Phase Transitions in Random Dyadic Tilings and Rectangular Dissections

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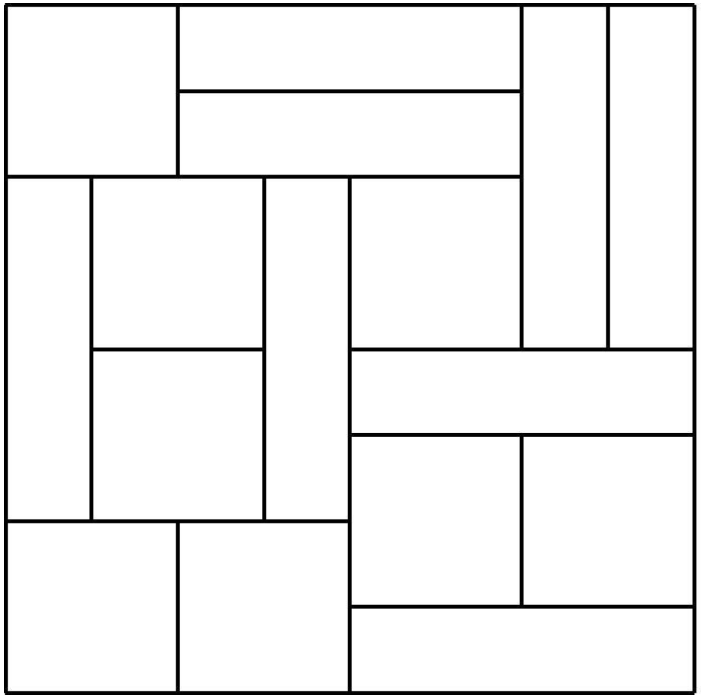
Rectangular Dissections

Rectangular Dissection: A partition of a lattice region into rectangles whose corners lie on lattice points.

Rectangular dissections arise in:

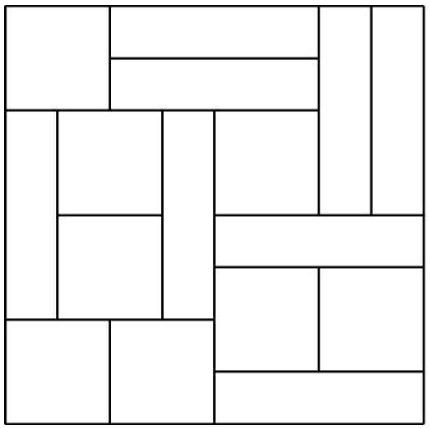
- VLSI layout
- Mapping graphs for floor layouts
- Routings and placements
- Combinatorics



Rectangular Dissections

Partition n x n lattice region into rectangles such that:

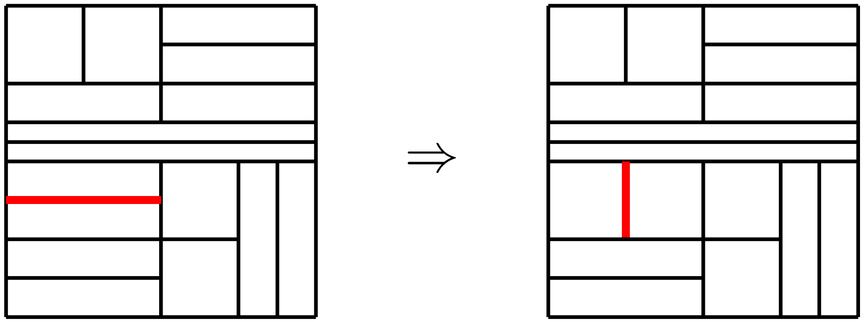
1. There are n rectangles each with area n
2. The corners of rectangles lie on lattice points
3. $n = 2^k$ for an even integer k



The Edge-Flip Chain

Repeat:

1. Pick an random edge e,
2. If e is flippable, flip edge e with probability ½



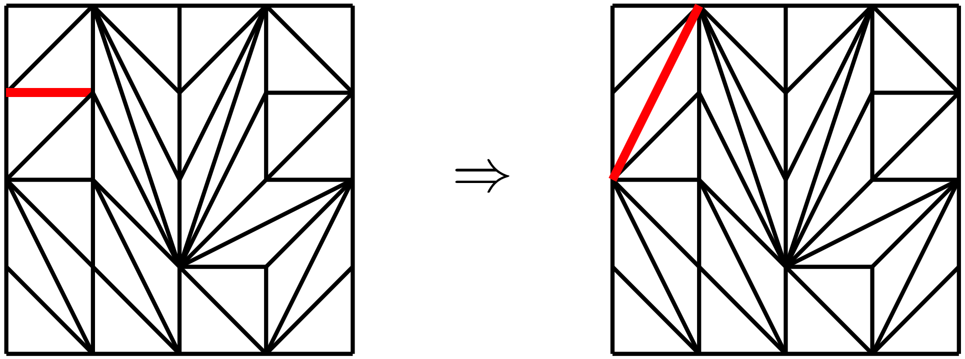
Open Question: Does the edge-flip chain mix rapidly?

Talk Outline

1. Background and Previous Work
2. Our Results
3. Proof Ideas

Related Work: Triangulations

The edge-flip chain:



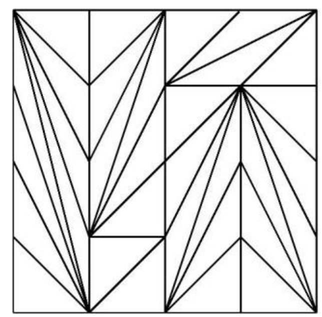
- Triangulations of general point sets: Open
- Triangulations of point sets in convex position: Fast
  [McShine, Tetali '98], [Molloy, Reed, Steiger '98]
- Triangulations on subsets of $\mathbf{Z}^2$: Open
- Weighted Triangulations on subsets of $\mathbf{Z}^2$
  [Caputo, Martinelli, Sinclair, Stauffer '13]

