Formal Languages and Compilers

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Contents

1 Formal Language Theory

Alphabet Σ: any finite set of symbols $\Sigma = \{a_1, a_2, ..., a_k\}$

String: a sequence of alphabeth elements

Language: a set (possibly infinite) of strings

$$\Sigma = \{a, b, c\} \quad L_1 = \{ab, ac\} \quad L_2 = \{ab, aab, aaab, aaaab, ...\}$$

Sentences/Phrases: strings belonging to a language

Language cardinality: number of sentences of the language

$$|L_1| = |\{ab, ab\}| = 2 \quad |L_2| = |\{ab, aab, aaab, aaaab, ...\}| = \infty$$

Number of occurrences of a symbol in a string: $|bbc|_b = 2$, $|bbc|_a = 0$

Length of a string: number of its elements

$$|bbc| = 3 \quad |abbc| = 4$$

String equality: two strings $x = a_1a_2...a_h$ and $y = a_1a_2...a_k$ are equal $\iff$

- have same length: $|x| = |y| \iff h = k$
- elements from left to right coincide: $a_i = b_i \quad \forall i \in \{1..h\}$

1.1 Operations on strings

Concatenation $x = a_1a_2...a_h \wedge y = b_1b_2...b_k \implies x \cdot y = a_1a_2...a_hb_1b_2...b_k$

- associative: $(xy)z = x(yz)$
- length: $|xy| = |x| + |y|$

Empty string ϵ is the neutral element for concatenation: $x\epsilon = \epsilon x = x \, \forall x$.

- length: $|\epsilon| = 0$
- **NB:** $\epsilon \neq \emptyset$

Substrings: if $x = uyv$ then

- y is a substring of x
- y is a proper substring of $x \quad \iff u \neq \epsilon \vee v \neq \epsilon$
- u is a prefix of x
- v is a suffix of y

Reflection: if $x = a_1a_2...a_h$ then $x^R = a_ha_{h-1}...a_1$

- $(x^R)^R = x$
- $(xy)^R = y^R x^R$
- $\epsilon^R = \epsilon$

Repetition: $x^m = \underbrace{xxx...x}_{m \text{ times}}$. Inductive definition:

- $x^0 = \epsilon$
- $x^m = x^{m-1}x \quad \text{if } m > 0$

1.2 Operations on Languages

Reflection: $L^R = \{x | \exists y \left(y \in L \land x = y^R \right)\}$

Prefixes(L): $\{y | y \neq \epsilon \land \exists x \exists z \left(x \in L \land z \neq \epsilon \land x = yz \right)\}$

- **Prefix-free language:** $L \cap Prefixes(L) = \emptyset$

Concatenation: $L'L'' = \{xy | x \in L' \land y \in L''\}$

Power: inductive definition:

- $L^0 = \{\epsilon\}$
- $L^m = L^{m-1}L \quad \text{for } m > 0$
- Consequences:
  - $\emptyset^0 = \{\epsilon\}$
  - $L \cdot \emptyset = \emptyset \cdot L = \emptyset$
  - $L \cdot \{\epsilon\} = \{\epsilon\} \cdot L = L$

Universal language: over alphabet Σ: $L_{\text{universal}} = \Sigma^0 \cup \Sigma^1 \cup ...$

Complement: of L over Σ: $\neg L = L_{\text{universal}} \setminus L$

Star: formally called **reflexive and transitive closure** or **Klenee star**

$$L^* = \bigcup_{h=0}^{\infty} L^h = L^0 \cup L^1 \cup ... = \epsilon \cup L^1 \cup L^2$$

$$\Sigma^* = L_{\text{universal}}$$

Monotonic: $L \subseteq L^*$

Close under concatenation: $x \in L^* \land y \in L^* \implies xy \in L^*$

Idempotent: $(L^*)^* = L^*$

Commutative with reflection: $(L^*)^R = (L^R)^*$

$\emptyset^* = \{\epsilon\}$

$\{\epsilon\}^* = \{\epsilon\}$

Cross: $L^+ = L \cdot L^*$

Quotient: $L_1/L_2 = \{y | \exists x \in L_1 \exists z \in L_2 (x = yz)\}$

- **Not set quotient!**
- Removes from L_1 suffixes contained in L_2

2 Regular Expressions and Languages

Regular languages are the simplest family of laguages.
They can be defined in three ways:

- Algebraically
- Using generative grammars
- Using recognizer automata

2.1 Algebraic definition

Regular expressions are expression on languages that composes languages operations.
Formally

- Is a string r
- Over the alphabet $\Sigma = \{a_1, a_2, ..., a_n\} \cup \{\emptyset, \cup, \cdot, *\}$

Moreover, assuming s and t are regular expressions, then r is a regular expression if any of the following rules applies:

- $r = \emptyset$
- $r = a, \quad a \in \Sigma$
- $r = s \cup t$ (alternative notation is $s|t$)
- $r = s \cdot t$ (the $\cdot$ can be omitted)
- $r = s^*$

The meaning of a r.e. is a **language** L_r of alphabet Σ according to the table:

Expression	**Language**
$\emptyset$	$\emptyset$
ϵ	$\{\epsilon\}$
$a \in \Sigma$	$\{a\}$
$s \cup t$	$L_s \cup L_t$
$s \cdot t$	$L_s \cdot L_t$
s^*	L_s^*

Regular Languages are languages denoted by a regular expression

2.2 Language Families

REG is the collection of all regular languages

FIN is the collection of all languages with finite cardinality

Every finite language is regular $FIN \subset REG$:

- $L \in FIN \implies L = \bigcup_{i=1}^{k \in \mathbb{N}} x_i \implies L \in FIN$
- $L = a^* \implies L \in REG \land L \notin FIN$

2.3 Derivation

Choice Union and Concatenation corresponds to possible choices. One obtains subexpressions by making a choice that identifies a sub language.

Regular expression	Choices
$e_1 \cup ... \cup e_k$	$e_i \quad \forall i \in \{1, 2, ..., k\}$
e^*	ϵ or $e^n \quad \forall n \geq 1$
e^+	$e^n \quad \forall n \geq 1$

Derivation among two r.e: $e_1 \Rightarrow e_2$ if

$$e_1 = \alpha\beta\gamma \land e_2 = \alpha\delta\gamma$$

where γ is a choice of β.
Derivation can be applied repeatedly, leading to $\overset{n}{\Longrightarrow}$ (deriving n times, $\overset{*}{\Longrightarrow}$ (0 or more times), $\overset{+}{\Longrightarrow}$ (1 or more times).

Language defined by an r.e. $L(r) = \{x \in \Sigma^* | r \overset{*}{\Rightarrow} x\}$

Equivalent r.e. defines the same language

2.4 Ambiguity of Regular Expressions

Numbered subexpressions of a R.E

- Add all passible parentheses to the r.e.
- number the elements of Σ
- identify all the subexpressions

Ambiguity happens when a phrase can be obtained through distinct derivations, which differ **not only for the order**.

Sufficient condition for ambiguity of the r.e. f having numbered version f' is that $\exists x \exists y \in L(f') | x \neq y$ but $x = y$ when numbers are removed

2.5 Extended Regular Expressions

Regular expressions extended with other operators:

Power: $a^n = \underbrace{aa...a}_{\text{n times}}$. NB: n is an actual number, cannot be a parameter.

Repetition: from k to $n > k$: $[a]_k^n = a^k \cup a^{k+1} \cup ... \cup a^n$

Optionality: $\epsilon \cup a$ or $[a]$

Ordered interval: $(0...9)$ $(a...z)$ $(A...Z)$

Intersection

Difference

Complement

It can be shown that Extended R.E. are not more powerful than standard R.E.

Closures REG is closed under

- Concatenation
- Union
- Star (*)
- Cross (+)
- Power
- Intersection
- Complement

Lists contains an unspecified number of elements of the same type. Lists can be represanted with regex:

$$ie(se)*f$$

where i,s,f are terminal symbols denoting the beginning of the list, a separator between elements, and the end of the string.

Nested lists are possible using regex if the nesting level is limited:

$$\begin{aligned}
list_1 &= i_1 \cdot list_2 \cdot (s_1 \cdot list_2)^* \cdot f_1 \\
list_2 &= i_2 \cdot list_3 \cdot (s_2 \cdot list_3)^* \cdot f_2 \\
&... \\
list_k &= i_k \cdot e_k \cdot (s_k \cdot e_k)^* \cdot f_k
\end{aligned}$$

3 Context Free Grammars

The language $L = \{a^n b^n | n > 0\}$ is **not** regular.

