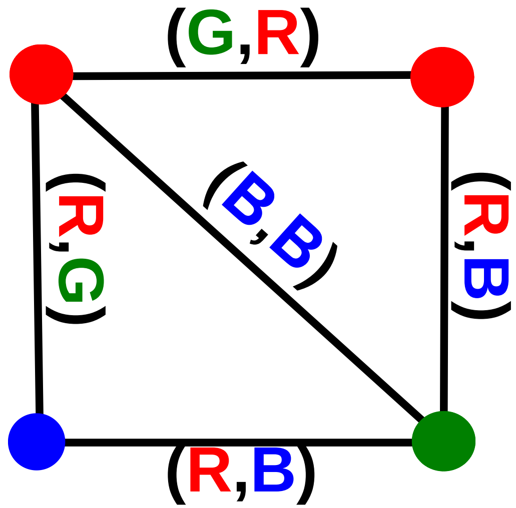
Algorithms to Approximately Count and Sample Conforming Colorings of Graphs



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“Conforming Colorings”

Given:

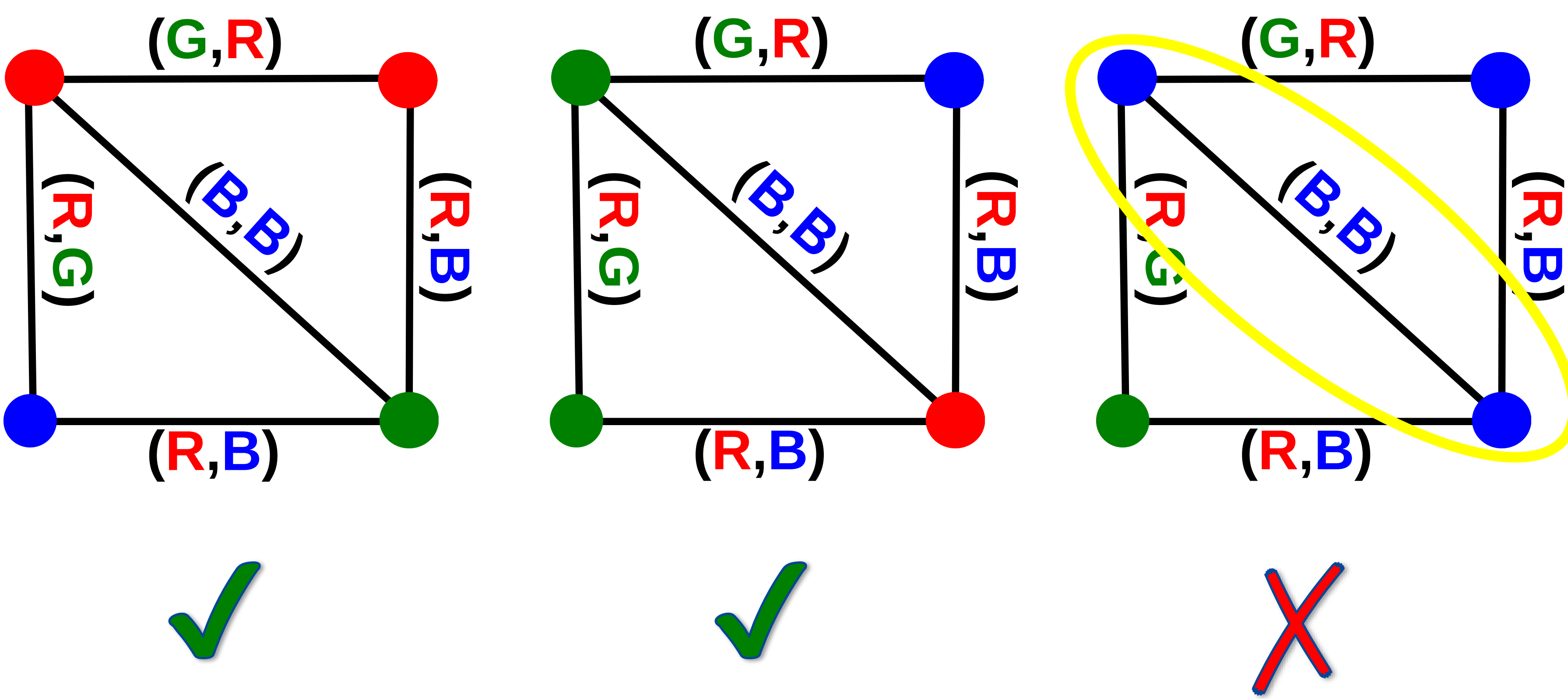
- a (multi) graph **$G = (V,E)$**
- a set of colors **$[k] = \{1, . . .,k\}$**
- a set of edge constraints **$F: E \rightarrow [k] \times [k]$**

A coloring **C** of the vertices of G is **conforming** to **F** if for every edge **$e=(u,v)$**,

$$F(e) \neq (C(v), C(u)).$$

Conforming Colorings

A coloring **C** of the vertices of G is **conforming** to F if for every edge e=(u,v), **F(e)≠(C(v),C(u))**.



Applications

1. Resource Allocation
   - Colors are Resources
   - Vertices are Jobs
   - Edge Constraints are Incompatible Scheduling Assignments
2. Generalizes Graph Theoretic Concepts

Examples in Graph Theory

Conforming colorings **generalize** the following:

- Independent sets
- Vertex colorings
- List colorings
- H-colorings
- Adapted colorings