Related Work: Weighted Triangulations

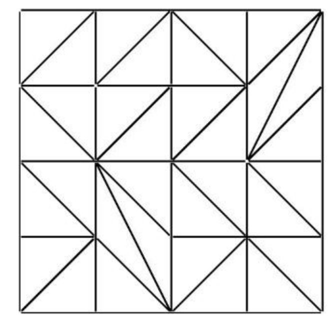
[Caputo, Martinelli, Sinclair, Stauffer '13]

$$\text{Weight}(\sigma) = \lambda^{(\text{total length of edges})}$$

E.g., for $\lambda > 1$,



larger weight



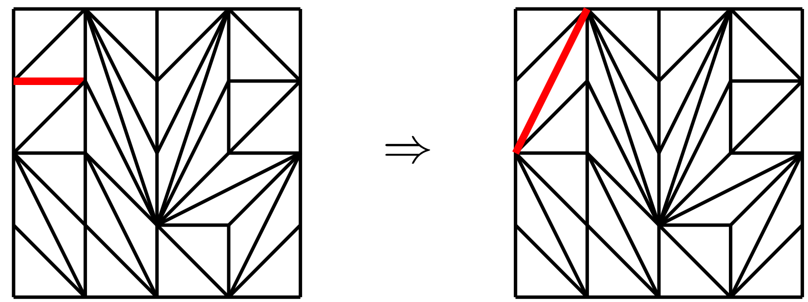
smaller weight

Related Work: Weighted Triangulations

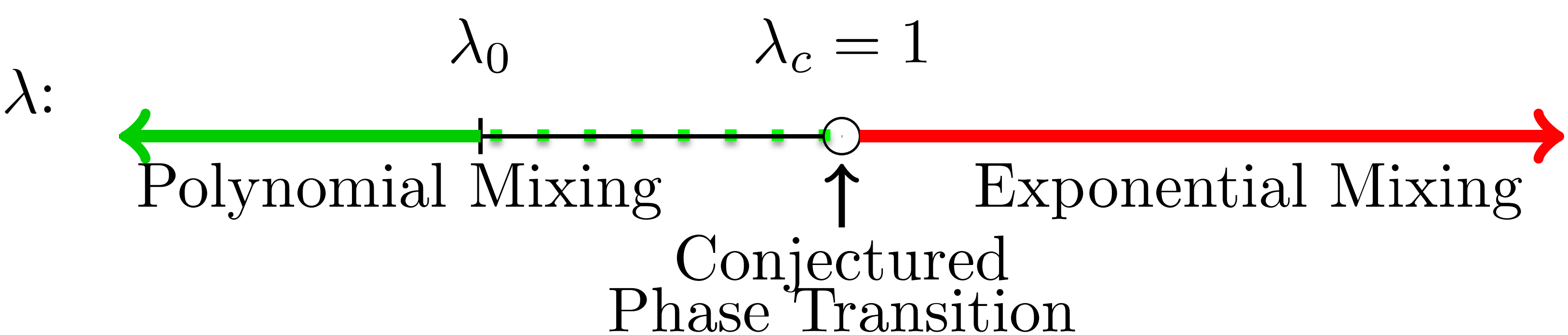
[Caputo, Martinelli, Sinclair, Stauffer '13]

Weight $(\sigma) = \lambda^{(\text{total length of edges})}$

The edge-flip chain:

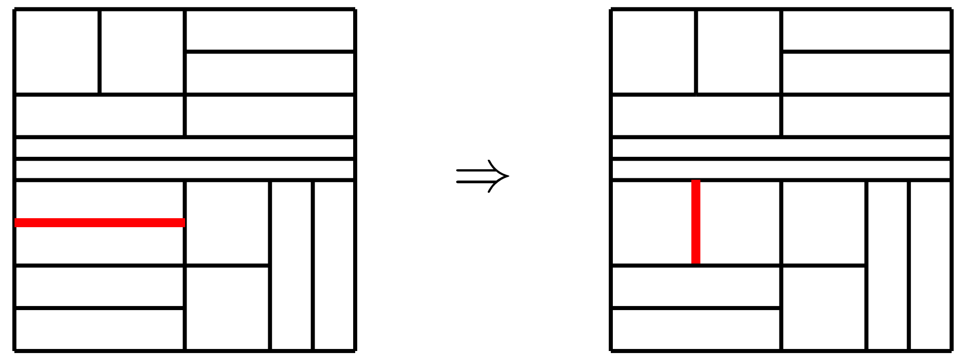


Results [CMSS]:



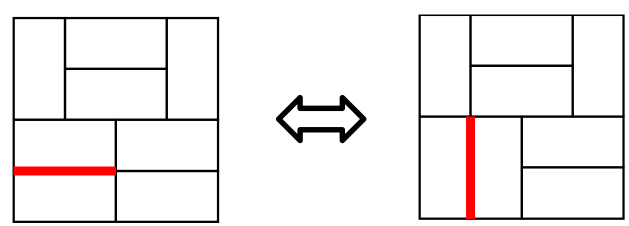
Previous Work: Rectangular Dissections

The edge-flip chain:



Special Cases:

1. Domino Tilings



Fast: [Luby, Randall, Sinclair ‘01], [Randall, Tetali ‘00]

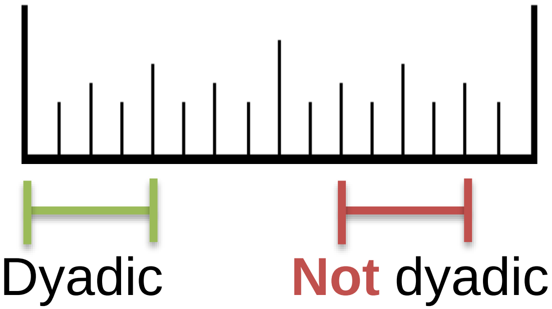
2. Dyadic Tilings

Special Cases: 2. Dyadic Tilings

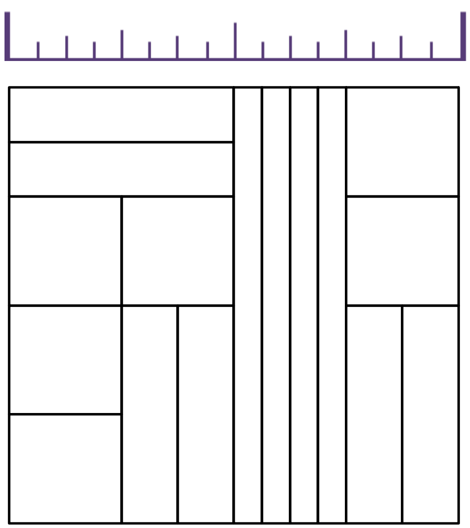
A dyadic rectangle is a region R with dimensions

$$R = \left[a2^s, (a+1)2^s\right] \times \left[b2^t, (b+1)2^t\right],$$

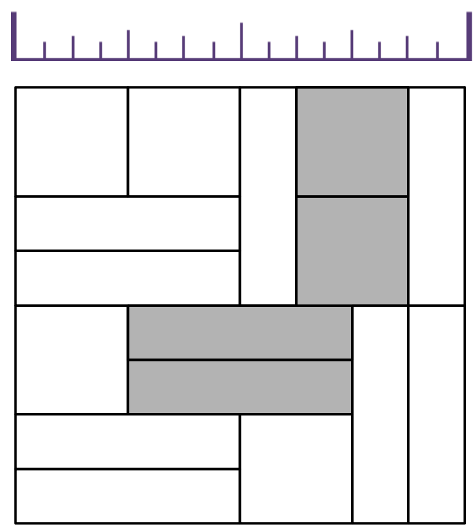
where a, b, s and t are nonnegative integers.



