CS CS6220 HW 1

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1 Problem 1

1.1 Trapezoidal Rule

The trapezoidal rule is given by the equation

$$y^{n+1} = y^n + \frac{\Delta t}{2}\left(f(t_n, y^n) + f(t_{n+1}, y^{n+1})\right) \tag{1}$$

The linear test equation we will use to show A-Stability is given by

$$\frac{\delta y}{\delta t} = \lambda y, \lambda \in \mathrm{C}, \Re\{\lambda\} < 0 \tag{2}$$

Plugging this equation into the trapezoidal equation we get

$$y^{n+1} = y^n + \frac{\Delta_t}{2}\lambda(y^n + y^{n+1}) \tag{3}$$

$$= y^n + \frac{\Delta_t}{2}\lambda y^n + \frac{\Delta_t}{2}\lambda y^{n+1} \tag{4}$$

Rearranging for y^{n+1} we get

$$y^{n+1} - \frac{\Delta_t}{2}\lambda y^{n+1} = \frac{\Delta_t}{2}\lambda y^n \tag{5}$$

$$y^{n+1}\left(1 - \frac{\Delta_t}{2}\lambda\right) = y^n\left(1 + \frac{\Delta_t}{2}\lambda\right) \tag{6}$$

$$y^{n+1} = \frac{y^n\left(1 + \frac{\Delta_t}{2}\lambda\right)}{\left(1 - \frac{\Delta_t}{2}\lambda\right)} \tag{7}$$

Giving g to be the stability function we can show A-stability by proving the inequality $\left|g\right| \leq 1$

$$g(\lambda\Delta_t) = \frac{\left(1+\frac{\Delta_t}{2}\lambda\right)}{\left(1-\frac{\Delta_t}{2}\lambda\right)} \tag{8}$$

$$|g| = \left|\frac{\left(1+\frac{\Delta_t}{2}\lambda\right)}{\left(1-\frac{\Delta_t}{2}\lambda\right)}\right| \leq 1 \tag{9}$$

From equation (9) it can be observed that given the conditions on λ, specifically $\Re\{\lambda\} < 0$, the inequality always holds true. Therefore, the trapezoidal rule is A-Stable.

1.2 Backward Euler

The equation for Backward Euler is given by (10)

$$y^{n+1} - \Delta t f(t_{n+1}, y^{n+1}) = y^n \tag{10}$$

Applying the linear test equation (2) to (10) results in (11).

$$y^{n+1} - \Delta t \lambda y^{n+1} = y^n \tag{11}$$

$$= y^{n+1}(1 - \Delta t \lambda) \tag{12}$$

After simplifying (12) g can be obtained by dividing to get y^{n+1} on its own.

$$y^{n+1} = \frac{y^n}{1-\Delta t\lambda} \tag{13}$$

$$g(\Delta_t\lambda) = \frac{1}{1-\Delta t\lambda} \tag{14}$$

given the constraints on λ it can be shown that the denominator is strictly increasing with λ decreasing. Therefore the condition for A-stability holds.

2 Problem 2

2.1 Fwd Euler

See code

2.2 RK2

See code

2.3 Accuracy

Both FWD Euler, and RK2 provided the same accuracy compared to the exact solution. This is likely due to the fact that RK2 is using FWD Euler to bootstrap its process. The error is plotted as the L2 loss between the exact solution and the estimated solutions in fig. 1

