

## Ejercicio 0

Calcular las derivadas de las siguientes funciones:

a)  $f(x) = (33 - 2x)^{\frac{4}{3}}$

$$f'(x) = (33 - 2x)^{\frac{4}{3} - \frac{3}{3}} \cdot (-2) \cdot \frac{4}{3} = (33 - 2x)^{\frac{1}{3}} \cdot -\frac{8}{3} = -\frac{8}{3} \cdot \sqrt[3]{(33 - 2x)}$$

$$f'(x) = -\frac{8\sqrt[3]{(33 - 2x)}}{3}$$

b)  $f(x) = e^{2x}$

$$f'(x) = e^{2x} \cdot \frac{d}{dx}[2x]$$

$$f'(x) = e^{2x} \cdot 2$$

c)  $f(x) = 2^x$

$$f(x) = 2^x \Rightarrow \ln(f(x)) = \ln(2^x) = x \ln(2) \Rightarrow$$

$$\Rightarrow \frac{f(x)'}{f(x)} = \ln(2) \Rightarrow f'(x) = \ln(2) \cdot f(x) = 2^x \cdot \ln(2)$$

$$\Rightarrow f'(x) = 2^x \cdot \ln(2)$$

d)  $f(x) = \ln(7 - x)$

$$f'(x) = \frac{1}{7 - x} \cdot (-1)$$

$$f'(x) = \frac{1}{x - 7}$$

e)  $f(x) = \ln(x^2 + 3x + 4)$

$$f'(x) = \frac{1}{x^2 + 3x + 4} \cdot (2x + 3)$$

$$f'(x) = \frac{2x + 3}{x^2 + 3x + 4}$$

f)  $f(x) = \ln(e^x + e^{-x})$

$$f'(x) = \frac{1}{e^x + e^{-x}} \cdot (e^x + e^{-x} \cdot (-1))$$

$$f'(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

g)  $f(x) = \ln(\cos(x) + \sin(x))$

$$f'(x) = \frac{1}{\cos(x) + \sin(x)} \cdot (-\sin(x) + \cos(x))$$

$$f'(x) = \frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)}$$

$$\text{h) } f(x) = \frac{\cos(x)}{\sin(x)}$$

$$f'(x) = \frac{\frac{d}{dx}[\cos x] \cdot \sin(x) - \cos(x) \cdot \frac{d}{dx}[\sin x]}{\sin^2(x)}$$

$$f'(x) = \frac{-\sin(x) \cdot \sin(x) - \cos(x) \cdot \cos(x)}{\sin^2(x)}$$

$$f'(x) = \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)}$$

$$f'(x) = -\frac{\cos^2(x) + \sin^2(x)}{\sin^2(x)}$$

$$f'(x) = -\frac{1}{\sin^2(x)}$$

## Ejercicio 1

Dar las primitivas de las siguientes funciones:

$$\text{a) } g(x) = x^3 - 5x$$

$$G(x) = \frac{x^4}{4} - 5\frac{x^2}{2} + c$$

$$\text{b) } g(x) = e^{0,3x}$$

$$G(x) = \frac{e^{0,3x}}{0,3} + c$$

Sino sale por sustitución.

$$\text{c) } g(x) = \sin(2x)$$

$$G(x) = -\frac{\cos(2x)}{2} + c$$

$$\text{d) } g(x) = 2x \cos(x^2)$$

$$\int 2x \cos(2x) dx, \quad \text{Sea } u = x^2, du = 2x dx$$

$$\int \cos(u) du = \sin(u) = \sin(x^2)$$

$$G(x) = \sin(x^2) + c$$

$$\text{e) } g(x) = x^{\frac{3}{2}}$$

$$G(x) = x^{\frac{5}{2}} \cdot \frac{2}{5} + c$$

$$\text{f) } g(x) = \sqrt{x+2} = (x+2)^{\frac{1}{2}}$$

$$G(x) = (x+2)^{\frac{3}{2}} \cdot \frac{2}{3} + c$$

## Ejercicio 2

Encontrar la primitiva  $F$  de  $f(x) = \frac{3}{x}$  tal que  $F(1) = 5$

$$\int \frac{3}{x} = 3 \int \frac{1}{x} = 3 \ln|x| + C \Rightarrow F(1) = 3 \ln|1| + C = 3 \cdot 0 + C = C \Rightarrow F(x) = 3 \ln|x| + 5$$

## Ejercicio 3

Calcular las siguientes integrales:

a)  $\int e^{2x} dx$

$$\int e^{2x} dx = \frac{1}{2} \int e^{2x} \cdot 2 dx \quad \text{Elemento neutro de la multiplicación y multiplico por } \frac{1}{2}$$

$$\int e^{2x} dx = \frac{1}{2} e^u + C \quad \text{Tomo } u = 2x \rightarrow du = 2 dx \text{ y resuelvo.}$$

$$\int e^{2x} dx = \frac{e^{2x}}{2} + C \quad \text{Reemplazo } u \text{ por lo que es.}$$

b)  $\int 2^x dx$

Utilizo el ejercicio 0.c.  $f(x) = 2^x \Rightarrow f'(x) = 2^x \cdot \ln(2)$ .

$$\int 2^x dx = \frac{\ln(2)}{\ln(2)} \int 2^x dx = \frac{1}{\ln(2)} \int 2^x \ln(2) dx = \frac{2^x}{\ln(2)} + C$$

c)  $\int \sqrt[3]{(33-2x)} dx$

$$\int \sqrt[3]{(33-2x)} dx = \int (33-2x)^{\frac{1}{3}} dx \quad \text{Reescribo utilizando propiedades.}$$

$$= -\frac{1}{2} \int (33-2x)^{\frac{1}{3}} \cdot (-2) dx \quad \text{Utilizo el neutro de la multiplicación.}$$

$$= -\frac{1}{2} \int u^{\frac{1}{3}} du = -\frac{1}{2} u^{\frac{4}{3}} \cdot \frac{3}{4} + C \quad \text{Tomo } u = (33-2x) \rightarrow du = -2 dx$$

$$\int \sqrt[3]{(33-2x)} dx = -\frac{3}{8} (33-2x)^{\frac{4}{3}} + C \quad \text{Reemplazo } u \text{ por lo que es:}$$

d)  $\int \frac{dx}{7-x}$

Es  $-\ln(7-x)$  por ejercicio 0.d. También se puede hacer por sustitución y da  $-\ln(|7-x|)$

e)  $\int \frac{2x+3}{x^2+3x+4} dx$

$$\int \frac{2x+3}{x^2+3x+4} dx = \int \frac{du}{u} = \ln|u| + C \quad \text{Tomo } u = (x^2 + 3x + 4) \rightarrow du = (2x + 3) dx$$

$$\int \frac{2x+3}{x^2+3x+4} dx = \ln|x^2 + 3x + 4| + C \quad \text{Reemplazo } u \text{ por lo que es.}$$

f)  $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

El resultado es  $\ln(e^x + e^{-x}) + C$  por ejercicio 0.f. Nos daría  $\ln|e^x + e^{-x}| + C$  usando sustitución.

g)  $\int \frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)} dx$

El resultado es  $\ln(\cos(x) + \sin(x))$  por ejercicio 0.h. Nos daría  $\ln|\cos(x) + \sin(x)|$  por sustitución. Hay que tener en cuenta el dominio de la función.

h)  $\int \frac{1}{\sin^2(x)} dx$

El resultado es  $-\frac{\cos(x)}{\sin(x)} + C$ . Se trata del ejercicio 0.h.

## Ejercicio 4

Si realizar el cálculo de la integral, justificar las siguientes igualdades y desigualdades.

a)  $\int_{-\pi}^{\pi} \sin(2x) dx = 0$

Esto es un hecho trivial. Ya que  $\sin(2x)$  es una función par. Probablemente podemos generalizar:

$$\int_{-a}^a \sin(2x) dx = \int_{-a}^0 \sin(2x) dx + \int_0^a \sin(2x) dx = 0, \forall a \in \mathbb{R}.$$

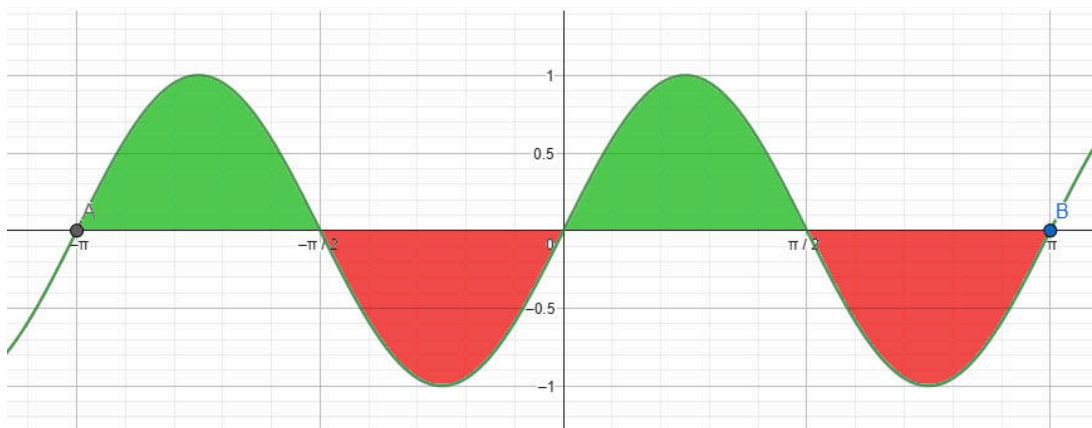


Figura 1:  $\sin(2x)$

$$b) \frac{\pi}{6} \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin(x) dx \leq \frac{\pi}{3}$$

Esto se puede ver aproximando el área por izquierda y por derecha. Tenemos que si tomo un rectángulo de ancho  $\frac{\pi}{3}$ , El menor área posible es  $\frac{\pi}{6}$ . Si tomamos el mayor área posible por derecha nos da  $\frac{\pi}{6}$ . Luego el valor real del área debe estar entre esos dos números.