A dyadic tiling of the 2^k x 2^k square is a set of 2^k dyadic rectangles, each with area 2^k (whose union is the full square).



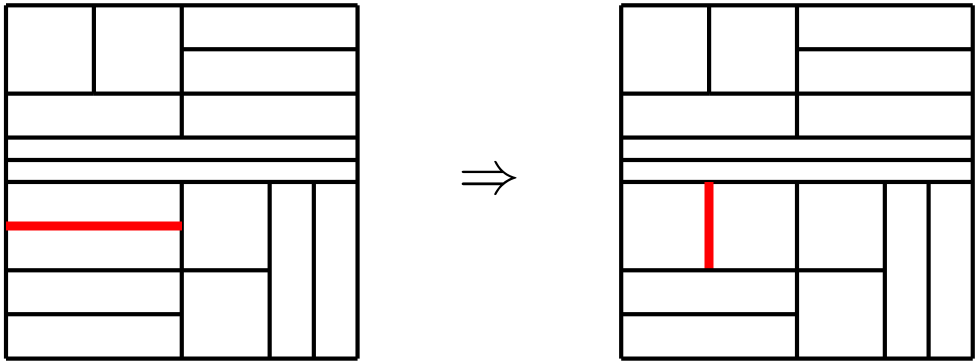
A dyadic tiling



Not a dyadic tiling

Previous Results – Dyadic Tilings

The edge-flip chain:



The edge-flip chain connects the set of dyadic tilings.

[Janson, Randall, Spencer '02]

There is a different Markov chain that converges quickly. [JRS]

Open Question: Does the edge-flip chain converge quickly?

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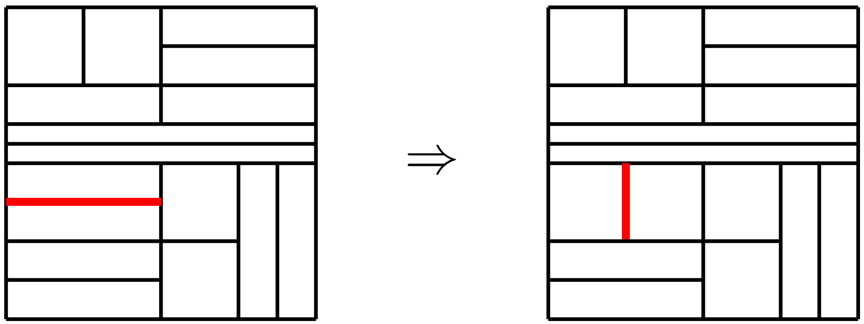
Our Results: Connectivity

The Edge-Flip Chain

Repeat:

1. Pick an random edge e,
2. If e is flippable, flip edge e with probability ½



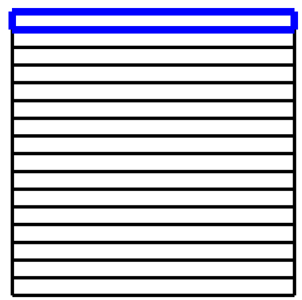
Theorem 1: The Edge-Flip Chain connects the set of all dissections of the n x n lattice region into n rectangles of size n.

Weighted Rectangular Dissections

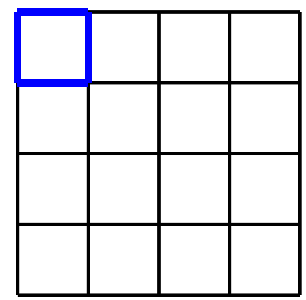
Given an input parameter $\lambda > 0$,

$$\text{Weight}(\sigma) = \lambda^{(\text{total length of edges})}.$$

For $\lambda > 1$,

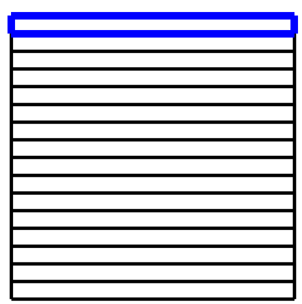


larger weight

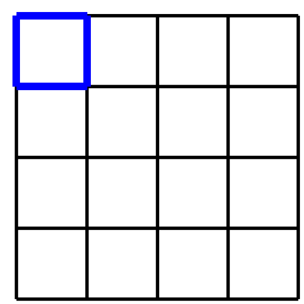


smaller weight

For $\lambda < 1$,



smaller weight



larger weight

Weighted Rectangular Dissections

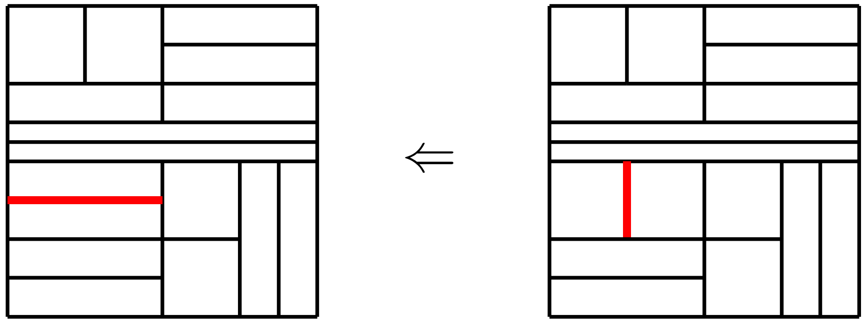
Given an input parameter $\lambda > 0$,

$$\text{Weight}(\sigma) = \lambda^{\text{(total length of edges)}}.$$

The Weighted Edge-Flip Chain

Repeat:

1. Pick a random edge e and $p \in_u (0,1)$
2. If e is flippable, let e' be the new edge it can be flipped to.
3. Flip edge e with probability ½ if $p < \lambda^{(|e'| - |e|)}$.



