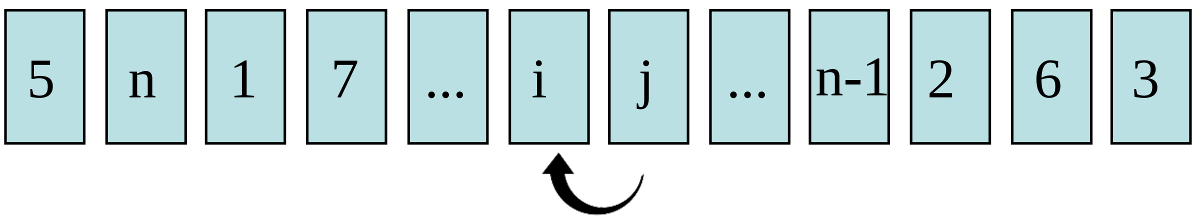
Mixing Times of Markov Chains for Self-Organizing Lists and Biased Permutations

Prateek Bhakta, Sarah Miracle,
Dana Randall and Amanda Streib

Sampling Permutations

- pick a pair of adjacent cards uniformly at random
- put j ahead of i with probability $p_{j,i} = 1 - p_{i,j}$



This is related to the "Move-Ahead-One Algorithm" for self-organizing lists.

Is **M** always rapidly mixing?

M is not always fast . . .

If $p_{i,j} \geq ½\ \forall\ i < j$ we say the chain is ***positively biased***.

Q: If the $\{p_{ij}\}$ are positively biased, is **M** always rapidly mixing?

Conjecture [Fill]: If $\{p_{ij}\}$ are positively biased and monotone then **M** is rapidly mixing.

What is already known?

- **Uniform bias**: If $p_{i,j} = ½ \;\; \forall\, i, j$ then **M** mixes in $\theta(n^3 \log n)$ time. [Aldous '83, Wilson '04]

- **Constant bias**: If $p_{i,j} = p > ½ \;\; \forall\, i < j$, then **M** mixes in $\theta(n^2)$ time. [Benjamini et al. '04, Greenberg et al. '09]

- **Linear extensions of a partial order**:
  If $p_{i,j} = ½$ or $1 \;\; \forall\, i < j$, then **M** mixes in $O(n^3 \log n)$ time. [Bubley, Dyer '98]

Our Results [Bhakta, M., Randall, Streib]

- **M** is **fast** for two new classes
  - **"Choose your weapon"**
  - **"League hierarchies"**
  - Both classes extend the uniform and constant bias cases
- **M** can be **slow** even when the $\{p_{ij}\}$ are positively biased

Talk Outline

1. Background
2. New Classes of Bias where **M** is fast
   - Choose your Weapon
   - League Hierarchies
3. **M** can be slow even when the $\{p_{ij}\}$ are positively biased

Choose Your Weapon

Given parameters $\frac{1}{2} \leq r_{1, \ldots,} r_{n-1} \leq 1.$

<u>Thm 1</u>: Let $p_{i,j} = r_i \quad \forall\, i < j.$ Then **M** is rapidly mixing.

<u>Definition</u>**:** The variation distance is

$$\Delta_x(t) = \frac{1}{2} \sum_{y \in \Omega} |P^t(x,y) - \pi(y)|.$$

<u>Definition</u>: Given ε, the mixing time is

$$\tau(\boldsymbol{\varepsilon}) = \max_x \min \{t: \Delta_x(t') < \boldsymbol{\varepsilon}, \forall\, t' \geq t\}.$$

A Markov chain is rapidly mixing if $\tau(\varepsilon)$ is poly(n, log(ε^{-1})).

Choose Your Weapon

Given parameters $\frac{1}{2} \le r_{1, \ldots ,} r_{n-1} \le 1$.

Thm 1: Let $p_{i,j} = r_i \quad \forall\, i < j$. Then **M** is rapidly mixing.

Proof sketch:

A. Define auxiliary Markov chain $\mathbf{M}_{inv}$

B. Show $\mathbf{M}_{inv}$ is rapidly mixing

C. Compare the mixing times of **M** and $\mathbf{M}_{inv}$

$\mathbf{M}_{inv}$ can swap pairs that are not nearest neighbors

- Maintains the same stationary distribution
- Allowed moves are based on inversion tables

Inversion Tables

Permutation σ:

5	6	3	1	2	7	4

$\leftrightarrow$

Inversion Table I_σ:

3 3 2 3 0 0 0

$I_\sigma(1)$ $I_\sigma(2)$

$I_\sigma(i)$ = # elements $j > i$ appearing before i in σ

The map I is a bijection from S_n to $T = \{(x_1, x_2, \ldots, x_n) : 0 \le x_i \le n-i\}$.