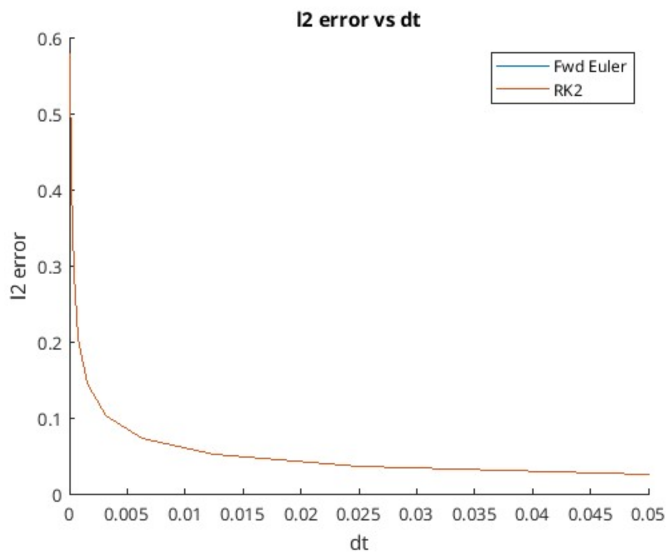

Figure 1: L2 Error

3 Problem 3

3.1 AB3 Code

See code

3.2 Stiff Problem

I would choose AB2 because it is more stable than an AB3 method which is of importance when dealing with stiff systems.

3.3 Imaginary Eigen Values

I would choose AB3 because the system is oscillatory so evaluation need happen over only a single cycle, reducing the need for stability over performance gains from the higher convergence rate of the AB3 system.

3.4 Convergence Comparison

I wasn't able to get a plot to demonstrate the convergence rate of AB2 over AB3, but my attempts are in the code.

4 Problem 4

General form of RK3 method

$$y^{n+1} = y^n + \Delta t \sum_{i=1}^{S} b_i k_i \tag{15}$$

$$k_1 = f(t_n, y_n) \tag{16}$$

$$k_2 = f(t_n + C_2 \Delta t, y_n + \Delta t a_{21} k_1) \tag{17}$$

$$k_3 = f(t_n + C_3 \Delta t, y_n + \Delta t(a_{31} k_1 + a_{32} k_2)) \tag{18}$$

$$y^{n+1} = y^n \Delta t(b_1 k_1 + b_2 k_2 + b_3 k_3) + \mathcal{O}\Delta t^4 \tag{19}$$

Solve for $C_2, C_3, a_{21}, a_{31}, a_{32}, b_1, b_2, b_3$

We need to match the Taylor expansion of y^{n+1} to the Taylor expansion of the general RK3 method.

Start with the tayor expansion of y^{n+1}

$$y^{n+1} = y^n + \Delta t \frac{\delta y^n}{\delta t} + \frac{\Delta t^2}{2} \frac{\delta^2 y^n}{\delta t^2} + \frac{\Delta t^3}{6} \frac{\delta^3 y^n}{\delta t^3} + \mathcal{O}\Delta t^4 \tag{20}$$

From the ODE, we have $y'(t) = f(t, y)$, from which follows:

$$y''(t) = y'(t) f_y(t, y) + f_t(t, y) \tag{21}$$

$$= f(t, y) f_y(t, y) + f_t(t, y) \tag{22}$$

$$y'''(t) = \frac{\delta f'(t, y)}{\delta t} = \tag{23}$$

$$\begin{aligned} y''' &= y'(t) f_{yy}(t, y) + 2y' f_{ty}(t, y) + y''(t) f_y(t, y) + f_{tt}(t, y) \\ &= f(t, y) f_{yy}(t, y) + 2f(t, y) f_{ty}(t, y) + f(t, y) f_y(t, y) + f_t(t, y) f_y(t, y) + f_{tt}(t, y) \end{aligned} \tag{24}$$

Plugging this expression back into the expression for y^{n+1} we get

$$\begin{aligned} y^{n+1} = {} & y^n + \Delta t f(t_n, y^n) + \frac{\Delta t^2}{2} \big(f(t, y) f_y(t, y) + f_t(t, y) \big) + \\ & \frac{\Delta t^3}{6} \big(f(t, y) f_{yy}(t, y) + 2f(t, y) f_{ty}(t, y) + f(t, y) f_y(t, y) + f_t(t, y) f_y(t, y) + f_{tt}(t, y) \big) + \mathcal{O}\Delta t^4 \end{aligned}$$

