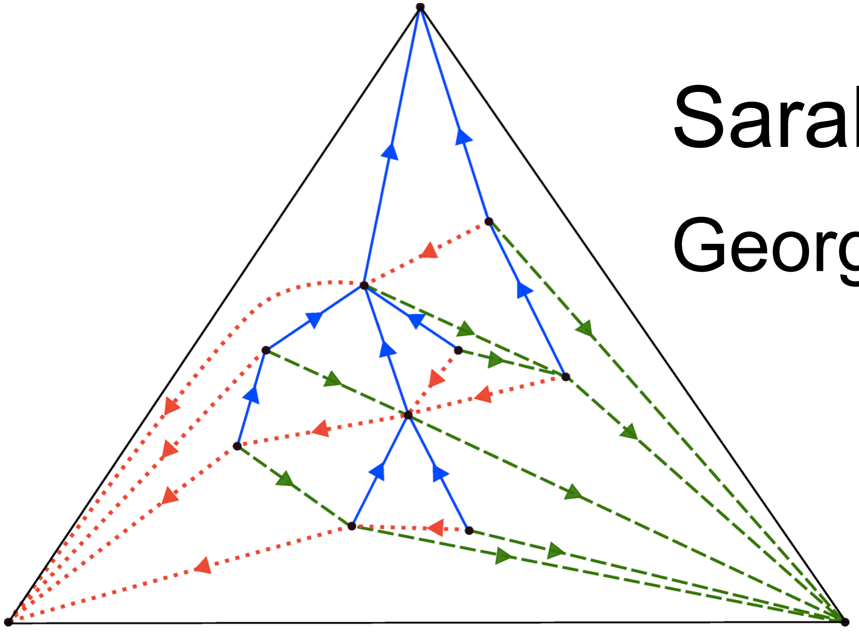
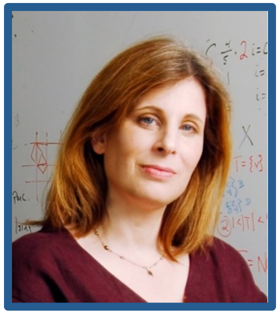
Mixing Times of Markov Chains on 3-Orientations of Planar Triangulations



Sarah Miracle

Georgia Institute of Technology

Joint work with . . . .



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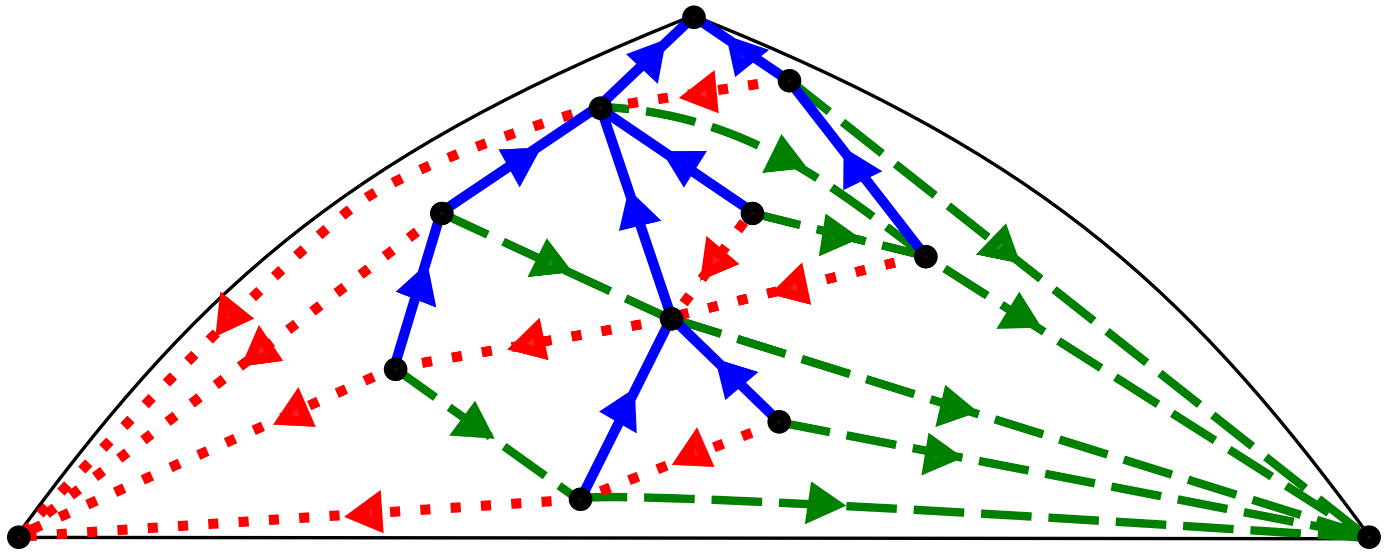
What is a 3-orientation?

An orientation of the internal edges of a planar triangulation such that

- The out-degree of each internal vertex is 3
- The out-degree of the 3 external vertices is 0

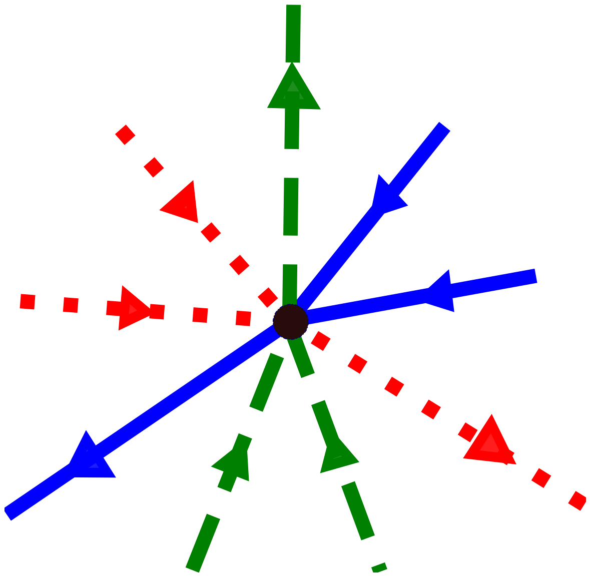


α-orientations

- (α-orientation) Given $G = (V,E)$ and $\alpha: V \rightarrow Z^+$, vertex v has out-degree $\alpha(v)$
- Bipartite perfect matchings, Eulerian orientations etc. are special instances of α-orientations
- Counting α-orientations is #P-complete
  - Let $\alpha(v) = d(v)/2$ and then α-orientations correspond to Eulerian orientations
  - Counting Eulerian orientations #P-complete [Mihail, Winkler]
  - Still #P-complete when restrict to planar graphs [Creed]

A Bijection with Schnyder Woods

- Each 3-orientation gives rise to a unique edge coloring known as a Schnyder wood where for each internal vertex v:
  - v has out-degree 1 in each of the 3 colors
  - the clockwise order of the edges incident to v is: outgoing green, incoming blue, outgoing red, incoming green, outgoing blue and incoming red
- Schnyder woods are used in
  - Graph drawing
  - Poset dimension theory
  - Counting Planar Maps
  - And many more ….



An Example Schnyder Wood

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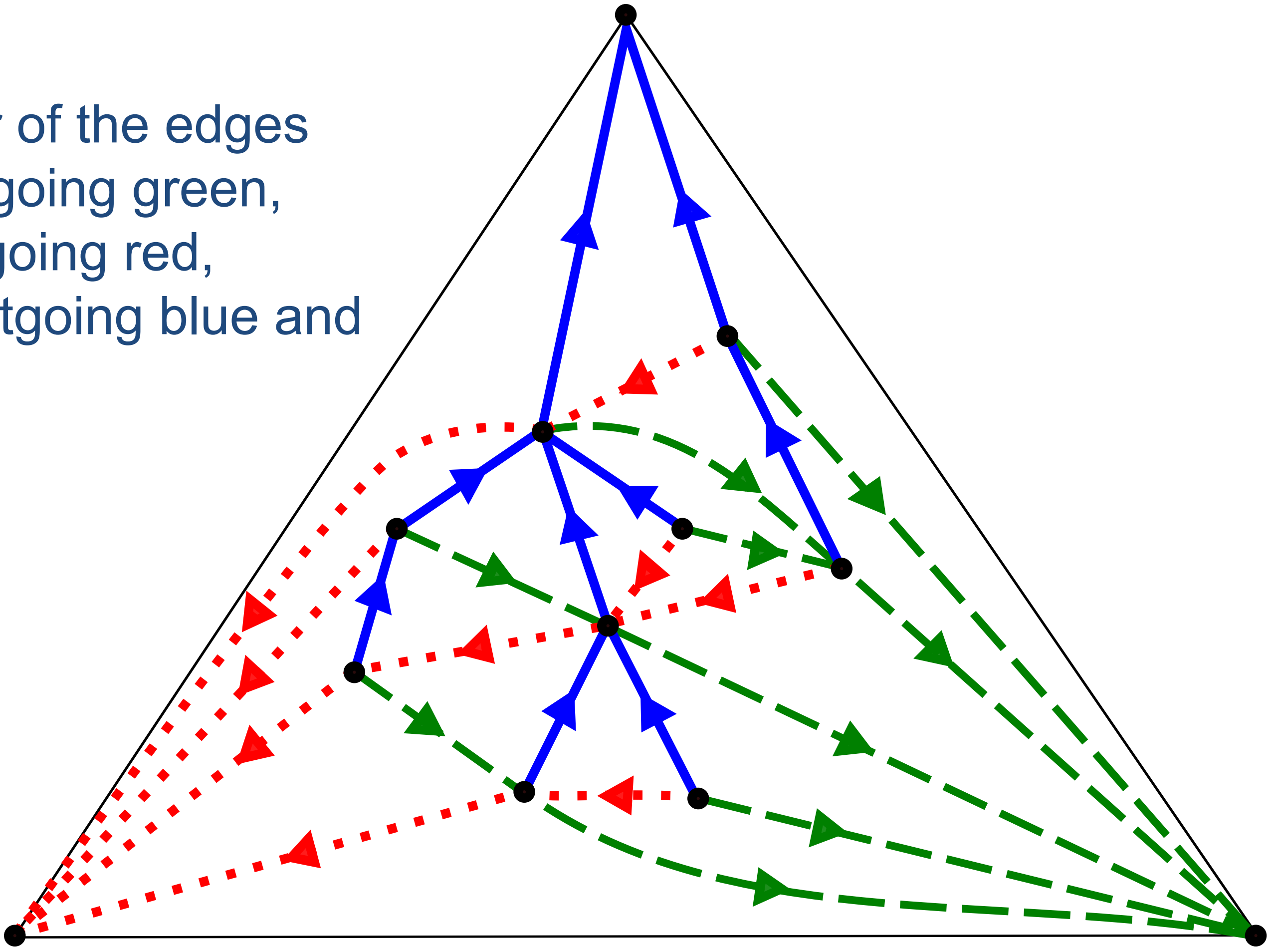

Two Sampling Problems

Sample from the set of all 3-orientations of a fixed triangulation.