Inversion Tables

Permutation σ:

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Inversion Table I_σ:

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$I_\sigma(i)$ = # elements j > i appearing before i in σ

$\mathbf{M}_{inv}$ on Permutations

- choose a card i uniformly
- swap element i with the first $j>i$ to the left w.p. r_i
- swap element i with the first $j>i$ to the right w.p. $1-r_i$

$\mathbf{M}_{inv}$ on Inversion Tables

- choose a column i uniformly
- w.p. r_i: subtract 1 from x_i (if $x_i > 0$)
- w.p. $1- r_i$: add 1 to x_i (if $x_i < n-i$)

Inversion Tables

Permutation σ:

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Inversion Table I_σ:

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M_{inv} on Inversion Tables

- choose a column i uniformly
- w.p. r_i: subtract 1 from x_i (if $x_i>0$)
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M$_{inv}$ is just a product of n independent biased random walks

$\Rightarrow$ **M**$_{inv}$ is rapidly mixing.

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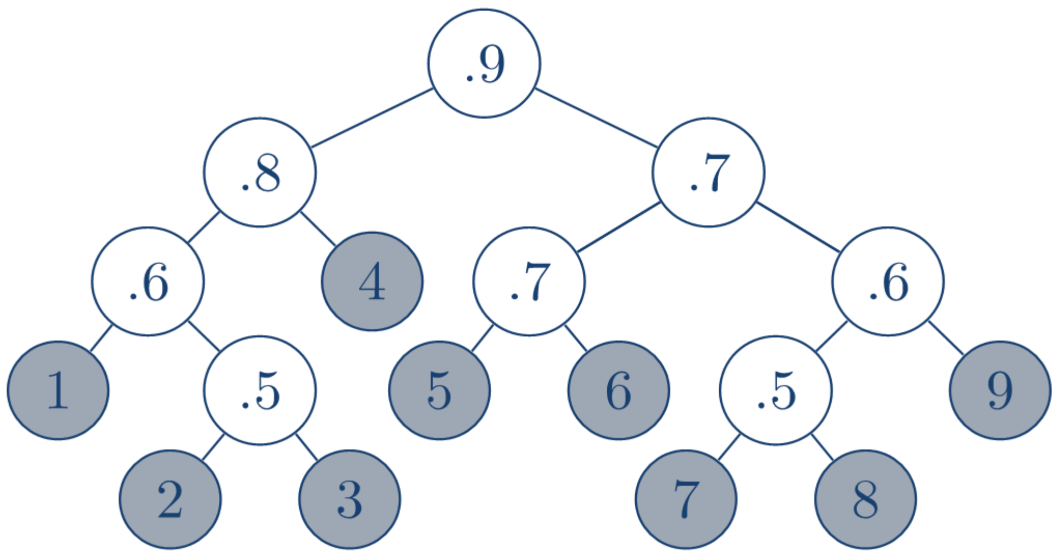
League Hierarchy

Let T be a binary tree with leaves labeled $\{1,\ldots,n\}$.
Given $q_v \geq 1/2$ for each *internal* vertex v.

<u>Thm 2:</u> Let $p_{i,j} = q_{i \wedge j}$ for all $i < j$. Then **M** is rapidly mixing.