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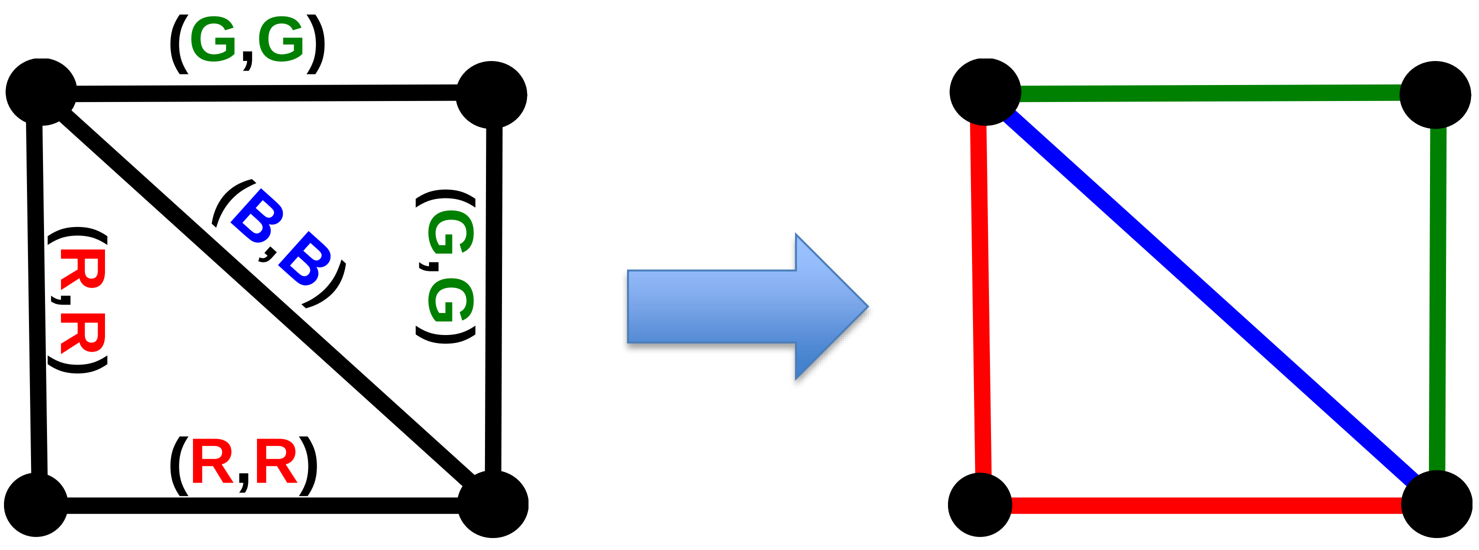
- **Adapted colorings**

Adapted (or Adaptable) Colorings

Given:

- a (multi) graph **G = (V,E)**
- an edge coloring **F**

A coloring **C** of the vertices of G is **adapted** to **F** if there is no edge **e = (u,v)** with **F(e) = C(v) = C(u).**



Adapted (or Adaptable) Colorings

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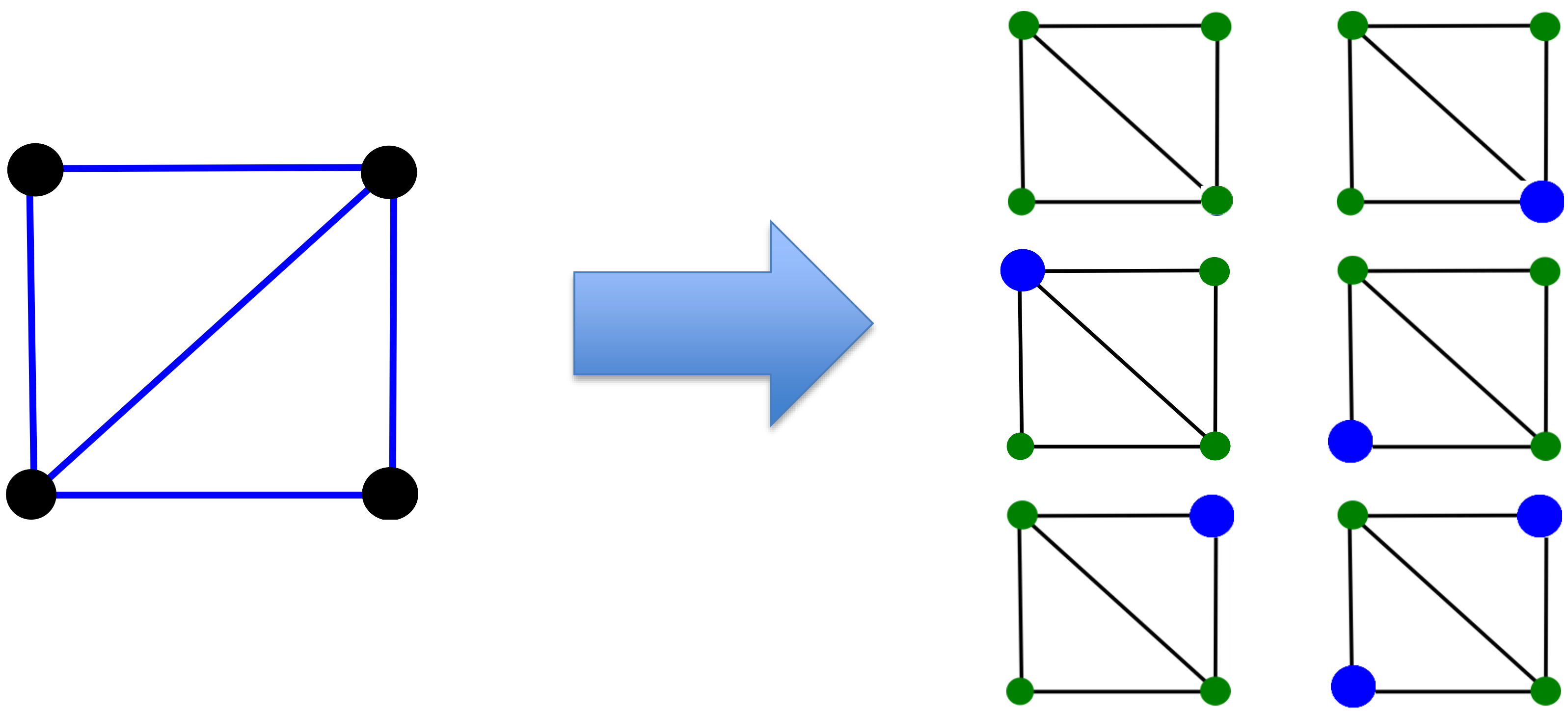
- Adaptable Chromatic Number introduced [Hell & Zhu, 2008]
- Adapted List colorings of Planar Graphs [Esperet et al., 2009]
- Adaptable chromatic number of graph products [Hell et al., 2009]
- Polynomial algorithm for finding adapted 3-coloring given edge 3-coloring of complete graph [Cygan et al., SODA ‘11]

Independent Sets

Conforming colorings generalize **Independent Sets**

- Set k = 2
- Color each edge **(B,B)**
- Each conforming coloring is an independent set



Approximately Counting and Sampling

- Extensive work using Monte Carlo approaches to count and sample for special cases.
- Design a Markov chain for sampling configurations that is rapidly mixing

  (i.e. independent sets, colorings)

- These chains can be slow. Non-local ones can be more effective but harder to analyze.