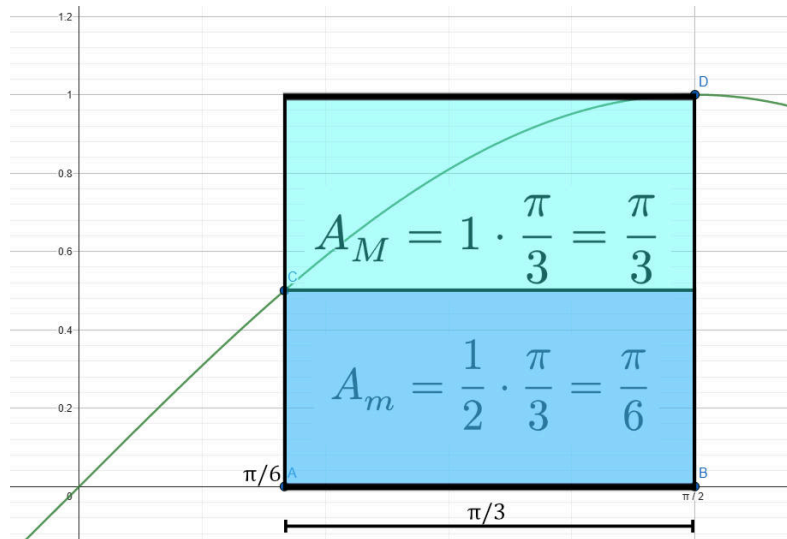


Figura 2:  $\sin(x)$

$$c) \int_1^2 \sqrt{5-x} dx \geq \int_1^2 \sqrt{x+1} dx$$

Basta con ver el gráfico. En ese intervalo, el área mínima de  $\sqrt{5-x}$  es igual al área máxima  $\sqrt{x+1}$ . Luego por una cadena desigualdades tenemos que el resultado es verdadero.

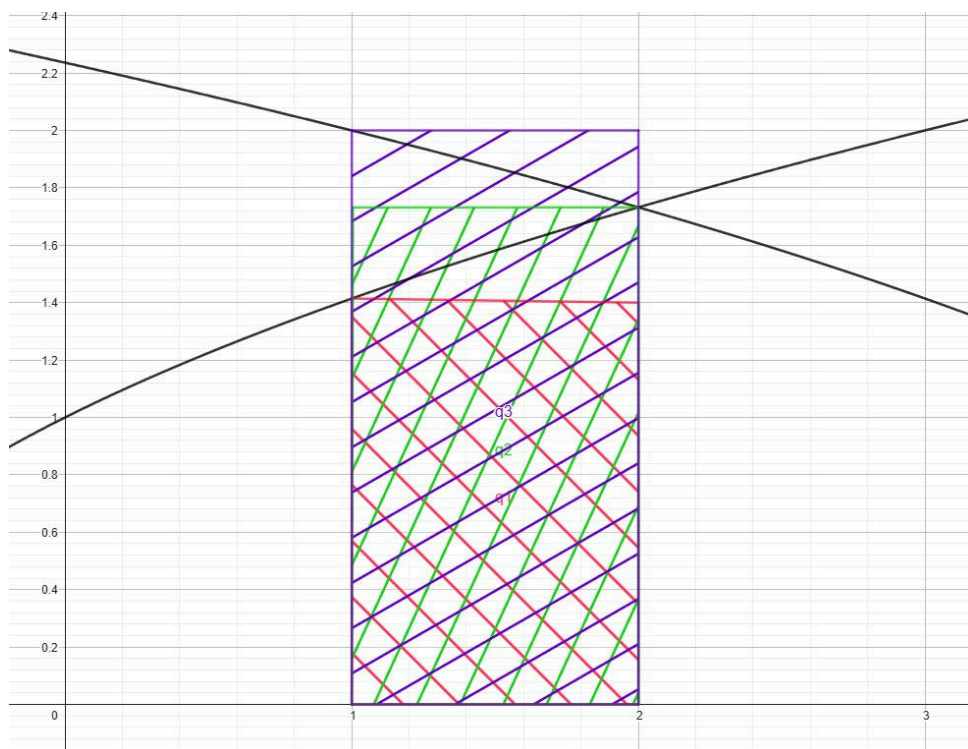


Figura 3: Aproximación del área.

## Ejercicio 5

Calcular la derivada de las siguientes funciones donde sea posible:

En estos problemas debemos usar la primera parte del teorema fundamental del cálculo.

$$\text{a) } f(x) = \int_0^x \frac{\sin(t^2)}{1 + \cos^2(t)} dt$$

$$f'(x) = \frac{\sin(x^2)}{1 + \cos^2(x)}$$

$$\text{b) } f(x) = \int_0^{x^2} \frac{e^{t^2} + 1}{\sqrt{1 - t^2}} dt$$

$$f(x) = \int_0^{x^2} \frac{e^{t^2} + 1}{\sqrt{1 - t^2}} dt$$

$$f(x) = \int_0^{u(x)} \frac{e^{t^2} + 1}{\sqrt{1 - t^2}} dt$$

$$u(x) = x^2$$

$$f(x) = g(u(x))$$

$$g(x) = \int_0^x \frac{e^{t^2} + 1}{\sqrt{1 - t^2}}$$

$$g(x) = \int_0^x \frac{e^{t^2} + 1}{\sqrt{1 - t^2}} \Rightarrow g'(x) = \frac{e^{x^2} + 1}{\sqrt{1 - x^2}} \quad \text{Utilizo el TFC para encontrar } g'(x)$$

$$f(x) = g(u(x)) \Rightarrow f'(x) = g'(u(x)) \cdot u'(x)$$

$$f'(x) = \frac{e^{x^4} + 1}{\sqrt{1 - x^4}} \cdot 2x$$

$$c) f(x) = \int_{\sqrt{x}}^{x^3} \frac{t+1}{\sqrt{1+2^t}} dt$$

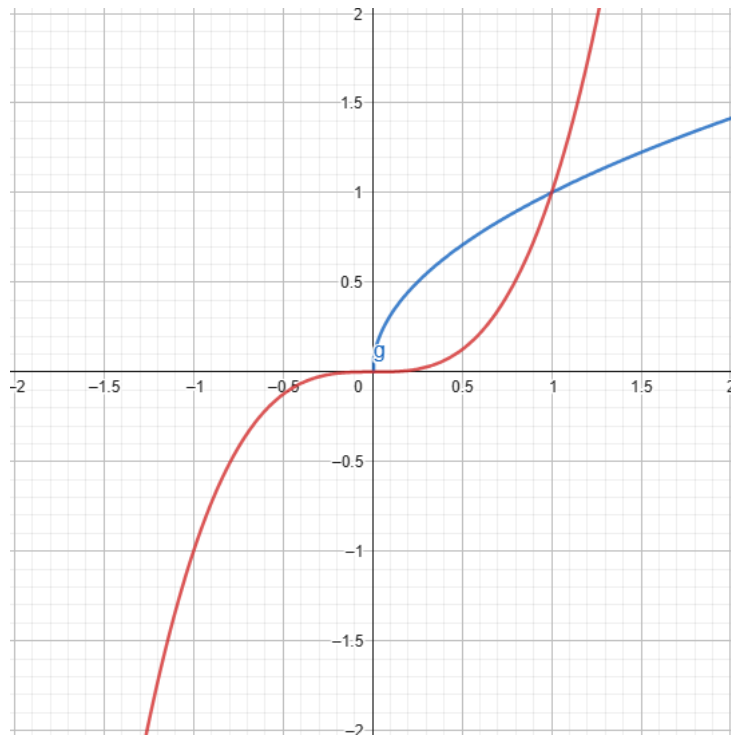


Figura 4:  $u(x)$  y  $v(x)$ .

