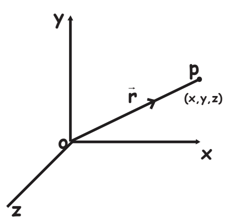
MOTION IN A PLANE

VECTORS

Position Vector

$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

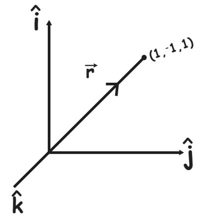
$|\vec{r}|=\sqrt{x^2+y^2+z^2}$



$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$|\vec{r}|$ or $r=\sqrt{x^2+y^2+z^2}$

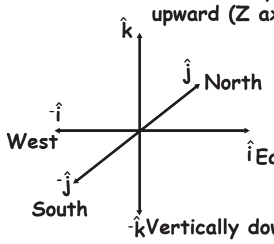
unit vector $=\hat{r}=\frac{\vec{r}}{|\vec{r}|}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$



$\vec{r}=\hat{i}-\hat{j}+\hat{k}$

$|\vec{r}|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$

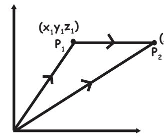
$\hat{r}=\frac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}$



5m north
3m down
2m east
$=5\hat{j}-3\hat{k}+2\hat{i}$
$=2\hat{i}+5\hat{j}-3\hat{k}$

Displacement Vector

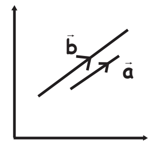
Particle displaces from position P_1 to position P_2



$\Delta\vec{r}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$

$|\Delta\vec{r}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

Parallel Vectors



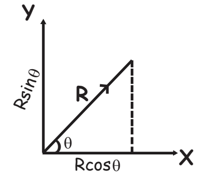
$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

$\vec{a}=m\vec{b}$

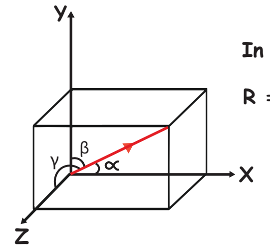
$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=m$

Components of Vector



In two dimentions

$\vec{R}=R\cos\theta\hat{i}+R\sin\theta\hat{j}$



In three dimentions

$R = R\cos\alpha\,\hat{i}+R\cos\beta\,\hat{j}+R\cos\gamma\,\hat{k}$

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

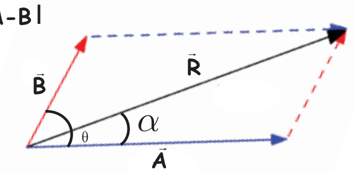
$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$

Addition Of Vectors

$\vec{R}=\sqrt{A^2+B^2+2AB\cos\theta}$

$R_{max}=A+B$ $\qquad R_{min}=|A-B|$

$\tan\alpha=\frac{B\sin\theta}{A+B\cos\theta}$



Vector product

$\vec{C}=\vec{A}\times\vec{B}$ $\qquad |\vec{A}\times\vec{B}|=AB\sin\theta$

$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\A_x&A_y&A_z\\B_x&B_y&B_z\end{vmatrix}=\hat{i}[A_yB_z-B_yA_z]-\hat{j}[A_xB_z-A_zB_x]+\hat{k}[A_xB_y-A_yB_x]$