  (i.e. Wang-Swendsen-Kotecký)

Our Results

$k \geq \max(\Delta,3)$

(Δ = max degree including multi-edges and self-loops)

– A polynomial time algorithm to find a conforming coloring
– The local chain M_L connects the state space
– M_L mixes rapidly
– A FPRAS for approximately counting

$k = 2$

– A new "component" chain M_C
– Provide conditions under which we have
  - a polynomial time algorithm to find a conforming coloring
  - M_C mixes rapidly
  - a FPRAS for approximately counting
– An example where M_C and M_L are slow

Definitions

Definition: Given ε, the mixing time is

$$\tau(\varepsilon) = \max_{x} \min \{t: \Delta_x(t') < \varepsilon, \text{ for all } t' \geq t\}.$$

($\Delta_x(t)$ is the total variation distance)

- A Markov chain is rapidly mixing if $\tau(\varepsilon)$ is poly(n, log(ε^{-1})).
- A Markov chain is slowly mixing if $\tau(\varepsilon)$ is at least exp(n).

Definition: A Fully Polynomial Randomized Approximation Scheme (FPRAS) is a randomized algorithm that given a graph G with n vertices, edge coloring F and error parameter $0 < \varepsilon \leq 1$ produces a number N such that

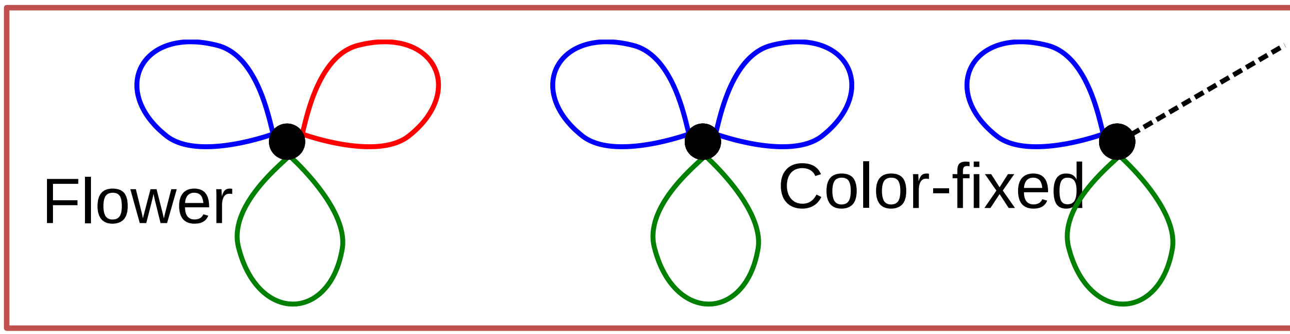
$$P[(1-\varepsilon)N \leq A(G,F) \leq (1+\varepsilon)N] \geq 3/4$$

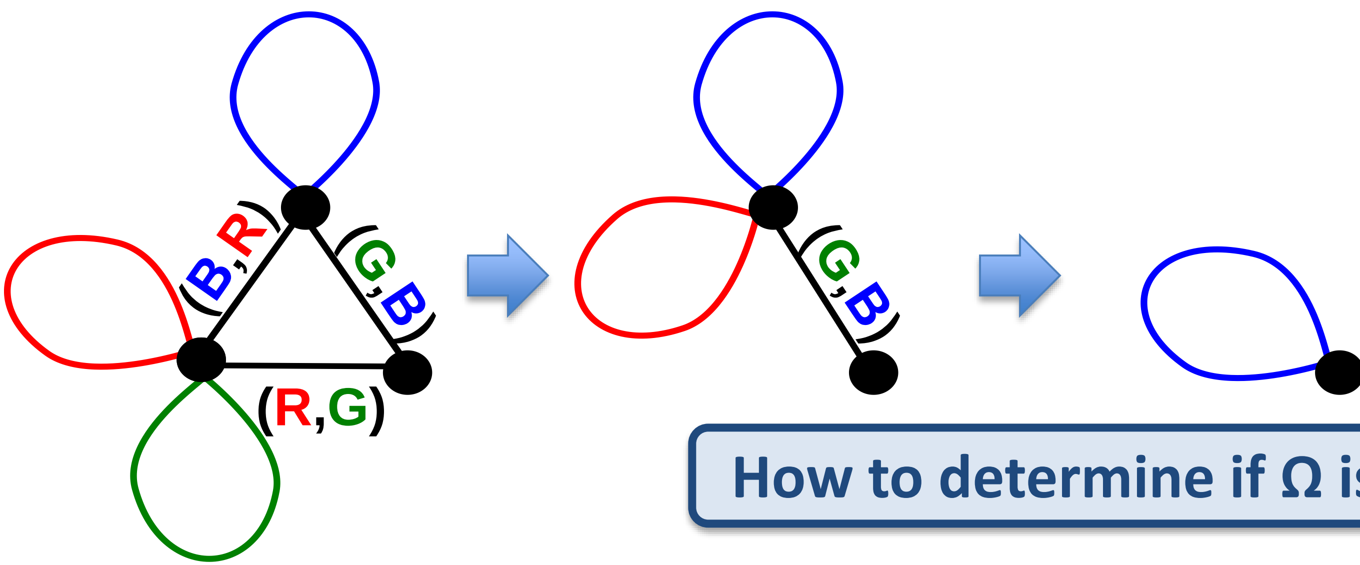
$k \geq \max(\Delta,3)$

Finding a Conforming Coloring

Thm: When $k \geq \max(\Delta,3)$ and Ω is not degenerate, there exists a conforming k-coloring and we give an $O(\Delta n^2)$ algorithm for finding one.

