Ejercicio 0

Calcular las derivadas de las siguientes funciones:

a) $f(x) = (33-2x)^{\frac{4}{3}}$

$$f'(x) = (33-2x)^{\frac{4}{3}-\frac{3}{3}} \cdot (-2) \cdot \frac{4}{3} = (33-2x)^{\frac{1}{3}} \cdot -\frac{8}{3} = -\frac{8}{3} \cdot \sqrt[3]{(33-2x)}$$

$$f'(x) = -\frac{8\sqrt[3]{(33-2x)}}{3}$$

b) $f(x) = e^{2x}$

$$f'(x) = e^{2x} \cdot \frac{d}{\mathrm{d}x}[2x]$$

$$f'(x) = e^{2x} \cdot 2$$

c) $f(x) = 2^x$

$$f(x) = 2^x \Rightarrow \ln(f(x)) = \ln(2^x) = x\ln(2) \Rightarrow$$

$$\Rightarrow \frac{f(x)'}{f(x)} = \ln(2) \Rightarrow f'(x) = \ln(2) \cdot f(x) = 2^x \cdot \ln(2)$$

$$\Rightarrow f'(x) = 2^x \cdot \ln(2)$$

d) $f(x) = \ln(7-x)$

$$f'(x) = \frac{1}{7-x} \cdot (-1)$$

$$f'(x) = \frac{1}{x-7}$$

e) $f(x) = \ln(x^2+3x+4)$

$$f'(x) = \frac{1}{x^2+3x+4} \cdot (2x+3)$$

$$f'(x) = \frac{2x+3}{x^2+3x+4}$$

f) $f(x) = \ln(e^x + e^{-x})$

$$f'(x) = \frac{1}{e^x + e^{-x}} \cdot (e^x + e^{-x} \cdot (-1))$$

$$f'(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

g) $f(x) = \ln(\cos(x) + \sin(x))$

$$f'(x) = \frac{1}{\cos(x)+\sin(x)} \cdot (-\sin(x) + \cos(x))$$

$$f'(x) = \frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)}$$

h) $f(x) = \frac{\cos(x)}{\sin(x)}$

$$f'(x) = \frac{\frac{d}{\mathrm{d}x}[\cos x] \cdot \sin(x) - \cos(x) \cdot \frac{d}{\mathrm{d}x}[\sin x]}{\sin^2(x)}$$

$$f'(x) = \frac{-\sin(x) \cdot \sin(x) - \cos(x) \cdot \cos(x)}{\sin^2(x)}$$

$$f'(x) = \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)}$$

$$f'(x) = -\frac{\cos^2(x) + \sin^2(x)}{\sin^2(x)}$$

$$f'(x) = -\frac{1}{\sin^2(x)}$$

Ejercicio 1

Dar las primitivas de las siguientes funciones:

a) $g(x) = x^3 - 5x$

$$G(x) = \frac{x^4}{4} - 5\frac{x^2}{2} + c$$

b) $g(x) = e^{0,3x}$

$$G(x) = \frac{e^{0,3x}}{0,3} + c$$

Sino sale por sustitución.

c) $g(x) = \sin(2x)$

$$G(x) = -\frac{\cos(2x)}{2} + c$$

d) $g(x) = 2x\cos(x^2)$

$$\int 2x \cos(2x)\,\mathrm{d}x, \quad \text{Sea } u = x^2, \mathrm{d}u = 2x\,\mathrm{d}x$$

$$\int \cos(u)\,\mathrm{d}u = \sin(u) = \sin(x^2)$$

$$G(x) = \sin(x^2) + c$$

e) $g(x) = x^{\frac{3}{2}}$

$$G(x) = x^{\frac{5}{2}} \cdot \frac{2}{5} + c$$

f) $g(x) = \sqrt{x+2} = (x+2)^{\frac{1}{2}}$

$$G(x) = (x+2)^{\frac{3}{2}} \cdot \frac{2}{3} + c$$

Ejercicio 2

Encontrar la primitiva F de $f(x) = \frac{3}{x}$ tal que $F(1) = 5$

$$\int \frac{3}{x} = 3 \int \frac{1}{x} = 3 \ln|x| + C \Rightarrow F(1) = 3 \ln|1| + C = 3 \cdot 0 + C = C \Rightarrow F(x) = 3 \ln|x| + 5$$

Ejercicio 3

Calcular las siguientes integrales:

a) $\displaystyle\int e^{2x} \, \mathrm{d}x$

$$\begin{aligned}
\int e^{2x} \, \mathrm{d}x &= \frac{1}{2} \int e^{2x} \cdot 2 \, \mathrm{d}x && \text{Elemento neutro de la multiplicación y multiplico por } \frac{1}{1} \\
\int e^{2x} \, \mathrm{d}x &= \frac{1}{2} e^{u} + C && \text{Tomo } u = 2x \to \mathrm{d}u = 2 \, \mathrm{d}x \text{ y resuelvo.} \\
\int e^{2x} \, \mathrm{d}x &= \frac{e^{2x}}{2} + C && \text{Reemplazo } u \text{ por lo que es.}
\end{aligned}$$

b) $\displaystyle\int 2^{x} \, \mathrm{d}x$

Utilizo el ejercicio 0.c. $f(x) = 2^x \Rightarrow f'(x) = 2^x \cdot \ln(2)$.

$$\int 2^x \, \mathrm{d}x = \frac{\ln(2)}{\ln(2)} \int 2^x \, \mathrm{d}x = \frac{1}{\ln(2)} \int 2^x \ln(2) \, \mathrm{d}x = \frac{2^x}{\ln(2)} + C$$

c) $\displaystyle\int \sqrt[3]{(33 - 2x)} \, \mathrm{d}x$

$$\begin{aligned}
\int \sqrt[3]{(33 - 2x)} \, \mathrm{d}x &= \int (33 - 2x)^{\frac{1}{3}} \, \mathrm{d}x && \text{Reescribo utilizando propiedades.} \\
&= -\frac{1}{2} \int (33 - 2x)^{\frac{1}{3}} \cdot (-2) \, \mathrm{d}x && \text{Utilizo el neutro de la multiplicación.} \\
&= -\frac{1}{2} \int u^{\frac{1}{3}} \, \mathrm{d}u = -\frac{1}{2} u^{\frac{4}{3}} \cdot \frac{3}{4} + C && \text{Tomo } \; u = (33 - 2x) \to \mathrm{d}u = -2 \, \mathrm{d}x \\
\int \sqrt[3]{(33 - 2x)} \, \mathrm{d}x &= -\frac{3}{8} (33 - 2x)^{\frac{4}{3}} + C && \text{Reemplazo } u \text{ por lo que es:}
\end{aligned}$$

d) $\displaystyle\int \frac{\mathrm{d}x}{7 - x}$

Es $-\ln(7 - x)$ por ejercicio 0.d. Tambien se puede hacer pos sustitución y da $-\ln(|7 - x|)$

e) $\displaystyle\int \frac{2x+3}{x^2+3x+4}\,\mathrm{d}x$

$$\int \frac{2x+3}{x^2+3x+4}\,\mathrm{d}x = \int \frac{\mathrm{d}u}{u} = \ln|u| + C \quad \text{Tomo } u = (x^2+3x+4) \to \mathrm{d}u = (2x+3)\,\mathrm{d}x$$

$$\int \frac{2x+3}{x^2+3x+4}\,\mathrm{d}x = \ln|x^2+3x+4| + C \qquad \text{Reemplazo } u \text{ por lo que es.}$$

f) $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,\mathrm{d}x$

El resultado es $\ln(e^x + e^{-x}) + C$ por ejercicio 0.f. Nos daría $\ln|e^x + e^{-x}| + C$ usando sustitución.

g) $\displaystyle\int \frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)}\,\mathrm{d}x$

El resultado es $\ln(\cos(x) + \sin(x))$ por ejercicio 0.h. Nos daría $\ln|\cos(x) + \sin(x)|$ por sustitución. Hay que tener en cuenta el dominio de la función.

h) $\displaystyle\int \frac{1}{\sin^2(x)}\,\mathrm{d}x$

El resultado es $-\frac{\cos(x)}{\sin(x)} + C$. Se trata del ejercicio 0.h.

Ejercicio 4

Si realizar el cálculo de la integral, justificar las siguientes igualdades y desigualdades.

a) $\displaystyle\int_{-\pi}^{\pi} \sin(2x)\,\mathrm{d}x = 0$

Esto es un hecho trivial. Ya que $\sin(2x)$ es una función par. Probablemente podemos generalizar:

$$\int_{-a}^{a} \sin(2x)\,\mathrm{d}x = \int_{-a}^{0} \sin(2x)\,\mathrm{d}x + \int_{0}^{a} \sin(2x)\,\mathrm{d}x = 0, \forall a \in \mathbb{R}.$$

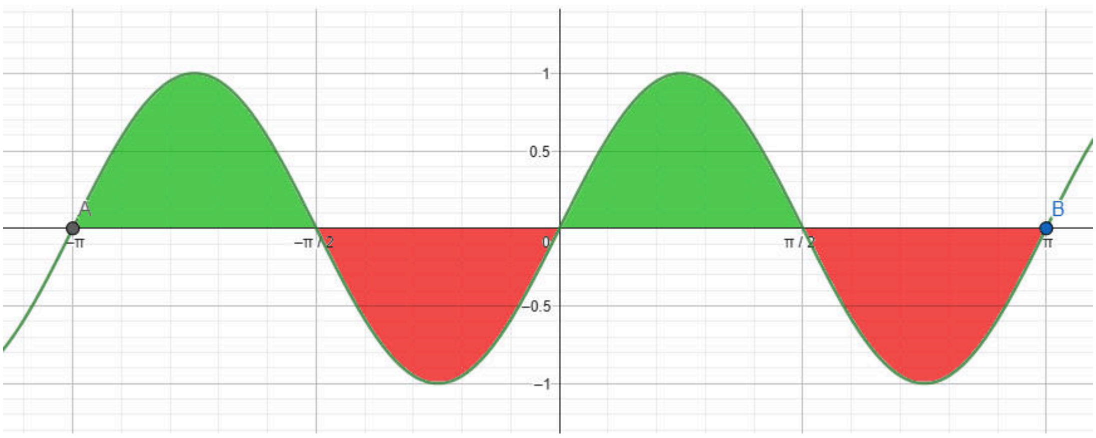

Figura 1: $\sin(2x)$

b) $\frac{\pi}{6} \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin(x)\,\mathrm{d}x \leq \frac{\pi}{3}$

Esto es se puede ver apróximando el area por izquierda y por derecha. Tenemos que si tomo un rectángulo de ancho $\frac{\pi}{3}$, El menor área posible es $\frac{\pi}{6}$. Si tomamos el mayor área posible por derecha nos da $\frac{\pi}{6}$. Luego el valor real del área debe estar entre esos dos números.