The Mixing Time

Definition: The total variation distance is

$$||P^t,\pi|| = \max_{x\in\Omega} \frac{1}{2} \sum_{y\in\Omega} |P^t(x,y) - \pi(y)|.$$

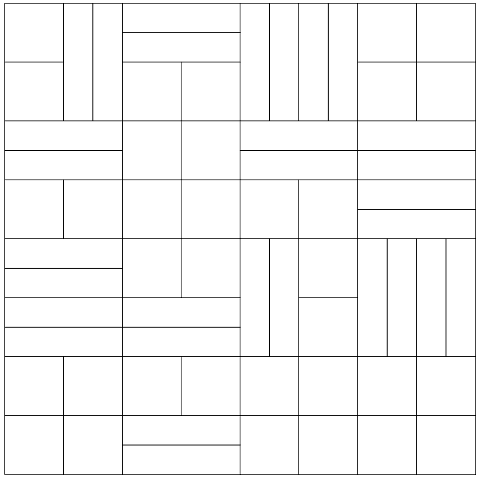
Definition: Given ε, the mixing time is

$$\tau(\varepsilon) = \min \{t: ||P^{t'},\pi|| < \varepsilon, \quad \forall t' \geq t\}.$$

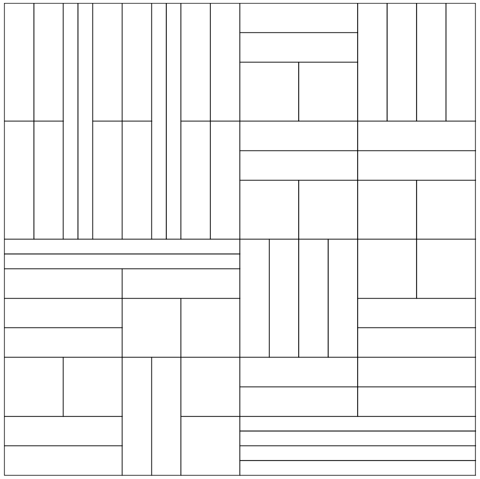
A Markov chain is polynomial mixing if $\tau(\varepsilon)$ is $poly(n, \log(\varepsilon^{-1}))$.
(n is the number of rectangles)

A Markov chain is exponential mixing if $\tau(\varepsilon)$ is at least $\exp(n)$.

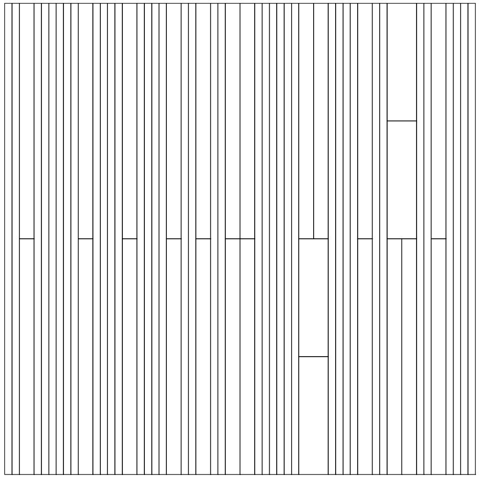
Our Results: Dyadic Tilings



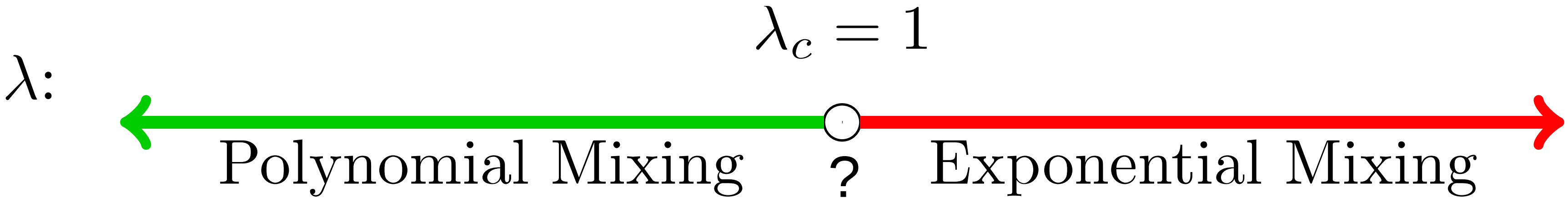
$\lambda = 0.80$



$\lambda = 1.00$

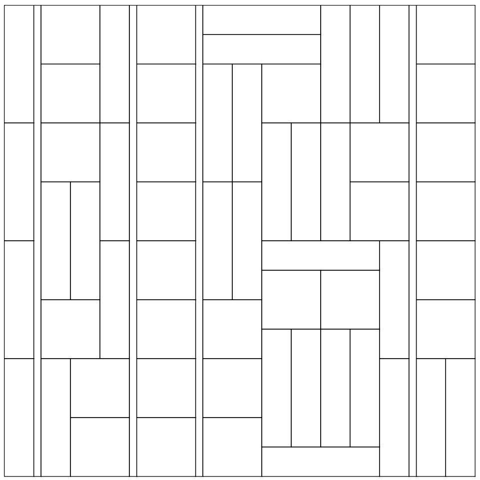


$\lambda = 1.03$



Rigorous proofs all the way to the critical point $\lambda_c = 1$!

Our Results: Rectangular Dissections



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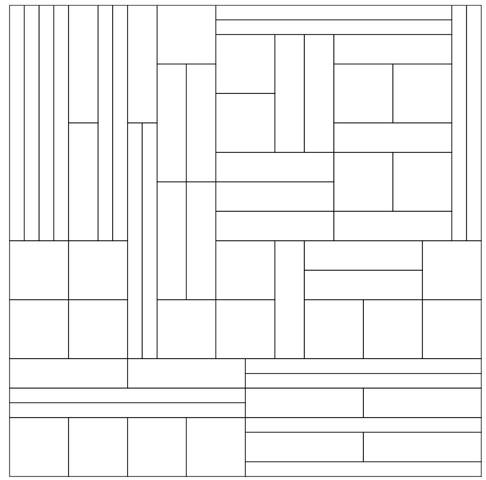

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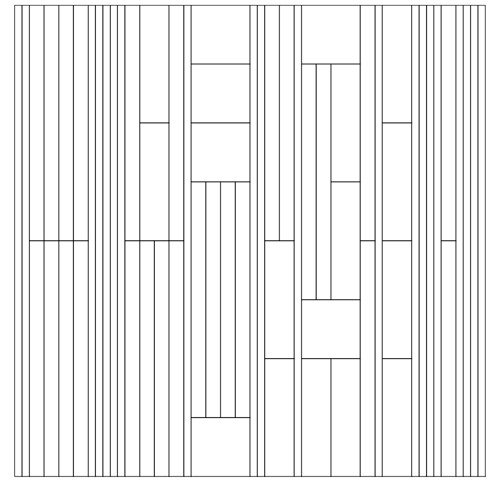

