CS CS6220 HW 1

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1 Problem 2

Derive a finite difference approximation to the second derivative of some arbitrary function f (x) at the point x1, assuming that f (x) is sampled at 3 points, x0, x1 and x2. To do this derivation, use polynomial interpolation (in Lagrange form). Derive the finite difference error term using the error estimate for derivatives of polynomial interpolants. Assume the three points are equispaced with spacing h

We begin with the polynomial definition

$$
\begin{aligned}
p(x) &= \sum_{k=0}^{N} y_k l_x \quad \text{Lagrange definition of a polynomial } p \\
p(x) &= y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x) \quad \text{Expansion around } N = 2 \\
&= y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\
&= y_0 \frac{x^2 - xx_1 - xx_2 + x_1x_2}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{x^2 - xx_0 - xx_2 + x_0x_2}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{x^2 - xx_0 - xx_1 + x_0x_1}{(x_2-x_0)(x_2-x_1)}
\end{aligned}
$$

The error for a polynomial approximation is given as

$$
E = \frac{f^{(N+1)}(\xi_x)}{(N+1)!} \prod_{k=0}^{N} (x - x_k)
$$

$$
\begin{aligned}
p'(x) &= y_0 \frac{2x - x_1 - x_2}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{2x - x_0 - x_2}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{2x - x_0 - x_1}{(x_2-x_0)(x_2-x_1)} \\
p''(x) &= y_0 \frac{2}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{2}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{2}{(x_2-x_0)(x_2-x_1)}
\end{aligned}
$$

Taking the second derivative of the error and expanding the e term we get

$$E'' = \frac{f^{(N+1)}(\xi_x)}{(N+1)!}e''(x) + 2e'(x)\frac{f^{(N+2)}(\xi_x)}{(N+1)!}\frac{d}{d_x}\xi_x + \frac{e(x)}{(N+1)!}f^{(N+3)}(\xi_x)\frac{d}{d_x}\xi_x$$
$$= \frac{f^{(3)}(\xi_x)}{(3)!}e''(x) + 2e'(x)\frac{f^{(4)}(\xi_x)}{(3)!}\frac{d}{d_x}\xi_x + \frac{e(x)}{(3)!}f^{(5)}(\xi_x)\frac{d}{d_x}\xi_x$$
$$= \frac{f^{(3)}(\xi_x)}{(3)!}\left[(x-x_0)(x-x_1)(x-x_2)\right]''$$
$$+2\left[(x-x_0)(x-x_1)(x-x_2)\right]'\frac{f^{(4)}(\xi_x)}{(3)!}\frac{d}{d_x}\xi_x$$
$$+\frac{\left[(x-x_0)(x-x_1)(x-x_2)\right]}{(3)!}f^{(5)}(\xi_x)\frac{d}{d_x}\xi_x$$

$$= \frac{f^{(3)}(\xi_x)}{(3)!}\left[x^3 - x^2x_2 - x^2x_1 + xx_1x_2 - x^2x_0 + xx_0x_2 + xx_0x_1 - x_0x_1x_2\right]''$$
$$+2\left[x^3 - x^2x_2 - x^2x_1 + xx_1x_2 - x^2x_0 + xx_0x_2 + xx_0x_1 - x_0x_1x_2\right]'\frac{f^{(4)}(\xi_x)}{(3)!}\frac{d}{d_x}\xi_x$$
$$+\frac{\left[x^3 - x^2x_2 - x^2x_1 + xx_1x_2 - x^2x_0 + xx_0x_2 + xx_0x_1 - x_0x_1x_2\right]}{(3)!}f^{(5)}(\xi_x)\frac{d}{d_x}\xi_x$$

$$= \frac{f^{(3)}(\xi_x)}{(3)!}(6x - 2x_2 - 2x_1 - 2x_0)$$
$$+ 2(3x^2 - 2xx_2 - 2xx_1 - 2xx_0 + x_2x_1 + x_2x_0 + x_1x_0)\frac{f^{(4)}(\xi_x)}{(3)!}\frac{d}{d_x}\xi_x$$
$$+ \frac{(x^3 - x^2x_2 - x^2x_1 + xx_1x_2 - x^2x_0 + xx_0x_2 + xx_0x_1 - x_0x_1x_2)}{(3)!}f^{(5)}(\xi_x)\frac{d}{d_x}\xi_x$$

2 Problem 3

Derive a quadrature-based approximation to $\int_a^b f(x)dx$ using a quadratic polynomial interpolant to f . Assume you have 3 equispaced points, including the end points a and b; that is, let $x0 = a, x1 = (a+b)/2 \text{and} x2 = b$ be the 3 points. Derive the corresponding error term for this quadrature rule using the error estimate for integrals of polynomials. What is this quadrature rule called?

$$
\begin{aligned}
p(x) &= \sum_{k=0}^{N} y_k l_x \\
&= y_0 \frac{x^2 - xx_1 - xx_2 + x_1 x_2}{(x_0 - x_1)(x_0 - x_2)} \\
&+ y_1 \frac{x^2 - xx_0 - xx_2 + x_0 x_2}{(x_1 - x_0)(x_1 - x_2)} \\
&+ y_2 \frac{x^2 - xx_0 - xx_1 + x_0 x_1}{(x_2 - x_0)(x_2 - x_1)} \\
\int_b^a f(x)dx &\approx \int_b^a p(x)dx \\
&= (y_0 \frac{x^3/3 - x^2/2 \cdot x_1 - x^2/2 \cdot x_2 + xx_1 x_2}{(x_0 - x_1)(x_0 - x_2)} \\
&+ y_1 \frac{x^3/3 - x^2/2 \cdot x_0 - x^2/2 \cdot x_2 + xx_0 x_2}{(x_1 - x_0)(x_1 - x_2)} \\
&+ y_2 \frac{x^3/3 - x^2/2 \cdot x_0 - x^2/2 \cdot x_1 + xx_0 x_1}{(x_2 - x_0)(x_2 - x_1)})|_a^b \\
&= (y_0 \frac{x^3/3 - x^2/2 \cdot \frac{a+b}{2} - x^2/2 \cdot b + x\frac{a+b}{2}b}{(a - (a+b)/2)(a-b)} \\
&+ y_1 \frac{x^3/3 - x^2/2 \cdot a - x^2/2 \cdot b + xab}{((a+b)/2 - a)((a+b)/2 - b)} \\
&+ y_2 \frac{x^3/3 - x^2/2 \cdot a - x^2/2 \cdot \frac{a+b}{2} + xa\frac{a+b}{2}}{(b-a)(b-(a+b)/2)})|_a^b
\end{aligned}
$$

$$
\boxed{= -a\frac{y_0}{6} - 2\frac{ay_1}{3} - a\frac{y_2}{6} + b\frac{y0}{6} + 2b\frac{y_1}{3} + b\frac{y_2}{6}}
$$