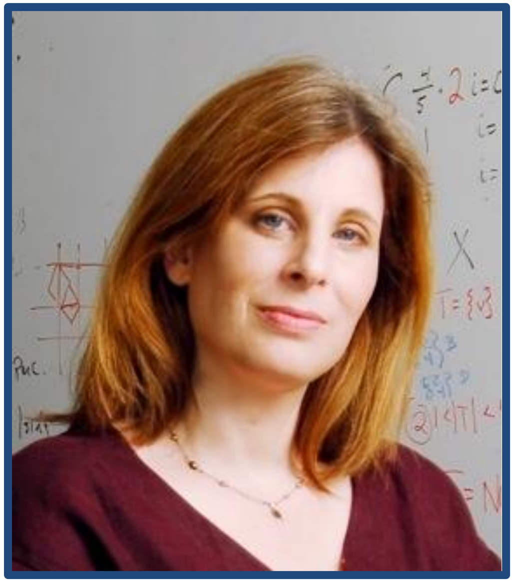
# Mixing Times of Self-Organizing Lists and Biased Permutations

Sarah Miracle

Georgia Institute of Technology

# Joint work with . . . .

Prateek Bhakta



Dana Randall

Amanda Streib

# *Biased* Card Shuffling

- pick a pair of adjacent cards uniformly at random
- put j ahead of i with probability $p_{j,i} = 1 - p_{i,j}$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | n | 1 | 7 | … | i | j | … | n-1 | 2 | 6 | 3 |

Converges to: $$\pi(\sigma) = \prod_{i<j:\sigma(i)>\sigma(j)} \frac{p_{ji}}{p_{ij}} / Z$$

$M_{nn}$

This is related to the "Move-Ahead-One Algorithm" for self-organizing lists.

# What is already known?

- **Uniform bias**: If $p_{i,j} = ½ \forall\ i, j$ then $M_{nn}$ mixes in $\theta(n^3 \log n)$ [Wilson]
- **Constant bias**: If $p_{i,j} = p > ½\ \forall\ i < j$, then $M_{nn}$ mixes in $\theta(n^2)$ time [Benjamini, Berger, Hoffman, Mossel]

If $p_{i,j} \geq ½\ \forall\ i < j$ we say the chain is *positively biased*.

Q: If the $\{p_{ij}\}$ are positively biased, is $M_{nn}$ always rapidly mixing?

Conjecture (Fill): If $\{p_{ij}\}$ satisfy a "regularity" condition, then the spectral gap is minimized when $p_{ij} = 1/2\ \ \forall\ i,j$

- proved this conjecture for $n \leq 4$

# What is already known?

- **Uniform bias**: If $p_{i,j} = ½ \forall i, j$ then $M_{nn}$ mixes in $\theta(n^3 \log n)$ [Wilson]
- **Constant bias**: If $p_{i,j} = p > ½ \; \forall i < j$, then $M_{nn}$ mixes in $\theta(n^2)$ time [Benjamini, Berger, Hoffman, Mossel]
- **Linear extensions of a partial order**: If $p_{i,j} = ½$ or $1 \; \forall i < j$, then $M_{nn}$ mixes in $O(n^3 \log n)$ time [Bubley, Dyer]
- $M_{nn}$ is fast for two new classes: **"Choose your weapon"** and **"League hierarchies"** [Bhakta, M., Randall, Streib]
- $M_{nn}$ can be slow even when the chain is positively biased [Bhakta, M., Randall, Streib]

# Talk Outline

✓ 1. Background
2. New Classes of Bias where $M_{nn}$ is fast
   - Choose your Weapon
   - League Hierarchies
3. $M_{nn}$ can be slow

# The mixing time

Definition: The total variation distance is

$$\|P^t,\pi\| = \max_{x\in\Omega} \frac{1}{2} \sum_{y\in\Omega} |P^t(x,y) - \pi(x)|.$$

Definition: Given $\varepsilon$, the mixing time is

$$\tau(\varepsilon) = \min \{t: \|P^{t'},\pi\| < \varepsilon, \ \forall\ t' \geq t\}.$$

A Markov chain is rapidly mixing if $\tau(\varepsilon)$ is poly(n, log($\varepsilon^{-1}$)).
(or polynomially mixing)

A Markov chain is slowly mixing if $\tau(\varepsilon)$ is at least exp(n).

# Choose your weapon

Given $r_{1, \ldots ,} r_{n-1,}$ $r_i \geq 1/2$.

Thm 1: Let $p_{i,j} = r_i$ $\forall\, i < j$. Then $M_{NN}$ is rapidly mixing.

**Proof sketch:**

A. Define auxiliary Markov chain M'
B. Show M' is rapidly mixing
C. Compare the mixing times of $M_{NN}$ and M'

M' can swap pairs that are not nearest neighbors

- Maintains the same stationary distribution
- Allowed moves are based on inversion tables

# Inversion Tables

Permutation $\sigma$: Inversion Table $I_\sigma$:

5 6 3 1 2 7 4 $\leftrightarrow$ 3 3 2 3 0 0 0

$I_\sigma(1)$ $I_\sigma(2)$

$I_\sigma(i)$ = # elements $j > i$ appearing before $i$ in $\sigma$

The map $I$ is a bijection from $S_n$ to $T = \{(x_1, x_2, \ldots, x_n) : 0 \le x_i \le n-i\}$.