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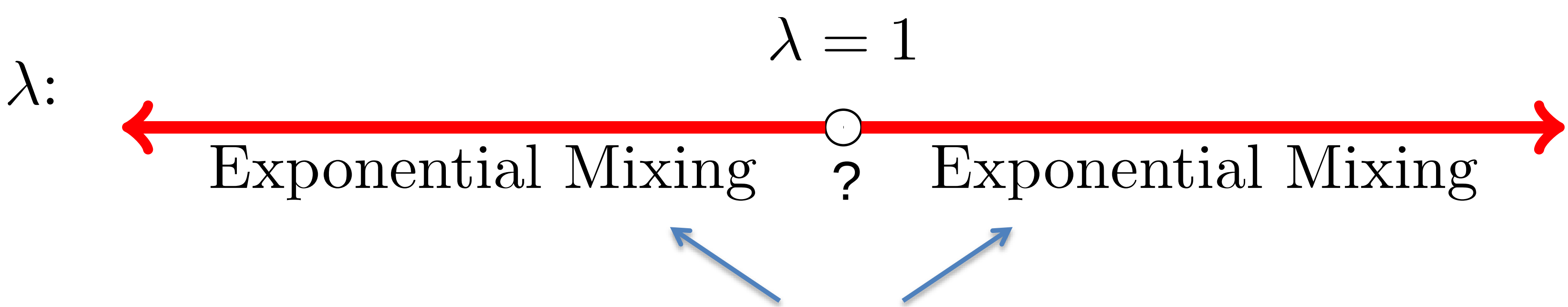

Exponential mixing for very different reasons

Talk Outline

1. Background and Previous Work
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3. Proof Ideas
    a. (General) The edge-flip chains connects.
    b. (Dyadic) When $\lambda < 1$, the edge-flip chain is poly.
    c. (Both) When $\lambda > 1$, the edge-flip chain is exp.
    d. (General) When $\lambda < 1$, the edge-flip chain is exp.

Proof Sketch: Connectivity

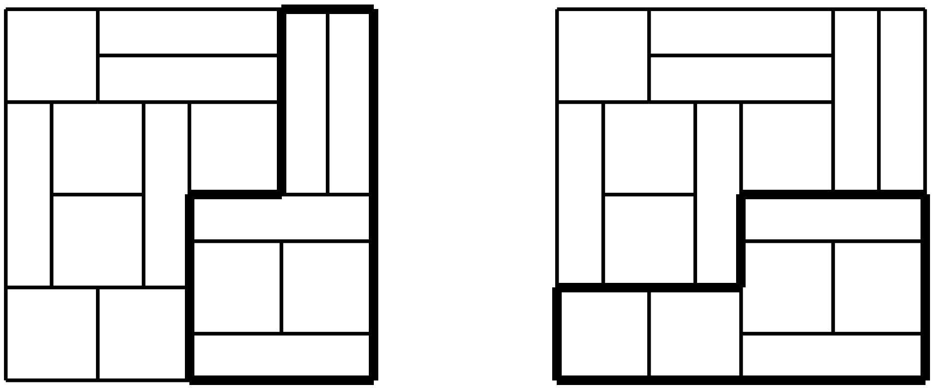
Thm 1: The Edge-Flip Chain connects the set of dissections of the n x n lattice region into n rectangles of area n.

It's not immediately obvious that a single valid move even exists!

Proof sketch: Double induction on "**h-regions**":

- Simply-connected subset of rectangles from a dissection
- All rectangles have height at most h
- All vertical sections on the boundary have height c·h (for some integer c)

For n =16, an 8-region and a 4-region.



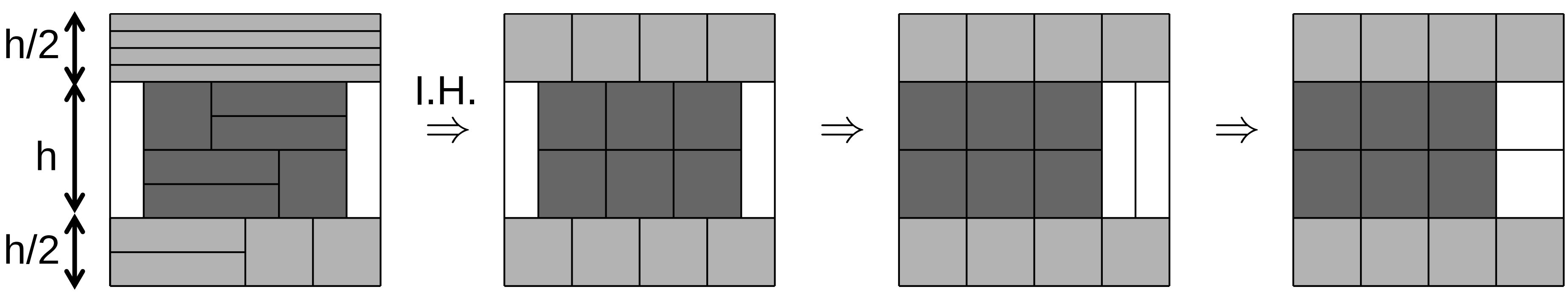
Proof Sketch: Connectivity

Thm 1: The Edge-Flip Chain connects the set of dissections of the n x n lattice region into n rectangles of size n.

It's not immediately obvious that a single valid move even exists!

Proof sketch: Double induction on "**h-regions**":

- Prove can tile every h-region with all rectangles of height h
- Inside every h-region, find an $^h/_2$-region or an h-region with smaller area



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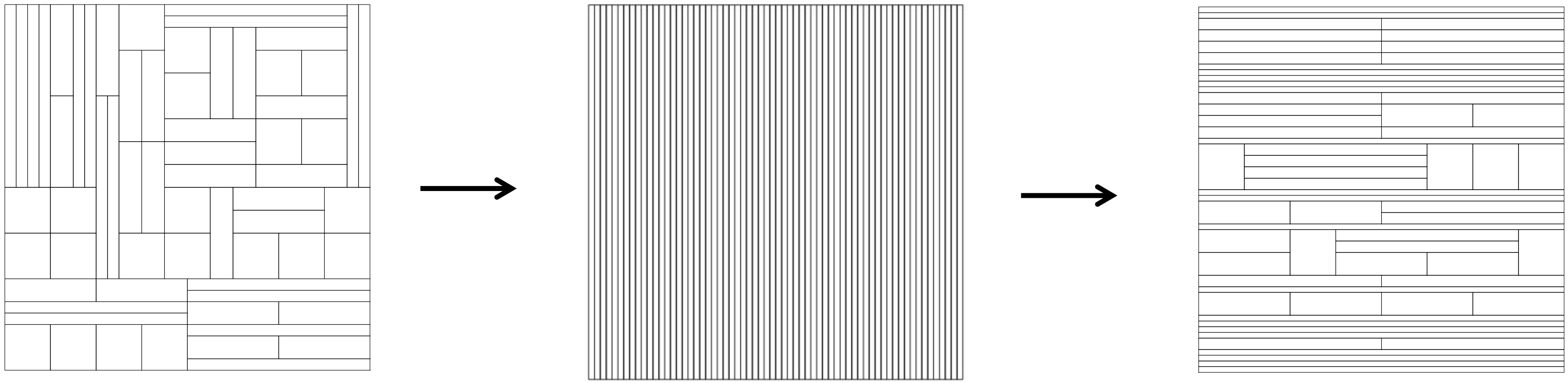

