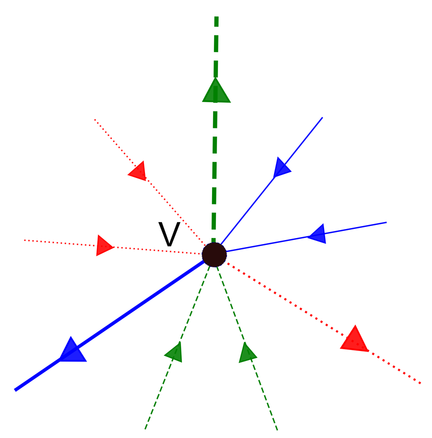
Sampling 3-Orientations of Triangulations

Sarah Miracle, Dana Randall, Amanda Streib and Prasad Tetali

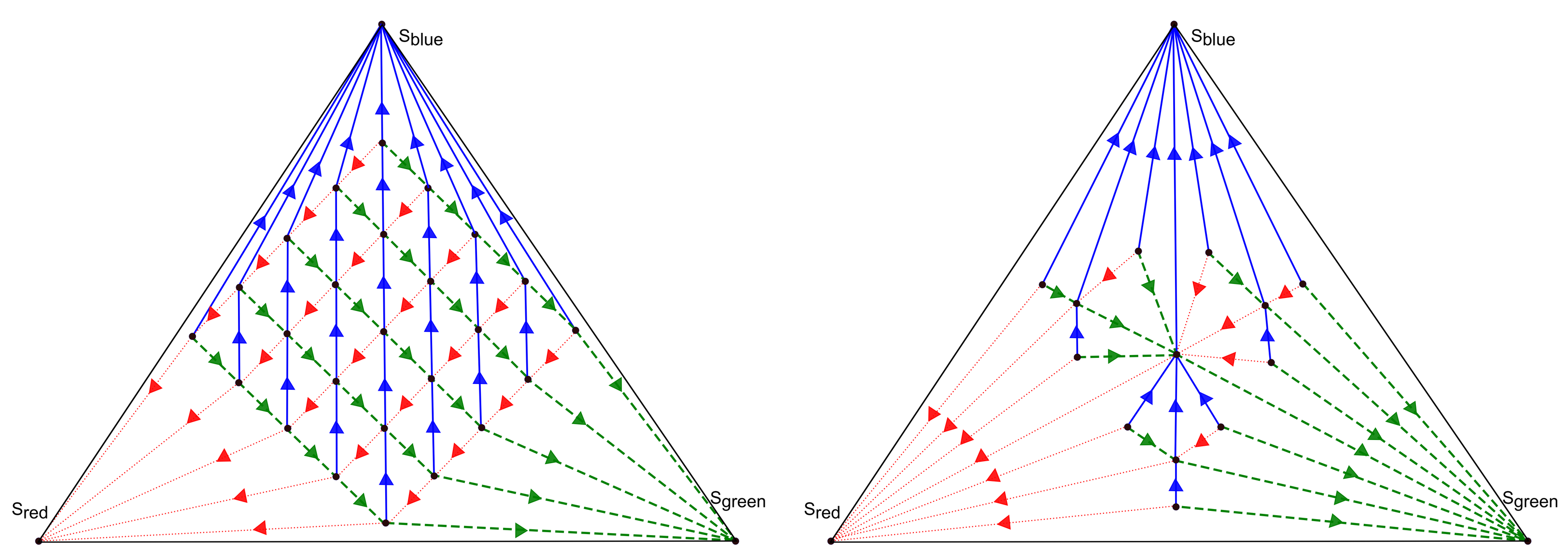
What is a 3-Orientation?

A **3-Orientation** of a triangulation is an orientation of the edges and an assignment of 3 colors to the edges such that each vertex v satisfies the following two properties:

- v has exactly 1 outgoing edge of each color.
- The clockwise order of the edges incident on v is: outgoing green, incoming blue, outgoing red, incoming green, outgoing blue and incoming red.



A fixed triangulation can have an exponential number of 3-orientations or only a single 3-orientation.



Why are 3-orientations important?

3-orientations are also called Schnyder realizers and were introduced by Walter Schnyder in 1989 and used to prove the following two important results:

Theorem [Schnyder]: A graph is planar if and only if its order dimension is at most three.