# Inversion Tables

Permutation $\sigma$:

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 | 6 | 3 | 1 | 2 | 7 | 4 |

$\leftrightarrow$

Inversion Table $I_\sigma$:

3 3 2 3 0 0 0

M' on Inversion Tables

- choose a column i uniformly
- w.p. $r_i$: subtract 1 from $x_i$ (if $x_i>0$)
- w.p. $1- r_i$: add 1 to $x_i$ (if $x_i<n-i$)

M' is just a product of n independent biased random walks

$\Rightarrow$ M' is rapidly mixing.

# Choose your weapon

Given $r_{1, \ldots, }r_{n-1,} \quad r_i \geq 1/2$.

Thm 1: Let $p_{i,j} = r_i \quad \forall\, i < j$. Then $M_{NN}$ is rapidly mixing.

**Proof sketch:**

✓ A. Define auxiliary Markov chain M’

✓ B. Show M’ is rapidly mixing

C. Compare the mixing times of $M_{NN}$ and M’

# Inversion Tables

Permutation $\sigma$:

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 | 6 | 3 | 1 | 2 | 7 | 4 |

$\leftrightarrow$

Inversion Table $I_\sigma$:

3 3 2 3 0 0 0

$I_\sigma(i)$ = # elements $j > i$ appearing before $i$ in $\sigma$

M' on Permutations

- choose a card i uniformly
- swap element i with the first $j>i$ to the left w.p. $r_i$
- swap element i with the first $j>i$ to the right w.p. $1-r_i$

$\leftrightarrow$

M' on Inversion Tables

- choose a column i uniformly
- w.p. $r_i$: subtract 1 from $x_i$ (if $x_i > 0$)
- w.p. $1- r_i$: add 1 to $x_i$ (if $x_i < n-i$)

# Comparing M' and $M_{nn}$

<u>M' on Permutations</u>

- choose a card i uniformly
- swap element i with the first j>i to the left w.p. $r_i$
- swap element i with the first j>i to the right w.p. $1-r_i$

Want stationary dist.:

$$\pi(\sigma) = \prod_{i<j:\sigma(i)>\sigma(j)} \frac{p_{ij}}{p_{ji}} / Z$$

<u>Permutation $\sigma$:</u>

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 | 6 | 3 | 1 | 2 | 7 | 4 |

<u>Permutation $\tau$:</u>

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 | 6 | 2 | 1 | 3 | 7 | 4 |

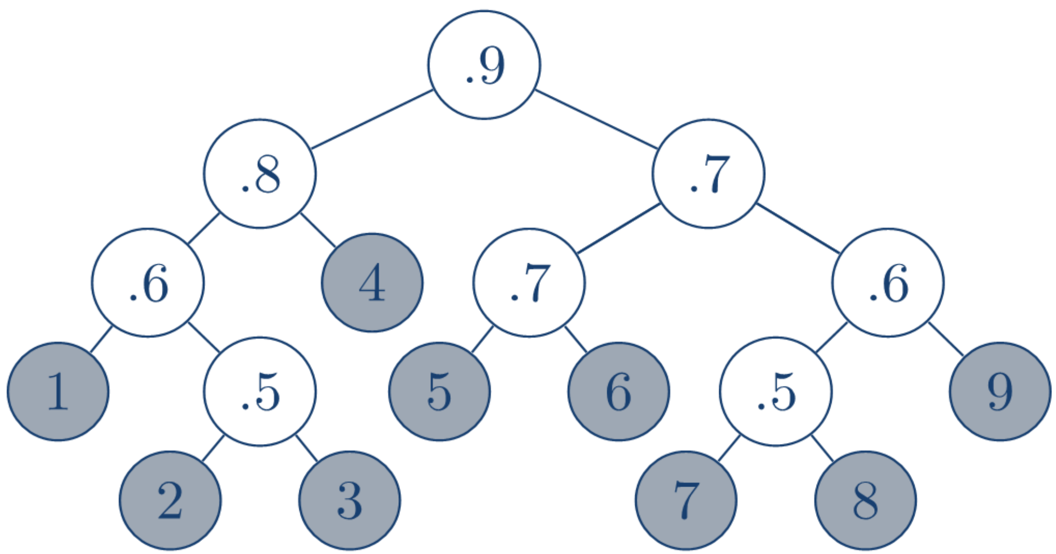
$$\frac{P'(\sigma,\tau)}{P'(\tau,\sigma)} = \frac{\pi(\tau)}{\pi(\sigma)} = \frac{\cancel{p_{2,1}}\, p_{2,3}\, \cancel{p_{1,3}}}{\cancel{p_{1,2}}\, p_{3,2}\, \cancel{p_{3,1}}} = \frac{p_{2,3}}{p_{3,2}}$$

$$\boxed{P'(\sigma,\tau) = r_2 = p_{2,3}}$$

# League Hierarchy

Let $T$ be a binary tree with leaves labeled $\{1,\ldots,n\}$.
Given $q_v \geq 1/2$ for each *internal* vertex $v$.