Grammars a tool to define language through **rewriting rules**. Phrases are generated hrough repeated application of the rules.

Context Free Grammar is defined by 4 entities:

Non-terminal aplhabet V

Terminal alphabet Σ, alphabeth of the resulting language

Rules/Productions P

Axiom/Start $S \in V$, from which derivation starts

Rules form: $X \to \alpha$ where $X \in V \wedge \alpha \in (V \cup \Sigma)^*$. Rules can be condensed:

$$X \to \alpha_1$$
$$X \to \alpha_2$$
$$...$$
$$X \to \alpha_k$$
$$\text{can be rewritten as}$$
$$X \to \alpha_1 | \alpha_2 | ... | \alpha_k$$

.

Safety conventions:

- $\{\to, |, \cup, \epsilon\} \cap \Sigma = \emptyset$
- $V \cap \Sigma = \emptyset$

Notation conventions: V elements can be distinguished using:

- <Angle brackets> surrounding elements of V
- Elements of Σ in **bold**, elements of V in *italic*
- Elements of Σ 'quoted'
- Elements of V in UPPERCASE

3.1 Types of rules

Terminal $\to u|\epsilon$

Empty/Null $\to \epsilon$

Initial/Axiomatic $S \to$

Recursive $A \to \alpha A \beta$

Left-Recursive $A \to A\beta$

Right-Recursive $A \to \alpha A$

Left-and-Right-Recursive $A \to A\beta A$

Copy/Categorization $A \to B$

Linear $\to uBv|w$

Right-linear $\to uB|w$

Left-Linear $\to Bv|w$

Homogeneous normal $\to A_1...A_n|a$

Chomsky normal $\to BC|a$

Greibach normal $\to a\sigma|b$ where $\sigma \in V^*$

Operator normal $\to AaB$

3.2 Derivation

Derivation $\implies$ Let $\beta, \gamma \in (V \cup \Sigma)^*$. Then $\beta \implies \gamma$ for grammar $G =< V, \Sigma, P, S >$ iff

$$\begin{aligned} &\beta = \delta A \eta && \wedge \\ &A \to \alpha \quad \in V && \wedge \\ &\gamma = \delta \alpha \eta \end{aligned}$$

Power, star and cross operators apply to derivation as usual

3.3 Erroneous Grammars and Useless Rules

Clean grammar $G =< V, \Sigma, P, S >$ is clean iff $\forall A \in V$

A is reachable: $S \xRightarrow{*} \alpha A \beta$ where $\alpha, \beta \in (V \cap \Sigma)^*$

A is defined: $L_A(G) \neq \emptyset$ (generates a non-empty language)

(G doesn't allow for circular derivations) optional, but useful

Algorithm 1 Undefined nonterminals identification

$NEW \leftarrow \{A | (A \to u) \in P \wedge u \in \Sigma^*\}$
repeat
  $DEF \leftarrow NEW$
  $NEW \leftarrow DEF \cup \{B | (B \to D_1 D_2 ... D_n) \in P \wedge \overbrace{\forall i (D_i \in DEF \cup \Sigma)}^{D_i \text{in DEF or a terminal}}\}$
until $NEW = DEF$
$UNDEF \leftarrow V \setminus DEF$

Produce relation A produce B iff $A \to (\alpha B \beta) \in P$, where $A \neq B \wedge \alpha, \beta$ are strings

Algorithm 2 Unreachable nonterminals identification

Write the graph of the **produce** relation
Delete states that are not reachable from S

3.4 Infinite Languages and Recursion

Interesting languages are infinite. Infinite languages require the grammar generating them to be recursive

Recursive derivation $A \overset{n}{\Longrightarrow} xAy$

Immediately recursive derivation $A \overset{1}{\Longrightarrow} xAy$

Left-recursive derivation $A \overset{n}{\Longrightarrow} Ay$

Right-recursive derivation $A \overset{n}{\Longrightarrow} xA$

Infinity condition $|L(G)| = \infty \iff G$ is clean $\wedge$ G avoids circular derivations $\wedge$ G allows recursive derivations