n-region

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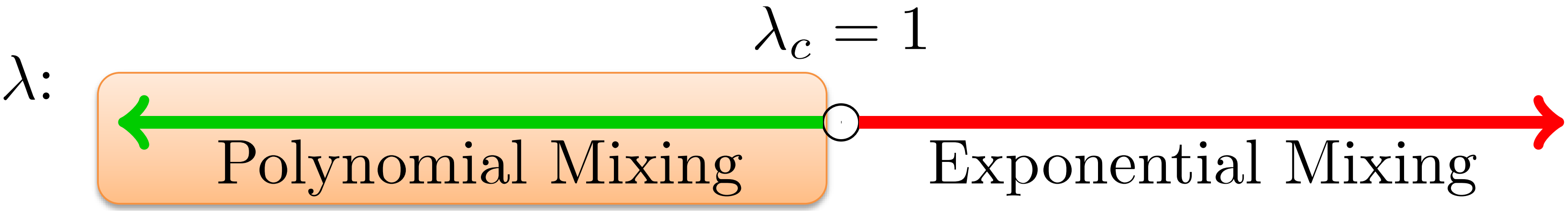
Fast Mixing for Dyadic Tilings



Thm: For any constant $\lambda < 1$, the edge-flip chain on the set of dyadic tilings converges in time $O(n^2 \log n)$.

Proof Technique:

Path coupling with an exponential metric

[Kenyon, Mossel, Perez '01][Greenberg, Pascoe, Randall '09]

For two configurations differing by flipping edge f to edge e, let the distance between them be $\lambda^{|f|-|e|}$.

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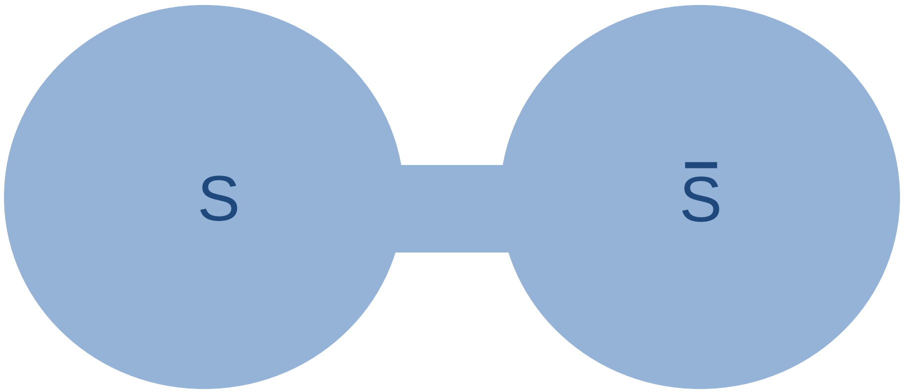
   d. (General) When $\lambda < 1$, the edge-flip chain is exp.

