest to an angular speed of 1.0 rad/s. What is the person's moment of inertia about an axis through her center of mass? Ignore friction in the turntable axle. (a) $2.5\ \text{kg}\cdot\text{m}^2$; (b) $6.0\ \text{kg}\cdot\text{m}^2$; (c) $7.5\ \text{kg}\cdot\text{m}^2$; (d) $9.0\ \text{kg}\cdot\text{m}^2$.

10.95 While the turntable is being accelerated, the person suddenly extends her legs. What happens to the turntable? (a) It suddenly speeds up; (b) it rotates with constant speed; (c) its acceleration decreases; (d) it suddenly stops rotating.

10.96 A doubling of the torque produces a greater angular acceleration. Which of the following would do this, assuming that the tension in the rope doesn't change? (a) Increasing the pulley diameter by a factor of $\sqrt{2}$; (b) increasing the pulley diameter by a factor of 2; (c) increasing the pulley diameter by a factor of 4; (d) decreasing the pulley diameter by a factor of $\sqrt{2}$.

10.97 If the body's center of mass were not placed on the rotational axis of the turntable, how would the person's measured moment of inertia compare to the moment of inertia for rotation about the center of mass? (a) The measured moment of inertia would be too large; (b) the measured moment of inertia would be too small; (c) the two moments of inertia would be the same; (d) it depends on where the body's center of mass is placed relative to the center of the turntable.

ANSWERS

Chapter Opening Question ?

(iv) A tossed pin rotates around its center of mass (which is located toward its thick end). This is also the point at which the gravitational force acts on the pin, so this force exerts no torque on the pin. Hence the pin rotates with constant angular momentum, and its angular speed remains the same.

Key Example √ARIATION Problems

VP10.3.1 (a) $10\ \text{rad/s}^2$ (b) $0.90\ \text{N}\cdot\text{m}$ (c) 15 N

VP10.3.2 (a) $a_y = \dfrac{g}{1 + M/m}$ (b) $T = \dfrac{mg}{1 + m/M} = \dfrac{Mg}{1 + M/m}$

VP10.3.3 (a) $T = m(g - a)$ (b) $I = mR^2\left(\dfrac{g}{a} - 1\right)$

VP10.3.4 (a) $T = m(g + a)$ (b) $\tau_{\text{cable on cylinder}} = mR(g + a)$ (c) $\tau_{\text{motor on cylinder}} = mR(g + a) + \frac{1}{2}MRa$

VP10.7.1 (a) $a_{\text{cm-}y} = \frac{8}{13}g$ (b) $T = \frac{5}{13}Mg$

VP10.7.2 (a) $a_{\text{cm-}x} = \frac{3}{5}g\sin\beta$ (b) $f = \frac{2}{5}Mg\sin\beta$ (c) $\tau = \frac{2}{5}MgR\sin\beta$

VP10.7.3 (a) $a_{\text{cm}} = \frac{1}{3}g$, downward (b) $\alpha_z = \frac{4g}{3R}$

VP10.7.4 (a) $f = \frac{1}{3}Mg\sin\beta$ (b) $\beta_{\text{critical}} = \arctan 3\mu_s$

VP10.12.1 (a) $\frac{9}{10}\omega_A$ (b) $\frac{81}{85} = 0.953$

VP10.12.2 (a) $\frac{7}{10}\omega_A$ (b) $\frac{49}{85} = 0.576$

VP10.12.3 (a) 0.80 rad/s (b) $0.0020 = 1/500$

VP10.12.4 (a) MRv before, $MRv/2$ after (b) $\omega = \dfrac{3v}{4R}$ (c) $\frac{5}{8}$

Bridging Problem

(a) $h = \dfrac{2R}{5}$

(b) $\frac{5}{7}$ of the speed it had just after the hit