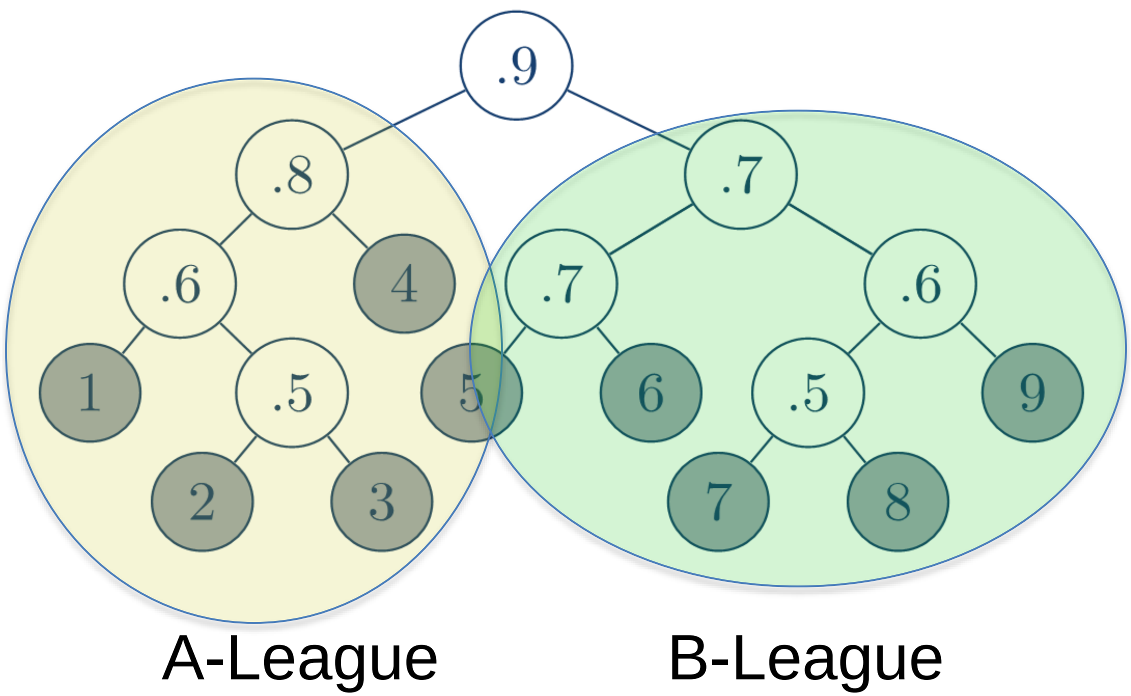
<u>Thm 2:</u> Let $p_{i,j} = q_{i \wedge j}$ for all $i < j$. Then $M_{NN}$ is rapidly mixing.



# League Hierarchy

Let $T$ be a binary tree with leaves labeled $\{1,\ldots,n\}$.
Given $q_v \geq 1/2$ for each *internal* vertex $v$.

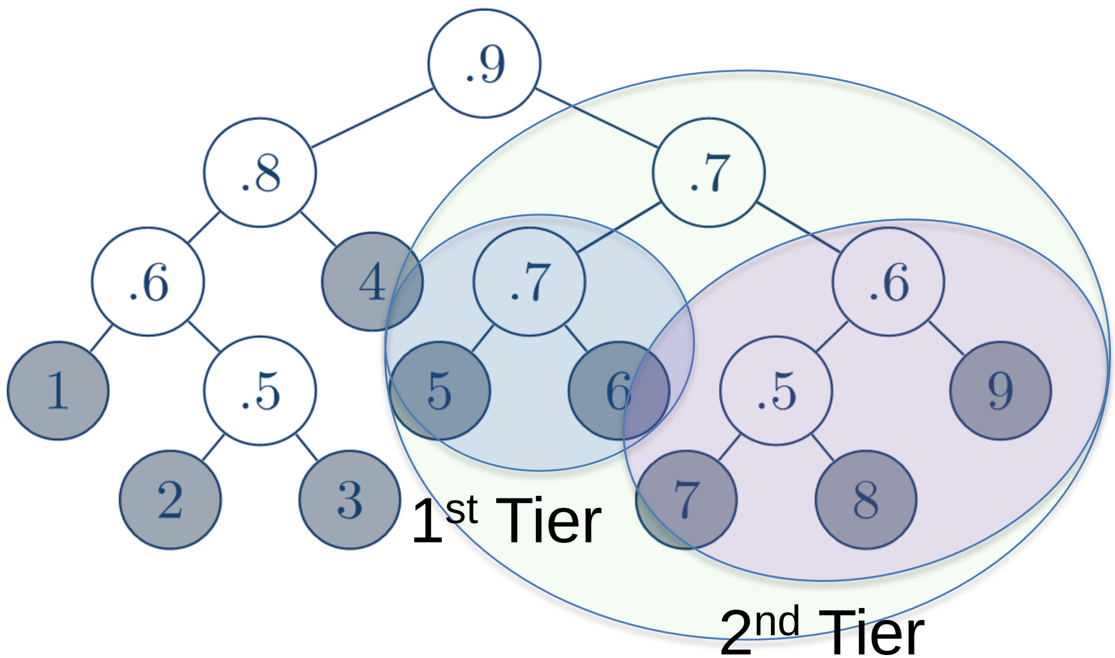
Thm 2: Let $p_{i,j} = q_{i \wedge j}$ for all $i < j$. Then $M_{NN}$ is rapidly mixing.



# League Hierarchy

Let $T$ be a binary tree with leaves labeled $\{1,\dots,n\}$.
Given $q_v \geq 1/2$ for each *internal* vertex $v$.

<u>Thm 2:</u> Let $p_{i,j} = q_{i \wedge j}$ for all $i < j$. Then $M_{NN}$ is rapidly mixing.



# League Hierarchy

Let $T$ be a binary tree with leaves labeled $\{1,\ldots,n\}$.
Given $q_v \geq 1/2$ for each *internal* vertex $v$.