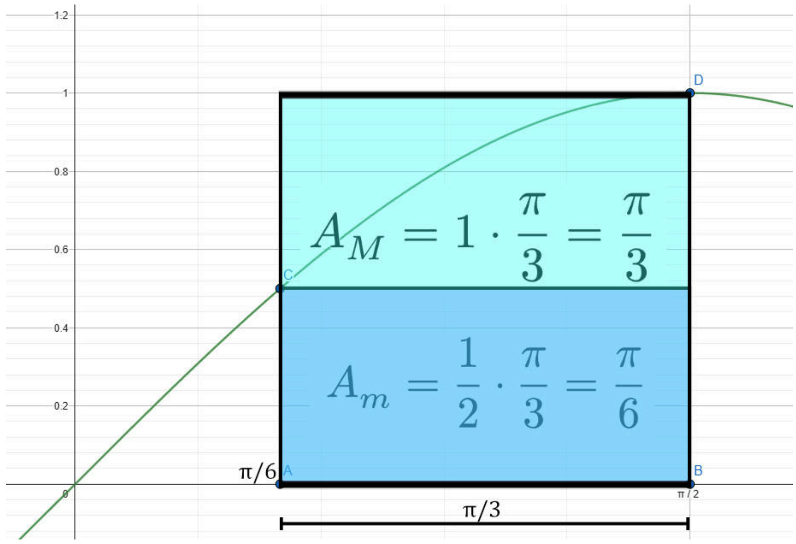

Figura 2: $\sin(x)$

c) $\int_1^2 \sqrt{5-x}\,\mathrm{d}x \geq \int_1^2 \sqrt{x+1}\,\mathrm{d}x$

Basta con ver el gráfico. En ese intervál o, el área mínima de $\sqrt{5-x}$ es igual al área máxima $\sqrt{x+1}$. Luego por una cadena desigualdades tenemos que el resultado es verdadero.

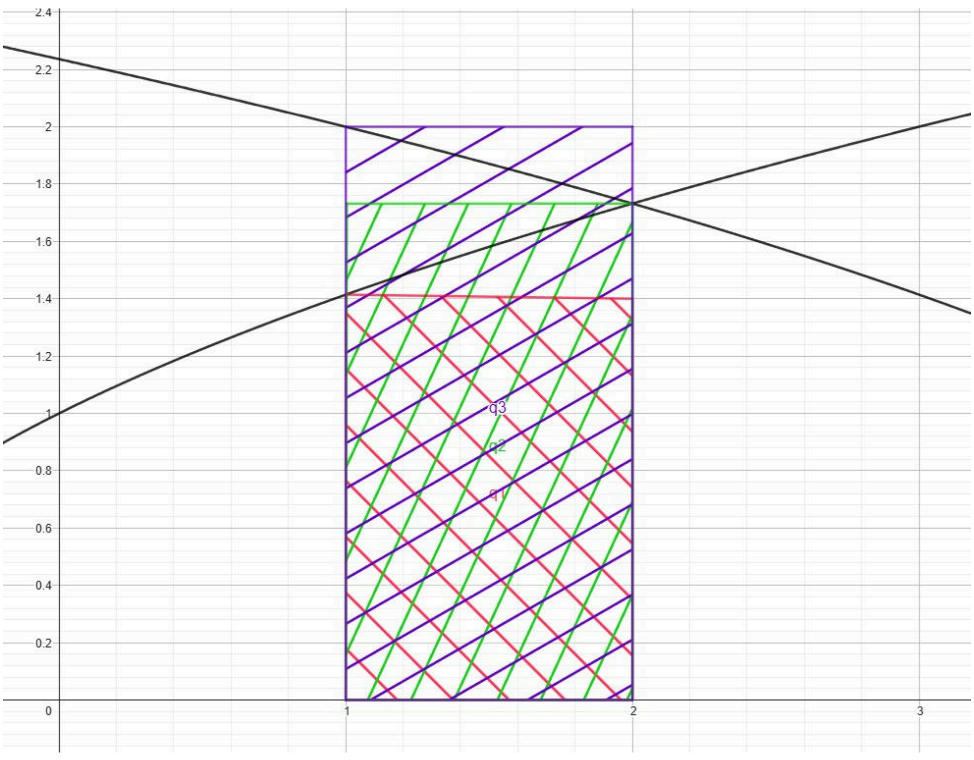

Figura 3: Aproximación del área.

Ejercicio 5

Calcular la derivada de las siguientes funciones donde sea posible:

En estos problemas debemos usar la primera parte del teorema fundamental del cálculo.

a) $f(x) = \displaystyle\int_0^x \frac{\sin(t^2)}{1 + \cos^2(t)}\,\mathrm{d}t$

$$f'(x) = \frac{\sin(x^2)}{1 + \cos^2(x)}$$

b) $f(x) = \displaystyle\int_0^{x^2} \frac{e^{t^2} + 1}{\sqrt{1 - t^2}}\,\mathrm{d}t$

$$f(x) = \int_0^{x^2} \frac{e^{t^2} + 1}{\sqrt{1 - t^2}}\,\mathrm{d}t$$

$$f(x) = \int_0^{u(x)} \frac{e^{t^2} + 1}{\sqrt{1 - t^2}}\,\mathrm{d}t \qquad u(x) = x^2$$

$$f(x) = g(u(x)) \qquad g(x) = \int_0^x \frac{e^{t^2} + 1}{\sqrt{1 - t^2}}$$

$$g(x) = \int_0^x \frac{e^{t^2} + 1}{\sqrt{1 - t^2}} \Longrightarrow g'(x) = \frac{e^{x^2} + 1}{\sqrt{1 - x^2}} \quad \text{Utilizo el TFC para encontrar } g'(x)$$

$$f(x) = g(u(x)) \Longrightarrow f'(x) = g'(u(x)) \cdot u'(x)$$

$$f'(x) = \frac{e^{x^4} + 1}{\sqrt{1 - x^4}} \cdot 2x$$

c) $f(x) = \int_{\sqrt{x}}^{x^3} \frac{t+1}{\sqrt{1+2^t}}\,\mathrm{d}t$

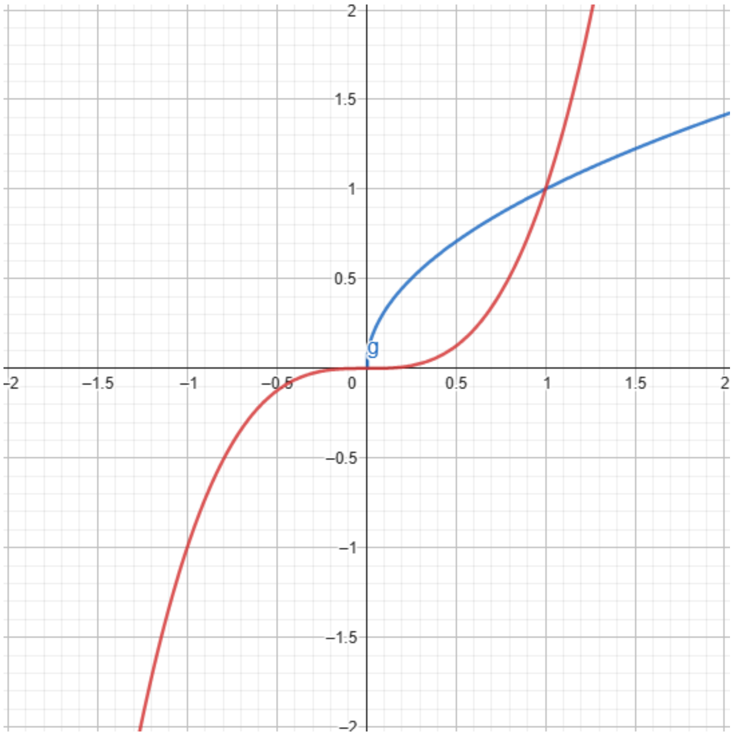

Figura 4: $u(x)$ y $v(x)$.