Sea  $g(x) = \sqrt{x}$ .

$$u(x) = x^3 \quad u(x) : \mathbb{R} \rightarrow \mathbb{R} \wedge v(x) = \sqrt{x} \quad v(x) : [0, \infty) \rightarrow \mathbb{R}$$

$$f(x) = \int_a^{x^3} \frac{t+1}{\sqrt{1+2^t}} dt + \int_{\sqrt{x}}^a \frac{t+1}{\sqrt{1+2^t}} dt$$

$$f(x) = \int_a^{x^3} \frac{t+1}{\sqrt{1+2^t}} dt - \int_a^{\sqrt{x}} \frac{t+1}{\sqrt{1+2^t}} dt$$

Luego se resuelve como el ejercicio anterior utilizando sustituciones.

$$f(x) = \int_a^{u(x)} \frac{t+1}{\sqrt{1+2^t}} dt - \int_a^{v(x)} \frac{t+1}{\sqrt{1+2^t}} dt$$

$$f(x) = g(u(x)) - g(v(x))$$

$$g(x) = \int_a^x \frac{t+1}{\sqrt{1+2^t}} dt$$

$$g(x) = \int_a^x \frac{t+1}{\sqrt{1+2^t}} dt \Rightarrow g'(x) = \frac{x+1}{\sqrt{1+2^x}}$$

$$f(x) = g(u(x)) - g(v(x)) \Rightarrow f'(x) = g'(u(x)) \cdot u'(x) - g'(v(x)) \cdot v'(x)$$

$$f'(x) = \frac{x^3+1}{\sqrt{1+2^{x^3}}} \cdot 3x^2 - \frac{\sqrt{x}+1}{\sqrt{1+2^{\sqrt{x}}}} \cdot \frac{1}{2\sqrt{x}}$$

Nota: Valido para  $\forall a \in (1, \infty)$

## Ejercicio 6

Calcular las siguientes integrales usando el Teorema Fundamental del Cálculo:

Para resolver estos ejercicios utilice las primitivas que encuentre previamente en el ejercicio 3.

a)  $\int_1^2 2^x \, dx$

$$\int_1^2 2^x \, dx = \frac{2^2}{\ln(2)} - \frac{2^1}{\ln(2)} = \frac{4}{\ln(2)} - \frac{2}{\ln(2)}$$

$$\int_1^2 2^x \, dx = \frac{2}{\ln(2)}$$

b)  $\int_3^5 \sqrt[3]{(33-2x)} \, dx$

$$\int_3^5 \sqrt[3]{(33-2x)} \, dx = \int_3^5 (33-2x)^{\frac{1}{3}} \, dx \quad \text{Definición de raíz}$$

Sea  $u = 33 - 2x \rightarrow du = -dx, u(3) = 27 \wedge u(5) = 23$

$$\begin{aligned} \int_3^5 \sqrt[3]{(33-2x)} \, dx &= \int_{27}^{23} (u)^{\frac{1}{3}} \frac{du}{-2} \\ &= -\frac{1}{2} \int_{27}^{23} (u)^{\frac{1}{3}} \, du \\ &= \frac{1}{2} \int_{23}^{27} (u)^{\frac{1}{3}} \, du \\ &= \frac{3}{8} 27^{\frac{4}{3}} - \frac{3}{8} 23^{\frac{4}{3}} \\ &= \frac{3 \cdot 27^{\frac{4}{3}} - 3 \cdot (23)^{\frac{4}{3}}}{8} \\ &= \frac{3 \cdot 3^4 - 3 \cdot (23)^{\frac{4}{3}}}{8} \\ &= \frac{3^5 - 3 \cdot 23^{\frac{4}{3}}}{8} \end{aligned}$$

$$\int_3^5 \sqrt[3]{(33-2x)} \, dx = \frac{243 - 3 \cdot 23^{\frac{4}{3}}}{8}$$



$$\text{c) } \int_1^5 \frac{dx}{7-x}$$

$$\text{Sea } u = 7 - x \rightarrow du = -dx, u(1) = 6 \wedge u(5) = 2$$

$$\begin{aligned} \int_1^5 \frac{dx}{7-x} &= \int_6^2 \frac{-du}{u} \\ &= - \int_6^2 \frac{du}{u} \\ &= \int_2^6 \frac{du}{u} \\ \int_1^5 \frac{dx}{7-x} &= \ln(6) - \ln(2) \end{aligned}$$

$$\text{d) } \int_0^1 \frac{2x+3}{x^2+3x+4} dx$$

$$\text{Sea } u = x^2 + 3x + 4 \rightarrow du = (2x+3) dx, u(0) = 4 \wedge u(1) = 8$$

$$\begin{aligned} \int_0^1 \frac{2x+3}{x^2+3x+4} dx &= \int_4^8 \frac{du}{u} \\ \int_0^1 \frac{2x+3}{x^2+3x+4} dx &= \ln(8) - \ln(4) = \ln\left(\frac{8}{4}\right) \\ \int_0^1 \frac{2x+3}{x^2+3x+4} dx &= \ln(2) \end{aligned}$$

$$\text{e) } \int_{\ln(2)}^{\ln(3)} \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$$\text{Sea } u = e^x + e^{-x} \rightarrow du = (e^x - e^{-x}) dx, u(\ln(2)) = 2 + \frac{1}{2} = \frac{5}{2} \wedge u(3) = 3 + \frac{1}{3} = \frac{10}{3}$$

$$\begin{aligned} \int_{\ln(2)}^{\ln(3)} \frac{e^x - e^{-x}}{e^x + e^{-x}} dx &= \int_{\frac{5}{2}}^{\frac{10}{3}} \frac{du}{u} \\ \int_{\ln(2)}^{\ln(3)} \frac{e^x - e^{-x}}{e^x + e^{-x}} dx &= \ln\left(\frac{10}{3}\right) - \ln\left(\frac{5}{2}\right) \\ \int_{\ln(2)}^{\ln(3)} \frac{e^x - e^{-x}}{e^x + e^{-x}} dx &= \ln\left(\frac{20}{15}\right) = \ln\left(\frac{4}{3}\right) \end{aligned}$$

$$f) \int_0^{\frac{\pi}{2}} \frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)} dx$$

$$\text{Sea } u = \cos(x) + \sin(x) \rightarrow du = (\cos(x) - \sin(x)) dx, u(0) = 1 \wedge u\left(\frac{\pi}{2}\right) = 1$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)} dx = \int_1^1 \frac{du}{u}$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)} dx = 0 \quad \text{Por propiedad de la integral definida.}$$

## Ejercicio 7

Calcular las siguientes integrales:

$$a) \int x e^x$$

$$\text{Sea } u = e^x \rightarrow du = e^x, v = x \rightarrow dv = 1$$

$$\int x e^x = e^x x - \int e^x dx$$

$$\int x e^x = e^x x - e^x + C$$

$$\int x e^x = e^x (x - 1) + C$$

$$b) \int_{-1}^1 (1 - 2x) e^{-2x} dx$$

$$\text{Sea } u = (1 - 2x) \rightarrow du = -2 dx, v = \frac{e^{-2x}}{-2} \rightarrow dv = e^{-2x}$$

$$\int_{-1}^1 (1 - 2x) e^{-2x} dx = (1 - 2x) \cdot \frac{e^{-2x}}{-2} \Big|_{-1}^1 - \int_{-1}^1 \frac{e^{-2x}}{-2} (-2) dx$$

$$= (1 - 2x) \cdot \frac{e^{-2x}}{-2} \Big|_{-1}^1 - \int_{-1}^1 e^{-2x} dx$$

$$= \frac{(1 - 2x) \cdot e^{-2x}}{-2} \Big|_{-1}^1 - \int_{-1}^1 e^{-2x} dx$$

$$= \frac{(1 - 2) \cdot e^{-2}}{-2} - \frac{(1 + 2) \cdot e^2}{-2} - \int_{-1}^1 e^{-2x} dx$$

$$= -\frac{e^{-2}}{-2} - \frac{3 \cdot e^2}{-2} - \int_{-1}^1 e^{-2x} dx$$

$$\int_{-1}^1 (1 - 2x) e^{-2x} dx = \frac{e^{-2} + 3 \cdot e^2}{2} - \int_{-1}^1 e^{-2x} dx$$

$$\text{Sea } u = -2x \rightarrow du = -2 dx \Rightarrow dx = \frac{du}{-2}, u(-1) = 2 \wedge u(1) = -2$$

$$\begin{aligned}
\int_{-1}^1 (1-2x)e^{-2x} dx &= \frac{e^{-2} + 3 \cdot e^2}{2} - \int_2^{-2} e^u \frac{du}{-2} \\
&= \frac{e^{-2} + 3 \cdot e^2}{2} + \frac{1}{2} \int_2^{-2} e^u du \\
&= \frac{e^{-2} + 3 \cdot e^2}{2} - \frac{1}{2} \int_{-2}^2 e^u du \\
&= \frac{e^{-2} + 3 \cdot e^2}{2} - \frac{1}{2} e^u \Big|_{-2}^2 \\
&= \frac{e^{-2} + 3 \cdot e^2}{2} - \frac{1}{2} (e^2 - e^{-2}) \\
&= \frac{e^{-2} + 3 \cdot e^2}{2} - \frac{e^2 - e^{-2}}{2} \\
&= \frac{e^{-2} + 3 \cdot e^2 - e^2 + e^{-2}}{2} \\
&= \frac{2 \cdot e^{-2} + 2 \cdot e^2}{2}
\end{aligned}$$

$$\int_{-1}^1 (1-2x)e^{-2x} dx = e^{-2} + e^2$$

c)  $\int x^2 \cos(x) dx$

Sea  $u = x^2 \rightarrow du = 2x dx, v = \sin(x) \rightarrow dv = \cos(x)$

$$\int x^2 \cos(x) dx = x^2 \sin(x) - \int 2x \sin(x) dx$$

Sea  $u = 2x \rightarrow du = 2 dx, v = -\cos(x) \rightarrow dv = \sin(x)$

$$\int x^2 \cos(x) dx = x^2 \sin(x) - \left( 2x \cdot (-\cos(x)) - \int -2 \cos(x) dx \right)$$