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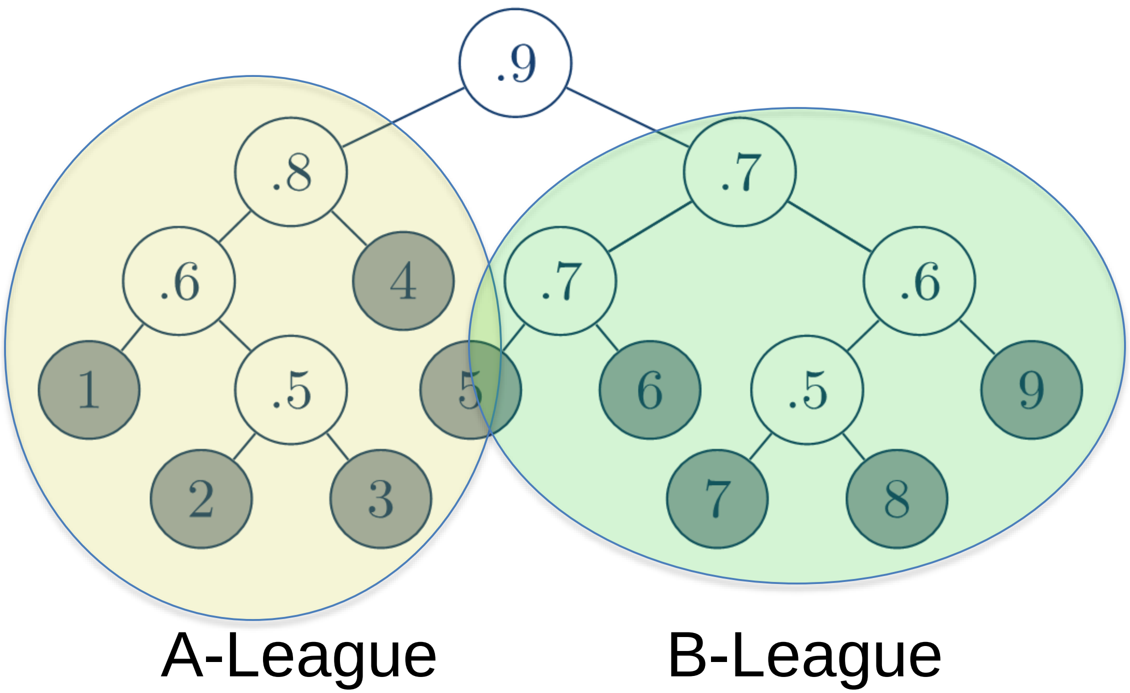

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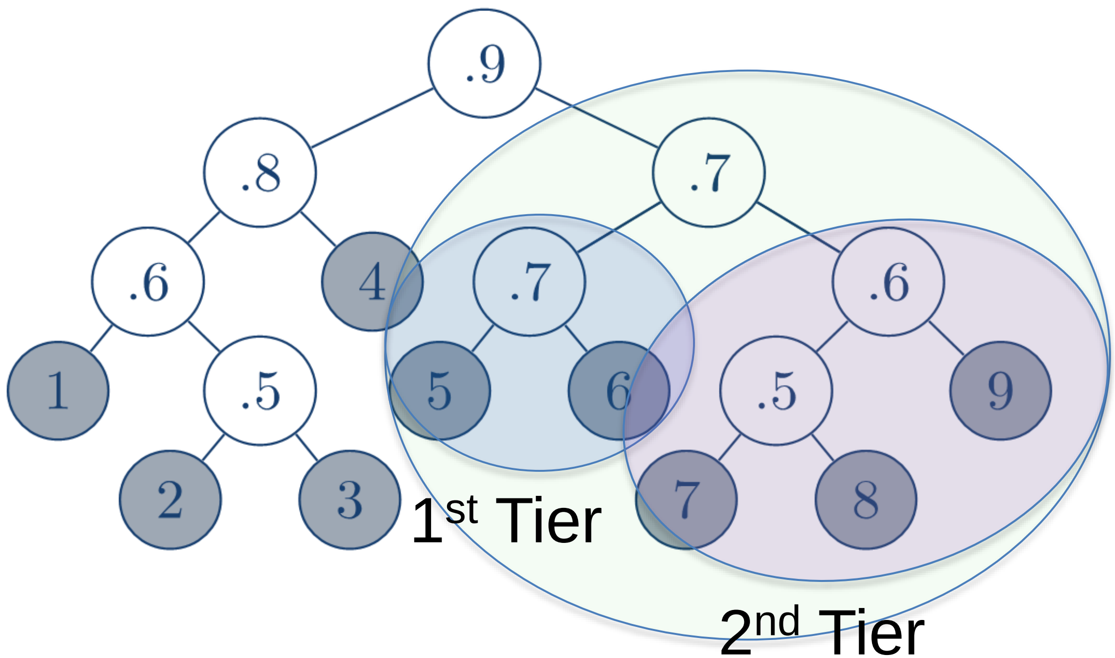

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Thm 2: Let $p_{i,j} = q_{i \wedge j}$ for all $i < j$. Then **M** is rapidly mixing*.