Sea $g(x) = \sqrt{x}$.

$$u(x) = x^3 \quad u(x) : \mathbb{R} \to \mathbb{R} \wedge v(x) = \sqrt{x} \quad v(x) : [0, \infty) \to \mathbb{R}$$

$$f(x) = \int_a^{x^3} \frac{t+1}{\sqrt{1+2^t}}\,\mathrm{d}t + \int_{\sqrt{x}}^{a} \frac{t+1}{\sqrt{1+2^t}}\,\mathrm{d}t$$

$$f(x) = \int_a^{x^3} \frac{t+1}{\sqrt{1+2^t}}\,\mathrm{d}t - \int_a^{\sqrt{x}} \frac{t+1}{\sqrt{1+2^t}}\,\mathrm{d}t$$

Luego se resuelve como el ejercio anterior utilizando sustituciones.

$$f(x) = \int_a^{u(x)} \frac{t+1}{\sqrt{1+2^t}}\,\mathrm{d}t - \int_a^{v(x)} \frac{t+1}{\sqrt{1+2^t}}\,\mathrm{d}t$$

$$f(x) = g(u(x)) - g(v(x)) \qquad g(x) = \int_a^{x} \frac{t+1}{\sqrt{1+2^t}}\,\mathrm{d}t$$

$$g(x) = \int_a^{x} \frac{t+1}{\sqrt{1+2^t}}\,\mathrm{d}t \Rightarrow g'(x) = \frac{x+1}{\sqrt{1+2^x}}$$

$$f(x) = g(u(x)) - g(v(x)) \Rightarrow f'(x) = g'(u(x)) \cdot u'(x) - g'(v(x)) \cdot v'(x)$$

$$f'(x) = \frac{x^3+1}{\sqrt{1+2^{x^3}}} \cdot 3x^2 - \frac{\sqrt{x}+1}{\sqrt{1+2^{\sqrt{x}}}} \cdot \frac{1}{2\sqrt{x}}$$

Nota: Valido para $\forall a \in (1, \infty)$

Ejercicio 6

Calcular las siguientes integrales usando el Teorema Fundamental del Cálculo:

Para resolver estos ejercicios utilize las primitivas que encontre previamente en el ejercicio 3.

a) $\displaystyle\int_1^2 2^x \,\mathrm{d}x$

$$\int_1^2 2^x \,\mathrm{d}x = \frac{2^2}{\ln(2)} - \frac{2^1}{\ln(2)} = \frac{4}{\ln(2)} - \frac{2}{\ln(2)}$$

$$\int_1^2 2^x \,\mathrm{d}x = \frac{2}{\ln(2)}$$

b) $\displaystyle\int_3^5 \sqrt[3]{(33-2x)}\,\mathrm{d}x$

$$\int_3^5 \sqrt[3]{(33-2x)}\,\mathrm{d}x = \int_3^5 (33-2x)^{\frac{1}{3}}\,\mathrm{d}x \text{ Definición de raíz}$$

Sea $u = 33 - 2x \to \mathrm{d}u = -\,\mathrm{d}x, u(3) = 27 \wedge u(5) = 23$

$$\begin{aligned}
\int_3^5 \sqrt[3]{(33-2x)}\,\mathrm{d}x &= \int_{27}^{23} (u)^{\frac{1}{3}} \frac{\mathrm{d}u}{-2} \\
&= -\frac{1}{2}\int_{27}^{23} (u)^{\frac{1}{3}}\,\mathrm{d}u \\
&= \frac{1}{2}\int_{23}^{27} (u)^{\frac{1}{3}}\,\mathrm{d}u \\
&= \frac{3}{8}27^{\frac{4}{3}} - \frac{3}{8}23^{\frac{4}{3}} \\
&= \frac{3 \cdot 27^{\frac{4}{3}} - 3 \cdot (23)^{\frac{4}{3}}}{8} \\
&= \frac{3 \cdot 3^4 - 3 \cdot (23)^{\frac{4}{3}}}{8} \\
&= \frac{3^5 - 3 \cdot 23^{\frac{4}{3}}}{8}
\end{aligned}$$

$$\int_3^5 \sqrt[3]{(33-2x)}\,\mathrm{d}x = \frac{243 - 3 \cdot 23^{\frac{4}{3}}}{8}$$

c) $\displaystyle\int_1^5 \frac{\mathrm{d}x}{7-x}$

$$\text{Sea } u = 7 - x \to \mathrm{d}u = -\,\mathrm{d}x, u(1) = 6 \wedge u(5) = 2$$

$$\begin{aligned}\int_1^5 \frac{\mathrm{d}x}{7-x} &= \int_6^2 \frac{-\,\mathrm{d}u}{u} \\ &= -\int_6^2 \frac{\mathrm{d}u}{u} \\ &= \int_2^6 \frac{\mathrm{d}u}{u} \\ \int_1^5 \frac{\mathrm{d}x}{7-x} &= \ln(6) - \ln(2)\end{aligned}$$

d) $\displaystyle\int_0^1 \frac{2x+3}{x^2+3x+4}\,\mathrm{d}x$

$$\text{Sea } u = x^2 + 3x + 4 \to \mathrm{d}u = (2x+3)\,\mathrm{d}x, u(0) = 4 \wedge u(1) = 8$$

$$\int_0^1 \frac{2x+3}{x^2+3x+4}\,\mathrm{d}x = \int_4^8 \frac{\mathrm{d}u}{u}$$

$$\int_0^1 \frac{2x+3}{x^2+3x+4}\,\mathrm{d}x = \ln(8) - \ln(4) = \ln\left(\frac{8}{4}\right)$$

$$\int_0^1 \frac{2x+3}{x^2+3x+4}\,\mathrm{d}x = \ln(2)$$

e) $\displaystyle\int_{\ln(2)}^{\ln(3)} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,\mathrm{d}x$

$$\text{Sea } u = e^x + e^{-x} \to \mathrm{d}u = (e^x - e^{-x})\,\mathrm{d}x, u(\ln(2)) = 2 + \frac{1}{2} = \frac{5}{2} \wedge u(3) = 3 + \frac{1}{3} = \frac{10}{3}$$

$$\int_{\ln(2)}^{\ln(3)} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,\mathrm{d}x = \int_{\frac{5}{2}}^{\frac{10}{3}} \frac{\mathrm{d}u}{u}$$

$$\int_{\ln(2)}^{\ln(3)} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,\mathrm{d}x = \ln\left(\frac{10}{3}\right) - \ln\left(\frac{5}{2}\right)$$

$$\int_{\ln(2)}^{\ln(3)} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,\mathrm{d}x = \ln\left(\frac{20}{15}\right) = \ln\left(\frac{4}{3}\right)$$

f) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)}\,\mathrm{d}x$

$$\text{Sea } u = \cos(x)+\sin(x) \to \mathrm{d}u = (\cos(x)-\sin(x))\,\mathrm{d}x, u(0) = 1 \wedge u\left(\frac{\pi}{2}\right) = 1$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)}\,\mathrm{d}x = \int_1^1 \frac{\mathrm{d}u}{u}$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)}\,\mathrm{d}x = 0 \qquad \text{Por propiedad de la integral definida.}$$

Ejercicio 7

Calcular las siguientes integrales:

a) $\displaystyle\int xe^x$

$$\text{Sea } u = e^x \to \mathrm{d}u = e^x, v = x \to \mathrm{d}v = 1$$

$$\int xe^x = e^x x - \int e^x\,\mathrm{d}x$$

$$\int xe^x = e^x x - e^x + C$$

$$\int xe^x = e^x(x-1) + C$$

b) $\displaystyle\int_{-1}^1 (1-2x)e^{-2x}\,\mathrm{d}x$

$$\text{Sea } u = (1-2x) \to \mathrm{d}u = -2\,\mathrm{d}x, v = \frac{e^{-2x}}{-2} \to \mathrm{d}v = e^{-2x}$$

$$\begin{aligned}
\int_{-1}^1 (1-2x)e^{-2x}\,\mathrm{d}x &= (1-2x)\cdot\frac{e^{-2x}}{-2}\Bigg|_{-1}^1 - \int_{-1}^1 \frac{e^{-2x}}{-2}(-2)\,\mathrm{d}x \\
&= (1-2x)\cdot\frac{e^{-2x}}{-2}\Bigg|_{-1}^1 - \int_{-1}^1 e^{-2x}\,\mathrm{d}x \\
&= \frac{(1-2x)\cdot e^{-2x}}{-2}\Bigg|_{-1}^1 - \int_{-1}^1 e^{-2x}\,\mathrm{d}x \\
&= \frac{(1-2)\cdot e^{-2}}{-2} - \frac{(1+2)\cdot e^{2}}{-2} - \int_{-1}^1 e^{-2x}\,\mathrm{d}x \\
&= -\frac{e^{-2}}{-2} - \frac{3\cdot e^{2}}{-2} - \int_{-1}^1 e^{-2x}\,\mathrm{d}x
\end{aligned}$$

$$\int_{-1}^1 (1-2x)e^{-2x}\,\mathrm{d}x = \frac{e^{-2}+3\cdot e^2}{2} - \int_{-1}^1 e^{-2x}\,\mathrm{d}x$$

Sea $u = -2x \to \mathrm{d}u = -2\,\mathrm{d}x \Rightarrow \mathrm{d}x = \frac{\mathrm{d}u}{-2}, u(-1) = 2 \wedge u(1) = -2$

$$
\begin{aligned}
\int_{-1}^{1} (1-2x)e^{-2x}\,\mathrm{d}x &= \frac{e^{-2}+3\cdot e^{2}}{2} - \int_{2}^{-2} e^{u}\frac{\mathrm{d}u}{-2} \\
&= \frac{e^{-2}+3\cdot e^{2}}{2} + \frac{1}{2}\int_{2}^{-2} e^{u}\,\mathrm{d}u \\
&= \frac{e^{-2}+3\cdot e^{2}}{2} - \frac{1}{2}\int_{-2}^{2} e^{u}\,\mathrm{d}u \\
&= \frac{e^{-2}+3\cdot e^{2}}{2} - \frac{1}{2}e^{u}\big|_{-2}^{2} \\
&= \frac{e^{-2}+3\cdot e^{2}}{2} - \frac{1}{2}(e^{2}-e^{-2}) \\
&= \frac{e^{-2}+3\cdot e^{2}}{2} - \frac{e^{2}-e^{-2}}{2} \\
&= \frac{e^{-2}+3\cdot e^{2}-e^{2}+e^{-2}}{2} \\
&= \frac{2\cdot e^{-2}+2\cdot e^{2}}{2}
\end{aligned}
$$

$$\int_{-1}^{1} (1-2x)e^{-2x}\,\mathrm{d}x = e^{-2}+e^{2}$$

c) $\displaystyle\int x^2\cos(x)\,\mathrm{d}x$

$$\text{Sea } u = x^2 \to \mathrm{d}u = 2x\,\mathrm{d}x, v = \sin(x) \to \mathrm{d}v = \cos(x)$$

$$\int x^2\cos(x)\,\mathrm{d}x = x^2\sin(x) - \int 2x\sin(x)\,\mathrm{d}x$$

$$\text{Sea } u = 2x \to \mathrm{d}u = 2\,\mathrm{d}x, v = -\cos(x) \to \mathrm{d}v = \sin(x)$$

$$\int x^2\cos(x)\,\mathrm{d}x = x^2\sin(x) - \left(2x\cdot(-\cos(x)) - \int -2\cos(x)\,\mathrm{d}x\right)$$