Sample from the set of all 3-orientations of triangulations with n internal vertices.

The Mixing Time

Definition: The total variation distance is

$$||P^t,\pi|| = \max_{x \in \Omega} \frac{1}{2} \sum_{y \in \Omega} |P^t(x,y) - \pi(x)|.$$

Definition: Given ε, the mixing time is

$$\tau(\varepsilon) = \min \{t: ||P^{t'},\pi|| < \varepsilon, \quad \forall\, t' \geq t\}.$$

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(or polynomially mixing)

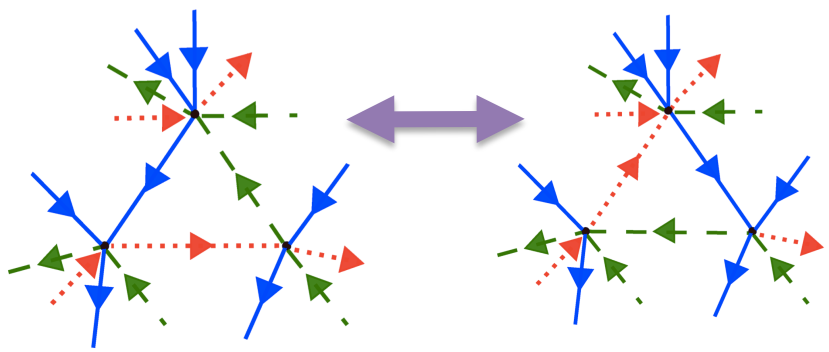
A Markov chain is slowly mixing if $\tau(\varepsilon)$ is at least exp(n).

Our Results

Sample from the set of all 3-orientations of a fixed triangulation.

1. The local chain is fast if max degree ≤ 6.

2. The local chain can take exponential time.



Sample from the set of all 3-orientations of triangulations with n internal vertices.

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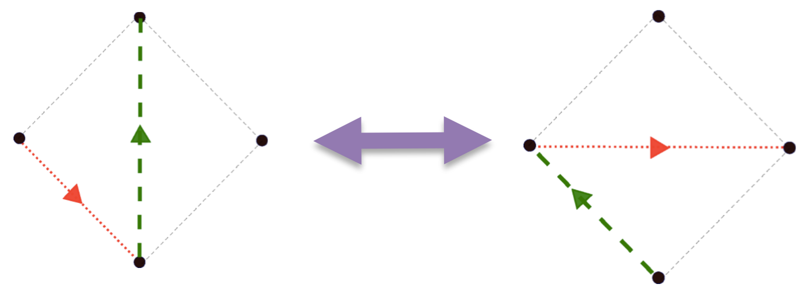

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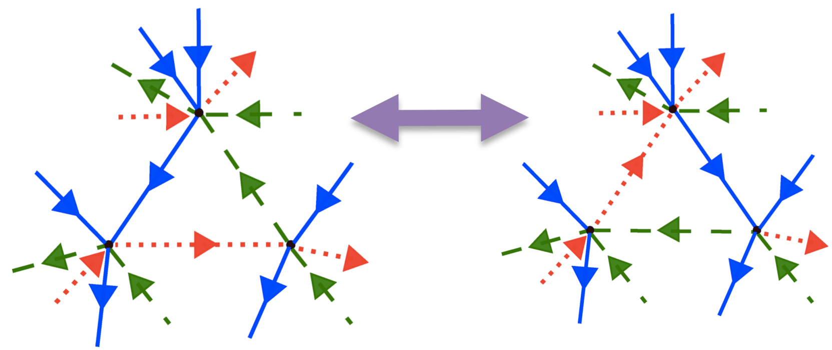

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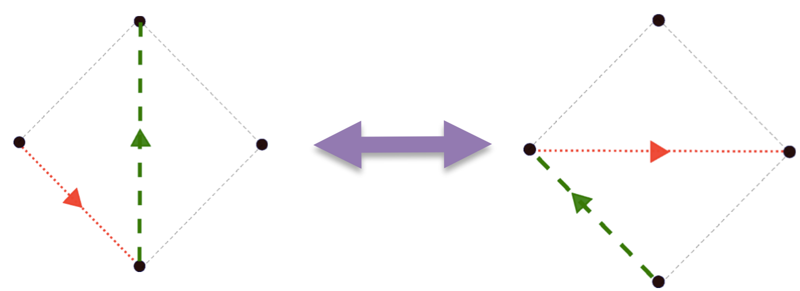
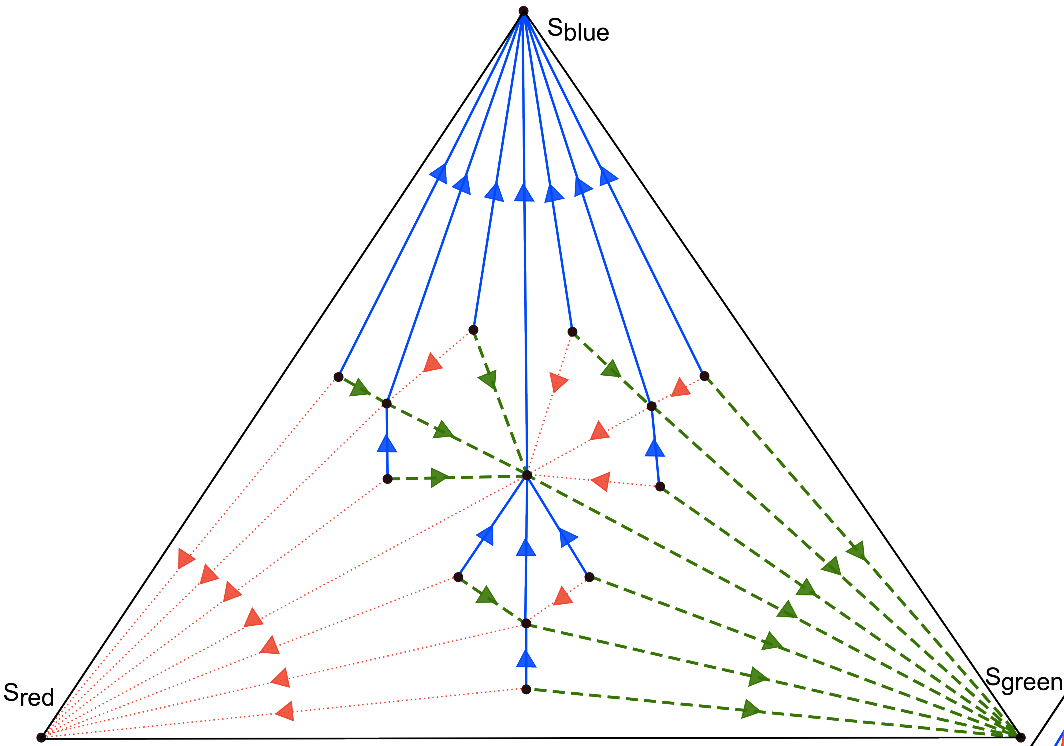
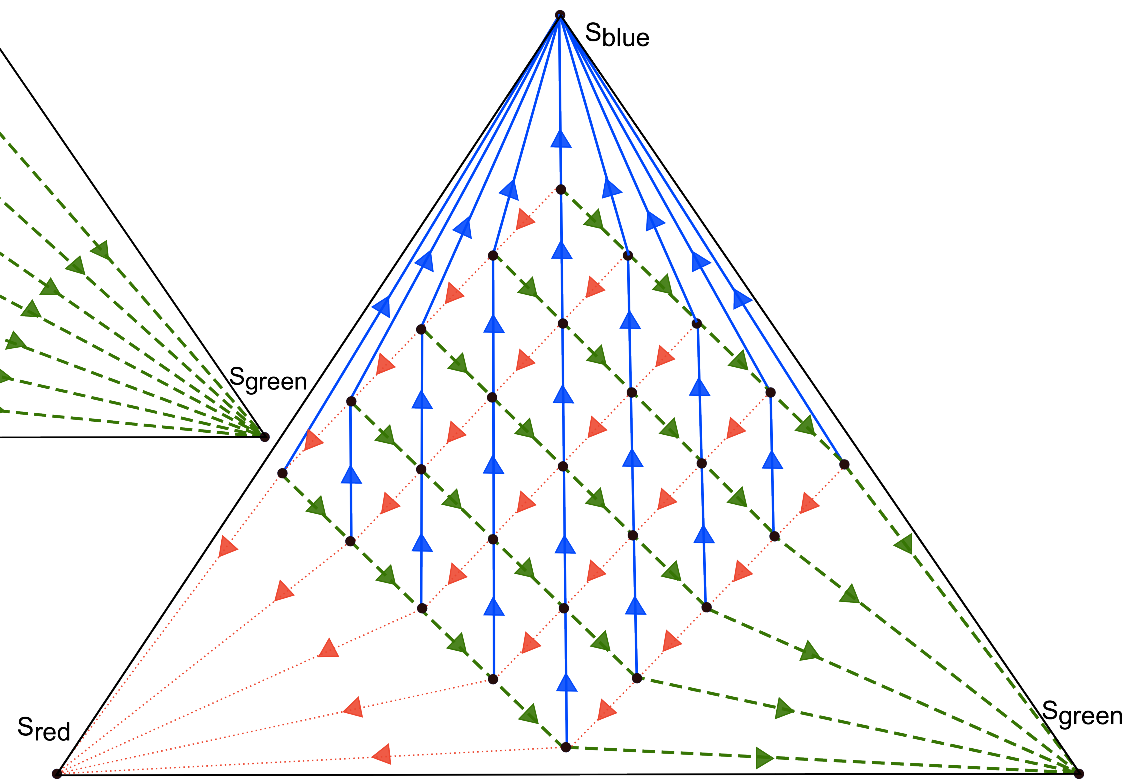


How Big is the Set of all 3-orientations of a Fixed Triangulation?