Example: k = 3





How to determine if Ω is degenerate?

k ≥ max(Δ,3)

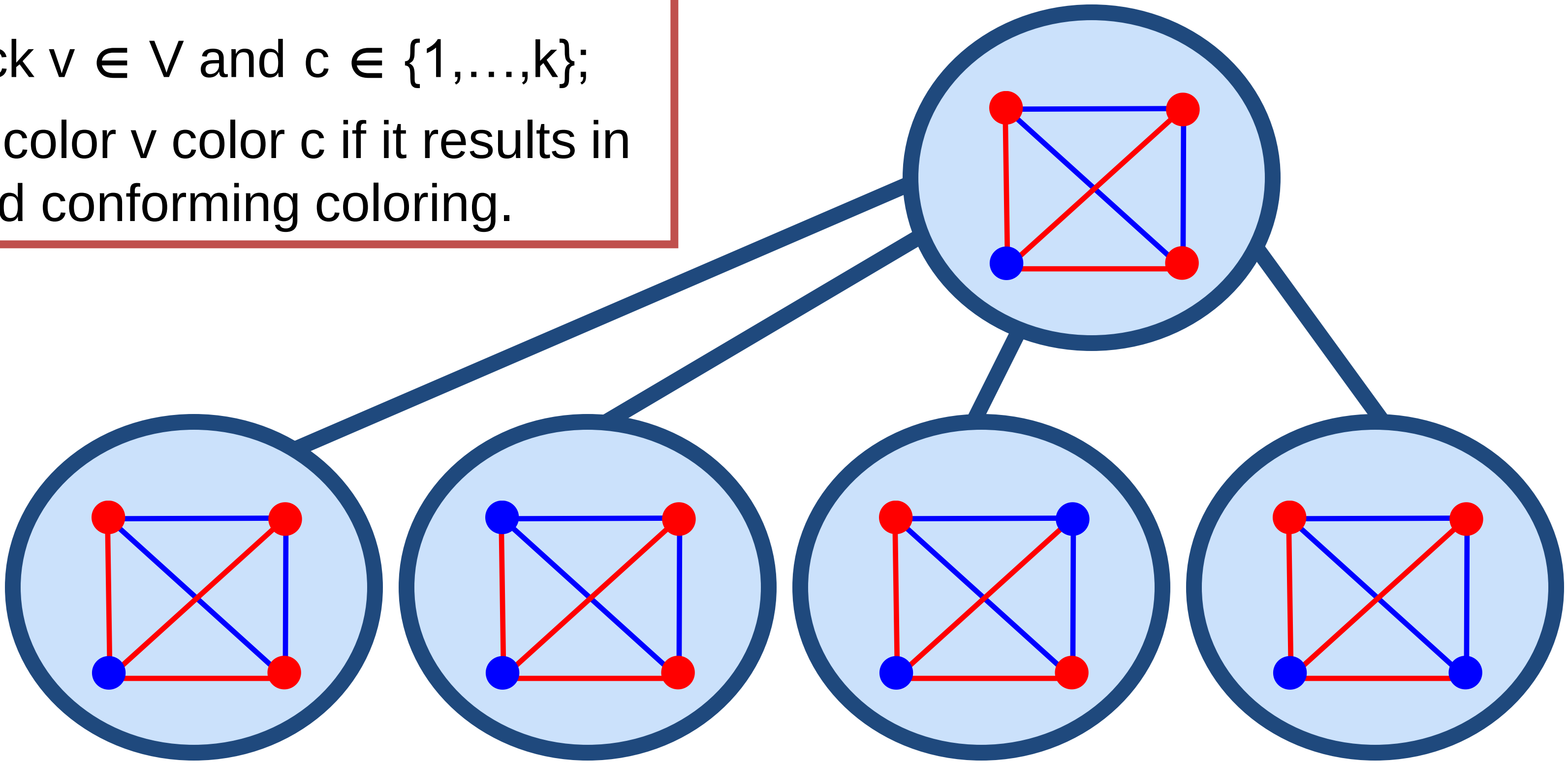
The Local Markov Chain M_L

The Markov chain M_L:

Starting at σ_0, Repeat:

- With prob. ½ do nothing;
- Pick $v \in V$ and $c \in \{1,\ldots,k\}$;
- Recolor v color c if it results in a valid conforming coloring.