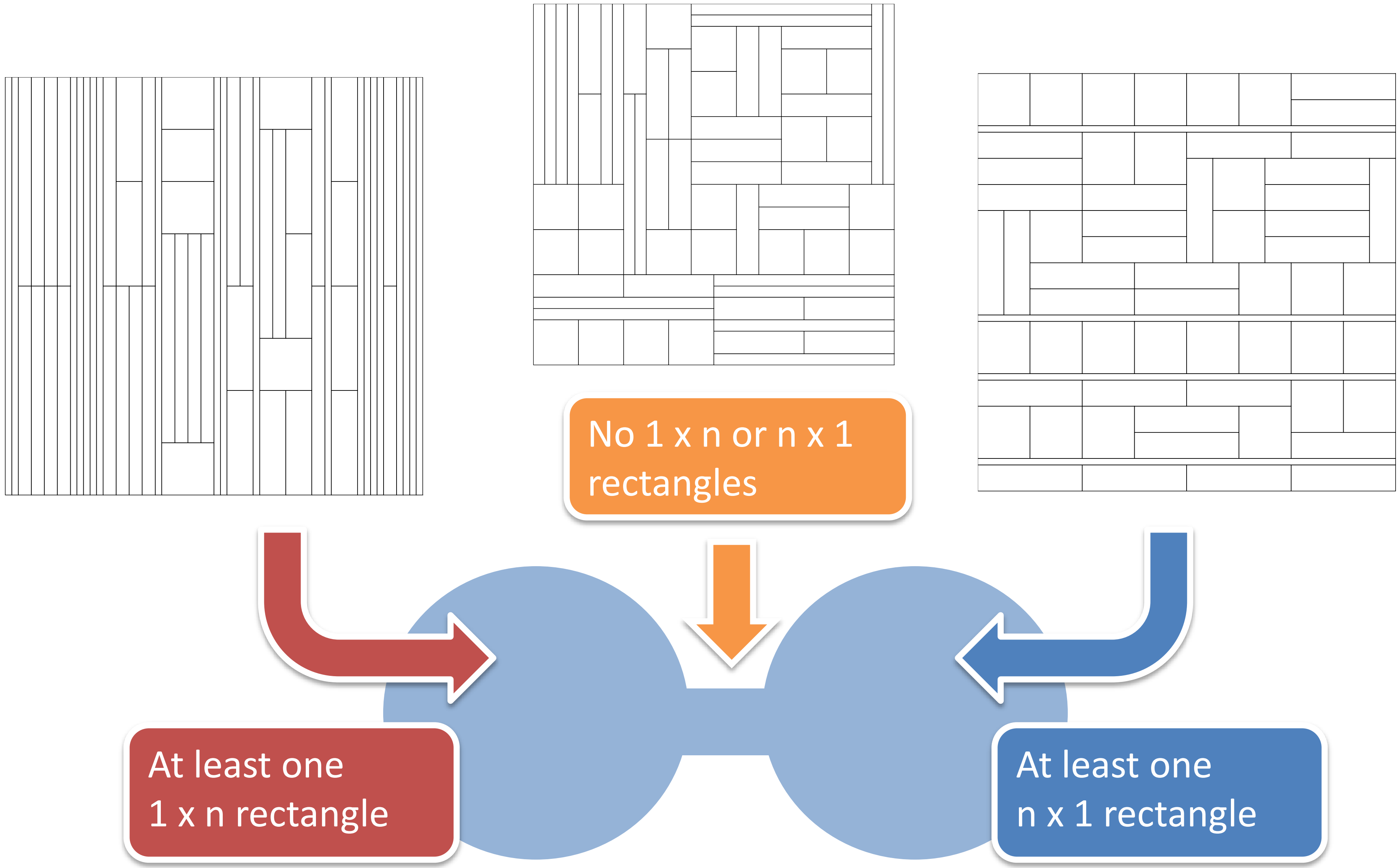
Slow Mixing when λ>1

Thm: For any constant $\lambda > 1$, the edge-flip chain requires time $\exp(\Omega(n^2))$.

Proof idea: Show that a "bottleneck" exists.



The “Bottleneck”



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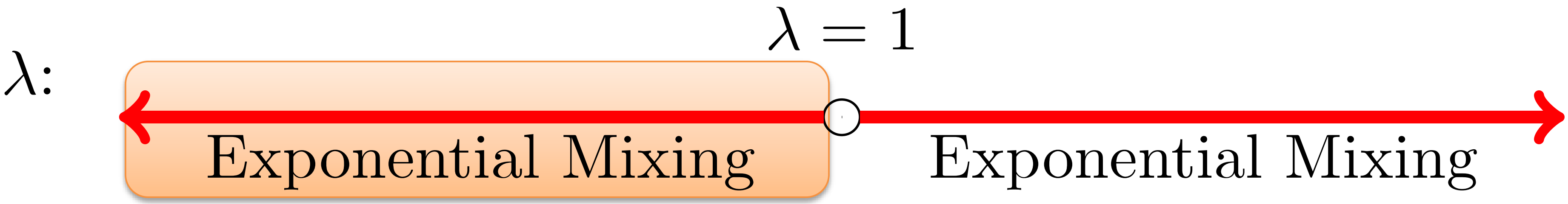
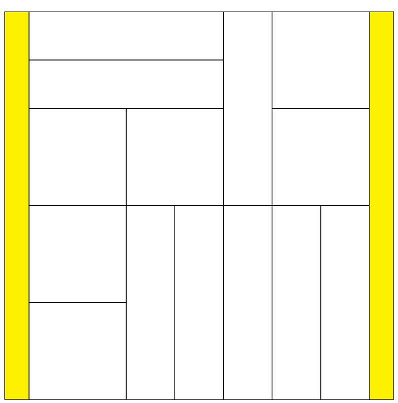


Thm: For any constant $\lambda < 1$, the edge-flip chain on rectangular dissections requires time $\exp(\Omega(n \log n))$.

Proof idea: Show that a "bottleneck" exists.

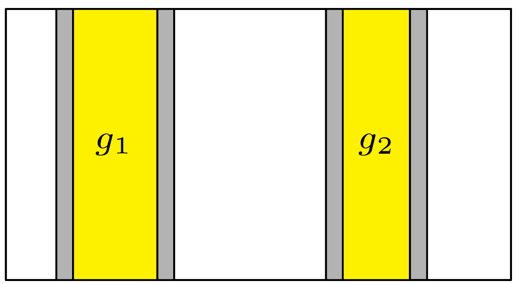


Key Ideas:

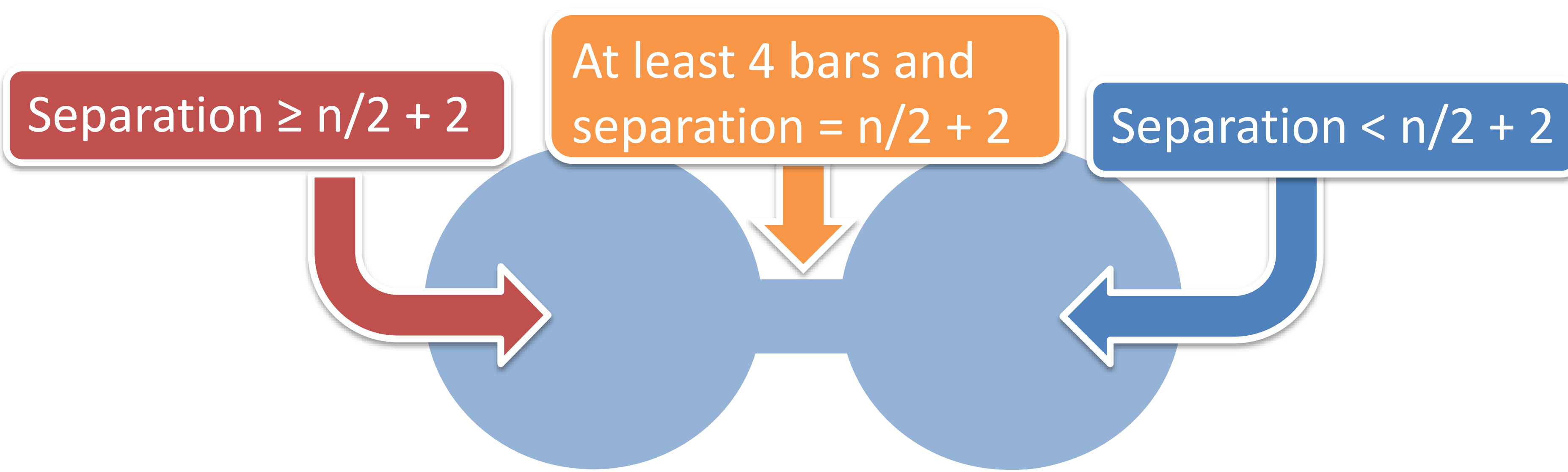
1. In order to remove a "bar" you need two bars next to each other.
2. If you have 2 bars you must also have lots of other thin rectangles.