$$\int x^2 \cos(x) dx = x^2 \sin(x) - \left( -2x \cos(x) + 2 \int \cos(x) dx \right)$$

$$\int x^2 \cos(x) dx = x^2 \sin(x) - (-2x \cos(x) + 2 \sin(x)) + C$$

$$\int x^2 \cos(x) dx = x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + C$$

d)  $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x dx}{\sin^2(x)}$

Sea  $u = x \rightarrow du = dx, v = -\frac{\cos(x)}{\sin(x)} \rightarrow dv = \frac{1}{\sin^2(x)}$

$$\begin{aligned}
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x \, dx}{\sin^2(x)} &= \left( x \cdot -\frac{\cos(x)}{\sin(x)} \right) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int -\frac{\cos(x)}{\sin(x)} \, dx \\
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x \, dx}{\sin^2(x)} &= \left( -\frac{x \cos(x)}{\sin(x)} \right) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot(x) \, dx \\
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x \, dx}{\sin^2(x)} &= -\frac{\frac{\pi}{2} \cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} + \frac{\frac{\pi}{4} \cos(\frac{\pi}{4})}{\sin(\frac{\pi}{4})} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot(x) \, dx \\
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x \, dx}{\sin^2(x)} &= -\frac{\frac{\pi}{2} \cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} + \frac{\frac{\pi}{4} \cos(\frac{\pi}{4})}{\sin(\frac{\pi}{4})} + \ln(\sin(x)) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x \, dx}{\sin^2(x)} &= -\frac{\frac{\pi}{2} \cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} + \frac{\frac{\pi}{4} \cos(\frac{\pi}{4})}{\sin(\frac{\pi}{4})} + \ln\left(\sin\left(\frac{\pi}{2}\right)\right) - \ln\left(\sin\left(\frac{\pi}{4}\right)\right) \\
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x \, dx}{\sin^2(x)} &= -\frac{\frac{\pi}{2} \cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} + \frac{\frac{\pi}{4} \cos(\frac{\pi}{4})}{\sin(\frac{\pi}{4})} + \ln(1) - \ln\left(\frac{\sqrt{2}}{2}\right) \\
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x \, dx}{\sin^2(x)} &= -\frac{\frac{\pi}{2} \cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} + \frac{\frac{\pi}{4} \cos(\frac{\pi}{4})}{\sin(\frac{\pi}{4})} + \ln\left(\frac{2}{\sqrt{2}}\right) \\
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x \, dx}{\sin^2(x)} &= -\frac{\frac{\pi}{2} \cdot 0}{1} + \frac{\frac{\pi}{4} \cos(\frac{\pi}{4})}{\sin(\frac{\pi}{4})} + \ln\left(\frac{2}{\sqrt{2}}\right) \\
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x \, dx}{\sin^2(x)} &= \frac{\frac{\pi}{4} \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} + \ln\left(\frac{2}{\sqrt{2}}\right) \\
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x \, dx}{\sin^2(x)} &= \frac{\pi}{4} + \ln\left(\frac{2}{\sqrt{2}}\right)
\end{aligned}$$

e)  $\int_3^9 x \ln(x-1) \, dx$

Sea  $u = x \rightarrow du = dx, v = (x-1) \ln(x-1) - (x-1) \rightarrow dv = \ln(x-1)$

$$\begin{aligned}
\int_3^9 x \ln(x-1) \, dx &= x \cdot ((x-1) \ln(x-1) - (x-1)) \Big|_3^9 - \int_3^9 ((x-1) \ln(x-1) - (x-1)) \, dx \\
&= (216 \ln(2) - 72 - 6 \ln(2) + 6) - \int_3^9 ((x-1) \ln(x-1) - (x-1)) \, dx \\
&= 210 \ln(2) - 66 - \int_3^9 ((x-1) \ln(x-1) - (x-1)) \, dx \\
&= 210 \ln(2) - 66 - \int_3^9 (x-1) \ln(x-1) \, dx + \int_3^9 (x-1) \, dx \\
&= 210 \ln(2) - 66 - \int_3^9 (x-1) \ln(x-1) \, dx + \int_2^8 u \, du \\
&= 210 \ln(2) - 66 - \int_3^9 (x-1) \ln(x-1) \, dx + \left( \frac{u^2}{2} \Big|_2^8 \right)
\end{aligned}$$

$$= 210 \ln(2) - 66 - \int_3^9 (x-1) \ln(x-1) \, dx + (32-2)$$

$$= 210 \ln(2) - 66 - \int_3^9 (x-1) \ln(x-1) \, dx + 30$$

$$= 210 \ln(2) - 36 - \int_3^9 (x-1) \ln(x-1) \, dx$$

$$= 210 \ln(2) - 36 - \int_2^8 u \ln(u) \, du$$

$$\text{Tomo } u = x-1 \, du = dx$$

$$\text{Sea } w = \ln(u) \rightarrow dw = \frac{1}{u} du, \quad v = \frac{u^2}{2} \rightarrow dv = u$$

$$= 210 \ln(2) - 36 - \left( \frac{u^2}{2} \cdot \ln(u) \Big|_2^8 - \int_2^8 \frac{1}{u} \frac{u^2}{2} \right)$$

$$= 210 \ln(2) - 36 - \left( \frac{u^2}{2} \cdot \ln(u) \Big|_2^8 - \int_2^8 \frac{u}{2} \right)$$

$$= 210 \ln(2) - 36 - \left( \frac{u^2}{2} \cdot \ln(u) \Big|_2^8 - \frac{1}{2} \int_2^8 u \right)$$

$$= 210 \ln(2) - 36 - \left( \frac{u^2}{2} \cdot \ln(u) \Big|_2^8 - \frac{1}{2} \left( \frac{u^2}{2} \Big|_2^8 \right) \right)$$

$$= 210 \ln(2) - 36 - \left( \frac{u^2}{2} \cdot \ln(u) \Big|_2^8 - \frac{1}{2} (32-2) \right)$$

$$= 210 \ln(2) - 36 - \left( \frac{u^2}{2} \cdot \ln(u) \Big|_2^8 - \frac{1}{2} 30 \right)$$

$$= 210 \ln(2) - 36 - \left( \frac{u^2}{2} \cdot \ln(u) \Big|_2^8 \right) + 15$$

$$= 210 \ln(2) - 36 - (96 \ln(2) - 2 \ln(2)) + 15$$

$$= 210 \ln(2) - 36 - 94 \ln(2) + 15$$

$$\int_3^9 x \ln(x-1) \, dx = 116 \ln(2) - 21$$

$$\text{f) } \int \ln(x^2 + 1) \, dx$$

$$\text{Sea } u = \ln(x^2 + 1) \rightarrow du = \frac{1}{x^2 + 1} 2x, v = x \rightarrow dv = dx$$

$$\begin{aligned} \int \ln(x^2 + 1) \, dx &= x \ln(x^2 + 1) - \int \frac{2x}{x^2 + 1} x \, dx \\ &= x \ln(x^2 + 1) - 2 \int \frac{x^2}{x^2 + 1} \, dx \\ &= x \ln(x^2 + 1) - 2 \int \frac{x^2 + 1 - 1}{x^2 + 1} \, dx \\ &= x \ln(x^2 + 1) - 2 \left( \int \frac{x^2 + 1}{x^2 + 1} \, dx - \int \frac{1}{x^2 + 1} \, dx \right) \\ &= x \ln(x^2 + 1) - 2 \left( \int 1 \, dx - \int \frac{1}{x^2 + 1} \, dx \right) \\ &= x \ln(x^2 + 1) - 2 \left( x - \int \frac{1}{x^2 + 1} \, dx \right) \\ &= x \ln(x^2 + 1) - 2x + \int \frac{1}{x^2 + 1} \, dx \\ \int \ln(x^2 + 1) \, dx &= x \ln(x^2 + 1) - 2x + \arctan(x) + C \end{aligned}$$

$$\text{g) } \int_0^2 x \ln(x^2 + 4) \, dx$$

$$\begin{aligned} \int_0^2 x \ln(x^2 + 4) \, dx &= \int_4^8 \frac{\ln(u)}{2} \, du \quad | \quad u = x^2 + 4 \rightarrow du = 2x \, dx \Rightarrow x \, dx = \frac{du}{2}, u(0) = 4 \wedge u(2) = 8 \\ &= \frac{1}{2} \int_4^8 \ln(u) \, du \\ &= \frac{1}{2} (u \ln(u) - u) \Big|_4^8 \\ &= \frac{1}{2} (8 \ln(8) - 8 - 4 \ln(4) + 4) \\ &= \frac{1}{2} (8 \cdot 3 \ln(2) - 2 \cdot 2 \ln(2) - 4) \\ &= \frac{1}{2} (24 \ln(2) - 4 \ln(2) - 4) \\ &= \frac{1}{2} (20 \ln(2) - 4) \\ &= 10 \ln(2) - 2 \end{aligned}$$

$$\text{h) } \int e^{-x} \sin(2x) \, dx$$

$$\text{Sea } u = \sin(2x) \rightarrow du = 2 \cos(2x) \, dx, v = -(e^{-x}) \rightarrow dv = e^{-x}$$

$$\int e^{-x} \sin(2x) \, dx = \sin(2x) \cdot (-(e^{-x})) + \int 2 \cos(2x) e^{-x} \, dx$$