<u>Thm 2:</u> Let $p_{i,j} = q_{i \wedge j}$ for all $i < j$. Then $M_{NN}$ is rapidly mixing.

**Proof sketch:**

A. Define auxiliary Markov chain M'
B. Show M' is rapidly mixing
C. Compare the mixing times of $M_{NN}$ and M'

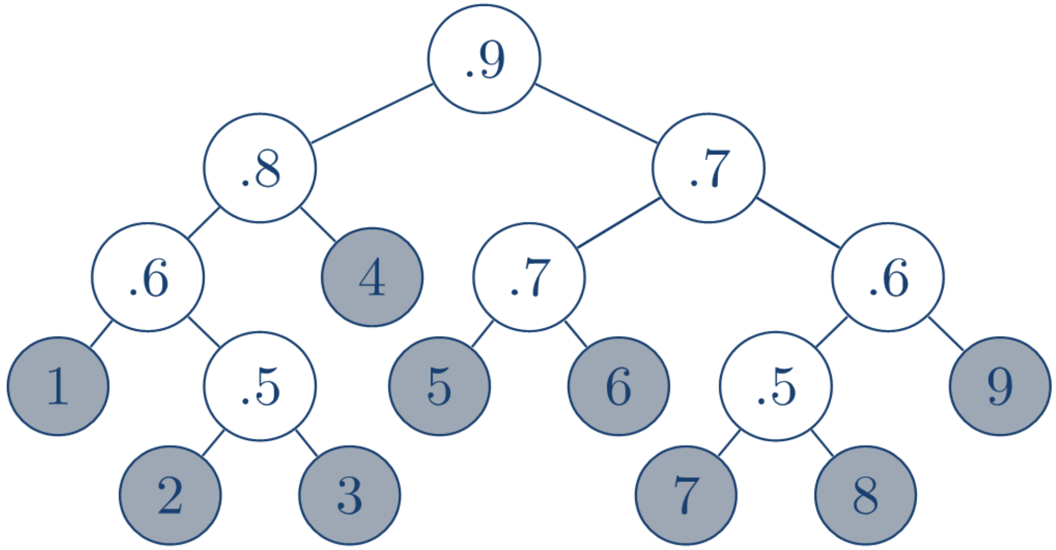
M' can swap pairs that are not nearest neighbors

- Maintains the same stationary distribution
- Allowed moves are based on the binary tree T

# League Hierarchy

Let $T$ be a binary tree with leaves labeled $\{1,\ldots,n\}$.
Given $q_v \geq 1/2$ for each *internal* vertex $v$.

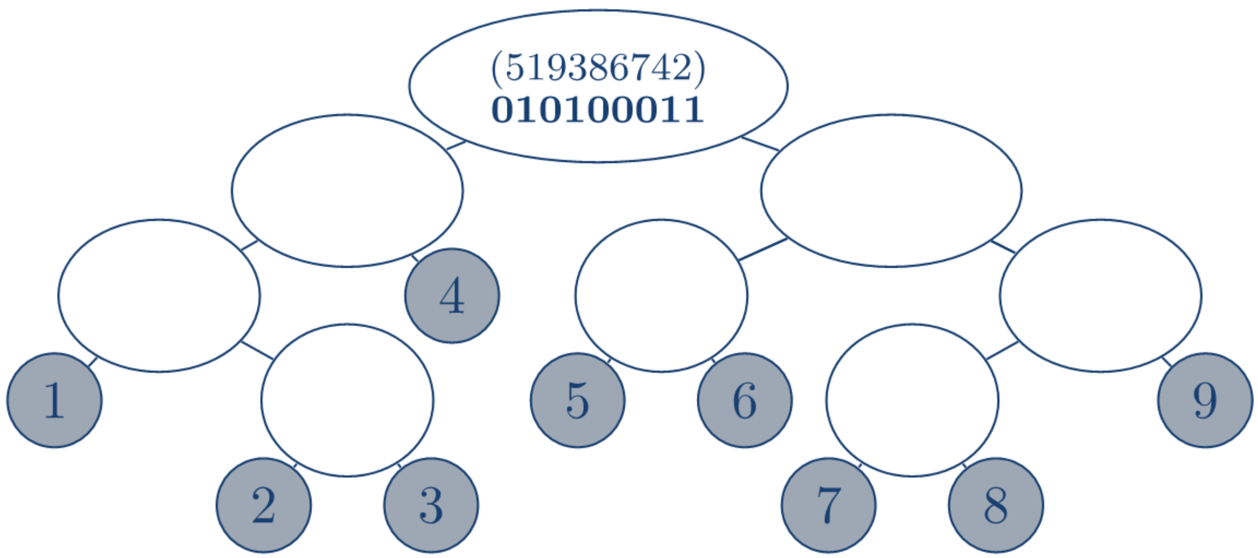
<u>Thm 2:</u> Let $p_{i,j} = q_{i \wedge j}$ for all $i < j$. Then $M_{NN}$ is rapidly mixing.



# Thm 2: Proof sketch

Let $T$ be a binary tree with leaves labeled $\{1,\ldots,n\}$.
Given $q_v \geq 1/2$ for each *internal* vertex $v$.

