F : Tree Game

Suppose that Takahashi initially puts the piece on vertex r. Let r be the root of the tree. For each vertex v, we define $state(v)$ as follows:

Consider a subtree rooted at v. The two players play the game using this subtree (they are not allowed to move the piece out of the subtree), and initially the piece is at vertex v. If the first player can win in this game, define $state(v) = W$. Otherwise $state(v) = L$.

Note that $state(r)$ gives the result of the entire game when the piece is initially at r.

We claim the following:

1. If there exists a child c of v such that $A_c < A_v$ and $state(c) = L$, then $state(v) = W$.
2. Otherwise, $state(v) = L$. (In particular, if v is a leaf, $state(v) = L$.)

The proof of 1. Suppose that this is your turn and the piece is currently at v. First you take a stone from v (this is possible because A_v is not zero) and move the piece to c. Whenever the opponent tries to move the piece from c to v, you can refuse to do that by moveing it back to c (this is possible because $A_v > A_c$). This way, your opponent is forced to play the game within the subtree rooted at c. Since $state(c) = L$, your opponent loses and you win. (Strictly speaking, your opponent can also reduce the value of A_c by moving the piece between c and v back and forth, but this doesn't change the state of vertex c.)

The proof of 2. If v is a leaf, you can't move the piece and you lose. Otherwise you can move the token from v to one of v's children, w. There are two cases: $state(w) = W$ or $A_w \geq A_v$. If $state(w) = W$, your opponent never moves the piece back to v and play the rest of the game completely within the subtree rooted at w. Since $state(w) = W$, this way your opponent wins and you lose. If $A_w \geq A_v$, your opponent refuses to move the piece from v to c by moving it back to v (this is possible because $A_w \geq A_v$). Thus, you lose in this case.

This way, for a fixed position of the initial piece, we can solve the problem in $O(N)$. In total the complexity of this solution is $O(N^2)$. (Exercise: can you improve it to $O(N)$?)