Theorem [Schnyder]: Every planar graph with n vertices admits a straight line drawing on the $(n-2)\times(n-2)$ grid.

(Straight line drawings are used extensively in microchip layout and design, software engineering diagrams and other areas.)

Since their introduction, Schnyder realizers have been used in many different areas including:

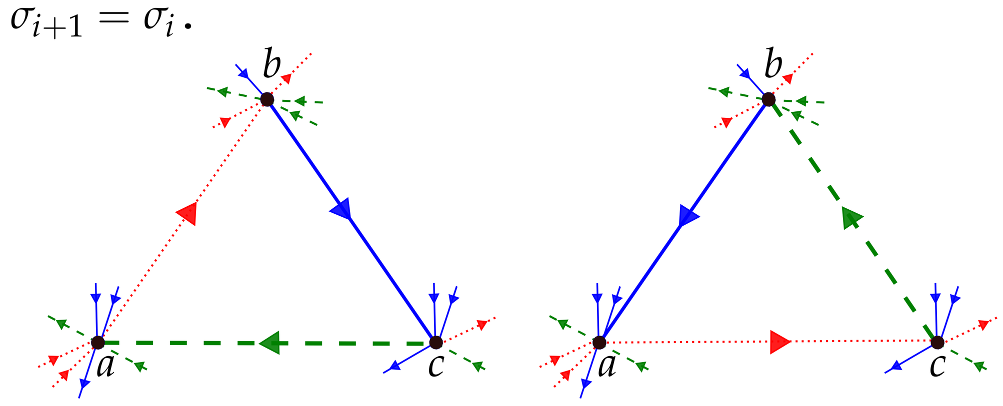
- **Visibility Representation:** Improved compact visibility representation of planar graph via Schnyder's realizer. (C-C. Lin, H-I. Lu, I-F. Sun)
- **Counting Planar Maps:** A unified bijective method for maps: application to two classes with boundaries. (O.Bernardi, E. Fusy)
- **Greedy Routing:** Schnyder Greedy Routing Algorithm. (X. He, H. Zhang)

Fix a Triangulation and Sample

The Markov chain $\mathcal{M}_{fixed}$

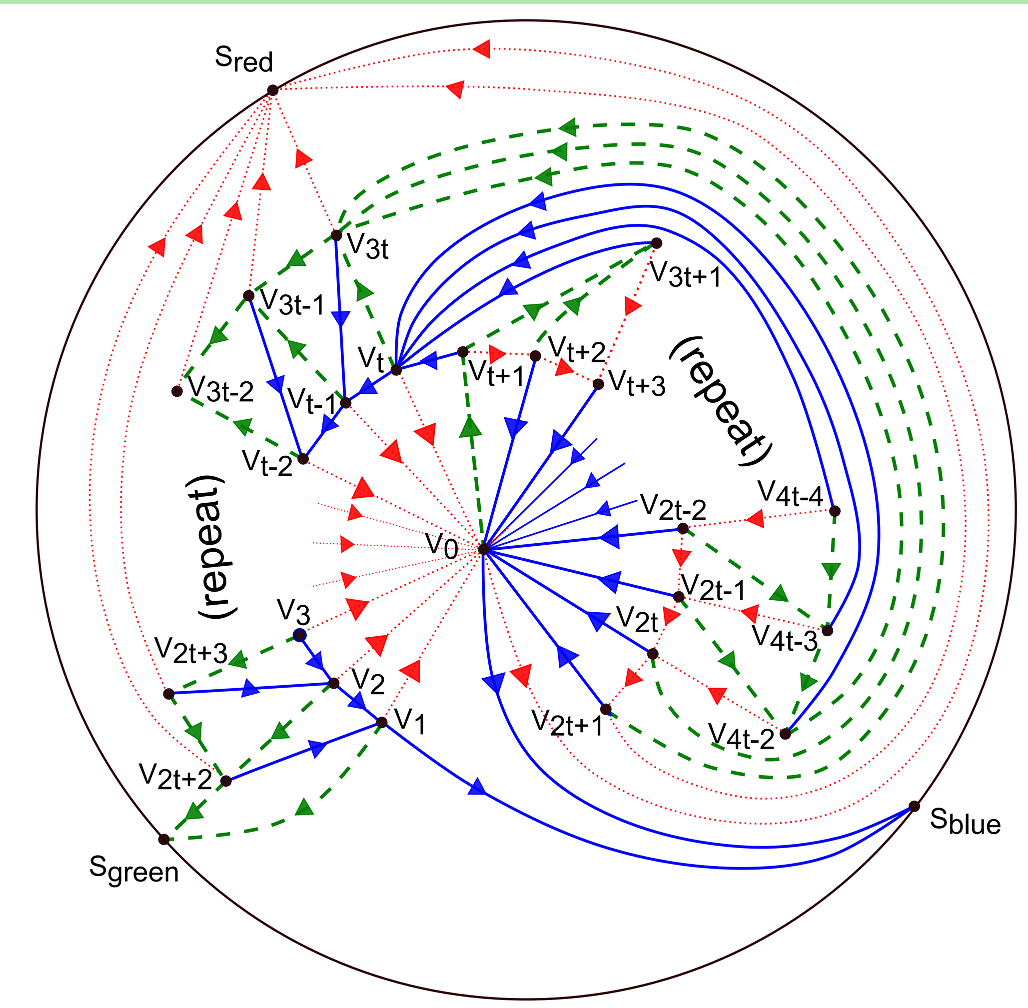
Starting at any σ_0, iterate the following:

- Choose a triangle T uniformly at random.
- With probability 1/2, reverse the orientation of the edges of T to obtain σ_{i+1}.
- Otherwise, $\sigma_{i+1} = \sigma_i$.



A move of $\mathcal{M}_{fixed}$. The triangle $\triangle abc$ reverses to $\triangle acb$.

Theorem: $\exists$ a triangulation on which $\mathcal{M}_{fixed}$ takes exponential time to converge.



Theorem: If the triangulation has max degree ≤ 6 then $\mathcal{M}_{fixed}$ converges in polynomial time.

We show that a related chain $\mathcal{M}_{tower}$ converges in polynomial time and then use this to show the original chain also converges in polynomial time.

The Tower Markov chain $\mathcal{M}_{tower}$

Starting at any σ_0, iterate the following:

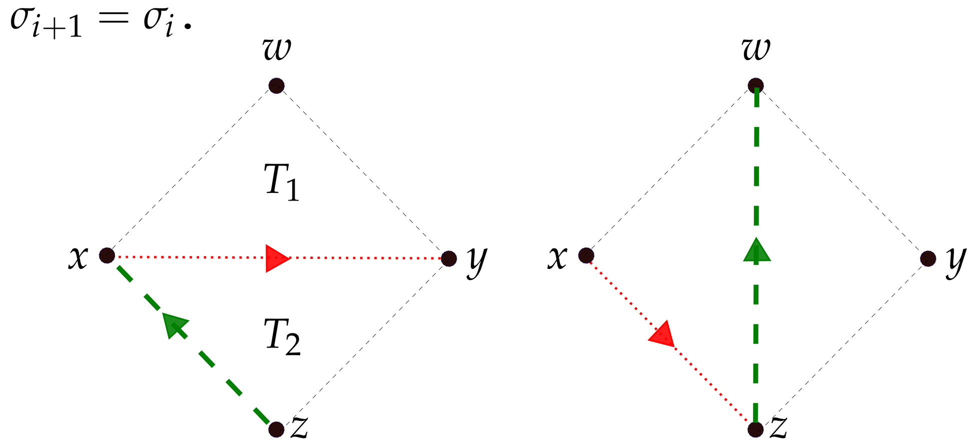
- Choose a face f uniformly at random.
- Assume that f is the beginning of a tower of length k. Then with probability $\begin{cases} \frac{1}{6k}: & k \geq 2 \\ \frac{1}{2}: & k = 1 \end{cases}$ reverse the orientation of this tower to obtain σ_{i+1}.
- Otherwise, $\sigma_{i+1} = \sigma_i$.