$$\int x^2\cos(x)\,\mathrm{d}x = x^2\sin(x) - \left(-2x\cos(x) + 2\int \cos(x)\,\mathrm{d}x\right)$$

$$\int x^2\cos(x)\,\mathrm{d}x = x^2\sin(x) - (-2x\cos(x) + 2\sin(x)) + C$$

$$\int x^2\cos(x)\,\mathrm{d}x = x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C$$

d) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x\,\mathrm{d}x}{\sin^2(x)}$

$$\text{Sea } u = x \to \mathrm{d}u = \mathrm{d}x, v = -\frac{\cos(x)}{\sin(x)} \to \mathrm{d}v = \frac{1}{\sin^2}(x)$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x\,\mathrm{d}x}{\sin^2(x)} = \left(x \cdot -\frac{\cos(x)}{\sin(x)}\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int -\frac{\cos(x)}{\sin(x)}\,\mathrm{d}x$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x\,\mathrm{d}x}{\sin^2(x)} = \left(-\frac{x\cos(x)}{\sin(x)}\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot(x)\,\mathrm{d}x$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x\,\mathrm{d}x}{\sin^2(x)} = -\frac{\frac{\pi}{2}\cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} + \frac{\frac{\pi}{4}\cos(\frac{\pi}{4})}{\sin(\frac{\pi}{4})} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot(x)\,\mathrm{d}x$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x\,\mathrm{d}x}{\sin^2(x)} = -\frac{\frac{\pi}{2}\cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} + \frac{\frac{\pi}{4}\cos(\frac{\pi}{4})}{\sin(\frac{\pi}{4})} + \ln(\sin(x))|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x\,\mathrm{d}x}{\sin^2(x)} = -\frac{\frac{\pi}{2}\cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} + \frac{\frac{\pi}{4}\cos(\frac{\pi}{4})}{\sin(\frac{\pi}{4})} + \ln\left(\sin\left(\frac{\pi}{2}\right)\right) - \ln\left(\sin\left(\frac{\pi}{4}\right)\right)$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x\,\mathrm{d}x}{\sin^2(x)} = -\frac{\frac{\pi}{2}\cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} + \frac{\frac{\pi}{4}\cos(\frac{\pi}{4})}{\sin(\frac{\pi}{4})} + \ln(1) - \ln\left(\frac{\sqrt{2}}{2}\right)$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x\,\mathrm{d}x}{\sin^2(x)} = -\frac{\frac{\pi}{2}\cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} + \frac{\frac{\pi}{4}\cos(\frac{\pi}{4})}{\sin(\frac{\pi}{4})} + \ln\left(\frac{2}{\sqrt{2}}\right)$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x\,\mathrm{d}x}{\sin^2(x)} = -\frac{\frac{\pi}{2}\cdot 0}{1} + \frac{\frac{\pi}{4}\cos(\frac{\pi}{4})}{\sin(\frac{\pi}{4})} + \ln\left(\frac{2}{\sqrt{2}}\right)$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x\,\mathrm{d}x}{\sin^2(x)} = \frac{\frac{\pi}{4}\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} + \ln\left(\frac{2}{\sqrt{2}}\right)$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x\,\mathrm{d}x}{\sin^2(x)} = \frac{\pi}{4} + \ln\left(\frac{2}{\sqrt{2}}\right)$$

e) $\displaystyle\int_3^9 x\ln(x-1)\,\mathrm{d}x$

Sea $u = x \to \mathrm{d}u = \mathrm{d}x, v = (x-1)\ln(x-1) - (x-1) \to \mathrm{d}v = \ln(x-1)$

$$\begin{aligned}
\int_3^9 x\ln(x-1)\,\mathrm{d}x &= x\cdot((x-1)\ln(x-1)-(x-1))|_3^9 - \int_3^9 (x-1)\ln(x-1)-(x-1)\,\mathrm{d}x \\
&= (216\ln(2) - 72 - 6\ln(2) + 6) - \int_3^9 (x-1)\ln(x-1)-(x-1)\,\mathrm{d}x \\
&= 210\ln(2) - 66 - \int_3^9 (x-1)\ln(x-1)-(x-1)\,\mathrm{d}x \\
&= 210\ln(2) - 66 - \int_3^9 (x-1)\ln(x-1)\,\mathrm{d}x + \int_3^9 (x-1)\,\mathrm{d}x \\
&= 210\ln(2) - 66 - \int_3^9 (x-1)\ln(x-1)\,\mathrm{d}x + \int_2^8 u\,\mathrm{d}u \\
&= 210\ln(2) - 66 - \int_3^9 (x-1)\ln(x-1)\,\mathrm{d}x + \left(\frac{u^2}{2}\Big|_2^8\right)
\end{aligned}$$

$$= 210\ln(2) - 66 - \int_3^9 (x-1)\ln(x-1)\,\mathrm{d}x + (32-2)$$

$$= 210\ln(2) - 66 - \int_3^9 (x-1)\ln(x-1)\,\mathrm{d}x + 30$$

$$= 210\ln(2) - 36 - \int_3^9 (x-1)\ln(x-1)\,\mathrm{d}x$$

$$= 210\ln(2) - 36 - \int_2^8 u\ln(u)\,\mathrm{d}u \qquad \text{Tomo } u = x - 1 y\,\mathrm{d}u = \mathrm{d}x$$

Sea $w = \ln(u) \to \mathrm{d}w = \frac{1}{u}\,\mathrm{d}u, \quad v = \frac{u^2}{2} \to \mathrm{d}v = u$

$$= 210\ln(2) - 36 - \left(\frac{u^2}{2}\cdot\ln(u)\Big|_2^8 - \int_2^8 \frac{1}{u}\frac{u^2}{2}\right)$$

$$= 210\ln(2) - 36 - \left(\frac{u^2}{2}\cdot\ln(u)\Big|_2^8 - \int_2^8 \frac{u}{2}\right)$$

$$= 210\ln(2) - 36 - \left(\frac{u^2}{2}\cdot\ln(u)\Big|_2^8 - \frac{1}{2}\int_2^8 u\right)$$

$$= 210\ln(2) - 36 - \left(\frac{u^2}{2}\cdot\ln(u)\Big|_2^8 - \frac{1}{2}\left(\frac{u^2}{2}\Big|_2^8\right)\right)$$

$$= 210\ln(2) - 36 - \left(\frac{u^2}{2}\cdot\ln(u)\Big|_2^8 - \frac{1}{2}(32-2)\right)$$

$$= 210\ln(2) - 36 - \left(\frac{u^2}{2}\cdot\ln(u)\Big|_2^8 - \frac{1}{2}30\right)$$

$$= 210\ln(2) - 36 - \left(\frac{u^2}{2}\cdot\ln(u)\Big|_2^8\right) + 15$$

$$= 210\ln(2) - 36 - (96\ln(2) - 2\ln(2)) + 15$$

$$= 210\ln(2) - 36 - 94\ln(2) + 15$$

$$\int_3^9 x\ln(x-1)\,\mathrm{d}x = 116\ln(2) - 21$$

f) $\int \ln(x^2+1)\,\mathrm{d}x$

$$\text{Sea } u = \ln(x^2+1) \to \mathrm{d}u = \frac{1}{x^2+1}2x, v = x \to \mathrm{d}v = \mathrm{d}x$$

$$\begin{aligned}
\int \ln(x^2+1)\,\mathrm{d}x &= x\ln(x^2+1) - \int \frac{2x}{x^2+1}x\,\mathrm{d}x \\
&= x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1}\,\mathrm{d}x \\
&= x\ln(x^2+1) - 2\int \frac{x^2+1-1}{x^2+1}\,\mathrm{d}x \\
&= x\ln(x^2+1) - 2\left(\int \frac{x^2+1}{x^2+1}\,\mathrm{d}x - \int \frac{1}{x^2+1}\,\mathrm{d}x\right) \\
&= x\ln(x^2+1) - 2\left(\int 1\,\mathrm{d}x - \int \frac{1}{x^2+1}\,\mathrm{d}x\right) \\
&= x\ln(x^2+1) - 2\left(x - \int \frac{1}{x^2+1}\,\mathrm{d}x\right) \\
&= x\ln(x^2+1) - 2x + \int \frac{1}{x^2+1}\,\mathrm{d}x \\
\int \ln(x^2+1)\,\mathrm{d}x &= x\ln(x^2+1) - 2x + \arctan(x) + C
\end{aligned}$$

g) $\int_0^2 x\ln(x^2+4)\,\mathrm{d}x$

$$\begin{aligned}
\int_0^2 x\ln(x^2+4)\,\mathrm{d}x &= \int_4^8 \frac{\ln(u)}{2}\,\mathrm{d}u \quad \mid u = x^2+4 \to \mathrm{d}u = 2x\,\mathrm{d}x \Rightarrow x\,\mathrm{d}x = \frac{\mathrm{d}u}{2}, u(0) = 4 \wedge u(2) = 8 \\
&= \frac{1}{2}\int_4^8 \ln(u)\,\mathrm{d}u \\
&= \frac{1}{2}(u\ln(u) - u)|_4^8 \\
&= \frac{1}{2}(8\ln(8) - 8 - 4\ln(4) + 4) \\
&= \frac{1}{2}(8\cdot 3\ln(2) - 2\cdot 2\ln(2) - 4) \\
&= \frac{1}{2}(24\ln(2) - 4\ln(2) - 4) \\
&= \frac{1}{2}(20\ln(2) - 4) \\
&= 10\ln(2) - 2
\end{aligned}$$