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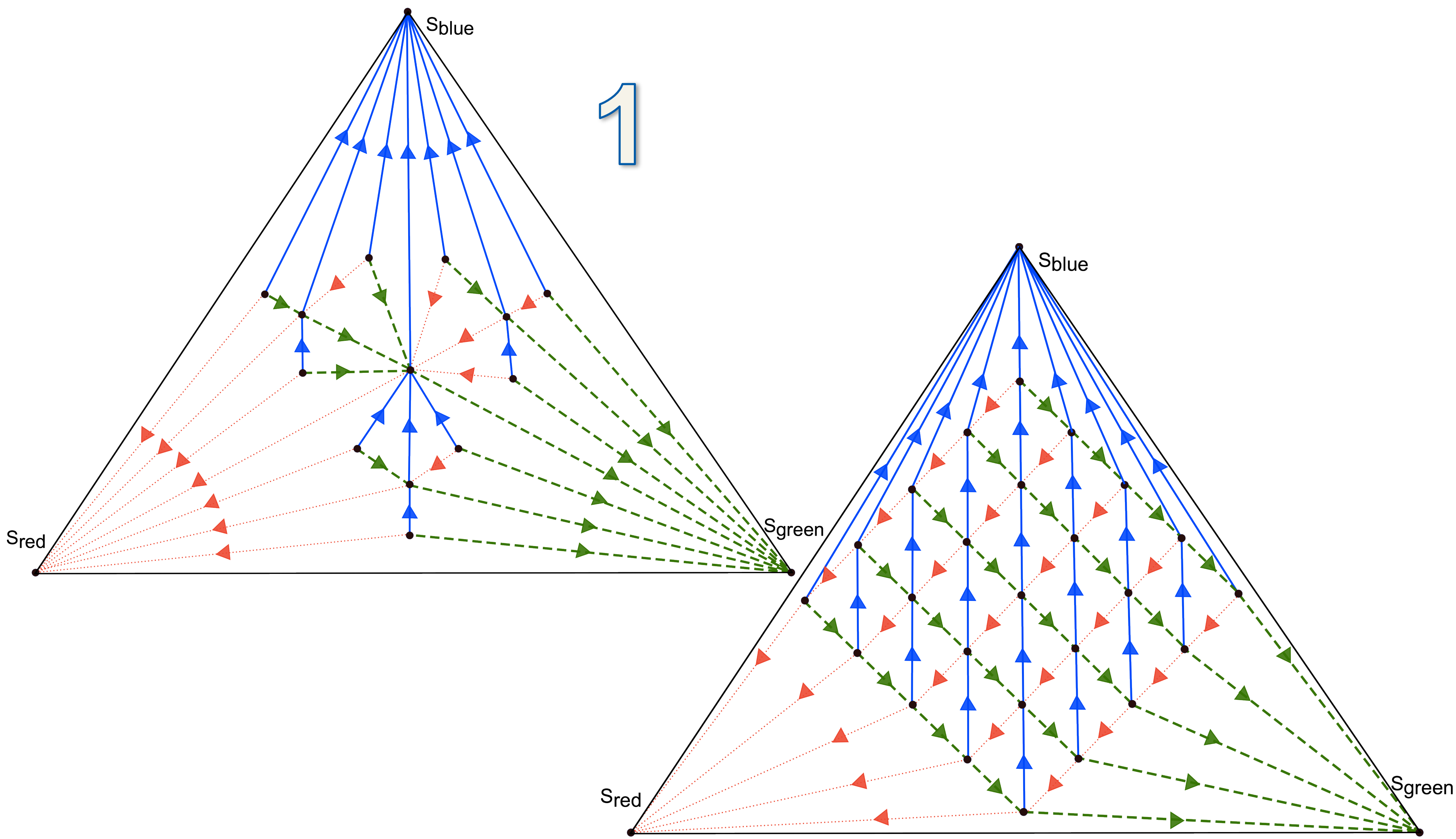

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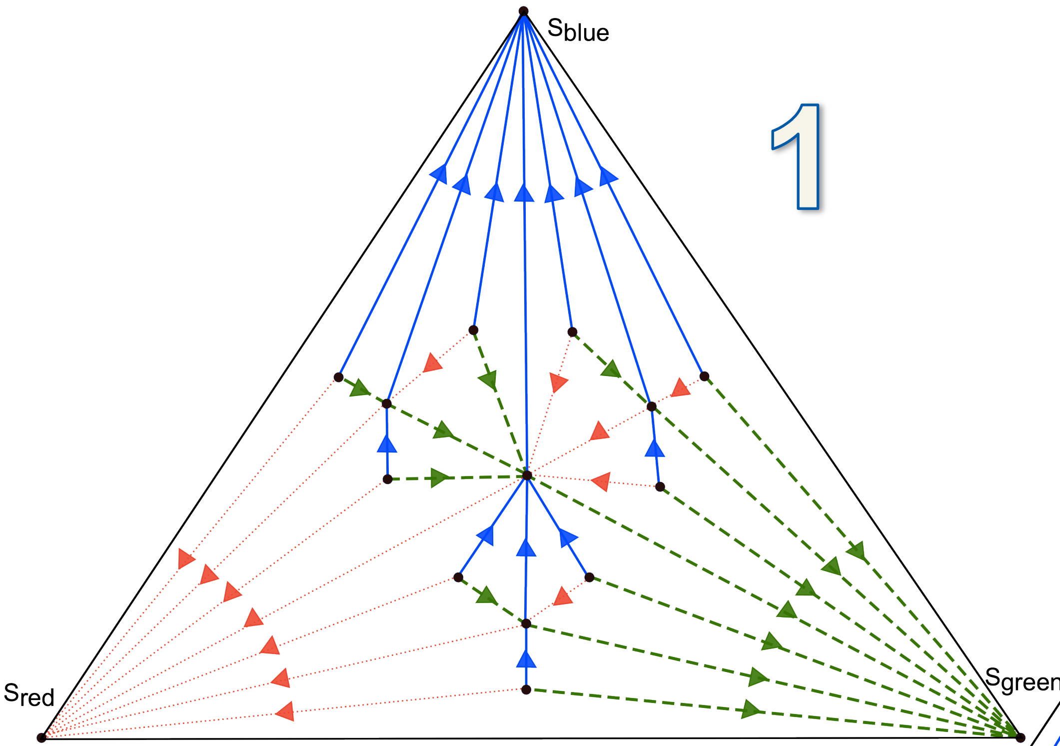
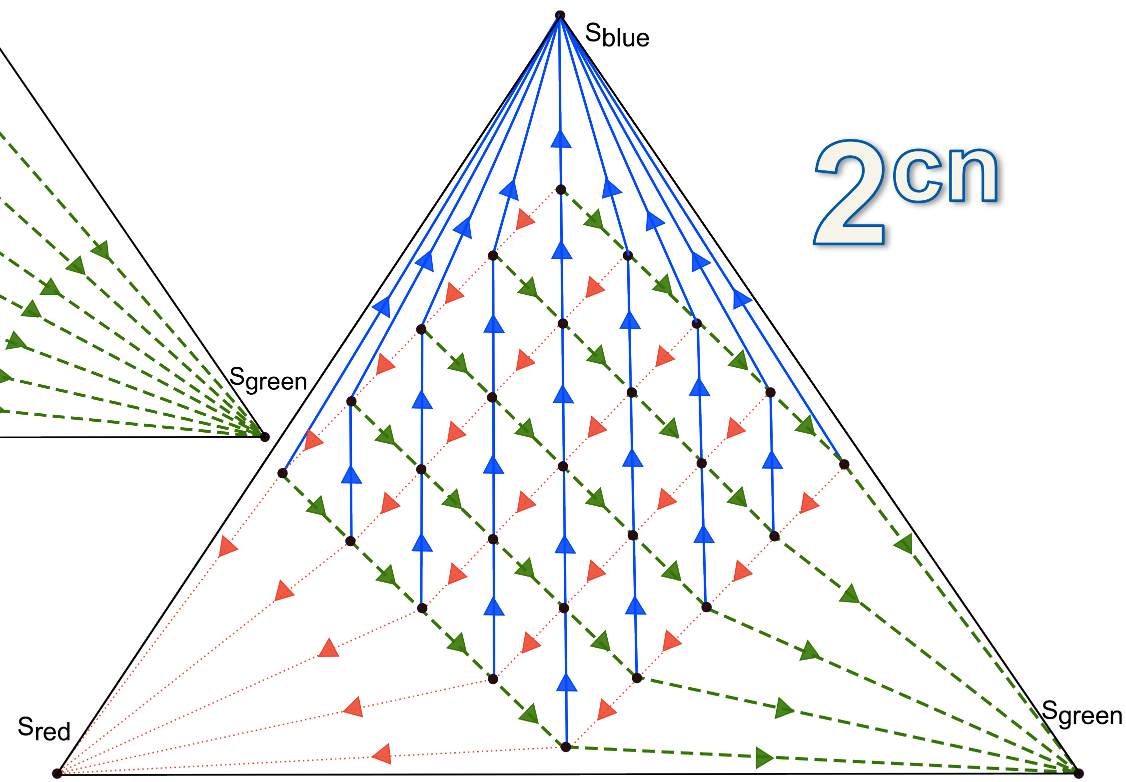




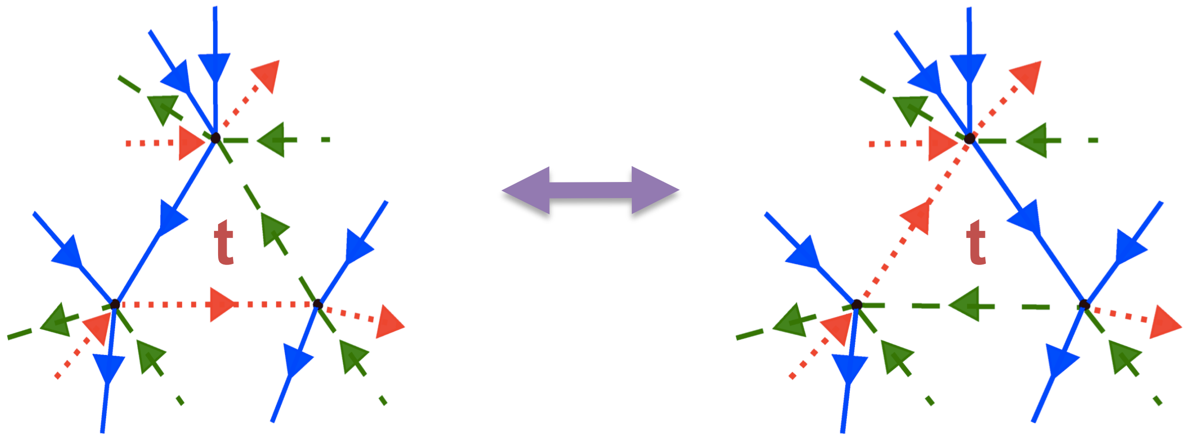
All 3-orientations of a Fixed Triangulation

- No known efficient method for exactly counting
- There is a known FPRAS based on a bijection with perfect matchings in bipartite graphs $O^*(n^7)$ [Bezáková et al.]
  - Improving on result by [Jerrum, Sinclair, Vigoda]
  - Implies an efficient sampling algorithm exists
- Special Case: triangular lattice $O(n^4)$ algorithm using a “tower chain” [Creed]

The Local Markov Chain $\mathcal{M}_{TR}$

Repeat:

- Pick a triangle **t**;
- If **t** is a directed cycle, reverse it with probability ½.



