CHAPTER

# 6 Rotational Motion

## CENTRE OF MASS OF A SYSTEM OF 'N' DISCRETE PARTICLES

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n}{m_1 + m_2 + \cdots + m_n}\ ;\ \vec{r}_{cm} = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{\sum_{i=1}^{n} m_i},\ \vec{r}_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i$$

## CENTRE OF MASS OF A CONTINUOUS MASS DISTRIBUTION

$$x_{cm} \aleph \frac{\int x\,dm}{\int dm},\quad y_{cm} \frac{\int y\,dm}{\int dm},\quad z_{cm} \frac{\int z\,dm}{\int dm}$$

$$\int dm = M(\text{mass of the body})$$

## MOTION OF CENTRE OF MASS AND CONSERVATION OF MOMENTUM

### Velocity of Centre of Mass of System

$$\vec{v}_{cm} = \frac{m_1\overrightarrow{\frac{dv_1}{dt}} + m_2\overrightarrow{\frac{dr_2}{dt}} + m_3\overrightarrow{\frac{dr_3}{dt}} \cdots + m_n\overrightarrow{\frac{dr_n}{dt}}}{M}$$

$$= \frac{m_1\overrightarrow{v_1} + m_2\overrightarrow{v_2} + m_3\overrightarrow{v_3} \ldots + m_n\overrightarrow{v_n}}{M}$$

$$\vec{P}_{sys} = M\vec{v}_{cm}$$

### Acceleration of Centre of Mass of System

$$\vec{a}_{cm} = \frac{m_1\overrightarrow{\frac{dV_1}{dt}} + m_2\overrightarrow{\frac{dV_2}{dt}} + m_3\overrightarrow{\frac{dV_3}{dt}} \cdots + m_n\overrightarrow{\frac{dV_n}{dt}}}{M}$$

$$= \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 \ldots + m_n\vec{a}_n}{M}$$

$$= \frac{\text{Net force on system}}{M}$$

$$= \frac{\text{Net external force + Net internal force}}{M}$$

$$= \frac{\text{Net External Force}}{M} \qquad (\because \Sigma \text{ Internal force} = 0)$$

$$\vec{F}_{ext} = M\vec{a}_{cm}$$

Impulse of a force F on a body is defined as:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}\,dt = \int_{t_i}^{t_f} d\vec{P} = \Delta\vec{P}$$

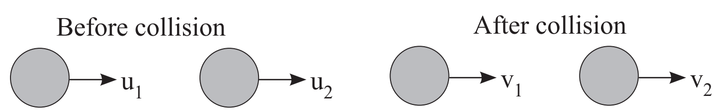
(Area under the Force vs time curve gives the impulse)

$\vec{J} = \Delta\vec{P}$ **(impulse – momentum theorem)**

### Principle of conservation of linear momentum

- ❖ If, $\left(\sum\vec{F}_{ext}\right)_{system} = 0 \Rightarrow (\vec{P}_i)_{system} = (\vec{P}_f)_{system}$
- ❖ $(KE)_{system} = \frac{1}{2}(m_1v_1^2 + m_2v_2^2 + \ldots m_nv_n^2) \neq \frac{1}{2}MV_{com}^2$

## COEFFICIENT OF RESTITUTION (E)




$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

$$e = \frac{\text{Impulse of reformation}}{\text{Impulse of deformation}} = \frac{\int F_r dt}{\int F_d dt}$$

$$= \frac{\text{Velocity of separation along line of impact}}{\text{Velocity of approach along line of impact}}$$

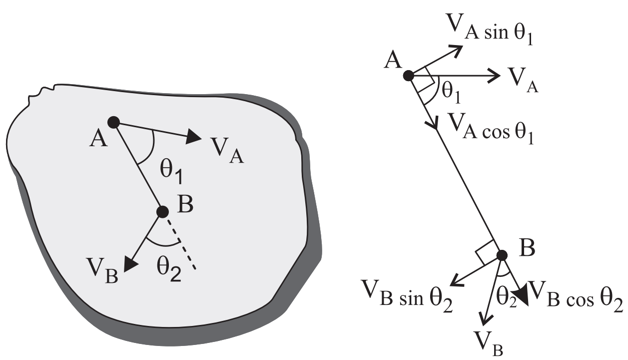
$$V_1 = \frac{P_i + m_2e(u_2 - u_1)}{m_1 + m_2},\quad V_2 = \frac{P_i + m_1e(u_1 - u_2)}{m_1 + m_2}$$

- ❖ e = 1
  - ✦ Impulse of Reformation = Impulse of Deformation
  - ✦ Velocity of separation = Velocity of approach
  - ✦ Kinetic Energy is conserved in elastic collision.
- ❖ e = 0

- Impulse of Reformation = 0
- Velocity of separation = 0
- Kinetic Energy is not conserved
- Perfectly Inelastic collision.