Dot product

$x=\vec{A}.\vec{B}=AB\cos\theta$

PROJECTILE MOTION

Projectile motion

Horizontal component = $U\cos\theta$

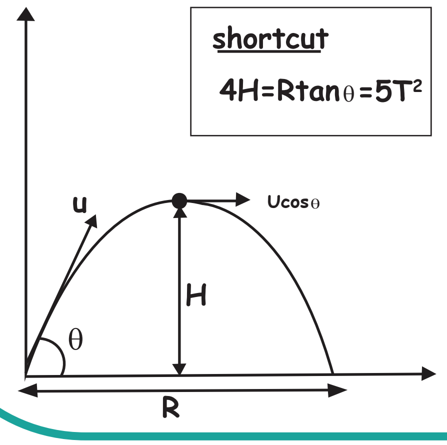
Vertical component = $U\sin\theta$

$H=\frac{U^2\sin^2\theta}{2g}=\frac{(U\sin\theta)^2}{2g}=\frac{U_y^2}{2g}$

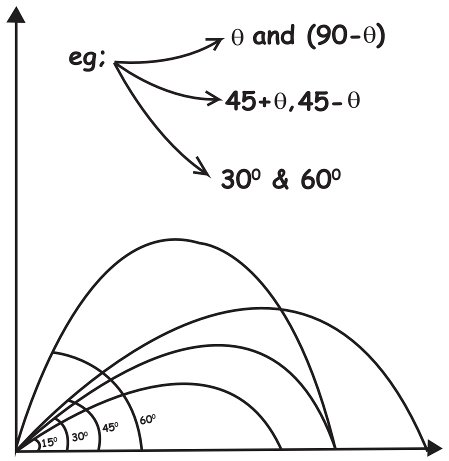
$R=\frac{U^2\sin2\theta}{g}=\frac{2U\sin\theta\,U\cos\theta}{g}=\frac{2U_xU_y}{g}$

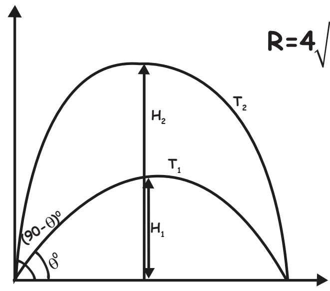
shortcut

$4H=R\tan\theta=5T^2$



Same range for θ and (90-θ)





$R=4\sqrt{H_1.H_2}$

- $\frac{T_1}{T_2}=\tan\theta$
- $\frac{H_1}{H_2}=\tan^2\theta$
- $T_1\times T_2=\frac{2R}{g}$
- $H_1\times H_2=\frac{R^2}{16}$

$T_1^2+T_2^2=\frac{4\,R_{max}}{g}$ $\qquad H_1+H_2=\frac{R_{max}}{2}$

Maximum range

For $\theta = 45°$

$R_{max}=\frac{U^2}{g}$

From the relation,

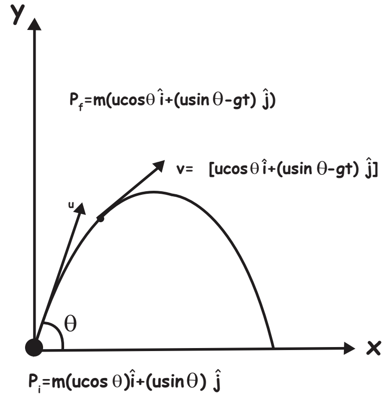
$4H=R\tan\theta=5T^2$

$4H=R_{max}\tan45 \Rightarrow H=\frac{R_{max}}{4}$

$4H=R_{max} \Rightarrow H=\frac{u^2}{4g}$

KE at maximum height $=K\cos^2\theta$

Momentum



$\Delta p=-mgt\,\hat{j}$

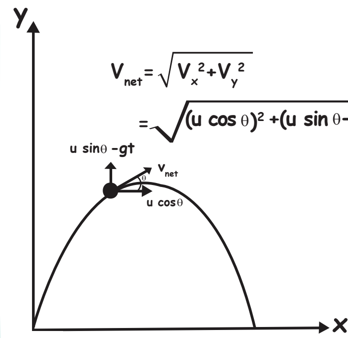
$=-mg\times\frac{u\sin\theta}{g}\hat{j}$