Proof sketch:

A. Define auxiliary Markov chain $\mathbf{M}_{tree}$

B. Show $\mathbf{M}_{tree}$ is rapidly mixing

C. Compare the mixing times of **M** and $\mathbf{M}_{tree}$

$\mathbf{M}_{tree}$ can swap pairs that are not nearest neighbors

- Maintains the same stationary distribution
- Allowed moves are based on the binary tree T

League Hierarchy

Let T be a binary tree with leaves labeled $\{1,\ldots,n\}$.
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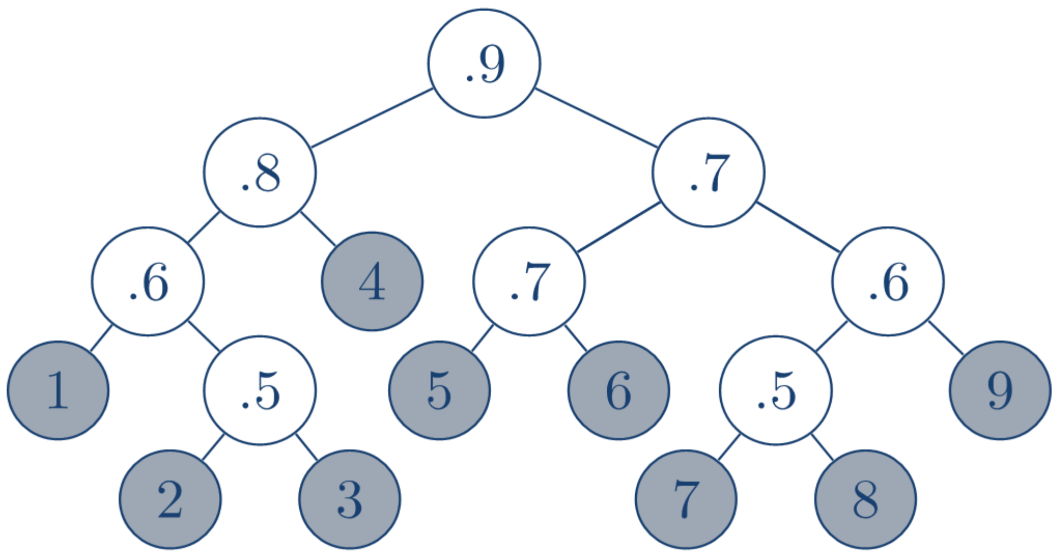
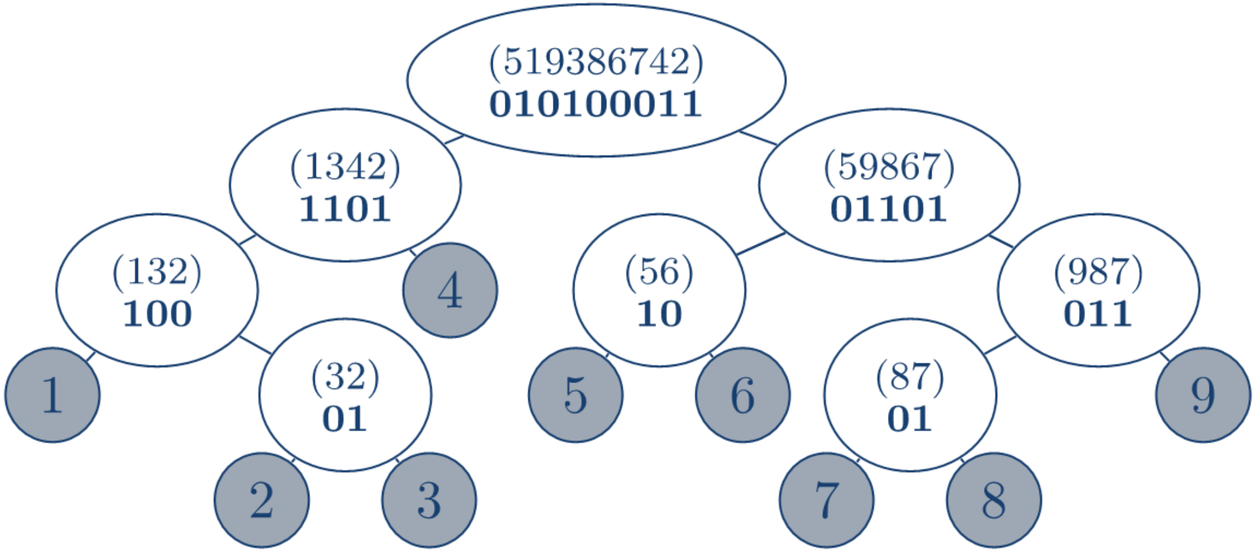