h) $\int e^{-x} \sin(2x) \,\mathrm{d}x$

$$
\begin{aligned}
&\text{Sea } u = \sin(2x) \to \mathrm{d}u = 2\cos(2x)\,\mathrm{d}x, v = -(e^{-x}) \to \mathrm{d}v = e^{-x} \\
\int e^{-x} \sin(2x)\,\mathrm{d}x &= \sin(2x) \cdot (-(e^{-x})) + \int 2\cos(2x)e^{-x}\,\mathrm{d}x \\
&= -\sin(2x) \cdot e^{-x} + 2\int \cos(2x)e^{-x}\,\mathrm{d}x \\
&\text{Sea } u = \cos(2x) \to \mathrm{d}u = -2\sin(2x)\,\mathrm{d}x, v = -(e^{-x}) \to \mathrm{d}v = e^{-x} \\
&= -\sin(2x) \cdot e^{-x} + 2\left(-\cos(2x) \cdot e^{-x} - \int -2\sin(2x)(-(e^{-x}))\right)\mathrm{d}x \\
&= -\sin(2x) \cdot e^{-x} + 2\left(-\cos(2x) \cdot e^{-x} - \int 2\sin(2x)e^{-x}\,\mathrm{d}x\right) \\
&= -\sin(2x) \cdot e^{-x} - 2\cos(2x) \cdot e^{-x} - 2\int 2\sin(2x)e^{-x}\,\mathrm{d}x) \\
\int e^{-x} \sin(2x)\,\mathrm{d}x &= -\sin(2x) \cdot e^{-x} - 2\cos(2x) \cdot e^{-x} - 4\int \sin(2x)e^{-x}\,\mathrm{d}x \Rightarrow \\
\Rightarrow \int e^{-x} \sin(2x)\,\mathrm{d}x + 4\int \sin(2x)e^{-x})\,\mathrm{d}x &= -\sin(2x) \cdot e^{-x} - 2\cos(2x) \cdot e^{-x} \Rightarrow \\
\Rightarrow 5\int \sin(2x)e^{-x})\,\mathrm{d}x &= -\sin(2x) \cdot e^{-x} - 2\cos(2x) \cdot e^{-x} \Rightarrow \\
\int \sin(2x)e^{-x})\,\mathrm{d}x &= \frac{-\sin(2x) \cdot e^{-x} - 2\cos(2x) \cdot e^{-x}}{5}
\end{aligned}
$$

i) $\int_0^{2\pi} \cos^4(x)\,\mathrm{d}x$

???

Ejercicio 8

Calcular las siguientes integrales:

a) $\displaystyle\int_0^1 e^{\sqrt{x}}\,\mathrm{d}x$

$$u = \sqrt{x} \Rightarrow \mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x \Rightarrow \mathrm{d}x = 2\sqrt{x}\,\mathrm{d}u = 2u\,\mathrm{d}u$$

$$\begin{aligned}
\int_0^1 e^{\sqrt{x}}\,\mathrm{d}x &= \int_0^1 e^u 2u\,\mathrm{d}u \\
&= 2\int_0^1 e^u u\,\mathrm{d}u \\
\text{Sea } f = u \to f' = 1 \text{ , } g' = e^u \to g = e^u & \\
&= 2\left(u \cdot e^u\Big|_0^1 - \int_0^1 e^u\,\mathrm{d}u\right) \\
&= 2\left(u \cdot e^u\Big|_0^1 - e^u\Big|_0^1\right) \\
&= 2\left(u \cdot e^u\Big|_0^1 - (e^1 - e^0)\right) = 2\left(u \cdot e^u\Big|_0^1 - (e-1)\right) \\
&= 2\left(u \cdot e^u\Big|_0^1\right) - 2e + 2 \\
&= 2(1 \cdot e^1 - 0 \cdot e^0) - 2e + 2 \\
&= 2e - 2e + 2 \\
\int_0^1 e^{\sqrt{x}}\,\mathrm{d}x &= 2
\end{aligned}$$

b) $\displaystyle\int \sin(\sqrt{x})\,\mathrm{d}x$

$$u = \sqrt{x} \to \mathrm{d}u = \frac{\mathrm{d}x}{2\sqrt{x}} \to \mathrm{d}x = 2\sqrt{x}\,\mathrm{d}u = 2u\,\mathrm{d}u$$

$$\int \sin(\sqrt{x})\,\mathrm{d}x = \int \sin(u) 2u\,\mathrm{d}u = 2\int \sin(u) u\,\mathrm{d}u$$

$$\text{Sea } f = u \to f' = 1, g' = \sin(u) \to g = -\cos(u)$$

$$\begin{aligned}
\int \sin(\sqrt{x})\,\mathrm{d}x &= 2\left(-u\cos(u) - \int -\cos(u)\,\mathrm{d}u\right) \\
&= 2\left(-u\cos(u) + \int \cos(u)\,\mathrm{d}u\right) \\
&= 2(-u\cos(u) + \sin(u)) + C \\
\int \sin(\sqrt{x})\,\mathrm{d}x &= -2\sqrt{x}\cos(\sqrt{x}) + 2\sin(\sqrt{x}) + C
\end{aligned}$$

c) $\displaystyle\int_0^1 (2x+1)\ln(x+1)\,\mathrm{d}x$

$$f = \ln(x+1) \to f' = \frac{1}{x+1}\,\mathrm{d}x, g' = 2x+1 \to g = x^2+x$$

$$\begin{aligned}
\int_0^1 (2x+1)\ln(x+1)\,\mathrm{d}x = &\ \ln(x+1)(x^2+x)\big|_0^1 - \int_0^1 \frac{(x^2+x)\,\mathrm{d}x}{x+1} \\
= &\ \ln(x+1)(x^2+x)\big|_0^1 - \int_0^1 \frac{x(x+1)\,\mathrm{d}x}{x+1} \\
= &\ \ln(x+1)(x^2+x)\big|_0^1 - \int_0^1 x\,\mathrm{d}x \\
= &\ln(2) - \int_0^1 x\,\mathrm{d}x \\
= &\ln(2) - \left(\frac{x^2}{2}\right)\Bigg|_0^1 \\
= &\,2\ln(2) - \frac{1}{2}
\end{aligned}$$

d) $\displaystyle\int \frac{1}{x\ln(x)}\,\mathrm{d}x$

Resuelvo por sustitución.

$$u = \ln(x) \to \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$$

$$\begin{aligned}
\int \frac{1}{x\ln(x)}\,\mathrm{d}x &= \int \frac{\mathrm{d}u}{u} \\
\int \frac{1}{x\ln(x)}\,\mathrm{d}x &= \ln(|u|) + C \\
\int \frac{1}{x\ln(x)}\,\mathrm{d}x &= \ln(|\ln(x)|) + C
\end{aligned}$$

e) $\displaystyle\int_0^1 \arccos(x)\,\mathrm{d}x$

$$f = \arccos(x) \to f' = -\frac{1}{\sqrt{1-x^2}}, g' = 1 \to g = x$$

$$\int_0^1 \arccos(x)\,\mathrm{d}x = x \cdot \arccos(x)|_0^1 - \int_0^1 -\frac{x}{\left(\sqrt{1-x^2}\right)}\,\mathrm{d}x$$

$$\int_0^1 \arccos(x)\,\mathrm{d}x = \arccos(1) - \int_0^1 -\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$$

$$u = 1 - x^2 \to \mathrm{d}u = -2x\,\mathrm{d}x \to -x\,\mathrm{d}x = \frac{\mathrm{d}u}{2}$$

$$\int_0^1 \arccos(x)\,\mathrm{d}x = 0 - \int_1^0 \frac{\mathrm{d}u}{2\sqrt{u}}$$

$$= \frac{1}{2}\int_0^1 \frac{\mathrm{d}u}{\sqrt{u}}$$

$$= \frac{1}{2}\left(2\sqrt{u}|_0^1\right)$$

$$= \frac{1}{2} \cdot 2$$

$$\int_0^1 \arccos(x)\,\mathrm{d}x = 1$$

f) $\int_0^1 x^3 e^{x^2}\,\mathrm{d}x$

Empiezo resolviendo por sustitución.

$$u = x^2 \to \mathrm{d}u = 2x\,\mathrm{d}x \to \frac{\mathrm{d}u}{2} = x\,\mathrm{d}x$$

$$\int_0^1 x^3 e^{x^2}\,\mathrm{d}x = \int_0^1 x^2 \cdot x \cdot e^{x^2}\,\mathrm{d}x = \int_0^1 u \cdot e^u \cdot \frac{\mathrm{d}u}{2}$$

$$\int_0^1 x^3 e^{x^2}\,\mathrm{d}x = \frac{1}{2}\int_0^1 u \cdot e^u \cdot \mathrm{d}u$$

$$f = u \to f' = 1, g' = e^u \to g = e^u$$

$$\int_0^1 x^3 e^{x^2}\,\mathrm{d}x = \frac{1}{2}\left(u \cdot e^u|_0^1 - \int_0^1 e^u\,\mathrm{d}u\right)$$

$$= \frac{1}{2}\left(u \cdot e^u|_0^1 - e^u|_0^1\right)$$

$$= \frac{1}{2}(e - e + 1)$$

$$\int_0^1 x^3 e^{x^2}\,\mathrm{d}x = \frac{1}{2}$$

g) $\displaystyle\int e^x(1-e^x)^{-1}\,\mathrm{d}x$

$$u = 1 - e^x \to \mathrm{d}u = -e^x\,\mathrm{d}x$$

$$\begin{aligned}
\int e^x(1-e^x)^{-1}\,\mathrm{d}x &= \int \frac{e^x}{1-e^x}\,\mathrm{d}x = \int \frac{-\mathrm{d}u}{u} \\
\int e^x(1-e^x)^{-1}\,\mathrm{d}x &= -\int \frac{\mathrm{d}u}{u} \\
\int e^x(1-e^x)^{-1}\,\mathrm{d}x &= -\ln(|1-e^x|) + C
\end{aligned}$$

h) $\displaystyle\int \frac{\mathrm{d}x}{x\sqrt{x-1}}$

$$u = \sqrt{x-1} \to \mathrm{d}u = \frac{1}{2\sqrt{x-1}}\,\mathrm{d}x \to \mathrm{d}x = 2\sqrt{x-1}\,\mathrm{d}u = 2u\,\mathrm{d}u$$