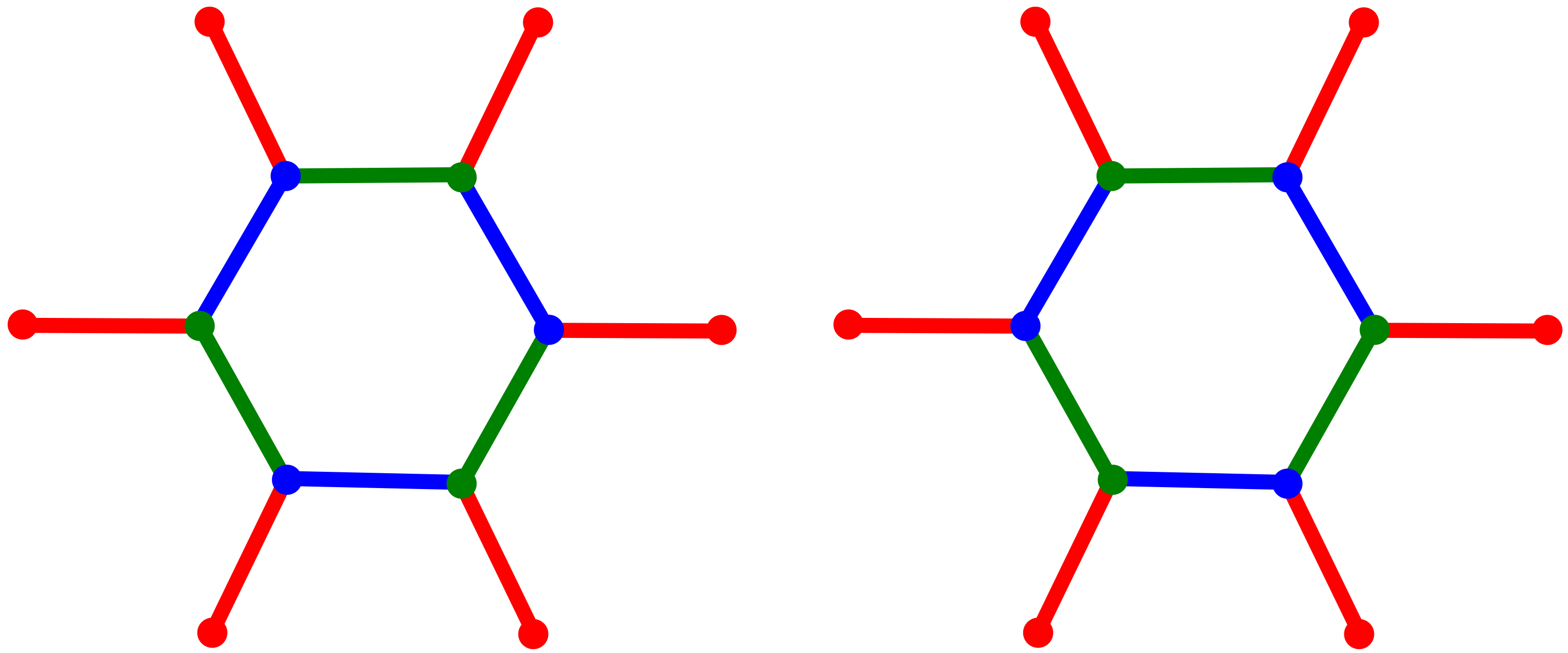
Example: k = 2



k ≥ max(Δ,3)

When does M_L connect Ω ?

Thm: If k ≥ max(Δ,3), M_L connects the state space Ω.



k ≥ max(Δ,3)

Rapid Mixing of M_L and a FPRAS

Thm: If $k \geq \max(\Delta,3)$, then M_L is rapidly mixing and there exists a FPRAS for counting the number of conforming k-colorings.

- Use path coupling [Dyer, Greenhill '98] to show M_L is rapidly mixing.
- Use M_L to design a FPRAS [Jerrum, Valiant, Vazirani '86]

Our Results

$k \geq \max(\Delta, 3)$

(Δ = max degree including multi-edges and self-loops)

- A polynomial time algorithm to find a conforming coloring
- The local chain M_L connects the state space
- M_L mixes rapidly
- A FPRAS for approximately counting

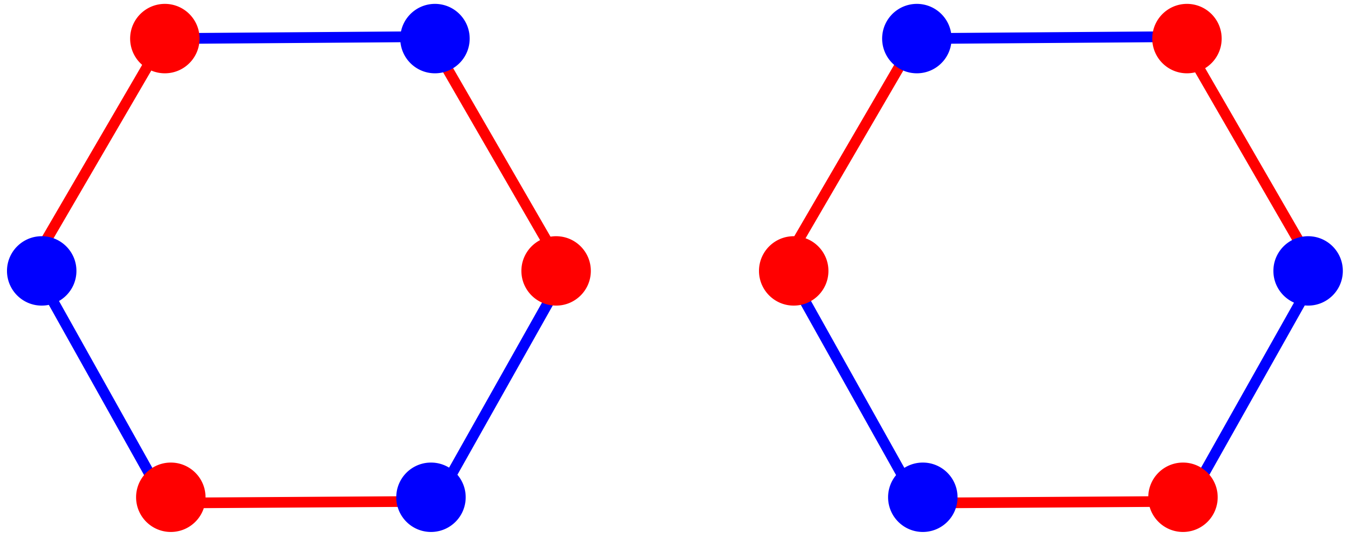
$k = 2$

- A new "component" chain M_C
- Provide conditions under which we have
  - a polynomial time algorithm to find a conforming coloring
  - M_C mixes rapidly
  - a FPRAS for approximately counting
- An example where M_C and M_L are slow

k = 2

What's Special about k = 2?

- Generalizes independent sets
- The local chain M_L does not connect the state space Ω