Theorem 2: Proof sketch

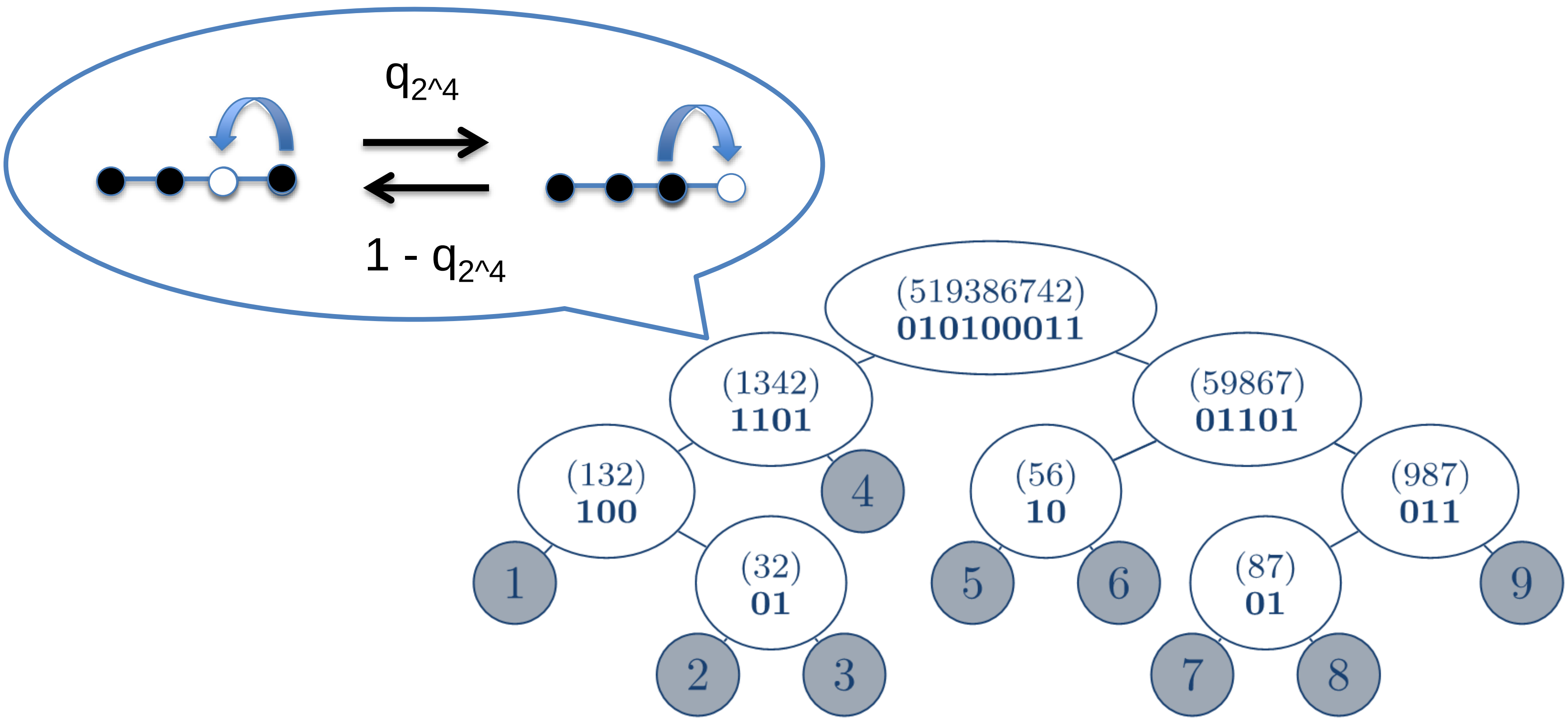
Let T be a binary tree with leaves labeled $\{1,\ldots,n\}$.
Given $q_v \geq 1/2$ for each *internal* vertex v.

<u>Thm 2:</u> Let $p_{i,j} = q_{i\wedge j}$ for all $i < j$. Then **M** is rapidly mixing.



Theorem 2: Proof sketch

Markov chain $\mathbf{M}_{tree}$ allows a transposition if it corresponds to an ASEP move on one of the internal vertices.



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But…..M can be slow

<u>Thm 3</u>: There are examples of positively biased $\{p_{ij}\}$ for which **M** is slowly mixing.