Now we can expand the RK3 scheme, and compare.

$$f(t_n + C_2\Delta t, y_n + \Delta t a_{21} k_1) \tag{25}$$

$$= f(t_n + C_2\Delta t, y_n + \Delta t a_{21} f(t_n, y_n)) \tag{26}$$

$$\begin{aligned} &= f\left(\mathrm{t_n}, \mathrm{y_n}\right) + \Delta \mathrm{t}\left(c_2\, f_t(t_n, y^n) + a_{21}\, f\left(\mathrm{t_n}, \mathrm{y_n}\right)\, f_y(t_n, y^n)\right) + \\ &\frac{\Delta t^2}{2}\left(c_2\left(c_2\, f_{tt}\left(\mathrm{t_n}, \mathrm{y_n}\right) + a_{21}\, f\left(\mathrm{t_n}, \mathrm{y_n}\right)\, f_{yt}\left(\mathrm{t_n}, \mathrm{y_n}\right)\right)\right. \\ &\left. + a_{21}\, f\left(\mathrm{t_n}, \mathrm{y_n}\right)\left(c_2\, f_{yt}\left(\mathrm{t_n}, \mathrm{y_n}\right) + a_{21}\, f\left(\mathrm{t_n}, \mathrm{y_n}\right)\, f_{yy}\left(\mathrm{t_n}, \mathrm{y_n}\right)\right)\right) \end{aligned} \tag{27}$$

$$f(t_n + C_2\Delta t, y_n + \Delta t a_{21} k_1) \tag{28}$$

$$\begin{aligned} &= \Delta \mathrm{t}\left(\left(a_{31} f\left(\mathrm{t_n}, \mathrm{y_n}\right) + a_{32} f\left(\mathrm{t_n}, \mathrm{y_n}\right)\right) f_y\left(\mathrm{t_n}, \mathrm{y_n}\right) + c_3 f_t\left(\mathrm{t_n}, \mathrm{y_n}\right)\right) \\ &+ f\left(\mathrm{t_n}, \mathrm{y_n}\right) \\ &+ \frac{\Delta \mathrm{t}^2}{2}\left(c_3\left(c_3 f_{tt}\left(\mathrm{t_n}, \mathrm{y_n}\right) + \left(a_{31} f\left(\mathrm{t_n}, \mathrm{y_n}\right) + a_{32} f\left(\mathrm{t_n}, \mathrm{y_n}\right)\right) f_{yt}\left(\mathrm{t_n}, \mathrm{y_n}\right)\right)\right. \\ &+ \left(c_3 f_{yt}\left(\mathrm{t_n}, \mathrm{y_n}\right) + \left(a_{31} f\left(\mathrm{t_n}, \mathrm{y_n}\right) + a_{32} f\left(\mathrm{t_n}, \mathrm{y_n}\right)\right) f_{yy}\left(\mathrm{t_n}, \mathrm{y_n}\right)\right)\left(a_{31} f\left(\mathrm{t_n}, \mathrm{y_n}\right) + a_{32} f\left(\mathrm{t_n}, \mathrm{y_n}\right)\right)\Big) \\ &+ a_{32}\left(c_2 f_t\left(\mathrm{t_n}, \mathrm{y_n}\right) + a_{21} f\left(\mathrm{t_n}, \mathrm{y_n}\right) f_y\left(\mathrm{t_n}, \mathrm{y_n}\right)\right) f_y\left(\mathrm{t_n}, \mathrm{y_n}\right) \end{aligned} \tag{29}$$

Now matching the expanded RK3 expression with the expanded numerical expression we can create a system of non-linear equations relating the unknowns to the final expression. We can choose these values such that the expression holds true.

One such solution, Huen's 3rd order method, comes out to be

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 \\ 2/3 & 0 & 2/3 & 0 \\ \hline & 1/4 & 0 & 3/4 \end{array} \tag{30}$$