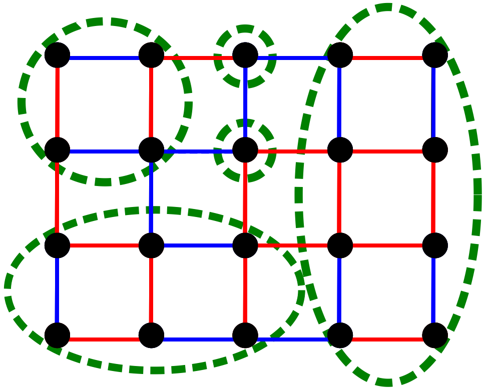


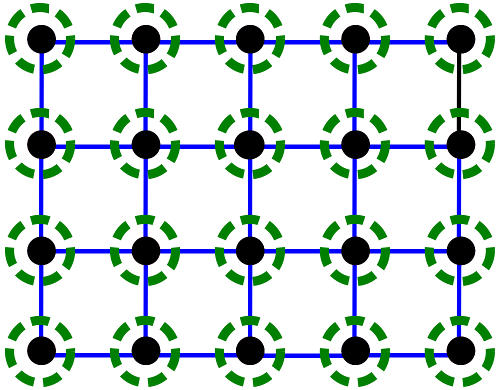
k = 2

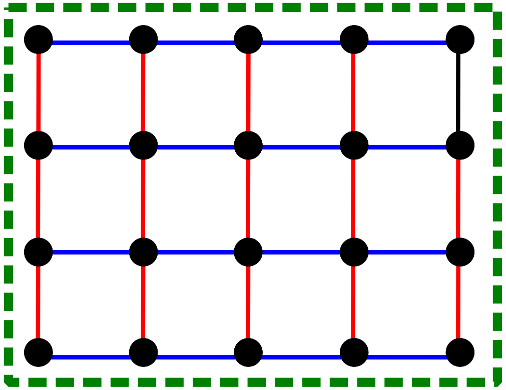
The Component Chain M_C

Defining Color-Implied Components

- A path $P = v_1, v_2, \ldots, v_x$ is b-alternating if
  1. $F_1(v_1,v_2) = b$
  2. For all $1 \le i \le x-2$, $F_{i+1}(v_i,v_{i+1}) \neq F_{i+1}(v_{i+1},v_{i+2})$
- Vertices u and v are color-implied if there is a 1-alternating and a 2-alternating path from u to v or u=v
- Color-implied is an equivalence relation and defines a partition of the vertices into color-implied components





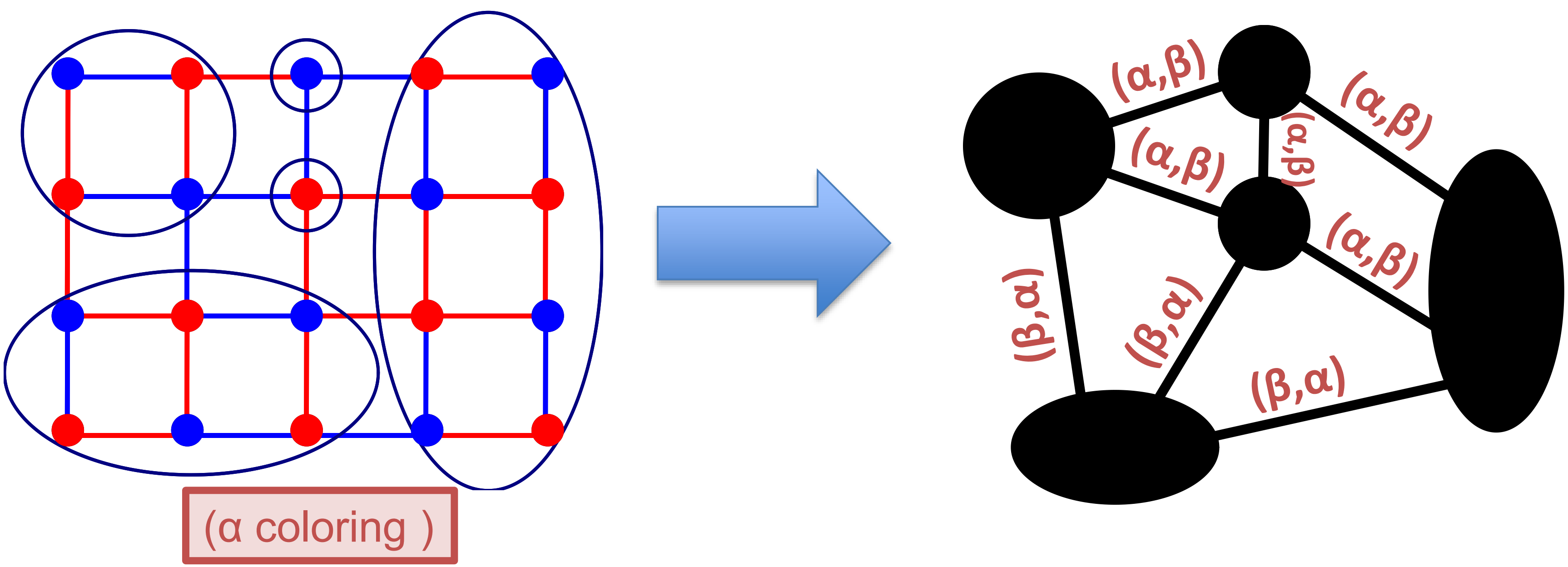


k = 2

The Component Chain M_c

The Color-Implied Component Graph

- Each color implied component C has at most 2 colorings α(C) and β(C).
- Components become vertices
- Edges reflect the edges and coloring constraints from F