3.5 Syntax Trees and Canonical Derivations

Syntax tree A graph representing the derivation process which is

- Oriented
- Sorted (Top-down, Left-to-right)
- Acyclical
- $\forall n_1, n_2 \exists!$ a path $n_1 \leftrightarrow n_2$

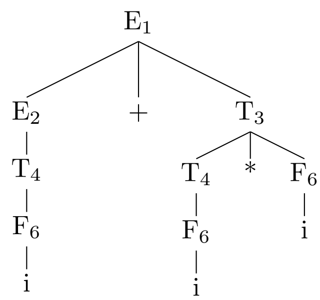
Subtree with root N is the tree having N as root, includes N and all its descendant

Example grammar

- $E \to E + T|T$
- $T \to T * F|F$
- $F \to (E)|i$

Example sentence $i + i * i$

Example tree

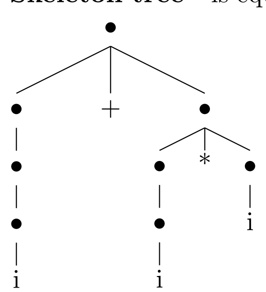


Left derivation the left-most rule is applied first

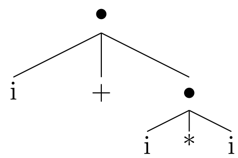
Right derivation the right-most rule is applied first

Unicity of derivations for a fixed syntax tree $\exists!$ the left and right derivations

Skeleton tree is equal to the syntax tree with all the non-terminals obscured



Condensed skeleton tree obtained from the skeleton tree by merging internal nodes and non-branching paths



3.5.1 Parenthesis languages

Are expressed by the Dyck language

- $\Sigma = \{a, c\}$
- $S \to aScS|\epsilon$

We can observe that $L_1 = \{a^n c^n | n \geq 1\} \subset L_{\text{DYCK}}$

3.6 Regular composition of (Context) Free Languages

The family of free languages is closed under union, concatenation, kleen star. Given $G_1 = (\Sigma_1, V_{N_1}, P_1, S_1)$ and $G_2 = (\Sigma_2, V_{N_2}, P_2, S_2$ such that $V_{N_1} \cap V_{N_2} = \emptyset \wedge S \notin (V_{N_1} \cup V_{N_2})$

Union $G_1 \cup G_2 = (\Sigma_1 \cup \Sigma_2, \quad V_{N_1} \cup V_{N_2} \cup \{S\}, \quad P_1 \cup P_2 \cup \underbrace{\{S \to S_1 | S_2\}}_{\text{execute concatenation at the beginning}}, \quad S)$

Concatenation $G_1 G_2 = (\Sigma_1 \cup \Sigma_2, \quad \{S\} \cup V_{N_1} \cup V_{N_2}, \quad P_1 \cup P_2 \cup \underbrace{\{S \to S_1 | S_2\}}_{\text{choose one of the languages at the beginning}}, \quad S)$

Kleen star $G_1^* = (\Sigma_1, \{S\} \cup V_{N_1}, \quad P_1 \cup \underbrace{\{S \to SS_1 | \epsilon\}}_{\text{Perform repetition}}, \quad S$

Cross $G_1^* = (\Sigma_1, \{S\} \cup V_{N_1}, \quad P_1 \cup \underbrace{\{S \to SS_1 | S_1\}}_{\text{Perform repetition}}, \quad S$

3.7 Ambiguity

Syntactic ambiguity A sentence x of a grammar G is ambiguous if it admits multiple distinct syntax trees

Degree of ambiguity DOA

of a sentence x of a language $L(G)$: $DOA(x)$ =number of distinct trees for x compatible with G

of a grammar $DOA(G) = max(\{DOA(x) | x \in L(G)\})$. It may happen that $DOA(G) = \infty$

Determining if a grammar is ambiguous is a semi-decidable problem. Can be proven only if the grammar is ambiguous.