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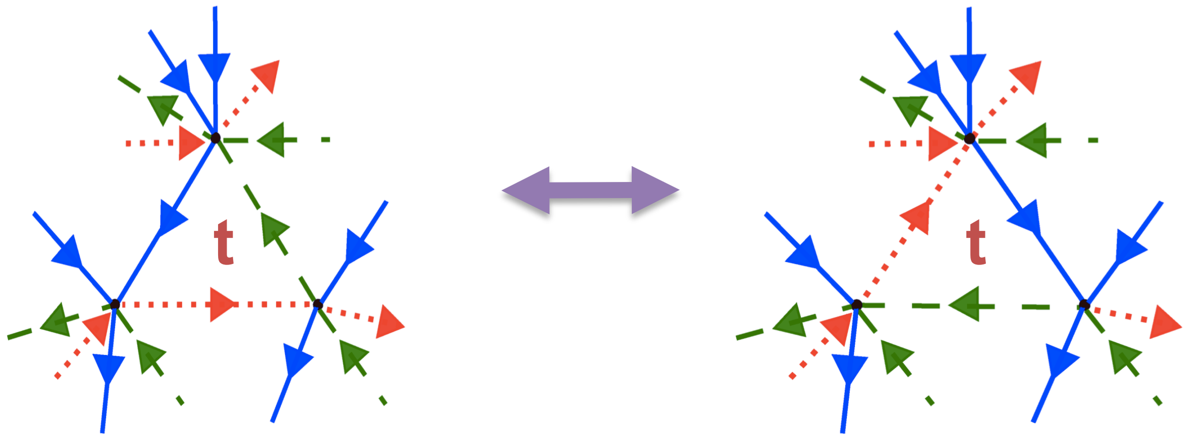

Thm: The local Markov chain $\mathcal{M}_{TR}$ connects the set of all 3-orientations of a fixed triangulations [Brehm].

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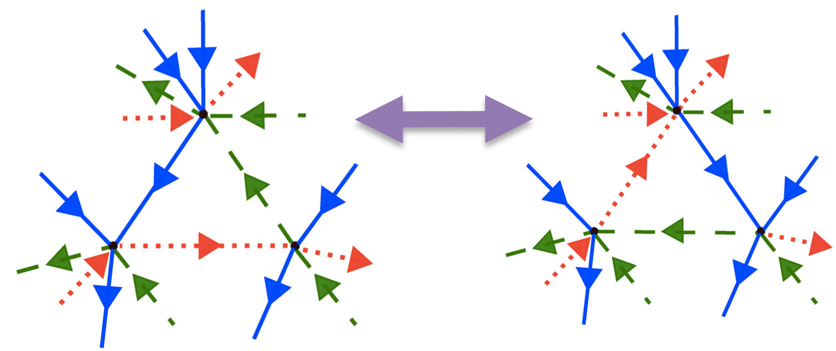
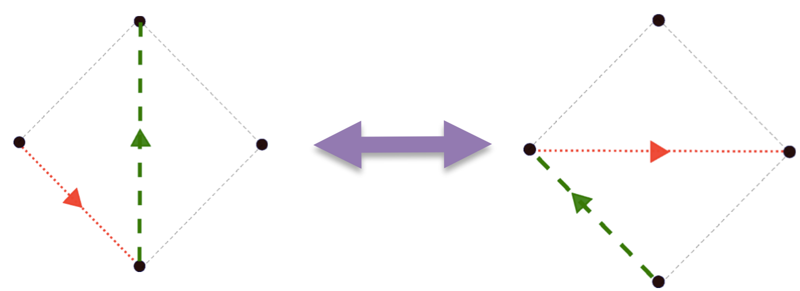


Sample from the set of all 3-orientations of triangulations with n internal vertices.

3. The local chain is fast.



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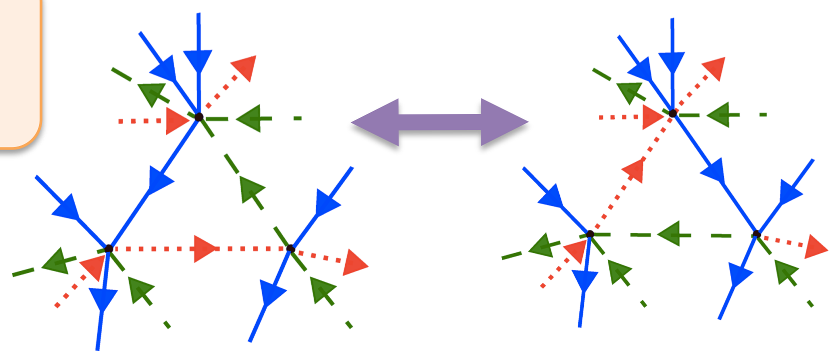

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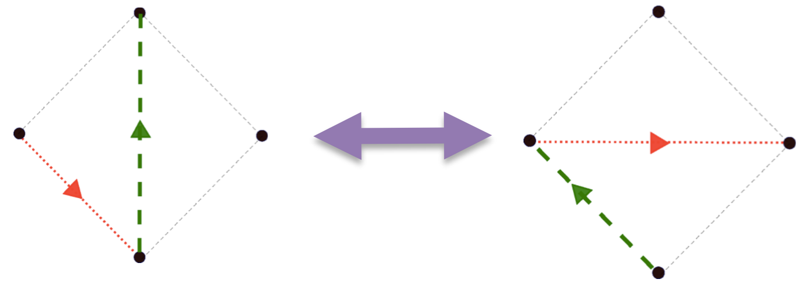

Fast Mixing for $\mathcal{M}_{TR}$

Let $\Delta_I(T)$ be the maximum degree of any internal vertex of T

Thm: If $\Delta_I(T) \leq 6$ then $\mathcal{M}_{TR}$ mixes in time $O(n^8)$

[M., Randall, Streib, Tetali]

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Proof sketch:

A. Define auxiliary Markov chain $\mathcal{M}_{CR}$

B. Show $\mathcal{M}_{CR}$ is rapidly mixing

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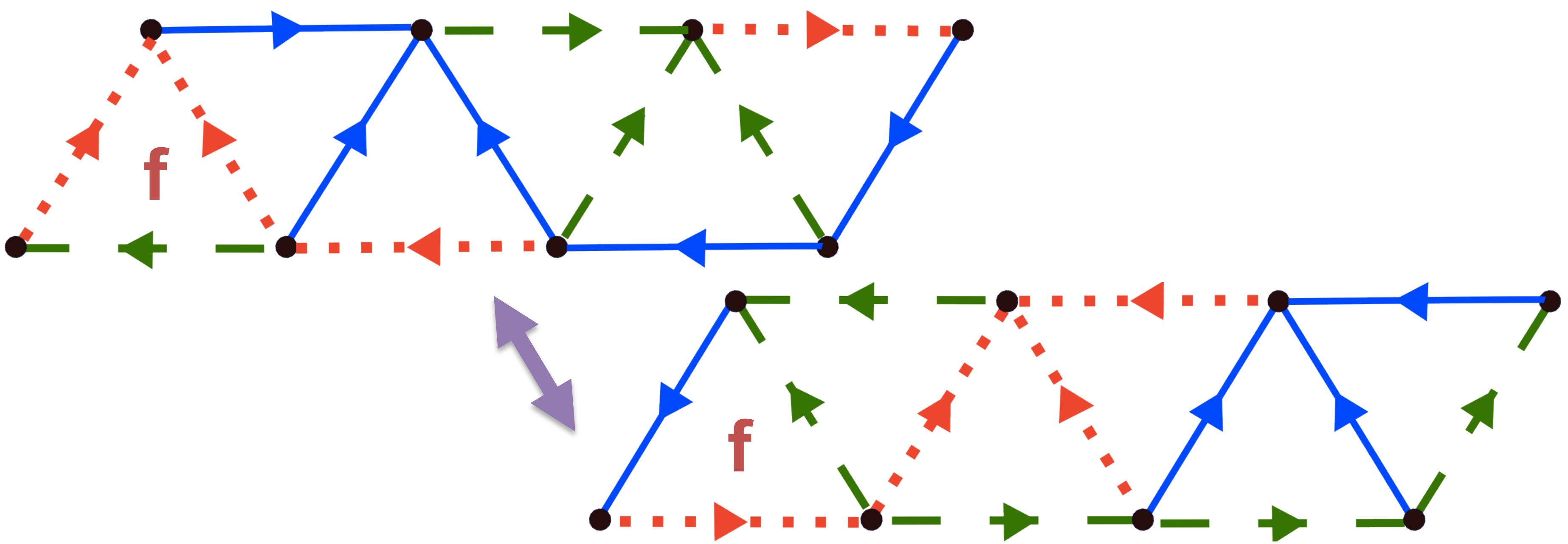
$\mathcal{M}_{CR}$ can reverse directed cycles which contain more than one triangle

- Maintains the same stationary distribution
- Moves are based on “tower moves” introduced by [Luby, Randall, Sinclair]

The Tower Chain $\mathcal{M}_{CR}$

Repeat:

- Pick a triangle **f**;
- If **f** is the beginning of a tower of length 1, reverse it with probability ½.
- If **f** is the beginning of a tower of length k ≥ 2, reverse it with probability 1/6k.

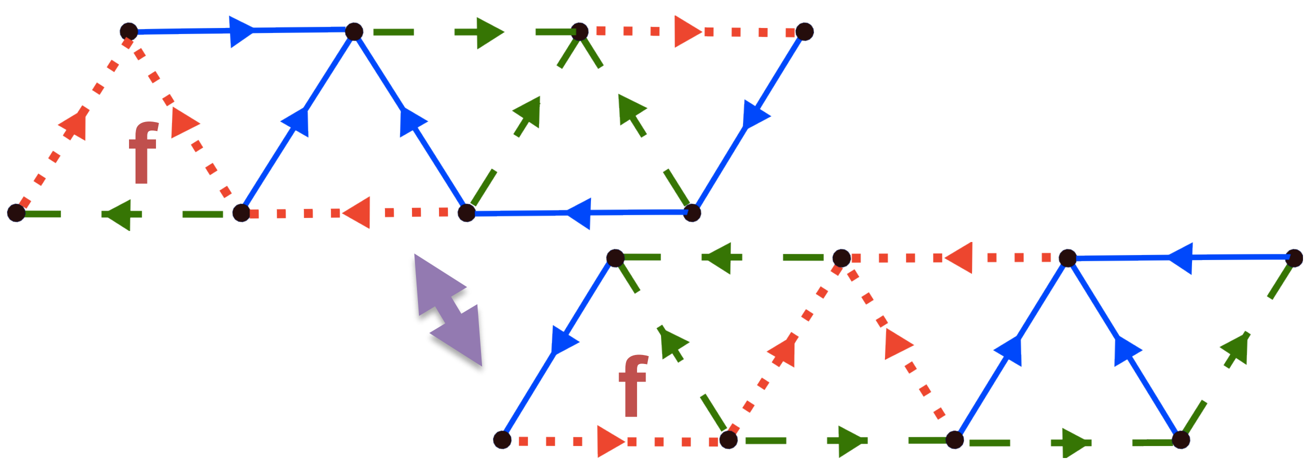


The Tower Chain $\mathcal{M}_{CR}$

A tower of length k is a path of faces

$f_1, f_2, \ldots, f_k$ such that:

1. f_k is the only face bounded by a directed cycle
2. For every $1 \le i < k$, the disagree edge of f_i is also incident with f_{i+1}
3. Every vertex v is incident to at most 3 consecutive faces in the path



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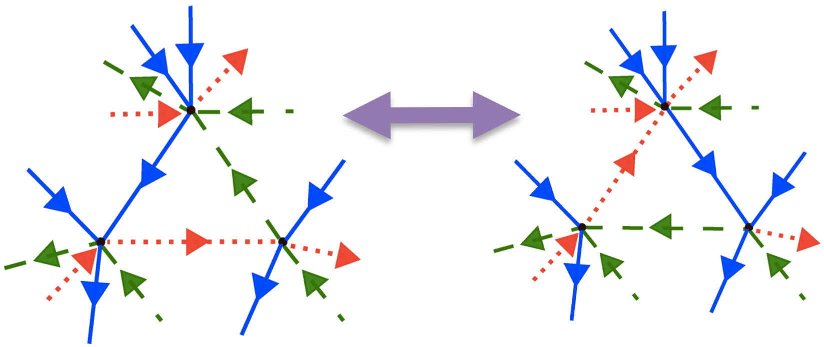

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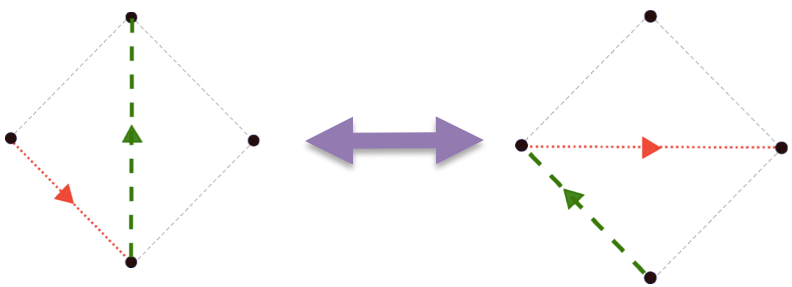

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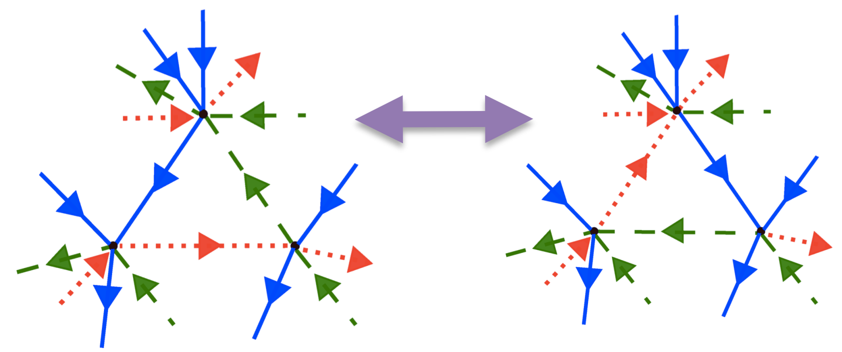

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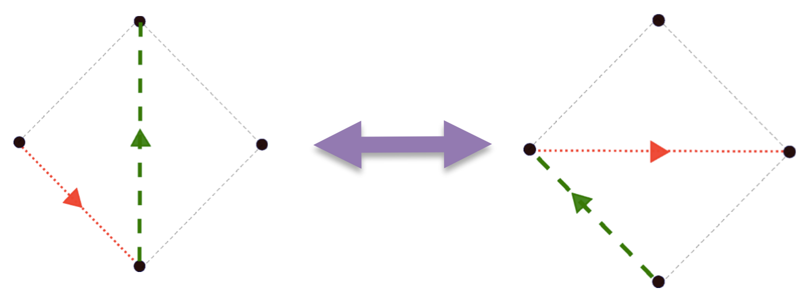

Slow Mixing for $\mathcal{M}_{\text{TR}}$

Thm: There exists a triangulation T on which $\mathcal{M}_{\text{TR}}$ requires exponential time. [M., Randall, Streib, Tetali]

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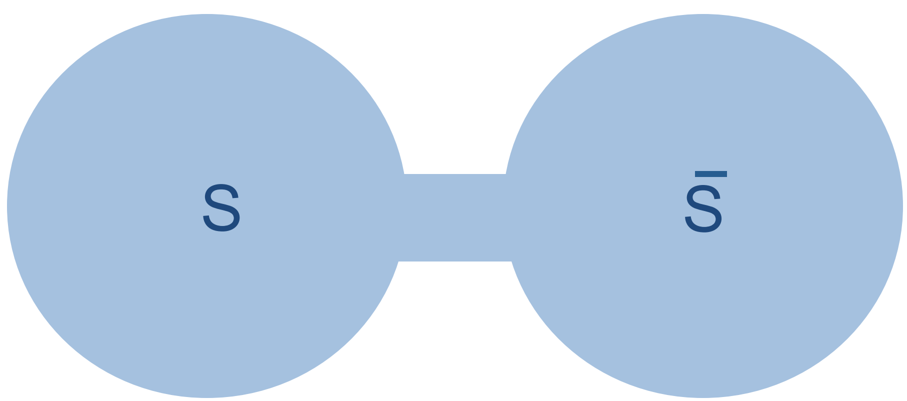

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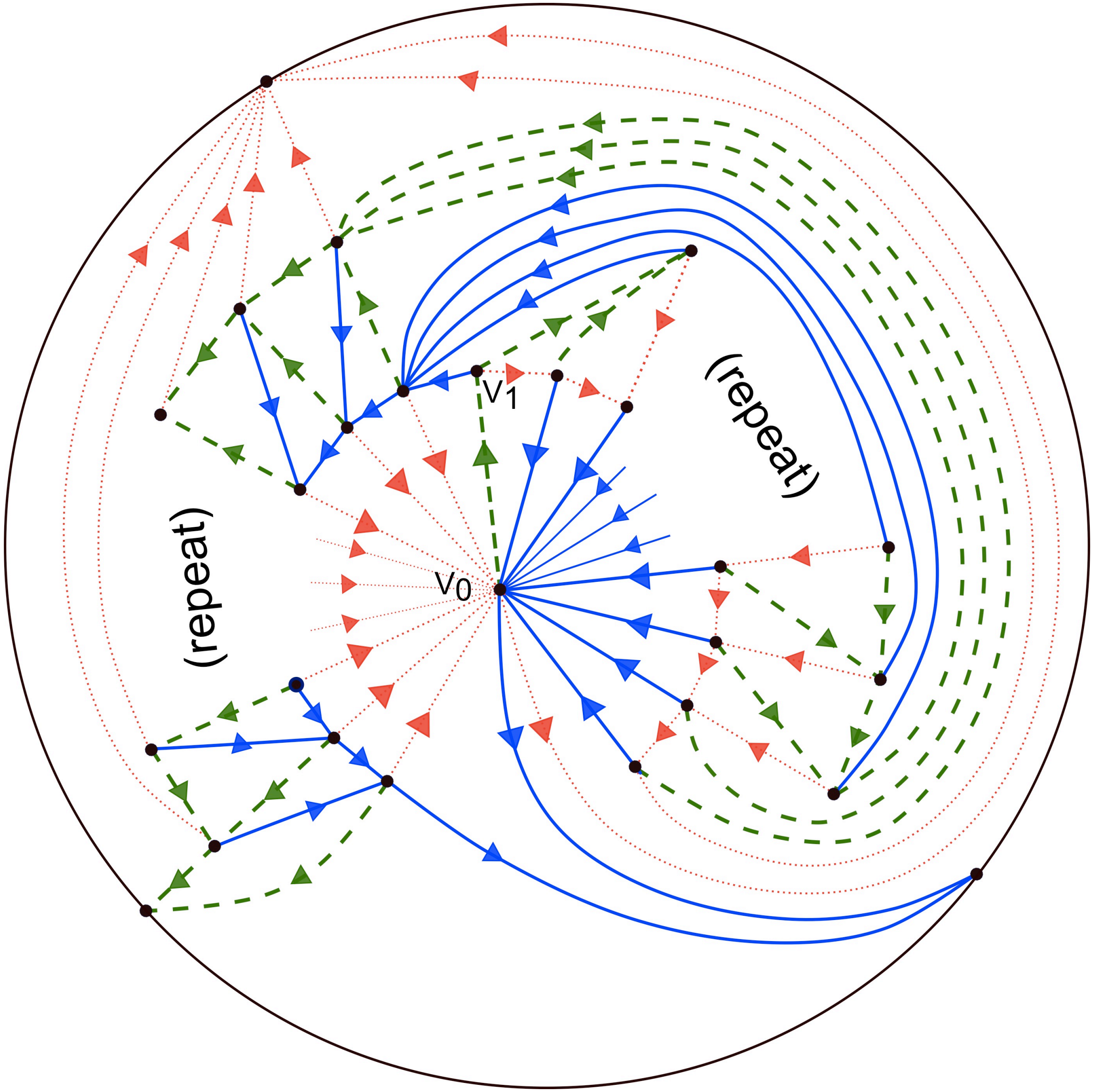
The Triangulation T