$=-mu\sin\theta\,\hat{j}$

Equation of Velocity

$V_{net}=\sqrt{V_x^2+V_y^2}$

$=\sqrt{(u\cos\theta)^2+(u\sin\theta-gt)^2}$



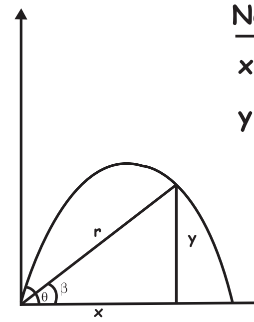
Net displacement

$x=U\cos\theta t$

$y=U\sin\theta t-\frac{1}{2}gt^2$

$r=\sqrt{x^2+y^2}$

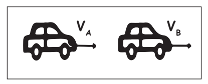
$\beta=\tan^{-1}(y/x)$



RELATIVE MOTION

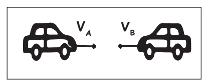
Relative Motion

1) Velocity of A with respect to B $V_{AB}=V_A-V_B$



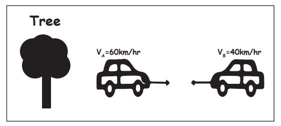
2) $V_{A/B}=V_A-V_B$

$=V_A-(-V_B)=V_A+V_B$



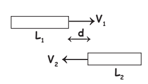
3) $V_{A/Tree}=V_A-V_{Tree}=60-0=60$

$V_{B/Tree}=V_B-V_{Tree}=^{-}40$

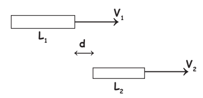


Relative Motion in one dimension (overtaking & chasing)

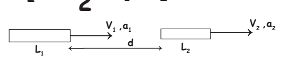
1) $t=\frac{d+L_1+L_2}{V_1+V_2}$



2) $t=\frac{d+L_1+L_2}{V_1-V_2}$



3) $d+L_1+L_2=(u_1-u_2)t+\frac{1}{2}(a_1-a_2)t^2$



Minimum separation to avoid collision

$u_{rel}=u_1-u_2$ $\qquad a_{rel}=-a_1$

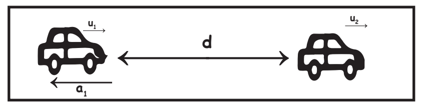
$0=u^2_{rel}+2a_{rel}s_{rel}$

$0=(u_1-u_2)^2-2a_1s$ $\qquad S=\frac{(u_1-u_2)^2}{2a_1}$

to avoid collision,

$s\leq d$

$\Rightarrow d\geq\frac{(u_1-u_2)^2}{2a_1}$



a_1 = retardation of car 1

MOTION IN A PLANE - 02

RELATIVE MOTION

Man-river problem

1) $\vec{V}_{MR}$ or $\vec{V}_{M/Still\ water}$ = velocity due to effort of man, OR velocity of man in still water

2) $\vec{V}_R$ = velocity of River

3) $\vec{V}_m$ = Resultant velocity of man with respect to ground

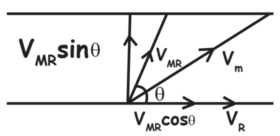
1) Upstream

$V_M = V_{M/R} - V_R$

2) Down stream

$V_M = V_{M/R} + V_R$

Swimming across the river



1) $V_{\perp} = V_{MR}\sin\theta$

2) $V_{\parallel} = V_{MR}\cos\theta + V_R$

$t_{cross} = \dfrac{d}{V_{MR}\cos\theta} = \dfrac{d}{V_{\perp}}$

$X_{drift} = (V_{MR}\cos\theta + V_R) \times t$

$= V_{MR}\cos\theta \times t \quad + \quad V_R t$

Due to effort of man — Additional due to push of river

Shortest time	Shortest Path (No drift)
$\theta = 90$, $\sin\theta = 1$	Horizontal component of V_m becomes 0
$t = \dfrac{d}{V_{MR}\sin\theta}$	$\Rightarrow V_{MR}\sin\theta = V_R$ $\quad V_{MR}\cos\theta = V_m$
$t_{min} = \dfrac{d}{V_{MR}}$	Condition for no drifting
$X_{drift} = V_R \times t$	$\Rightarrow \sin\theta = \dfrac{V_R}{V_{MR}}$
$V_m = \sqrt{(V_{MR})^2 + (V_R)^2}$	$\Rightarrow t_{cross} = \dfrac{d}{V_m}$
	$\Rightarrow V_m = \sqrt{V_{MR}^2 - V_R^2}$
	$\Rightarrow$ Drift = 0