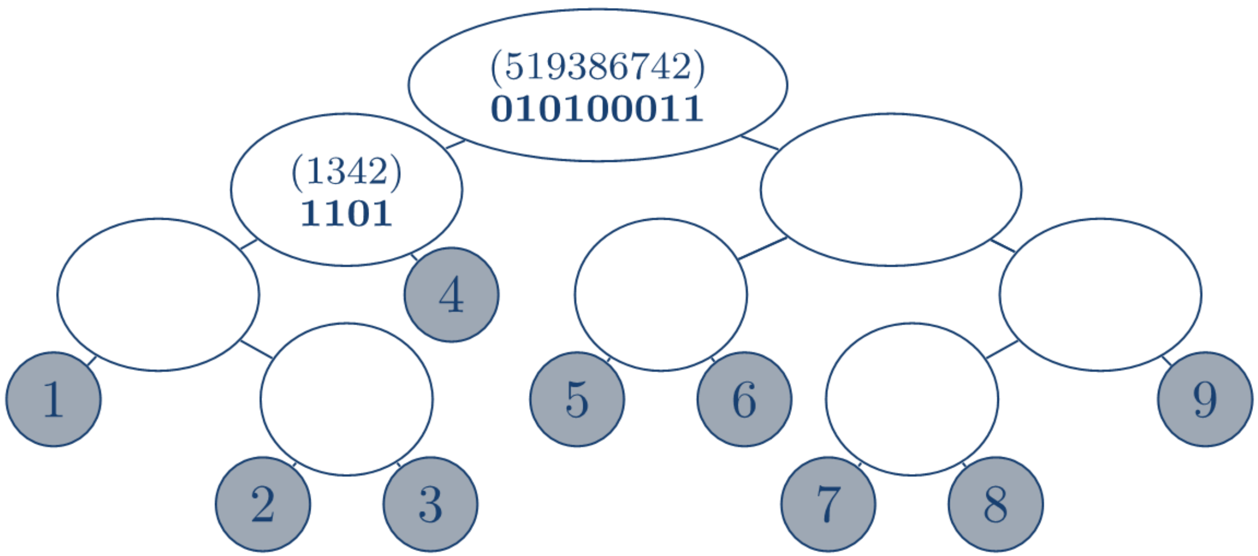
Thm 2: Let $p_{i,j} = q_{i\wedge j}$ for all $i < j$. Then $M_{NN}$ is rapidly mixing.



# Thm 2: Proof sketch

Let $T$ be a binary tree with leaves labeled $\{1,\ldots,n\}$.
Given $q_v \geq 1/2$ for each *internal* vertex $v$.

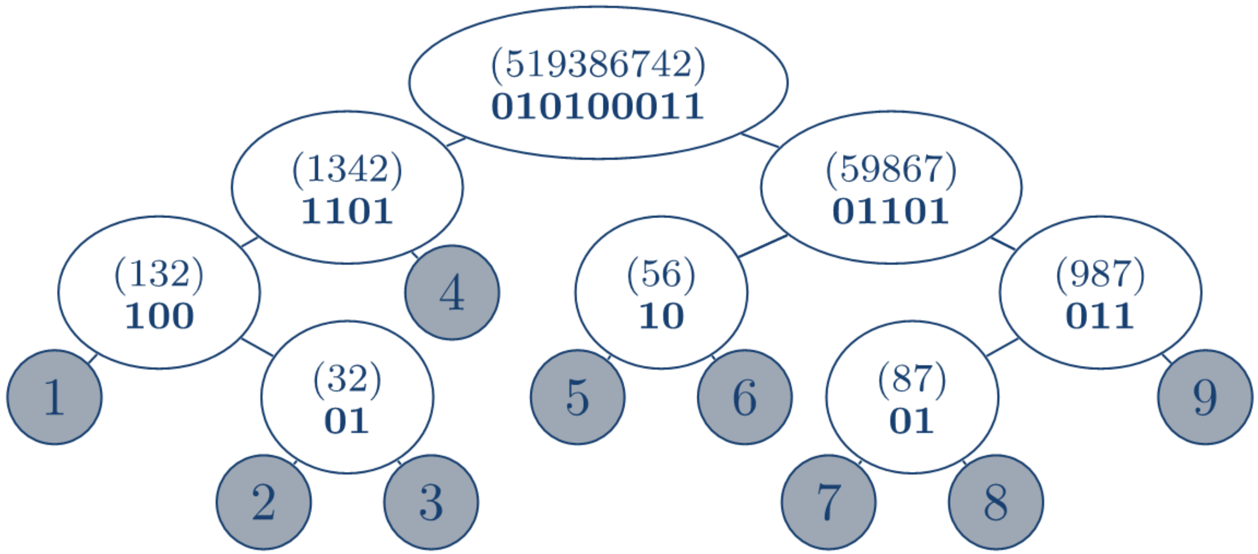
<u>Thm 2:</u> Let $p_{i,j} = q_{i \wedge j}$ for all $i < j$. Then $M_{NN}$ is rapidly mixing.