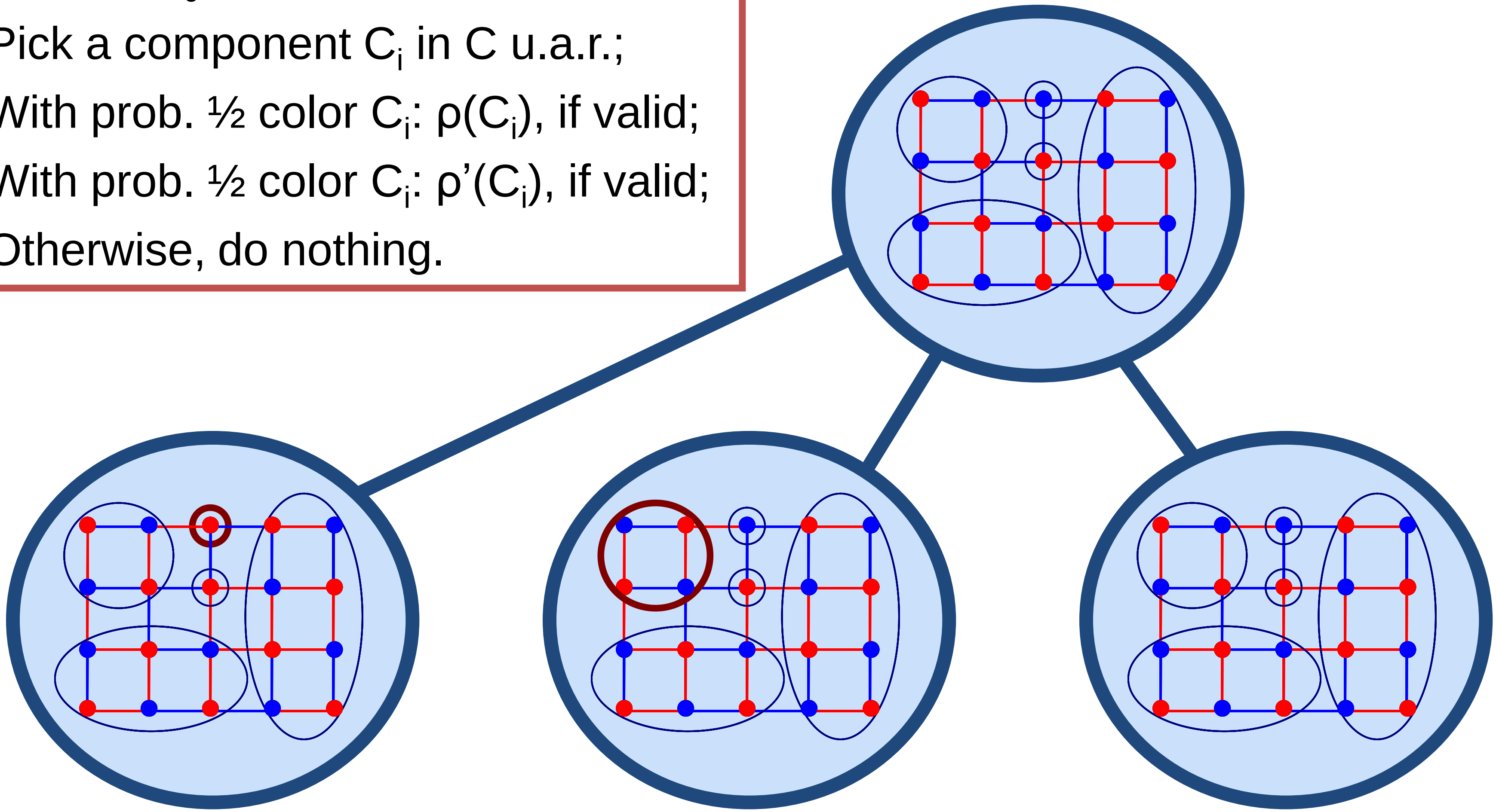
k = 2

The Component Chain M_C

The Markov chain M_C:

Starting at σ_0, Repeat:

- Pick a component C_i in C u.a.r.;
- With prob. ½ color C_i: $\rho(C_i)$, if valid;
- With prob. ½ color C_i: $\rho'(C_i)$, if valid;
- Otherwise, do nothing.



k = 2

Positive Results

If every vertex v in the color-implied component graph satisfies one of the following

- $d(v) \leq 2$ (d(v) = the degree of v)
- $d(v) \leq 4$ and v is monochromatic

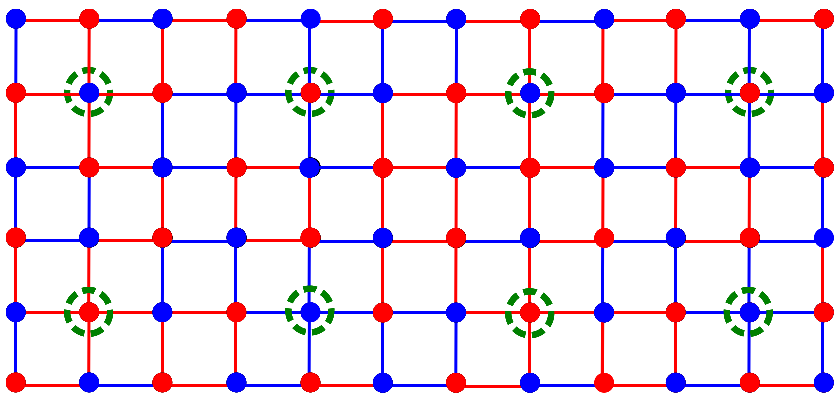
1. We give an $O(n^3)$ algorithm for finding a 2-coloring
2. The chain M_C is rapidly mixing
3. We give a FPRAS for approximately counting

Rapid Mixing Proof: Uses path coupling and comparison with a similar chain which selects an edge in the component graph and recolors both vertices.