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x\sqrt{x-1}} &= \int \frac{2u\,\mathrm{d}u}{(u^2+1)\cdot u} = \int \frac{2\,\mathrm{d}u}{u^2+1} \\
&= 2\int \frac{\mathrm{d}u}{u^2+1} \\
&= 2\arctan(u) \\
\int \frac{\mathrm{d}x}{x\sqrt{x-1}} &= 2\arctan\left(\sqrt{x-1}\right)
\end{aligned}$$

i) $\displaystyle\int \sin(x)^3\,\mathrm{d}x$

$$\begin{aligned}
\int \sin(x)^3\,\mathrm{d}x &= \int \sin(x)\cdot\sin(x)^2\,\mathrm{d}x \\
&= \int \sin(x)\cdot\left(1-\cos(x)^2\right)\mathrm{d}x
\end{aligned}$$

$$u = \cos(x) \to \mathrm{d}u = -\sin(x)\,\mathrm{d}x$$

$$\begin{aligned}
&= \int -(1-u^2)\,\mathrm{d}u = \int u^2 - 1\,\mathrm{d}u \\
&= \int u^2\,\mathrm{d}u - \int 1\,\mathrm{d}u \\
&= \frac{u^3}{3} - u + C \\
\int \sin(x)^3\,\mathrm{d}x &= \frac{\cos(x)^3}{3} - \cos(x) + C
\end{aligned}$$

Ejercicio 9

Trazar la región limitada por las curvas dadas y calcular su área:

a) $y = 4x^2, \quad y = x^2 + 3$

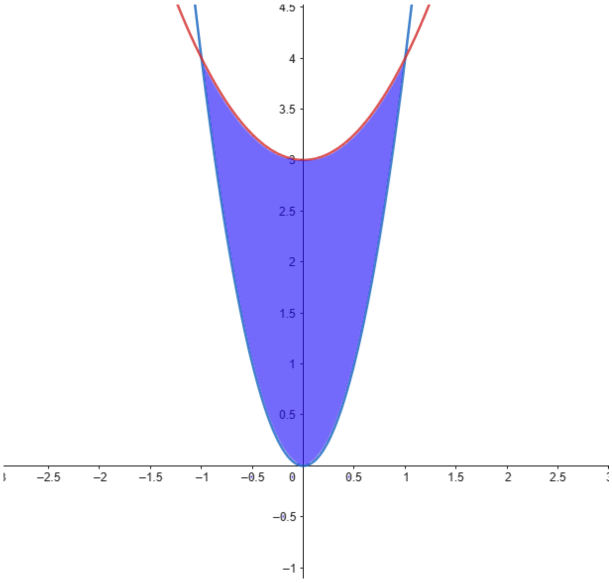

Figura 5: Reprentación del área.

$$4x^2 = x^2 + 3 \to 0 = 3x^2 - 3 = 3(x^2 - 1)$$

Luego sabemos que las curvas se intersectan en $x = 1$ y $x = -1$.

$$\begin{aligned} \int_{-1}^{1} x^2 + 3 \, \mathrm{d}x - \int_{-1}^{1} 4x^2 \, \mathrm{d}x &= \int_{-1}^{1} x^2 - 4x^2 + 3 \\ &= \int_{-1}^{1} -3x^2 + 3 \\ &= -x^3 + 3x\Big|_{-1}^{1} \\ &= 2 - (-2) \end{aligned}$$

$$\int_{-1}^{1} x^2 + 3 \, \mathrm{d}x - \int_{-1}^{1} 4x^2 \, \mathrm{d}x = 4$$

b) $y = \cos(x), \quad y = \sin(x), \quad x = 0, \quad x = \dfrac{\pi}{2}$

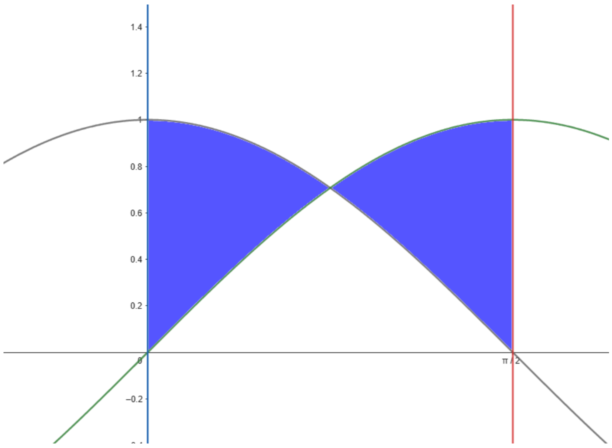

Figura 6: Reprentación del área.

Sabemos que ambas curvas se intersectan en $\frac{\pi}{4}$

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \cos(x) - \sin(x) &= \sin(x) + \cos(x)\Big|_0^{\frac{\pi}{4}} \\
&= \left(\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\right) - (\sin(0) + \cos(0)) \\
&= \sqrt{2} - 1
\end{aligned}$$

$$\begin{aligned}
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x) - \cos(x) &= -\cos(x) - \sin(x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \left(-\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right) - \left(-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right) \\
&= -\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)) \\
&= 0 - 1 + \sqrt{2} \\
&= \sqrt{2} - 1
\end{aligned}$$

Luego el área es $2\sqrt{2} - 2$.

c) $y = |x|, \quad y = (x+1)^2 - 7, \quad x = -4$

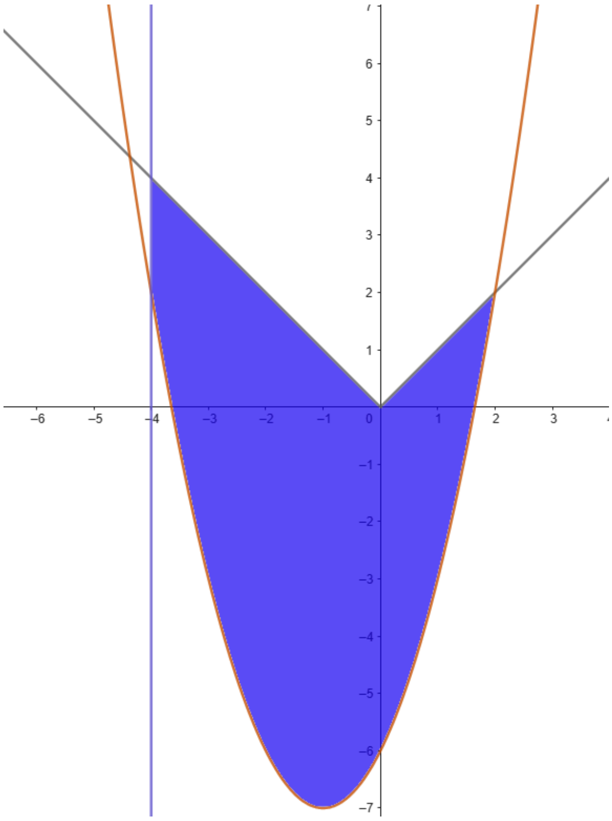

Figura 7: Reprentación del área.

$$
\begin{aligned}
\int_{-4}^{0} -x - ((x+1)^2 - 7) &= \int_{-4}^{0} -x - (x^2 + 2x - 6) \\
&= \int_{-4}^{0} -x - x^2 - 2x + 6 \\
&= \int_{-4}^{0} -x^2 - 3x + 6 \\
&= \left. -\frac{x^3}{3} - \frac{3x^2}{2} + 6x \right|_{-4}^{0} \\
&= -\left(-\frac{(-4)^3}{3} - \frac{3(-4)^2}{2} + 6(-4) \right) \\
&= -\left(\frac{64}{3} - 24 - 24 \right) \\
&= -\left(\frac{64}{3} - 48 \right) \\
&= \frac{144 - 64}{3} \\
\int_{-4}^{0} -x - ((x+1)^2 - 7) &= \frac{80}{3}
\end{aligned}
$$

$$
\begin{aligned}
\int_{0}^{2} x - ((x+1)^2 - 7) &= \int_{0}^{2} x - (x^2 + 2x - 6) \\
&= \int_{0}^{2} -x^2 - x + 6 \\
&= \left. -\frac{x^3}{3} - \frac{x^2}{2} + 6x \right|_{0}^{2} \\
&= -\frac{2^3}{3} - \frac{2^2}{2} + 6 \cdot 2 \\
&= -\frac{8}{3} - 2 + 12 \\
&= -\frac{8}{3} + 10 \\
&= -\frac{8}{3} + \frac{30}{3} \\
&= \frac{22}{3}
\end{aligned}
$$

Luego el área total es la suma de $\frac{22}{3} + \frac{80}{3}$, es decir $\frac{102}{3}$.

Ejercicio 10

Calcular las siguientes integrales:

a) $\displaystyle\int_2^4 \frac{x^2+4x+24}{x^2-4x+8}\,\mathrm{d}x$

Primero hago una división ya que el númerador y el denominador son del mismo grado.