1. Reduce to "biased staircase walks"

always in order

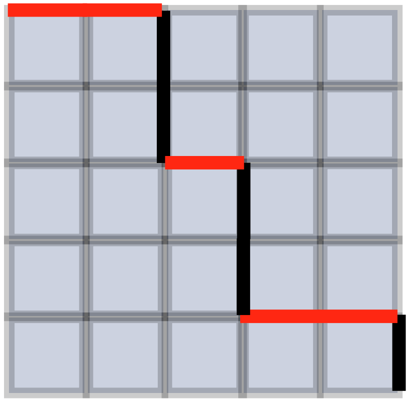
$$\underbrace{1\ 2\ 3\ \ldots\ \tfrac{n}{2}}\ \ \underbrace{\tfrac{n}{2}+1\ \ldots\ n}$$

$$p_{ij} = \begin{cases} 1 & \text{if } i < j \le \frac{n}{2} \quad \text{or} \quad \frac{n}{2} < i < j \end{cases}$$

<u>Permutation σ:</u>

1 2 6 7 3 8 9 4 5 10

1 1 0 0 1 0 0 1 1 0 $\longrightarrow$

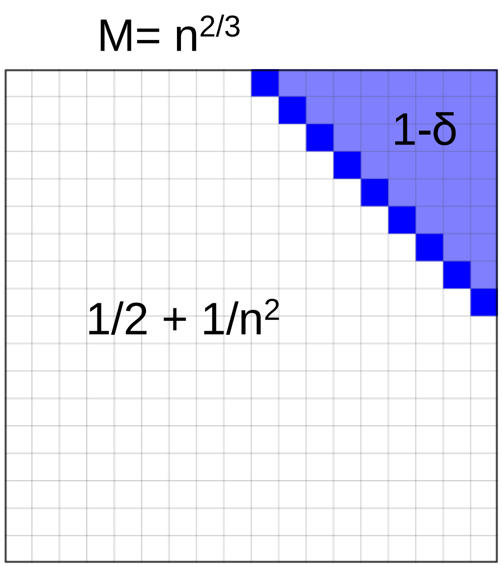


$\frac{n}{2}$ 1's and $\frac{n}{2}$ 0's

Slow Mixing Example

Thm 3: There are examples of positively biased {p} for which **M** is slowly mixing.

1. Reduce to biased staircase walks
2. Define bias on individual cells (non-uniform growth proc.)



$$p_{ij} = \begin{cases} 1 & \text{if } i < j \leq \frac{n}{2} \text{ or } \frac{n}{2} < i < j \\ 1/2 + 1/n^2 & \text{if } i+(n-j+1) < M \\ 1-\delta & \text{otherwise} \end{cases}$$

Each choice of p_{ij} where $i \leq \frac{n}{2} < j$ determines the bias on square $(i, n-j+1)$

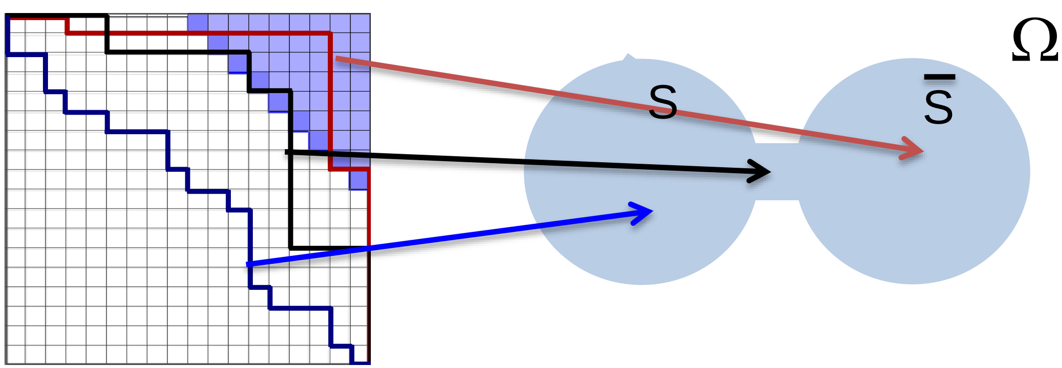
("fluctuating bias")

[Greenberg, Pascoe, Randall]

Slow Mixing Example

<u>Thm 3</u>: There are examples of positively biased {p} for which **M** is slowly mixing.

1. Reduce to biased staircase walks
2. Define bias on individual cells
3. Show that there is a “bad cut” in the state space



Implies that M can take exponential time to reach stationarity.

Therefore biased permutations can be slow too!

Open Problems

1. Is **M** always rapidly mixing when $\{p_{i,j}\}$ are positively biased and satisfy a *monotonicity condition*? (i.e., $p_{i,j}$ is decreasing in i and j)

2. When does bias speed up or slow down a chain?

Thank you!