Escalator

$t_3 = \dfrac{d}{V_E + V_{M/E}} = \dfrac{d}{\dfrac{d}{t_2} + \dfrac{d}{t_1}} = \dfrac{t_1 t_2}{t_1 + t_2}$

t_1 = Time taken by a man to move distance d on a stationary escalator

t_2 = Time taken by a stationary man to move distance d along with moving escalator

t_3 = Time taken by a man to move distance d while walking along a moving escalator

V_E = Velocity of escalator

$V_{M/E}$ = Velocity of man w.r.t escalator

MAN RAIN PROBLEM

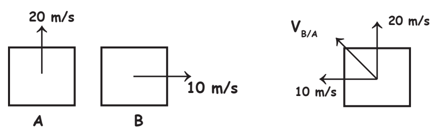
Man-rain problem

In order to find the relative velocity of B with respect to A we have to reverse the direction of vector A and add it with vector B

w.r.t. A

$V_{B/A} = V_B - V_A$, $\quad V_B$ w.r.t A

$V_{A/B} = V_B - V_A$, $\quad V_A$ w.r.t B



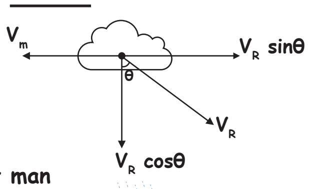
Terms

$V_R \Rightarrow$ Velocity of rain w.r.t stationary man

$V_m \Rightarrow$ Velocity of man

$V_{R/m} \Rightarrow$ Velocity of Rain w.r.t man

Method



$V_{R/m} = \sqrt{(V_R\sin\theta - V_m)^2 + (V_R\cos\theta)^2}$

Case 1

$V_R\sin\theta > V_m$

$V_{R/m} = \sqrt{(V_R\cos\theta)^2 + (V_R\sin\theta - V_m)^2}$

$\tan\alpha = \dfrac{V_R\sin\theta - V_m}{V_R\cos\theta}$

$\alpha = \tan^{-1}\left(\dfrac{V_R\sin\theta - V_m}{V_R\cos\theta}\right)$

Case 2

$V_m > V_R\sin\theta$

$V_{R/m} = \sqrt{(V_R\cos\theta)^2 + (V_m - V_R\sin\theta)^2}$

$\alpha = \tan^{-1}\left(\dfrac{V_m - V_R\sin\theta}{V_R\cos\theta}\right)$

Case 3

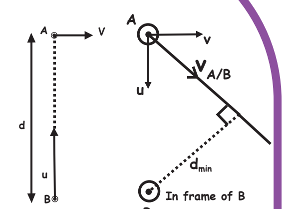
$V_m = V_R\sin\theta$

$V_m = V_R\sin\theta$

$V_{R/m} = V_R\sin\theta$

$\alpha = 0$

minimum distance



$d_{min} = \dfrac{d \times V}{\sqrt{V^2 + U^2}}$

Time taken to reach minimum distance $= \dfrac{d \times U}{V^2 + U^2}$

CIRCULAR MOTION

$1° = \dfrac{\pi}{180}$ rad

Angular velocity

$\omega = \dfrac{\Delta\theta}{\Delta t} = \dfrac{2\pi}{T}$ (in uniform circular motion), $V = \omega r$

$\vec{V} = \vec{\omega} \times \vec{r}$

Angular acceleration

$\alpha = \dfrac{d\omega}{dt}$ $\quad a_t = r\alpha$, $\quad \vec{a}_t = \vec{\alpha} \times \vec{r}$