3.7.1 Ambiguous forms and remedies

Bilateral recursion $S \to SxS|y$ where $x, y \in \Sigma \cup V$

[Right-recursive] $S \to yS|y$

[Left-recursive] $S \to Sy|y$

Left and right recursion in different rules $S \to Sa|bS|c$

[Separate] $S \to AcB$, $A \to Aa$, $B \to bB$

[Enforce order] $S \to aS|B$, $B \to Xb|c$

Union If $G = G_1 \cup G_2$ and $L(G_1) \cap L(G_2) \neq \emptyset$ then some sentences in G can be derived using both the rules of G_1 or the rules of G_2

[Disjoint] provide disjointed set of rules: $G = (G_1 \cap G_2) \cup (G_1 \setminus G_2) \cup (G_2 \setminus G_1)$ and the rules of these subsets are disjointed

Concatenation $G = G_1G_2$ is ambiguous if

$$\exists x_1, u \in L_1 \exists x_2, z \in L_2 \exists v \neq \epsilon | x_1 = uv \wedge x_2 = vz$$

$$S \Rightarrow S_1S_2 \stackrel{+}{\Longrightarrow} uS_2 \stackrel{+}{\Longrightarrow} uvz \qquad \wedge \qquad S \Rightarrow S_1S_2 \stackrel{+}{\Longrightarrow} uvS_2 \stackrel{+}{\Longrightarrow} uvz$$

Inherent ambiguity L is inherently ambiguous if any grammar G for L is ambiguous.

[Avoidance] inherent ambiguity is rare and can be avoided

Others See slides

For practical purposes, it is also possible to modify the language (and for programming languages this may be desirable)

3.8 Grammar equivalence

Weak equivalence G_1 and G_2 are weakly equivalent if $L(G_1) = L(G_2)$. Semi decidable

Strong equivalence G_1 is strongly/structurally equivalent to G_2 if $L(G_1) = L(G_2) \wedge G_1$ and G_2 have the same **condensed skeleton tree**. Decidable

3.9 Normal forms and Transformations

Expansion of a non-terminal allows to eliminate it from the rules where it appears

$$A \to xBy \quad B \to b_1|b_2|...|b_n$$

$$A \to xb_1y|xb_2y|...|xb_ny$$

Elimination of the axiom S from right parts obtained by introducing a replacement axiom:

$$S_{new} \rightarrow S_{old}$$

3.9.1 Normal form without nullable nonterminals

Grammar such that $\forall A \in V \setminus \{S\} \quad \neg A \stackrel{+}{\Longrightarrow} \epsilon$

Nullable non-terminals A is nullable if $\exists A \stackrel{+}{\Longrightarrow} \epsilon$

Nullables set $Null \subseteq V$

Algorithm 3 Compute $Null$

```
repeat
  for all A ∈ V do
    if A → ϵ ∈ P then
      A ∈ Null
    else if A → A_1 A_2 ... A_n ∈ P | A_i ∈ V \ {A} and ∀A_i(A_i ∈ Null) then
      A ∈ Null
    end if
  end for
until convergence is reached
```

Algorithm 4 Construction of non-nullable normal form

```
Compute Null
for all R ∈ P do
  for all A ∈ Null do
    for all Occurrencies of A in the right part of R do
      Remove A in the given occurrence of R
      Add the resulting rule to P
    end for
  end for
end for
for all R = A → ϵ ∈ P | A ≠ S do
  P ← P \ R
end for
Clean the grammar
Remove circularities
```

3.9.2 Copy rules elimination

Copy rules reduce grammar size but increase derivation length

Example: $loop \rightarrow while|for|repeat$

Copy set $Copy(A) = \{B \in V | \exists A \overset{*}{\Rightarrow} B\}$

Algorithm 5 Computation of $Copy$

Require: $A \overset{+}{\Rightarrow} \epsilon \implies A = S$ (no empty rules)

```
repeat
  for all A ∈ V do
    A ∈ Copy(A)
    for all B, C ∈ V do
      if B ∈ Copy(A) and B → C ∈ P then
        C ∈ Copy(A)
      end if
    end for
  end for
until convergence is reached
```

Algorithm 6 Definition of equivalent grammar without copy rules

---

$P' \leftarrow P \setminus \{A \rightarrow B | A, B \in V\}$ {delete copy rules}
$P' \leftarrow P' \cup \{A \rightarrow a | \exists B(B \in Copy(A) \wedge (B \rightarrow a) \in P)\}$ {add compensating rules}

3.10 Conversion from left to right recursion