# Thm 2: Proof sketch

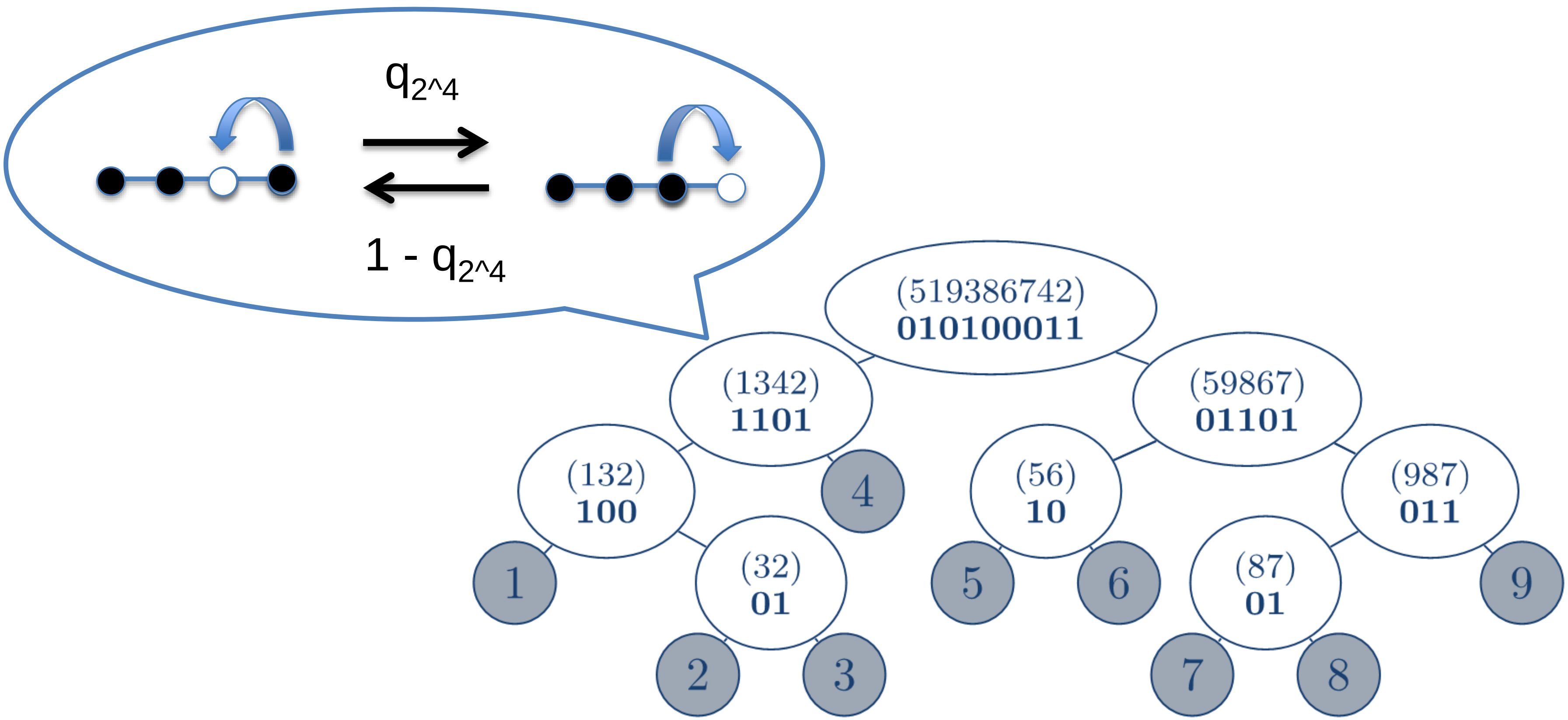
Let $T$ be a binary tree with leaves labeled $\{1,\ldots,n\}$.
Given $q_v \geq 1/2$ for each *internal* vertex $v$.

<u>Thm 2:</u> Let $p_{i,j} = q_{i \wedge j}$ for all $i < j$. Then $M_{NN}$ is rapidly mixing.

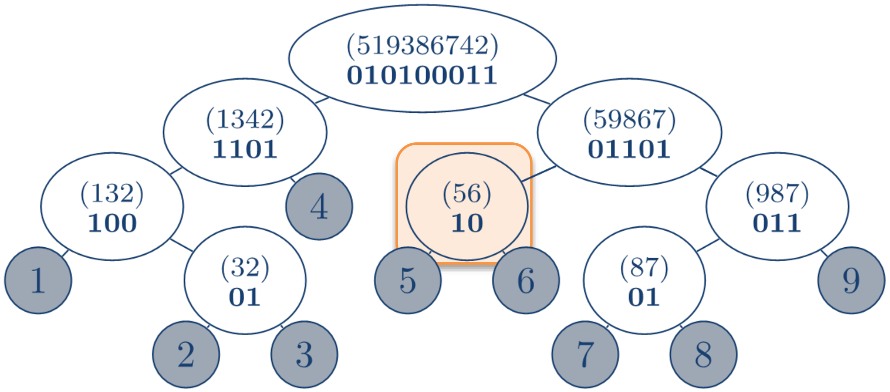


# Thm 2: Proof sketch

Markov chain M' allows a transposition if it corresponds to an ASEP move on one of the internal vertices.



# What about the Stationary Distribution?




Want stationary distribution:

$$\pi(\sigma) = \prod_{i<j:\sigma(i)>\sigma(j)} \frac{p_{ij}}{p_{ji}} \cancel{/\, Z}$$