Equation of angular motion

1) Constant angular velocity : ω = constant

2) Constant angular acceleration

$\Rightarrow \omega = \omega_o + \alpha t$

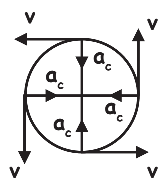
$\Rightarrow \Delta\theta = \omega_o t + 1/2\, \alpha t^2$

$\Rightarrow \omega^2 = \omega_o^2 + 2\alpha(\Delta\theta)$

Centripetal acceleration

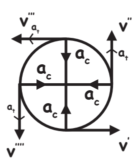
Directed towards centre

Not a constant vector



- a_c is due to change in the direction of velocity
- $a_c = \dfrac{v^2}{R} = a_c = r\omega^2$
- $\vec{a}_c \perp \vec{v}$
- $\vec{F}_c \perp \vec{s}$

Tangential acceleration



In direction of tangent

$a_t = \dfrac{dv}{dt} = r\alpha \rightarrow \alpha = \dfrac{d\omega}{dt}$

Tangential Force

$F_t = ma_t$

a_t is due to change in magnitude of velocity

Resultant acceleration

$a_r = \sqrt{a_c^2 + a_t^2}$

Velocity is tangent to the circle

Circular Motion

Uniform Circular Motion

- Speed Constant
- Direction of velocity changes
- ω = Constant
- $a_c = \dfrac{v^2}{r} = r\omega^2$
- $a_t = 0$

Non-uniform Circular Motion

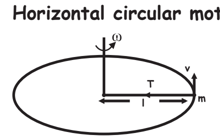
- Speed not Constant
- Velocity changes in direction and magnitude
- a_c = Centripetal acceleration
- a_t = tangential acceleration

$= \dfrac{dv}{dt}$

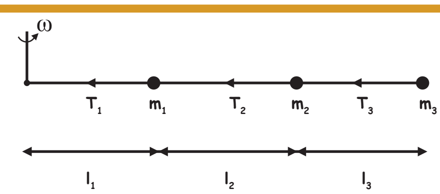
$- \alpha = \dfrac{d\omega}{dt}$

$- \omega$ = Changes $\longrightarrow \alpha$ angular acceleration

Horizontal circular motion



$T = \dfrac{mv^2}{r} = mr\omega^2 = ml\omega^2$

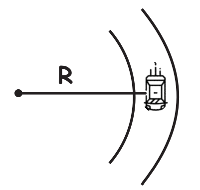


$T_1 = m_1 l_1 \omega^2 + m_2(l_1 + l_2)\omega^2 + m_3(l_1 + l_2 + l_3)\omega^2$

$T_2 = m_2(l_1 + l_2)\omega^2 + m_3(l_1 + l_2 + l_3)\omega^2$

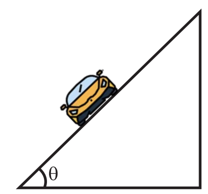
$T_3 = m_3(l_1 + l_2 + l_3)\omega^2$

Flat circular track



$V_{max} = \sqrt{\mu_s R g}$

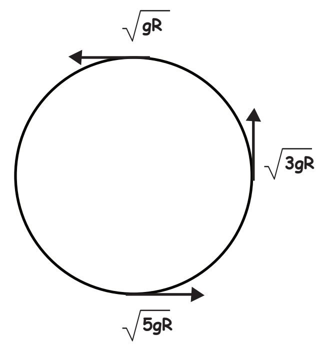
Banking of Road



$V_{max} = \sqrt{\dfrac{rg(\tan\theta + \mu)}{1 - \mu\tan\theta}}$

$V_{opt} = \sqrt{rg\tan\theta}$ (for smooth road)

$V_{min} = \sqrt{\dfrac{rg(\tan\theta - \mu)}{1 + \mu\tan\theta}}$



At Bottom

a) $T_{max} = \dfrac{mv^2}{r} + mg$

b) min velocity at bottom to complete circle $= \sqrt{5gR}$

At Top

a) $T_{min} = \dfrac{mv^2}{r} - mg$

b) min velocity at top to complete the circle $= \sqrt{gR}$