Sample 3-Orientations with n Vertices

The Markov chain $\mathcal{M}_{var}$

Starting at any $\sigma_0 \in \Omega$, iterate the following:

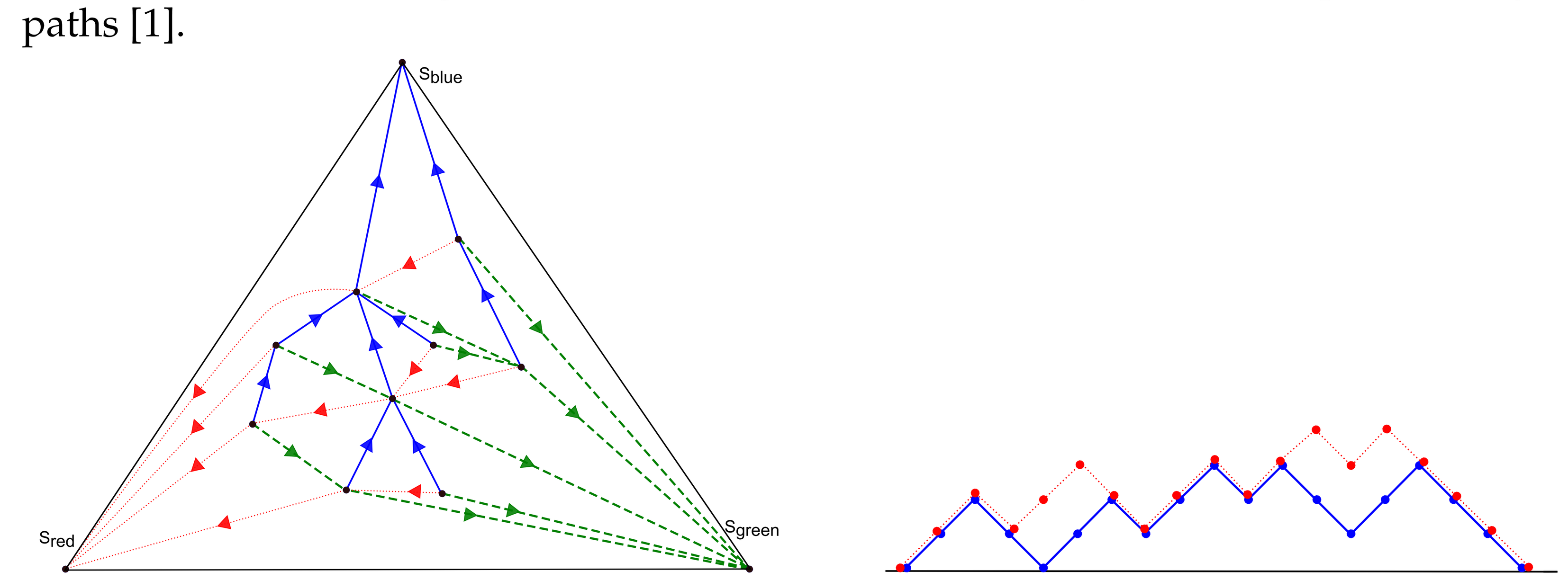
- Choose two adjacent facial triangles T_1 and T_2 uniformly at random. Let (x,y) be the shared edge of T_1 and T_2.
- Choose an edge (z,x) from $T_1 \cup T_2$ uniformly, if one exists. Let w be the remaining vertex of $T_1 \cup T_2$. With probability 1/2 replace the path $\{(z,x),(x,y)\}$ by the path $\{(z,w),(x,z)\}$.
- Otherwise, $\sigma_{i+1} = \sigma_i$.



A move of $\mathcal{M}_{var}$.

Theorem: The Markov chain $\mathcal{M}_{var}$ converges in polynomial time.

Proof. (Sketch)
3-Orientations of triangulations with n vertices are in bijection with pairs of Dyck paths [1].



The natural chain on Dyck paths is known to converge in polynomial time [4] and we use this to show that $\mathcal{M}_{var}$ converges in polynomial time by using the comparison method [3,5]. □

References

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[4] M. Luby, D. Randall, and A.J. Sinclair. Markov Chains for Planar Lattice Structures. *SIAM Journal on Computing*, **31:** 167–192, 2001.

[5] D. Randall and P. Tetali. Analyzing Glauber dynamics by comparison of Markov chains. *Journal of Mathematical Physics*, **41:** 1598–1615, 2000.