**Permutation $\sigma$:**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5 | 1 | 9 | 3 | 8 | 6 | 7 | 4 | 2 |

**Permutation $\tau$:**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6 | 1 | 9 | 3 | 8 | 5 | 7 | 4 | 2 |

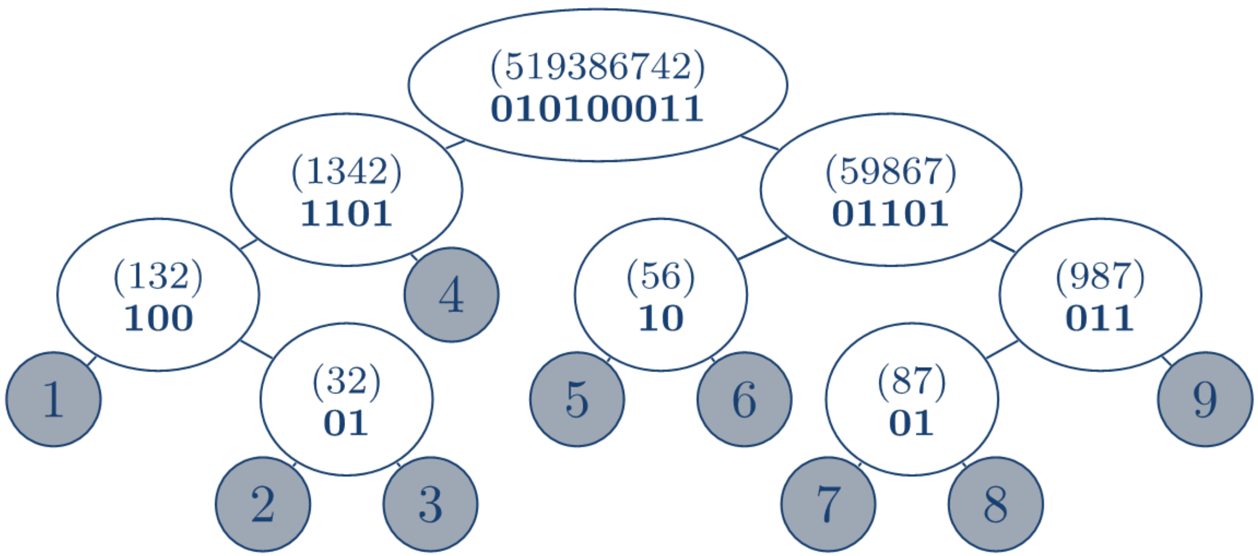
$$\frac{P'(\sigma,\tau)}{P'(\tau,\sigma)} = \frac{\pi(\tau)}{\pi(\sigma)} = \frac{\cancel{p_{5,1}}\ \cancel{p_{5,3}}\ p_{8,6}\ p_{9,6}\ \cancel{p_{6,1}}\ \cancel{p_{6,3}}\ p_{8,5}\ p_{9,6}\ p_{6,5}}{\cancel{p_{1,5}}\ \cancel{p_{3,5}}\ p_{6,8}\ p_{6,9}\ \cancel{p_{1,6}}\ \cancel{p_{3,6}}\ p_{5,8}\ p_{6,9}\ p_{5,6}} = \frac{p_{6,5}}{p_{5,6}}$$

$$\boxed{P'(\sigma,\tau) = 1 - v_{5\wedge 6} = p_{6,5}}$$

# Thm 2: Proof sketch

Markov chain M' allows a transposition if it corresponds to an ASEP move on one of the internal vertices.

Each ASEP is rapidly mixing $\Rightarrow$ M' is rapidly mixing.




$M_{NN}$ is also rapidly mixing if {p} is *weakly regular*.

i.e., for all i, $p_{i,j} < p_{i,j+1}$ if $j > i$. (by comparison)

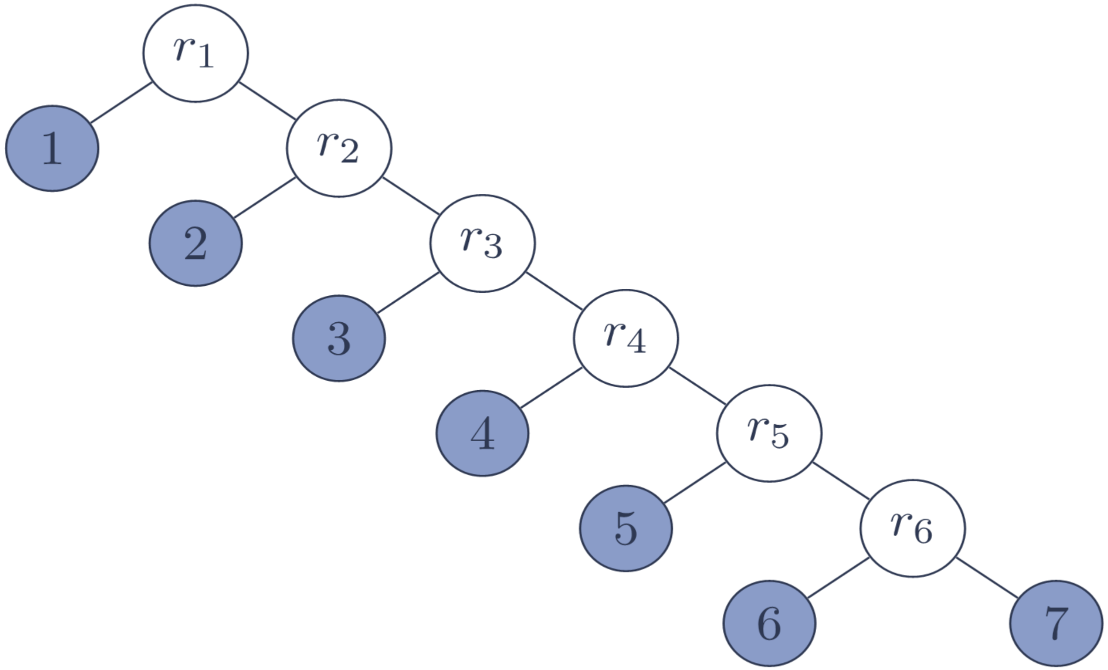
# So $M_{NN}$ is rapidly mixing when

**Choose your weapon:**

$p_{i,j} = r_i \geq 1/2 \quad \forall\, i < j$

**Tree hierarchies:**

$p_{i,j} = q_{i \wedge j} > 1/2 \quad \forall\, i < j$



# Talk Outline

1. Background
2. New Classes of Bias where $M_{nn}$ is fast
    - Choose your Weapon
    - League Hierarchies
3. $M_{nn}$ can be slow

# But.....$M_{NN}$ can be slow

<u>Thm 3</u>: There are examples of positively biased $\{p_{ij}\}$ for which $M_{NN}$ is slowly mixing.