❖ $0 < e < 1$
- Impulse of Reformation < Impulse of Deformation
- Velocity of seperation < Velocity of approach
- Kinetic Energy is not conserved
- Inelastic collision.

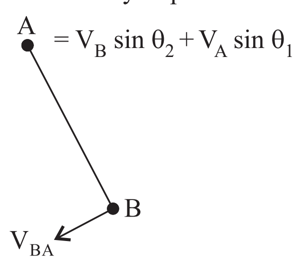
## RIGID BODY




If the above body is rigid

$V_A \cos\theta_1 = V_B \cos\theta_2$

$V_{BA}$ = relative velocity of point $B$ with respect to point $A$.




Types of Motion of rigid body

- Pure Translational Motion
- Pure Rotational Motion
- Combined Translational and Rotational Motion

## MOMENT OF INERTIA (I)

**Definition:** Moment of Inertia is defined as the capability of system to oppose the change produced in the rotational inertia of a body.

Moment of Inertia is a scalar (positive quantity).

$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots$$
$$= I_1 + I_2 + I_3 + \ldots$$

SI unit of Moment of Inertia is Kgm$^2$.

**Moment of Inertia of**

**A single particle**

$$I = mr^2$$

where $m$ = mass of the particle

$r$ = perpendicular distance of the particle from the axis about which moment of Inertia is to be calculated

**For a continuous object**

$$I = \int dI = \int r^2 dm$$

where, $dI$ = moment of inertia of a small element

$dm$ = mass of a small element

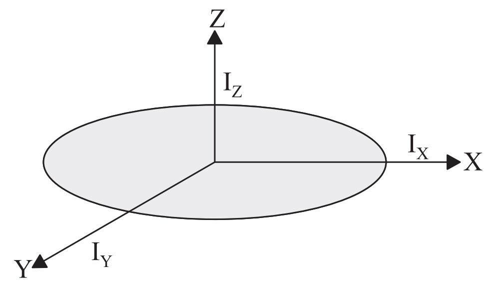
$r$ = perpendicular distance of the particle from the axis

## TWO IMPORTANT THEOREMS ON MOMENT OF INERTIA

### Perpendicular Axis Theorem

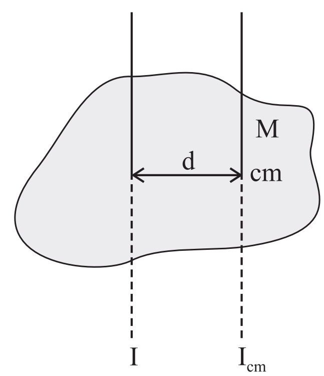
Only applicable to plane lamina (that means for 2-D objects only)

$I_Z = I_X + I_Y$ (when object is in $x - y$ plane).



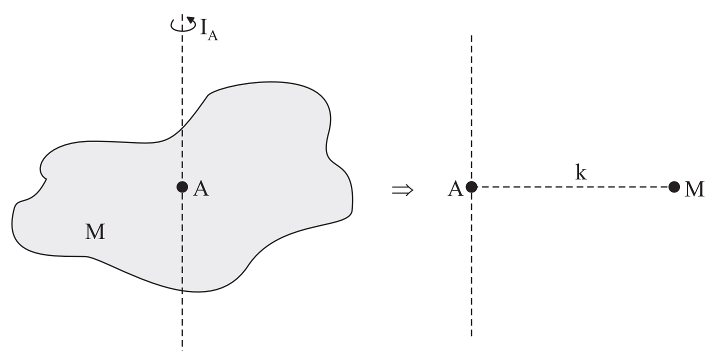

### Parallel Axis Theorem

(Applicable to any type of object) :