$$= -\sin(2x) \cdot e^{-x} + 2 \int \cos(2x) e^{-x} \, dx$$

$$\text{Sea } u = \cos(2x) \rightarrow du = -2 \sin(2x) \, dx, v = -(e^{-x}) \rightarrow dv = e^{-x}$$

$$= -\sin(2x) \cdot e^{-x} + 2 \left( -\cos(2x) \cdot e^{-x} - \int -2 \sin(2x) (-(e^{-x})) \, dx \right)$$

$$= -\sin(2x) \cdot e^{-x} + 2 \left( -\cos(2x) \cdot e^{-x} - \int 2 \sin(2x) e^{-x} \, dx \right)$$

$$= -\sin(2x) \cdot e^{-x} - 2 \cos(2x) \cdot e^{-x} - 2 \int 2 \sin(2x) e^{-x} \, dx$$

$$\int e^{-x} \sin(2x) \, dx = -\sin(2x) \cdot e^{-x} - 2 \cos(2x) \cdot e^{-x} - 4 \int \sin(2x) e^{-x} \, dx \Rightarrow$$

$$\Rightarrow \int e^{-x} \sin(2x) \, dx + 4 \int \sin(2x) e^{-x} \, dx = -\sin(2x) \cdot e^{-x} - 2 \cos(2x) \cdot e^{-x} \Rightarrow$$

$$\Rightarrow 5 \int \sin(2x) e^{-x} \, dx = -\sin(2x) \cdot e^{-x} - 2 \cos(2x) \cdot e^{-x} \Rightarrow$$

$$\int \sin(2x) e^{-x} \, dx = \frac{-\sin(2x) \cdot e^{-x} - 2 \cos(2x) \cdot e^{-x}}{5}$$

$$\text{i) } \int_0^{2\pi} \cos^4(x) \, dx$$

???

## Ejercicio 8

Calcular las siguientes integrales:

$$\text{a) } \int_0^1 e^{\sqrt{x}} dx$$

$$u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx \Rightarrow dx = 2\sqrt{x} du = 2u du$$

$$\begin{aligned} \int_0^1 e^{\sqrt{x}} dx &= \int_0^1 e^u 2u du \\ &= 2 \int_0^1 e^u u du \end{aligned}$$

$$\text{Sea } f = u \rightarrow f' = 1, g' = e^u \rightarrow g = e^u$$

$$\begin{aligned} &= 2 \left( u \cdot e^u \Big|_0^1 - \int_0^1 e^u du \right) \\ &= 2 \left( u \cdot e^u \Big|_0^1 - e^u \Big|_0^1 \right) \\ &= 2 \left( u \cdot e^u \Big|_0^1 - (e^1 - e^0) \right) = 2 \left( u \cdot e^u \Big|_0^1 - (e - 1) \right) \\ &= 2 \left( u \cdot e^u \Big|_0^1 \right) - 2e + 2 \\ &= 2(1 \cdot e^1 - 0 \cdot e^0) - 2e + 2 \\ &= 2e - 2e + 2 \end{aligned}$$

$$\int_0^1 e^{\sqrt{x}} dx = 2$$

$$\text{b) } \int \sin(\sqrt{x}) dx$$

$$u = \sqrt{x} \rightarrow du = \frac{dx}{2\sqrt{x}} \rightarrow dx = 2\sqrt{x} du = 2u du$$

$$\int \sin(\sqrt{x}) dx = \int \sin(u) 2u du = 2 \int \sin(u) u du$$

$$\text{Sea } f = u \rightarrow f' = 1, g' = \sin(u) \rightarrow g = -\cos(u)$$

$$\begin{aligned} \int \sin(\sqrt{x}) dx &= 2 \left( -u \cos(u) - \int -\cos(u) du \right) \\ &= 2 \left( -u \cos(u) + \int \cos(u) du \right) \\ &= 2(-u \cos(u) + \sin(u)) + C \end{aligned}$$

$$\int \sin(\sqrt{x}) dx = -2\sqrt{x} \cos(\sqrt{x}) + 2 \sin(\sqrt{x}) + C$$



$$\text{c) } \int_0^1 (2x+1) \ln(x+1) \, dx$$

$$f = \ln(x+1) \rightarrow f' = \frac{1}{x+1} \, dx, g' = 2x+1 \rightarrow g = x^2 + x$$

$$\begin{aligned} \int_0^1 (2x+1) \ln(x+1) \, dx &= \ln(x+1)(x^2+x) \Big|_0^1 - \int_0^1 \frac{(x^2+x) \, dx}{x+1} \\ &= \ln(x+1)(x^2+x) \Big|_0^1 - \int_0^1 \frac{x(x+1) \, dx}{x+1} \\ &= \ln(x+1)(x^2+x) \Big|_0^1 - \int_0^1 x \, dx \\ &= \ln(2) - \int_0^1 x \, dx \\ &= \ln(2) - \left( \frac{x^2}{2} \right) \Big|_0^1 \\ &= 2\ln(2) - \frac{1}{2} \end{aligned}$$

$$\text{d) } \int \frac{1}{x \ln(x)} \, dx$$

Resuelvo por sustitución.

$$u = \ln(x) \rightarrow du = \frac{1}{x} \, dx$$

$$\int \frac{1}{x \ln(x)} \, dx = \int \frac{du}{u}$$

$$\int \frac{1}{x \ln(x)} \, dx = \ln(|u|) + C$$

$$\int \frac{1}{x \ln(x)} \, dx = \ln(|\ln(x)|) + C$$

$$\text{e) } \int_0^1 \arccos(x) \, dx$$

$$f = \arccos(x) \rightarrow f' = -\frac{1}{\sqrt{1-x^2}}, g' = 1 \rightarrow g = x$$

$$\int_0^1 \arccos(x) \, dx = x \cdot \arccos(x) \Big|_0^1 - \int_0^1 -\frac{x}{(\sqrt{1-x^2})} \, dx$$

$$\int_0^1 \arccos(x) \, dx = \arccos(1) - \int_0^1 -\frac{x}{\sqrt{1-x^2}} \, dx$$

$$u = 1 - x^2 \rightarrow du = -2x \, dx \rightarrow -x \, dx = \frac{du}{2}$$

$$\int_0^1 \arccos(x) \, dx = 0 - \int_1^0 \frac{du}{2\sqrt{u}}$$

$$= \frac{1}{2} \int_0^1 \frac{du}{\sqrt{u}}$$

$$= \frac{1}{2} \left( 2\sqrt{u} \Big|_0^1 \right)$$

$$= \frac{1}{2} \cdot 2$$

$$\int_0^1 \arccos(x) \, dx = 1$$

$$\text{f) } \int_0^1 x^3 e^{x^2} \, dx$$

Empiezo resolviendo por sustitución.

$$u = x^2 \rightarrow du = 2x \, dx \rightarrow \frac{du}{2} = x \, dx$$

$$\int_0^1 x^3 e^{x^2} \, dx = \int_0^1 x^2 \cdot x \cdot e^{x^2} \, dx = \int_0^1 u \cdot e^u \cdot \frac{du}{2}$$

$$\int_0^1 x^3 e^{x^2} \, dx = \frac{1}{2} \int_0^1 u \cdot e^u \cdot du$$

$$f = u \rightarrow f' = 1, g' = e^u \rightarrow g = e^u$$

$$\int_0^1 x^3 e^{x^2} \, dx = \frac{1}{2} \left( u \cdot e^u \Big|_0^1 - \int_0^1 e^u \, du \right)$$

$$= \frac{1}{2} \left( u \cdot e^u \Big|_0^1 - e^u \Big|_0^1 \right)$$

$$= \frac{1}{2} (e - e + 1)$$

$$\int_0^1 x^3 e^{x^2} \, dx = \frac{1}{2}$$

$$\text{g)} \int e^x(1 - e^x)^{-1} \mathrm{d}x$$

$$u = 1 - e^x \rightarrow \mathrm{d}u = -e^x \mathrm{d}x$$

$$\int e^x(1 - e^x)^{-1} \mathrm{d}x = \int \frac{e^x}{1 - e^x} \mathrm{d}x = \int \frac{-\mathrm{d}u}{u}$$

$$\int e^x(1 - e^x)^{-1} \mathrm{d}x = - \int \frac{\mathrm{d}u}{u}$$

$$\int e^x(1 - e^x)^{-1} \mathrm{d}x = -\ln(|1 - e^x|) + C$$

$$\text{h)} \int \frac{\mathrm{d}x}{x\sqrt{x-1}}$$

$$u = \sqrt{x-1} \rightarrow \mathrm{d}u = \frac{1}{2\sqrt{x-1}} \mathrm{d}x \rightarrow \mathrm{d}x = 2\sqrt{x-1} \mathrm{d}u = 2u \mathrm{d}u$$

$$\int \frac{\mathrm{d}x}{x\sqrt{x-1}} = \int \frac{2u \mathrm{d}u}{(u^2 + 1) \cdot u} = \int \frac{2 \mathrm{d}u}{u^2 + 1}$$