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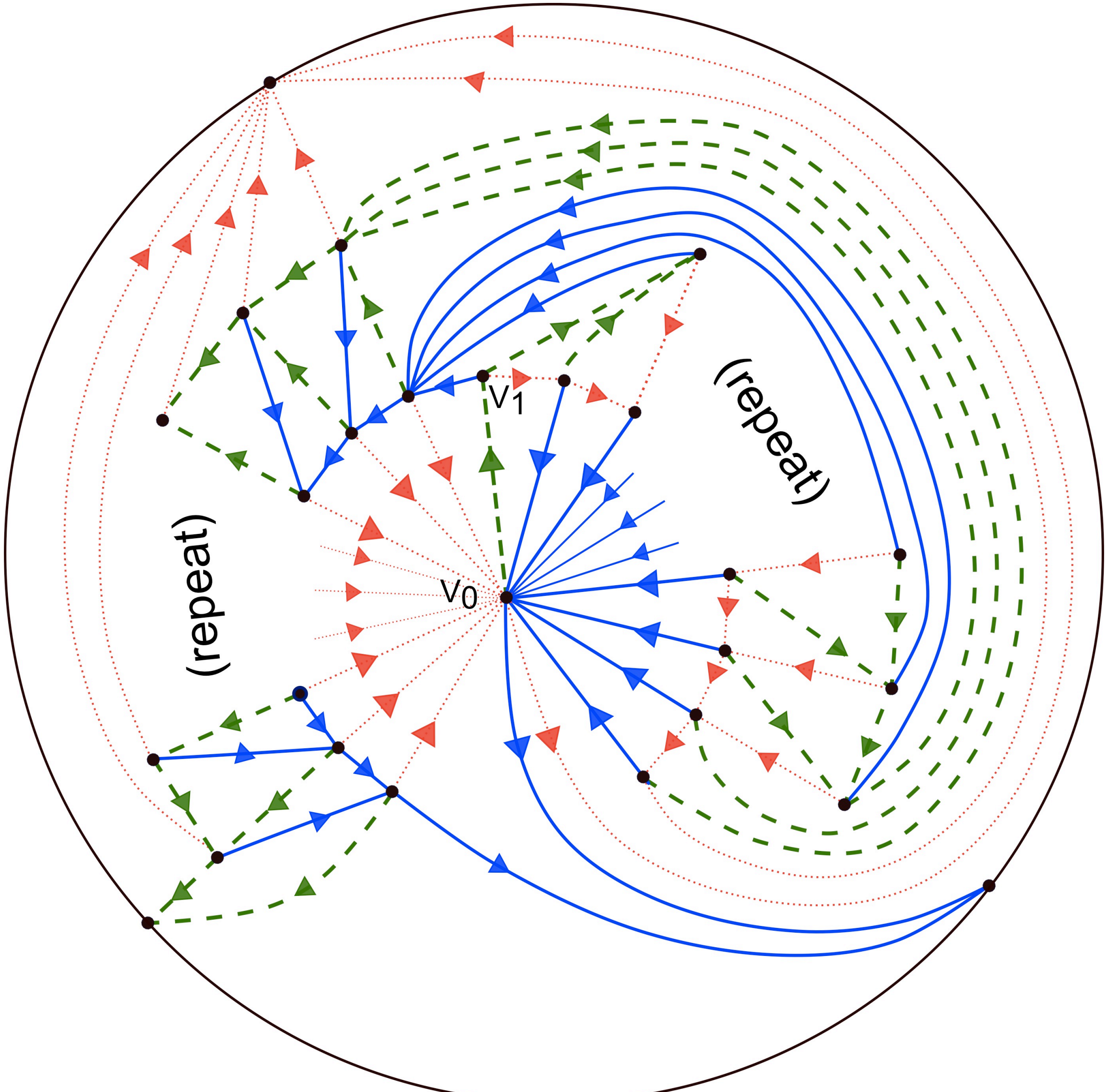
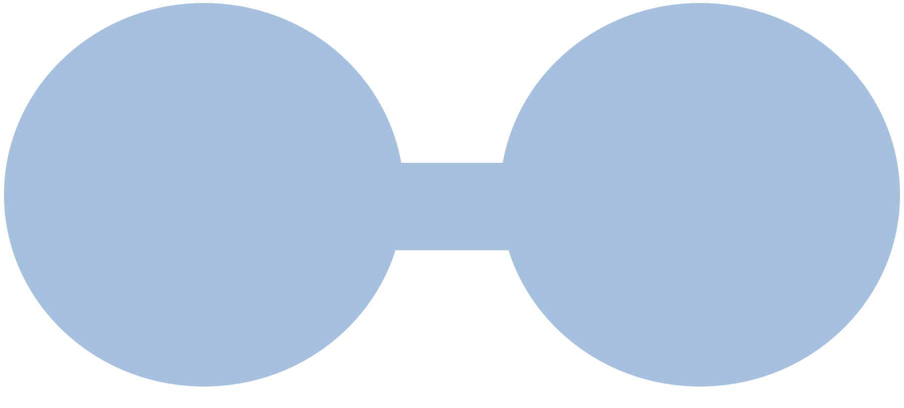


There is only **one** 3-orientation of T with edge (v_0, v_1) colored **green**!

The “Bottleneck



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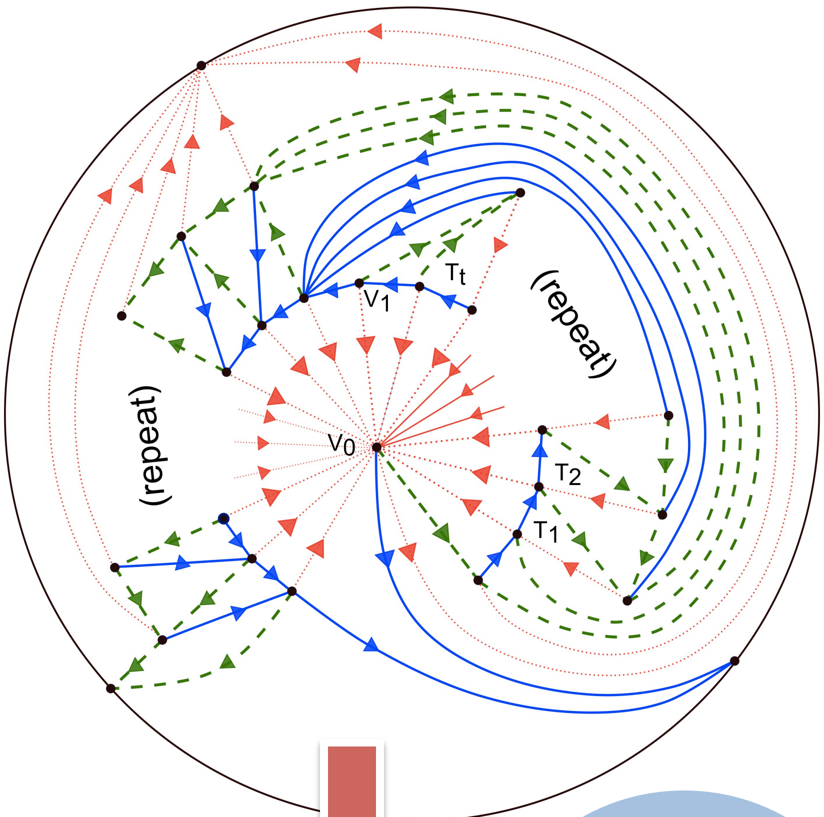

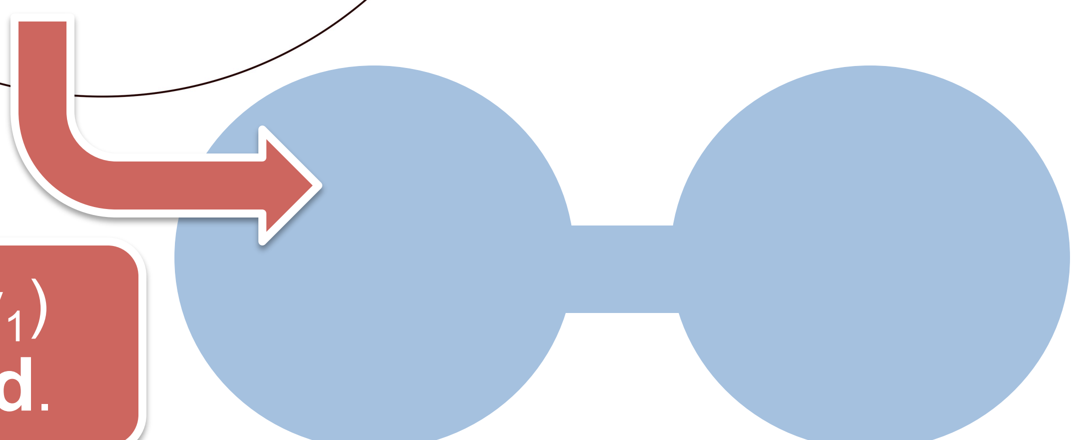

Edge (v_0, v_1) colored **red**.

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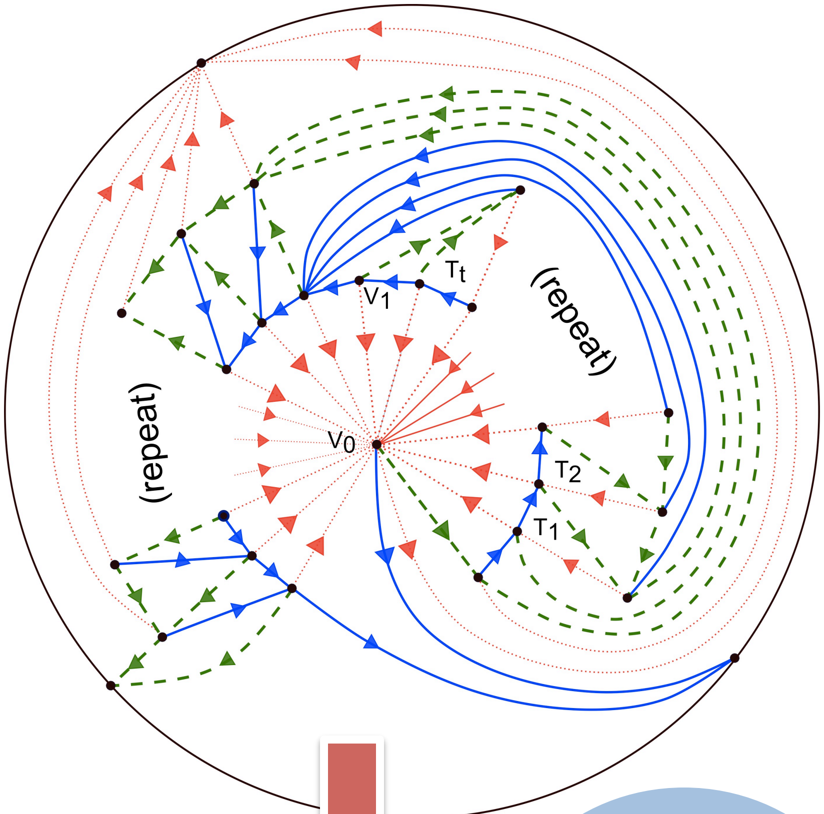