The "Bottleneck"

- Pair up the bars left to right
- The *separation* is the sum of the "gaps"



Separation = $g_1 + g_2 + 4$



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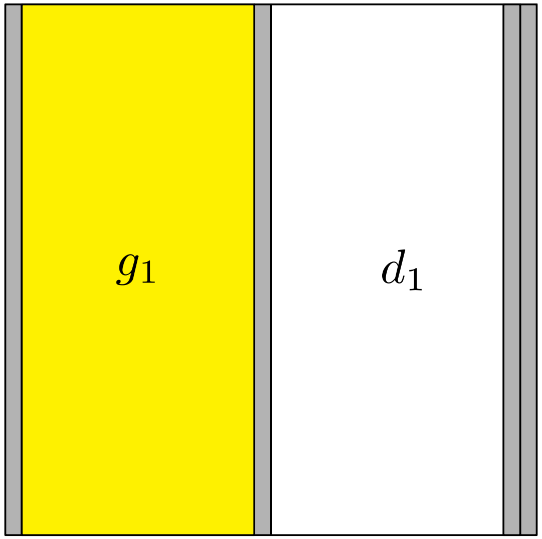
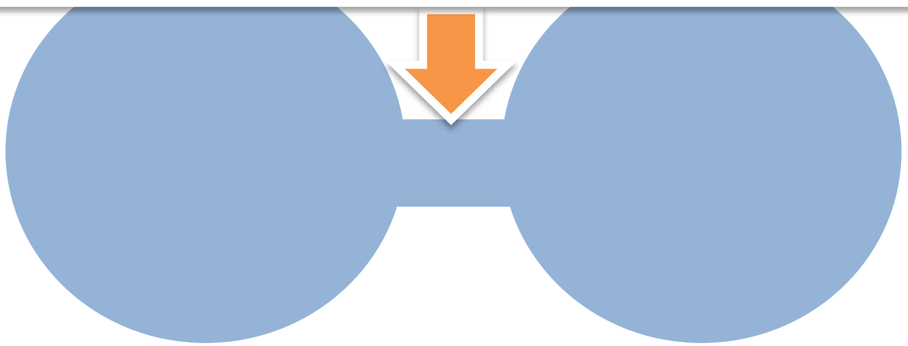


$g_1 = d_1 = n/2 - 2$

$= 0111 \ldots 1110$ (in binary)

At least 4 bars and separation = n/2 + 2



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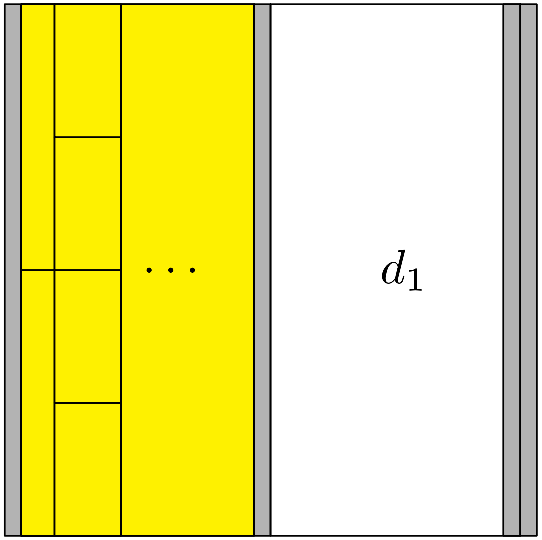

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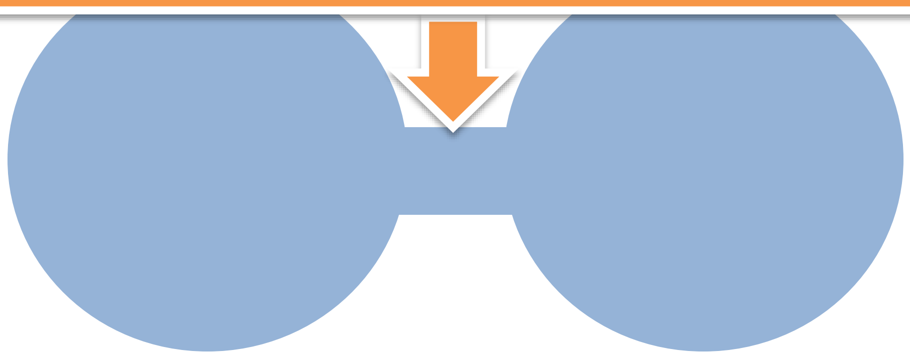
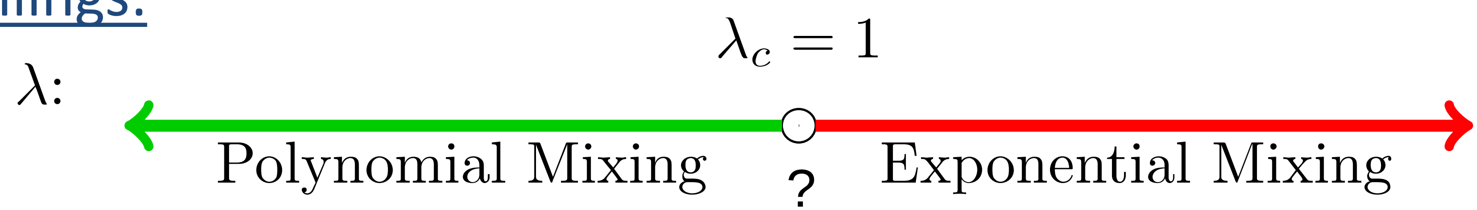
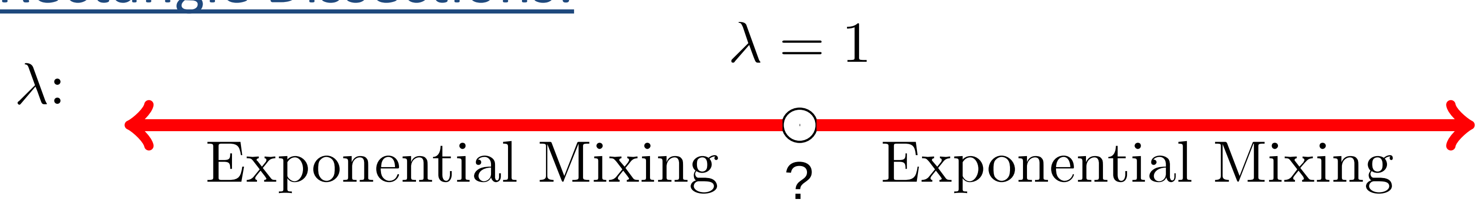


Summary and Open Problems

Dyadic Tilings:



General Rectangle Dissections:



1. What happens when λ = 1 for dyadic and general tilings?
2. When does bias speed up or slow down a chain?
3. Is there a MC that is polynomial when the EF chain is not?

Thank you!