$$= 2 \int \frac{\mathrm{d}u}{u^2 + 1}$$

$$= 2 \arctan(u)$$

$$\int \frac{\mathrm{d}x}{x\sqrt{x-1}} = 2 \arctan(\sqrt{x-1})$$

$$\text{i)} \int \sin(x)^3 \mathrm{d}x$$

$$\int \sin(x)^3 \mathrm{d}x = \int \sin(x) \cdot \sin(x)^2 \mathrm{d}x$$

$$= \int \sin(x) \cdot (1 - \cos(x)^2) \mathrm{d}x$$

$$u = \cos(x) \rightarrow \mathrm{d}u = -\sin(x) \mathrm{d}x$$

$$= \int -(1 - u^2) \mathrm{d}u = \int u^2 - 1 \mathrm{d}u$$

$$= \int u^2 \mathrm{d}u - \int 1 \mathrm{d}u$$

$$= \frac{u^3}{3} - u + C$$

$$\int \sin(x)^3 \mathrm{d}x = \frac{\cos(x)^3}{3} - \cos(x) + C$$

## Ejercicio 9

Trazar la región limitada por las curvas dadas y calcular su área:

a)  $y = 4x^2$ ,  $y = x^2 + 3$

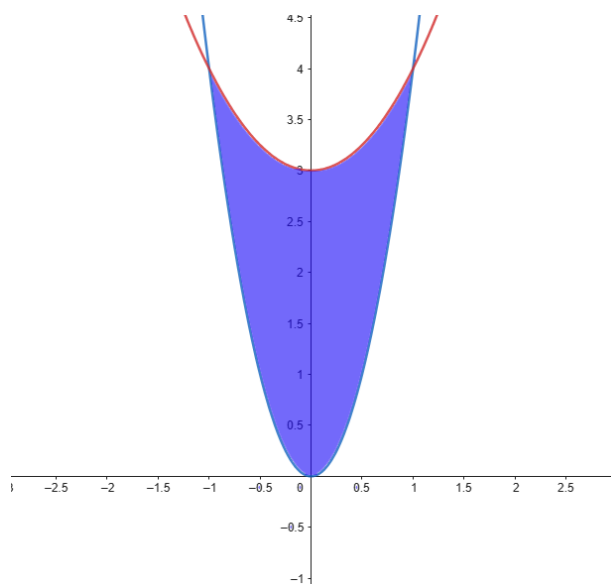


Figura 5: Reprerentación del área.

$$4x^2 = x^2 + 3 \rightarrow 0 = 3x^2 - 3 = 3(x^2 - 1)$$

Luego sabemos que las curvas se intersectan en  $x = 1$  y  $x = -1$ .

$$\begin{aligned} \int_{-1}^1 x^2 + 3 \, dx - \int_{-1}^1 4x^2 \, dx &= \int_{-1}^1 x^2 - 4x^2 + 3 \\ &= \int_{-1}^1 -3x^2 + 3 \\ &= -x^3 + 3x \Big|_{-1}^1 \\ &= 2 - (-2) \end{aligned}$$

$$\int_{-1}^1 x^2 + 3 \, dx - \int_{-1}^1 4x^2 \, dx = 4$$

b)  $y = \cos(x)$ ,  $y = \sin(x)$ ,  $x = 0$ ,  $x = \frac{\pi}{2}$

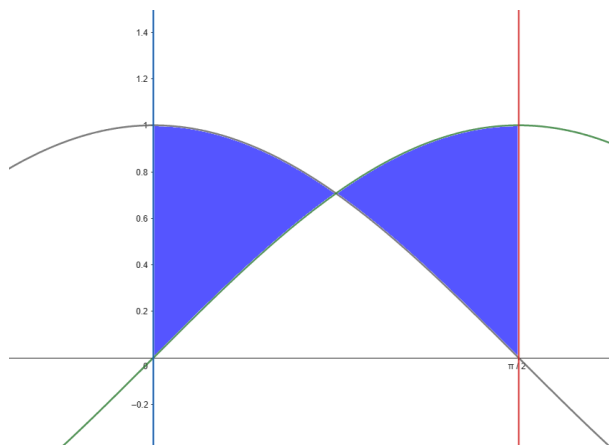


Figura 6: Representación del área.

Sabemos que ambas curvas se intersectan en  $\frac{\pi}{4}$

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \cos(x) - \sin(x) &= \sin(x) + \cos(x) \Big|_0^{\frac{\pi}{4}} \\ &= \left( \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) \right) - (\sin(0) + \cos(0)) \\ &= \sqrt{2} - 1 \end{aligned}$$

$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x) - \cos(x) &= -\cos(x) - \sin(x) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= \left( -\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) \right) - \left( -\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) \right) \\ &= -\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) \\ &= 0 - 1 + \sqrt{2} \\ &= \sqrt{2} - 1 \end{aligned}$$

Luego el área es  $2\sqrt{2} - 2$ .

c)  $y = |x|$ ,  $y = (x + 1)^2 - 7$ ,  $x = -4$

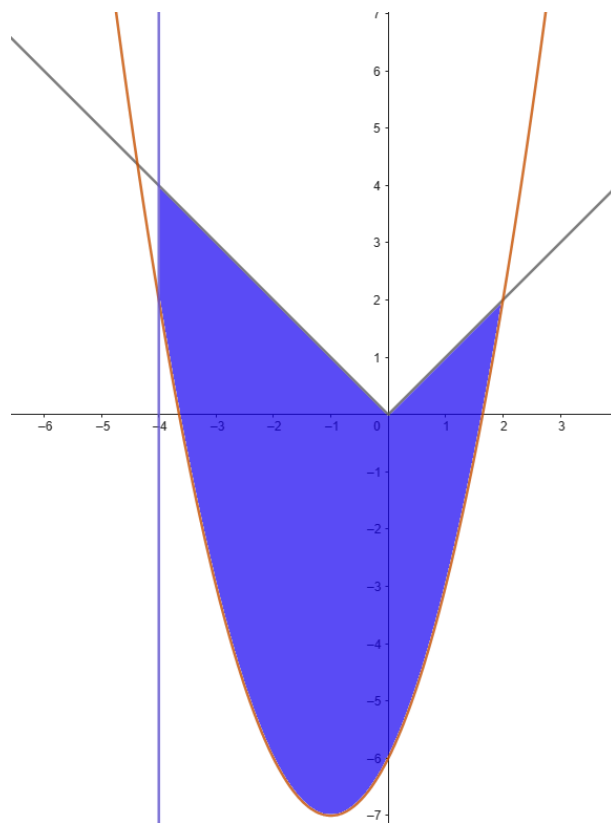


Figura 7: Reprerentación del área.

$$\begin{aligned}
\int_{-4}^0 -x - ((x+1)^2 - 7) &= \int_{-4}^0 -x - (x^2 + 2x - 6) \\
&= \int_{-4}^0 -x - x^2 - 2x + 6 \\
&= \int_{-4}^0 -x^2 - 3x + 6 \\
&= -\frac{x^3}{3} - \frac{3x^2}{2} + 6x \Big|_{-4}^0 \\
&= -\left(-\frac{(-4)^3}{3} - \frac{3(-4)^2}{2} + 6(-4)\right) \\
&= -\left(\frac{64}{3} - 24 - 24\right) \\
&= -\left(\frac{64}{3} - 48\right) \\
&= \frac{144 - 64}{3}
\end{aligned}$$

$$\int_{-4}^0 -x - ((x+1)^2 - 7) = \frac{80}{3}$$

$$\begin{aligned}
\int_0^2 x - ((x+1)^2 - 7) &= \int_0^2 x - (x^2 + 2x - 6) \\
&= \int_0^2 -x^2 - x + 6 \\
&= -\frac{x^3}{3} - \frac{x^2}{2} + 6x \Big|_0^2 \\
&= -\frac{2^3}{3} - \frac{2^2}{2} + 6 \cdot 2 \\
&= -\frac{8}{3} - 2 + 12 \\
&= -\frac{8}{3} + 10 \\
&= -\frac{8}{3} + \frac{30}{3} \\
&= \frac{22}{3}
\end{aligned}$$

Luego el área total es la suma de  $\frac{22}{3} + \frac{80}{3}$ , es decir  $\frac{102}{3}$ .