$$\begin{aligned}
\int_2^4 \frac{x^2+4x+24}{x^2-4x+8}\,\mathrm{d}x &= \int_2^4 1+\frac{8x+16}{x^2-4x+8}\,\mathrm{d}x \\
&= \int_2^4 1\,\mathrm{d}x + \int_2^4 \frac{8x+16}{x^2-4x+8}\,\mathrm{d}x \\
&= \int_2^4 1\,\mathrm{d}x + \int_2^4 4\frac{2x-4}{x^2-4x+8}\,\mathrm{d}x + \int_2^4 32\frac{1}{x^2-4x+8}\,\mathrm{d}x \\
&= \int_2^4 1\,\mathrm{d}x + 4\int_2^4 \frac{2x-4}{x^2-4x+8}\,\mathrm{d}x + 32\int_2^4 \frac{1}{x^2-4x+8}\,\mathrm{d}x \\
&= \int_2^4 1\,\mathrm{d}x + 4\int_4^8 \frac{\mathrm{d}u}{u} + 32\int_2^4 \frac{1}{x^2-4x+8}\,\mathrm{d}x \\
&= \int_2^4 1\,\mathrm{d}x + 4\int_4^8 \frac{\mathrm{d}u}{u} + 32\int_2^4 \frac{1}{(x-2)^2+4}\,\mathrm{d}x \\
&= \int_2^4 1\,\mathrm{d}x + 4\int_4^8 \frac{\mathrm{d}u}{u} + 32\int_0^2 \frac{1}{u^2+2^2}\,\mathrm{d}x \\
&= x|_2^4 + 4(\ln(u)|_4^8) + 32\left(\frac{1}{2}\arctan\left(\frac{u}{2}\right)\Big|_0^2\right) \\
&= (4-2) + 4(\ln(8)-\ln(4)|) + 32\left(\frac{1}{2}\arctan(1) - \frac{1}{2}\arctan(0)\right) \\
&= 2 + 4\ln(2) + 16(\arctan(1)-\arctan(0)) \\
&= 2 + 4\ln(2) + 16\left(\frac{\pi}{4}-0\right) \\
&= 2 + 4\ln(2) + 4\pi
\end{aligned}$$

$$\int_2^4 \frac{x^2+4x+24}{x^2-4x+8}\,\mathrm{d}x = 2+4\ln(2)+4\pi$$

b) $\int_0^2 \frac{x-1}{x^2+4}\,\mathrm{d}x$

$$\begin{aligned}
\int_0^2 \frac{x-1}{x^2+4}\,\mathrm{d}x &= \int_0^2 \frac{1}{2}\frac{2x}{x^2+4} - \frac{1}{x^2+4}\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^2 \frac{2x}{x^2+4}\,\mathrm{d}x - \int_0^2 \frac{1}{x^2+4}\,\mathrm{d}x \\
&= \frac{1}{2}\int_4^8 \frac{\mathrm{d}u}{u} - \int_0^2 \frac{1}{x^2+2^2}\,\mathrm{d}x \\
&= \frac{1}{2}(\ln(u)|_4^8) - \left(\frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big|_0^2\right) \\
&= \frac{1}{2}(\ln(8)-\ln(4)) - \left(\frac{1}{2}\arctan(1) - \frac{1}{2}\arctan(0)\right) \\
&= \frac{1}{2}\ln(2) - \left(\frac{1}{2}\frac{\pi}{4} - \frac{1}{2}0\right) \\
&= \frac{1}{2}\ln(2) - \frac{\pi}{8}
\end{aligned}$$

c) $\int_2^4 \frac{x}{x^3-3x+2}\,\mathrm{d}x$

Necesito expresar (x^3-3x+2) como polinomios de producto de grado 1 y 2. A ojo veo que 1 es una raíz, por lo tanto divido (x^3-3x+2) entre $(x-1)$ para encontrar otra forma de expresarlo.

$$\int_2^4 \frac{x}{(x^2+x-2)(x-1)}\,\mathrm{d}x$$

Sin embargo (x^2+x-2) tiene raíces reales asi que todavía no terminamos.

$$\int_2^4 \frac{x}{(x-1)(x-1)(x+2)}\,\mathrm{d}x = \int_2^4 \frac{x}{(x-1)^2(x+2)}\,\mathrm{d}x$$

Finalmente llegamos al caso 3.

$$\begin{aligned}
\int_2^4 \frac{x}{(x-1)^2(x+2)}\,\mathrm{d}x &= \int_2^4 \frac{A_1}{x+2} + \frac{A_2}{x-1} + \frac{A_3}{(x-1)^2}\,\mathrm{d}x \\
&= \int_2^4 \frac{A_1(x-1)^2 + A_2(x-1)(x+2) + A_3(x+2)}{(x+2)(x-1)^2}\,\mathrm{d}x \\
&= \int_2^4 \frac{A_1(x^2-2x+1) + A_2(x^2+x-2) + A_3(x+2)}{(x+2)(x-1)^2}\,\mathrm{d}x \\
&= \int_2^4 \frac{A_1x^2 - A_1\cdot 2x + A_1 + A_2x^2 + A_2x - A_2\cdot 2 + A_3x + A_3\cdot 2}{(x+2)(x-1)^2}\,\mathrm{d}x \\
&= \int_2^4 \frac{x^2(A_1+A_2) + x(-2A_1+A_2+A_3) + A_1 - 2A_2 + 2A_3}{(x+2)(x-1)^2}\,\mathrm{d}x
\end{aligned}$$

$$0 = A_1 + A_2 \to A_1 = -A_2$$
$$1 = -2A_1 + A_2 + A_3 = -2(-A_2) + A_2 + A_3 = 3A_2 + A_3 \to A_3 = 1 - 3A_2$$
$$0 = A_1 - 2A_2 + 2A_3 = -A_2 - 2A_2 + 2(1 - 3A_2) = -3A_2 + 2 - 6A_2 = 2 - 9A_2 \to A_2 = \frac{2}{9}$$

$$A_1 = -\frac{2}{9}$$

$$A_2 = \frac{2}{9}$$

$$A_3 = \frac{1}{3}$$

$$\begin{aligned}
\int_2^4 \frac{x}{(x-1)^2(x+2)}\,\mathrm{d}x &= \int_2^4 \frac{-\frac{2}{9}}{x+2} + \frac{\frac{2}{9}}{x-1} + \frac{\frac{1}{3}}{(x-1)^2}\,\mathrm{d}x \\
&= \int_2^4 \frac{-\frac{2}{9}}{x+2}\,\mathrm{d}x + \int_2^4 \frac{\frac{2}{9}}{x-1}\,\mathrm{d}x + \int_2^4 \frac{\frac{1}{3}}{(x-1)^2}\,\mathrm{d}x \\
&= -\frac{2}{9}\int_2^4 \frac{1}{x+2}\,\mathrm{d}x + \frac{2}{9}\int_2^4 \frac{1}{x-1}\,\mathrm{d}x + \frac{1}{3}\int_2^4 \frac{1}{(x-1)^2}\,\mathrm{d}x \\
&= -\frac{2}{9}\int_4^6 \frac{1}{u}\,\mathrm{d}u + \frac{2}{9}\int_1^3 \frac{1}{u}\,\mathrm{d}u + \frac{1}{3}\int_1^3 \frac{1}{u^2}\,\mathrm{d}u \\
&= -\frac{2}{9}\left(\ln(u)\big|_4^6\right) + \frac{2}{9}\left(\ln(u)\big|_1^3\right) + \frac{1}{3}\left(-\frac{1}{u}\Big|_1^3\right) \\
&= -\frac{2}{9}\ln(6) + \frac{2}{9}\ln(4) + \frac{2}{9}\ln(3) + \frac{1}{3}\cdot\frac{2}{3} \\
&= -\frac{2}{9}\ln(6) + \frac{2}{9}\ln(4) + \frac{2}{9}\ln(3) + \frac{2}{9} \\
&= \frac{2}{9}\ln\left(\frac{2}{3}\right) + \frac{2}{9}\ln(3) + \frac{2}{9} \\
&= \frac{2}{9}\ln(2) + \frac{2}{9}
\end{aligned}$$

Ejercicio 11

La sustitución $t = \tan(\frac{x}{2})$, o equivalentemente, $x = 2\arctan(t)$, transforma cualquier integral que involucre sólo senos y cosenos vinculados por suma, producto o cociente, en la integral de una función racional. Verificar que con esta sustitución resulta:

$$\cos(x) = \frac{1-t^2}{1+t^2}, \quad \sin(x) = \frac{2t}{1+t^2} \quad y \quad \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t.$$

Utilizar esta sutitución en los siguientes casos:

a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{2}{1+\cos(x)}\,\mathrm{d}x$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{2}{1+\cos(x)}\,\mathrm{d}x &= \int_0^1 \frac{2}{1+\left(\frac{1-t^2}{1+t^2}\right)}\frac{2}{1+t^2}\,\mathrm{d}t \quad t=\frac{\tan(x)}{2}\\
&= \int_0^1 \frac{2}{\frac{1+t^2+1-t^2}{1+t^2}}\frac{2}{1+t^2}\,\mathrm{d}t\\
&= \int_0^1 \frac{2}{\frac{2}{1+t^2}}\frac{2}{1+t^2}\,\mathrm{d}t\\
&= \int_0^1 \frac{4}{2}\,\mathrm{d}t\\
&= \int_0^1 2\,\mathrm{d}t\\
&= 2t|_0^1\\
\int_0^{\frac{\pi}{2}} \frac{2}{1+\cos(x)}\,\mathrm{d}x &= 2
\end{aligned}$$

b) $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin(x)}\,\mathrm{d}x$

$$\begin{aligned}
\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin(x)}\,\mathrm{d}x &= \int_{\frac{\sqrt{3}}{3}}^1 \frac{1+t^2}{2t}\cdot\frac{2}{1+t^2}\,\mathrm{d}t \qquad t=\frac{\tan(x)}{2}\\
&= \int_{\frac{\sqrt{3}}{3}}^1 \frac{1}{t}\,\mathrm{d}t\\
&= \ln(t)|_{\frac{\sqrt{3}}{3}}^1\\
&= \ln(1)-\left(\ln\left(\frac{\sqrt{3}}{3}\right)\right) = \ln(1)-\left(\ln\left(\sqrt{3}\right)-\ln(3)\right)\\
\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin(x)}\,\mathrm{d}x &= \ln\left(\sqrt{3}\right)+\ln(3)
\end{aligned}$$

Ejercicio 12

Calcular las siguientes integrales:

a) $\displaystyle\int \tan(x)^2 \,\mathrm{d}x$

$$\begin{aligned}
\int \tan(x)^2 \,\mathrm{d}x &= \int \frac{\sin(x)^2}{\cos(x)^2} \,\mathrm{d}x = \int \frac{1-\cos(x)^2}{\cos(x)^2} \,\mathrm{d}x \\
\int \tan(x)^2 \,\mathrm{d}x &= \int \frac{1}{\cos(x)^2} \,\mathrm{d}x - \int 1 \,\mathrm{d}x \\
\int \tan(x)^2 \,\mathrm{d}x &= \tan(x) - x + C
\end{aligned}$$

b) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{9-4x^2}}$

Sale por sustitución:

$$9u^2 = 4x^2 \to u^2 = \frac{4}{9}x^2 \to u = \sqrt{\frac{4}{9}x^2} = \frac{2}{3}x \to \mathrm{d}u = \frac{2}{3}\,\mathrm{d}x$$