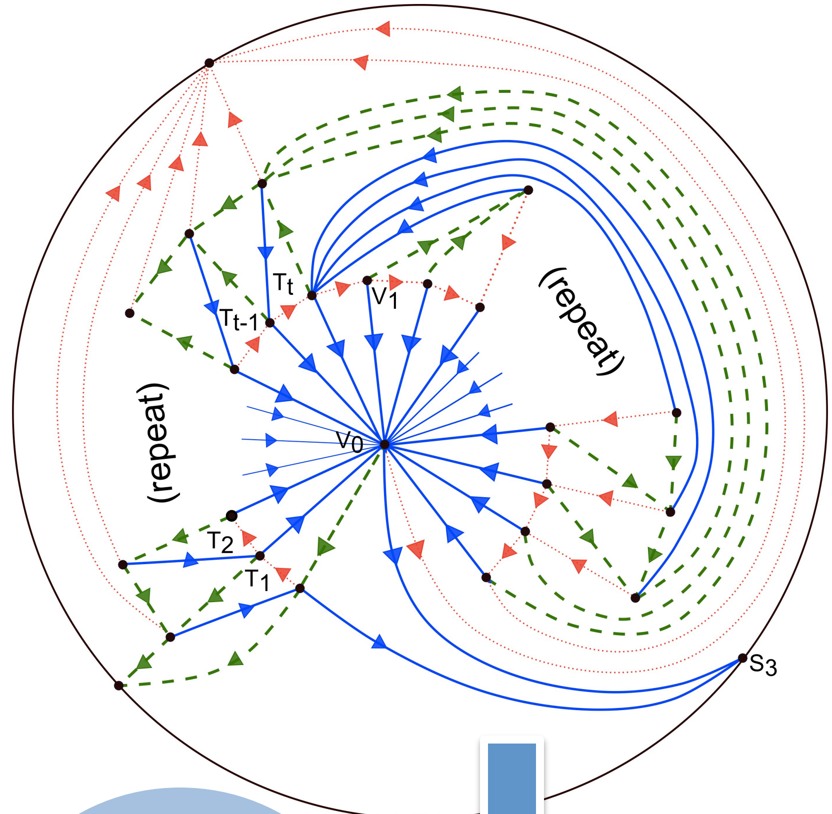

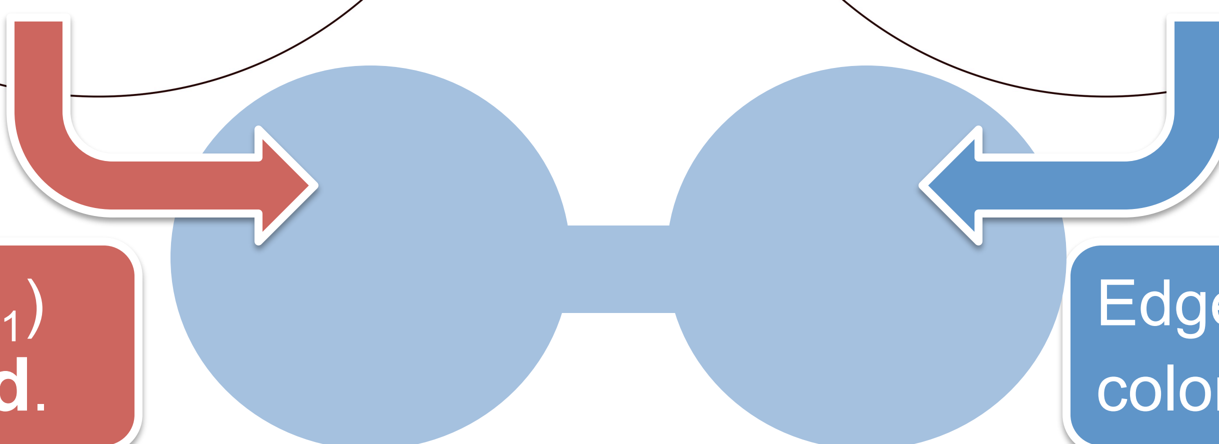

Edge (v_0, v_1) colored **red**.

Edge (v_0, v_1) colored **blue**.

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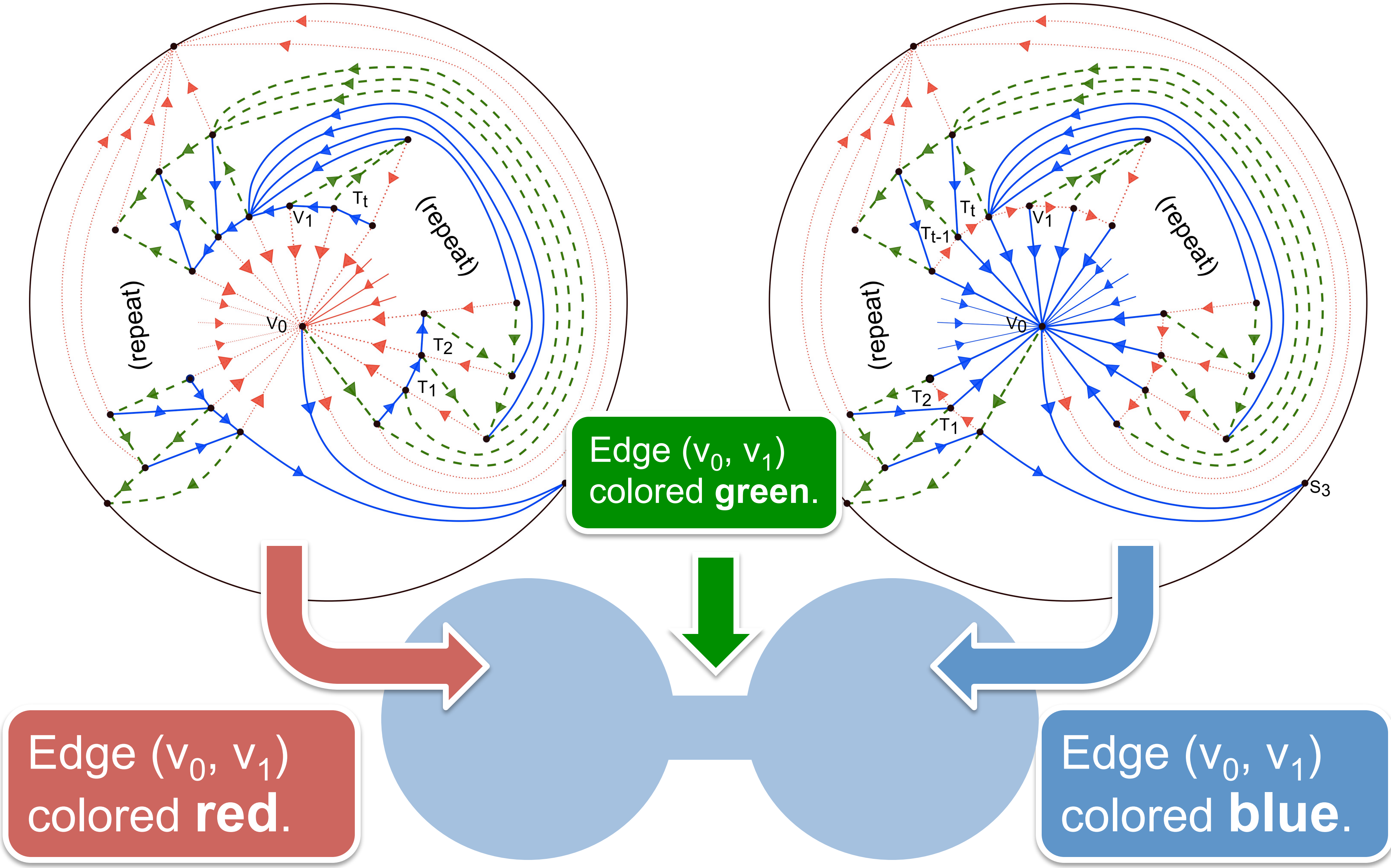

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All 3-orientations of Triangulations with n Internal Vertices

- In bijection with pairs of non-crossing Dyck paths

  (string with equal # of 1's and -1's where the sum is always greater than 0)

- Has size $C_{n+2}C_n - C_{n+1}^2$

  (C_n is the nth Catalan number)

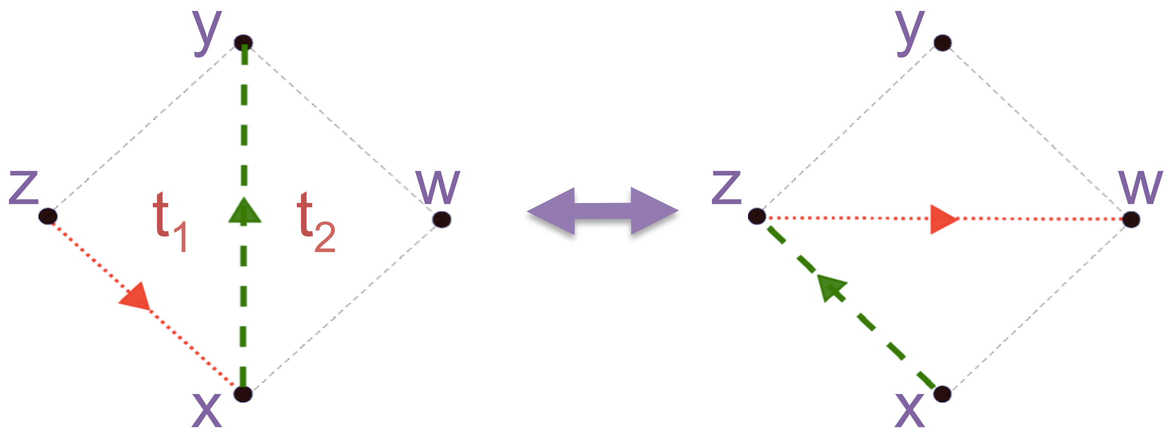
- Can sample using the reduction to counting

  (Explicitly worked out by [Bonichon, Mosbah])

The Local Markov Chain $\mathcal{M}_{EF}$

Repeat:

- Pick two adjacent triangles t_1 and t_2 with shared edge $\overrightarrow{xy}$;
- Pick an edge $\overrightarrow{zx}$ from $t_1 \cup t_2$, if possible, replace $(\overrightarrow{zx}, \overrightarrow{xy})$ by $(\overrightarrow{xz}, \overrightarrow{zw})$ with probability ½.



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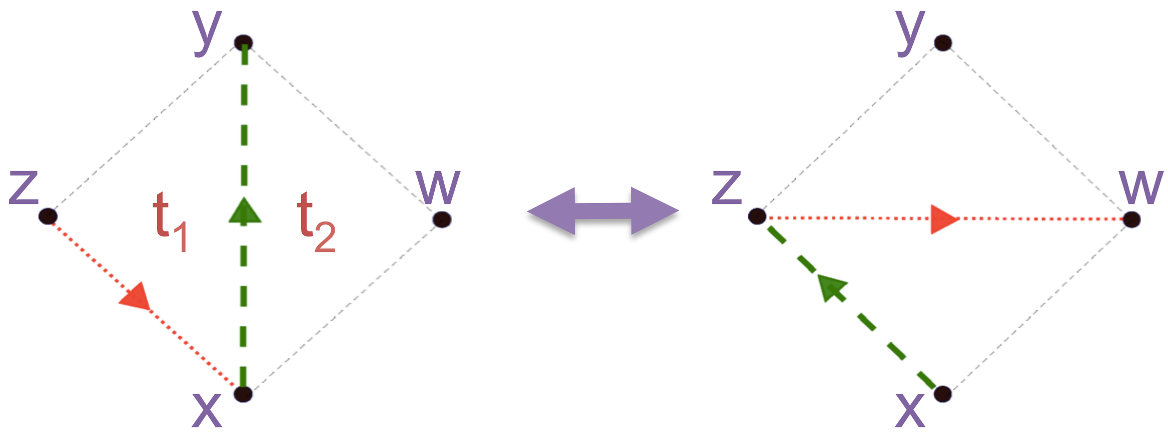

Thm: The local Markov chain $\mathcal{M}_{EF}$ connects the set of all 3-orientations with n internal vertices [Bonichon, Le Saec, Mosbah].