## Ejercicio 10

Calcular las siguientes integrales:

a)  $\int_2^4 \frac{x^2 + 4x + 24}{x^2 - 4x + 8} dx$

Primero hago una división ya que el numerador y el denominador son del mismo grado.

$$\begin{aligned}\int_2^4 \frac{x^2 + 4x + 24}{x^2 - 4x + 8} dx &= \int_2^4 1 + \frac{8x + 16}{x^2 - 4x + 8} dx \\&= \int_2^4 1 dx + \int_2^4 \frac{8x + 16}{x^2 - 4x + 8} dx \\&= \int_2^4 1 dx + \int_2^4 4 \frac{2x - 4}{x^2 - 4x + 8} dx + \int_2^4 32 \frac{1}{x^2 - 4x + 8} dx \\&= \int_2^4 1 dx + 4 \int_2^4 \frac{2x - 4}{x^2 - 4x + 8} dx + 32 \int_2^4 \frac{1}{x^2 - 4x + 8} dx \\&= \int_2^4 1 dx + 4 \int_4^8 \frac{du}{u} + 32 \int_2^4 \frac{1}{x^2 - 4x + 8} dx \\&= \int_2^4 1 dx + 4 \int_4^8 \frac{du}{u} + 32 \int_2^4 \frac{1}{(x - 2)^2 + 4} dx \\&= \int_2^4 1 dx + 4 \int_4^8 \frac{du}{u} + 32 \int_0^2 \frac{1}{u^2 + 2^2} dx \\&= x|_2^4 + 4(\ln(u)|_4^8) + 32 \left( \frac{1}{2} \arctan\left(\frac{u}{2}\right) \Big|_0^2 \right) \\&= (4 - 2) + 4(\ln(8) - \ln(4)) + 32 \left( \frac{1}{2} \arctan(1) - \frac{1}{2} \arctan(0) \right) \\&= 2 + 4 \ln(2) + 16(\arctan(1) - \arctan(0)) \\&= 2 + 4 \ln(2) + 16 \left( \frac{\pi}{4} - 0 \right) \\&= 2 + 4 \ln(2) + 4\pi\end{aligned}$$
$$\int_2^4 \frac{x^2 + 4x + 24}{x^2 - 4x + 8} dx = 2 + 4 \ln(2) + 4\pi$$



$$\text{b) } \int_0^2 \frac{x-1}{x^2+4} dx$$

$$\begin{aligned} \int_0^2 \frac{x-1}{x^2+4} dx &= \int_0^2 \frac{1}{2} \frac{2x}{x^2+4} - \frac{1}{x^2+4} dx \\ &= \frac{1}{2} \int_0^2 \frac{2x}{x^2+4} dx - \int_0^2 \frac{1}{x^2+4} dx \\ &= \frac{1}{2} \int_4^8 \frac{du}{u} - \int_0^2 \frac{1}{x^2+2^2} dx \\ &= \frac{1}{2} (\ln(u)|_4^8) - \left( \frac{1}{2} \arctan\left(\frac{x}{2}\right) \Big|_0^2 \right) \\ &= \frac{1}{2} (\ln(8) - \ln(4)) - \left( \frac{1}{2} \arctan(1) - \frac{1}{2} \arctan(0) \right) \\ &= \frac{1}{2} \ln(2) - \left( \frac{1}{2} \frac{\pi}{4} - \frac{1}{2} 0 \right) \\ &= \frac{1}{2} \ln(2) - \frac{\pi}{8} \end{aligned}$$

$$\text{c) } \int_2^4 \frac{x}{x^3-3x+2} dx$$

Necesito expresar  $(x^3 - 3x + 2)$  como polinomios de producto de grado 1 y 2. A ojo veo que 1 es una raíz, por lo tanto divido  $(x^3 - 3x + 2)$  entre  $(x - 1)$  para encontrar otra forma de expresarlo.

$$\int_2^4 \frac{x}{(x^2+x-2)(x-1)} dx$$

Sin embargo  $(x^2 + x - 2)$  tiene raíces reales así que todavía no terminamos.

$$\int_2^4 \frac{x}{(x-1)(x-1)(x+2)} dx = \int_2^4 \frac{x}{(x-1)^2(x+2)} dx$$

Finalmente llegamos al caso 3.

$$\begin{aligned} \int_2^4 \frac{x}{(x-1)^2(x+2)} dx &= \int_2^4 \frac{A_1}{x+2} + \frac{A_2}{x-1} + \frac{A_3}{(x-1)^2} dx \\ &= \int_2^4 \frac{A_1(x-1)^2 + A_2(x-1)(x+2) + A_3(x+2)}{(x+2)(x-1)^2} dx \\ &= \int_2^4 \frac{A_1(x^2-2x+1) + A_2(x^2+x-2) + A_3(x+2)}{(x+2)(x-1)^2} dx \\ &= \int_2^4 \frac{A_1x^2 - A_1 \cdot 2x + A_1 + A_2x^2 + A_2x - A_2 \cdot 2 + A_3x + A_3 \cdot 2}{(x+2)(x-1)^2} dx \\ &= \int_2^4 \frac{x^2(A_1 + A_2) + x(-2A_1 + A_2 + A_3) + A_1 - 2A_2 + 2A_3}{(x+2)(x-1)^2} dx \end{aligned}$$

$$0 = A_1 + A_2 \rightarrow A_1 = -A_2$$

$$1 = -2A_1 + A_2 + A_3 = -2(-A_2) + A_2 + A_3 = 3A_2 + A_3 \rightarrow A_3 = 1 - 3A_2$$

$$0 = A_1 - 2A_2 + 2A_3 = -A_2 - 2A_2 + 2(1 - 3A_2) = -3A_2 + 2 - 6A_2 = 2 - 9A_2 \rightarrow A_2 = \frac{2}{9}$$

$$A_1 = -\frac{2}{9}$$

$$A_2 = \frac{2}{9}$$

$$A_3 = \frac{1}{3}$$

$$\begin{aligned} \int_2^4 \frac{x}{(x-1)^2(x+2)} dx &= \int_2^4 \frac{-\frac{2}{9}}{x+2} + \frac{\frac{2}{9}}{x-1} + \frac{\frac{1}{3}}{(x-1)^2} dx \\ &= \int_2^4 \frac{-\frac{2}{9}}{x+2} dx + \int_2^4 \frac{\frac{2}{9}}{x-1} dx + \int_2^4 \frac{\frac{1}{3}}{(x-1)^2} dx \\ &= -\frac{2}{9} \int_2^4 \frac{1}{x+2} dx + \frac{2}{9} \int_2^4 \frac{1}{x-1} dx + \frac{1}{3} \int_2^4 \frac{1}{(x-1)^2} dx \\ &= -\frac{2}{9} \int_4^6 \frac{1}{u} du + \frac{2}{9} \int_1^3 \frac{1}{u} du + \frac{1}{3} \int_1^3 \frac{1}{u^2} du \\ &= -\frac{2}{9} (\ln(u)|_4^6) + \frac{2}{9} (\ln(u)|_1^3) + \frac{1}{3} \left( -\frac{1}{u} \Big|_1^3 \right) \\ &= -\frac{2}{9} \ln(6) + \frac{2}{9} \ln(4) + \frac{2}{9} \ln(3) + \frac{1}{3} \cdot \frac{2}{3} \\ &= -\frac{2}{9} \ln(6) + \frac{2}{9} \ln(4) + \frac{2}{9} \ln(3) + \frac{2}{9} \\ &= \frac{2}{9} \ln\left(\frac{2}{3}\right) + \frac{2}{9} \ln(3) + \frac{2}{9} \\ &= \frac{2}{9} \ln(2) + \frac{2}{9} \end{aligned}$$

## Ejercicio 11

La sustitución  $t = \tan\left(\frac{x}{2}\right)$ , o equivalentemente,  $x = 2 \arctan(t)$ , transforma cualquier integral que involucre sólo senos y cosenos vinculados por suma, producto o cociente, en la integral de una función racional. Verificar que con esta sustitución resulta:

$$\cos(x) = \frac{1-t^2}{1+t^2}, \quad \sin(x) = \frac{2t}{1+t^2} \quad y \quad dx = \frac{2}{1+t^2} dt.$$

Utilizar esta sustitución en los siguientes casos:

$$\text{a) } \int_0^{\frac{\pi}{2}} \frac{2}{1 + \cos(x)} \, dx$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{2}{1 + \cos(x)} \, dx &= \int_0^1 \frac{2}{1 + \left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{1+t^2} \, dt \quad t = \frac{\tan(x)}{2} \\ &= \int_0^1 \frac{2}{\frac{1+t^2+1-t^2}{1+t^2}} \frac{2}{1+t^2} \, dt \\ &= \int_0^1 \frac{2}{\frac{2}{1+t^2}} \frac{2}{1+t^2} \, dt \\ &= \int_0^1 \frac{4}{2} \, dt \\ &= \int_0^1 2 \, dt \\ &= 2t \Big|_0^1 \end{aligned}$$

$$\int_0^{\frac{\pi}{2}} \frac{2}{1 + \cos(x)} \, dx = 2$$

$$\text{b) } \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin(x)} \, dx$$

$$\begin{aligned} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin(x)} \, dx &= \int_{\frac{\sqrt{3}}{3}}^1 \frac{1+t^2}{2t} \cdot \frac{2}{1+t^2} \, dt \quad t = \frac{\tan(x)}{2} \\ &= \int_{\frac{\sqrt{3}}{3}}^1 \frac{1}{t} \, dt \\ &= \ln(t) \Big|_{\frac{\sqrt{3}}{3}}^1 \\ &= \ln(1) - \left( \ln\left(\frac{\sqrt{3}}{3}\right) \right) = \ln(1) - (\ln(\sqrt{3}) - \ln(3)) \end{aligned}$$

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin(x)} \, dx = \ln(\sqrt{3}) + \ln(3)$$