1. Reduce to biased lattice paths

always in order

$$\overbrace{1\ \ 2\ \ 3\ \ \dots\ \ \tfrac{n}{2}}\quad \overbrace{\tfrac{n}{2}+1\ \ \dots\ \ n}$$

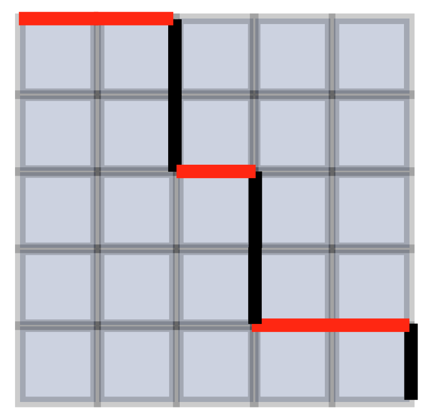
$$p_{ij} = \begin{cases} 1 & \text{if } \ i < j \le \frac{n}{2} \quad \text{or} \quad \frac{n}{2} < i < j \end{cases}$$

<u>Permutation $\sigma$:</u>

1 2 6 7 3 8 9 4 5 10

1 1 0 0 1 0 0 1 1 0

$\longrightarrow$

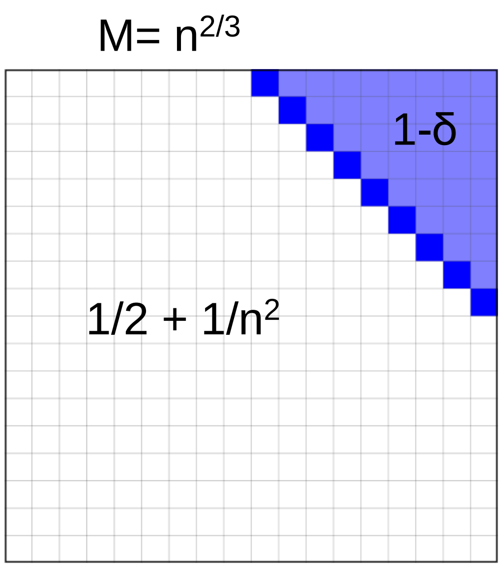


$\frac{n}{2}$ 1's and $\frac{n}{2}$ 0's

# Slow mixing example

Thm 3: There are examples of positively biased {p} for which $M_{NN}$ is slowly mixing.

1. Reduce to biased lattice paths
2. Define bias on individual cells (non-uniform growth proc.)




$$p_{ij} = \begin{cases} 1 & \text{if } i < j \le \frac{n}{2} \text{ or } \frac{n}{2} < i < j \\ 1/2 + 1/n^2 & \text{if } i + (n-j+1) < M \\ 1-\delta & \text{otherwise} \end{cases}$$

Each choice of $p_{ij}$ where $i \le \frac{n}{2} < j$ determines the bias on square $(i, n-j+1)$

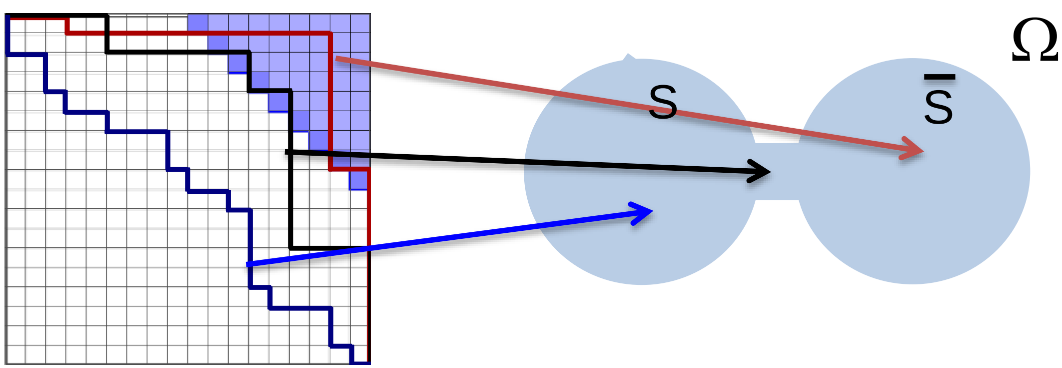
("fluctuating bias")

[Greenberg, Pascoe, Randall]

# Slow mixing example

Thm 3: There are examples of positively biased {p} for which $M_{NN}$ is slowly mixing.

1. Reduce to biased lattice paths
2. Define bias on individual cells
3. Show that there is a “bad cut” in the state space




Implies that $M_{NN}$ can take exp. time to reach stationarity.

Therefore biased card shuffling can be slow too!

# Open Problems

1. Is $M_{NN}$ always rapidly mixing when $\{p_{i,j}\}$ are positively biased and satisfy a *regularity condition*? (i.e., $p_{i,j}$ is decreasing in i and j)

2. When does bias speed up or slow down a chain?

Thank you!