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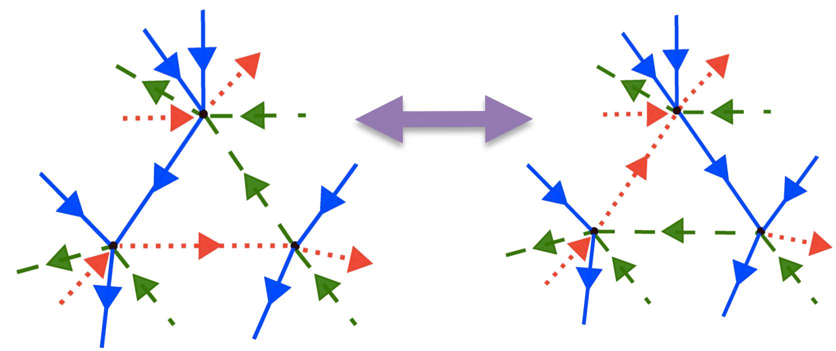

Sample from the set of all 3-orientations of triangulations with n internal vertices.

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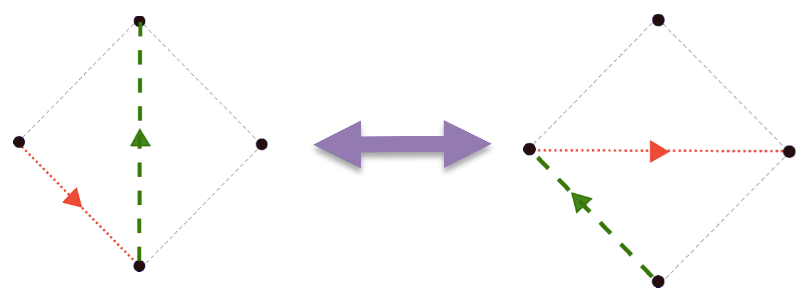

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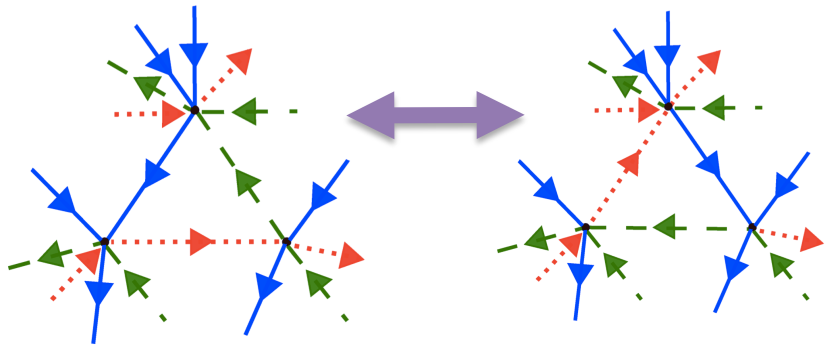

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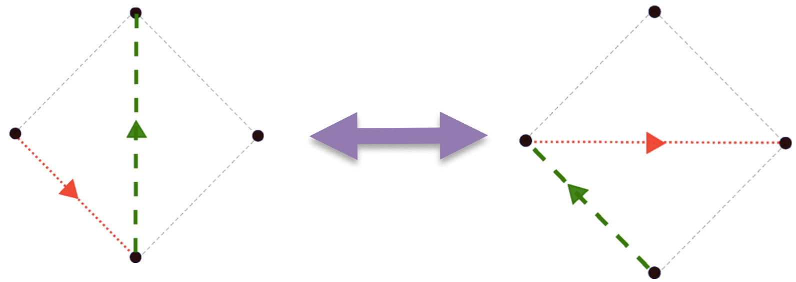

Fast Mixing for $\mathcal{M}_{\text{EF}}$

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Proof sketch:

A. Let $\mathcal{M}_{DK}$ be the "mountain to valley" chain on Dyck paths
B. $\mathcal{M}_{DK}$ is known to mix rapidly [Wilson]
C. Compare the mixing times of $\mathcal{M}_{EF}$ and $\mathcal{M}_{DK}$

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Although $\mathcal{M}_{DK}$ is local in the setting of Dyck paths, in the context of 3-orientations it can make global changes in a single step.

The Bijection with Dyck Paths

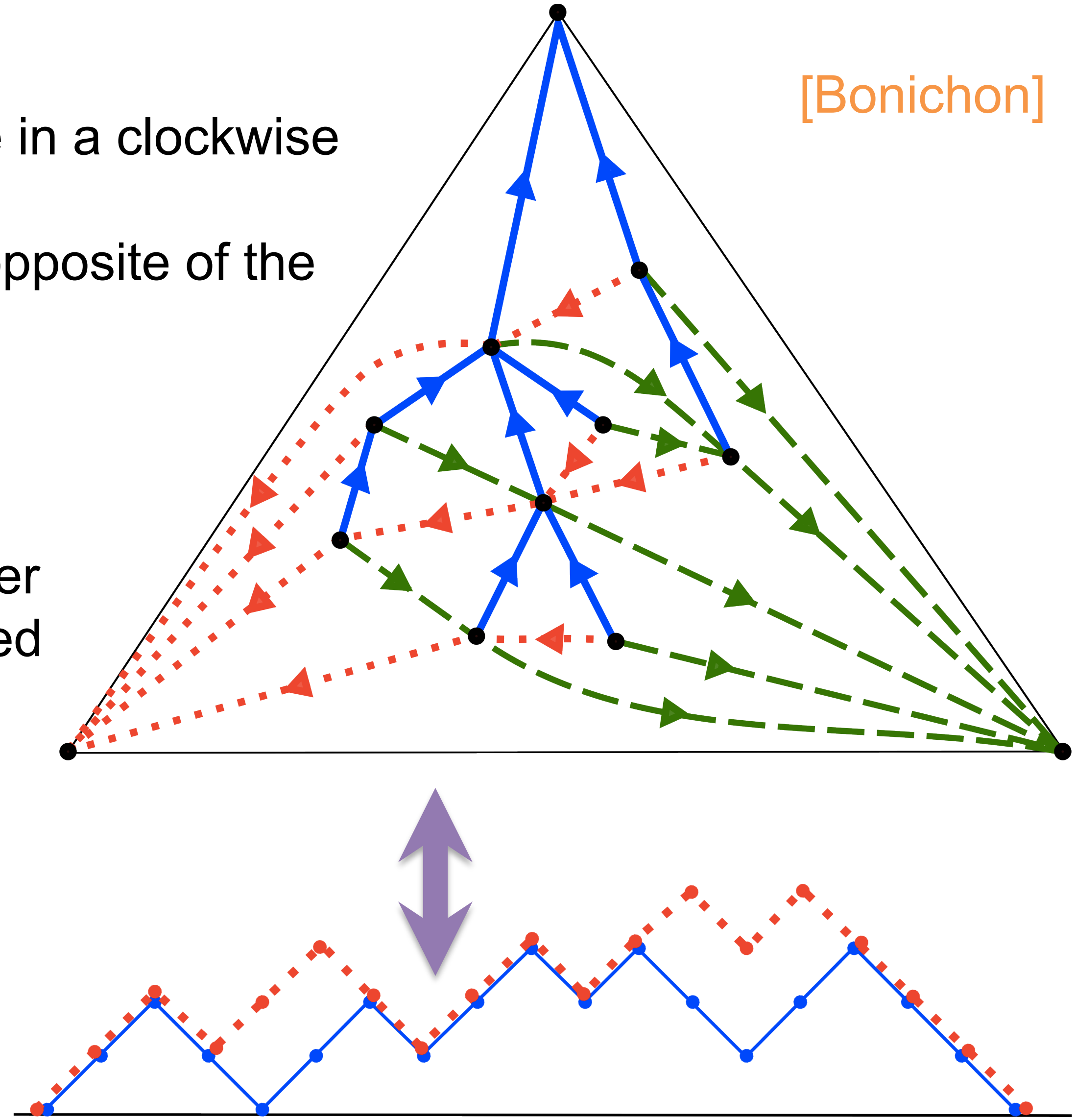
Bottom Path:

1. Trace around the **blue** tree in a clockwise direction
2. Add a 1 if the edge is the opposite of the trace direction
3. Otherwise add a 0

Top Path:

1. Let $v_1, v_2, \ldots, v_n$ be the order the vertices are encountered in step 1 above
2. Let d_i be the # of incoming **red** edges to vertex v_i
3. Let r be the # of incoming **red** edges to s_{red}
4. The top path is:

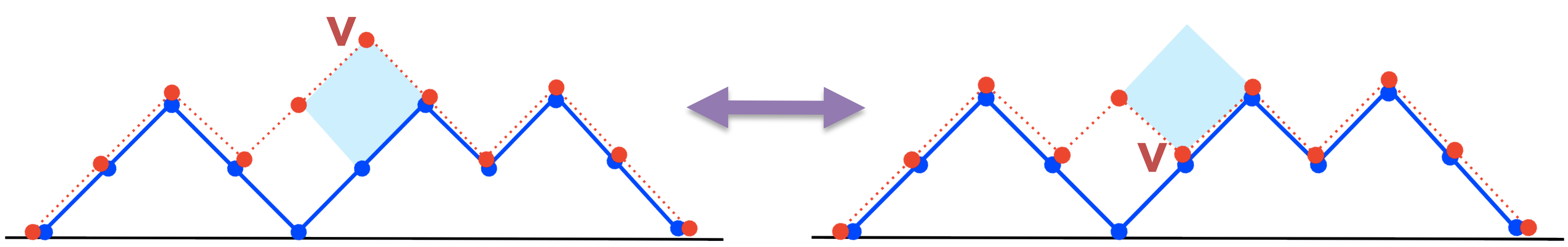
$1(-1)^{d_2},1, (-1)^{d_3},\ldots 1(-1)^{d_n} (-1)^r$



The "Mountain to Valley" chain $\mathcal{M}_{\mathrm{DK}}$

Repeat:

- Pick **v** on one of the paths;
- If **v** marks a mountain/valley, invert with probability ½, if possible.



Open Problems

1. Can the fast mixing proof for $\mathcal{M}_{TR}$ be extended to triangulations with $\Delta_I(T) > 6$?
2. Is there a family of triangulations with bounded degree where the mixing time of $\mathcal{M}_{TR}$ is exponentially large but has bounded degree?
   – Recently [Felsner, Heldt] created a, somewhat simpler, family of graphs based on our construction but the maximum degree still grows with n.
3. Is there an alternative local chain which can sample efficiently from the set of 3-orientations of a fixed triangulation without recourse to the bipartite perfect matching sampler of [Bezáková et al.]?

Thank you!