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{9-4x^2}} &= \int \frac{3}{2\sqrt{9-9u^2}} \,\mathrm{d}u = \int \frac{3}{2\cdot 3\sqrt{1-u^2}} \,\mathrm{d}u \\
&= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}} \,\mathrm{d}u \\
&= \frac{1}{2}\arccos(u) \\
&= \frac{1}{2}\arccos\left(\frac{2}{3}x\right) + C
\end{aligned}$$

b) $\displaystyle\int \frac{x+1}{\sqrt{1-x^2}} \,\mathrm{d}x$

$$\begin{aligned}
\int \frac{x+1}{\sqrt{1-x^2}} \,\mathrm{d}x &= \int \frac{x}{\sqrt{1-x^2}} \,\mathrm{d}x + \int \frac{1}{\sqrt{1-x^2}} \,\mathrm{d}x \\
&= \int \frac{x}{\sqrt{1-x^2}} \,\mathrm{d}x + \arcsin(x) \\
&= \int -\frac{1}{2}\frac{1}{\sqrt{u}} \,\mathrm{d}u + \arcsin(x) \qquad\qquad u = 1-x^2 \to \mathrm{d}u = -2x\,\mathrm{d}x \\
&= -\frac{1}{2}\int u^{-\frac{1}{2}} \,\mathrm{d}u + \arcsin(x) \\
&= -\frac{1}{2}\cdot\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + \arcsin(x) + C \\
&= -\sqrt{u} + \arcsin(x) + C \\
\int \frac{x+1}{\sqrt{1-x^2}} \,\mathrm{d}x &= -\sqrt{1-x^2} + \arcsin(x) + C
\end{aligned}$$

Ejercicio 13

Determinar si las siguientes integrales impropias convergen y en tal caso calcularlas.

a) $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt{s+1}}\,\mathrm{d}s$

Para empezar, la «función» es continua en $(0, +\infty)$

$$\begin{aligned}\int_0^{+\infty} \frac{1}{\sqrt{s+1}}\,\mathrm{d}s &= \lim_{t\to+\infty} \int_0^t \frac{1}{\sqrt{s+1}}\,\mathrm{d}s \\ &= \lim_{t\to+\infty} \int_0^t (s+1)^{-\frac{1}{2}}\,\mathrm{d}s \qquad u = s+1 \to \mathrm{d}u = \mathrm{d}s \\ &= \lim_{t\to+\infty} \left(2\sqrt{t+1}\Big|_0^t\right) \\ &= \lim_{t\to+\infty} \left(2\sqrt{t+1} - 2\sqrt{1}\right) \\ &= \lim_{t\to+\infty} \left(2\sqrt{t+1} - 2\right) \\ &= 2\underbrace{\lim_{t\to+\infty} \left(\sqrt{t+1}\right)}_{\to+\infty} - 2 \end{aligned}$$

$$\int_0^{+\infty} \frac{1}{\sqrt{s+1}}\,\mathrm{d}s = +\infty$$

Luego la integral diverge.

b) $\displaystyle\int_0^{2} \frac{1}{(1-y)^{\frac{2}{3}}}\,\mathrm{d}y$

Lo primero que tenemos que notar es que la función no es continua en (0,2). Es por esto que parto la integral en dos partes.

$$\int_0^2 \frac{1}{(1-y)^{\frac{2}{3}}}\,\mathrm{d}y = \int_0^1 \frac{1}{(1-y)^{\frac{2}{3}}}\,\mathrm{d}y + \int_1^2 \frac{1}{(1-y)^{\frac{2}{3}}}\,\mathrm{d}y$$

Luego veo cada una de las integrales por separado. Si una de las dos diverge eso me es suficiente.

$$\begin{aligned}\int_0^1 \frac{1}{(1-y)^{\frac{2}{3}}}\,\mathrm{d}y &= \lim_{t\to1^-} \int_0^t \frac{1}{(1-y)^{\frac{2}{3}}}\,\mathrm{d}y \\ &= \lim_{t\to1^-} \int_0^t (1-y)^{-\frac{2}{3}}\,\mathrm{d}y \\ &= -3\sqrt[3]{(1-y)}\Big|_0^1 \end{aligned}$$

$$\int_0^1 \frac{1}{(1-y)^{\frac{2}{3}}}\,\mathrm{d}y = 0 - (-3) = 3$$

$$
\begin{aligned}
\int_1^2 \frac{1}{(1-y)^{\frac{2}{3}}}\,\mathrm{d}y &= \lim_{t\to 1^+} \int_t^2 (1-y)^{-\frac{2}{3}}\,\mathrm{d}y \\
&= \lim_{t\to 1^+} \int_t^2 (1-y)^{-\frac{2}{3}}\,\mathrm{d}y \\
&= -3\sqrt[3]{(1-y)}\Big|_1^2 \\
\int_1^2 \frac{1}{(1-y)^{\frac{2}{3}}}\,\mathrm{d}y &= 3-0=3
\end{aligned}
$$

Finalmente:

$$
\begin{aligned}
&\int_0^2 \frac{1}{(1-y)^{\frac{2}{3}}}\,\mathrm{d}y = \int_0^1 \frac{1}{(1-y)^{\frac{2}{3}}}\,\mathrm{d}y + \int_1^2 \frac{1}{(1-y)^{\frac{2}{3}}}\,\mathrm{d}y \\
&\int_0^2 \frac{1}{(1-y)^{\frac{2}{3}}}\,\mathrm{d}y = 3+3 \\
&\int_0^2 \frac{1}{(1-y)^{\frac{2}{3}}}\,\mathrm{d}y = 6
\end{aligned}
$$

Luego la integral converge.

c) $\displaystyle\int_{-\infty}^{0} xe^{-x^2}\,\mathrm{d}x$

$$\begin{aligned}
\int_{-\infty}^{0} xe^{-x^2}\,\mathrm{d}x &= \lim_{t\to-\infty}\left(\int_{t}^{0} xe^{-x^2}\,\mathrm{d}x\right) \\
u = -x^2 &\to \mathrm{d}u = -2x\,\mathrm{d}x \\
&= \lim_{t\to-\infty}\left(-\frac{1}{2}\int_{-t^2}^{0} e^{u}\,\mathrm{d}u\right) \\
&= -\frac{1}{2}\lim_{t\to-\infty}\left(\int_{-t^2}^{0} e^{u}\,\mathrm{d}u\right) \\
&= -\frac{1}{2}\lim_{t\to-\infty}\left(e^{u}\Big|_{-t^2}^{0}\right) \\
&\text{Deshago la sustitución} \\
&= -\frac{1}{2}\lim_{t\to-\infty}\left(e^{-x^2}\Big|_{t}^{0}\right) \\
&= -\frac{1}{2}\lim_{t\to-\infty}\left(e^{0}-e^{-t^2}\right) \\
&= -\frac{1}{2}\lim_{t\to-\infty}\left(1-e^{-t^2}\right) \\
&= -\frac{1}{2}-\frac{1}{2}\lim_{t\to-\infty}\left(e^{-t^2}\right) \\
t^2\to\infty \Rightarrow -t^2 &\to -\infty \Rightarrow e^{-t^2}\to 0 \\
&= -\frac{1}{2}-\frac{1}{2}0 \\
\int_{-\infty}^{0} xe^{-x^2}\,\mathrm{d}x &= -\frac{1}{2}
\end{aligned}$$

Por lo tanto la integral converge.

Ejercicio 14

Determinar si cada una de las siguientes integrales impropias converge o no.

Nota: Creo que se supone que use los criterios de comparación para integrales impropias.

a) $\displaystyle\int_{4}^{+\infty} \frac{1}{\sqrt{s}-1}\,\mathrm{d}s$

$$\sqrt{s} > \sqrt{s}-1 \Rightarrow \underbrace{\frac{1}{\sqrt{s}}\cdot\frac{1}{\sqrt{s}-1}\cdot\sqrt{s} > \left(\sqrt{s}-1\right)\frac{1}{\sqrt{s}}\cdot\frac{1}{\sqrt{s}-1}}_{\text{Verdadero ya que. } s\in[4,+\infty) \text{ y compatibilidad del prod. con el orden}} \Rightarrow \frac{1}{\sqrt{s}-1} > \frac{1}{\sqrt{s}}$$

Ahora basta con ver que si $\int_4^{+\infty}\frac{1}{\sqrt{s}}\,\mathrm{d}s$ diverge o no.

$$\int_4^{+\infty} \frac{1}{\sqrt{s}} \,\mathrm{d}s = \lim_{t \to +\infty} \int_4^t \frac{1}{\sqrt{s}} \,\mathrm{d}s = \lim_{t \to +\infty} \int_4^t s^{-\frac{1}{2}} \,\mathrm{d}s$$

$$\int_4^{+\infty} \frac{1}{\sqrt{s}} \,\mathrm{d}s = \lim_{t \to +\infty} 2\sqrt{s}\Big|_4^t$$

$$\int_4^{+\infty} \frac{1}{\sqrt{s}} \,\mathrm{d}s = \lim_{t \to +\infty} \left(2\sqrt{t} - 2\sqrt{4}\right)$$

$$\int_4^{+\infty} \frac{1}{\sqrt{s}} \,\mathrm{d}s = \lim_{t \to +\infty} \left(2\sqrt{t} - 4\right)$$

$$\int_4^{+\infty} \frac{1}{\sqrt{s}} \,\mathrm{d}s = \lim_{t \to +\infty} \underbrace{\left(2\sqrt{t}\right)}_{2\sqrt{t} \to +\infty} - 4$$

$$\int_4^{+\infty} \frac{1}{\sqrt{s}} \,\mathrm{d}s = +\infty$$

Luego la integral $\int_4^{+\infty} \frac{1}{\sqrt{s}} \,\mathrm{d}s$ diverge y por ende $\int_4^{+\infty} \frac{1}{\sqrt{s}-1} \,\mathrm{d}s$ tambien.

b) $\displaystyle\int_0^4 \frac{\mathrm{d}x}{(x-3)^{\frac{2}{3}}}$

c) $\displaystyle\int_0^4 \frac{\mathrm{d}x}{x^2 - x - 2}$