$$I = I_{cm} + Md^2$$

## RADIUS OF GYRATION (k)

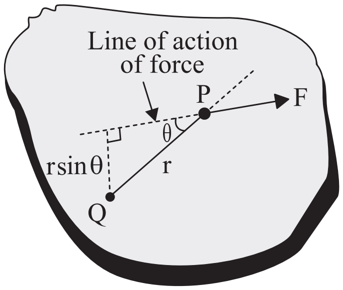



$$I_A = Mk^2$$

$$k = \sqrt{\frac{I_A}{M}}$$

## TORQUE

$\vec{\tau} = \vec{r} \times \vec{F}$




$\vec{P} = M\vec{v}_{CM}$ $\qquad$ $\vec{F}_{external} = M\vec{a}_{CM}$

Net external force acting on the body has two components tangential and centripetal.

$$F_C = ma_C = m\frac{v^2}{r_{CM}} = m\omega^2 r_{CM} \qquad F_t = ma_t = m\alpha\, r_{CM}$$

## ROTATIONAL EQUILIBRIUM

For translational equilibrium.

$\sum F_x = 0$

and $\sum F_y = 0$

The condition of rotational equilibrium is

$\sum \vec{\tau} = 0$

## ANGULAR MOMENTUM ( $\vec{L}$ )

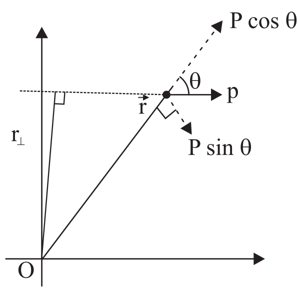
### Angular Momentum of a Particle About a Point

$\vec{L} = \vec{r} \times \vec{P}$

$L = rP \sin\theta$

$|L| = r_\perp \times P$

$|L| = P_\perp \times r$




### Angular Momentum of a Rigid Body Rotating about Fixed Axis

$L_H = I_H\omega$

$L_H$ = angular momentum of object about axis H.

$I_H$ = Moment of Inertia of rigid object about axis H.

$\omega$ = angular velocity of the object.

### Conservation of Angular Momentum

Angular momentum of a particle or a system remains constant if $\tau_{ext} = 0$ about that point or axis of rotation.

$L_i = L_f \Rightarrow I_i\omega_i = I_f\omega_f$

### Relation Between Torque and Angular Momentum

$$\tau = \frac{dL}{dt}$$

Torque is change in angular momentum

## IMPULSE OF TORQUE

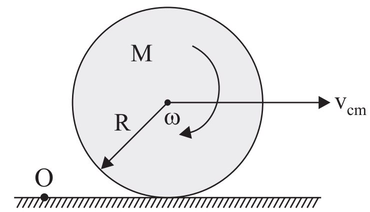
$\int \vec{\tau}dt = \Delta\vec{J}$

$\overrightarrow{\Delta J}$ = Change in angular momentum.

### Rolling Motion

Total kinetic energy = $\frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{cm}\omega^2$

Total angular momentum about O = $Mv_{CM}R + I_{cm}\omega$, ⊗




### Pure Rolling (or Rolling without Slipping) On Stationary Surface

**Condition:** $v_{cm} = R\omega$

If $v_{cm} > R\omega$ then rolling with forward slipping.

If $v_{cm} < R\omega$ then rolling with backward slipping.

Total kinetic energy in pure rolling

$$K_{total} = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}(Mk^2)\left(\frac{v_{cm}^2}{R^2}\right) = \frac{1}{2}Mv_{cm}^2\left(1 + \frac{k^2}{R^2}\right)$$

**Dynamics:**

$\vec{\tau}_{cm} = I_{cm}\vec{\alpha},\ \vec{F}_{ext} = M\vec{a}_{cm},\ \vec{P}_{system} = M\vec{v}_{cm}$

$$\text{Total K.E.} = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\left(\frac{v_{cm}}{R}\right)^2$$

**Pure rolling motion on an inclined plane**

$$\text{Acceleration } a = \frac{g\sin\theta}{1 + k^2/R^2}$$

$$\text{Minimum frictional coefficient } \mu_{min} = \frac{\tan\theta}{1 + R^2/k^2}$$

**Angular momentum about axis O** = $\vec{L}$ about C.M. + $\vec{L}$ of C.M. about O

$\vec{L}_O = I_{CM}\vec{\omega} + \vec{r}_{CM} \times M\vec{v}_{CM}$