## Ejercicio 12

Calcular las siguientes integrales:

a)  $\int \tan(x)^2 \, dx$

$$\begin{aligned}\int \tan(x)^2 \, dx &= \int \frac{\sin(x)^2}{\cos(x)^2} \, dx = \int \frac{1 - \cos(x)^2}{\cos(x)^2} \, dx \\ \int \tan(x)^2 \, dx &= \int \frac{1}{\cos(x)^2} \, dx - \int 1 \, dx \\ \int \tan(x)^2 \, dx &= \tan(x) - x + C\end{aligned}$$

b)  $\int \frac{dx}{\sqrt{9 - 4x^2}}$

Sale por sustitución:

$$\begin{aligned}9u^2 &= 4x^2 \rightarrow u^2 = \frac{4}{9}x^2 \rightarrow u = \sqrt{\frac{4}{9}x^2} = \frac{2}{3}x \rightarrow du = \frac{2}{3} \, dx \\ \int \frac{dx}{\sqrt{9 - 4x^2}} &= \int \frac{3}{2\sqrt{9 - 9u^2}} \, du = \int \frac{3}{2 \cdot 3\sqrt{1 - u^2}} \, du \\ &= \frac{1}{2} \int \frac{1}{\sqrt{1 - u^2}} \, du \\ &= \frac{1}{2} \arccos(u) \\ &= \frac{1}{2} \arccos\left(\frac{2}{3}x\right) + C\end{aligned}$$

b)  $\int \frac{x + 1}{\sqrt{1 - x^2}} \, dx$

$$\begin{aligned}\int \frac{x + 1}{\sqrt{1 - x^2}} \, dx &= \int \frac{x}{\sqrt{1 - x^2}} \, dx + \int \frac{1}{\sqrt{1 - x^2}} \, dx \\ &= \int \frac{x}{\sqrt{1 - x^2}} \, dx + \arcsin(x) \\ &= \int -\frac{1}{2} \frac{1}{\sqrt{u}} \, du + \arcsin(x) \quad u = 1 - x^2 \rightarrow du = -2x \, dx \\ &= -\frac{1}{2} \int u^{-\frac{1}{2}} \, du + \arcsin(x) \\ &= -\frac{1}{2} \cdot \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + \arcsin(x) + C \\ &= -\sqrt{u} + \arcsin(x) + C \\ \int \frac{x + 1}{\sqrt{1 - x^2}} \, dx &= -\sqrt{1 - x^2} + \arcsin(x) + C\end{aligned}$$

### Ejercicio 13

Determinar si las siguientes integrales impropias convergen y en tal caso calcularlas.

a)  $\int_0^{+\infty} \frac{1}{\sqrt{s+1}} ds$

Para empezar, la «función» es continua en  $(0, +\infty)$

$$\begin{aligned}\int_0^{+\infty} \frac{1}{\sqrt{s+1}} ds &= \lim_{t \rightarrow +\infty} \int_0^t \frac{1}{\sqrt{s+1}} ds \\&= \lim_{t \rightarrow +\infty} \int_0^t (s+1)^{-\frac{1}{2}} ds \quad u = s+1 \rightarrow du = ds \\&= \lim_{t \rightarrow +\infty} \left( 2\sqrt{s+1} \Big|_0^t \right) \\&= \lim_{t \rightarrow +\infty} (2\sqrt{t+1} - 2\sqrt{1}) \\&= \lim_{t \rightarrow +\infty} (2\sqrt{t+1} - 2) \\&= 2 \underbrace{\lim_{t \rightarrow +\infty} (\sqrt{t+1})}_{\rightarrow +\infty} - 2 \\&= +\infty\end{aligned}$$

Luego la integral diverge.

b)  $\int_0^2 \frac{1}{(1-y)^{\frac{2}{3}}} dy$

Lo primero que tenemos que notar es que la función no es continua en  $(0,2)$ . Es por esto que parto la integral en dos partes.

$$\int_0^2 \frac{1}{(1-y)^{\frac{2}{3}}} dy = \int_0^1 \frac{1}{(1-y)^{\frac{2}{3}}} dy + \int_1^2 \frac{1}{(1-y)^{\frac{2}{3}}} dy$$

Luego veo cada una de las integrales por separado. Si una de las dos diverge eso me es suficiente.

$$\begin{aligned}\int_0^1 \frac{1}{(1-y)^{\frac{2}{3}}} dy &= \lim_{t \rightarrow 1^-} \int_0^t \frac{1}{(1-y)^{\frac{2}{3}}} dy \\&= \lim_{t \rightarrow 1^-} \int_0^t (1-y)^{-\frac{2}{3}} dy \\&= -3\sqrt[3]{(1-y)} \Big|_0^1 \\&= 0 - (-3) = 3\end{aligned}$$

$$\begin{aligned}
\int_1^2 \frac{1}{(1-y)^{\frac{2}{3}}} dy &= \lim_{t \rightarrow 1^+} \int_t^2 (1-y)^{-\frac{2}{3}} dy \\
&= \lim_{t \rightarrow 1^+} \int_t^2 (1-y)^{-\frac{2}{3}} dy \\
&= -3 \sqrt[3]{(1-y)} \Big|_1^2 \\
\int_1^2 \frac{1}{(1-y)^{\frac{2}{3}}} dy &= 3 - 0 = 3
\end{aligned}$$

Finalmente:

$$\begin{aligned}
\int_0^2 \frac{1}{(1-y)^{\frac{2}{3}}} dy &= \int_0^1 \frac{1}{(1-y)^{\frac{2}{3}}} dy + \int_1^2 \frac{1}{(1-y)^{\frac{2}{3}}} dy \\
\int_0^2 \frac{1}{(1-y)^{\frac{2}{3}}} dy &= 3 + 3 \\
\int_0^2 \frac{1}{(1-y)^{\frac{2}{3}}} dy &= 6
\end{aligned}$$

Luego la integral converge.

$$c) \int_{-\infty}^0 x e^{-x^2} dx$$

$$\int_{-\infty}^0 x e^{-x^2} dx = \lim_{t \rightarrow -\infty} \left( \int_t^0 x e^{-x^2} dx \right)$$

$$u = -x^2 \rightarrow du = -2x dx$$

$$= \lim_{t \rightarrow -\infty} \left( -\frac{1}{2} \int_{-t^2}^0 e^u du \right)$$

$$= -\frac{1}{2} \lim_{t \rightarrow -\infty} \left( \int_{-t^2}^0 e^u du \right)$$

$$= -\frac{1}{2} \lim_{t \rightarrow -\infty} (e^u \big|_{-t^2}^0)$$

Deshago la sustitución

$$= -\frac{1}{2} \lim_{t \rightarrow -\infty} (e^{-x^2} \big|_t^0)$$

$$= -\frac{1}{2} \lim_{t \rightarrow -\infty} (e^0 - e^{-t^2})$$

$$= -\frac{1}{2} \lim_{t \rightarrow -\infty} (1 - e^{-t^2})$$

$$= -\frac{1}{2} - \frac{1}{2} \lim_{t \rightarrow -\infty} (e^{-t^2})$$

$$t^2 \rightarrow \infty \Rightarrow -t^2 \rightarrow -\infty \Rightarrow e^{-t^2} \rightarrow 0$$

$$= -\frac{1}{2} - \frac{1}{2} 0$$

$$\int_{-\infty}^0 x e^{-x^2} dx = -\frac{1}{2}$$

Por lo tanto la integral converge.

## Ejercicio 14

Determinar si cada una de las siguientes integrales impropias converge o no.

Nota: Creo que se supone que use los criterios de comparación para integrales impropias.

$$a) \int_4^{+\infty} \frac{1}{\sqrt{s}-1} ds$$

$$\sqrt{s} > \sqrt{s}-1 \Rightarrow \underbrace{\frac{1}{\sqrt{s}} \cdot \frac{1}{\sqrt{s}-1} \cdot \sqrt{s}}_{\text{Verdadero ya que } s \in [4, +\infty) \text{ y compatibilidad del prod. con el orden}} > (\sqrt{s}-1) \frac{1}{\sqrt{s}} \cdot \frac{1}{\sqrt{s}-1} \Rightarrow \frac{1}{\sqrt{s}-1} > \frac{1}{\sqrt{s}}$$

Verdadero ya que.  $s \in [4, +\infty)$  y compatibilidad del prod. con el orden

Ahora basta con ver que si  $\int_4^{+\infty} \frac{1}{\sqrt{s}} ds$  diverge o no.

$$\int_4^{+\infty} \frac{1}{\sqrt{s}} \, ds = \lim_{t \rightarrow +\infty} \int_4^t \frac{1}{\sqrt{s}} \, ds = \lim_{t \rightarrow +\infty} \int_4^t s^{-\frac{1}{2}} \, ds$$

$$\int_4^{+\infty} \frac{1}{\sqrt{s}} \, ds = \lim_{t \rightarrow +\infty} 2\sqrt{s} \Big|_4^t$$

$$\int_4^{+\infty} \frac{1}{\sqrt{s}} \, ds = \lim_{t \rightarrow +\infty} (2\sqrt{t} - 2\sqrt{4})$$

$$\int_4^{+\infty} \frac{1}{\sqrt{s}} \, ds = \lim_{t \rightarrow +\infty} (2\sqrt{t} - 4)$$

$$\int_4^{+\infty} \frac{1}{\sqrt{s}} \, ds = \lim_{t \rightarrow +\infty} \underbrace{(2\sqrt{t})}_{2\sqrt{t} \rightarrow +\infty} - 4$$

$$\int_4^{+\infty} \frac{1}{\sqrt{s}} \, ds = +\infty$$

Luego la integral  $\int_4^{+\infty} \frac{1}{\sqrt{s}} \, ds$  diverge y por ende  $\int_4^{+\infty} \frac{1}{\sqrt{s}-1} \, ds$  tambien.

b)  $\int_0^4 \frac{dx}{(x-3)^{\frac{2}{3}}}$

c)  $\int_0^4 \frac{dx}{x^2 - x - 2}$