The Fundamental Theorems of Calculus

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1 The Fundamental Theorem of Calculus, Part I

Here is a statement of the Fundamental Theorem of Calculus, Part I:

Theorem 1 (Fundamental Theorem of Calculus, Part I)**.** *If f is continuous on $[a, b]$, then the function g defined by*

$$g(x) = \int_a^x f(t)dt \qquad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b) with $g'(x) = f(x)$.

We now present a proof of Theorem 1.

Proof. If x and $x + h$ are in (a, b), then

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \left(\int_a^x f(t)dt + \int_x^{x+h} f(t)dt \right) - \int_a^x f(t)dt \\ &= \int_x^{x+h} f(t)dt \end{aligned}$$

and so, for $h \neq 0$,

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)dt. \tag{1}$$

For now, let's assume that $h > 0$. Since f is continuous on $[x, x + h]$, the Extreme Value Theorem says that there are numbers u and v in $[x, x+h]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x, x + h]$.

By a property of integrals, we have

$$mh \leq \int_x^{x+h} f(t)dt \leq Mh.$$

That is,

$$f(u)h \leq \int_x^{x+h} f(t)dt \leq f(v)h.$$

Since $h > 0$, we can divide this inequality by h:

$$f(u) \leq \frac{1}{h}\int_x^{x+h} f(t)dt \leq f(v).$$

Now we use Equation (1) to replace the middle part of the inequality:

$$f(u) \leq \frac{g(x+h) - g(x)}{h} \leq f(v) \tag{2}$$

Inequality (2) can be proved in a similar manner for the case where $h < 0$.

Now, we let $h \to 0$. Then $u \to x$ and $v \to x$, since u and v lie between x and $x + h$. Therefore

$$\lim_{h\to 0} f(u) = \lim_{u\to x} f(u) = f(x) \quad \text{and} \quad \lim_{h\to 0} f(v) = \lim_{v\to x} f(v) = f(x)$$

because f is continuous at x. We conclude from equation (2) and the Squeeze Theorem, that

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f(x). \tag{3}$$

If $x = a$ or b, then equation (3) can be interpreted as a one-sided limit. □

Using Leibnitz notation for derivatives, we can write the Fundamental Theorem of Calculus, Part I as

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \tag{4}$$

when f is continuous. Roughly speaking, equation (4) says that if we first integrate f and then differentiate the result, we get back to the original function f.

2 The Fundamental Theorem of Calculus, Part II

We first learned to compute integrals from the definition as a limit of Riemann sums and we saw that this procedure is sometimes long and difficult. The second part of the Fundamental Theorem of Calculus, which follows easily from the first part, provides us with a much simpler method for the evaluation of integrals. We need the following corollary before proceeding:

Corollary 2. *If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + C$ where C is a constant.*

We now state the Fundamental Theorem of Calculus, Part II:

Theorem 3 (Fundamental Theorem of Calculus, Part II). *If f is continuous on $[a, b]$, then*

$$\int_a^b f(x) = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Here is a proof of the second fundamental theorem of calculus:

Proof. Let $g(x) = \int_a^x f(t)dt$. We know by Theorem 1 that $g'(x) = f(x)$; that is, g is an antiderivative of f. If F is any other antiderivative of f on $[a, b]$, we know from Corollary 2 that F and g differ by a constant:

$$F(x) = g(x) + C \tag{5}$$

for $a < x < b$. But both F and g are continuous on $[a, b]$ and so, by taking limits of both sides of (5) (as $x \to a^+$ and $x \to b^-$), we see that it also holds when $x = a$ and $x = b$.

If we put $x = a$ in the formula for $g(x)$, we get

$$g(a) = \int_a^a f(t)dt = 0.$$

So using (5) with $x = b$ and $x = a$, we have

$$\begin{aligned} F(b) - F(a) &= (g(b) + C) - (g(a) + C) \\ &= g(b) - g(a) \\ &= g(b) - 0 \\ &= g(b) \\ &= \int_a^b f(t)dt \end{aligned}$$

$\square$

Theorem 3 states that if we know an antiderivative F of f, then we can evaluate $\int_a^b f(x)dx$ simply by subtracting the values of F at the endpoints of the interval $[a, b]$. It's very surprising that $\int_a^b f(x)dx$, which was defined by a complicated procedure involving *all* of the values of $f(x)$ for $a \leq x \leq b$, can be found by knowing the values of $F(x)$ at only two points, a and b.