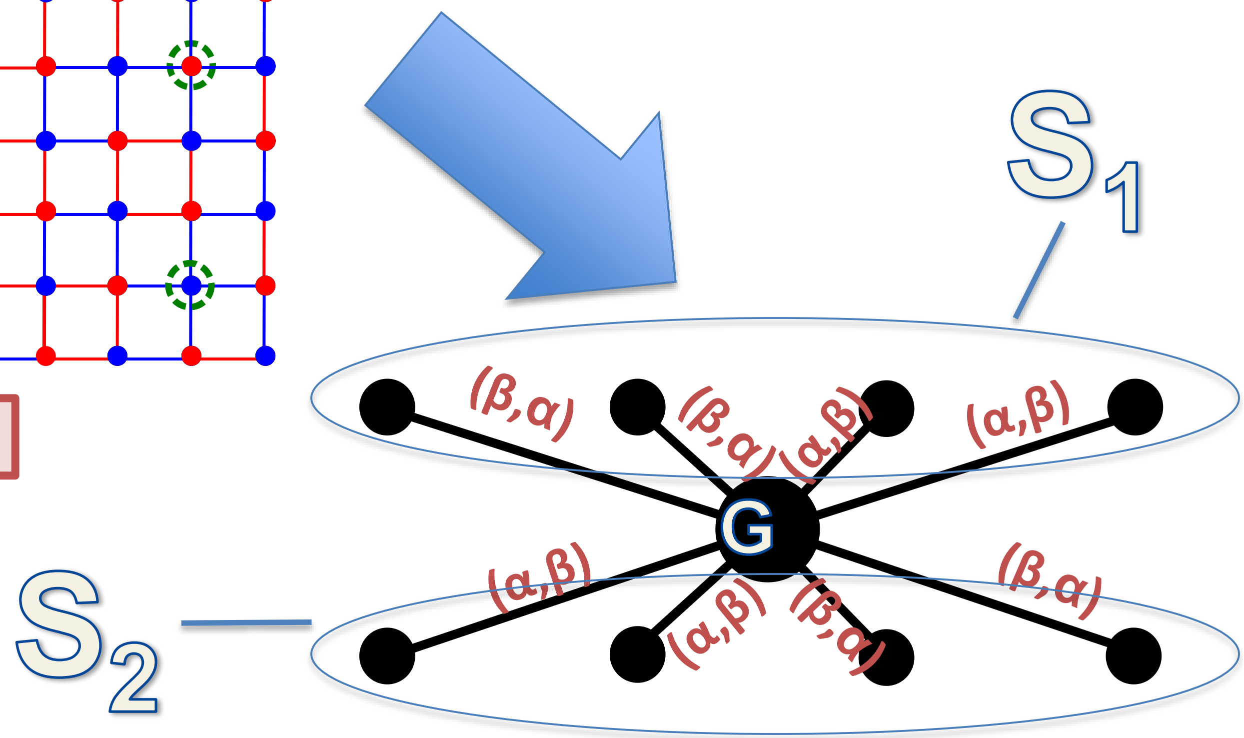
k = 2

Slow Mixing of M_C on Other Graphs

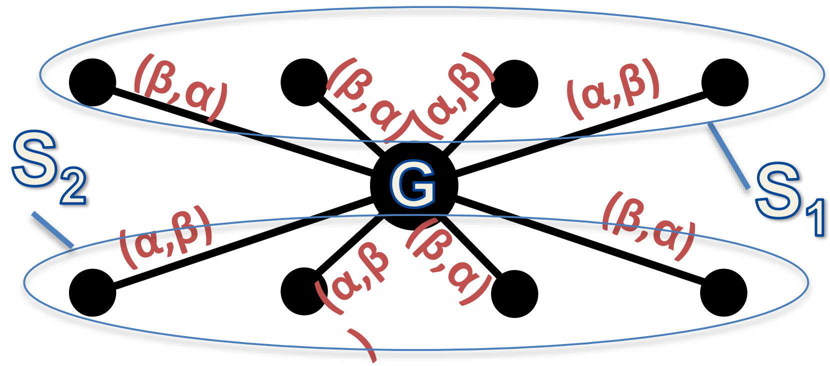
<u>Thm</u>: There exists a bipartite graph G with $\Delta = 4$ and edge 2-coloring F for which M_C and M_L take exponential time to converge.

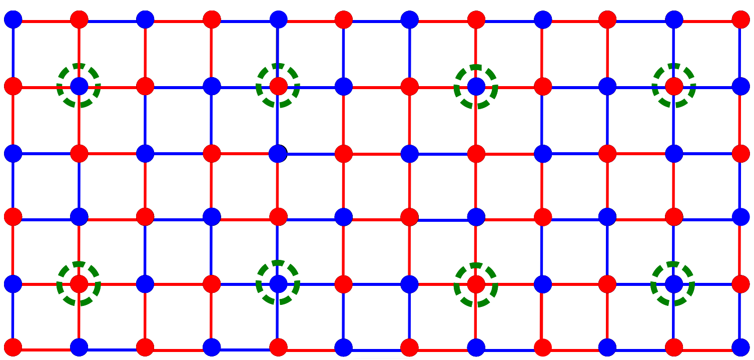


(α coloring)

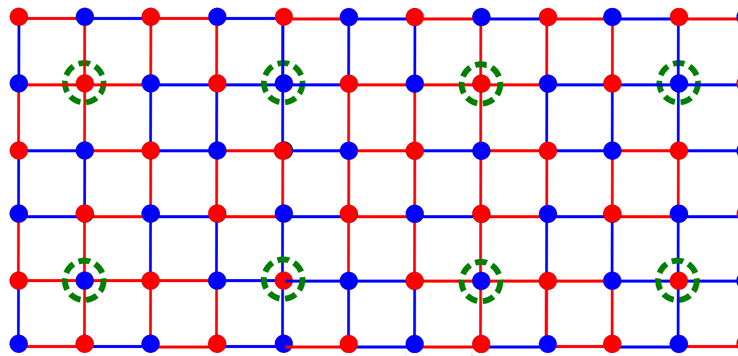


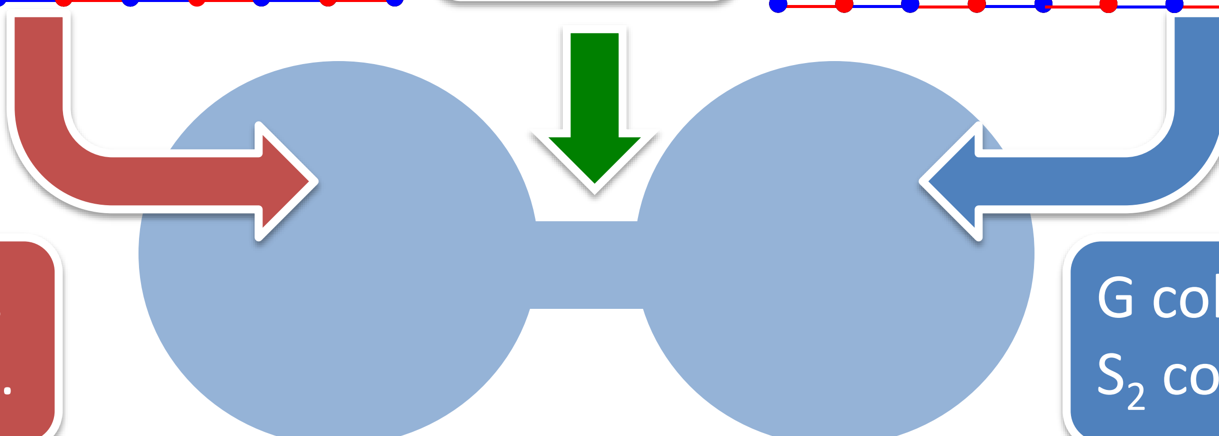
The "Bottleneck"





S_1 colored α.
S_2 colored β.





G colored α.
S_1 colored α.

G colored β.
S_2 colored β.

Open Problems

1. When $k > 2$, is there an analog to M_C that is faster?
2. Other approaches to sampling when $k \geq 2$?
3. Can conforming/adapted colorings give insights into phase transitions for independent sets and colorings?

Thank you!